Ch. 3.1 Vector Spaces and Subspaces

- The most important vector spaces we work with are $\mathbb{R}^1$, $\mathbb{R}^2$, $\mathbb{R}^3$, ..., $\mathbb{R}^n$.
- Think of $\mathbb{R}^3$ as all possible (x, y, z) vectors in 3-dimensional space.
- A vector space must have a zero vector. For $\mathbb{R}^3$ it's $(0,0,0)$.
- A vectorspace is closed under addition & scalar multiplication.
    - In other words, linear combinations of vectors in $\mathbb{R}^3$ stay in $\mathbb{R}^3$.

- A subspace of a vectorspace is a set of vectors that live in a host space such that linear combinations in the subspace stay in the subspace.

For $\mathbb{R}^3$, the only subspaces are: ___

- For two vectors $\vec{v}, \vec{w}$, a <u>linear combination</u> is $c\vec{v}+d\vec{w}$ $\forall c,d \in \mathbb{R}$.

<u>Example</u>: Is $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$?

<u>Example</u>: Do the linear combinations of $\vec{v}=(1,2,2,4)$ and $\vec{w}=(e,\pi,3,4)$ form a subspace of $\mathbb{R}^4$?

<u>yes</u>. $c\vec{v}+d\vec{w}$ will form a <u>plane</u> in $\mathbb{R}^4$.

Does it go thru the origin?

<u>yes</u>. $0\vec{v}+0\vec{w}=\vec{0}=(0,0,0,0)$

<u>Example</u>: Which of the following are <u>subspaces</u> of $\mathbb{R}^3$?

A.) The plane of vectors (b_1, b_2, b_3) with $b_1=0$?

① Does it go thru the origin? yes.

② Does the sum of two vectors retain the form?

yes. $\begin{bmatrix} 0 \\ b_2 \\ b_3 \end{bmatrix} + \begin{bmatrix} 0 \\ b_2^* \\ b_3^* \end{bmatrix} = \begin{bmatrix} 0 \\ b_2+b_2^* \\ b_3+b_3^* \end{bmatrix}$ ✓

3. Does a scalar multiple retain the form?

yes. $c \begin{bmatrix} 0 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 \\ cb_2 \\ cb_3 \end{bmatrix}$ ✓ yes. A subspace.

B.) Vectors (b_1, b_2, b_3) such that $b_1 b_2 b_3 = 0$?

1. Does it go thru origin $(0,0,0)$? yes.
2. Is it closed under addition?

No. For example $\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} e \\ 0 \\ \pi \end{bmatrix} = \begin{bmatrix} 2+e \\ 2 \\ \pi \end{bmatrix}$

does not retain its form.

No. Not a subspace.

C.) Vectors (b_1, b_2, b_3) such that $b_1 = b_2$?

works

1. Origin? yes.
2. Scalar multiplication? $c \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} cb_1 \\ cb_2 \\ cb_3 \end{bmatrix}$
3. Vector addition?

$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + \begin{bmatrix} b_1^* \\ b_2^* \\ b_3^* \end{bmatrix} = \begin{bmatrix} b_1+b_1^* \\ b_2+b_2^* \\ b_3+b_3^* \end{bmatrix}$ Works ✓

Other vector spaces that are not related to $\mathbb{R}^n$:

1.) 7×17 matrices ✓
2.) The integers ✗
3.) The polynomial functions ✓
4.) 4×4 inventible matrices ✗
5.) 4×4 matrices ✓
6.) Real-valued functions, $f(x)$. ✓
7.) The number 0 ✓
8.) The differentiable functions ✓

Note: A subset is a collection of vectors inside a space, but it doesn't meet the requirements of a subspace.

Example: In $\mathbb{R}^3$, the collection of vectors with non-negative entries is a subset of $\mathbb{R}^3$, but not a subspace of $\mathbb{R}^3$.

Columnspace of A

- Denoted $C(A)$, it is the collection of all vectors $A\vec{x}$ for all $\vec{x}$.
- It is the set of linear combinations of the columns of A.
- For $A\vec{x} = \vec{b}$ to be solvable, we need $\vec{b} \in C(A)$.

Example: Prove that $C(A)$ is a subspace of $\mathbb{R}^m$.

Proof: Let A be $m \times n$.

1. Clearly $\vec{0}$ is in $C(A)$, just take 0 times each column of A to generate RHS $\vec{b} = \vec{0}$.
2. Let $\vec{b}, \vec{b}^*$ be in $C(A)$.

   Then $A\vec{x} = \vec{b}$ and $A\vec{x}^* = \vec{b}^*$ for some $\vec{x}, \vec{x}^*$.

   Take the sum →

$$\begin{array}{l} A\vec{x} = \vec{b} \\ A\vec{x}^* = \vec{b}^* \\ \hline A(\vec{x}+\vec{x}^*) = \vec{b}+\vec{b}^* \end{array}$$

The sum of vectors in $C(A)$ remains in the $C(A)$.

(3.) Must show scalar multiples of $\vec{b}$ remain in $C(A)$.

We know $A\vec{x} = \vec{b}$ for some $\vec{x}$.

Take $c\vec{b}$.

Then $A(c\vec{x}) = c\vec{b}$ shows that $c\vec{b}$ is also in $C(A)$.

Thus $C(A)$ is a subspace of $\mathbb{R}^m$.

<u>Notes</u>: $C(A)$ can be all of $\mathbb{R}^m$ or just a part of $\mathbb{R}^m$.
It depends on how many pivot rows A has.

<u>Example</u>: Describe the col. space of each matrix.

A.) $I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

We have a pivot in each row.
$C(I_4)$ will be all of $\mathbb{R}^4$.

B.) $A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

$C(A)$ is all of $\mathbb{R}^4$.

C.) $B = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\begin{array}{ccccc} P & F & F & F & F \end{array}$

We only have one pivot.

$C(B)$ is a line in $\mathbb{R}^2$.

$C(B)$ is all multiples of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

D.) $D = \begin{bmatrix} e & 0 \\ 0 & \pi \\ 0 & 0 \end{bmatrix}$

We have 2 pivots, so $C(D)$ is a 2-dim. subspace of $\mathbb{R}^3$.

$C(D)$ is the plane $c\begin{bmatrix} e \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ \pi \\ 0 \end{bmatrix}$ for any $c, d \in \mathbb{R}$.

E.)

$E = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim$

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{array}{ccc} P & P & F \end{array}$

$C(E)$ is a 2-dim subspace of $\mathbb{R}^3$.

it is a plane spanned by $\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$

Example: For which RHS is $A\vec{x} = b$ solvable?

$$\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$\boxed{\vec{b} \text{ has to be in } C(A)}$

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 2 & 8 & 4 & b_2 \\ -1 & -4 & -2 & b_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 + b_1 \end{array}\right]$$

we need $b_3 = -b_1$

we need $b_2 = 2b_1$

For $A\vec{x} = \vec{b}$ to be solvable, RHS $\vec{b}$ must look like this:

$$\vec{b} = \begin{bmatrix} b_1 \\ 2b_1 \\ -b_1 \end{bmatrix} \quad \text{or} \quad \vec{b} = c\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$C(A)$ is a line in $\mathbb{R}^3$, thru origin, pointing in $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ direction.

Example: If the 9×12 system $A\vec{x} = \vec{b}$ is solvable for every $\vec{b}$, what is the $C(A)$?

$$\underset{9\times 12}{\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}} \underset{12\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{11} \\ x_{12} \end{bmatrix}} = \underset{9\times 1}{\begin{bmatrix} \\ \vec{b} \\ \\ \end{bmatrix}}$$

$C(A)$ is all of $\mathbb{R}^9$.

Example: Find the colspace for:

$$A = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 3 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 2 & 2 & 2 & 4 & 4 & 6 \end{bmatrix} \sim \begin{array}{c} \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 3 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} P & F & P & F & F & F \end{matrix} \end{array}$$

$C(A)$ is a plane in $\mathbb{R}^3$.

$C(A)$ is all linear combinations of

$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Example: A is 24×24 and invertible.

$C(A)$ is __________.

Example: Explain why matrices

A and $[A \;\; AB]$ (with extra columns)

have the same columnspace.

Welp, AB is a matrix filled up with linear combinations of the columns of A. Thus the matrix $[A \;\; AB]$ has the same columnspace as A.

Example: Suppose $A\vec{x} = \vec{b}$ and $A\vec{y} = \vec{b}^*$. Then $A\vec{z} = \vec{b} + \vec{b}$.

What is $\vec{z}$?

Clearly $A\vec{x} + A\vec{y} = \vec{b} + \vec{b}^*$

$A(\vec{x} + \vec{y}) = \vec{b} + \vec{b}^*$

so $\vec{z} = \vec{x} + \vec{y}$.

Thus, if $\vec{b}, \vec{b}^* \in C(A)$

then $\vec{b} + \vec{b}^* \in C(A)$ too.

Section 3.2 : N(A)

The set of solutions to $A\vec{x} = \vec{0}$ form a subspace of $\mathbb{R}^n$.

Proof: Let A be an $m \times n$ matrix:

$$\underset{m \times n}{\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}} \underset{n \times 1}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}} = \underset{m \times 1}{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}$$

✓

① Clearly $\vec{x} = 0$ is in the nullspace of A.

② Let $\vec{x}, \vec{y}$ be in $N(A)$.

Then $A\vec{x} = \vec{0}$ and $A\vec{y} = \vec{0}$.

Sum:
$$\begin{array}{r} A\vec{x} = \vec{0} \\ + \; A\vec{y} = \vec{0} \\ \hline A(\vec{x}+\vec{y}) = \vec{0} \end{array}$$
and $\vec{x}+\vec{y}$ is in $N(A)$. ✓

③ Let $\vec{x} \in N(A)$.

Then $A\vec{x} = \vec{0}$. Take scalar multiple:

$$A(c\vec{x}) = c\vec{0} = \vec{0}$$

and $c\vec{x} \in N(A)$ too.

Since the 3 properties are met, $N(A)$ is a subspace of $\mathbb{R}^n$.

Example: Find the $C(A)$, $N(A)$ of this 2×6 matrix A.

$$A = \begin{bmatrix} 4 & 1 & 0 & 1 & 5 & 11 \\ 8 & 1 & 1 & 3 & 5 & 12 \end{bmatrix} \sim$$

$$\begin{bmatrix} 4 & 1 & 0 & 1 & 5 & 11 \\ 0 & -1 & 1 & 1 & -5 & -10 \end{bmatrix} \sim$$

$$\begin{bmatrix} 4 & 0 & 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & 1 & -5 & -10 \end{bmatrix} \sim$$

$$\begin{array}{cccccc} \begin{bmatrix} 1 & 0 & 1/4 & 1/2 & 0 & 1/4 \\ 0 & 1 & -1 & -1 & 5 & 10 \end{bmatrix} = \operatorname{rref}(A). \\ \end{array}$$

$$\begin{array}{cccccc} P & P & F & F & F & F \end{array}$$

$C(A)$ is all of $\mathbb{R}^2$.

$C(A)$ is all linear comb. of $\begin{bmatrix} 4 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$N(A)$ is all solutions $\vec{x}$ to $A\vec{x} = \vec{0}$.

$N(A)$ is a subspace of $\mathbb{R}^6$.

$N(A)$ is 4-dimensional.

We will describe $N(A)$ by getting the special solutions to $A\vec{x} = \vec{0}$.

$$x_1 \qquad + \tfrac{1}{4}x_3 + \tfrac{1}{2}x_4 \qquad + \tfrac{1}{4}x_6 = 0$$
$$x_2 \quad - x_3 - \quad x_4 + 5x_5 + 10x_6 = 0$$

Set each free variable to 1 with others set to 0. Solve. Repeat 4 times.

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(1) $x_3 = 1, \; x_4 = 0, \; x_5 = 0, \; x_6 = 0$

$\Rightarrow x_1 = -1/4$

$x_2 = 1$

$\vec{S}_1 = (-1/4, 1, 1, 0, 0, 0)$

(2.) $x_3 = 0, \; x_4 = 1, \; x_5 = 0, \; x_6 = 0$

$\Rightarrow x_1 = -1/2$

$x_2 = 1$

$\vec{S}_2 = (-1/2, 1, 0, 1, 0, 0)$

(3.) $x_3 = 0, \; x_4 = 0, \; x_5 = 1, \; x_6 = 0$

$\Rightarrow x_1 = 0$

$x_2 = -5$

$\vec{S}_3 = (0, -5, 0, 0, 1, 0)$

(4.) $x_3 = 0, \; x_4 = 0, \; x_5 = 0, \; x_6 = 1$

$\Rightarrow x_1 = -1/4$

$x_2 = -10$

$\vec{S}_4 = (-1/4, -10, 0, 0, 0, 1)$

The vectors $\vec{s}_1, \vec{s}_2, \vec{s}_3, \vec{s}_4$ form a basis for $N(A)$.

Any linear combination

$$\vec{x} = c_1\vec{s}_1 + c_2\vec{s}_2 + c_3\vec{s}_3 + c_4\vec{s}_4$$ also solves $A\vec{x} = \vec{0}$

$\dim(C(A)) = \#$ pivot variables

$\dim(N(A)) = \#$ free variables

$= n - \#$ pivots.

Note: $N(A)$ is identical to

$N(\mathrm{rref}(A)) = N(R(A))$

So $A\vec{x} = \vec{0}$ or

$R\vec{x} = \vec{0}$ have same nullspace.

<u>Example:</u> Find $C(A)$, $N(A)$ for

$$A = \begin{bmatrix} 12 & \pi & -e & 2 \\ 1 & 1 & 2 & 1 \\ -\pi & 0 & 0 & 0 \\ \log(2) & 4 & 6 & 8 \end{bmatrix}$$

$\mathrm{rref}(A) = I$ and A is inventible.

$C(A)$ is all of $\mathbb{R}^4$. We have 4 pivots.

$N(A)$ is just $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Only one solution to $A\vec{x} = \vec{0}$.

The easy trick to find the special solutions to $A\vec{x} = \vec{0}$.

Example: $A = \begin{bmatrix} 1 & 5 & 4 & 3 & 2 \\ 1 & 6 & 6 & 6 & 6 \\ 1 & 7 & 8 & 10 & 12 \\ 1 & 6 & 6 & 7 & 8 \end{bmatrix}$.

Describe $C(A)$, $N(A)$.

$$\mathrm{rref}(A) = \begin{array}{c} \begin{bmatrix} 1 & 0 & -6 & 0 & 6 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} P & \;P & \;F & \;P & \;F \end{matrix} \end{array}.$$

Pivot columns reveal $C(A)$. It is all linear comb. of $(1,1,1,1)$, $(5,6,7,6)$, $(3,6,10,7)$

- Not all RHS $\vec{b}$ are soluable for $A\vec{x} = \vec{b}$.

$N(A)$, take the two free columns and do the trick:

$$\vec{S}_1 = (6, -2, 1, 0, 0)$$

$$\vec{S}_2 = (-6, 2, 0, -2, 1)$$

and $A\vec{S}_1 = \vec{0}$, $A\vec{S}_2 = \vec{0}$

and $A(c_1\vec{S}_1 + c_2\vec{S}_2) = \vec{0}$

$N(A)$ is a 2-dim. subspace of $\mathbb{R}^5$.

Example: Let A be an invertible 2×2 matrix. Describe all vectors in the nullspace of 2×4 matrix $B = [A \ A]$.

Well, $rref(A) = I_{2\times2}$ and

$$rref(B) = \begin{array}{c} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \\ \begin{array}{cccc} P & P & F & F \end{array} \end{array}$$

$N(A)$ is two-dimensional.

Basis vectors are $\vec{s}_1 = (-1, 0, 1, 0)$
$\vec{s}_2 = (0, -1, 0, 1)$

$N(A)$ is all linear comb. of $\vec{s}_1, \vec{s}_2$.

Example: Construct a matrix whose nullspace consists of all linear comb. of $\vec{s}_1 = (5,4,1,0)$ and $\vec{s}_2 = (2,2,0,1)$.

$$A \underset{4\times1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}} = \vec{0}$$ and A must be $m\times4$

$$A = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 1 & 0 & -5 & -2 \\ 0 & 1 & -4 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccc} P & P & F & F \end{array} \end{array}$$ Works. So does

$$A = \begin{bmatrix} 1 & 0 & -5 & -2 \\ 0 & 1 & -4 & -2 \end{bmatrix}$$

Example: 1. Show $A\vec{x} = \vec{b}$ has no solution when the 9×9 matrix $[A \ \vec{b}]$ is invertible.

The system is 9×8. Augmenting RHS $\vec{b}$ gives an invertible 9×9 matrix with 9 pivots. This shows vector $\vec{b}$ is independent of the 8 columns of A. RHS is not a linear comb. of columns of A!

$A\vec{x} = \vec{b}$ will have no solution!

$\vec{b} \notin C(A)$!

2. So then $A\vec{x} = \vec{b}$ is solvable if the 9×9 matrix $[A \ \vec{b}]$ is singular!

Example: Give the complete solution to

$$x + y + z = 7$$
$$x - y + z = 7.$$

Solution: We have two planes in $\mathbb{R}^3$. Should intersect at a line (unless parallel).

$$[A \mid \vec{b}] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 1 & -1 & 1 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & -2 & 0 & 0 \end{array}\right] \sim$$

$$\begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 7 \\ 0 & 1 & 0 & 0 \end{array}\right] \\ \begin{array}{ccc} P & P & F \end{array} \end{array} \Rightarrow \begin{array}{l} x + z = 7 \\ y = 0 \end{array}$$

So, need a particular solution on the line, we choose $z = 0$.

$\vec{x}_p = (7, 0, 0)$

Now, need special solution $\vec{s}_1$. We choose $z = 1$.

$\vec{s}_1 = (-1, 0, 1)$. (And any multiple.)

Complete Solution: $\vec{x} = \vec{x}_p + c\vec{s}_1 = \begin{bmatrix} 7 - c \\ 0 \\ c \end{bmatrix}$

Definitions:

- **Rank** of a matrix is the number of pivots.
- $\text{rank}(A)$ is the true size of a matrix.
- $\text{rank}(A) = \#$ of independent columns
  $= \#$ of independent rows
- $\dim(C(A)) = r$
- $\dim(N(A)) = n - r$
- If $n > m$, there must be at least one free variable and there must be at least one special solution to $A\vec{x} = \vec{0}$.

Example: Why does no 3×3 matrix have a columnspace that equals its nullspace?

Example: The matrix A reveals two special solutions,

$\vec{s}_1 = \begin{bmatrix} c \\ 1 \\ 0 \end{bmatrix}$, $\vec{s}_2 = \begin{bmatrix} d \\ 0 \\ 1 \end{bmatrix}$ (These solve $A\vec{x} = \vec{0}$).

(a.) Describe all possibilities for the size of A.

Since solutions $\vec{s}_1, \vec{s}_2$ are 3×1,

A must have 3 columns.

Number of rows could be anything.

$$\underset{m\times 3}{\begin{bmatrix} & A & \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{m\times 1}{\begin{bmatrix} \vec{b} \end{bmatrix}}$$

and A is $m \times 3$.

(b.) Give a 1×3 example for A.

Let $A = \begin{bmatrix} 1 & -c & -d \end{bmatrix}$. Then

$$A\vec{s}_1 = \begin{bmatrix} 1 & -c & -d \end{bmatrix} \begin{bmatrix} c \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$A\vec{s}_2 = \begin{bmatrix} 1 & -c & -d \end{bmatrix} \begin{bmatrix} d \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

as required

c. Give a 3×3 example for A.

Let $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -c & -d \\ -\pi & \pi c & \pi d \end{bmatrix}$

d. Describe all possibilities for rank(A).

r must be 1 for any size A.

We have 3 columns, 1 pivot, 2 free.

$r = \#\text{pivots} = 1$ no matter how many rows we add to A.

Example: A is $m\times n$. $A\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ has no solutions.

$A\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ has one solution.

a. Give all information about m, n, r.

$A\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ has one solution tells us

$N(A)$ is just the $\vec{x} = \vec{0}$ vector.

There are no free variables.

Each column has a pivot.

Thus $r = n$.

Also, RHS $\vec{b}$ is 3×1, so $m = 3$.

Also, $A\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ has no solutions tells us $C(A)$ is not all of $\mathbb{R}^3$

So $m = 3$, $r = n < 3$.

We have two possibilities:

(i) $m=3$, $r=1$, $n=1$

(ii) $m=3$, $r=2$, $n=2$

(b.) Give an example matrix A for each (i) (ii)

(i) $A=\begin{bmatrix}0\\1\\0\end{bmatrix}$ works. $m=3$, $n=1$, $r=1$

(ii) $A=\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}$ works $m=3$, $n=2$, $r=2$

Section 3.3 Complete Solution to $A\vec{x}=\vec{b}$

$A\vec{x}=\vec{b}$ has complete solution

$$\vec{x}=\vec{x}_p+\vec{x}_n$$

(1.) To get $\vec{x}_p$, set all free variables to 0 and solve $A\vec{x}=\vec{b}$.

(2.) To get $\vec{x}_n$, set all free variables to 1 (in turn) and solve $A\vec{x}=\vec{0}$.

(3.) The $\vec{x}_n$ part gets scalars c_1, c_2, etc.... The $\vec{x}_p$ does not.

Example: Give the complete solution to $A\vec{x} = \vec{b}$. Also, describe $C(A)$, $N(A)$.

$$A = \begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} -8 \\ -12 \\ -4 \end{bmatrix}$$

$$[A|\vec{b}] = \left[\begin{array}{cccc|c} 2 & 4 & 6 & 4 & -8 \\ 2 & 5 & 7 & 6 & -12 \\ 2 & 3 & 5 & 2 & -4 \end{array}\right] \sim$$

$$\left[\begin{array}{cccc|c} 2 & 4 & 6 & 4 & -8 \\ 0 & 1 & 1 & 2 & -4 \\ 0 & -1 & -1 & -2 & 4 \end{array}\right] \sim$$

$$\left[\begin{array}{cccc|c} 2 & 4 & 6 & 4 & -8 \\ 0 & 1 & 1 & 2 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim$$

$$\left[\begin{array}{cccc|c} 2 & 0 & 2 & -4 & 8 \\ 0 & 1 & 1 & 2 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -2 & 4 \\ 0 & 1 & 1 & 2 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$C(A)$ is a 2-dimensional subspace of $\mathbb{R}^2$.

It is the plane spanned by $(2,2,2)$ and $(4,5,3)$.

Get special solutions to $A\vec{x} = \vec{0}$.

$\vec{s}_1 = (-1, -1, 1, 0)$

$\vec{s}_2 = (2, -2, 0, 1)$

$N(A)$ is a 2-dim subspace of $\mathbb{R}^4$.

It is the plane spanned by $\vec{s}_1, \vec{s}_2$.

Solution $\vec{x}_p$ to $A\vec{x} = \vec{b}$ is same as $R\vec{x} = \vec{d}$.

$$\begin{aligned} x_1 \quad + x_3 - 2x_4 &= 4 \\ x_2 + x_3 + 2x_4 &= -4 \end{aligned}$$

Set $x_3 = 0$, $x_4 = 0$, $\Rightarrow$

$x_1 = 4$, $x_2 = -4$

$\vec{x}_p = (4, -4, 0, 0)$

$\vec{x}_n = c_1 \vec{s}_1 + c_2 \vec{s}_2$

Complete Solution to $A\vec{x} = \vec{b}$ is

$$\vec{x} = \begin{bmatrix} 4 \\ -4 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 - c_1 + 2c_2 \\ -4 - c_1 - 2c_2 \\ c_1 \\ c_2 \end{bmatrix}$$

For any $c_1, c_2 \in \mathbb{R}$.

Example: The complete solution to a square, invertible matrix A with $m=n=r$.

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 4 & 1 & 5 \\ 1 & 4 & 6 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 2 \\ 8 \\ 1 \end{bmatrix}$

The solution will be unique.
The $N(A)$ is just $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$\text{rref}(A) = I$ (3 pivots)

$\vec{x}_p = A^{-1}\vec{b} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$

$\vec{x}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\vec{x} = \vec{x}_p + \vec{x}_n = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$

Matrices With Full Column Rank

- $r = n$
- All columns are pivot columns
- $N(A)$ is just $\vec{0}$.
- <u>If</u> $A\vec{x} = \vec{b}$ has a solution, it is unique.
- Columns of A are independent.

<u>Example:</u> $A = \begin{bmatrix} 2 & 1 \\ 4 & 1 \\ 0 & 1 \end{bmatrix}$ with $m = 3$, $n = 2$, $r = 2$.

(a.) Find solvability conditions for RHS $\vec{b}$.

$$[A|\vec{b}] = \left[\begin{array}{cc|c} 2 & 1 & b_1 \\ 4 & 1 & b_2 \\ 0 & 1 & b_3 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 1 & b_1 \\ 0 & -1 & b_2 - 2b_1 \\ 0 & 1 & b_3 \end{array}\right] \sim$$

$$\left[\begin{array}{cc|c} 2 & 0 & b_2 - b_1 \\ 0 & -1 & b_2 - 2b_1 \\ 0 & 0 & b_3 + b_2 - 2b_1 \end{array}\right] \sim$$

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{b_2 - b_1}{2} \\ 0 & 1 & 2b_1 - b_2 \\ 0 & 0 & b_3 + b_2 - 2b_1 \end{array}\right]$$

For $\vec{b} \in C(A)$ need

$$0 = b_3 + b_2 - 2b_1 \quad \text{or}$$

$$b_3 = 2b_1 - b_2$$

$C(A)$ is a plane in $\mathbb{R}^3$ through origin.
Most $\vec{b}$ not on that plane!
Most $A\vec{x} = \vec{b}$ not solvable.

$N(A)$ is just $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0}$ in $\mathbb{R}^2$.
(No free variables)

Complete Solution to $A\vec{x} = \vec{b}$ is

$$\vec{x} = \vec{x}_p + \vec{x}_n = \begin{bmatrix} \frac{b_2 - b_1}{2} \\ 2b_1 - b_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Check that $\vec{x}_p$ works:

$$A\vec{x}_p = \underbrace{\begin{bmatrix} 2 & 1 \\ 4 & 1 \\ 0 & 1 \end{bmatrix}}_{3 \times 2} \underbrace{\begin{bmatrix} \frac{b_2 - b_1}{2} \\ 2b_1 - b_2 \end{bmatrix}}_{2 \times 1} = \begin{bmatrix} b_2 - b_1 + 2b_1 - b_2 \\ 2b_2 - 2b_1 + 2b_1 - b_2 \\ 2b_1 - b_2 \end{bmatrix}$$

$$A\vec{x}_p = \begin{bmatrix} b_1 \\ b_2 \\ 2b_1 - b_2 \end{bmatrix}$$

Matrices with Full Row Rank

- $r = m$
- All rows are pivot rows, no zero rows.
- $A\vec{x} = \vec{b}$ has a solution for every $\vec{b}$.
- There are $n - r = n - m$ special solutions to $A\vec{x} = \vec{0}$.
- $C(A)$ is all of $\mathbb{R}^m$.
- rows of A are linearly independent.

Example:

$$3x_1 + 6x_2 + 9x_3 + 5x_4 + 25x_5 = 53$$
$$7x_1 + 14x_2 + 21x_3 + 9x_4 + 53x_5 = 97$$
$$-4x_1 - 8x_2 - 12x_3 + 5x_4 + 10x_5 = 46$$

$$\left[\begin{array}{ccccc|c} 3 & 6 & 9 & 5 & 25 & 53 \\ 7 & 14 & 21 & 9 & 53 & 97 \\ -4 & -8 & -12 & 5 & 10 & 46 \end{array}\right] \sim$$

$$\begin{array}{c} \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \\ \begin{array}{ccccc} P & F & F & P & P \end{array} \end{array}$$

Every RHS $\vec{b}$ is soluable since

$m = r = 3$.

To get $\vec{x}_p$, set $x_2 = x_3 = 0$ and solve.

$$\begin{aligned} x_1 \qquad\qquad &= 1 \\ x_4 \quad &= 10 \\ x_5 &= 0 \end{aligned}$$

$\vec{x}_p = (1, 0, 0, 10, 0)$

$\vec{x}_n = c_1(-2, 1, 0, 0, 0) +$
$c_2(-3, 0, 1, 0, 0)$

$$\vec{x} = \vec{x}_p + \vec{x}_n = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 10 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Ex: Write down all known relationships for r, m, n if $A\vec{x} = \vec{b}$ has.....

① No solution for some $\vec{b}$.

$r < m$, $r \leq n$

② Infinitely many solutions for every $\vec{b}$.

$r < n$, $r = m$

③ Exactly 1 solution for some $\vec{b}$, 0 solutions for other $\vec{b}$'.

$r = n$, $r < m$

④ 1 solution for every $\vec{b}$

$r = m = n$

Ex: Find the complete solution to

$$x + y + z = \pi$$

$$\begin{array}{ccc|c} 1 & 1 & 1 & \pi \\ P & F & F & \end{array}$$

$$\vec{x} = \begin{bmatrix} \pi \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \pi - c_1 - c_2 \\ c_1 \\ c_2 \end{bmatrix}$$

Section 3.4 Independence, Basis, Dimension.

<u>Linear</u> <u>Independence</u> : A sequence of vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ is linearly independent iff

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0} \text{ only}$$

when all $c_i = 0$.

Vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are <u>linearly independent</u> iff $\text{rref}\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}$ produces a pivot in each column.

Columns of A are linearly independent iff the only solution to $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$.

Example: Are (1,1,1,1), (1,2,3,4) and (1,4,7,10) linearly independent? NO.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 7 \\ 1 & 4 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \\ 0 & 3 & 9 \end{bmatrix} \sim \begin{array}{c} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} P & P & F \end{matrix} \end{array}$$

Example: Are $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ linearly independent? No!

Too many vectors!

$\mathbb{R}^2$ is two-dimensional.

Cannot have more than 2 linearly independent vectors!

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{bmatrix} \sim \begin{array}{c} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \end{bmatrix} \\ \begin{matrix} P & P & F & F \end{matrix} \end{array}$$

Pro Tip: Any sequence with a repeated vector is linearly dependent.

Pro Tip: Any sequence with the zero vector is linearly dependent.

Def: A set of vectors <u>span</u> a space if the linear comb. fill the space.

<u>Example:</u> Let $A = \begin{bmatrix} 7 & 0 & 7 \\ 4 & 0 & 4 \\ 1 & 6 & 61 \end{bmatrix}$

* The <u>three</u> columns of A <u>span</u> C(A).
* Note: $\vec{v}_3 = \vec{v}_1 + 10\vec{v}_2$, so columns are dependent.
* $\vec{v}_1$ and $\vec{v}_2$ also span C(A).
* How many vectors do I need to totally describe C(A)?

Def: A <u>basis</u> is a set of <u>linearly</u> independent vectors that <u>span</u> the space.

<u>EX:</u> Give two bases for $\mathbb{R}^3$.

<u>Example</u>: Give two bases for the rowspace of $A = \begin{bmatrix} 1 & 0 & 4 & 4 \\ 2 & 7 & 4 & 4 \end{bmatrix}$.

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 4 & 4 \\ 0 & 7 & -4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 & 4 \\ 0 & 1 & -4/7 & -4/7 \end{bmatrix}$$

Rows 1,2 are pivot rows.

One basis is $[1\ 0\ 4\ 4], [2\ 7\ 4\ 4]$

Another basis is $[1\ 0\ 4\ 4], [0\ 1\ -4/7\ -4/7]$

Another basis is $[3\ 7\ 8\ 8], [2\ 7\ 4\ 4]$

<u>Fact</u>: A basis has exactly the right number of vectors: Enough to span the space, but still linearly independent!

<u>Fact</u>: Vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ form a basis for $\mathbb{R}^n$ exactly when they are the columns of an invertible $n \times n$ matrix.

Theorem: For vector $\vec{v}$ in vectorspace V, there is only one way to write $\vec{v}$ as a linear combination of a set of basis vectors.

Quick Illustration: In $\mathbb{R}^3$, a basis is
$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ and}$$
$$\begin{bmatrix} \pi \\ e \\ \sqrt{2} \end{bmatrix} = \pi \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + e \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \sqrt{2} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Proof: Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be a basis for V.
Suppose $\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$
and also $\vec{v} = b_1\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n$.
(Now subtract)
$$\vec{0} = \vec{v} - \vec{v} = (a_1 - b_1)\vec{v}_1 + (a_2 - b_2)\vec{v}_2 + \cdots + (a_n - b_n)\vec{v}_n$$

By def, a basis must have independent vectors, so the only comb. of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ to get the $\vec{0}$ must have all coeff. equal to 0.

Thus, $a_1 - b_1 = 0$ or $a_1 = b_1$

$\vdots$

$a_n - b_n = 0$ or $a_n = b_n$.

Therefore the combination of $\vec{v}_1 \ldots \vec{v}_n$ to give $\vec{v}$ is unique.

Example: If $\vec{w}_1, \vec{w}_2, \vec{w}_3$ are independent, show that

$$\vec{v}_1 = \vec{w}_2 - \vec{w}_3$$
$$\vec{v}_2 = \vec{w}_1 - \vec{w}_3 \quad \text{and}$$
$$\vec{v}_3 = \vec{w}_1 - \vec{w}_2 \quad \text{are dependent.}$$

Solution: Find a combination that goes to $\vec{0}$.

$$\vec{v}_1 - \vec{v}_2 + \vec{v}_3 = (\vec{w}_2 - \vec{w}_3) - (\vec{w}_1 - \vec{w}_3) + (\vec{w}_1 - \vec{w}_2)$$
$$= \vec{0}$$

Example: Give a basis for the space of all 2×3 matrices whose columns add to 0.

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

Example: Give a basis for the plane $x-2y+3z=0$ in $\mathbb{R}^3$.

$A = \begin{bmatrix} 1 & -2 & 3 \end{bmatrix}$ has nullspace basis

$\vec{s}_1 = (2, 1, 0)$

$\vec{s}_2 = (-3, 0, 1)$

Find a basis for the line of vectors perpendicular to that plane: $\begin{bmatrix} 1 & -2 & 3 \end{bmatrix}$

Find a basis for the intersection of $x-2y+3z=0$ with the xy-plane.

xy-plane has $z=0$, so

$x-2y+0z=0$ is the line of intersection.

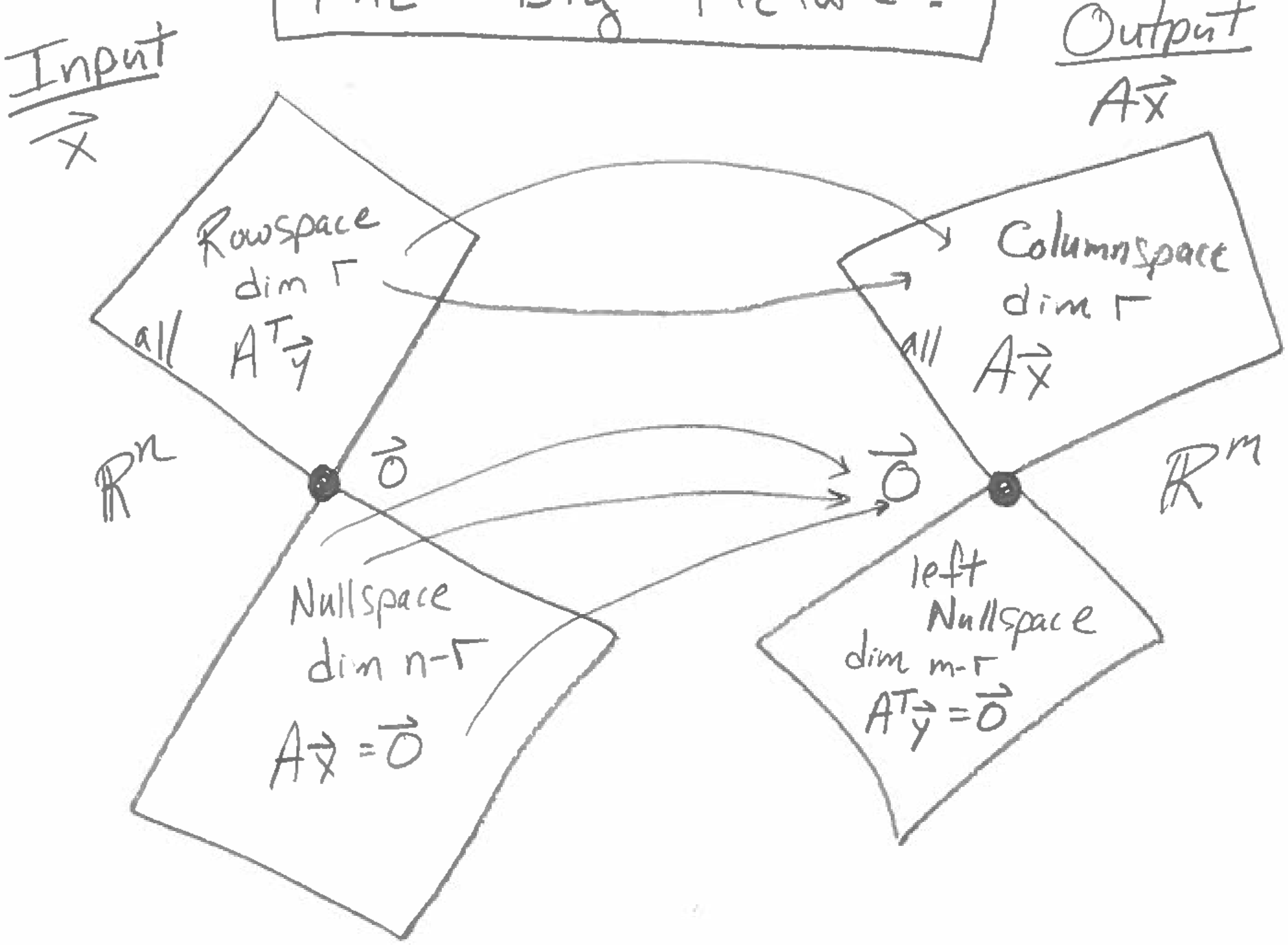
A basis for that line is $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

Fundamental Theorem of Linear Algebra

$\dim(\text{Columnspace}(A)) = r$
$\dim(\text{left Null}(A)) = m - r$
> Carve up $\mathbb{R}^m$

$\dim(\text{rowspace}(A)) = r$
$\dim(\text{nullspace}(A)) = n - r$
> Carve up $\mathbb{R}^n$

The Big Picture.



What is....

<u>Columnspace(A)</u>: Set of all linear comb. of columns of A.
Set of all vectors $A\vec{x}$ $\forall \vec{x}$.

<u>Left Nullspace (A)</u>: Set of solutions

to $A^T\vec{y} = \vec{0}$ OR
$(\vec{y}^T A)^T = \vec{0}^T = \vec{0}$

It is the set of linear comb. of the rows of A to generate the zero row.

<u>Rowspace(A)</u>: Set of all linear comb. of the rows of A.
Set of all vectors $A^T\vec{y}$ $\forall \vec{y}$.

<u>Nullspace(A)</u>: Set of all solutions

to $A\vec{x} = \vec{0}$.

It is the set of all linear combinations of the columns to generate the zero column.

EXAMPLE : GIVE BASES FOR THE 4 FUNDAMENTAL SUBSPACES FOR INVERTIBLE $A = \begin{bmatrix} 12 & 1 & 2 \\ 15 & 0 & 15 \\ 0 & 7 & 14 \end{bmatrix}$

Solution: $\text{rank}(A) = 3$

$\dim(C(A)) = r = 3$ with basis $\begin{bmatrix} 12 \\ 15 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 15 \\ 14 \end{bmatrix}$

$\dim(N(A^T)) = m - r = 3 - 3 = 0$
with basis $\emptyset$.

$\dim(C(A^T)) = r = 3$ with basis
$[12 \;\; 1 \;\; 2]$,
$[15 \;\; 0 \;\; 15]$,
$[0 \;\; 7 \;\; 14]$

$\dim(N(A)) = n - r = 3 - 3 = 0$
with basis $\emptyset$.

EX: Give bases for the four fund....

$$A = \begin{bmatrix} 12 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 15 & 0 & 0 & 15 \\ 0 & 7 & 0 & 14 \end{bmatrix}$$ now 4×4.

$C(A)$ is 3-dimensional subspace of $\mathbb{R}^4$.

A basis is $\begin{bmatrix} 12 \\ 0 \\ 15 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 15 \\ 14 \end{bmatrix}$

$N(A^T)$ is a 1-dimensional subspace of $\mathbb{R}^4$.

A basis is $\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$

Check: $\underset{1 \times 4}{\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}} \underset{4 \times 4}{\begin{bmatrix} 12 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 15 & 0 & 0 & 15 \\ 0 & 7 & 0 & 14 \end{bmatrix}} = \underset{1 \times 4}{\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}}$

$C(A^T)$ is a 3-dim. subspace of $\mathbb{R}^4$.

A basis is $\begin{bmatrix} 12 & 1 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 15 & 0 & 0 & 15 \end{bmatrix},$
$\begin{bmatrix} 0 & 7 & 0 & 14 \end{bmatrix}$

$N(A)$ is a 1-dim. subspace of $\mathbb{R}^4$.

A basis is $(0, 0, 1, 0)$

Tip: One way to find a basis for the left nullspace is to tack on RHS $\vec{b}$ and do elimination.

Example: Give bases for the $C(A)$, $N(A^T)$.

for $A = \begin{bmatrix} 3 & 6 & 6 & 9 & 12 \\ 9 & 4 & 1 & 0 & 0 \\ 15 & 2 & -4 & -9 & -12 \end{bmatrix}$

$m=3$, $n=5$, $r=\underline{2}$.

$$[A|\vec{b}] = \left[\begin{array}{ccccc|c} 3 & 6 & 6 & 9 & 12 & b_1 \\ 9 & 4 & 1 & 0 & 0 & b_2 \\ 15 & 2 & -4 & -9 & -12 & b_3 \end{array}\right] \sim$$

$$\left[\begin{array}{ccccc|c} 3 & 6 & 6 & 9 & 12 & b_1 \\ 0 & -14 & -17 & -27 & -36 & b_2-3b_1 \\ 0 & -28 & -34 & -54 & -72 & b_3-5b_1 \end{array}\right] \sim$$

$$\begin{array}{c} \left[\begin{array}{ccccc|c} 1 & 2 & 2 & 3 & 4 & b_1/3 \\ 0 & 1 & 17/14 & 27/14 & 36/14 & (3b_1-b_2)/14 \\ 0 & 0 & 0 & 0 & 0 & b_3-2b_2+b_1 \end{array}\right] \\ \begin{array}{ccccc} P & P & F & F & F \end{array} \end{array}$$

$C(A)$ is 2 dim. with basis $\begin{bmatrix} 3 \\ 9 \\ 15 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$

$N(A^T)$ is 1 dim. with basis $\begin{bmatrix} 1 & -2 & 1 \end{bmatrix}$

$C(A)$ is a plane in $\mathbb{R}^3$. $N(A^T)$ is the perpendicular line!!

Example: Give bases for the 4 fund. subspaces for

$$A = \begin{bmatrix} 1 & 3 & 1 & 2 & 4 \\ 0 & 0 & 1 & 2 & 1 \\ 2 & 6 & 2 & 4 & 8 \\ 0 & 0 & 2 & 4 & 2 \end{bmatrix}$$

with $m=4$, $n=5$

$r = \underline{2}$.

$$[A|\vec{b}] \sim \left[\begin{array}{ccccc|c} 1 & 3 & 1 & 2 & 4 & b_1 \\ 0 & 0 & 1 & 2 & 1 & b_2 \\ 2 & 6 & 2 & 4 & 8 & b_3 \\ 0 & 0 & 2 & 4 & 2 & b_4 \end{array}\right] \sim$$

$$\left[\begin{array}{ccccc|c} 1 & 3 & 1 & 2 & 4 & b_1 \\ 0 & 0 & 1 & 2 & 1 & b_2 \\ 0 & 0 & 0 & 0 & 0 & b_3-2b_1 \\ 0 & 0 & 2 & 4 & 2 & b_4 \end{array}\right] \sim$$

$$\left[\begin{array}{ccccc|c} 1 & 3 & 1 & 2 & 4 & b_1 \\ 0 & 0 & 1 & 2 & 1 & b_2 \\ 0 & 0 & 0 & 0 & 0 & b_3-2b_1 \\ 0 & 0 & 0 & 0 & 0 & b_4-2b_2 \end{array}\right]$$

$$\text{rref}(A) = \begin{array}{c} \begin{bmatrix} 1 & 3 & 0 & 0 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} P & F & P & F & F \end{array} \end{array}$$

$\dim(C(A)) = r = 2$ with basis $\begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$

$\dim(N(A^T)) = m - r = 4-2 = 2$ with basis
$\begin{bmatrix} -2 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 & 0 & 1 \end{bmatrix}$

We've carved up $\mathbb{R}^4$ into two perp. subspaces !!

$\dim(C(A^T)) = r = 2$, basis is
$\begin{bmatrix} 1 & 3 & 0 & 0 & 3 \end{bmatrix}$,
$\begin{bmatrix} 0 & 0 & 1 & 2 & 1 \end{bmatrix}$

$\dim(N(A)) = n - r = 5-2 = 3$, a basis is $\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

We've carved up $\mathbb{R}^5$ into two perp. subspaces !!
