A Tale of a Boundary-Layer Problem

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Abstract

I will demonstrate the solution with all its steps (and missteps) to the boundary layer problem

$$\varepsilon y'' = y \cdot y' - y, \tag{1}$$

such that

$$y(0) = 1 \quad \text{and} \quad y(1) = -1 \quad \text{and where} \quad \varepsilon \ll 1. \tag{2}$$

The journey towards the solution will take us on an excurse through some basics of *perturbation theory* and *asymptotic approximation matching*. I will demonstrate visualizations that will give you insight into the reasoning behind the steps taken towards the solution serving as an exhibition for the leverage in problem solving accessible to us through the computational power at our fingertips.

Outline

Classifying the problem

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first-degree highest degree of the unknown function is 1

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It is *non-linear* !

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How did we manage to solve these equations?

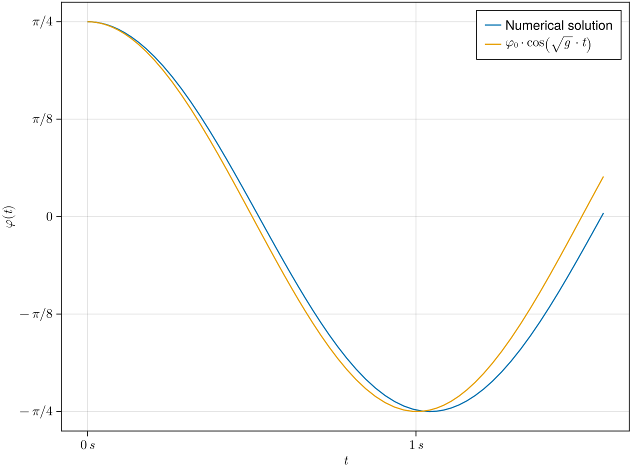
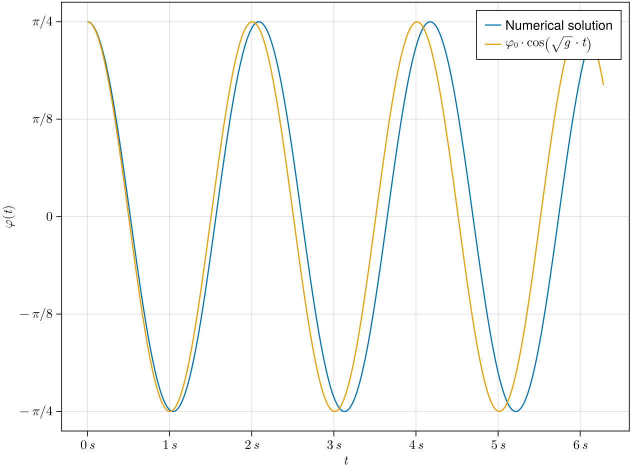


Figure: Comparison of the numerical solution and the small angle approximation of the simple pendulum DE for the initial angle $\varphi(0) = \pi/4$ and $\dot{\varphi}(0) = 0$.



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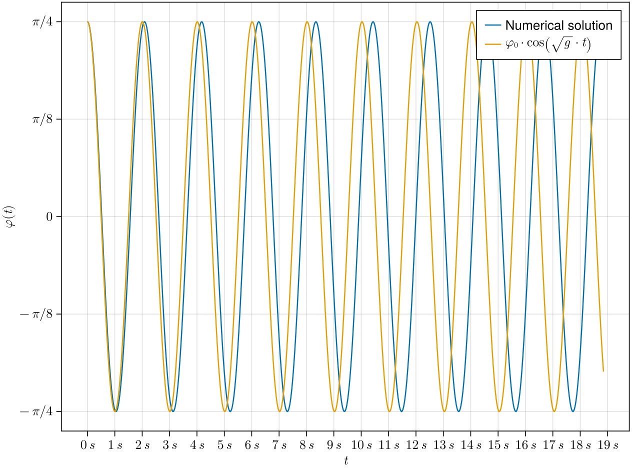

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singularly-perturbed contains one small term

How good is our numerical solution?

Note (Analytical solution - Salas et al. 2014)

There is an analytical solution to the pendulum DE in the form

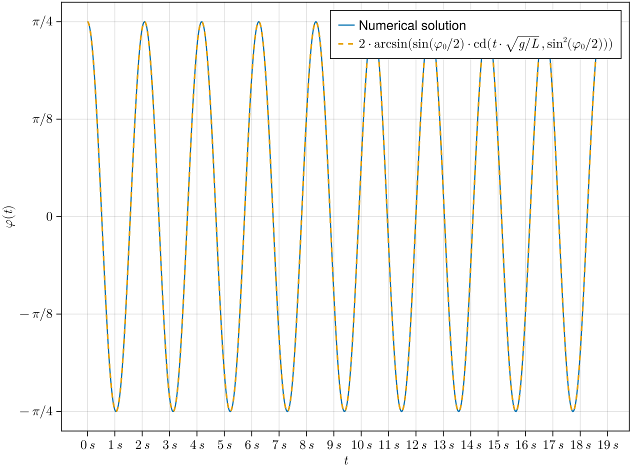
$$\varphi(t) = 2 \cdot \arcsin\Big(\sin(\varphi_0/2) \cdot \mathrm{cd}(t \cdot \sqrt{g/L},\, \sin^2(\varphi_0/2))\Big), \text{ where}$$

$$\mathrm{cd}(u,m) = \mathrm{cn}(u,m)/\,\mathrm{dn}(u,m)$$

$$\mathrm{cn}(u,m) = \cos(\mathrm{am}(u,m)) \quad \mathrm{dn}(u,m) = \frac{\mathrm{d}}{\mathrm{d}u}(\mathrm{am}(u,m))$$

and $\mathrm{am}(u,m) = \varphi$ *is the Jacobi amplitude which is the upper-bound of the integral such that*

$$u = \int_0^{\varphi} \frac{\mathrm{d}\vartheta}{\sqrt{1 - m\sin^2(\vartheta)}}.$$



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- Numerical solution is not guaranteed to be correct
- *Size of the latent space for the solution!* *(we cannot search the whole space and find all the solutions numerically)*

How precise is a perturbation solution?

Note (Perturbation solution - Beléndez et al. 2006)

There is an approximation to the simple pendulum problem based on a homotopy perturbation method. It is uniformly convergent and is precise from the first few terms of the expansion.

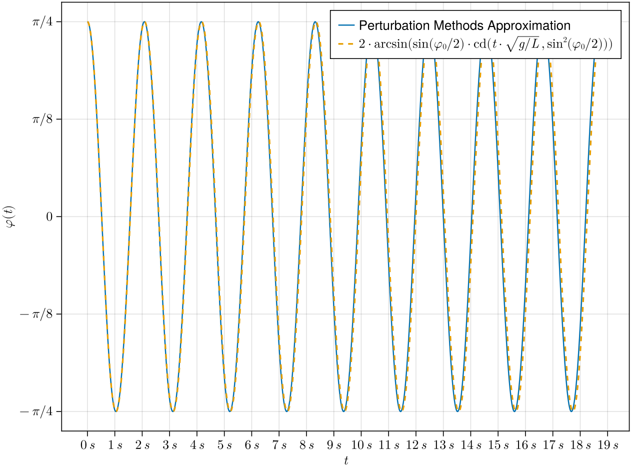

Figure: Comparison of the perturbation solution and the analytical solution of the simple pendulum DE for the initial angle $\varphi(0) = \pi/4$ and $\dot{\varphi}(0) = 0$.

2 Theory
- Mathematical Basics
- Perturbation Theory
- Boundary Layers

Theorem (Taylor's theorem)

Given a function $f(\varepsilon)$, suppose its $(n+1)$st derivative $f^{(n+1)}$ is continuous for $\varepsilon_a < \varepsilon < \varepsilon_b$. Suppose that the points $\varepsilon, \varepsilon_0 \in (\varepsilon_a, \varepsilon_b)$. Then

$$f(\varepsilon) = f(\varepsilon_0) + (\varepsilon - \varepsilon_0) f'(\varepsilon_0) + \ldots + \frac{1}{n!}(\varepsilon - \varepsilon_0)^n f^{(n)}(\varepsilon_0) + R_{n+1},$$

where the error term R_{n+1} is given by

$$R_{n+1} = \frac{1}{(n+1)!}(\varepsilon - \varepsilon_0)^{n+1} f^{(n+1)}(\xi),$$

and $\xi \in]\varepsilon, \varepsilon_0[$

Definition (Bachmann-Landau Symbols - Holmes 2012)

1. $f = O(\varphi)$ as $\varepsilon \to \varepsilon_0^+$ if there is a constant k and $\tilde{\varepsilon}$ so that

$$|f(\varepsilon)| \leq k\,|\varphi(\varepsilon)| \quad \text{for} \quad \varepsilon_0 < \varepsilon < \tilde{\varepsilon}$$

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2. $f = o(\varphi)$ as $\varepsilon \to \varepsilon_0^+$ if $\forall \delta > 0$ there is a $\tilde{\varepsilon}$ so that

$$|f(\varepsilon)| \leq \delta\,|\varphi(\varepsilon)| \quad \text{for} \quad \varepsilon_0 < \varepsilon < \tilde{\varepsilon}$$

Definition (Asymptotic approximation - Holmes 2012)

Given a function $f(\varepsilon)$ and $\varphi(\varepsilon)$ we say that $\varphi(\varepsilon)$ is an *asymptotic approximation* to $f(\varepsilon)$ as $\varepsilon \to \varepsilon_0^+$ whenever $f = \varphi + o(\varphi)$ as $\varepsilon \to \varepsilon_0^+$. We write $f \sim \varphi$ as $\varepsilon \to \varepsilon_0^+$.

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1. The sequence of functions $\varphi_1(\varepsilon), \varphi_2(\varepsilon), \ldots$ form an *asymptotic sequence* (are *well ordered*) as $\varepsilon \to \varepsilon_0^+$ if $\varphi_{m+1} = o(\varphi_m)$ as $\varepsilon \to \varepsilon_0^+$ for $\forall n$.

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2. Let $\varphi_1(\varepsilon), \varphi_2(\varepsilon), \ldots$ be basis functions. Then $f(\varepsilon)$ has an *asymptotic expansion* to n terms for this sequence if

$$f = \sum_{k=1}^{m} a_k \varphi_k + o(\varphi_m) \quad \text{for} \quad m \in \hat{n} \quad \text{as} \quad \varepsilon \to \varepsilon_0^+,$$

where a_k are independent of ε. We can write

$$f \sim a_1\varphi_1(\varepsilon) + a_2\varphi_2(\varepsilon) + \ldots + a_n\varphi_n(\varepsilon) \quad \text{as} \quad \varepsilon \to \varepsilon_0^+.$$

Asymptotic Solution of the Projectile Problem

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\frac{g \cdot R^2}{(x+R)^2} \quad \text{for} \quad 0 < t.$$

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Doing so transforms the problem into

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\tau^2} = -\frac{1}{(1+\varepsilon y)^2} \quad \text{for} \quad 0 < \tau,$$

where

$$y(0) = 0 \quad \text{and} \quad y'(0) = 1 \quad \text{and} \quad \varepsilon = v_0^2 / Rg.$$

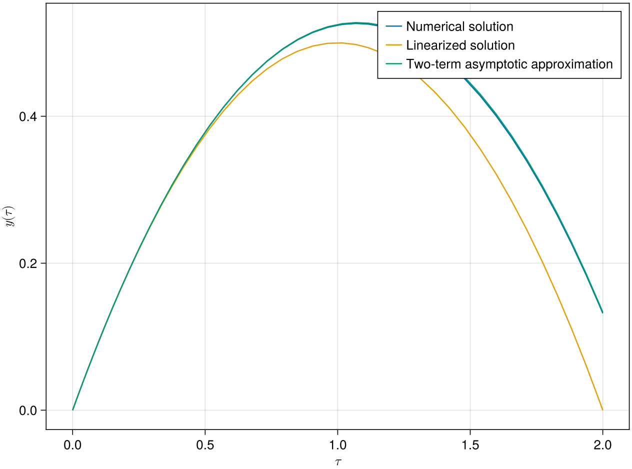

Figure: Comparison of the two term approximation, the linearised problem (one-term approximation) and the numerical solution for $\varepsilon = 1/10$

Boundary Layer

A boundary layer in lay terms is an interval of the solution of a DE where the derivative is large compared with surrounding times.

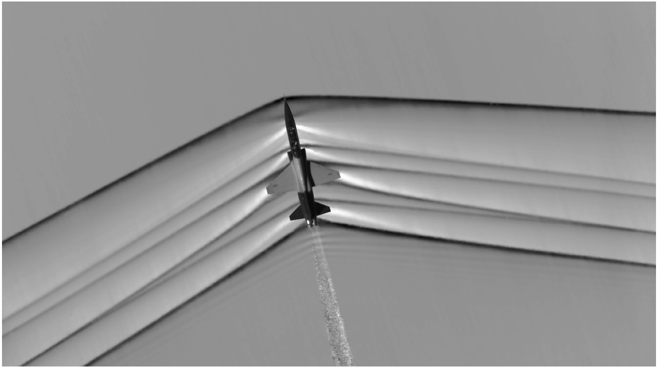

Figure: Shlieren image of the air waves of a supersonic airflow produced by a fighter jet *[Source: NASA (2015)]*

Plausibility Argument of no Border Boundary Layers - Holmes 2012

If we are looking for a boundary layer at $x = 0$, then $y' < 0$ and if we assume that $y'' > 0$, then $\varepsilon y'' > 0$, but $y \cdot (y' - 1)$ on the left hand side is both > 0 on some interval and < 0. Therefore we can rule out a boundary layers at the interval borders for $y'' > 0$.

Is there any symmetry in the DE?

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$$(x, y) \to (1 - x, -y)$$

Phase Portrait of the DE

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Phase Portrait of the DE

$$
\begin{aligned}
y'(t) &= z(t) \\
z'(t) &= \frac{1}{\varepsilon} \cdot y(t) \cdot (z(t) - 1)
\end{aligned}
$$

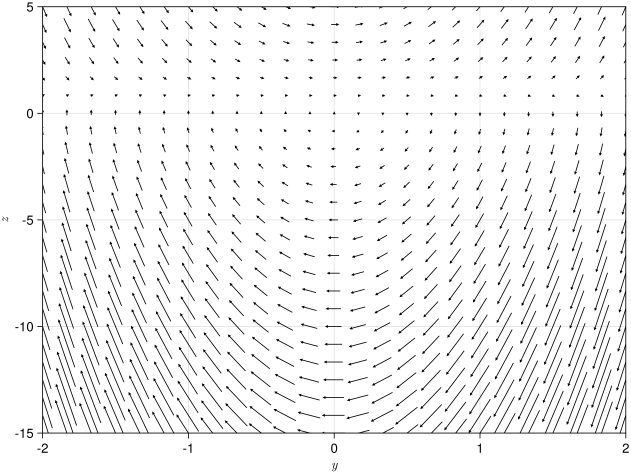

Figure: Arrows representing the vector field of our system of differential equations

Phase Portrait

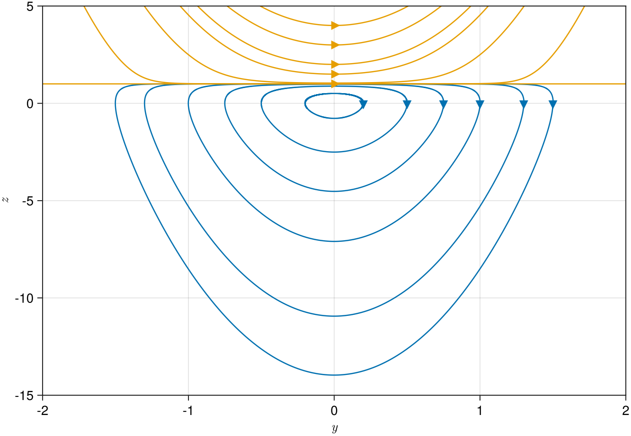

Figure: Trajectories of imaginary particles in the phase portrait of our system of differential equations for $\varepsilon = 0.1$

Phase Portrait

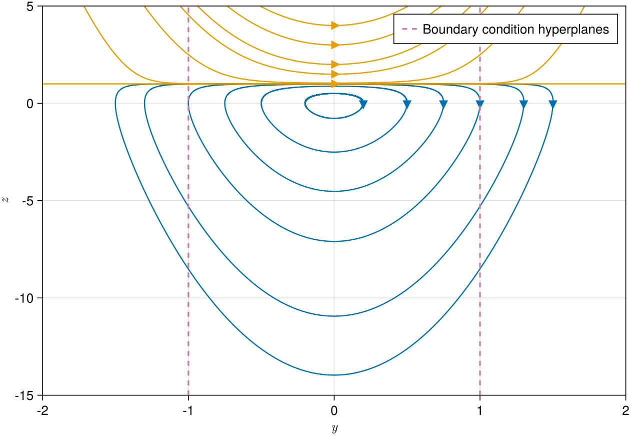

Figure: Trajectories of imaginary particles in the phase portrait of our system of differential equations for $\varepsilon = 1$ with the boundary condition hyperplanes

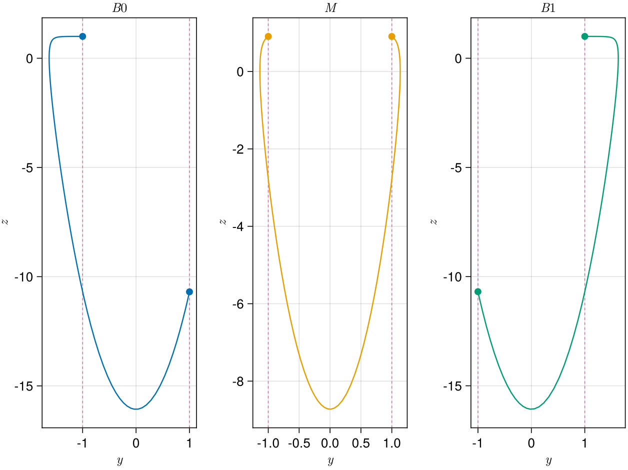

Figure: Solutions B_0, M and B_1 in the phase plane of the DE

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$$y_0^L(x) = 1 + x \quad \text{and} \quad y_0^R(x) = x - 2$$

Scaling

For the boundaries, we have to scale the solutions as

$$X \equiv \frac{x - x_0}{\delta},$$

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$$\frac{\mathrm{d}^2 Y}{\mathrm{d}X^2} = Y \frac{\mathrm{d}\,Y}{\mathrm{d}X} - \varepsilon\, Y.$$

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When we create these composite solutions, we get

$$\tilde{y}^M = x - \frac{1}{2} - \frac{3}{2} \cdot \tanh\left[\frac{3}{4\varepsilon}(x - \frac{1}{2})\right] + O(\varepsilon)$$

$$\tilde{y}^{B0} = x - 2 \cdot \tanh\left[\frac{x}{\varepsilon} - \tanh^{-1}(1/2)\right] + O(\varepsilon)$$

$$\tilde{y}^{B1} = x - 1 - 2 \cdot \tanh\left[\frac{x-1}{\varepsilon} + \tanh^{-1}(1/2)\right] + O(\varepsilon)$$

add the final solution figure and compare it with the numerical solution

4 Interactive Solution

References

- Beléndez, Augusto et al. (2006). "Application of the homotopy perturbation method to the non-linear pendulum". In: *European Journal of Physics* 28.1, p. 93.
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