PDEs – Astrophysical Fluid Dynamics

Lecture 11, Computational Astrophysics (ASTR660)

Hsiang-Yi Karen Yang, NTHU, 11/24/2022

Announcements

- *HW4* is due TODAY. Late submission within one week will receive *75%* of the credit
- Please help to provide comments/feedback to improve this course! The questionnaire can be found via this link or by scanning this QR code:

Previous lecture…

- *Partial differential equations (PDEs)* involve partial derivatives w.r.t more than one independent variable
- Three types of 2nd-order PDEs:
  - *Hyperbolic*: time-dependent systems, conservative physical processes (e.g., advection, hydro), not evolving toward steady state, IVP+BVP
  - *Parabolic*: time-dependent systems, dissipative physical processes (e.g., diffusion), are evolving toward steady state, IVP+BVP
  - *Elliptic*: time-independent systems (e.g., gravitational potential for a point mass), already reached steady state, BVP

Previous lecture...

- Two ways to discretize PDEs:
  - *Finite-difference* methods – replacing derivatives using finite-difference approximations at fixed *mesh points*
  - *Finite-volume* methods – solving *cell-averaged* quantities on *grid cells*, key is to approximate *fluxes* at cell edges
- For time-dependent PDEs, stability criterion can be obtained using the von Neumann stability analysis
- *CFL criterion*: Courant number $C \equiv \frac{c\Delta t}{\Delta x} \leq C_{max} \approx 1$
- Commonly used BCs: periodic, outflow, and reflect
- *Operator splitting* is often used for multi-D simulations

This lecture…

- In-class exercise -- finite-volume method
- Crash course on astrophysical fluid dynamics
  - Introduction to astrophysical fluids
  - Hydrodynamical equations
  - Fluid properties & phenomena

Finite-volume methods for PDEs

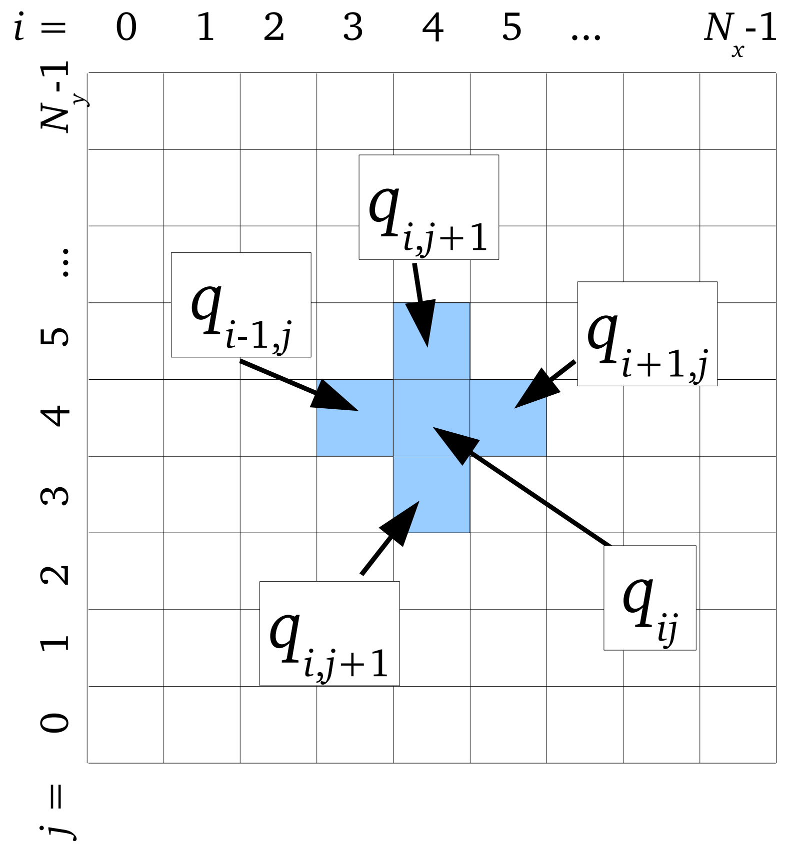
Finite-volume methods

- This method is widely used *grid-based* hydro codes
- Mesh points -> *grid cells*
- $q_i^n = q(t = t_n, x = x_i)$ now represents *cell-averaged* values instead of cell-centered values:

$$q_{ij} \equiv \frac{1}{\Delta x \Delta y} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} q(x, y)\, dx\, dy$$

- We can write down a general *conservation* law:

$$\frac{\partial q}{\partial t} + \nabla \cdot \boldsymbol{F}(q, t) = 0$$



Finite-volume methods

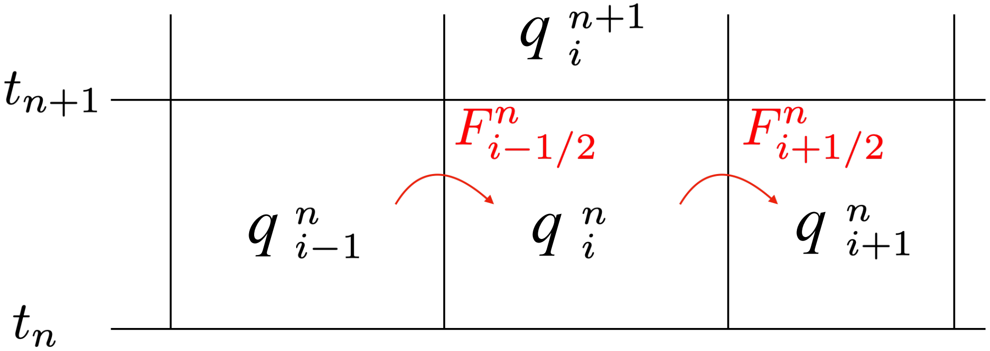
- Cell values can be updated by evaluating ***fluxes*** at cell edges (in 1D):

★ *Step 2*

$$\boxed{q_i^{n+1} = q_i^{n} - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^{n} - F_{i-1/2}^{n}\right)}$$

Recall

$$\boxed{\frac{\partial q}{\partial t} + \nabla \cdot \boldsymbol{F}(q,t) = 0}$$



- The above equation is exact. So the problem becomes how to ★ ***approximate the fluxes at cell edges!!***

Step 1

Example: advection equation $\boxed{u_t = -cu_x}$

- Rewrite the advection equation in the conservative form $\dfrac{\partial q}{\partial t} + \nabla \cdot \boldsymbol{F}(q,t) = 0$

  then $\boldsymbol{q = u, F(q,t) = cu}$

- Now use the *finite-volume* method: $q_i^{n+1} = q_i^{n} - \dfrac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right)$

- The simplest approach to approximate fluxes at cell edges as simple *arithmetic averages*:

$$F_{i-1/2}^n = \frac{1}{2}[F(q_{i-1}^n) + F(q_i^n)]$$

-> This method is unstable!

Algorithms for evaluating the fluxes

Lax-Friedrichs method:

$$F_{i-1/2}^{n} = \frac{1}{2}[F(q_{i-1}^{n}) + F(q_{i}^{n})] - \frac{\Delta x}{2\Delta t}(q_{i}^{n} - q_{i-1}^{n})$$

Upwind method

$$F_{i-1/2}^{n} = F(q_{i-1}^{n})$$

Lax-Wendroff method:

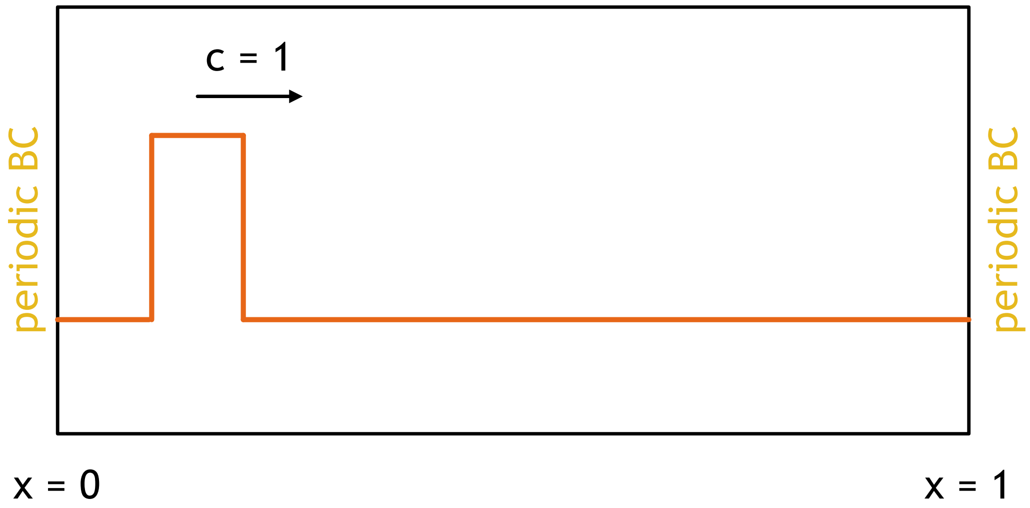
$$q_{i-1/2}^{n+1/2} = \frac{1}{2}(q_{i}^{n} + q_{i-1}^{n}) - \frac{\Delta t}{2\Delta x}[F(q_{i}^{n}) - F(q_{i-1}^{n})]$$

$$q_{i}^{n+1} = q_{i}^{n} - \frac{\Delta t}{\Delta x}[F\left(q_{i+1/2}^{n+1/2}\right) - F(q_{i-1/2}^{n+1/2})]$$

Exercise #1 – solving advection equation using finite-volume methods

$\boxed{u_t = -cu_x}$

- Let's propagate the top-hat function using the ***finite-volume*** methods



$$\frac{\partial q}{\partial t} + \nabla \cdot \boldsymbol{F}(q,t) = 0$$

Rewrite the advection equation in the conservative form then

$\boldsymbol{q = u, F(q,t) = cu}$

Exercise #1 – solving advection equation using finite-volume methods

- Create a directory fvm under L10exercise:

```
mkdir L10exercise/fvm
```

- Copy the *.f90 files and Makefile you wrote from the fdm directory to fvm
- Modify the update subroutine in evolution.f90 in order to implement the ***finite-volume*** method

evolution.f90 – where the update subroutine & fluxes are evaluated

Update solutions using fluxes at cell edges

This is where the fluxes can be evaluated using different methods

```
subroutine update(time, dt)
    use Simulation_data
    implicit none
    real, intent(in) :: time ,dt
    integer :: i
    real    :: FL, FR

    uold  = u
    do i = istart, iend
        call flux(i,dt,FL,FR)
        u(i) = uold(i) - dt/dx*(FR-FL)
    enddo

end subroutine update

!
! Routine to evaluate flux cell edges
!
subroutine flux(i,dt,FL, FR)
    use Simulation_data
    implicit none
    integer, intent(in) :: i
    real, intent(in)    :: dt
    real, intent(out)   :: FL, FR

    ! Arithmetic average method
    FL = ! TODO
    FR = ! TODO

    ! The Lax-Friedrichs Method
    !FL = ! TODO
    !FR = ! TODO

    ! The upwind method
    !FL = ! TODO
    !FR = ! TODO

    ! The Lax-Wendroff Method
    !FL = ! TODO
    !FR = ! TODO

end subroutine flux
```

Exercise #1 – solving advection equation using finite-volume methods

- Implement the ***arithmetic average*** method by modifying the update and flux subroutines in evolution.f90
- Compile and run the codes by

```
make
```

```
./advection
```

- Use your favorite plotting routine to plot the results and see whether the solution is stable or not

Arithmetic average method

$$\boxed{F_{i-1/2}^{n} = \frac{1}{2}\left[F(q_{i-1}^{n}) + F(q_{i}^{n})\right]}$$

Exercise #1b – solving advection equation using finite-volume methods

- Implement the ***Lax-Friedrichs*** method by modifying the update and flux subroutines in evolution.f90
- Compile and run the codes by

```
make
```

```
./advection
```

- Use your favorite plotting routine to plot the results and see whether the solution is stable or not
- To get the bonus credits, please submit the code and both plots to the TAs by end of today (11/24/2022)

Lax-Friedrichs method

$$\boxed{F^n_{i-1/2} = \frac{1}{2}[F(q^n_{i-1}) + F(q^n_i)] - \frac{\Delta x}{2\Delta t}(q^n_i - q^n_{i-1})}$$

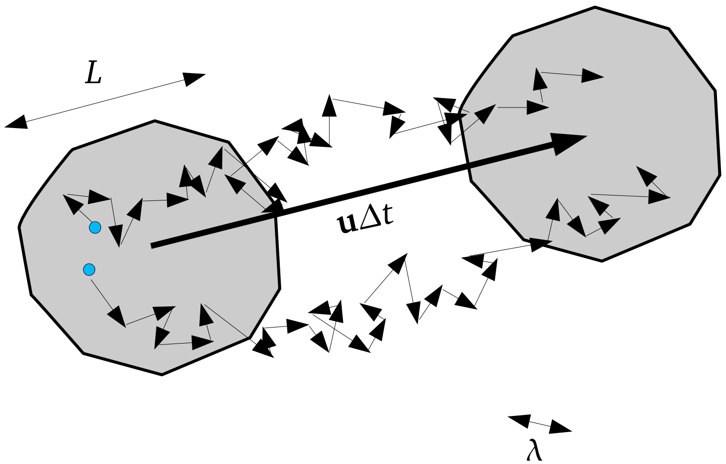
Astrophysical fluids

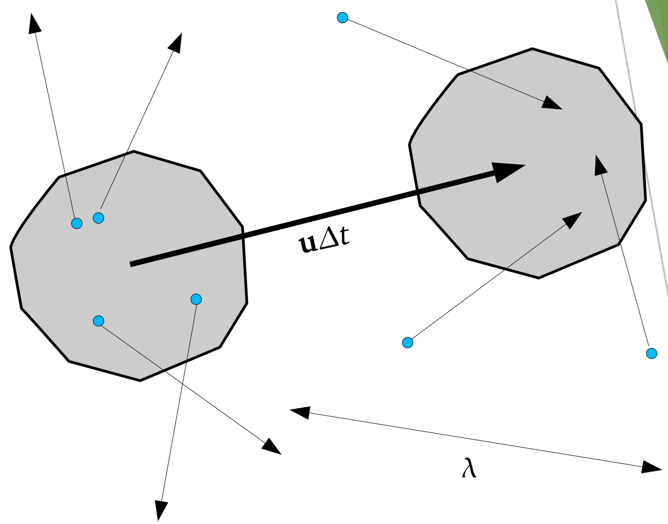
What is a “fluid”?

- A *fluid* is a liquid, gas, or other material that *continuously* deforms/flows under an external force
- The particles contained within a fluid are *collisional*, so that they can be described by some collective properties (e.g., density, pressure, temperature, etc)
- The system is collisional if the *Knudsen number* is small:

$$\boxed{K_n \equiv \frac{\lambda}{L} = \frac{mean\ free\ path}{typical\ scale} \ll 1} \quad \text{or} \quad \boxed{\lambda \ll L}$$

Collisionality of a gas





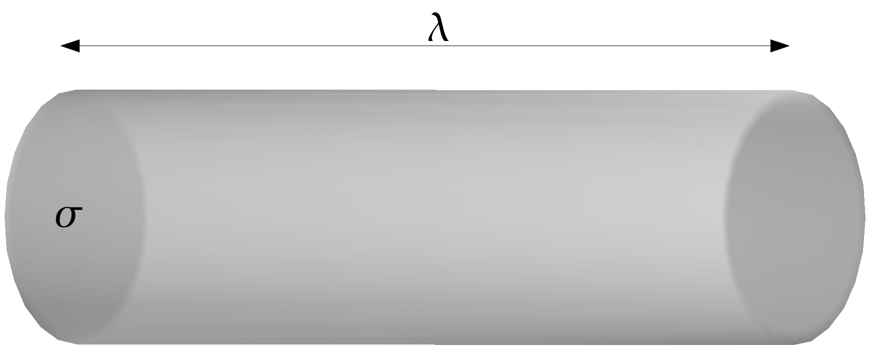
Collisional gas (fluid): Kn << 1

- Mean free path λ << typical scale L
- Random motions do not carry particles far from mean trajectory
- Solve *fluid* equations for motion of fluid elements

Collisionless gas: Kn >> 1

- Mean free path λ >> typical scale L
- Random motions carry particles far from mean trajectory
- Solve *kinetic* equations for motion of particles (or distribution)

Mean free path for neutral particles



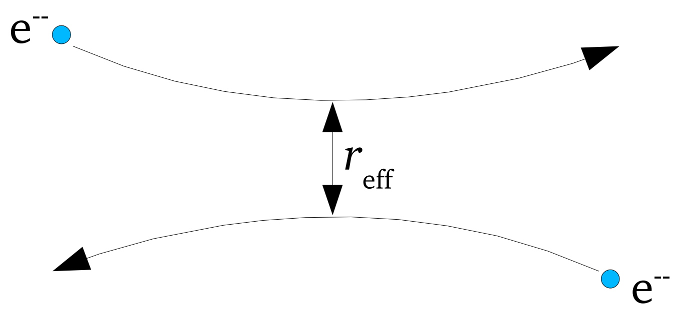
$$\boxed{\lambda = \frac{1}{n\sigma}}$$

Example: air in the room

- $\rho \sim 10^{-3}\ \mathrm{g}/cm^3, \mu \sim 28 m_p, \sigma \sim 3\times10^{-15} cm^2$
- $\lambda = (n\sigma)^{-1} \sim 3\times10^{-5}\mathrm{cm}$

Mean free path for charged particles

- Coulomb interaction is long-range => total cross section is infinite
- Finite cross section can be defined using an ***effective radius for particle scattering***
- Large-angle scattering requires



$$\frac{e^2}{r_{eff}} \sim m_e v_{th}^2 \sim kT$$

$$\Rightarrow \lambda = \frac{1}{n\sigma} \sim \frac{1}{n_e \pi r_{eff}^2} \sim \frac{m_e^2 v_{th}^4}{n_e e^4}$$

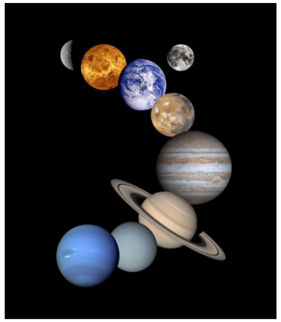
Mean free path for photons

- Under many conditions, the most important interaction for photons is *scattering by free electrons*
- The cross section involved is the *Thomson cross-section*:

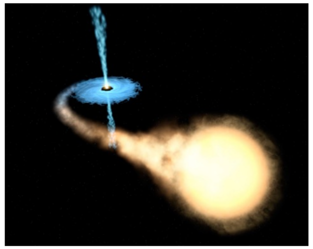
$$\sigma_T \equiv \frac{8\pi}{3} r_e^2 = \frac{8\pi e^4}{3 m_e^2 c^4} = 6.65 \times 10^{-25}\, cm^2$$

- This is generally << matter cross-section except in degenerate matter (e.g., white dwarfs)
- Radiation is generally less efficient than matter in transporting momentum

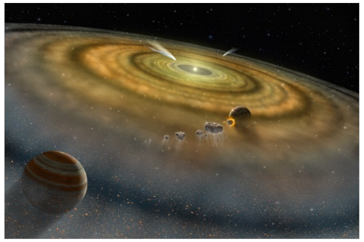
Astrophysical fluids



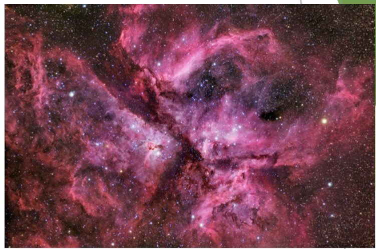
Planets
(~ Earth radius)



Stars/Binaries
(~0.1-1000 AU)



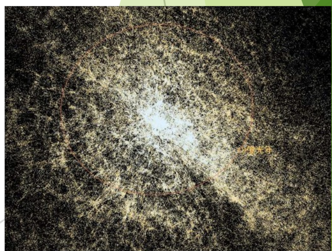
Planetary disk
(~100-1000 AU)



Interstellar medium
(~10-100 pc)

Galaxy
(~50 kpc)

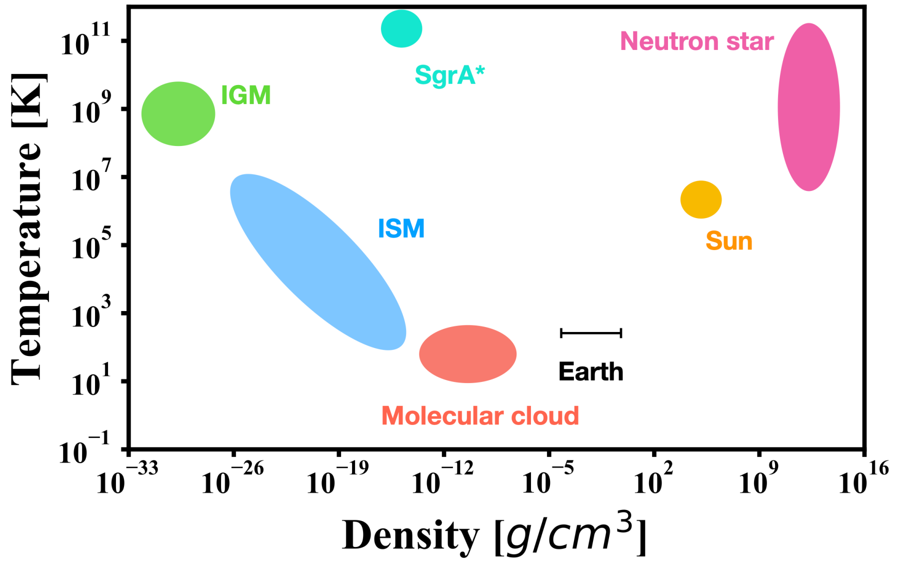
Galaxy clusters
(~1 Mpc)



Large scale structure
(~1 Gpc)

Properties of astrophysical fluids

- Different from Earth-bound fluids
- Have a wide range of physical conditions
- Magnetic fields are often important *-- MHD*
- Compressibility is often important -- $\nabla \cdot v \neq 0$
- Self-gravity is sometimes important -- $\nabla^2 \Phi = 4\pi\rho$
- Relativity is sometimes important *- SRHD/GRHD*
- Non-ideal processes (e.g., conduction, viscosity) are often unimportant *- ideal HD/MHD*
- Sometimes involve interactions with non-thermal particles (i.e., cosmic rays) and strong radiation fields *- CR-HD or radiation-HD*



Hydrodynamic equations

Hydrodynamic equations

- *Assumptions: collisional fluids (Kn << 1), ideal fluids (no viscosity/conduction/diffusion)*
- Can be derived from integrating the Boltzmann equation (which describes the particle distribution function $f(x, p, t)$) multiplied by different moments $(m, mv_i, 0.5mv^2)$
- The (ideal) hydrodynamic/Euler's equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

Continuity equation

Unit tensor

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}\boldsymbol{v} + P \cdot \boldsymbol{I}) = 0$$

Momentum equation

$$\frac{\partial \rho E}{\partial t} + \nabla \cdot [(\rho E + P)\boldsymbol{v}] = 0$$

Total energy equation

Hydrodynamic equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} + \nabla \cdot (\rho \boldsymbol{v} \boldsymbol{v} + P \cdot \boldsymbol{I}) = 0$$

$$\frac{\partial \rho E}{\partial t} + \nabla \cdot [(\rho E + P) \boldsymbol{v}] = 0$$

6 independent variables, 5 equations
=> need a *closure*

ρ Gas density
$\boldsymbol{v}$ Gas velocity
p Gas pressure
ϵ Gas specific internal energy
E Gas specific total energy
u Gas internal energy

$$u = \rho \epsilon$$

$$E = \frac{1}{2} v^2 + \epsilon$$

Equation of state (EOS)

- One can use the EOS of an ideal gas as the closure of hydrodynamic equations
- For ideal gas, particle distributions can be approximated by the Maxwellian distribution, and one can derive the EOS:

$$\boxed{u = \rho\epsilon = \frac{P}{\gamma - 1} = \frac{nk_B T}{\gamma - 1}}$$

$n \equiv \frac{\rho}{m}$ = number density

$\gamma \equiv c_v/c_p$= ratio of specific heats = adiabatic index

- Special cases when *P* is only a function of ρ:
  - ***Isothermal gas*** $(\gamma = 1)$: $\boldsymbol{P \propto \rho}$
  - Adiabatic gas ("***polytropic EOS***"; entropy = constant): $\boldsymbol{P \propto \rho^{\gamma}}$

Adiabatic index γ

- For particles with *d* degrees of freedom (d.o.f.): $\gamma = 1 + 2/d$
- For large γ: "***stiff***" EOS, adiabatic compression causes large pressure increase
- For small γ: "***soft***" EOS, adiabatic compression causes small pressure increase
- Typical values:

$\gamma = 1.6667$	monatomic gas (no internal d.o.f.)
$\gamma = 1.3333$	relativistic monatomic gas
$\gamma = 1.4$	diatomic gas (rotational d.o.f. only)
$\gamma = 1.3333$	diatomic gas (rotational + vibrational d.o.f.)
$\gamma = 1$	isothermal gas (compression cannot heat, infinite d.o.f.)
$\gamma \approx 1.4$	air (mostly N_2 and O_2)

Hydrodynamic equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}\boldsymbol{v} + P \cdot \boldsymbol{I}) = 0$$

$$\frac{\partial \rho E}{\partial t} + \nabla \cdot [(\rho E + P)\boldsymbol{v}] = 0$$

$$u = \rho\epsilon = \frac{P}{\gamma - 1} = \frac{n k_B T}{\gamma - 1}$$ EOS

ρ	Gas density
$\boldsymbol{v}$	Gas velocity
p	Gas pressure
ϵ	Gas specific internal energy
E	Gas specific total energy
u	Gas internal energy

$$u = \rho\epsilon$$

$$E = \frac{1}{2}v^2 + \epsilon$$

Hydrodynamic equations + gravity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}\boldsymbol{v} + P \cdot \boldsymbol{I}) = -\rho \nabla \Phi$$

$$\frac{\partial \rho E}{\partial t} + \nabla \cdot [(\rho E + P)\boldsymbol{v}] = -\rho \boldsymbol{v} \cdot \nabla \Phi$$

$$u = \rho\epsilon = \frac{P}{\gamma - 1} = \frac{n k_B T}{\gamma - 1}$$

$$\nabla^2 \Phi = 4\pi G \rho$$

Poisson equation

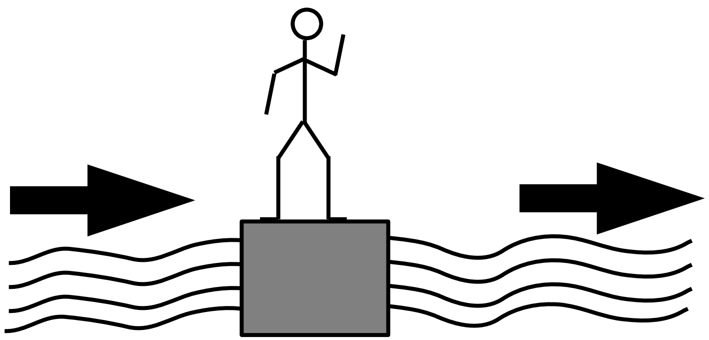
- ρ Gas density
- $\boldsymbol{v}$ Gas velocity
- p Gas pressure
- ϵ Gas specific internal energy
- E Gas specific total energy
- u Gas internal energy
- Φ Gravitational Potential

$$u = \rho\epsilon$$

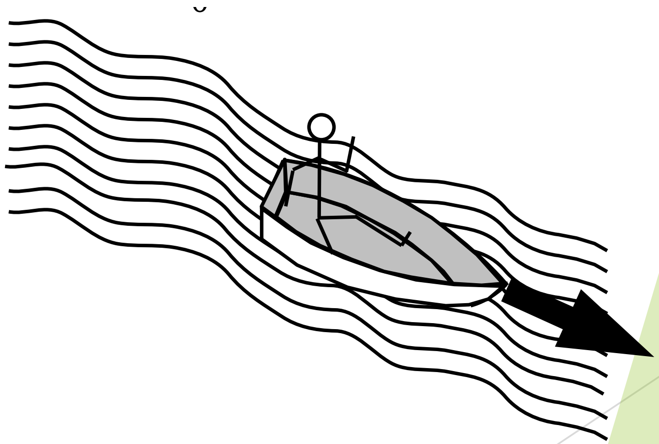
$$E = \frac{1}{2}v^2 + \epsilon$$

Eulerian vs. Lagrangian viewpoints

- *Eulerian*: stand still as fluid moves by
- Fluid quantities are functions of position $\boldsymbol{x}$ and time t



- *Lagrangian*: move with the fluid
- Fluid quantities are functions of initial position $\boldsymbol{x}(t_0)$ and time t



Eulerian vs. Lagrangian viewpoints

- From the chain rule: $$\frac{df(r,t)}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial t} = \underbrace{\frac{\partial f}{\partial t}}_{\text{change of f at fixed location}} + \underbrace{(\boldsymbol{v} \cdot \nabla) f}_{\text{change of f due to fluid motions (advection)}} \equiv \underbrace{\frac{Df}{Dt}}_{\text{change of f with the fluid}}$$

- By defining the ***convective derivative***: $\boxed{\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla}$

one could rewrite the continuity equation in Lagrangian form:

$$\partial_t \rho = -\nabla \cdot (\rho \boldsymbol{v})$$
$$= -(v \cdot \nabla)\rho - \rho(\nabla \cdot v)$$

$$\longrightarrow \partial_t \rho + (v \cdot \nabla)\rho = -\rho(\nabla \cdot v)$$

$$\longrightarrow \frac{D\rho}{Dt} = -\rho(\nabla \cdot v)$$

Hydrodynamic equations in Lagrangian form

$$\frac{D\rho}{Dt} = -\rho\nabla \cdot v$$

$$\frac{Dv}{Dt} = -\frac{\nabla p}{\rho} - \nabla\Phi$$

$$\frac{Du}{Dt} = -(u+p)\nabla \cdot v$$

$$u = \rho\epsilon = \frac{P}{\gamma - 1} = \frac{nk_B T}{\gamma - 1}$$

$$\nabla^2\Phi = 4\pi G\rho$$

ρ	Gas density
$\boldsymbol{v}$	Gas velocity
p	Gas pressure
ϵ	Gas specific internal energy
E	Gas specific total energy
u	Gas internal energy
Φ	Gravitational Potential

$$u = \rho\epsilon$$

$$E = \frac{1}{2}v^2 + \epsilon$$

Navier-Stokes equations in Lagrangian form

Non-ideal hydrodynamic equations including diffusive terms (viscosity/conduction)

$$\frac{D\rho}{Dt} = -\rho \nabla \cdot \boldsymbol{v}$$

$$\pi_{ij} = \mu D_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \nabla \cdot \boldsymbol{u} \right)$$

Viscous stress tensor

$$\rho \frac{D\boldsymbol{v}}{Dt} = -\rho \nabla \phi - \nabla P + \nabla \cdot \underline{\overleftrightarrow{\pi}}$$

$$\rho \frac{D\epsilon}{Dt} = -P \nabla \cdot \boldsymbol{v} - \nabla \cdot \underline{\boldsymbol{F}} + \underline{\Psi}$$

Conductive heat flux

Rate of viscous dissipation

$$F_k = -K_H \frac{\partial T}{\partial x_k} \qquad \Psi \equiv \pi_{ik} \frac{\partial u_i}{\partial x_k}$$

Dimensionless quantities governing the relative importance of physical processes

- *Mach number*: ratio of bulk velocity v to sound speed a: $M \equiv \frac{v}{a}$

- *Reynolds number*: ratio of internal and viscous terms in the momentum equation:

$$R_e \equiv \frac{\frac{\partial(\rho v_i v_k)}{\partial x_k}}{\frac{\partial \pi_{ik}}{\partial x_k}} \sim \frac{\frac{\rho U^2}{L}}{\frac{\mu U}{L^2}} = \frac{UL}{\nu}$$

kinematic viscosity
$\nu \equiv \mu/\rho$

- *Prandtl number*: ratio of kinematic viscosity ν to thermal diffusivity χ:

$$Pr \equiv \frac{\nu}{\chi} \qquad \chi \equiv \frac{K_H}{\rho c_p}$$

In astrophysics, usually $M \sim 1, Re \gg 1, Pr \sim 1$ => advection ~ pressure >> diffusive effects

Fluid properties and phenomena

Sound waves

- Consider a uniform fluid with density ρ_0 and pressure P_0, subject to a small density ***perturbation*** $\delta\rho \ll \rho_0$. The resulting velocity perturbation δv is also small. The corresponding adiabatic pressure perturbation is

$$\delta P = \left(\frac{\partial P}{\partial \rho}\right)_s \delta \rho$$

- The continuity and momentum equations to ***linear*** order of perturbations are:

$$\frac{\partial \delta \rho}{\partial t} + \rho_0 \nabla \cdot \boldsymbol{\delta v} = 0$$

$$\rho_0 \frac{\partial \boldsymbol{\delta v}}{\partial t} + \left(\frac{\partial P}{\partial \rho}\right)_s \nabla \delta \rho = 0$$

- Take $\partial/\partial t$ of continuity equation and substitute from momentum equation:

$$\frac{\partial^2 \delta \rho}{\partial t^2} - \left(\frac{\partial P}{\partial \rho}\right)_s \nabla^2 \delta \rho = 0$$

This is the wave equation with propagation speed $\boldsymbol{a} \equiv [\left(\frac{\partial P}{\partial \rho}\right)_s]^{1/2}$

Characteristics

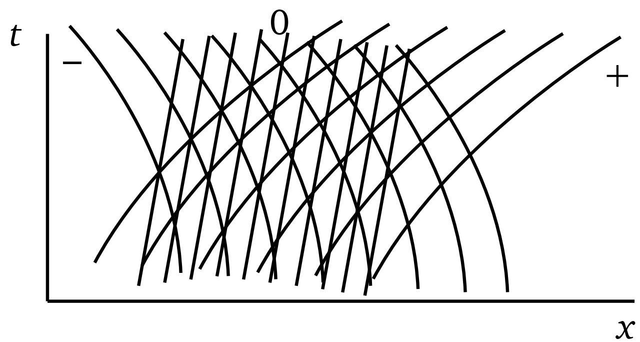
- Note that the wave equation (in 1D) can be written as

$$\left(\frac{\partial}{\partial t}+a\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}-a\frac{\partial}{\partial x}\right)\delta\rho=0$$

- Thus arbitrary smooth function of the arguments $x+at$ and $x-at$ are solutions:

$$\delta\rho \equiv \delta\rho(x+at) \qquad \text{wave moving to left}$$
$$\delta\rho \equiv \delta\rho(x-at) \qquad \text{wave moving to right}$$

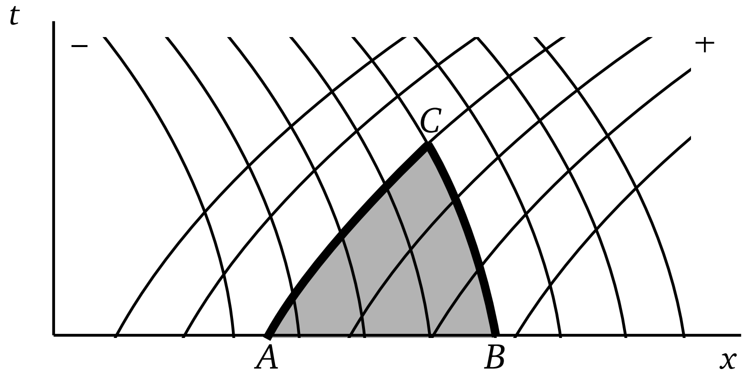
- These solutions are called ***characteristics***
- More generally, for an adiabatic fluid moving with speed *v*, there are three families of characteristics:



+: sound wave moving to right
-: sound wave moving to left
0: adiabatic motion of fluid

Characteristics & information propagation

- The characteristic speeds $(v, v \pm a)$ are the speeds at which information propagates in the solution



- Solution at C can be written as a function of solution A and solution B
- Behavior of characteristics AC and BC depends on solution along AB
- Thus the region ABC is called the ***domain of dependence*** of C. Solution at C is independent of the solutions outside ABC

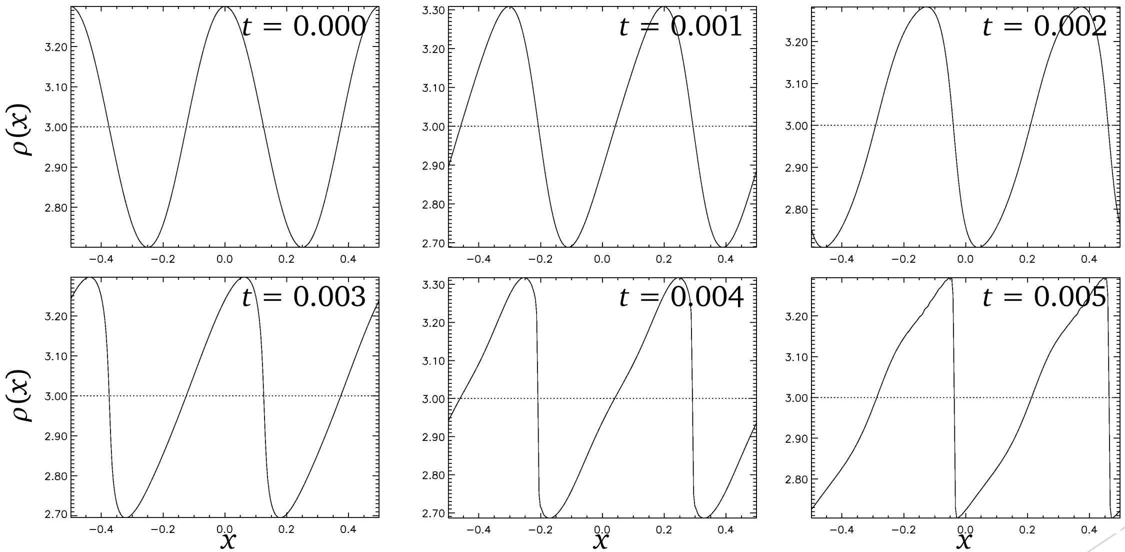
*Recall how this is related to the CFL criterion (last lecture)

Steepening of sound waves

- For sound waves, the propagation speed *a* is not constant but faster in ***compressions (where $\delta\rho > 0$)*** and slower in ***rarefactions (where $\delta\rho < 0$):***

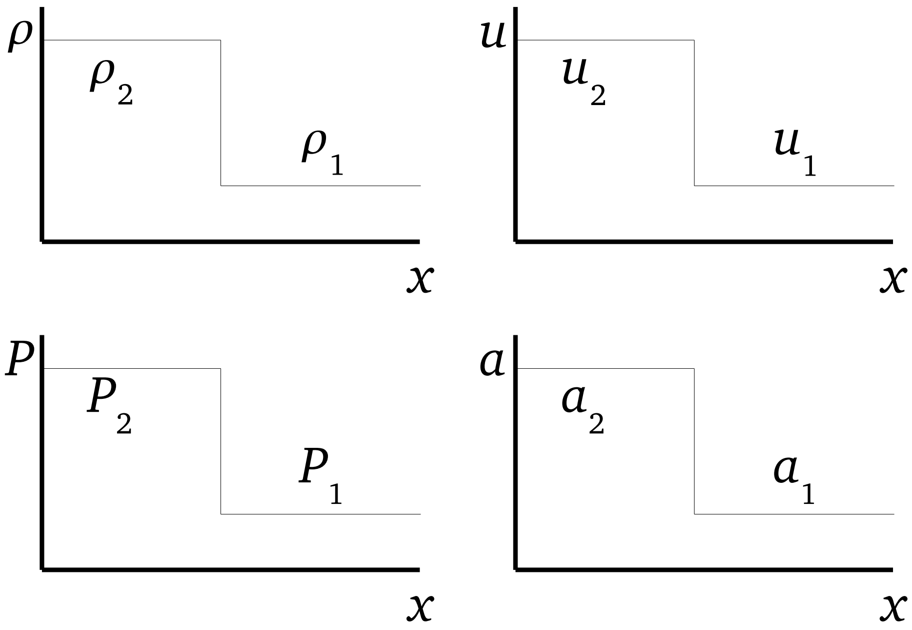
$$\boxed{a \equiv \left[\left(\frac{\partial P}{\partial \rho} \right)_s \right]^{1/2} = \sqrt{\frac{\gamma P}{\rho}} \propto \rho^{(\gamma-1)/2}}$$

- As a result, a sound wave with finite amplitudes would ***steepen*** over time:



Shock waves

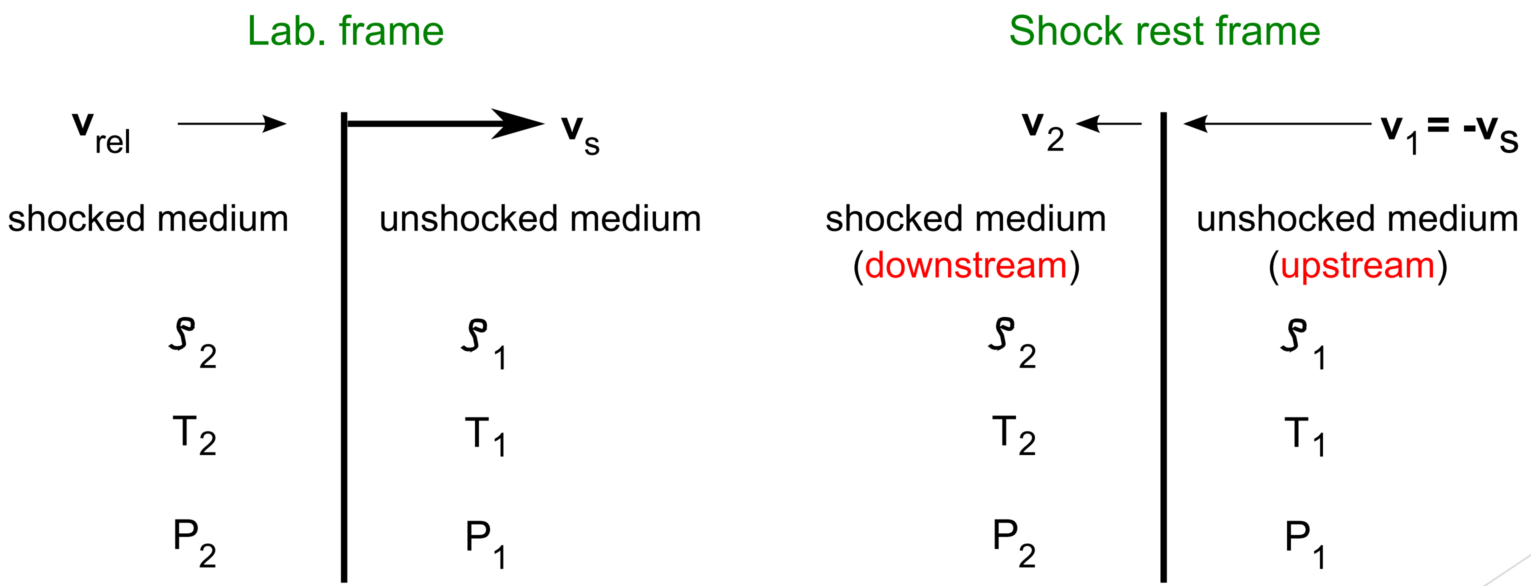
- The steepening of sound waves of finite amplitudes (i.e., *nonlinear* fluctuations) would eventually form a *shock*
- At the shock front, there are *discontinuities* in density, pressure, and velocity



Shock waves

- Define ***Mach number*** = (shock velocity)/(sound speed of unshocked medium)
- Shock occurs when Mach > 1

$$\boxed{M \equiv \frac{v_s}{c_s}}$$



Use fluid dynamics to study gas properties across the shock

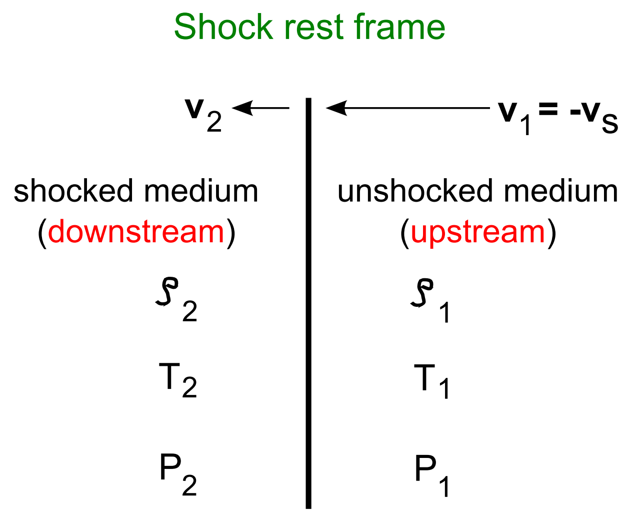
- Hydrodynamic equations are conservation laws for fluids
- Across the shock, the fluid quantities should still be conserved, independent of what happens on microscopic scales
- Recall the hydrodynamic equations:

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0}$$

$$\boxed{\rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v} \cdot \nabla)\vec{v} = -\nabla P}$$

$$\boxed{\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho U\right) + \nabla \cdot \left[\vec{v}\left(\frac{1}{2}\rho v^2 + \rho U + P\right)\right] = 0}$$

Conservation of fluid quantities across the shock

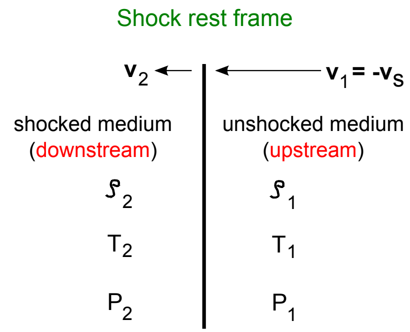


- Consider 1D ***steady*** shock (time derivatives = 0), no B field, no gravitational field
- Apply conservation laws in the rest frame of the shock
- Mass: $\frac{d}{dx}(\rho v) = 0 \quad \Rightarrow \quad \rho_1 v_1 = \rho_2 v_2$
- Momentum: $\frac{d}{dx}\left(P + \rho v^2\right) = 0 \quad \Rightarrow \quad P_1 + \rho_1 v_1^2 = P_2 + \rho_2 v_2^2$
- Energy: $\frac{d}{dx}\left(\rho v \left[\frac{v^2}{2} + U + \frac{P}{\rho}\right]\right) = 0$

$$\Rightarrow \quad \rho_1 v_1 \left(\frac{v_1^2}{2} + U_1 + \frac{P_1}{\rho_1}\right) = \rho_2 v_2 \left(\frac{v_2^2}{2} + U_2 + \frac{P_2}{\rho_2}\right)$$

3 equations for 3 unknowns!

Rankine-Hugoniot jump conditions



- Solutions to the last 3 equations give the ***jump conditions*** that are widely used to connect upstream and downstream fluid properties across the shocks

- Density and velocity discontinuity: $$\boxed{\text{compression ratio r} := \frac{v_1}{v_2} = \frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2+2}}$$

- Pressure discontinuity: $$\boxed{\frac{P_2}{P_1} = \frac{2\gamma M_1^2-(\gamma-1)}{\gamma+1}}$$

- Temperature discontinuity: $$\boxed{\frac{T_2}{T_1} = \frac{P_2/\rho_2}{P_1/\rho_1} = \frac{P_2}{P_1}\frac{\rho_1}{\rho_2} = \frac{[2\gamma M_1^2-(\gamma-1)]\,[(\gamma-1)M_1^2+2]}{(\gamma+1)^2M_1^2}}$$

Rankine-Hugoniot jump conditions

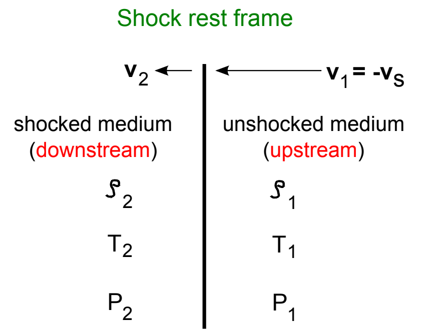
- ***For strong shocks ($M \to \infty$),*** the density jump is limited, but pressure & temperature increase without limit:

$$\frac{\rho_2}{\rho_1} \to \frac{\gamma+1}{\gamma-1}$$

$$\frac{P_2}{P_1} \to \frac{2\gamma}{\gamma+1} M^2 \to \infty$$

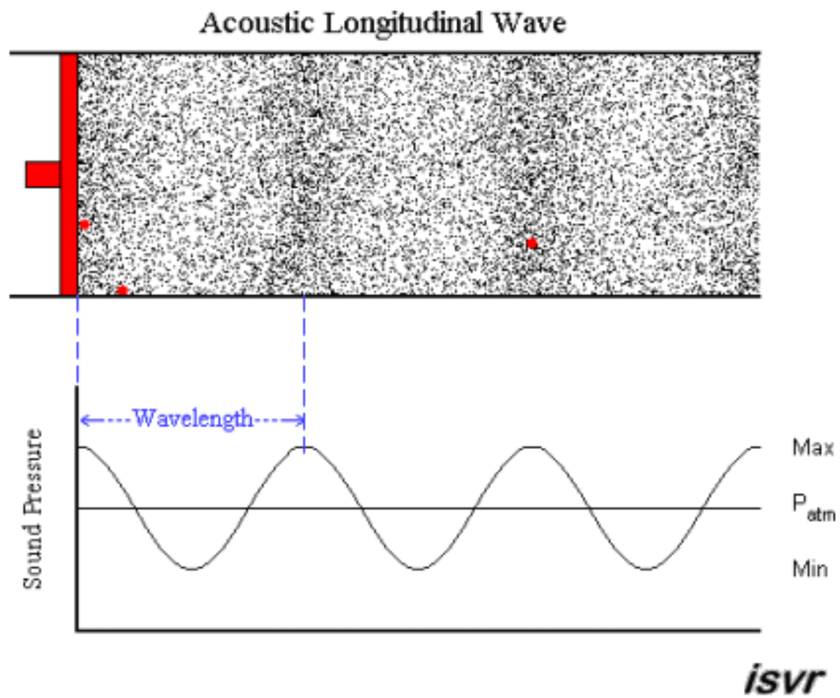
$$\frac{T_2}{T_1} \to 2\gamma \frac{\gamma-1}{(\gamma+1)^2} M^3 \to \infty$$

- ***For $\gamma = 5/3$, compression ratio $r = \rho_2/\rho_1 \to 4$***



Rarefactions (疏波)

- This is another solution of *nonlinear* waves that can exist without steepening into a shock
- Imagine a 1D piston that accelerates quickly to a constant velocity –U away from the fluid to its right



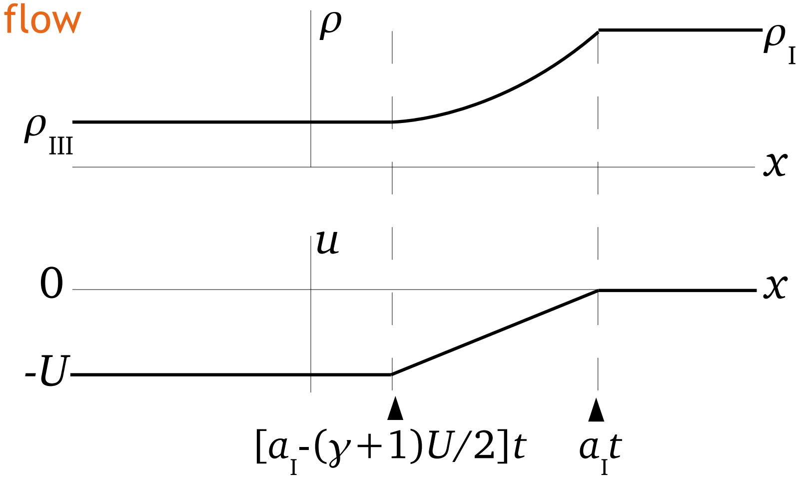
Rarefactions (疏波)

- This is another solution of ***nonlinear*** waves that can exist without steepening into a shock
- Imagine a 1D piston that accelerates quickly to a constant velocity –U away from the fluid to its right

Region III: again constant flow

Region II: rarefaction fan

Region I: undisturbed flow



Analytical solution for region II

$$\rho=\rho_I\left[1-\frac{\gamma-1}{2}\frac{|u|}{a_I}\right]^{2/(\gamma-1)}$$

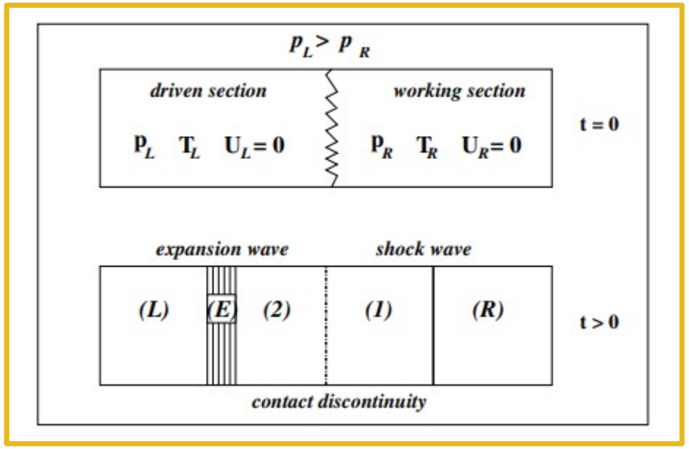
$$P=P_I\left[1-\frac{\gamma-1}{2}\frac{|u|}{a_I}\right]^{2\gamma/(\gamma-1)}$$

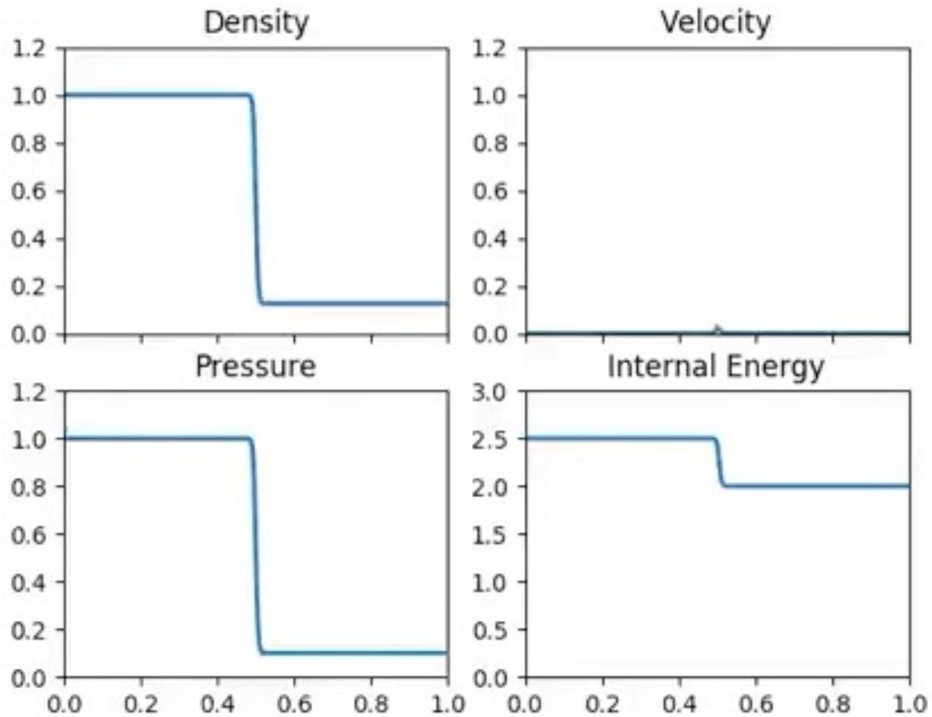
$$|u|=\frac{2}{\gamma+1}\left(a_I-\frac{x}{t}\right)$$

Contact discontinuities

- Discontinuity in ρ only; $P, v_{\perp}$ are continuous
- **Example**: the *Sod shock-tube test*, where all three types of nonlinear waves are present

Schematic of the shock-tube test

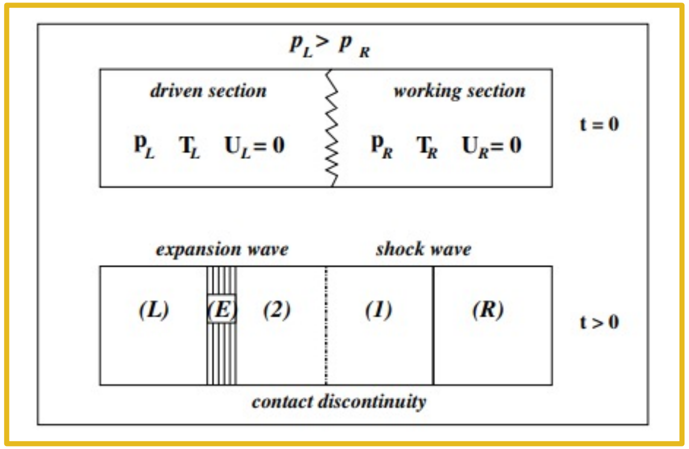


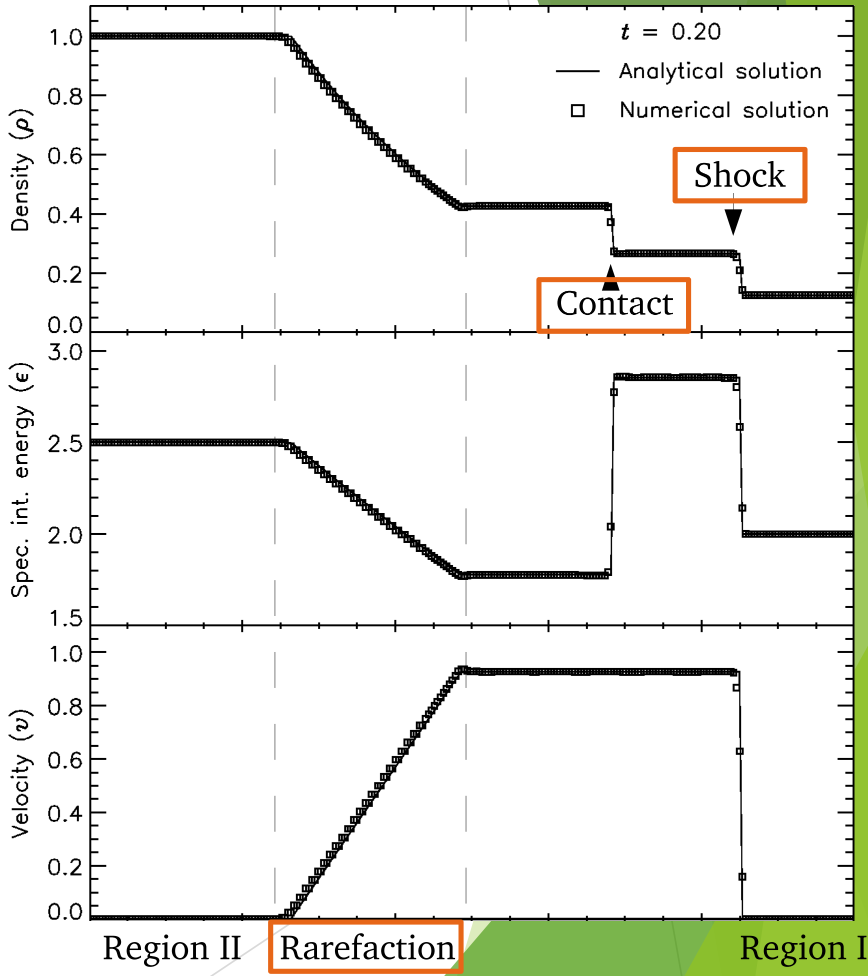


Contact discontinuities

- Discontinuity in ρ only; $P, v_\perp$ are continuous
- **Example:** the *Sod shock-tube test*, where all three types of nonlinear waves are present
- Analytical solution can be obtained and used to verify hydrodynamic simulations

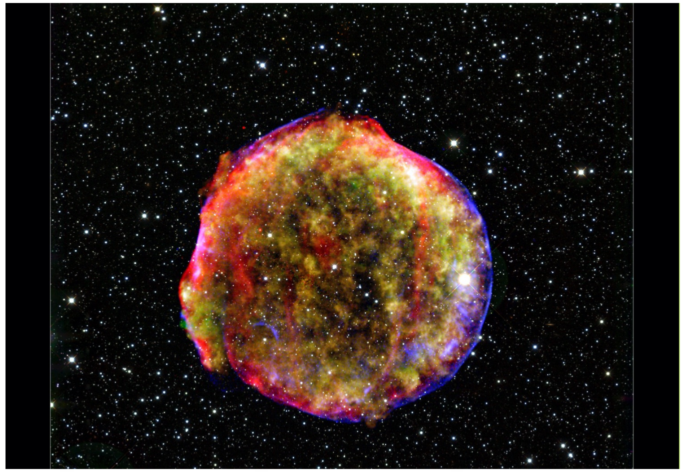
Schematic of the shock-tube test

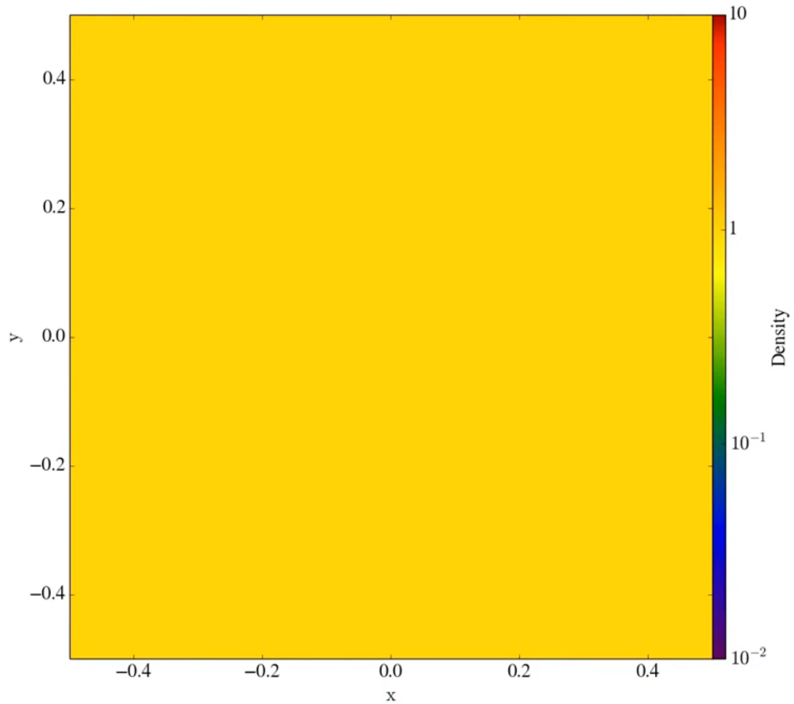




Sedov-Taylor blast wave test

- Point explosion with energy *E* expands into uniform medium with density ρ_1 and negligible pressure P_1
- This describes the energy-conserving stage of the supernova remnant evolution





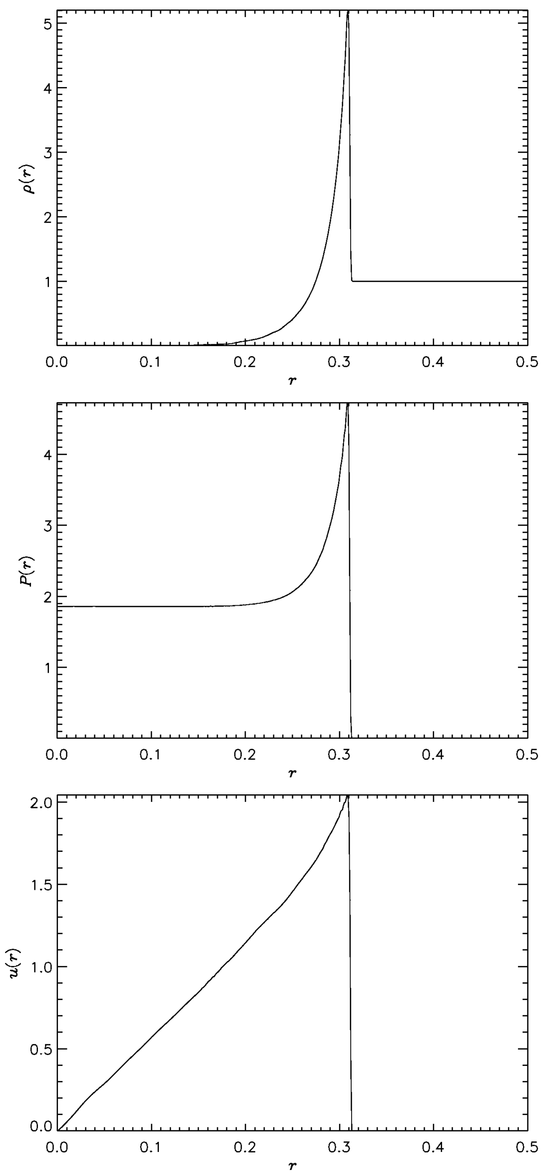
Sedov-Taylor blast wave test

- Solution is ***self-similar***:

shock radius
$$R(t)=\beta\left[\frac{Et^2}{\rho_1}\right]^{1/5}$$

shock velocity
$$u_1=\frac{dR}{dt}=\frac{2}{5}\beta\left(\frac{E}{\rho_1 t^3}\right)^{1/5}$$

- Post-shock profiles can also be derived analytically

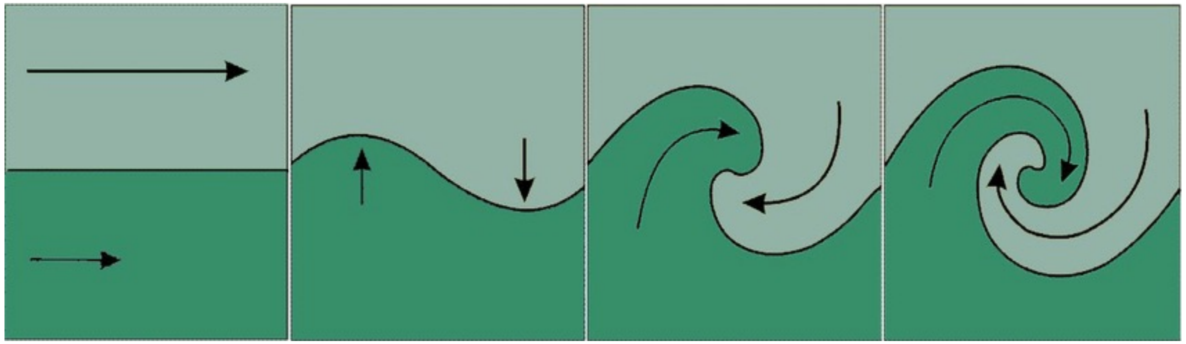


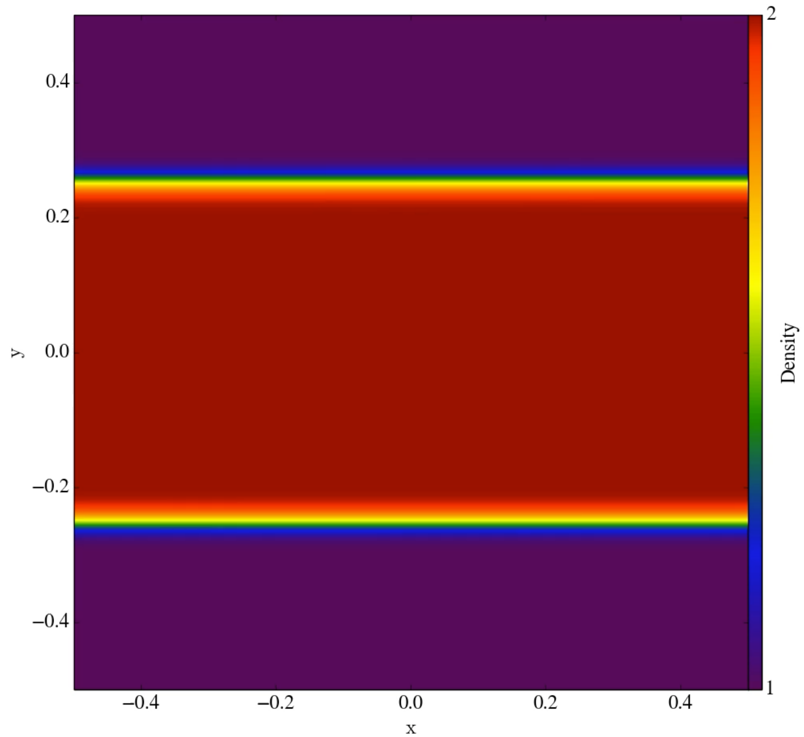
Fluid instabilities

- A fluid is ***unstable*** if small perturbations grow with time rather than damping out
- Many different types of instabilities associated with different forcing mechanisms, flow regimes, geometries, etc
- Key questions regarding fluid instabilities:
  - What is the criterion for instability?
  - How fast do linear perturbations grow? Is the growth rate a function of wavenumber?
  - What is the nonlinear development of perturbations like?
  - What additional effects can stabilize the fluid?

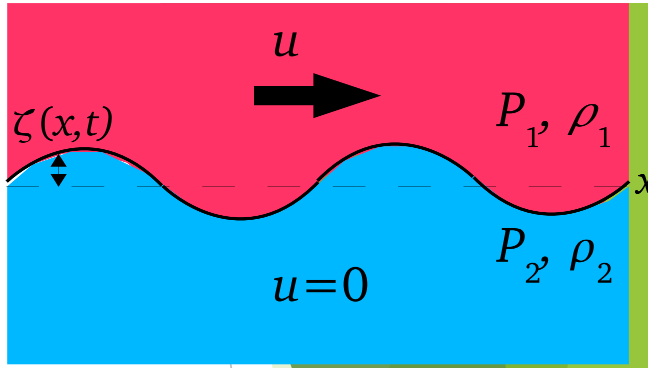
Kelvin-Helmholtz instability

- Arises in *shear flow along a contact discontinuity*
- Often produces *wave/ripple-like* structures





Kelvin-Helmholtz instability



- Let $y = \zeta(x,t)$ be the displacement of the surface
- Linearized Euler equations for an incompressible ($\nabla \cdot v = 0$) fluid are:

$$\nabla \cdot \boldsymbol{\delta u} = 0$$

$$\frac{\partial \boldsymbol{\delta u}}{\partial t} + u \frac{\partial \boldsymbol{\delta u}}{\partial x} = -\frac{1}{\rho} \nabla \delta P$$

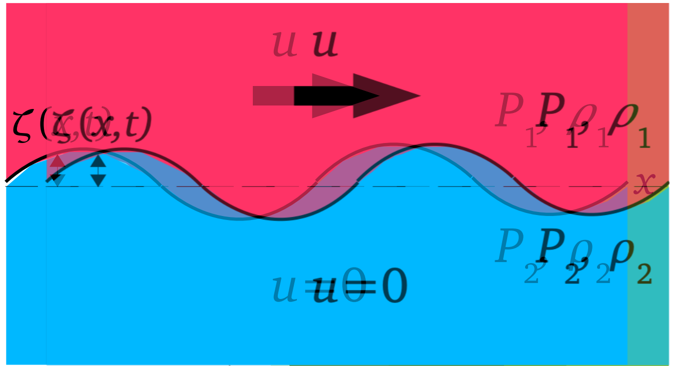
- Thus δP satisfies $\nabla^2 \delta P = 0$. Look for solution $\delta P = f(y) e^{i(kx-\omega t)}$:

$$\frac{d^2 f}{dy^2} - k^2 f = 0 \quad \rightarrow \quad f(y) \propto e^{\pm ky}$$

- Let $y > 0$ on side 1: $\delta P_1 \propto e^{i(kx-\omega t)} e^{-ky}$

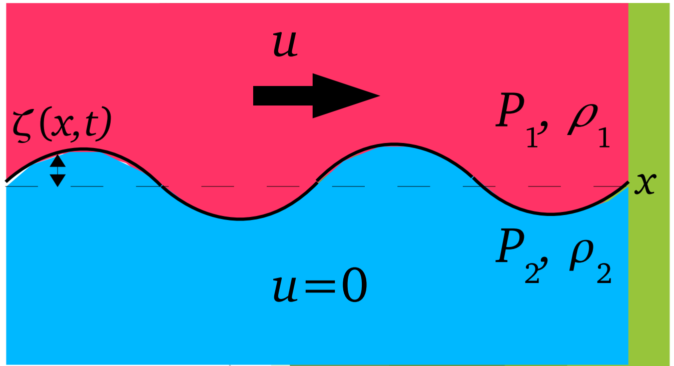
$$\delta u_y = \frac{k \, \delta P_1}{i \rho_1 (ku - \omega)}$$

Kelvin-Helmholtz instability



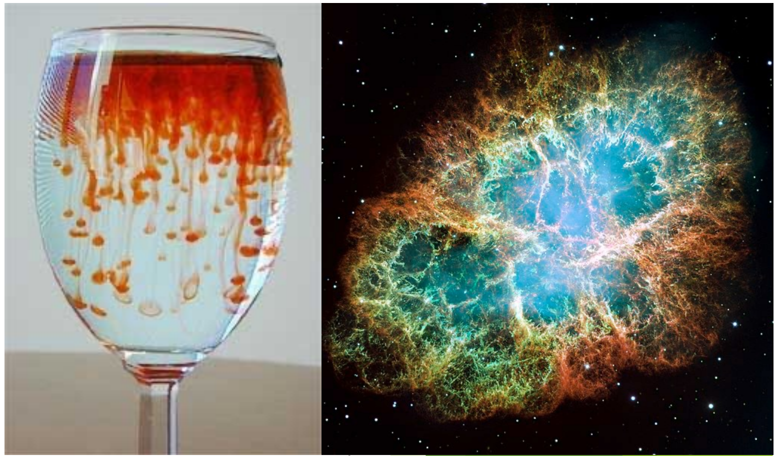
- Note that (from the movement of $\zeta(x,t)$): $$\frac{\partial \zeta}{\partial t} = \delta u_y - u \frac{\partial \zeta}{\partial x}$$
- If $\delta u_y = i\zeta(ku - \omega)$, $\zeta \propto e^{i(kx-\omega t)}$ is solution to the above equation. Thus
$$\delta P_1 = -\zeta \rho_1 \frac{(ku-\omega)^2}{k}$$
- Similarly for side 2: $$\delta P_2 = \zeta \rho_2 \frac{\omega^2}{k}$$
- Requiring $\delta P_1 = \delta P_2$ on the boundary gives: $$\omega = ku \frac{\rho_1 \pm i\sqrt{\rho_1 \rho_2}}{\rho_1 + \rho_2}$$

Kelvin-Helmholtz instability

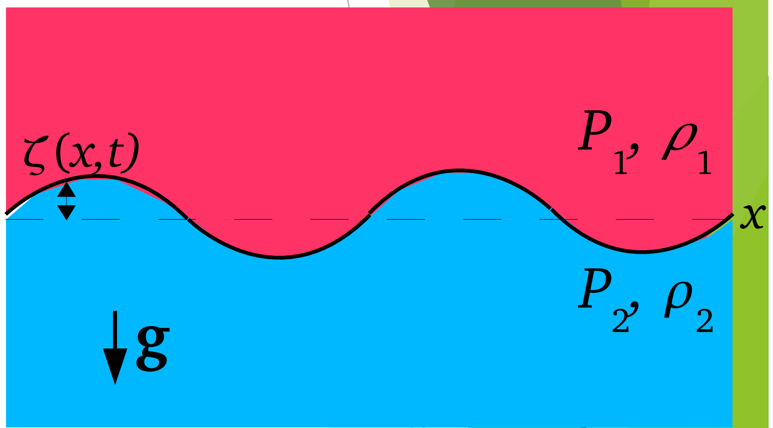


- So we have $\zeta \propto e^{i(kx-\omega t)}$, $\quad \omega = ku \dfrac{\rho_1 \pm i\sqrt{\rho_1 \rho_2}}{\rho_1 + \rho_2}$
- There are always frequencies with a positive imaginary part => *all modes are unstable*
- Modes with larger *k* or *shorter wavelengths grow fastest*
- If there is some *surface tension* on the interface (e.g., magnetic field, viscosity) as a restoring force, small-wavelength perturbations can be stabilized

Rayleigh-Taylor instability



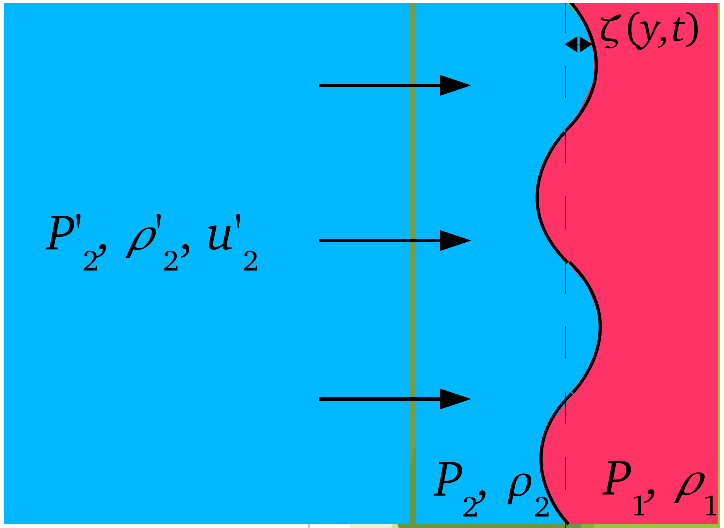
- Arises when a ***denser fluid overlies a less dense fluid***
- Often produces ***plume/finger-like*** structures
- Again, one can consider two incompressible fluids with sinusoidal perturbation at the interface. Both are at rest, and $\rho_1 > \rho_2$. A constant gravitational field $\boldsymbol{g}$ is present.
- This configuration is always unstable if there is no surface tension

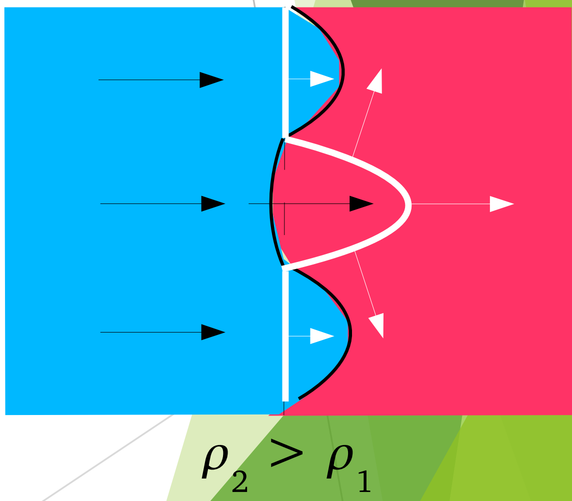


Rayleigh-Taylor instability & non-linear development

Richtmyer-Meshkov instability

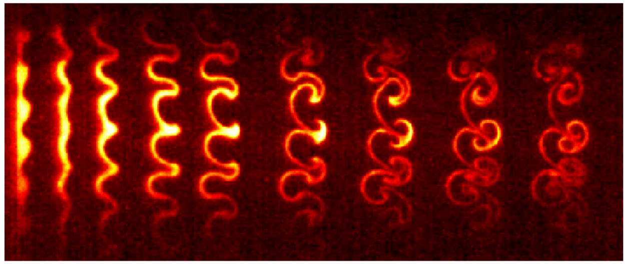
- Arises when a ***shock passes through a perturbed interface between fluids of different densities***
- What happens?
  - Different parts of the shock hit the interface at different times
  - The parts that have reached the interface will travel at a different speed than other parts
  - Multiple intersecting shocks will form on the other side of the interface
  - The interface will also stretch in a way similar to Rayleigh-Taylor instability



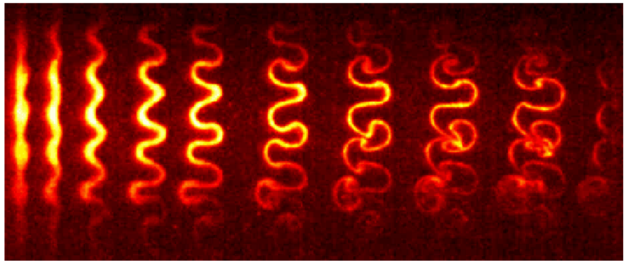


Richtmyer-Meshkov instability

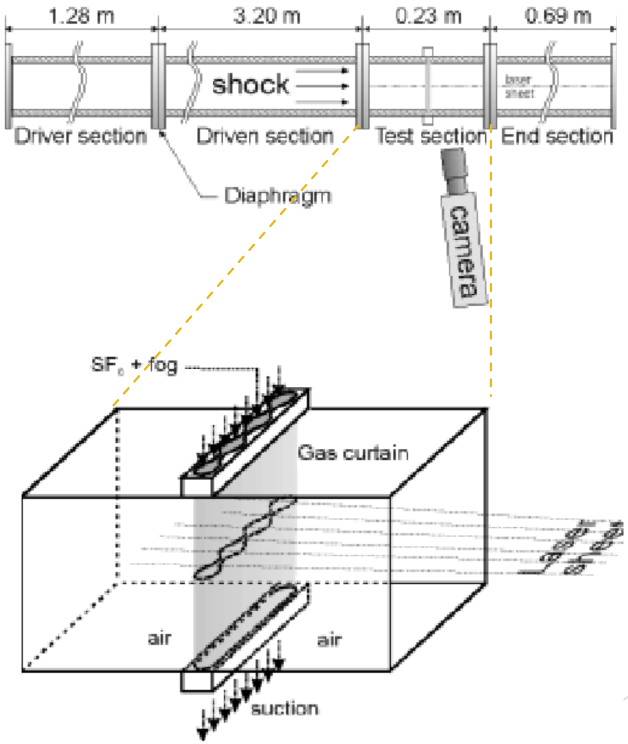
- Often produces *mushroom-like* structures
- Demonstration of Richtmyer-Meshkov instability using the gas-curtain experiment:



Downstream Mushrooms



Sinuous Mode



Rightley, Benjamin, & Vorobieff (1997)

Astrophysical fluid dynamics -- summary

- A system can be described as a fluid if it continuously flows and is ***collisional***, i.e., if the ***Knudsen number*** is small:

$$\boxed{K_n \equiv \frac{\lambda}{L} = \frac{mean\ free\ path}{typical\ scale} \ll 1} \quad \text{or} \quad \boxed{\lambda \ll L}$$

- Hydrodynamic equations are conservation laws for mass, momentum, and energy:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \qquad \text{Continuity equation}$$

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} + \nabla \cdot (\rho \boldsymbol{v} \boldsymbol{v} + P \cdot \boldsymbol{I}) = 0 \qquad \text{Momentum equation}$$

$$\frac{\partial \rho E}{\partial t} + \nabla \cdot [(\rho E + P) \boldsymbol{v}] = 0 \qquad \text{Total energy equation}$$

Astrophysical fluid dynamics -- summary

- The *equation of state (EOS)* of ideal gas is often used as a *closure* for the hydrodynamic equations
- Hydrodynamic equations can be described in different views
  - *Eulerian*: stand still as fluid moves by
  - *Lagrangian*: move with the fluid
- Fluid phenomena discussed in this lecture:
  - Solution for linear perturbations: *sound waves*
  - Solutions for nonlinear waves: *shocks, rarefactions, and contact discontinuities*
  - Fluid instabilities: *Kelvin-Helmholtz instability, Rayleigh-Taylor instability, Richtmyer-Meshkov instability*

$$\boxed{u = \rho\epsilon = \frac{P}{\gamma - 1} = \frac{nk_BT}{\gamma - 1}}$$

Convective derivative

$$\boxed{\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla}$$

References & acknowledgements

- Course materials of Computational Astrophysics from Prof. Kuo-Chuan Pan (NTHU)
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- Course materials of Computational Astrophysics and Cosmology from Prof. Paul Ricker (UIUC)
- “Computational Physics” by Rubin H. Landau, Manuel Jose Paez and Cristian C. Bordeianu
- “Scientific Computing – An Introductory Survey” by Michael T. Heath