7.1, 7.2 SVD

- If we have a complete set of eigenvectors,

$$A = \underline{X}\,\underline{\Lambda}\,\underline{X}^{-1}$$

indep. but not orthogonal.

- If $A = S$ (symmetric),

$$S = Q\,\underline{\Lambda}\,Q^{-1} = Q\,\underline{\Lambda}\,Q^{T}$$

orthogonal eigenvectors (can be chosen normal)

- If S is positive definite,

$$S = Q\,\underline{\Lambda}\,Q^{T}$$

eigenvalues are positive

- Now, for any matrix A, $m \times n$, rank r

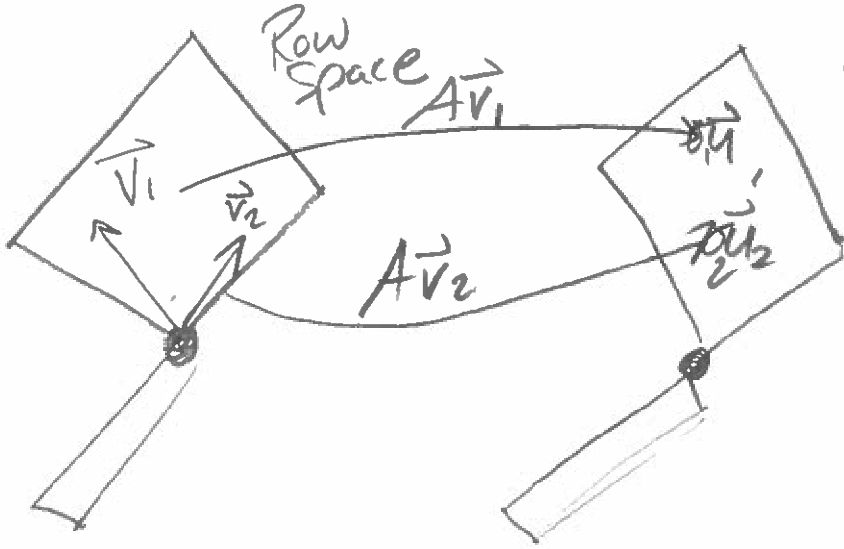
Diagonal, positive numbers.

$$A = U\,\Sigma\,V^{T}$$

orthonormal

From Video 29 / Text

Want an orthonormal basis for row space to go to an orthogonal basis for columnspace



Rank is r

so . . .

$A\vec{v}_r = \vec{u}_r$

σ_r

Main Equation of SVD:

$$A\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_r \end{bmatrix} = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_r \end{bmatrix}\begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \ddots & \\ & & & \sigma_r \end{bmatrix}$$

$$\underset{(m\times n)}{A}\ \underset{(n\times r)}{V} = \underset{(m\times r)}{U}\ \underset{(r\times r)}{\Sigma}$$

$A\vec{v}_1 = \sigma_1\vec{u}_1$

$A\vec{v}_2 = \sigma_2\vec{u}_2$

$\vdots$

$A\vec{v}_r = \sigma_r\vec{u}_r$

$$\sigma_1, \sigma_2, \ldots, \sigma_r > 0$$

Now, fill in with orthonormal basis vectors for the nullspaces:

$$A\begin{bmatrix} v_1 & v_2 & \cdots & v_r & \underbrace{v_{r+1} & \cdots & v_n}_{\text{null}} \end{bmatrix} = \begin{bmatrix} \underbrace{u_1 & u_2 & \cdots & u_r}_{\text{col}} & \underbrace{u_{r+1} & \cdots & u_m}_{\text{left null}} \end{bmatrix}\begin{bmatrix} \sigma_1 & & & 0 \\ & \ddots & & \\ & & \sigma_r & \\ 0 & & & 0 \end{bmatrix}$$

Row

$$\underset{(m\times n)}{A}\ \underset{(n\times n)}{V} = \underset{(m\times m)}{U}\ \underset{(m\times n)}{\Sigma}$$

$$AV = U\Sigma$$

$$A = U\Sigma V^{-1} = U\Sigma V^T$$

$$\underset{\substack{\text{pos. def.} \\ \text{or} \\ \text{semi def.}}}{A^TA} = (V\Sigma^T U^T)(U\Sigma V^T) = \underset{\substack{\text{eigenvectors} \\ \text{of} \\ A^TA}}{V}\begin{bmatrix} \sigma_1^2 & & \\ & \sigma_2^2 & \\ & & \ddots \end{bmatrix}V^T$$

Do the same for U.

For any rank r matrix A, $m\times n$.

V has eigenvectors of A^TA

U has eigenvectors of AA^T

Σ has positive $\sigma_i = \sqrt{\lambda_i}$

Calculating the SVD:

Want: $A = U \Sigma V^T$

1. Compute $A^TA = V \Sigma^T \Sigma V^T$
   get orthonormal $\vec{v}_1 \cdots \vec{v}_r$ eigenvectors.
   get $\sigma_1 \ldots \sigma_r$ singular values
2. Compute $AV = U\Sigma$
   to find $\vec{u}_1 \cdots u_r$
3. Make sure $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$

$$\vec{u}_i = \frac{A\vec{v}_i}{\sigma_i}$$

Example: Compute $A = U \Sigma V^T$

for $A = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$, rank$(A) = 2$

① $$A^T A = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

(is diagonal) $\lambda_1 = 4$, $\lambda_2 = 1$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (orthonormal)

$\sigma_1 = \sqrt{4} = 2$, $\sigma_2 = \sqrt{1} = 1$

$$V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

② $AV = U\Sigma$

$$AV = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$$

← has eigenvectors of AA^T need to pull out σ_i

← Want this to be U.

Make unit length

$$\begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}}_{\Sigma}$$

③ $$A = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example: Calculate the SVD for

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$ given. Want

$$\underset{4\times 2}{A} = \underset{4\times 4}{U}\ \underset{4\times 2}{\Sigma}\ \underset{2\times 2}{V^T}$$

U (4×4): orthonormal bases for $C(A)$, $N(A^T)$

V (2×2): ortho. normal basis for $C(A^T)$, $N(A)$

$$A^TA = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 4 & 3 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 1 & 3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 11 \\ 11 & 25 \end{bmatrix}$$

A^TA has $\lambda_1 = 29.866$ $\qquad \lambda_2 = 0.134$

$$\vec{v}_1 = \begin{bmatrix} 0.442 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} -2.261 \\ 1 \end{bmatrix}$$

or $\qquad$ OR

$$\vec{v}_1 = \begin{bmatrix} 0.404 \\ 0.915 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} -0.915 \\ 0.404 \end{bmatrix}$$

$$\sigma_1 = \sqrt{\lambda_1} \qquad \sigma_2 = \sqrt{\lambda_2}$$

$$\sigma_1 = 5.465 \qquad \sigma_2 = 0.366$$

$$V = \underset{2\times 2}{\begin{bmatrix} 0.404 & -0.915 \\ 0.915 & 0.404 \end{bmatrix}}, \quad \Sigma = \underset{4\times 2}{\begin{bmatrix} 5.465 & 0 \\ 0 & 0.366 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}$$

(630)

$$U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \vec{u}_4 \end{bmatrix}$$

ortho-Basis for $C(A)$

ORTHO. BASIS FOR $N(A^T)$

$$\vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \vec{u}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$AV = \underbrace{\begin{bmatrix} 2 & 4 \\ 1 & 3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{4\times 2} \underbrace{\begin{bmatrix} 0.404 & -0.915 \\ 0.915 & 0.404 \end{bmatrix}}_{2\times 2} = \begin{array}{cc} \vec{u}_1 & \vec{u}_2 \\ \end{array} \begin{bmatrix} 4.468 & -0.214 \\ 3.149 & 0.297 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

want this to be U, factor out σ

divide $\vec{u}_1$ by σ_1

divide $\vec{u}_2$ by σ_2

$$\vec{u}_1 = \frac{1}{\sigma_1}\begin{bmatrix} 4.468 \\ 3.149 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{u}_2 = \frac{1}{\sigma_2}\begin{bmatrix} -0.214 \\ 0.297 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{u}_1 = \frac{1}{5.465}\begin{bmatrix} 4.468 \\ 3.149 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.818 \\ 0.576 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{u}_2 = \begin{bmatrix} -0.585 \\ 0.811 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = U \Sigma V^T$$

$$= \underbrace{\begin{bmatrix} 0.818 & -0.585 & 0 & 0 \\ 0.576 & 0.811 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{4\times4} \underbrace{\begin{bmatrix} 5.465 & 0 \\ 0 & 0.366 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{4\times2} \underbrace{\begin{bmatrix} 0.404 & 0.915 \\ -0.915 & 0.404 \end{bmatrix}}_{2\times2}$$

<u>Extension:</u> SVD gives us

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T + \cdots + \sigma_r \vec{u}_r \vec{v}_r^T$$

which decomposes A into the sum of r rank 1 matrices.

$$A = 5.465 \begin{bmatrix} 0.818 \\ 0.576 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0.404 & 0.915 \end{bmatrix} + 0.366 \begin{bmatrix} -0.585 \\ 0.811 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -0.915 & 0.404 \end{bmatrix}$$

$$= \begin{bmatrix} 1.806 & 4.090 \\ 1.272 & 2.880 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0.196 & -0.087 \\ -0.272 & 0.120 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2.002 & 4.003 \\ 1 & 3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Example: Every square invertible matrix A can be factored into

$$A = HQ = (\text{Symmetric Pos. Def.})(\text{Orthogonal}).$$

For $A = \begin{bmatrix} 1 & 3 \\ -1 & 3 \end{bmatrix}$, obtain $A = U\Sigma V^T$.

Then choose $Q = UV^T$ and find H.

$$A^TA = \begin{bmatrix} 1 & -1 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 18 \end{bmatrix}$$

$\lambda_1 = 18 \qquad \lambda_2 = 2$

$\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\sigma_1 = 3\sqrt{2} \qquad \sigma_2 = \sqrt{2}$

$$V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

$$\vec{u}_1 = \frac{A\vec{v}_1}{\sigma_1} = \frac{\begin{bmatrix} 1 & 3 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}}{3\sqrt{2}} = \frac{1}{3\sqrt{2}}\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\vec{u}_2 = \frac{A\vec{v}_2}{\sigma_2} = \frac{\begin{bmatrix} 1 & 3 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}}{\sqrt{2}} = \frac{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}{\sqrt{2}} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

(233.)

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{V^T}$$

Choose $Q = UV^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$$A = HQ = H(UV^T)$$

$$= (U \Sigma U^T)(UV^T)$$

$U \Sigma U^T = \begin{bmatrix} 2\sqrt{2} & \sqrt{2} \\ \sqrt{2} & 2\sqrt{2} \end{bmatrix} = H$ (symmetric Pos. Def.)

$UV^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = Q$ (orthogonal)

Ch. 8.1 Linear Transformations

- Transformation T is applied to vector $\vec{v}$. In goes $\vec{v}$, out comes $T(\vec{v})$.
- Analogy: Function f is applied to x. In goes x, out comes $f(x)$.
- Linear transformations will take us from input space $\mathbb{R}^n$ to output space $\mathbb{R}^m$ using matrix A.

  $$T(\vec{v}) = A\vec{v}.$$

- Linear transformations:

  $$T(c\vec{v} + d\vec{w}) = cT(\vec{v}) + dT(\vec{w})$$

  and always $T(\vec{0}) = \vec{0}$.
- Rotations, projections, shearings, stretchings are all linear.

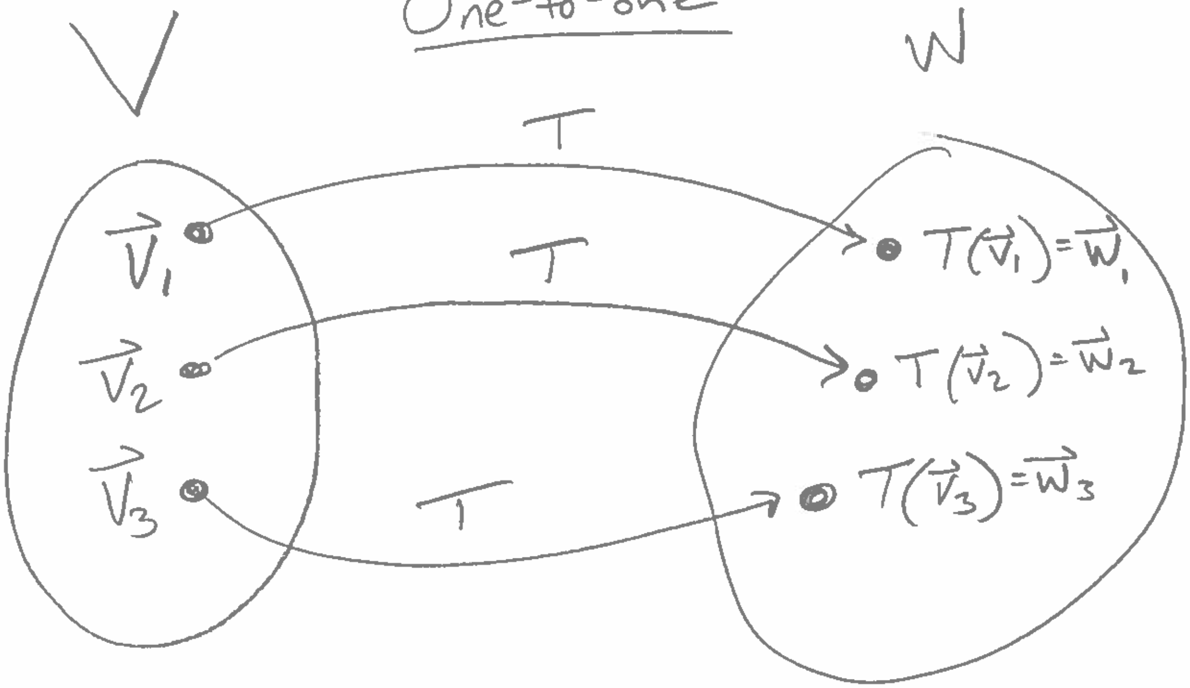
Definitions:

$T : V \to W$

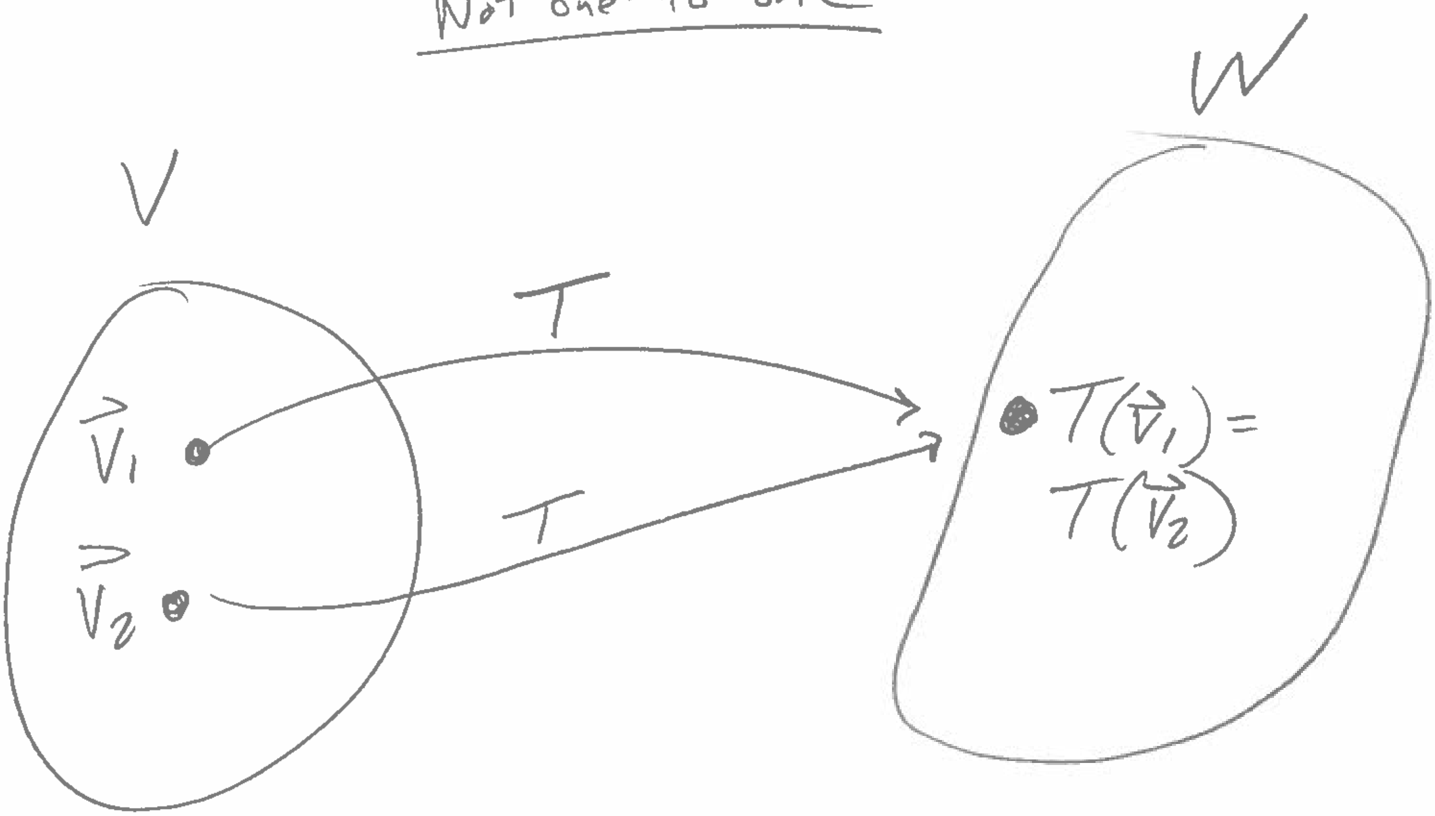
- V is the input space $(\dim(V) = n)$
- W is the output space $(\dim(W) = m)$
- kernal(T) is the set of all inputs for which $T(\vec{v}) = \vec{0}$ (nullspace(A))
- range(T) is the set of all outputs $T(\vec{v})$ (columnspace(A)) (range or image)
- range(T) could be all of W or just part of it.
- dim(input space) or $\dim(V) = \dim(\text{range}) + \dim(\text{kernal})$
- Linear transformation T has matrix A behind it, but there are infinite choices for A. Matrix A is determined by choice of basis for V and W.

- Transformation T is <u>one-to-one</u> if the <u>pre-image</u> of every $\vec{w} \in W$ consists of <u>one</u> vector $\vec{v} \in V$.

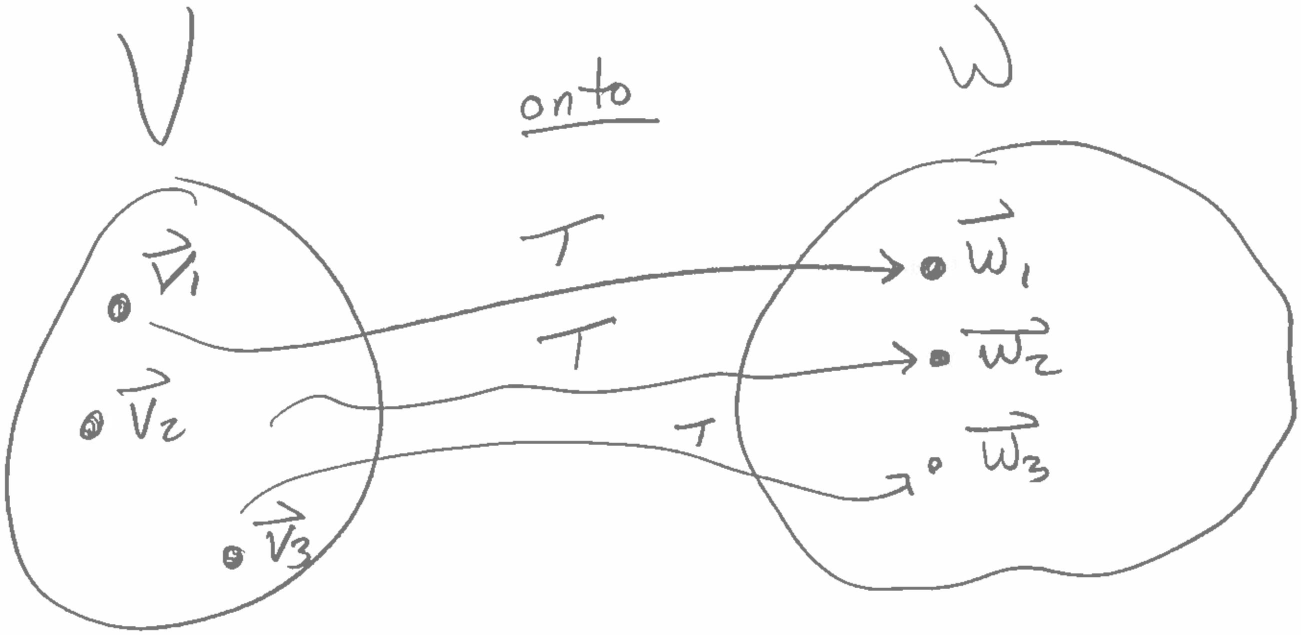
<u>One-to-one</u>

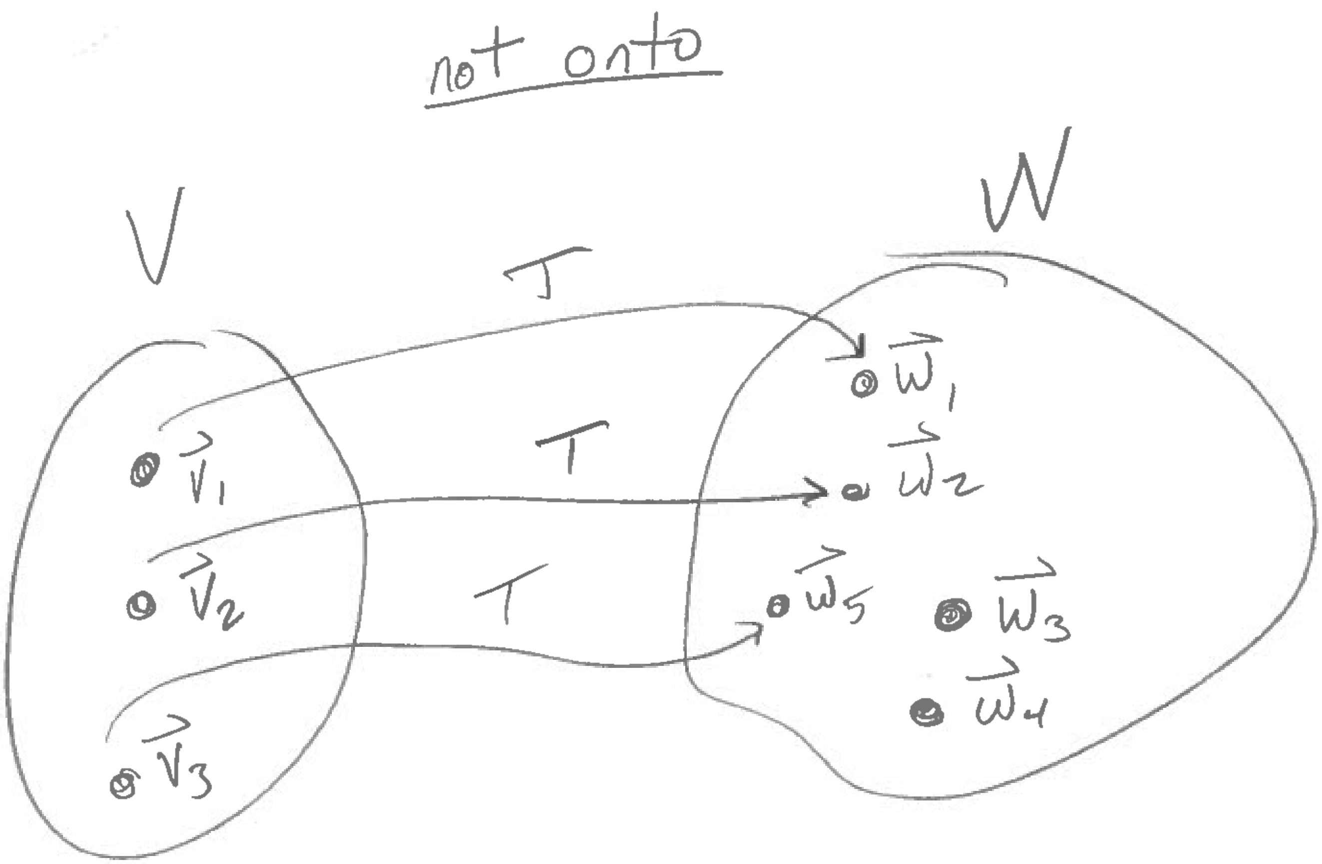


<u>Not one-to-one</u>



- Transformation T is <u>onto</u> if every $\vec{w} \in W$ has a preimage $\vec{v} \in V$.





Example: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be given by $T(\vec{x}) = A\vec{x}$.

- Give rank(A), dim(input space), dim(output), dim(kernal), dim(range), one-to-one, onto, etc....

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a.) $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

* $T: \mathbb{R}^3 \to \mathbb{R}^3$
* dim(input space) = 3
* dim(output space) = 3
* rank(A) = 3
* dim(range) = 3
* dim(kernal) = 0
* T is <u>one-to-one</u> because Kernal(T) is just $\vec{0}$ (Proof to come).
* T is <u>onto</u> because dim(output space) = dim(range)

  (no vectors in $\mathbb{R}^3$ we can't get to through $A\vec{x}$).
* T is called an <u>isomorphism</u>
  (Square, full rank, one-to-one + onto)

b.) $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

* $T: \mathbb{R}^2 \to \mathbb{R}^3$
* dim(input space) = 2
* dim(output space) = 3
* rank(B) = 2
* dim(range) = 2
* dim(kernal) = 0

* T is one-to-one since kernal(T) is just the zero vector.

* T is <u>not</u> onto because output space has vectors we can't get to!
dim(output) = 3 but
dim(range) = 2

---

c.) $C = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

* $T: \mathbb{R}^3 \to \mathbb{R}^2$
* dim(input space) = 3
* dim(output space) = 2
* rank(C) = 2
* dim(range) = 2
* dim(kernal) = 1

* T is <u>not</u> one-to-one, because kernal(T) contains more than just the zero vector.

* T is <u>onto</u>.
dim(output space) = 2
dim(range) = 2

d.) $D = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix}$

* $T: \mathbb{R}^3 \to \mathbb{R}^3$
* dim(input space) = 3
* dim(output space) = 3
* rank(D) = 2
* dim(range) = 2
* dim(kernal) = 1
* T is <u>not</u> one-to-one since kernal(T) has more than just $\vec{0}$.
* T is <u>not</u> onto since dim(output) = 3, dim(range) = 2

Theorem: Let T be a linear transformation from V to W.

T is one-to-one iff kernal(T) is just $\vec{0}$.

Proof: Let T be one-to-one.

Then $T(\vec{v}) = \vec{0}$ can have only one solution (by one-to-one property), namely $\vec{v} = \vec{0}$ (by linear transformation property).

Therefore, kernal(T) is just $\vec{0}$.

Let kernal(T) be just $\vec{0}$. #1

Let $T(\vec{v}_1) = T(\vec{v}_2)$. #2

Because T is linear,

$$T(\vec{v}_1 - \vec{v}_2) = T(\vec{v}_1) - T(\vec{v}_2) \overset{\text{by \#2}}{=} \vec{0}.$$

Thus, $\vec{v}_1 - \vec{v}_2$ is also in the kernal(T).

Thus, $\vec{v}_1 - \vec{v}_2 = \vec{0}$, by #1

OR $\vec{v}_1 = \vec{v}_2$.

So, assuming $T(\vec{v}_1) = T(\vec{v}_2)$ implied $\vec{v}_1 = \vec{v}_2$.

Thus T is one-to-one.

<u>Theorem</u>: Let $T: V \to W$ be linear,
W be finite-dimensional,
and $T(\vec{v}) = A\vec{v}$.
Then T is onto iff
$\text{rank}(A) = \dim(W)$.

<u>Theorem</u>: Let $T: V \to W$ be linear,
$\dim(V) = \dim(W) = n$.
Then T is <u>one-to-one</u> iff
T is <u>onto</u>.

<u>Theorem</u>: Let T is <u>one-to-one</u> and <u>onto</u>, $T(\vec{v}) = A\vec{v}$.

* Then T is an <u>isomorphism</u>.
* $\text{kernal}(T) = \vec{0}$ and $\text{range}(T) = W$
* A^{-1} exists

Finding the matrix for T

* Input vectors $\vec{v} \in V = \mathbb{R}^n$
* Output vectors $\vec{w} \in W = \mathbb{R}^m$
* Matrix A is $m \times n$ (not unique)
  - Need a basis for V and W

__Procedure__: T transforms space V to W

- Find a basis $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ for V
- Find a basis $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m$ for W
- The jth column of A is T applied to the jth basis vector $\vec{v}_j$.

The __Standard__ __Bases__ are fine (columns of I), but eigenvectors or singular vectors may be better (Linear Algebra II).

Example 1 : T takes

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ to } T(\vec{v}_1) = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ to } T(\vec{v}_2) = \begin{bmatrix} 1 \\ 7 \\ 1 \end{bmatrix}$$

If T is linear, matrix A exists, is 3×2

— Outputs $T(\vec{v}_1), T(\vec{v}_2)$ go into columns of A.

$$A = \begin{bmatrix} -1 & 1 \\ -1 & 7 \\ -1 & 1 \end{bmatrix}$$

Every other vector in $\mathbb{R}^2$ falls into place (linearly) onto that plane in $\mathbb{R}^3$.

For example, $\dfrac{\vec{v}_1 + \vec{v}_2}{2} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ goes to

$$A\vec{v} = \begin{bmatrix} -1 & 1 \\ -1 & 7 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

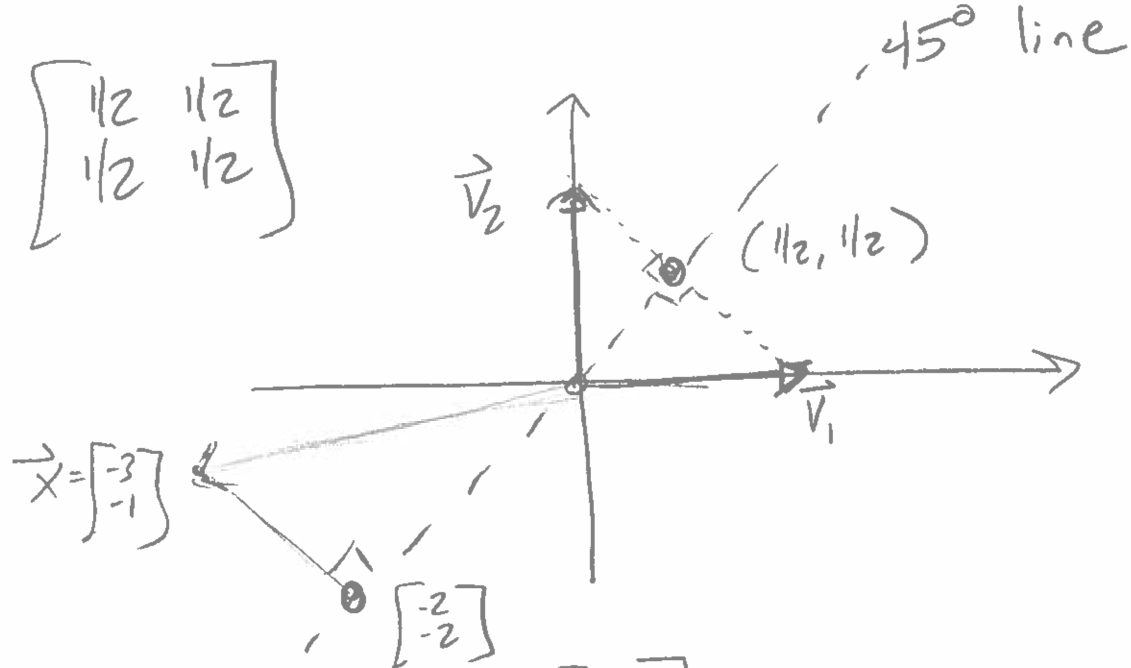
Example 2: Project vectors in $\mathbb{R}^2$ onto the 45° line.

WAY I: Standard basis for $\mathbb{R}^2$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ goes to $T(\vec{v}_1) = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$

$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ goes to $T(\vec{v}_2) = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$

$$A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$



Test it: Let $\vec{x} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$

$$A\vec{x} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} \checkmark$$

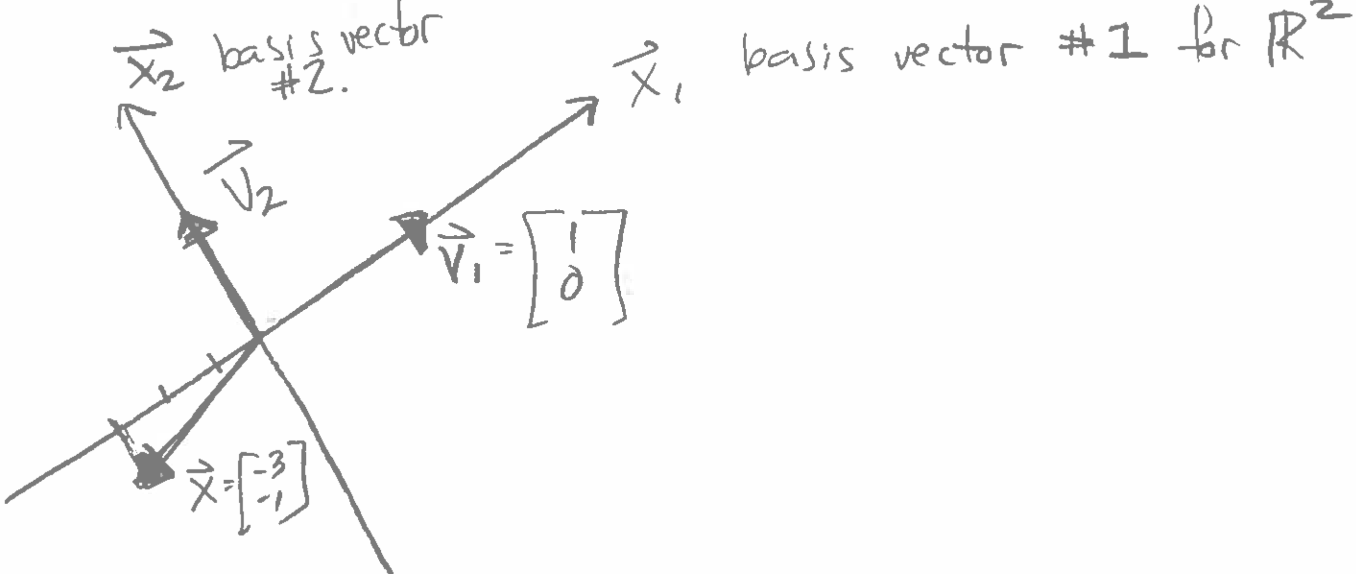
WAY 2 : Eigenvector basis :

The eigenvectors of $A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ are

$\lambda_1 = 1 \qquad \lambda_2 = 0$

$\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \vec{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Use $\vec{x}_1$, $\vec{x}_2$ as a basis for $\mathbb{R}^2$



Let $\vec{v}_1$ be on the 45° line. $T(\vec{v}_1) = \vec{v}_1$ gives column 1 of A^*, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Let $\vec{v}_2$ be on the 135° line $T(\vec{v}_2) = \vec{0}$ gives column 2 of A^*, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$A^* = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ for this basis.

Test on $\vec{x} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$ $\quad A^*\vec{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$

247.

Math 240 – Review Problems for Exam 4 December 6, 2017

1a. Find the three eigenvalues and all the real eigenvectors for matrix A. PS, it is symmetric, Markov, and has a repeated eigenvalue.

$$S = \begin{bmatrix} \frac{2}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{2}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{2}{4} \end{bmatrix} \text{ with characteristic polynomial } -\lambda^3 + \frac{3}{2}\lambda^2 - \frac{9}{16}\lambda + \frac{1}{16}$$

1b. Find the limit of S^k as $k \to \infty$.

1a.) We know $\lambda_1 = 1$ since S is Markov.

Since $\text{trace}(S) = \frac{3}{2} = \lambda_1 + \lambda_2 + \lambda_3$,

We must have $\lambda_2 = \lambda_3 = 1/4$

We could do polynomial long division on

$$\begin{array}{r} -\lambda^2 + 1/2\,\lambda - 1/16 \\ (\lambda - 1)\overline{\big)\, -\lambda^3 + 3/2\,\lambda^2 - 9/16\,\lambda + 1/16} \end{array}$$

$\leftarrow$ get other 2 roots

For $\lambda_1 = 1$ we observe $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Since S is symmetric, other two eigenvectors must be perp. to $\vec{x}_1$.

$$S - \frac{1}{4}I = \begin{bmatrix} 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 \end{bmatrix}$$

can choose $\vec{X}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$\vec{X}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

(248.)

1a.) Make eigen-vectors normal and put into Q.

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ \vec{q}_1 & \vec{q}_2 & \vec{q}_3 \end{bmatrix}$$

1b.) To find $\lim_{k\to\infty} S^k$, diagonalize!

$$S^k = Q \Lambda^k Q^T$$

$$S^k = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}^k Q^T$$

$$\lim_{k\to\infty} S^k = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

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2. We have populations of hawks and rats out in the desert. At time k, $\vec{x}_k = \begin{bmatrix} H_k \\ R_k \end{bmatrix}$ is the state vector with k measured in months and rats measured in the 1000s. Two equations govern the behavior of the populations:

$$H_{k+1} = (0.5)H_k + (0.4)R_k$$
$$R_{k+1} = (-0.104)H_k + (1.1)R_k$$

German websites give us $\lambda_1 = 1.02$ with $\vec{x}_1 = \begin{bmatrix} 10 \\ 13 \end{bmatrix}$ and $\lambda_2 = 0.58$ with $\vec{x}_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

2a. We observe a starting population of $\vec{x}_0 = \begin{bmatrix} 15 \\ 4 \end{bmatrix}$. Derive closed-form solutions for the number of hawks and rats at time k.

2b. For these starting population sizes, what is the ultimate fate of the animals?

2a.) Need to write $\vec{x}_0$ as a linear combination of $\vec{x}_1, \vec{x}_2$.

$$\begin{bmatrix} 15 \\ 4 \end{bmatrix} = c_1 \begin{bmatrix} 10 \\ 13 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 1 \end{bmatrix} \quad \text{has } c_1 = 1/11, \quad c_2 = 31/11$$

$$\vec{X}_k = A^k \vec{X}_0 = A^k \left[\frac{1}{11} \begin{bmatrix} 10 \\ 13 \end{bmatrix} + \frac{31}{11} \begin{bmatrix} 5 \\ 1 \end{bmatrix} \right]$$

$$\vec{X}_k = \frac{1}{11} A^k \begin{bmatrix} 10 \\ 13 \end{bmatrix} + \frac{31}{11} \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$\vec{X}_k = \frac{1}{11}(1.02)^k \begin{bmatrix} 10 \\ 13 \end{bmatrix} + \frac{31}{11}(0.58)^k \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$H_k = \frac{10}{11}(1.02)^k + \frac{155}{11}(0.58)^k$$

$$R_k = \frac{13}{11}(1.02)^k + \frac{31}{11}(0.58)^k$$

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2b.) As $k \to \infty$,

$$H_k \to \frac{10}{11}(1.02)^k$$

$$R_k \to \frac{13}{11}(1.02)^k$$

The ratio of Hawks to Rats will be 10 to 13(000)

($\vec{x}_1$ direction)

Both populations will experience unlimited 2% monthly growth from the $\lambda_1 = 1.02$.

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3. Perform SVD on $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix}$. ($A$ is rank 1)

$$A^TA = \begin{bmatrix} 14 & 14 & 14 & 14 \\ 14 & 14 & 14 & 14 \\ 14 & 14 & 14 & 14 \\ 14 & 14 & 14 & 14 \end{bmatrix}$$ with rank 1

$\lambda_1 = 56$ and

$\lambda_2 = \lambda_2 = \lambda_3 = 0$

For $\lambda_1 = 56$, $\vec{v}_1 = (1, 1, 1, 1)$ or $\vec{v}_1 = (1/2, 1/2, 1/2, 1/2)$

(Basis for rowspace(A))

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$\quad$ P F F F

has 3-dim nullspace

$\vec{v}_2 = (-1, 1, 0, 0)$

$\vec{v}_3 = (-1, 0, 1, 0)$

$\vec{v}_4 = (-1, 0, 0, 1)$

(will make unit).

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{56} = 2\sqrt{14}$$

$$\vec{u}_1 = \frac{A\vec{v}_1}{\sigma_1} = \frac{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix}}{2\sqrt{14}} \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \frac{1}{2\sqrt{14}} \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

$$= \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

(Basis for Col. space)

Still need basis for left Nullspace

$$\vec{u}_2 = \begin{bmatrix} -2 & 1 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} & 0 \end{bmatrix}$$

$$\vec{u}_3 = \begin{bmatrix} -3 & 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} -3/\sqrt{10} & 0 & 1/\sqrt{10} \end{bmatrix}$$

$$A = \underset{3\times 4}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix}} = \underset{(3\times 3)}{U} \; \underset{(3\times 4)}{\Sigma} \; \underset{(4\times 4)}{V^T} =$$

$$A = \underset{3\times 3}{\begin{bmatrix} 1/\sqrt{14} & -2/\sqrt{5} & -3/\sqrt{10} \\ 2/\sqrt{14} & 1/\sqrt{5} & 0 \\ 3/\sqrt{14} & 0 & 1/\sqrt{10} \end{bmatrix}} \underset{3\times 4}{\begin{bmatrix} 2\sqrt{14} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \underset{4\times 4}{\begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{bmatrix}}$$

GRAM-SCHMIDT EXTRA COLUMNS FOR NULLSPACES

$$A = \underset{U}{\begin{bmatrix} 1/\sqrt{14} & -2/\sqrt{5} & -3/\sqrt{70} \\ 2/\sqrt{14} & 1/\sqrt{5} & -6/\sqrt{70} \\ 3/\sqrt{14} & 0 & 5/\sqrt{70} \end{bmatrix}} \underset{\Sigma}{\begin{bmatrix} 2\sqrt{14} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \underset{V^T}{\begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} & 0 \\ -1/4\sqrt{3} & -1/4\sqrt{3} & -1/4\sqrt{3} & 3/4\sqrt{3} \end{bmatrix}}$$

ALL ORTHO-NORMAL

ALL ORTHO-NORMAL

4. Matrix $A = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & -9 & 0 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 7 & k \end{bmatrix}$ is behind a linear transformation, $T: \mathbb{R}^4 \to \mathbb{R}^4$.

4a. For what values of k is T one-to-one? All k except $k = -\frac{1}{2}$

4b. For what values of k is T onto? All k except $k = -1/2$

$$A \sim \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & -9 & 0 \\ 0 & 0 & 28 & -2 \\ 0 & 0 & 7 & k \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & -9 & 0 \\ 0 & 0 & 28 & -2 \\ 0 & 0 & 0 & k+\frac{1}{2} \end{bmatrix}$$

5. Linear transformation T is defined by $T(\vec{v}) = A\vec{v}$ with

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 12 & 13 & 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 \\ 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

5a. rank(A) = 3

5b. dim(input space) = 11

5c. dim(output space) = 3

5d. dim(kernel) = 8

5e. dim(range) = 3

5f. Can T produce the output $\vec{w} = \begin{bmatrix} e \\ \pi \\ \ln 2 \end{bmatrix}$? YES!

5g. Is T one-to-one? No

5h. Is T onto? YES

5i. Is T an isomorphism? No