Chapter 5. Determinants

Where did determinants come from? Example:

Solve: $a_{11}x + a_{12}y = b_1$
$a_{21}x + a_{22}y = b_2$

or

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Has solution $x = \dfrac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}$,

$$y = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}$$

And denominator of x and y is $\det(A)$

and numerator of x, y is also a determina[nt]

(This extends to 3×3, $n\times n$).

The system is uniquely solvable iff.
$\det(A) \neq 0$.

As expected, solution is not random,
but always depends on coefficients
and RHS b values in same config.

Section 5.1 Properties of det(A)

For $n \times n$ matrix A.

1. $\det(I) = 1$
2. det changes signs if rows swapped.
3. det is a linear function by row:

3a. $\det\begin{bmatrix} ta & tb \\ c & d \end{bmatrix} = t \det\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

3b. $\det\begin{bmatrix} a+a' & b+b' \\ c & d \end{bmatrix} =$

$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \det\begin{bmatrix} a' & b' \\ c & d \end{bmatrix}$

There are 7 other Strang properties of note:

$\det(AB) = \det(A)\det(B)$

$\det(A^{-1}) = \dfrac{1}{\det(A)}$

If A is triangular, $\det(A) = a_{11} a_{22} \cdots a_{nn}$

Row operations do not change $\det(A)$.

Example: We know $\det(A) = 1/3$ and A is 3×3.

a. $\det(3A) = (3)^3 \det(A) = 27 \times \frac{1}{3} = 9$

b. $\det(-A) = (-1)^3 \det(A) = -\frac{1}{3}$

c. $\det(A^2) = \det(A)\det(A) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$

d. $\det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{1/3} = 3$

Example: $\det \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 6 & 6 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 7 \end{bmatrix} =$

$\det \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 2 & 3 & 3 \\ 0 & 2 & 0 & 7 \end{bmatrix} =$

$\det \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 6 \end{bmatrix} = 1 \times 2 \times 3 \times 6$

$= 36$

Check on TI: ✓

Example: $\det \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} =$

$\cos^2\theta + \sin^2\theta = 1$ for all θ.

Example: Prove that orthogonal matrices Q have determinant of 1 or -1.

Proof: If Q is orthogonal,

$$Q^T Q = I.$$

Take det. of both sides.

$$\det(Q^T Q) = \det(I)$$

$$\det(Q^T)\det(Q) = 1$$

$$[\det(Q)]^2 = 1$$

$$\sqrt{[\det(Q)]^2} = \pm 1$$

$$\det(Q) = \pm 1$$

Example:- $\det\begin{bmatrix}101 & 201 & 301\\ 102 & 202 & 302\\ 103 & 203 & 303\end{bmatrix} =$

$$\det\begin{bmatrix}1 & 1 & 1\\ 102 & 202 & 302\\ 103 & 203 & 303\end{bmatrix} + \det\begin{bmatrix}100 & 200 & 300\\ 102 & 202 & 302\\ 103 & 203 & 303\end{bmatrix} =$$

$$\cancelto{0}{\det\begin{bmatrix}1 & 1 & 1\\ 2 & 2 & 2\\ 103 & 203 & 303\end{bmatrix}} + \det\begin{bmatrix}100 & 200 & 300\\ 2 & 2 & 2\\ 103 & 203 & 303\end{bmatrix} +$$

$$\det\begin{bmatrix}1 & 1 & 1\\ 100 & 200 & 300\\ 103 & 203 & 303\end{bmatrix} + \cancelto{0}{\det\begin{bmatrix}100 & 200 & 300\\ 100 & 200 & 300\\ 103 & 203 & 303\end{bmatrix}}$$

$$= \cancelto{0}{\det\begin{bmatrix}100 & 200 & 300\\ 2 & 2 & 2\\ 3 & 3 & 3\end{bmatrix}} + \cancelto{0}{\det\begin{bmatrix}100 & 200 & 300\\ 2 & 2 & 2\\ 100 & 200 & 300\end{bmatrix}} +$$

$$\cancelto{0}{\det\begin{bmatrix}1 & 1 & 1\\ 100 & 200 & 300\\ 3 & 3 & 3\end{bmatrix}} + \cancelto{0}{\det\begin{bmatrix}1 & 1 & 1\\ 100 & 200 & 300\\ 100 & 200 & 300\end{bmatrix}} =$$

$$0$$

<u>Example:</u>

$$\det\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} =$$

$$\det\begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{bmatrix} =$$

$$\det\begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & 0 & c^2-a^2-\left[\frac{c-a}{b-a}(b^2-a^2)\right] \end{bmatrix}$$

$$c^2-a^2-\frac{c-a}{\cancel{b-a}}\cancel{(b-a)}(b+a) =$$

$$(c-a)(c+a)-(c-a)(b+a) =$$

$$(c-a)\left[c+a-b-a\right] =$$

$$(c-a)(c-b)$$

$$=\det\begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & 0 & (c-a)(c-b) \end{bmatrix} = (b-a)(c-a)(c-b)$$

Example: For $n \geq 3$, show $\det(A) = 0$ if the i,j entry of A is $i+j$.

$$A = \begin{bmatrix} 2 & 3 & 4 & 5 & \cdots & n+1 \\ 3 & 4 & 5 & 6 & \cdots & n+2 \\ 4 & 5 & 6 & 7 & \cdots & n+3 \\ \vdots & \vdots & & & & \\ n+1 & n+2 & n+3 & \cdots & & n+n \end{bmatrix}$$

Notice Row 3 $= 2(\text{row } 2) - \text{row } 1$

Since row 3 is a linear comb. of above rows, matrix has dependent rows, not inventible, $\det(A) = 0$

Example: $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$. Find $\det(A - \lambda I)$.

$$A - \lambda I = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = 12 - 7\lambda + \lambda^2 - 2$$
$$= \lambda^2 - 7\lambda + 10$$

Section 5.2

det(A) = Product of the pivots
(any row swaps change signs

From earlier in course:

$A = LU$ but possibly need

$PA = LU$

$$\det(P)\det(A) = \det(L)\det(U)$$

identity matrix with row swaps

lower triangular with ones on diag.

upper triangular with pivots on diagonal

$\det(P) = \pm 1$

$\det(L) = 1$

$\det(U) = \pm\det(A)$

The Big Formula:

See book or video.

det(A) = sum over n! column permutations

$$P = (\alpha, \beta, \ldots, \omega)$$

$$= \sum \mathrm{Det}(P)\, a_{1\alpha} a_{2\beta} \cdots a_{n\omega}$$

2x2 has 2! = 2 terms.

3x3 has 3! = 6 terms

4x4 has 4! = 24 terms.

Terms are positive if the permutation takes an even # of column exchanges

<u>Example</u>: $\det \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & d & 1 \end{bmatrix} =$

$(1)(1)(c)(1) + (0)(b)(0)(0) + (a)(0)(0)(0) + (0)(0)(0)(d)$

+ 20 other terms:

Answer $\Rightarrow \det(A) = c$

Cofactors :

Take $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$\det(A) = (a)\det\begin{bmatrix} e & f \\ h & i \end{bmatrix}$, $\text{Cofactor}(a) = +\det\begin{bmatrix} e & f \\ h & i \end{bmatrix}$

$-(b)\det\begin{bmatrix} d & f \\ g & i \end{bmatrix}$, $\text{Cofactor}(b) = -\det\begin{bmatrix} d & f \\ g & i \end{bmatrix}$

$+(c)\det\begin{bmatrix} d & e \\ g & h \end{bmatrix} =$ $\text{Cofactor}(c) = +\det\begin{bmatrix} d & e \\ g & h \end{bmatrix}$

$= a(ei - fh) - b(di - fg) + c(dh - eg)$

$= aei + bfg + cdh$
$\quad - afh - bdi - ceg$

The + or − sign follows

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ & & & \ddots & \end{bmatrix}$$

Examples...

Example: Use cofactors to find $\det(A)$.

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

$$\det(A) = (1)\det\begin{bmatrix} 3 & 4 & 5 \\ 4 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$-(0)\det\begin{bmatrix} 0 & 4 & 5 \\ 5 & 0 & 3 \\ 2 & 0 & 1 \end{bmatrix}$$

$$+(0)\det\begin{bmatrix} 0 & 3 & 5 \\ 5 & 4 & 3 \\ 2 & 0 & 1 \end{bmatrix}$$

$$-(2)\det\begin{bmatrix} 0 & 3 & 4 \\ 5 & 4 & 0 \\ 2 & 0 & 0 \end{bmatrix} =$$

$$\det\begin{bmatrix} 3 & 4 & 5 \\ 4 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} + 2\det\begin{bmatrix} 0 & 3 & 4 \\ 5 & 4 & 0 \\ 2 & 0 & 0 \end{bmatrix} =$$

$$3\det\begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix} - 4\det\begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} + 5\det\begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$$

$$-2\left[0\det\begin{bmatrix}4&0\\0&0\end{bmatrix}-3\det\begin{bmatrix}5&0\\2&0\end{bmatrix}+4\det\begin{bmatrix}5&4\\2&0\end{bmatrix}\right]$$

$$=0-4(4)+0-2(0-0+4(-8))$$

$$=-16-2(-32)=48$$

<u>Example</u>: Use determinants to determine for which k matrix A is invertible.

$$A=\begin{bmatrix}1&1&k\\1&k&k\\k&k&k\end{bmatrix}$$

$$\det(A)=(1)\det\begin{bmatrix}k&k\\k&k\end{bmatrix}-(1)\det\begin{bmatrix}1&k\\k&k\end{bmatrix}$$

$$+(k)\det\begin{bmatrix}1&k\\k&k\end{bmatrix}=$$

$$-\left[k-k^2\right]+k\left[k-k^2\right]=$$

$$-k+k^2+k^2-k^3=-k^3+2k^2-k$$

$\det(A)=0$ iff $-k^3+2k^2-k=0$

$$-k(k^2-2k+1)=0$$

$$-k(k-1)(k-1)=0$$

A^{-1} exists $\forall k$ except $k=0,$

Example: Cook up a 4 by 4 matrix A with no 0 entries such that $\det(A) = 11$.

Start with $A = \begin{bmatrix} 11 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$A \sim \begin{bmatrix} 11 & 0 & 0 & 0 \\ 11 & 1 & 0 & 0 \\ 11 & 1 & 1 & 0 \\ 11 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 22 & 1 & 1 & 1 \\ 22 & 2 & 1 & 1 \\ 22 & 2 & 2 & 1 \\ 11 & 1 & 1 & 1 \end{bmatrix}$

Example: Find $f'(x)$ if

$$f(x) = \det \begin{bmatrix} 1 & 1 & 2 & 3 & 4 \\ 9 & 0 & 2 & 3 & 4 \\ 9 & 0 & 0 & 3 & 4 \\ x & 1 & 2 & 9 & 1 \\ 7 & 0 & 0 & 0 & 4 \end{bmatrix} =$$

$$-(x)\det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix} + (1)\det\left[\text{all numbers}\right]$$

$$-(2)\det\left[\text{all nums}\right]$$

$$+(9)\det\left[\text{all nums}\right]$$

$$-(1)\det\left[\text{all nums}\right]$$

And so $f'(x) =$

$$-\det\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix} + 0 - 0 + 0 - 0 =$$

$-(1)(2)(3)(4) = -24$

<u>Example:</u> Find det. for any $(2n)$ by $(2n)$ matrix

$$A = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}.$$

e.g. $n=4$, $A = \begin{bmatrix} 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \\ 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \end{bmatrix}$

needs 4 row swaps to be transformed to I.

If n was 3 we'd need 3 row swaps. $\det(A) = (-1)^n$

Section 5.3 Applications of Determinants

Solve $A\vec{x} = \vec{b}$ with determinants

$$\det\left[I\right] = 1 \qquad A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\det\begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{bmatrix} = x_1$$

$$\det\begin{bmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{bmatrix} = x_2$$

$$\det\begin{bmatrix} 1 & 0 & x_1 \\ 0 & 1 & x_2 \\ 0 & 0 & x_3 \end{bmatrix} = x_3$$

$$\underset{3\times 3}{\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}}\underset{3\times 3}{\begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{bmatrix}} = \underset{3\times 3}{\begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}}$$

Now take determinants.

$$\det(A)\det\begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{bmatrix} = \det\begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{23} & a_{33} \end{bmatrix}$$

$$x_1 = \frac{\det(A \text{ with col. 1 replaced by } \vec{b})}{\det(A)}$$

and likewise $x_n = \dfrac{\det(A \text{ with col. } n \text{ replaced by } \vec{b})}{\det(A)}$

This is Cramer's rule.

Example: Solve using Cramer's Rule:

$$x_1 - 7x_2 + x_3 = 4$$
$$3x_1 + x_2 + x_3 = 0$$
$$3x_1 - 2x_2 + 10x_3 = 10$$

has $A = \begin{bmatrix} 1 & -7 & 1 \\ 3 & 1 & 1 \\ 3 & -2 & 10 \end{bmatrix}$, $\det(A) = 192$

$B_1 = \begin{bmatrix} 4 & -7 & 1 \\ 0 & 1 & 1 \\ 10 & -2 & 10 \end{bmatrix}$, $B_2 = \begin{bmatrix} 1 & 4 & 1 \\ 3 & 0 & 1 \\ 3 & 10 & 10 \end{bmatrix}$, $B_3 = \begin{bmatrix} 1 & -7 & 4 \\ 3 & 1 & 0 \\ 3 & -2 & 10 \end{bmatrix}$

$\det(B_1) = -32$ $\det(B_2) = -88$ $\det(B_3) = 184$

$$x_1 = \frac{\det(B_1)}{\det(A)} = \frac{-32}{192} = -\frac{1}{6}$$

$$x_2 = \frac{\det(B_2)}{\det(A)} = \frac{-88}{192} = -\frac{11}{24}$$

$$x_3 = \frac{\det(B_3)}{\det(A)} = \frac{184}{192} = \frac{23}{24}$$

$$A\begin{bmatrix} -1/6 \\ -11/24 \\ 23/24 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 10 \end{bmatrix}$$

Cofactors to get A^{-1}

$$(A^{-1})_{ij} = \frac{C_{ji}}{\det(A)}, \quad A^{-1} = \frac{C^T}{\det(A)}$$

Example: Give the 2, 1 entry of A^{-1} if $A = \begin{bmatrix} 4 & -1 \\ 11 & -1 \end{bmatrix}$

$\det(A) = -4 + 11 = 7$

$$A^{-1} = \frac{1}{7}\begin{bmatrix} -1 & 1 \\ -11 & 4 \end{bmatrix}$$

$C_{ji} = C_{1,2} = -(11)$

$(A^{-1})_{21} = -11/7$

Example: Find A^{-1} using cofactors.

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix}$$

$$\det(A) = (1)\det\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} - (1)\det\begin{bmatrix} 1 & 2 \\ 1 & 5 \end{bmatrix} + 4\det\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

$$\det(A) = 6 - 3 + 0 = 3$$

$C_{11} = 6$, $C_{12} = -(3)$, $C_{13} = 0$

$C_{21} = -(-3) = 3$, $C_{22} = 1$, $C_{23} = -(1) = -1$

$C_{31} = -6$, $C_{32} = -(-2)$, $C_{33} = 1$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 6 & 3 & -6 \\ -3 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -2 \\ -1 & 1/3 & 2/3 \\ 0 & -1/3 & 1/3 \end{bmatrix}$$

Yikes!

Exam 3 Review

Example: Derive a formula for the number that shows up in the (1,3) position for A^{-1} if

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

$$A^{-1}_{(1,3)} = \frac{C_{31}}{\det A}$$

$$\det(A) = +a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + 0$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) =$$

$$a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31}$$

$$C_{31} = +\det \begin{bmatrix} a_{12} & 0 \\ a_{22} & a_{23} \end{bmatrix} = a_{12}a_{23}$$

$$A^{-1}_{(1,3)} = \frac{a_{12}a_{23}}{a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31}}$$

Test your Formula on

$$A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 7 \\ 3 & 6 & 8 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} x & x & 14/9 \\ x & x & x \\ x & x & x \end{bmatrix}$$

$$A^{-1}_{(1,3)} = \frac{4 \times 7}{40 - 42 - 64 + 84} = \frac{28}{18} = \frac{14}{9}$$

<u>Example</u>: Fit a parabola to the following 5 data points:

$(-2, 8), (-1, 1), (0, -4), (1, 2), (2, 9)$

Equation is $b = C + Dt + Et^2$

$$\vec{b} = \begin{bmatrix} 8 \\ 1 \\ -4 \\ 2 \\ 9 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

$$\hat{X} = \begin{bmatrix} \hat{C} \\ \hat{D} \\ \hat{E} \end{bmatrix} = (A^T A)^{-1} A^T \vec{b}$$

$$(A^TA) = \begin{bmatrix} 5 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{bmatrix}$$

$$(A^TA)^{-1} = \begin{bmatrix} 17/35 & 0 & -1/7 \\ 0 & 1/10 & 0 \\ -1/7 & 0 & 1/14 \end{bmatrix}$$

$$(A^TA)^{-1}A^T\vec{b} = \begin{bmatrix} -83/35 \\ 3/10 \\ 39/14 \end{bmatrix} = \begin{bmatrix} \hat{C} \\ \hat{D} \\ \hat{E} \end{bmatrix}$$

$$\hat{b} = \left(-83/35\right) + \left(3/10\right)t + \left(39/14\right)t^2$$

Give the error vector:

$$\vec{e} = \vec{b} - \vec{p} = \begin{bmatrix} 8 \\ 1 \\ -4 \\ 2 \\ 9 \end{bmatrix} - A(A^TA)^{-1}A^T\vec{b} = \begin{bmatrix} -6/35 \\ 31/35 \\ -57/35 \\ 45/35 \\ -13/35 \end{bmatrix}$$

What space does $\vec{e}$ live in? Left Null

What space does $\vec{p}$ live in? Columnspace.

What space does $\vec{b}$ live in? $\mathbb{R}^m$, but not one of the big 4.

Example: Use Cramer's Rule to give only x_3 in the solution to

$$\begin{aligned}
\tfrac{1}{2}x_1 + \tfrac{1}{2}x_2 + \tfrac{\sqrt{2}}{2}x_3 \qquad\qquad &= 0 \\
\tfrac{1}{2}x_1 + \tfrac{1}{2}x_2 - \tfrac{\sqrt{2}}{2}x_3 \qquad\qquad &= 0 \\
\tfrac{1}{2}x_1 - \tfrac{1}{2}x_2 \qquad\qquad + \tfrac{\sqrt{2}}{2}x_4 &= 0 \\
\tfrac{1}{2}x_1 - \tfrac{1}{2}x_2 \qquad\qquad - \tfrac{\sqrt{2}}{2}x_4 &= 2
\end{aligned}$$

$$x_3 = \frac{\det(B_3)}{\det(A)}$$

$$A = \frac{1}{2}\begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{bmatrix}$$

$$\det(A) = \left(\frac{1}{2}\right)^4 \left[+\sqrt{2}\det\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & \sqrt{2} \\ 1 & -1 & -\sqrt{2} \end{bmatrix} + \sqrt{2}\det\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & \sqrt{2} \\ 1 & -1 & -\sqrt{2} \end{bmatrix} \right]$$

$$\det\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & \sqrt{2} \\ 1 & -1 & -\sqrt{2} \end{bmatrix} = +1\det\begin{bmatrix} -1 & \sqrt{2} \\ -1 & -\sqrt{2} \end{bmatrix}$$

$$-1\det\begin{bmatrix} 1 & \sqrt{2} \\ 1 & -\sqrt{2} \end{bmatrix} =$$

$$\sqrt{2} + \sqrt{2} - (-\sqrt{2} - \sqrt{2}) = 4\sqrt{2}$$

$$\det(A) = \frac{1}{16}\left[\sqrt{2}(4\sqrt{2}) + \sqrt{2}(4\sqrt{2})\right] = 1$$

$$X_3 = \frac{\det(B_3)}{\det(A)} = \det(B_3) = \det\begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & -1/2 & 0 & \sqrt{2}/2 \\ 1/2 & -1/2 & 2 & -\sqrt{2}/2 \end{bmatrix}$$

$$= \left(\frac{1}{2}\right)^4 \det\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 4 & -\sqrt{2} \end{bmatrix} =$$

$$= \frac{1}{16}\left[(-4)\det\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & \sqrt{2} \end{bmatrix}\right] = 0$$

$$X_3 = 0$$

Example: Give an orthonormal basis for the left nullspace of

$$A = \begin{bmatrix} 2 & 2 & 0 & 2 & 2 & 7 \\ 0 & 1 & 4 & 7 & 7 & 1 \\ 0 & 2 & 8 & 14 & 14 & 2 \\ -6 & -6 & 0 & -6 & -6 & -21 \\ -4 & -1 & 12 & 17 & 17 & -11 \end{bmatrix}$$

By inspection, $\dim(N(A^T)) = 3$

Basis is $\vec{y}_1 = (0, 2, -1, 0, 0)$

$\vec{y}_2 = (3, 0, 0, 1, 0)$

$\vec{y}_3 = (1, 1, 1, 1, -1)$

$\vec{y}_1 \perp \vec{y}_2$ already, but $\vec{y}_3$ is not perp. to either $\vec{y}_1$ or $\vec{y}_2$.

Gram-Schmidt

$$\vec{Y}_3 = \vec{y}_3 - \frac{\vec{y}_1^T \vec{y}_3}{\vec{y}_1^T \vec{y}_1} \vec{y}_1 - \frac{\vec{y}_2^T \vec{y}_3}{\vec{y}_2^T \vec{y}_2} \vec{y}_2$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} - \frac{1}{5} \begin{bmatrix} 0 \\ 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} - \frac{4}{10} \begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/5 \\ 3/5 \\ 6/5 \\ 3/5 \\ -1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -1 \\ 3 \\ 6 \\ 3 \\ -5 \end{bmatrix}$$

Make Unit

$$\vec{Y}_1^* = \frac{1}{\sqrt{5}}(0, 2, -1, 0, 0)$$

$$\vec{Y}_2^* = \frac{1}{\sqrt{10}}(3, 0, 0, 1, 0)$$

$$\|\vec{Y}_3\| = \sqrt{\frac{1}{25} + \frac{9}{25} + \frac{36}{25} + \frac{9}{25} + \frac{25}{25}} = \sqrt{\frac{80}{25}} = \frac{4\sqrt{5}}{5}$$

$$\|\vec{Y}_3^*\| = \frac{\cancel{5}}{4\sqrt{5}\cdot\cancel{5}}(-1, 3, 6, 3, -5)$$

$$= \frac{1}{4\sqrt{5}}(-1, 3, 6, 3, -5)$$

Give a basis for $C(A)$.

$$\operatorname{rref}(A) = \begin{array}{c} \begin{bmatrix} 1 & 0 & -4 & -6 & -6 & 5/2 \\ 0 & 1 & 4 & 7 & 7 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccccc} P & P & F & F & F & F \end{array} \end{array}$$

Cols 1, 2 of A.

Basis: $\begin{bmatrix} 2 \\ 0 \\ 0 \\ -6 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ -6 \\ -1 \end{bmatrix}$

(These vectors are perp. to $\vec{Y}_1^*$, $\vec{Y}_2^*$, $\vec{Y}_3^*$)

Example: For matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$,

we know $\text{rank}(A) = 2$

a. $A\vec{x} = \vec{b}$ has a solution whenever

$\vec{b} \in$ ____________________,

or when $\vec{b}$ is onthogenal to any

vector $\vec{c}$ in the ____________________

of A. (We are saying the same thing in two different ways).

b. Find a basis for the left nullspace.

By inspection, $\vec{c} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

(Could take $\text{rref}(A^T)$ and get special sol.)

(Could tack on $\left[A \;\middle|\; \begin{matrix} b_1 \\ b_2 \\ b_3 \end{matrix} \right]$ and do elim.)

c. Project $\vec{b} = \begin{bmatrix} 9 \\ 9 \\ 9 \end{bmatrix}$ onto the $C(A)$.

Note: $\vec{p} = A(A^TA)^{-1}A^T\vec{b}$ fails because the columns of A are not independent.

WAY #1: HACK OFF COLUMN 3 of A.

$$A_{\text{ADJUSTED}} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$$

$$\vec{p} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ 9 \\ 9 \end{bmatrix}$$

$$\vec{p} = \begin{bmatrix} 6 \\ 6 \\ 12 \end{bmatrix}$$

WAY #2 Get projection matrix onto left Nullspace: $\vec{c} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

$$P = \frac{\vec{c}\,\vec{c}^T}{\vec{c}^T\vec{c}} = \frac{1}{3}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\begin{bmatrix} -1 & -1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

To project onto orthogonal complement (i.e. Col.space (A)), use $I - P$

$$I - P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1/3 & 1/3 & -1/3 \\ 1/3 & 1/3 & -1/3 \\ -1/3 & -1/3 & 1/3 \end{bmatrix}$$

$$I - P = \begin{bmatrix} 2/3 & -1/3 & 1/3 \\ -1/3 & 2/3 & 1/3 \\ 1/3 & 1/3 & 2/3 \end{bmatrix}$$

$$\vec{p} = (I-P)\vec{b} = \frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 9 \\ 9 \\ 9 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 18 \\ 18 \\ 36 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 12 \end{bmatrix}$$

Example: Find the determinant of

$$A = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 2 & 1 & 0 & 3 \\ 3 & 0 & 1 & 2 \\ 4 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -10 \\ 0 & 0 & 0 & -15 \end{bmatrix}$$

$$\det(A) = -15$$

Example: Let $\vec{q}_1, \vec{q}_2, \vec{q}_3$ be orthonormal vectors in $\mathbb{R}^3$.

(a.) $\det\begin{bmatrix}\vec{q}_1 & \vec{q}_2 & \vec{q}_3\end{bmatrix} = \pm 1$ (see by def.

Recall $Q^TQ = I$

$$\det(Q^TQ) = 1$$

$$\det(Q^T)\det(Q) = 1$$

$$(\det(Q))^2 = 1$$

$$\det Q = \pm 1$$

(b.) $\det\begin{bmatrix}\vec{q}_1 + q_2 & \vec{q}_2 + \vec{q}_3 & \vec{q}_3 + \vec{q}_1\end{bmatrix} =$

(det are linear by row/column)

$\det\begin{bmatrix}\vec{q}_1 & \vec{q}_2+\vec{q}_3 & \vec{q}_3+\vec{q}_1\end{bmatrix} + \cancel{de}\det\begin{bmatrix}\vec{q}_2 & \vec{q}_2+\vec{q}_3 & \vec{q}_3+\vec{q}_1\end{bmatrix} =$

$\det\begin{bmatrix}\vec{q}_1 & \vec{q}_2+\vec{q}_3 & \vec{q}_3\end{bmatrix} + \cancel{\det\begin{bmatrix}\vec{q}_1 & \vec{q}_2+\vec{q}_3 & \vec{q}_1\end{bmatrix}}$ 0

$+ \cancel{\det\begin{bmatrix}\vec{q}_2 & \vec{q}_2 & \vec{q}_3+\vec{q}_1\end{bmatrix}} + \det\begin{bmatrix}\vec{q}_2 & \vec{q}_3 & \vec{q}_3+\vec{q}_1\end{bmatrix}$

0

$$= \det\begin{bmatrix} \vec{q}_1 & \vec{q}_2 & \vec{q}_3 \end{bmatrix} + \cancel{\det\begin{bmatrix} \vec{q}_1 & \vec{q}_3 & \vec{q}_3 \end{bmatrix}} \to 0$$

$$+ \cancel{\det\begin{bmatrix} \vec{q}_2 & \vec{q}_3 & \vec{q}_3 \end{bmatrix}} + \det\begin{bmatrix} \vec{q}_2 & \vec{q}_3 & \vec{q}_1 \end{bmatrix} =$$

$$0$$

$$= \det\begin{bmatrix} \vec{q}_1 & \vec{q}_2 & \vec{q}_3 \end{bmatrix} + \det\begin{bmatrix} \vec{q}_1 & \vec{q}_2 & \vec{q}_3 \end{bmatrix} =$$

$$2\det\begin{bmatrix} \vec{q}_1 & \vec{q}_2 & \vec{q}_3 \end{bmatrix} = \pm 2$$

<u>Example</u>: Show $\det(P) = 0$ or 1 for P = Projection matrix.

Welp, Projection matrices are idempotent,

So $P^2 = P$

$$\det(P^2) = \det(P)$$

$$(\det(P))^2 = \det(P)$$

$$(\det(P))^2 - \det(P) = 0$$

$$\det(P)(\det(P) - 1) = 0$$

$$\det(P) = 0 \quad \text{or} \quad \det(P) = 1$$

Example: let $\vec{q}_1, \vec{q}_2$ be orthonormal (177)
in $\mathbb{R}^4$ and $\vec{v}$ is not a linear combination of $\vec{q}_1, \vec{q}_2$.

Find $\vec{q}_3$ by Gram-Schmidt.

$$\vec{q}_3 = \vec{v} - \frac{\vec{q}_1 \cdot \vec{v}}{\vec{q}_1 \cdot \vec{q}_1}\vec{q}_1 - \frac{\vec{q}_2 \cdot \vec{v}}{\vec{q}_2 \cdot \vec{q}_2}\vec{q}_2$$

$$\vec{q}_3^{\,*} = \frac{\vec{v} - (\vec{v}^T\vec{q}_1)\vec{q}_1 - (\vec{v}^T\vec{q}_2)\vec{q}_2}{\|\vec{v} - (\vec{v}^T\vec{q}_1)\vec{q}_1 - (\vec{v}^T\vec{q}_2)\vec{q}_2\|}$$

Now, project $\vec{b}$ onto the space spanned by $\vec{q}_1, \vec{q}_2, \vec{q}_3^{\,*}$

$$\vec{p} = A(A^TA)^{-1}A^T\vec{b} =$$

$A A^T \vec{b}$ and since A is filled with orthonormal $\vec{q}_1, \vec{q}_2, \vec{q}_3^{\,*}$

$$\vec{p} = \vec{q}_1(\vec{q}_1^{\,T}\vec{b}) + \vec{q}_2(\vec{q}_2^{\,T}\vec{b}) + \vec{q}_3(\vec{q}_3^{\,T}\vec{b})$$