Chapter 1 Notes

Example: Take vectors $\vec{v}, \vec{w}$ in the plane $\mathbb{R}^2$.

Combinations $c\vec{v} + d\vec{w}$ fill the whole plane unless ....

Example: Take $\vec{v}+\vec{w} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$ and

$\vec{v}-\vec{w} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$.

Compute and draw $\vec{v}, \vec{w}$.

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\vec{v}+\vec{w} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix} \text{ or } \begin{array}{l} v_1 + w_1 = 6 \\ v_2 + w_2 = 2 \end{array}$$

$$\vec{v}-\vec{w} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} - \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \text{ or } \begin{array}{l} v_1 - w_1 = 3 \\ v_2 - w_2 = 3 \end{array}$$

$$\begin{aligned} v_1 + w_1 &= 6 \\ v_1 - w_1 &= 3 \\ 2v_1 &= 9 \end{aligned}$$

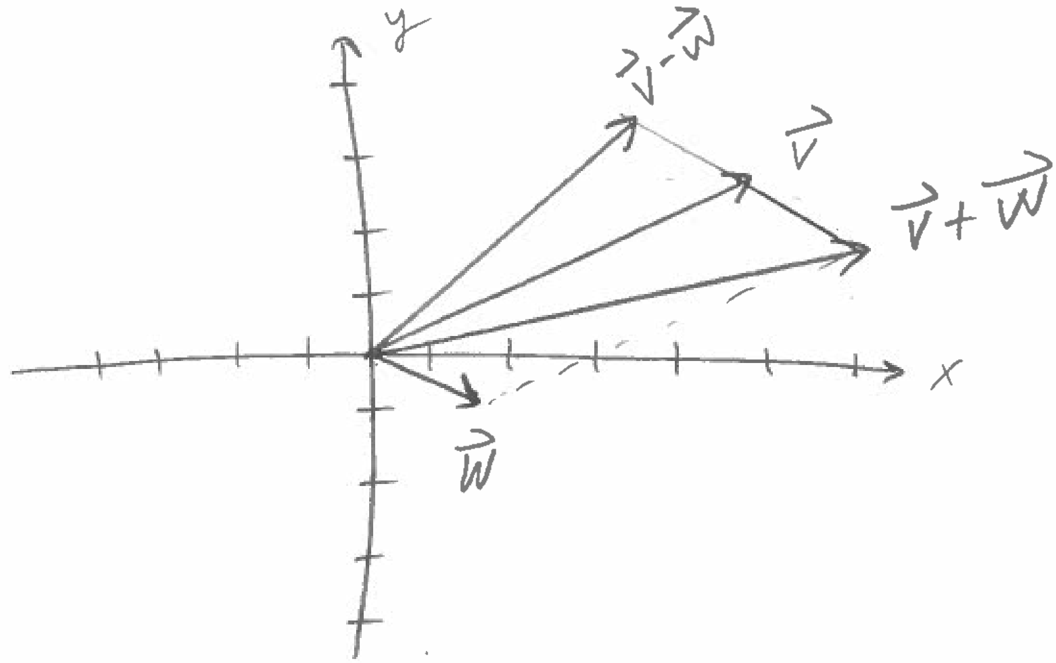
$$\boxed{v_1 = 9/2, \; w_1 = 3/2}$$

$$\begin{aligned} v_2 + w_2 &= 2 \\ v_2 - w_2 &= 3 \\ 2v_2 &= 5 \end{aligned}$$

$$\boxed{v_2 = 5/2, \; W_2 = -1/2}$$

So $\vec{v} = (9/2, 5/2)$ $\quad \vec{v}+\vec{w} = (6,2)$

$\vec{w} = (3/2, -1/2)$ $\quad \vec{v}-\vec{w} = (3,3)$

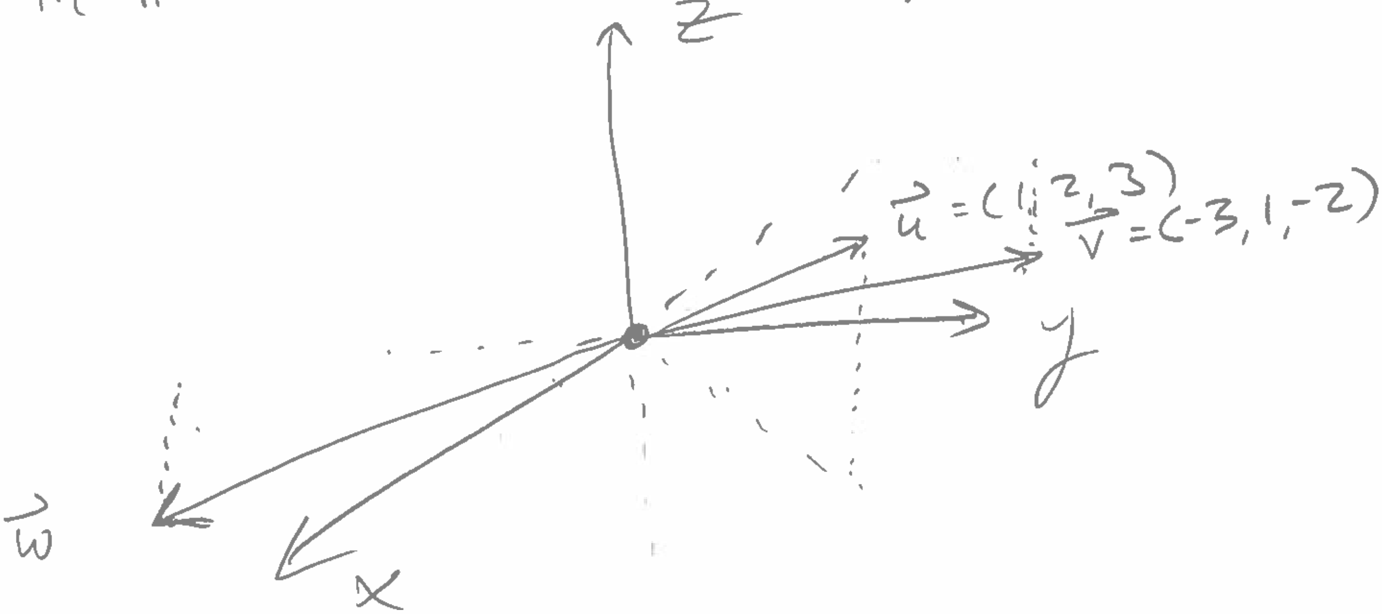


<u>Example:</u> Take $\vec{u} = (1,2,3)$

$\vec{v} = (-3,1,-2)$

$\vec{w} = (2,-3,-1)$.

How can you show these 3 vectors in $\mathbb{R}^3$ lie on the same plane?



Solution: one vector must be a linear combination of the other two.

For instance: $c\vec{u} + d\vec{v} = \vec{w}$ for some c,d

$$c\begin{bmatrix}1\\2\\3\end{bmatrix} + d\begin{bmatrix}-3\\1\\-2\end{bmatrix} = \begin{bmatrix}2\\-3\\-1\end{bmatrix}$$

$$\begin{aligned} c - 3d &= 2 \\ 2c + d &= -3 \\ 3c - 2d &= -1 \end{aligned} \longrightarrow \begin{aligned} c - 3d &= 2 \\ 7d &= -7 \\ 7d &= -7 \end{aligned}$$

So $d = -1$ and then $c = -1$.

Go back to picture. Also note

$-\vec{u} - \vec{v} = \vec{w}$.

$\vec{w}$ is a linear combination of $\vec{u}$ & $\vec{v}$.

Example: Take five random vectors in $\mathbb{R}^3$. Their combinations will certainly fill all of $\mathbb{R}^3$ ________.

How would they not? ________

Do we need 5 vectors to do this?

<u>Example</u>: The linear combinations of $\vec{v} = (1, 1, 0, 0)$ and $\vec{w} = (0, 1, 1, 0)$ will fill a ________________. (We see $\vec{v}$ is not a multiple of $\vec{w}$)

$$c\vec{v} + d\vec{w} = c\begin{bmatrix}1\\1\\0\\0\end{bmatrix} + d\begin{bmatrix}0\\1\\1\\0\end{bmatrix} = \begin{bmatrix}c\\c+d\\d\\0\end{bmatrix}$$

Find another vector on that plane.

Now find one <u>not</u> on that plane.

Does this plane goes thru origin in $\mathbb{R}^4$?

This system of equations will have a solution iff RHS vector $\vec{b}$ is on that plane →

$$\begin{aligned} 1c + 0d &= b_1 \\ 1c + 1d &= b_2 \\ 0c + 1d &= b_3 \\ 0c + 0d &= b_4 \end{aligned}$$

(*) $c\vec{v} + d\vec{w} = \vec{b}$

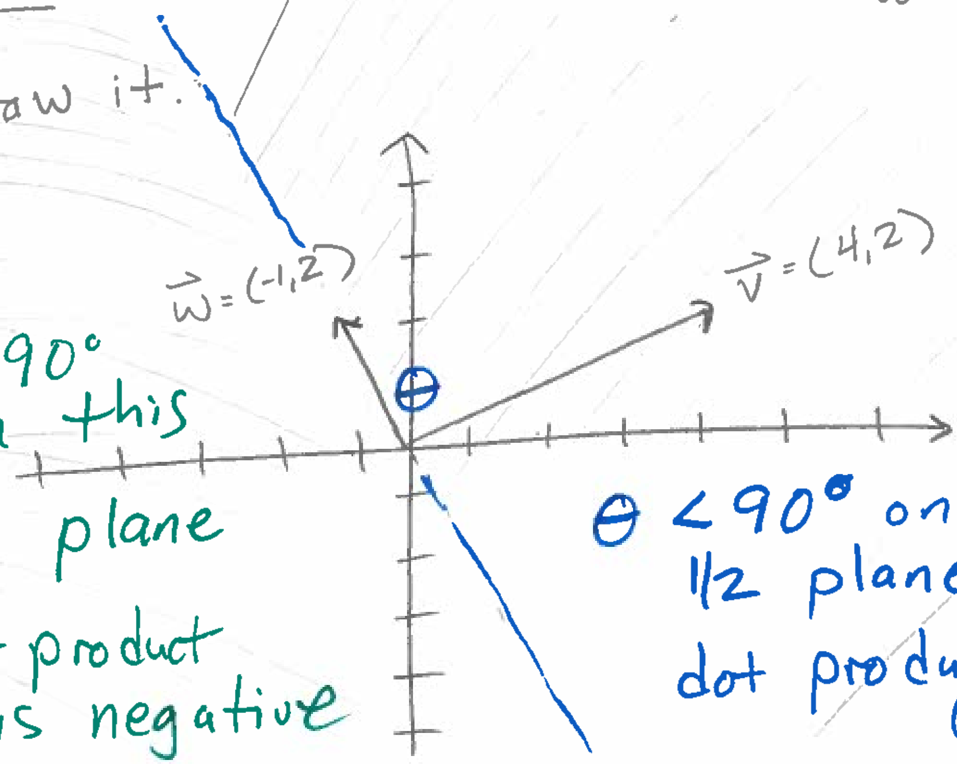
The solution to (*) is the vector $\begin{bmatrix}c\\d\end{bmatrix}$

Dot Product reveals the angle between two vectors.

e.g. in $\mathbb{R}^3$, $\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$

Example: Take $\mathbb{R}^2$ case with $\vec{v} = (4, 2)$, $\vec{w} = (-1, 2)$

Draw it.



$\theta > 90°$ on this 1/2 plane
dot product is negative

$\theta < 90°$ on this 1/2 plane
dot product is + (positive)

Now, $\vec{v} \cdot \vec{w} = 0$ and the vectors are perpendicular.

Take the half planes.
Fold $\vec{w}$ onto $\vec{v}$ so both point in same direction.
(Sum of squares)

Now dot product will blow up if vectors increase in length!

Length: $\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}}$

Cosine Formula: $\cos\theta = \dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\,\|\vec{w}\|}$

And so $\vec{v}\cdot\vec{w} = v_1w_1 + v_2w_2 + \cdots$
$= \|\vec{v}\|\,\|\vec{w}\|\cos\theta$

Example: How long is $\vec{v} = (1, 1, \ldots, 1)$ in 9-dimensions?

$\|\vec{v}\| = \sqrt{1+1+\cdots+1} = \sqrt{9} = 3$

$\vec{v}^* = (1/3, 1/3, \ldots, 1/3)$ is unit and points in the same direction as $\vec{v}$.

Example: Find the angle θ between $\vec{v} = (2, 2, -1)$ and $\vec{w} = (-2, -1, 2)$

Solution: $\cos\theta = \dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\,\|\vec{w}\|} = \dfrac{-4-2-2}{\sqrt{9}\sqrt{9}}$

$\cos\theta = \dfrac{-8}{9}$, $\theta \approx 152.74°$ by TI.

Example: For any unit $\vec{v}, \vec{w}$, find the dot product of $\vec{v}-2\vec{w}$ and $\vec{v}+2\vec{w}$.

Solution: $(\vec{v}-2\vec{w})\cdot(\vec{v}+2\vec{w}) =$

$$\vec{v}\cdot\vec{v} + 2\vec{v}\cdot\vec{w} - 2\vec{w}\cdot\vec{v} - 4\vec{w}\cdot\vec{w} =$$

$$\vec{v}\cdot\vec{v} - 4\vec{w}\cdot\vec{w} = 1-4(1) = 1-4 = -3$$

Is it good?

$$\vec{v}-2\vec{w} = (v_1-2w_1, v_2-2w_2, \dots, v_n-2w_n)$$

$$\vec{v}+2\vec{w} = (v_1+2w_1, v_2+2w_2, \dots, v_n+2w_n)$$

$$(\vec{v}-2\vec{w})\cdot(\vec{v}+2\vec{w}) = (v_1-2w_1)(v_1+2w_1) + (v_2-2w_2)(v_2+2w_2) + \dots + (v_n-2w_n)(v_n+2w_n) =$$

$$(v_1^2-4w_1^2) + (v_2^2-4w_2^2) + \dots + (v_n^2-4w_2^2) =$$

$$(v_1^2+v_2^2+\dots+v_n^2) - 4(w_1^2+w_2^2+\dots+w_n^2) =$$

$$1 - 4(1) = -3 \checkmark$$

Example: $\vec{v} = (1,1)$, $\vec{w} = (1,5)$.

Find c such that $\vec{w} - c\vec{v}$ is perp. to $\vec{v}$.

Need $(\vec{w} - c\vec{v}) \cdot \vec{v} = 0$

$$\left(\begin{bmatrix} 1 \\ 5 \end{bmatrix} - c\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1-c \\ 5-c \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$

$$(1-c) + (5-c) = 0$$

$$6 - 2c = 0$$

$$c = 3$$

$\vec{w} - 3\vec{v} = (-2, 2)$ is perp. to $(1,1)$.

Now find c for any nonzero $\vec{v}, \vec{w}$.

$\vec{v} = (v_1, v_2)$
$\vec{w} = (w_1, w_2)$

$$\rightarrow \left(\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} - c\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

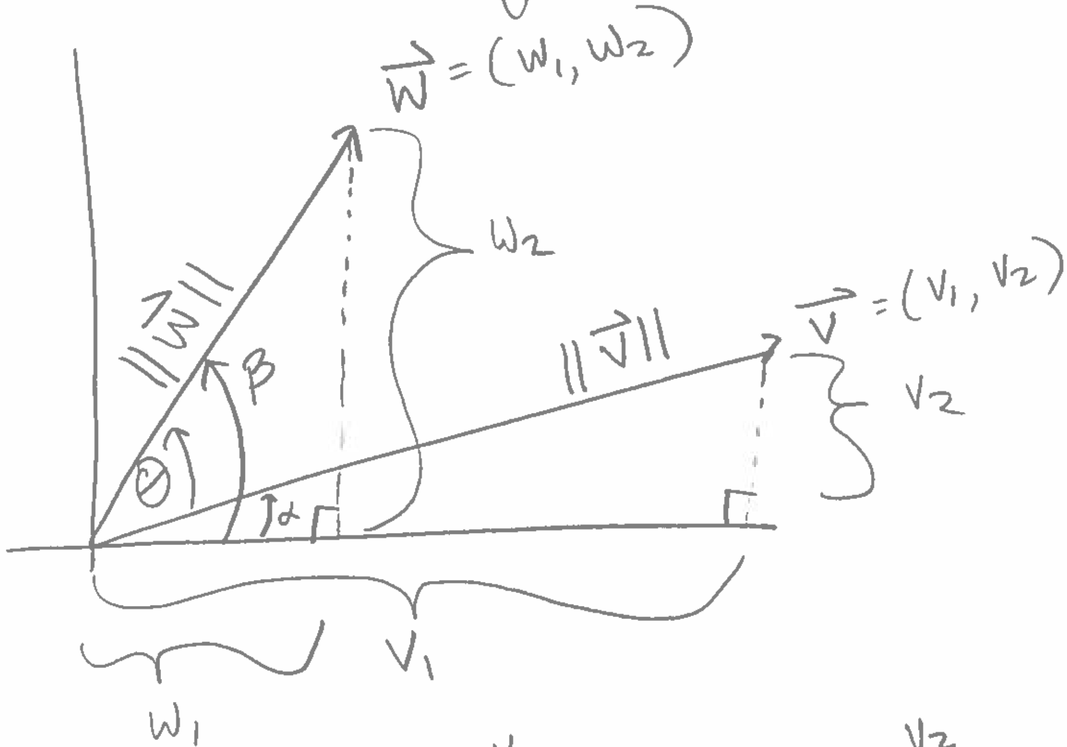
$$\begin{bmatrix} w_1 - cv_1 \\ w_2 - cv_2 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$v_1 w_1 - cv_1^2 + v_2 w_2 - cv_2^2 = 0$$

$$v_1 w_1 + v_2 w_2 = c(v_1^2 + v_2^2)$$

$$c = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|^2}$$

Example: Derive Dot Product Angle Rule:



Observe: $\cos\alpha = \dfrac{v_1}{\|\vec{v}\|}$, $\sin\alpha = \dfrac{v_2}{\|\vec{v}\|}$

$\cos\beta = \dfrac{w_1}{\|\vec{w}\|}$, $\sin\beta = \dfrac{w_2}{\|\vec{w}\|}$

$$\theta = \beta - \alpha$$

Recall from Trig: $\cos(\beta - \alpha) =$

$$\cos\beta\cos\alpha + \sin\beta\sin\alpha$$

And $\cos(\theta) = \dfrac{v_1 w_1}{\|\vec{v}\|\|\vec{w}\|} + \dfrac{w_1 w_2}{\|\vec{v}\|\,\|\vec{w}\|}$

$$\cos(\theta) = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\,\|\vec{w}\|}$$

Example: Given this system of equations:

$$x_1 + 2x_2 + 3x_3 = b_1$$
$$4x_1 + 5x_2 + 6x_3 = b_2$$
$$7x_1 + 8x_2 + 9x_3 = b_3$$

The solution is the unknown vector (x_1, x_2, x_3)

The RHS $(b_1, b_2, b_3) = \vec{b}$ is not the solution. (Here it is not specified)

Write with vectors:

$$x_1 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Write in matrix form:

$$A\vec{x} = \underset{3\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{3\times 1}{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}} = \vec{b}$$

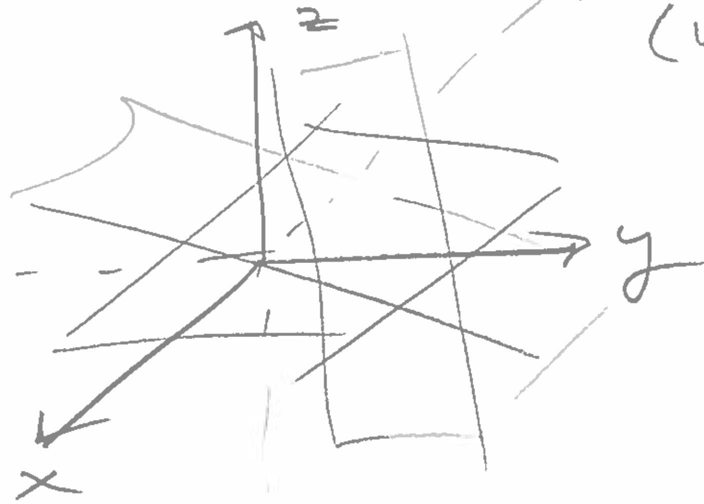
Note: Matrix times a vector gives a vector!

$$A\vec{x} = \vec{b}$$

The "row" picture of this system is 3 planes in 3-dim. space.

"Which planes" depends on the values of b_1, b_2, b_3.

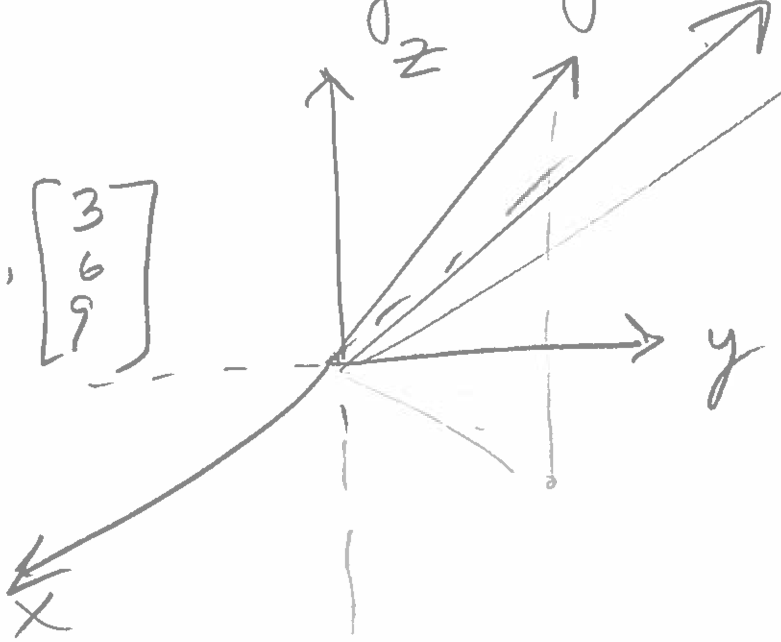
(will shift planes vertically)



The "column" picture of this system is 3 vectors combining to give RHS $\vec{b}$.

vectors are

$$\begin{bmatrix}1\\4\\7\end{bmatrix}, \begin{bmatrix}2\\5\\8\end{bmatrix}, \begin{bmatrix}3\\6\\9\end{bmatrix}$$



If I give you a particular $\vec{b} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$, can you get there?

Is this system solvable for any $\vec{b}$ I give you?

$A\vec{x} = \vec{b}$ has solution

$\vec{x} = A^{-1}\vec{b}$ if A^{-1} exists.

If A^{-1} exists, there will be exactly one unique $\vec{x}$ for any $\vec{b}$ given to you.

What if $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$?

$$x_1 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

One solution is $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

are there other solutions $\vec{x}$?

Example:

$$A\vec{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \vec{b}$$

(a.) Find the four components x_1, x_2, x_3, x_4.

$$x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$$\begin{aligned} x_1 &= b_1 \\ -x_1 + x_2 &= b_2 \\ -x_2 + x_3 &= b_3 \\ -x_3 + x_4 &= b_4 \end{aligned} \quad \longrightarrow \quad \begin{aligned} x_1 &= b_1 \\ x_2 &= b_1 + b_2 \\ x_3 &= b_1 + b_2 + b_3 \\ x_4 &= b_1 + b_2 + b_3 + b_4 \end{aligned}$$

(b.) Write the solution $\vec{x}$ as $\vec{x} = A^{-1}\vec{b}$ to reveal the inverse matrix A^{-1}.

$$\underset{4\times 1}{\vec{x}} = \underset{4\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}} = \underset{4\times 1}{\begin{bmatrix} b_1 \\ b_1 + b_2 \\ b_1 + b_2 + b_3 \\ b_1 + b_2 + b_3 + b_4 \end{bmatrix}} = \underset{4\times 4}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}} \underset{4\times 1}{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}} = A^{-1}\vec{b}$$

(c.) If RHS $\vec{b} = \vec{0}$ the only solution will be $\vec{x} = \vec{0}$. Can you see there are no other solutions $\vec{x}$?

Example: 3 equations, 3 unknowns, 1 solution.

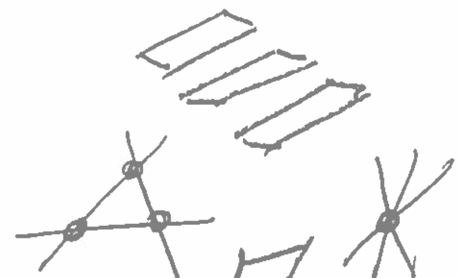
$$
\begin{aligned}
4x + 2y + z &= 1 \\
-4x + 6y + z &= 3 \\
9x + 4y - 5z &= 2
\end{aligned}
$$

Row picture: Each equation describes a plane in 3-D space.

Do any planes go thru the origin?

Possible outcomes:

0, 1, ∞ solutions.



Now, as long as the planes are not parallel, any two intersect in a line, all three intersect at a point (x, y, z), i.e. the solution.

Column Picture:

$A\vec{x} = \vec{b}$ or

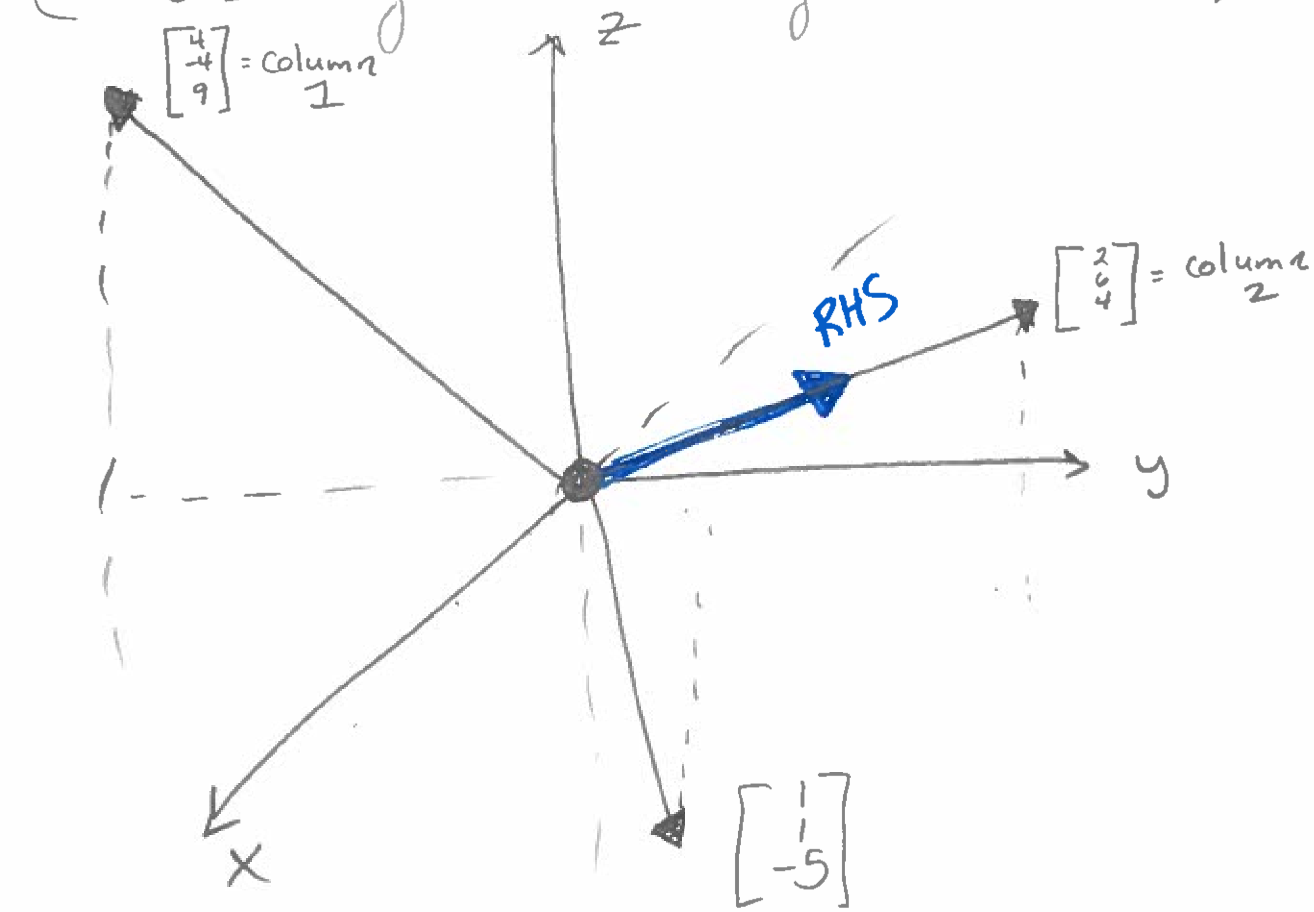
$$\begin{bmatrix} 4 & 2 & 1 \\ -4 & 6 & 1 \\ 9 & 4 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

RHS is a linear comb. of the columns of A.

$$x\begin{bmatrix} 4 \\ -4 \\ 9 \end{bmatrix} + y\begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} + z\begin{bmatrix} 1 \\ 1 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

All problems not this easy, but take $x = 0$, $y = 1/2$, $z = 0$

(Doesn't guarantee only 1 solution)



<u>Example</u>: The sum of equations ①, ② gives ③

① $x + y + z = 2$
② $x + 2y + z = 3$
③ $2x + 3y + 2z = 5$

① and ② intersect in a ______.

The plane ③ contains that line.

Find a few solutions on that line.

$$\begin{aligned} x + y + z &= 2 \\ x + 2y + z &= 3 \end{aligned} \rightarrow \begin{aligned} x + y + z &= 2 \\ y &= 1 \end{aligned} \rightarrow x + z = 1$$

Here is the line.

How about $x = 7$, $z = -6$ and $y = 1$

Point is $(7, 1, -6)$

How about $x = 0$, $z = 1$ and $y = 1$

Point is $(0, 1, 1)$

Now, if you shifted plane ③ up,

Say ③* $2x + 3y + 2z = 9$

Now system has no solutions.

<u>Example</u>: Multiply by dot products and then as combination of the columns.

$$\underset{4\times 5}{\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix}} \underset{5\times 1}{\begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ -2 \end{bmatrix}} = \underset{4\times 1}{\begin{bmatrix} 2+2+0+0+0 \\ 0+4+1+0+0 \\ 0+2+2+2+0 \\ 0+0+1+4-2 \end{bmatrix}} = \underset{4\times 1}{\begin{bmatrix} 4 \\ 5 \\ 6 \\ 3 \end{bmatrix}}$$

OR

$$1\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \\ 3 \end{bmatrix}$$

<u>Important</u>: Matrices act on vectors.

<u>Example</u>: Find a matrix that does each required job.

(a.) 2×2 matrix that leaves a vector unchanged.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x\begin{bmatrix} a \\ c \end{bmatrix} + y\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b.) 2×2 matrix that exchanges entries.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$x\begin{bmatrix} a \\ c \end{bmatrix} + y\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$

(c.) 2×2 matrix rotates vectors by 90° ccw.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix} \quad \text{has } A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

(d.) 3×3 matrix projects vectors to the xy-plane.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x\begin{bmatrix} a \\ d \\ g \end{bmatrix} + y\begin{bmatrix} b \\ e \\ h \end{bmatrix} + z\begin{bmatrix} c \\ f \\ i \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

(e.) 3×3 matrix multiplies (x, y, z) to give $(x, y, x+y+z)$?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ x+y+z \end{bmatrix}$$

<u>Example</u>: Given $-x_{i+1} + 2x_i - x_{i-1} = i$

for $i = 1, 2, 3, 4$ with $x_0 = x_5 = 0$.

Write out the equations in matrix form $A\vec{x} = \vec{b}$.

$i=1$ has $-x_2 + 2x_1 - x_0 = 1$

$i=2$ has $-x_3 + 2x_2 - x_1 = 2$

$i=3$ has $-x_4 + 2x_3 - x_2 = 3$

$i=4$ has $-x_5 + 2x_4 - x_3 = 4$

$$\begin{aligned} 2x_1 - x_2 &= 1 \\ -x_1 + 2x_2 - x_3 &= 2 \\ -x_2 + 2x_3 - x_4 &= 3 \\ -x_3 + 2x_4 &= 4 \end{aligned}$$

$$A\vec{x} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \vec{b}$$

Can you solve for $\vec{x}$?

Solution is $\vec{x} = (4, 7, 8, 6)$

Elimination in section 2.2 upcoming.

Section 2.2. Elimination

Goal: Take original system

$$A\vec{x} = \vec{b}$$

and eliminate to get

$$U\vec{x} = \vec{c} \quad \text{(upper triangular)}$$

Both systems have the same solution.

Example:

$$\begin{aligned} 2x - 3y \phantom{{}+z} &= 3 \\ 4x - 5y + z &= 7 \\ 2x - y - 3z &= 5 \end{aligned} \qquad \begin{bmatrix} 2 & -3 & 0 \\ 4 & -5 & 1 \\ 2 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix}$$

$a_{11} = 2$ in the 1,1 position is the first pivot.
use a_{11} to eliminate $a_{21} = 4$.
Multiplier is $\ell_{21} = 2$

$$\begin{aligned} 2x - 3y \phantom{{}+z} &= 3 \\ y + z &= 1 \\ 2x - y - 3z &= 5 \end{aligned} \qquad \begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & 1 \\ 2 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$$

Now use $a_{11} = 2$ to remove $a_{31} = 2$.
Multiplier $l_{31} = 1$

$$\begin{aligned} 2x - 3y \quad &= 3 \\ y + z &= 1 \\ 2y - 3z &= 2 \end{aligned} \qquad \begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

Now use $a_{22} = 1$ to remove $a_{32} = 2$.
Multiplier is $l_{32} = 2$

$$\begin{aligned} 2x - 3y \quad &= 3 \\ y + z &= 1 \\ -5z &= 0 \end{aligned} \qquad \begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

Now $A\vec{x} = \vec{b}$ is $U\vec{x} = \vec{c}$

Row Picture
3 Planes
intersecting
at (x, y, z)

Row Picture
3 different planes
intersecting at
<u>same</u> (x, y, z).

$U\vec{x} = \vec{c}$ is easy to solve by back-substitution:

$$z = 0, \quad y = 1, \quad x = 3$$

The solution is $\vec{x} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

3×3 system, 3 pivots, exactly one-solution.

If I change RHS $\vec{b}$ to $\vec{0}$, only solution will be $\vec{x} = \vec{0}$.

Does A^{-1} exist? It does.

<u>Example:</u> Solve using elimination:

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 2 \\ 4x_1 + 9x_2 - 3x_3 &= 8 \\ -2x_1 - 3x_2 + 7x_3 &= 10 \end{aligned} \qquad \left[\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 5 & 12 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 4 & 8 \end{array}\right]$$

Pivots: $a_{11} = 2$, $a_{22} = 1$, $a_{33} = 4$

Multipliers: $l_{21} = 2$, $l_{31} = -1$, $l_{32} = 1$

New system is: $2x_1 + 4x_2 - 2x_3 = 2$

$$x_2 + x_3 = 4$$

$$4x_3 = 8$$

So $x_3 = 2$, $x_2 = 2$, $x_1 = -1$

Solution: $\vec{x} = (-1, 2, 2)$

Example: Choose b so the system is singular. Then choose g to make it soluable.

→ 0 or ∞ solutions.

$$2x + by = 16$$
$$4x + 8y = g$$

$\boxed{\text{let } b = 4}$

$$2x + 4y = 16$$
$$4x + 8y = g$$

↓

$$2x + 4y = 16$$
$$0 + 0 = g - 32$$

if $g = 32$ we have ∞ solutions.

if $g \neq 32$ we have 0 solutions.

Example: Derive a test on b_1, b_2 to decide if the system has a solution.

$$\begin{aligned} 3x - 2y &= b_1 \\ 6x - 4y &= b_2 \end{aligned} \rightarrow \begin{aligned} 3x - 2y &= b_1 \\ 0x + 0y &= b_2 - 2b_1 \end{aligned}$$

and the test is:

$$0 = b_2 - 2b_1 \quad \text{or}$$

$$\boxed{b_2 = 2b_1}$$

Example: A system of linear equations has <u>two</u> solutions, (x, y, z) and (X^*, Y^*, Z^*)

It must have ∞ solutions. Find a 3^{rd}.

Solution: We know $A\vec{x} = \vec{b}$ and

$$\begin{aligned} &A\vec{x} = \vec{b} \\ + \; &A\vec{X}^* = \vec{b} \end{aligned}$$

(add)

$$A\vec{x} + A\vec{X}^* = \vec{b} + \vec{b}$$

$$A(\vec{x} + \vec{X}^*) = 2\vec{b}$$

$$A\left(\frac{\vec{x} + \vec{X}^*}{2}\right) = \vec{b}$$

↑

Here is a third solution.

Example: Elimination fails for what values of a?

$$A = \begin{bmatrix} a & 5 & 7 \\ a & a & 6 \\ a & a & a \end{bmatrix}$$

Could do elim.

$a = 0$, $a = 6$, $a = 5$

Example: If the last corner entry is $a_{55} = 11$ and the last pivot is $u_{55} = 4$, what <u>different</u> entry a_{55} would have made A singular?

Solution:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & 11 \end{bmatrix} \sim U = \begin{bmatrix} \cdot & \cdots & \cdot \\ \vdots & & \vdots \\ \cdot & \cdots & 4 \end{bmatrix}$$

How does this change?

Elimination on the bottom row only.

So, to eliminate a_{51} to 0,
the multiplier is $\frac{a_{51}}{a_{11}}$

How does this affect the 11?

$$11 - \frac{a_{51}}{a_{11}} a_{15}$$

Now eliminate a_{52}, a_{53}, a_{54} in turn.

$$11 - \frac{a_{51}}{a_{11}} a_{15} - \frac{a_{52}}{a_{22}} a_{25} - \frac{a_{53}}{a_{33}} a_{35} - \frac{a_{54}}{a_{44}} a_{45} = 4$$

to kill off the last pivot $= 4$,
11 would needed to have been $\underline{7}$

Example: If elimination led to:

$$x + y = 1$$
$$2y = 3,$$

Find an original problem.

Just add some multiple of row 1 to row 2. e.g.

$$x + y = 1$$
$$7x + 9y = 10$$

Section 2.3 Elimination Using Matrices

Example:

$$\begin{aligned} x_1 + x_2 \quad\quad &= 3 \\ 4x_1 + 6x_2 + x_3 &= 15 \\ -2x_1 + 2x_2 \quad\quad &= -2 \end{aligned}$$

Written as $A\vec{x} = \vec{b}$ is

$$\begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 15 \\ -2 \end{bmatrix}$$

① First elimination step is remove $a_{21} = 4$ using pivot $a_{11} = 1$. Multiplier is $l_{21} = 4$.

A matrix multiplication will do step 1.

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

New system of equations / same solution!

$$E_{21} A \vec{x} = E_{21} \vec{b}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 15 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ -2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -2 \end{bmatrix}$$

2. Next elim. step. Remove $a_{31} = -2$ using pivot $a_{11} = 1$. Multiplier is $l_{31} = -2$.

Obtain new system of equations!

$$E_{31} E_{21} A \vec{x} = E_{31} E_{21} \vec{b}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ -2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$$

3. One last step to get upper-triangular system. $A\vec{x} = \vec{b} \longrightarrow U\vec{x} = \vec{c}$

Remove $a_{32} = 4$ using pivot $a_{22} = 2$ with multiplier $l_{32} = 2$.

$$E_{32} E_{31} E_{21} A \vec{x} = E_{32} E_{31} E_{21} \vec{b}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -2 \end{bmatrix}$$

$U\vec{x} = \vec{c}$ is easily solvable:

$$\begin{aligned} x_1 + x_2 \quad &= 3 \\ 2x_2 + x_3 &= 3 \\ -2x_3 &= -2 \end{aligned}$$

$x_3 = 1, \quad x_2 = 1, \quad x_1 = 2$

Solution is $\vec{x} = (2, 1, 1)$

Now, all in one step, take

$$E = E_{32} E_{31} E_{21}$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix}$$

notice the 3 multipliers don't simply copy!

So start at $\quad A\vec{x} = \vec{b}$

Elimination $\quad EA\vec{x} = E\vec{b}$

New system $\quad U\vec{x} = \vec{c} \quad$ soluable by back sub.

Now, since $\vec{x}$ never changed, we can do the work on $[A \mid \vec{b}]$ for simplicity.

Example: Give E_{21}, E_{32}, E_{43}, l_{21}, l_{32}, l_{43}, and U

for $A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$

$l_{21} = -1/2$ and $E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$E_{21}A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$

$l_{32} = -2/3$ and $E_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2/3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$E_{32}E_{21}A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$

$l_{43} = -3/4$ and $E_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3/4 & 1 \end{bmatrix}$

$U = E_{43}E_{32}E_{21}A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}$

Question: Will $A\vec{x} = \vec{b}$ have a solution for any $\vec{b}$? Is it unique?

Row exchanges with P on the left.

$P_{ij}A = A$ with rows i, j swapped.

Take I and swap rows to get desired P.

Example: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 4 & 4 & 4 & 4 \\ 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 \end{bmatrix}$

Multiplying Matrices:

Associate Law: $A(BC) = (AB)C$ ✓

Commutative Law: $AB \neq BA$ ✓
unless you luck out.

Example: Explain. If column 3 of B is all zeros, column 3 of EB must be all zeros, for any B.

Answer: $EB = E\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \cdots & \vec{b}_n \end{bmatrix} = \begin{bmatrix} E\vec{b}_1 & E\vec{b}_2 & E\vec{b}_3 & \cdots \end{bmatrix}$

matrix mult.

Thought of as repeated matrix vector mult.

So column 3 of $E\vec{b}$ is matrix
E times vector $\vec{b}_3 = \vec{0}$ which results
in the $\vec{0}$ for any E.

OK, now let <u>row</u> 3 of B be all zeros.
Explain why <u>row</u> 3 of EB might
not be all zeros.

$$EB = \begin{bmatrix} \text{row } 1 & E \\ \text{row } 2 & E \\ \vdots & \\ \text{row } m & E \end{bmatrix} \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \\ & & & \end{bmatrix}$$

take entry $(3,1)$ in EB.
That is the dot product of row 3 E
with column 1 of B.
This dot product has no guarantee of
being zero, since only component
3 of $\vec{b}_1$ is 0.
Therefore row 3 of EB has
no guarantee to be all 0.

Example: The determinant of

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ is } \det(A) = ad - bc$$

Subtract l times row 1 away from row 2 to produce A^*.

Show $\det(A^*) = \det(A)$.

Solution:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \sim \begin{bmatrix} a & b \\ c - la & d - lb \end{bmatrix} = A^*$$

$$\det(A^*) = ad - lab - bc + lab$$
$$\det(A^*) = ad - bc$$

If this was an elimination step,

$$l = c/a \text{ and } A^* = \begin{bmatrix} a & b \\ 0 & d - \frac{cb}{a} \end{bmatrix}.$$

Product of pivots = $\det(A)$

Example: Parabola $y = a + bx + cx^2$ goes thru $(x,y) = (1,4)$ and $(2,8)$ and $(3,14)$

Find and solve a matrix equation for a, b, and c.

We have:
$$\begin{aligned} 4 &= a + b + c \\ 8 &= a + 2b + 4c \\ 14 &= a + 3b + 9c \end{aligned} \rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 14 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 8 & 10 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

System is now:
$$\begin{aligned} a \quad\quad &= 2 \\ b \quad &= 1 \\ c &= 1 \end{aligned}$$

parabola: $y = 2 + x + x^2$

Section 2.4 : Rules for Matrix Operations

Four ways to multiply matrices:

Example:

$$AB = \underset{3\times 2}{\begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 2 & 1 \end{bmatrix}} \underset{2\times 3}{\begin{bmatrix} 3 & 3 & 0 \\ 1 & 2 & 1 \end{bmatrix}}$$

(1.) Dot Products / Row times column / inner product.

$$AB = \begin{bmatrix} 1\times3+0\times1 & 1\times3+0\times2 & 1\times0+0\times1 \\ 2\times3+4\times1 & 2\times3+4\times2 & 2\times0+4\times1 \\ 2\times3+1\times1 & 2\times3+1\times2 & 2\times0+1\times1 \end{bmatrix} = \underset{3\times3}{\begin{bmatrix} 3 & 3 & 0 \\ 10 & 14 & 4 \\ 7 & 8 & 1 \end{bmatrix}}$$

3x3 filled with 9 dot products

18 multiplications and

9 additions

(2.) Column times row / outer product / sum of n rank 1 matrices

$$AB = \underset{3\times1}{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}} \underset{1\times3}{\begin{bmatrix} 3 & 3 & 0 \end{bmatrix}} + \underset{3\times1}{\begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}} \underset{1\times3}{\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}} =$$

$$= \begin{bmatrix} 3 & 3 & 0 \\ 6 & 6 & 0 \\ 6 & 6 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 4 & 8 & 4 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 0 \\ 10 & 14 & 4 \\ 7 & 8 & 1 \end{bmatrix}$$

3×3 9 mult. | 3×3 9 mult | 3×3 9 additions

3. Matrix A times each column of B.

$$AB = A\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} = \left[\begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 3 & 3 & 0 \\ 10 & 14 & 4 \\ 7 & 8 & 1 \end{bmatrix}$$

linear combination of columns of A

4. each row of A times matrix B

$$AB = \begin{bmatrix} \text{row } 1 \\ \text{row } 2 \\ \text{row } 3 \end{bmatrix} B = \begin{bmatrix} 1 \times \text{row } 1 \text{ of } B + 0 \times \text{row } 2 \text{ of } B \\ 2 \times \text{row } 1 \text{ of } B + 4 \times \text{row } 2 \text{ of } B \\ 2 \times \text{row } 1 \text{ of } B + 1 \times \text{row } 2 \text{ of } B \end{bmatrix}$$

= same 3×3 result.

Notation: If A is square,

$A^p = AAA \cdots A$ (p of them)

A^{-1} is the inverse of A (if it exists)

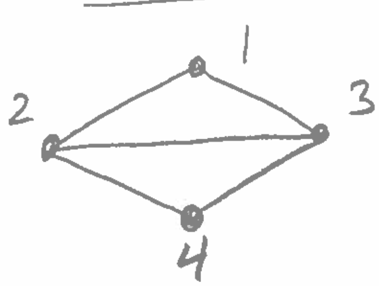
A^{-1} is not $\frac{I}{A}$

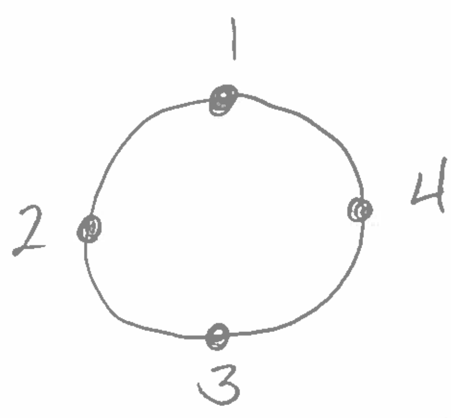
There is no such thing as a negative exponent.

$(A^{-1})^3$ is fine, but not A^{-3}.

Application: A graph/network has n nodes. Its adjacency matrix S has $s_{ij} = 1$ if those nodes are connected, and $s_{ij} = 0$ if not.

Examples:

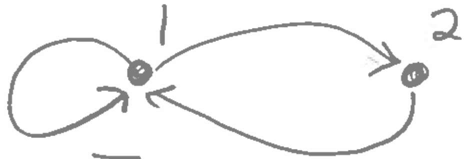




$$S = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \qquad S = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Amazingly, A^2 will count the 2-step paths from i to j, and A^k will count the k-step paths from i to j.

Take:



$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$

List: 1 to 1 in 3 steps: 1-1-1-1
1-2-1-1
1-1-2-1

1 to 2 in 3 steps: 1-1-1-2
1-2-1-2

2 to 1 in 3 steps: 2-1-1-1
2-1-2-1

2 to 2 in 3 steps: 2-1-1-2

Example: a_{ij} is the entry in row i, column j of A.

(a.) Assuming no zeros, give an expression for the 2nd pivot.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \sim \begin{bmatrix} a_{11} & a_{12} & \cdots \\ 0 & a_{22} - \frac{a_{21}}{a_{11}} a_{12} & \\ \vdots & \vdots & \end{bmatrix}$$

$l_{21} = \dfrac{a_{21}}{a_{11}}$, 2nd pivot: $a_{22} - \dfrac{a_{21} a_{12}}{a_{11}}$

(b.) What is the matrix if $a_{ij} = i/j$

$$A = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 2 & 1 & 2/3 \\ 3 & 3/2 & 1 \end{bmatrix}$$

(c.) What is the matrix if $a_{ij} = \min(i,j)$?

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Example: Suppose you solve $A\vec{x} = \vec{b}$ for 3 special RHS $\vec{b}$s.

$$A\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad A\vec{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad A\vec{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Put $\vec{x}_1, \vec{x}_2, \vec{x}_3$ in matrix X.

What is AX?

Solution: $$A\begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$AX = I$$

$$X = A^{-1}$$

Example: If above, we had $\vec{x}_1 = (1,1,1)$, $\vec{x}_2 = (0,1,1)$ and $\vec{x}_3 = (0,0,1)$,

Solve $A\vec{x} = \vec{b}$ when $\vec{b} = (3,5,8)$.

Solution: $$\vec{b} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix} = 3(A\vec{x}_1) + 5(A\vec{x}_2) + 8(A\vec{x}_3)$$

$$= A(3\vec{x}_1 + 5\vec{x}_2 + 8\vec{x}_3)$$

$$= A\left(3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 8\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

$$= A\begin{bmatrix} 3 \\ 8 \\ 16 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}$$

$$\boxed{\vec{x} = \begin{bmatrix} 3 \\ 8 \\ 16 \end{bmatrix}}$$

Section 2.5 Inverse Matrices

- If A is square, A^{-1} (if it exists) is the matrix such that:

$$AA^{-1} = I \quad \text{and} \quad A^{-1}A = I$$

- Tests for invertibility

  1. A has n (nonzero) pivots
  2. $\det(A)$ is not 0
  3. $A\vec{x} = \vec{0}$ has only one solution, $\vec{x} = \vec{0}$

- We can solve:

$$A\vec{x} = \vec{b}$$
$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$
$$I\vec{x} = A^{-1}\vec{b}$$
$$\vec{x} = A^{-1}\vec{b}$$

- Inverses are unique!

<u>Proof</u>: Let $BA = I$ and $AC = I$.

Well, certainly
$$BAC = BAC$$
$$B(AC) = (BA)C$$
$$BI = IC$$
$$B = C.$$

- $(AB)^{-1} = B^{-1}A^{-1}$

  $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

- For a 2×2 matrix, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

  $$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

  The inverse exists iff $\det(A) \neq 0$

  Now show the A^{-1} is correct.

  $$AA^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

  $$= \frac{1}{ad-bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

  $$= \frac{1}{ad-bc}\begin{bmatrix} ad-bc & -ab+ab \\ cd-cd & ad-bc \end{bmatrix}$$

  $$= I \text{ as expected.}$$

- Diagonal matrices D have inverse

  $$D^{-1} = \begin{bmatrix} 1/d_{11} & & 0 \\ & 1/d_{22} & \\ 0 & & \ddots \; 1/d_{nn} \end{bmatrix}$$

Example: Find A^{-1}, B^{-1}, if they exist.

$$A = \begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix} \text{ has } A^{-1} = \frac{1}{4-0}\begin{bmatrix} 2 & 0 \\ -4 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1/2 & 0 \\ -1 & 1/2 \end{bmatrix}$$

$B = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$ has determinant $24-24=0$, so not invertible.

Also $B \sim \begin{bmatrix} 4 & 3 \\ 0 & 0 \end{bmatrix}$ does not have $n=2$ pivots.

Also $B\vec{x} = \vec{0}$ or $\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

has $\vec{x} = (3,-4)$ a nonzero solution to $B\vec{x} = \vec{0}$!

Example: If B is the inverse of A^2, Show AB is the inverse of A.

Proof: We know $A^2B = I$

or $AAB = I$

$A^{-1}AAB = A^{-1}I$

$AB = A^{-1}$ done.

Finding A^{-1} by Gauss-Jordan

- Solve $AA^{-1} = I$ by finding A^{-1} one column at a time.
- Take 3×3 case for example:

$$\underbrace{\begin{bmatrix} | & | & | \\ \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \\ | & | & | \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} | & | & | \\ \vec{a}_1^{-1} & \vec{a}_2^{-1} & \vec{a}_3^{-1} \\ | & | & | \end{bmatrix}}_{A^{-1}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Take this as your unknown solution to $A\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Then repeat for $A\vec{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $A\vec{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

But do it all at once.

To solve any $A\vec{x} = \vec{b}$, we could create an augmented matrix

$$\underset{3\times 4}{[A \mid \vec{b}]} \underset{\text{elimination}}{\sim} [I \mid \vec{x}]$$

$\vec{x}$: Solution to $A\vec{x} = \vec{b}$.

So, take

$$\underset{3\times 6}{[A \mid I]} \underset{\text{elimination}}{\sim} [I \mid A^{-1}]$$

<u>Example</u>: For $A_{3\times 3}$ with $a_{ij} = \min(i,j)$, Find A^{-1}.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$ Now Gauss-Jordan.

Start: $$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] = [I \mid A^{-1}]$$

Example: Go again. $A=\begin{bmatrix}1&1&1\\2&3&2\\3&8&2\end{bmatrix}$

$$[A\,|\,I]=\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\2&3&2&0&1&0\\3&8&2&0&0&1\end{array}\right]\sim$$

$$\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&1&0&-2&1&0\\0&5&-1&-3&0&1\end{array}\right]\sim$$

$$\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&1&0&-2&1&0\\0&0&-1&7&-5&1\end{array}\right]\sim$$

$$\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&1&0&-2&1&0\\0&0&1&-7&5&-1\end{array}\right]\sim$$

$$\left[\begin{array}{ccc|ccc}1&0&0&10&-6&1\\0&1&0&-2&1&0\\0&0&1&-7&5&-1\end{array}\right]\sim$$

$$[I\,|\,A^{-1}]$$

Example: Recall that matrices do things to vectors. Derive the 2×2 ccw rotation-by-θ matrix.

Solution: Strategy: We need two starting vectors and, after rotating by θ, we need to see the result.

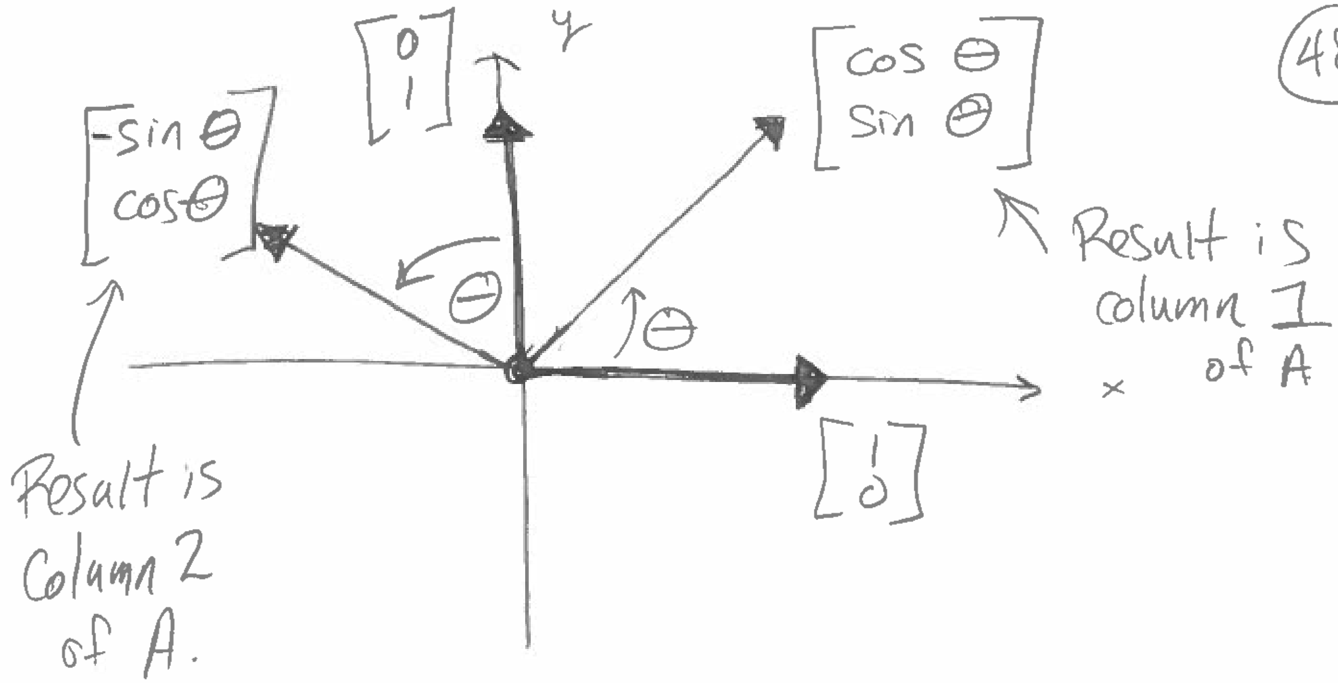
$$A \begin{bmatrix} \text{Vector 1} \end{bmatrix} = \begin{bmatrix} \text{Resulting} \\ \text{vector 1} \end{bmatrix}$$

$$A \begin{bmatrix} \text{Vector 2} \end{bmatrix} = \begin{bmatrix} \text{Resulting} \\ \text{Vector 2} \end{bmatrix}$$

Pick easy starting vectors.

where does $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ end up?

where does $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ end up?



So the 2×2 ccw rotation by θ matrix is $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Test it. Take starting vector $\begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$ and rotate by $45°$ ccw.

$$\begin{bmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} =$$

$$\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{2}{4}+\frac{2}{4} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

OK, so the inverse of ccw rotation is cw rotation. (49.)

Find A^{-1} using Gauss-Jordan.

$$[A|I] = \left[\begin{array}{cc|cc} \cos\theta & -\sin\theta & 1 & 0 \\ \sin\theta & \cos\theta & 0 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{cc|cc} \cos\theta & -\sin\theta & 1 & 0 \\ 0 & \cos\theta + \frac{\sin^2\theta}{\cos\theta} & -\frac{\sin\theta}{\cos\theta} & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{cc|cc} \cos\theta & -\sin\theta & 1 & 0 \\ 0 & \frac{\cos^2\theta + \sin^2\theta}{\cos\theta} & -\frac{\sin\theta}{\cos\theta} & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{cc|cc} \cos\theta & -\sin\theta & 1 & 0 \\ 0 & 1 & -\sin\theta & \cos\theta \end{array}\right] \sim$$

$$\left[\begin{array}{cc|cc} \cos\theta & 0 & 1-\sin^2\theta & \sin\theta\cos\theta \\ 0 & 1 & -\sin\theta & \cos\theta \end{array}\right] \sim$$

$$\left[\begin{array}{cc|cc} 1 & 0 & \cos\theta & \sin\theta \\ 0 & 1 & -\sin\theta & \cos\theta \end{array}\right] = [I|A^{-1}]$$

<u>Example</u>: Suppose A^{-1} exists.

Exchange two rows to obtain B.

How do you find B^{-1} from A^{-1}?

<u>Solution</u>: $PA = B$ where P exchanged rows.

$$(PA)^{-1} = B^{-1}$$

$$A^{-1}P^{-1} = B^{-1}$$

So B^{-1} will be A^{-1} with columns i, j exchanged.

<u>Example</u>: Could a 4×4 matrix A be invertible if we knew each row had $0, 1, 2, -3$ in some order?

<u>Answer</u>: This A is guaranteed to have $\vec{x} = (1,1,1,1)$ as a solution to $A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. A is <u>not</u> invertible.

<u>Example:</u> A is 4×4 with 1's on diag, and $-a, -b, -c$ on the diag. above. Find A^{-1}.

<u>Solution:</u> $[A \mid I] = \left[\begin{array}{cccc|cccc} 1 & -a & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -b & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -c & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & a & ab & abc \\ 0 & 1 & 0 & 0 & 0 & 1 & b & bc \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] = [I / A^{-1}]$$

Test: $AA^{-1} =$

$$\begin{bmatrix} 1 & -a & 0 & 0 \\ 0 & 1 & -b & 0 \\ 0 & 0 & 1 & -c \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a & ab & abc \\ 0 & 1 & b & bc \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \checkmark$$

presume $A^{-1}A$ works.

<u>Example</u>: Let P, Q be any row-permutation matrices.

Show that $P-Q$ is <u>not</u> invertible.

<u>Solution</u>: Recall $A\vec{x} = \vec{0}$ with a nonzero solution $\vec{x}$ is a test for invertibility.

Take a carefully chosen $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

Then $P\vec{x} = \vec{x}$ (row swaps won't alter $\vec{x}$)

Also $Q\vec{x} = \vec{x}$ (same logic).

Thus $P\vec{x} = Q\vec{x}$

$$P\vec{x} - Q\vec{x} = \vec{0}$$

$(P-Q)\vec{x} = \vec{0}$ and the matrix $P-Q$ has a nonzero solution to $(P-Q)\vec{x} = \vec{0}$.

Thus, $P-Q$ is <u>not</u> invertible.

Section 2.6 Elimination = Factorization.

- We will factor A into the product of two matrices, $A = LU$.
- U is upper triangular, pivots on diagonal.
- L is lower triangular, 1's on diagonal, multipliers from elimination below.

Example: 2×2 case.

$$A = \begin{bmatrix} 1 & 1 \\ 7 & 2 \end{bmatrix} \quad \text{has } \ell_{21} = 7$$

$$E_{21} A = \begin{bmatrix} 1 & 0 \\ -7 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -5 \end{bmatrix} = U$$

pivots revealed.

But we want $A = LU$ so

$$E_{21}^{-1} E_{21} A = E_{21}^{-1} U$$

$$A = E_{21}^{-1} U = LU$$

$$A = \begin{bmatrix} 1 & 0 \\ 7 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -5 \end{bmatrix} = LU$$

Example: 3×3 Case

$$\underbrace{E_{32} E_{31} E_{21}}_{\text{elimination steps}} A = U$$

← Result is upper triangular

Now get A by itself:

$$E_{21}^{-1} E_{31}^{-1} \underbrace{E_{32}^{-1} E_{32}}_{I} E_{31} E_{21} A = \underbrace{E_{21}^{-1} E_{31}^{-1} E_{32}^{-1}}_{L} U$$

$$\underbrace{E_{31}^{-1} E_{32}^{-1} E_{32} E_{31}}_{I} \qquad \underbrace{E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} E_{32} E_{31} E_{21}}_{I}$$

Resulting in:

$$A = LU$$

Lucky for us, L has 1's on diagonal, multipliers ℓ exactly in the right place.

Example: Factor $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}$

into $A = LU$.

$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

$\ell_{21} = 2$ $\qquad$ $\ell_{31} = 3$

$E_{31} E_{21} A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 4 & 2 \end{bmatrix}$ and now

$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ with $\ell_{32} = 2$

$E_{32} E_{31} E_{21} A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = U$

$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Example: Factor A into $A = LDU$

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix} \quad \text{has} \quad \begin{array}{l} l_{21} = 1 \\ l_{31} = 1 \\ l_{41} = 1 \end{array}$$

$$E_{41} E_{31} E_{21} A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{bmatrix} \quad \text{has} \quad \begin{array}{l} l_{32} = 1 \\ l_{42} = 1 \end{array}$$

$$E_{42} E_{32} \cdots A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & c-b & d-b \end{bmatrix} \quad \text{has} \quad l_{43} = 1$$

$$U = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

Now matrix D is diagonal with pivots on d_{ii}.

Factor out of U by row:

$$A = LDU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b-a & 0 & 0 \\ 0 & 0 & c-b & 0 \\ 0 & 0 & 0 & d-c \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\qquad\qquad L \qquad\qquad\qquad D \qquad\qquad\qquad U$$

For what values of a, b, c, d is A invertible?

$a \neq 0, \; b \neq a, \; c \neq b, \; d \neq c.$

<u>Example:</u> Factor A into $A = LDL^T$

$$A = \begin{bmatrix} a & a & 0 \\ a & a+b & b \\ 0 & b & b+c \end{bmatrix} \qquad l_{21} = 1$$

$$E_{21}A = \begin{bmatrix} a & a & 0 \\ 0 & b & b \\ 0 & b & b+c \end{bmatrix} \qquad l_{31} = 1$$

$$E_{32}E_{21}A = \begin{bmatrix} a & a & 0 \\ 0 & b & b \\ 0 & 0 & c \end{bmatrix} = U$$

$$A = LDU = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= LDL^T$$

<u>Example</u>: When a 0 appears in a pivot position, $A = LU$ is not possible. Show it here:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l & 1 & 0 \\ m & n & 1 \end{bmatrix} \begin{bmatrix} d & e & g \\ 0 & f & h \\ 0 & 0 & i \end{bmatrix}$$

(not possible)

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} d & e & g \\ dl & el+f & gl+h \\ dm & em+fn & gm+hn+i \end{bmatrix}$$

will have 0 in a_{22} after one step of elim.

By matching entries:

$d = 1$, $e = 1$, $g = 0$ from row 1

$l = 1$, $f = 0$, $h = 2$ from row 2

$m = 1$, $em + fn = 2$

$1 + 0 = 2$ contradiction.

original A cannot be factored into $A = LU$

Section 2.7 Transposes & Permutations

Transpose rules:

$(A^T)_{ij} = A_{ji}$ (Exchanges rows & columns)

$(AB)^T = B^T A^T$

$(A\vec{x})^T = \vec{x}^T A$

$(A^{-1})^T = (A^T)^{-1}$ (must be square)

Symmetric matrices have $S^T = S$

Orthogonal matrices have $Q^T = Q^{-1}$

$\vec{x} \cdot \vec{y} = \vec{x}^T \vec{y}$ (a number)

But $\vec{x}\vec{y}^T$ = a matrix

If S is symmetric, $S = LDU$ is

$S = LDL^T$

$A^T A = 0$ matrix not possible

unless ____________________.

Example: Prove that $(A^{-1})^T = (A^T)^{-1}$

Proof:

$$AA^{-1} = I$$
$$(AA^{-1})^T = I^T \quad \text{(transpose both sides)}$$
$$(A^{-1})^T A^T = I \quad \text{(product rule for trans.)}$$
$$(A^{-1})^T A^T (A^T)^{-1} = I (A^T)^{-1}$$
$$(A^{-1})^T = (A^T)^{-1}$$

Symmetric matrices have $A^T = A$ or $a_{ij} = a_{ji}$

Example: If $A = A^T$ and $B = B^T$, which are also symmetric?

a. $ABAB$? Take transpose: $(ABAB)^T = B^T A^T B^T A^T = BABA$

not symmetric

b. $A^2 - B^2$? Take transpose: $(A^2 - B^2)^T = (A^2)^T - (B^2)^T = A^T A^T - B^T B^T = AA - BB$

yes, symmetric.

Example: Prove $A^T A$ is symmetric for any A.

Proof: $(A^T A)^T = A^T (A^T)^T = A^T A$

Example: Find $A^T, A^{-1}, (A^{-1})^T, (A^T)^{-1}$ for

$$A = \begin{bmatrix} 1 & c \\ c & 0 \end{bmatrix}.$$ Note $A = A^T$

$$A^{-1} = \frac{1}{-c^2} \begin{bmatrix} 0 & -c \\ -c & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1/c \\ 1/c & -1/c^2 \end{bmatrix}$$

$$A^{-1} = (A^{-1})^T = (A^T)^{-1}$$

Example: Factor $S = \begin{bmatrix} 1 & b \\ b & c \end{bmatrix}$ into $S = LDL^T$

$$l_{21} = b, \quad E_{21} S = \begin{bmatrix} 1 & b \\ 0 & c-b^2 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$$

$$S = LDL^T = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & c-b^2 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$

For what c, b is S invertible?
Any c, b except $c = b^2$

Permutation Matrices

P is a permutation matrix if it has the rows of I in any order.

$P^{-1} = P^T$

Example: Explain why $\vec{x} \cdot \vec{y}$ will equal $(P\vec{x}) \cdot (P\vec{y})$ for any $\vec{x}, \vec{y}, P$.

Solution: P applied to $\vec{x}, \vec{y}$ will rearrange the components in each vector (in the same order).

The dot product will remain unchanged except for the order of adding terms.

OR: $(P\vec{x}) \cdot (P\vec{y}) =$

$(P\vec{x})^T (P\vec{y}) =$

$\vec{x}^T P^T P \vec{y} = \vec{x}^T \vec{y} = \vec{x} \cdot \vec{y}$

Since $P^T = P^{-1}$

Example: Find a 4×4 permutation matrix P where $P^4 = I$, $P \neq I$.

Find a 5×5 P with $P^5 \neq I$.

Recall: $A = LU$. Sometimes row exchanges are needed, so $PA = LU$

Example: Factor $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ into $PA = LU$.

Start with $P_{23}A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 4 & 1 \end{bmatrix}$.

Now $l_{21} = 1$, $l_{31} = 2$ and two steps of elimin. leaves $\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = U$

So $PA = LU$ will be

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

<u>Example</u>: Suppose $Q^T = Q^{-1}$

(a.) Show the columns of Q are unit in length.

Well, $Q^T = Q^{-1}$ so $Q^T Q = I$

The diagonal entries of I are all 1's, coming from $\vec{q}_1^{\,T}\vec{q}_1 = 1$,

$\vec{q}_2^{\,T}\vec{q}_2 = 1$, ...

(length squared) $\longrightarrow$ $\vec{q}_n^{\,T}\vec{q}_n = 1$

(b.) Show all columns in Q are mutually perpendicular:

Well, all off diagonal entries in $Q^T Q = I$ are 0, so

$\vec{q}_i^{\,T}\vec{q}_j = 0$ for all $i \neq j$,

and vectors are perp.

(c.) An amazing 2x2 example is

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Example: Block matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$

has transpose $M^T = \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix}$.

Show this for

$$M = \left[\begin{array}{cc|cc} 2 & 4 & 1 & 7 \\ 6 & 9 & 0 & 4 \\ \hline 7 & 9 & 1 & 2 \\ 9 & 7 & 3 & 4 \end{array}\right]$$

$$M^T = \left[\begin{array}{cc|cc} 2 & 6 & 7 & 9 \\ 4 & 9 & 9 & 7 \\ \hline 1 & 0 & 1 & 3 \\ 7 & 4 & 2 & 4 \end{array}\right]$$

When would M be symmetric?

$M = M^T$ requires $A = A^T$
$D = D^T$
and $B^T = C$

Followup: Is $M = \begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$ symmetric?

Not always. Need $A = A^T$.