1)

$$P(X_S^1=1|X_{\partial S}^1)=\frac{e^{\beta\sum_{<1,t>}1(1=X_t^1)}}{e^{\beta\sum_{<1,t>}1(1=X_t^1)}+e^{\beta\sum_{<1,t>}1(0=X_t^1)}}$$
$$P(X_S^2=1|X_{\partial S}^2)=\frac{e^{\beta\sum_{<1,t>}1(1=X_t^2)}}{e^{\beta\sum_{<1,t>}1(1=X_t^2)}+e^{\beta\sum_{<1,t>}1(0=X_t^2)}}$$

We generate a uniform random variable r from $[0,1]$. If $0<r\le P(X_S^1=1|X_{\partial S}^1)$, $X_S^1=1$ and if $P(X_S^1=1|X_{\partial S}^1)<r\le 1$, $X_S^1=0$. If $0<r\le P(X_S^2=1|X_{\partial S}^2)$, $X_S^2=1$ and if $P(X_S^2=1|X_{\partial S}^2)<r\le 1$, $X_S^2=0$.Now we prove Q1 by induction. Initially we know $X_S^1\ge X_S^2$ for every site.

Assume after n states, it's still true that $X_S^1\ge X_S^2$. For state $n+1$, we need to prove there is no case that $X_S^1=0$ and $X_S^2=1$. If $X_S^1=0$ and $X_S^2=1$, $r>P(X_S^1=1|X_{\partial S}^1)=\frac{e^{\beta\sum_{<1,t>}1(1=X_t^1)}}{e^{\beta\sum_{<1,t>}1(1=X_t^1)}+e^{\beta\sum_{<1,t>}1(0=X_t^1)}}$ and $r\le P(X_S^2=1|X_{\partial S}^2)=\frac{e^{\beta\sum_{<1,t>}1(1=X_t^2)}}{e^{\beta\sum_{<1,t>}1(1=X_t^2)}+e^{\beta\sum_{<1,t>}1(0=X_t^2)}}$. So if we can prove $P(X_S^1=1|X_{\partial S}^1)\ge P(X_S^2=1|X_{\partial S}^2)$, $X_S^1=0$ and $X_S^2=1$ cannot happen simultaneously.

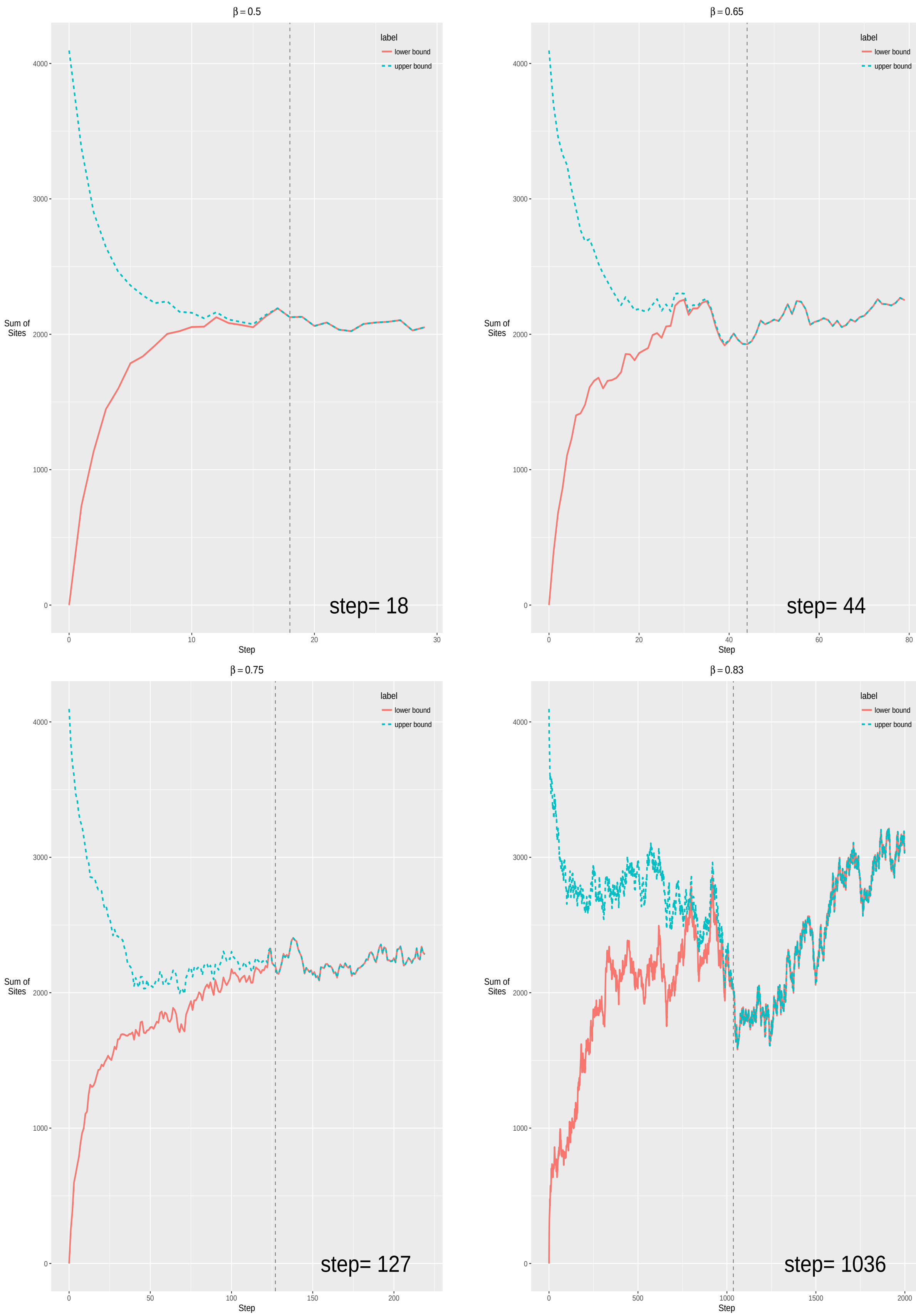
$$\frac{e^{\beta\sum_{<1,t>}1(1=X_t^1)}}{e^{\beta\sum_{<1,t>}1(1=X_t^1)}+e^{\beta\sum_{<1,t>}1(0=X_t^1)}}>\frac{e^{\beta\sum_{<1,t>}1(1=X_t^2)}}{e^{\beta\sum_{<1,t>}1(1=X_t^2)}+e^{\beta\sum_{<1,t>}1(0=X_t^2)}}\Rightarrow$$
$$e^{\beta\sum 1(1=X_t^1)+\beta\sum 1(1=X_t^2)}+e^{\beta\sum 1(1=X_t^1)+\beta\sum 1(0=X_t^2)}\ge$$
$$e^{\beta\sum 1(1=X_t^1)+\beta\sum 1(1=X_t^2)}+e^{\beta\sum 1(0=X_t^1)+\beta\sum 1(1=X_t^2)}\Rightarrow$$
$$\sum 1(1=X_t^1)+\sum 1(0=X_t^2)\ge\sum 1(0=X_t^1)+\sum 1(1=X_t^2)\quad(*)$$

$$(*)=\begin{cases}X_t^1=1,X_t^2=1 & 1(1=X_t^1)+1(0=X_t^2)=1\ge 1=1(0=X_t^1)+1(1=X_t^2)\\ X_t^1=1,X_t^2=0 & 1(1=X_t^1)+1(0=X_t^2)=2\ge 0=1(0=X_t^1)+1(1=X_t^2)\\ X_t^1=0,X_t^2=0 & 1(1=X_t^1)+1(0=X_t^2)=1\ge 1=1(0=X_t^1)+1(1=X_t^2)\\ X_t^1=0,X_t^2=1 & \text{This case won't happen based on n-state induction}\end{cases}$$

Now, the proof is finished.

2)

I attach R code and C++ code on CCLE.


β = 0.5
label
lower bound
upper bound
Sum of Sites
Step
step= 18
β = 0.65
step= 44
β = 0.75
step= 127
β = 0.83
step= 1036


β = 0.84
label
lower bound
upper bound
Sum of Sites
step= 1729
Step
β = 0.85
label
lower bound
upper bound
Sum of Sites
step= 2935
Step
β = 0.9
label
lower bound
upper bound
Sum of Sites
Step
β = 1
label
lower bound
upper bound
Sum of Sites
Step

3)
For the second graph, we scale y-axis with log function. So even the increasing rate of *tau* increases exponentially with respect to *beta*.