Matrix Calculus and Optimization

Statistics 202B

Professor: Mark S. Handcock

Homework 2

Due Tuesday, January 26, 2016

1) Show that, if $\mathbf{X}$ is orthonormal and $n \times m$, then the projector $\mathbf{P} = \mathbf{X}\mathbf{X}^{\mathbf{T}}$ satisfies $\mathrm{tr}(\mathbf{P}) = m$.

2) Let $\mathbf{A}$ be an $n \times n$ symmetric matrix, and let $d_1, d_2, \ldots, d_n$ represent the (not necessarily distinct) eigenvalues of $\mathbf{A}$. Show that

$$\lim_{k \to \infty} \mathbf{A}^k = \mathbf{0}$$

if, and only if, $|d_i| < 1$ for $i = 1, \ldots, n$.

3) For symmetric matrices $\mathbf{A}$ and $\mathbf{B}$, define $\mathbf{A} \triangleright \mathbf{0}$ to mean that $\mathbf{A}$ is positive semi-definite and $\mathbf{A} \triangleright \mathbf{B}$ to mean that $\mathbf{A} - \mathbf{B} \triangleright \mathbf{0}$.

a) Show that $\mathbf{A} \triangleright \mathbf{B}$ and $\mathbf{B} \triangleright \mathbf{C}$ imply that $\mathbf{A} \triangleright \mathbf{C}$.

b) Show that $\mathbf{A} \triangleright \mathbf{B}$ and $\mathbf{B} \triangleright \mathbf{A}$ imply that $\mathbf{A} = \mathbf{B}$.

Thus, $\triangleright$ induces a partial order on the set of symmetric matrices.

4) In the notation of the last question, suppose that $\mathbf{A}$ and $\mathbf{B}$ are positive definite matrices.

a) Show that $\mathbf{A} \triangleright \mathbf{B}$ if, and only if, $\mathbf{B}^{-1} \triangleright \mathbf{A}^{-1}$.

b) Show that $\mathbf{A} \triangleright \mathbf{B}$ implies that $\det\mathbf{A} \geq \det\mathbf{B}$ and $\mathrm{tr}\mathbf{A} \geq \mathrm{tr}\mathbf{B}$.

5) Denote by $X \sim N_n(\mu, \Sigma)$ the statement that the $n-$ random vector X is multivariate Gaussian with mean μ and $n \times n$ covariance matrix Σ. Show that $X \sim N_n(\mu, \Sigma)$ if and only if $\alpha^T X \sim N_n(\alpha^T \mu, \alpha^T \Sigma \alpha)$ for all $n-$ vectors α.

6) Question 5.5 in Gentle (2007), page 199.

7) Question 5.9, part a) only in Gentle (2007), page 200.

8) Let $\mathbf{A}$ be a square matrix with singular value decomposition

$$\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^T.$$

Verify the (spectral) decomposition:

$$\begin{bmatrix} \mathbf{0} & \mathbf{A}^T \\ \mathbf{A} & \mathbf{0} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{V} & \mathbf{V} \\ \mathbf{U} & -\mathbf{U} \end{bmatrix} \begin{bmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & -\mathbf{D} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{V}^T & \mathbf{U}^T \\ \mathbf{V}^T & -\mathbf{U}^T \end{bmatrix}$$

Thus, any algorithm producing a spectral decomposition will produce an singular value decomposition of $\mathbf{A}$ without computing $\mathbf{A}\mathbf{A}^T$ or $\mathbf{A}^T\mathbf{A}$.

9) Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 10 & 7 & 8 & 7 \\ 7 & 6 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{bmatrix}$$

a) Use `R` to find the eigenvalues and eigenvectors of $\mathbf{A}$.

b) Use `R` and the iterative power method to find the largest eigenvalue and eigenvector of $\mathbf{A}$ to an accuracy of 0.001. How many iterations were required?

c) Use `R` to find the singular value decomposition of $\mathbf{A}$.

d) Use `R` to find the singular value decomposition of the first three rows of $\mathbf{A}$.