Section 6.1 Eigenvalues, Eigenvectors

Main Equation: $A\vec{x} = \lambda\vec{x}$

- $\lambda\vec{x}$ is a scalar multiple of $\vec{x}$
  (stretch, shrink, reverse directions, go to 0)
- Most vectors change directions when multiplied by A.

Facts: $n \times n$ matrix A has n eigenvalues

λ could be 0, real, or imaginary

λ could have duplicates

$\det(A)$ = product of eigenvalues

$\text{trace}(A) = a_{11} + a_{22} + \cdots + a_{nn}$

$= \sum \lambda_i$

Get eigenvalues first.

Get eigenvectors second.

How to find λ:

$$A\vec{x} = \lambda\vec{x}$$

$$A\vec{x} - \lambda\vec{x} = \vec{0}$$

$$A\vec{x} - \lambda I\vec{x} = \vec{0}$$

$$(A - \lambda I)\vec{x} = \vec{0}$$

Vectors $\vec{x}$ are in the nullspace of $(A - \lambda I)$. We want the nonzero vectors $\vec{x}$. Thus, solve

$$\det(A - \lambda I) = 0$$ to get λ.

Once we get λ, find the nonzero eigenvectors $\vec{x}$.

How to find $\vec{x}$:

Take $A - \lambda I$ and find a non-zero vector in its nullspace.

λ_i and $\vec{x}_i$ are a pair.

Example: Find all eigenvalues/eigenvectors:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} =$$

$$(2-\lambda)^2 - (1) = \lambda^2 - 4\lambda + 3 \overset{\text{set}}{=} 0$$

$$(\lambda - 3)(\lambda - 1) = 0$$

$$\lambda_1 = 3, \quad \lambda_2 = 1$$

For $\lambda_1 = 3$, find $\vec{x}_1$.

$$A - 3I = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

This matrix is singular with a nonzero vector in $N(A-3I)$.

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\vec{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Check: $A\vec{x}_1 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \lambda_1 \vec{x}_1$

For $\lambda_2 = 1$, find $\vec{x}_2$

$$A - 1I = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

(find a nonzero vector in its nullspace)

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Check: $A\vec{x}_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda_2 \vec{x}_2$

Now, since we have a full set of eigenvectors, we have a basis for $\mathbb{R}^2$.

Any vector, e.g. $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$, can be written as a linear combination of $\vec{x}_1$ & $\vec{x}_2$:

$$\begin{bmatrix} 3 \\ 7 \end{bmatrix} = 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1 \vec{x}_1 + c_2 \vec{x}_2$$

For $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $\det(A) = 4-1 = 3$

and

$\lambda_1 \lambda_2 = 3(1) = 3$ ✓

$\text{trace}(A) = 2+2 = 4$ ✓

$\lambda_1 + \lambda_2 = 3+1 = 4$

$\det(A - \lambda I)$ is an n^{th} degree polynomial in λ, and solving $\det(A-\lambda I) = 0$ amounts to finding the n roots.

Example: Find all eigenvalues, eigenvectors for

$$A = \begin{bmatrix} 0 & 1 & 3 & 0 \\ -2 & 3 & 0 & 4 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 1 & 6 \end{bmatrix}$$

$$\det(A-\lambda I) = \det\begin{bmatrix} -\lambda & 1 & 3 & 0 \\ -2 & 3-\lambda & 0 & 4 \\ 0 & 0 & 6-\lambda & 1 \\ 0 & 0 & 1 & 6-\lambda \end{bmatrix} =$$

$$-\lambda \det\begin{bmatrix} 3-\lambda & 0 & 4 \\ 0 & 6-\lambda & 1 \\ 0 & 1 & 6-\lambda \end{bmatrix} + 2\det\begin{bmatrix} 1 & 3 & 0 \\ 0 & 6-\lambda & 1 \\ 0 & 1 & 6-\lambda \end{bmatrix} =$$

$$-\lambda(3-\lambda)\det\begin{bmatrix} 6-\lambda & 1 \\ 1 & 6-\lambda \end{bmatrix} + 2(1)\det\begin{bmatrix} 6-\lambda & 1 \\ 1 & 6-\lambda \end{bmatrix}$$

$$\det\begin{bmatrix} 6-\lambda & 1 \\ 1 & 6-\lambda \end{bmatrix}\left(\lambda^2 - 3\lambda + 2\right) =$$

$$\left((6-\lambda)^2 - 1\right)(\lambda-2)(\lambda-1) =$$

$$(\lambda^2 - 12\lambda + 35)(\lambda-2)(\lambda-1) =$$

$$(\lambda-7)(\lambda-5)(\lambda-2)(\lambda-1) \stackrel{set}{=} 0$$

$$\lambda_1 = 7, \quad \lambda_2 = 5, \quad \lambda_3 = 2, \quad \lambda_4 = 1$$

$$\text{trace}(A) = \sum \lambda_i = 15 \checkmark$$

$$\det(A) = 70 = \prod_i (\lambda_i) = 7 \times 5 \times 2 \times 1$$

For $\lambda_1 = 7$

$$A - 7I = \begin{bmatrix} -7 & 1 & 3 & 0 \\ -2 & -4 & 0 & 4 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -8/15 \\ 0 & 1 & 0 & -11/15 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

has $\vec{x}_1 = \begin{bmatrix} 8/15 \\ 11/15 \\ 1 \\ 1 \end{bmatrix}$ in its nullspace.

or $\vec{x}_1 = \begin{bmatrix} 8 \\ 11 \\ 15 \\ 15 \end{bmatrix}$

For $\lambda_2 = 5$

$$A - 5I = \begin{bmatrix} -5 & 1 & 3 & 0 \\ -2 & -2 & 0 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1/6 \\ 0 & 1 & 0 & -13/6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

has $\vec{x}_2 = \begin{bmatrix} -1/6 \\ 13/6 \\ -1 \\ 1 \end{bmatrix}$ or $\vec{x}_2 = \begin{bmatrix} -1 \\ 13 \\ -6 \\ 6 \end{bmatrix}$

Recall:

$$A\vec{x}_1 = 7\vec{x}_1$$

$$A\vec{x}_2 = 5\vec{x}_2$$

For $\lambda_3 = 2$,

$$A - 2I = \begin{bmatrix} -2 & 1 & 3 & 0 \\ -2 & 1 & 0 & 4 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix} \text{ has } \vec{x}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

For $$A - 1I = \begin{bmatrix} -1 & 1 & 3 & 0 \\ -2 & 2 & 0 & 4 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 1 & 5 \end{bmatrix} \text{ has } \vec{x}_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

And remember $A\vec{x}_3 = 2\vec{x}_3$ and

$A\vec{x}_4 = 1\vec{x}_4$

If A gets raised to a power,

1. Eigenvectors $\vec{x}$ stay in same direction
2. Eigenvalues λ get raised to that power.

Proof:

$$A\vec{x} = \lambda\vec{x}$$

$$A(A\vec{x}) = \lambda(A)\vec{x}$$

$$A^2\vec{x} = \lambda A\vec{x} = \lambda(\lambda\vec{x})$$

$A^2\vec{x} = \lambda^2\vec{x}$ with same $\vec{x}$.

Theorem: λ^{-1} is an eigenvalue of A^{-1}.

Proof:
$$A\vec{x} = \lambda\vec{x}$$
$$A^{-1}A\vec{x} = \lambda A^{-1}\vec{x}$$
$$\vec{x} = \lambda A^{-1}\vec{x}$$
$$\lambda^{-1}\vec{x} = A^{-1}\vec{x}$$

Theorem: $\lambda+1$ is an eigenvalue of $A+I$.

Proof:
$$A\vec{x} = \lambda\vec{x}$$
$$A\vec{x} + I\vec{x} = \lambda\vec{x} + I\vec{x}$$
$$(A+I)\vec{x} = \lambda\vec{x} + \vec{x} = (\lambda+1)\vec{x}$$

Theorem: Product of λ_i = $\det(A)$

Proof: $\det(A-\lambda I)$ is a factorable polynomial in λ:
$$\det(A-\lambda I) = (\lambda_1-\lambda)(\lambda_2-\lambda)\cdots(\lambda_n-\lambda)$$
now set $\lambda = 0$
$$\det(A) = \lambda_1\lambda_2\cdots\lambda_n$$

Theorem: $\text{trace}(A) = \sum \lambda_i$

Prove for 2×2 case: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = \lambda^2 - (a+d)\lambda + ad - bc \overset{\text{set}}{=} 0$$

Solve for λ using quadratic formula:

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(1)(ad-bc)}}{2(1)}$$

$$\lambda = \frac{(a+d) \pm \sqrt{a^2 + 2ad + d^2 - 4ad + 4bc}}{2}$$

and

$$\lambda_1 = \frac{(a+d) + \sqrt{\text{Mess}}}{2}$$

$$\lambda_2 = \frac{(a+d) - \sqrt{\text{Mess}}}{2}$$

$$\lambda_1 + \lambda_2 = a + d$$

This extends to $n \times n$.

Non-Real Eigenvalues

Example: The ccw rotation matrix

$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ certainly changes every $\mathbb{R}^2$ vector's direction!

First, take $\theta = 90°$.

Then $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Get eigenvalues, eigenvectors.

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1 \overset{\text{set}}{=} 0.$$

$$\lambda^2 = -1$$
$$\sqrt{\lambda^2} = \pm\sqrt{-1}$$
$$\lambda = \pm i$$

And we will get non-real eigenvectors as well!

(They don't rotate when multiplied by A).

Find $\vec{x}_1$ for $\lambda_1 = i$

$$A - iI = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \text{ has } \vec{x}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix} \text{ in its nullspace}$$

$$A\vec{x}_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ i \end{bmatrix} = i\begin{bmatrix} i \\ 1 \end{bmatrix} = \lambda_1 \vec{x}_1$$

Find $\vec{x}_2$ for $\lambda_2 = -i$

$$A + iI = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \text{ has } \vec{x}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix} \text{ in its nullspace}$$

$$A\vec{x}_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix} = -i\begin{bmatrix} 1 \\ i \end{bmatrix} = \lambda_2 \vec{x}_2$$

$$\det(A) = \det\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = 1$$

$$\lambda_1 \lambda_2 = i(-i) = -(i^2) = -(-1) = 1 \checkmark$$

$$\text{trace}(A) = 0 + 0 = 0$$

$$\lambda_1 + \lambda_2 = i - i = 0 \checkmark$$

Now, do this for any θ.

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \text{ find } \lambda_1, \lambda_2, \vec{x}_1, \vec{x}_2$$

$$\det(Q - \lambda I) =$$

$$\det \begin{bmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{bmatrix} =$$

$$\lambda^2 - 2\cos\theta\,\lambda + \cos^2\theta + \sin^2\theta =$$

$$\lambda^2 - 2\cos\theta\,\lambda + 1 \overset{\text{set}}{=} 0.$$

Quadratic Formula:

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4(1)(1)}}{2(1)}$$

$$\lambda = \frac{2\cos\theta \pm \sqrt{4(\cos^2\theta - 1)}}{2}$$

$$\lambda = \cos\theta \pm \sqrt{-\sin^2\theta}$$

$$\lambda = \cos\theta \pm i\sin\theta$$

$$\lambda_1 = \cos\theta + i\sin\theta$$

$$\lambda_2 = \cos\theta - i\sin\theta$$

For $\lambda_1 = \cos\theta + i\sin\theta$, get $\vec{x}_1$.

$$Q - \lambda_1 I = \begin{bmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{bmatrix}$$

Need a nonzero vector in its nullspace.

$$\begin{bmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\vec{x}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$ for λ_1

For $\lambda_2 = \cos\theta - i\sin\theta$, get $\vec{x}_2$.

$$Q - \lambda_2 I = \begin{bmatrix} i\sin\theta & -\sin\theta \\ \sin\theta & i\sin\theta \end{bmatrix}$$

Need a nonzero vector in nullspace.

$$\begin{bmatrix} i\sin\theta & -\sin\theta \\ \sin\theta & i\sin\theta \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\vec{x}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$ for λ_2

Strang 6.2 Diagonalizing a Matrix

For an $n \times n$ matrix with n linearly independent eigenvectors:

Λ = matrix with n eigenvalues on diagonal

X = matrix with n independent eigenvectors

$$A = X \Lambda X^{-1}$$

$$A^k = X \Lambda^k X^{-1}$$

$$A^{-1} = X \Lambda^{-1} X^{-1}$$

Example: Diagonalize $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$.

(1) Get λ first.

$$\det\left[A - \lambda I\right] = \det\begin{bmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{bmatrix} = (1-\lambda)(3-\lambda) \overset{\text{set}}{=} 0$$

$\lambda_1 = 1$

$\lambda_2 = 3$

For $\lambda_1 = 1$, get $\vec{x}_1$. For $\lambda_2 = 3$, get $\vec{x}_2$.

$A - 1I = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}$ has $\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in its nullspace.

$A - 3I = \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix}$ has $\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in its nulspace

$$X = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$X^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = X \Lambda X^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

If we needed A^5, $A^5 = X \Lambda^5 X^{-1}$

$$A^5 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}^5\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$A^5 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1^5 & 0 \\ 0 & 3^5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 243 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$A^5 = \begin{bmatrix} 1 & 243 \\ 0 & 243 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 242 \\ 0 & 243 \end{bmatrix}$$

Difference Equations / Discrete Steps

$$\vec{u}_{k+1} = A\vec{u}_k$$

"Values at $(k+1)^{th}$ step are the matrix A times values at the k^{th} step".

Example First: We start with $\vec{x}_0 = \begin{bmatrix} 1000 & \text{Comcast} \\ 1000 & \text{Fios} \end{bmatrix}$

On Jan 1^{st} each year, 50% with Comcast stay and 50% swith to Fios. Also, 70% with Fios switch to Comcast and 30% stay with Fios.

(#1) Get matrix A that governs the system:

$$A = \begin{array}{c|cc} & \multicolumn{2}{c}{\text{Start}} \\ \text{End} & C & F \\ C & 0.50 & 0.70 \\ F & 0.50 & 0.30 \end{array}$$

(#2) Show what happens after years 1, 2, n.

After 1 year: $\begin{bmatrix} 0.50 & 0.70 \\ 0.50 & 0.30 \end{bmatrix}\begin{bmatrix} 1000 \\ 1000 \end{bmatrix} = \begin{bmatrix} 1200 & \text{Comcast} \\ 800 & \text{Fios} \end{bmatrix}$

After 2 years: $\begin{bmatrix} 0.50 & 0.70 \\ 0.50 & 0.30 \end{bmatrix}\begin{bmatrix} 1200 \\ 800 \end{bmatrix} = \begin{bmatrix} 1160 & \text{Com.} \\ 840 & \text{Fios} \end{bmatrix}$

$$AA\vec{x} = A^2\vec{x}$$

After n years, $A^n \vec{x}$ and after a very long time, $A^{\infty}\vec{x} = \vec{x}_{\infty}$.

#3 Determine the long run behavior.
- Get λ, eigenvectors.

$$\det\left[A-\lambda I\right] = \begin{vmatrix} 0.50-\lambda & 0.70 \\ 0.50 & 0.30-\lambda \end{vmatrix} =$$

$$\lambda^2 - 0.80\lambda - 0.20 \overset{\text{set}}{=} 0$$

$$(\lambda-1)(\lambda+0.20) = 0$$

$$\lambda_1 = 1 \qquad \lambda_2 = -0.20$$

Get $\vec{x}_1$

$A-\lambda_1 I = \begin{bmatrix} -0.50 & 0.70 \\ 0.50 & -0.70 \end{bmatrix}$ has $\vec{x}_1 = \begin{bmatrix} 0.70 \\ 0.50 \end{bmatrix}$

$A-\lambda_2 I = \begin{bmatrix} 0.70 & 0.70 \\ 0.50 & 0.50 \end{bmatrix}$ has $\vec{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(#4) Now, write $\vec{x}_0$ as a linear combination of the eigenvectors.

T1

$$\begin{bmatrix} 1000 \\ 1000 \end{bmatrix} = C_1 \begin{bmatrix} 0.70 \\ 0.50 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$C_1 = \frac{5000}{3}, \quad C_2 = -\frac{500}{3}$$

(#5) Determine closed form solutions for $\vec{u}_K$ using $\vec{x}_1$, $\vec{x}_2$, λ_1, λ_2.

We know $\vec{u}_K = A^k \vec{u}_0$

$$\vec{u}_K = A^k \left[\frac{5000}{3} \vec{X}_1 - \frac{500}{3} \vec{X}_2 \right]$$

$$\vec{u}_K = \frac{5000}{3} A^k \vec{X}_1 - \frac{500}{3} A^k \vec{X}_2$$

$$\vec{u}_K = \frac{5000}{3} (\lambda_1)^k \vec{X}_1 - \frac{500}{3} (\lambda_2)^k \vec{X}_2$$

$$\vec{u}_K = \frac{5000}{3} (1)^k \begin{bmatrix} 0.70 \\ 0.50 \end{bmatrix} - \frac{500}{3} (-0.20)^k \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\vec{u}_K = \begin{bmatrix} 3500/3 \\ 2500/3 \end{bmatrix} - (-0.2)^k \begin{bmatrix} -500/3 \\ 500/3 \end{bmatrix}$$

Let $K \to \infty$ (long run)

$$\vec{u}_\infty = \begin{bmatrix} 3500/3 \\ 2500/3 \end{bmatrix}$$

The eigenvector associated with $\lambda_1 = 1$ is the steady state vector: $\vec{X}_1 = \begin{bmatrix} 0.70 \\ 0.50 \end{bmatrix}$

$\vec{X}_1$ could be $\begin{bmatrix} 7 \\ 5 \end{bmatrix}$ or $\begin{bmatrix} 7/12 \\ 5/12 \end{bmatrix} = \begin{bmatrix} 7/12 \text{ Com} \\ 5/12 \text{ Fros} \end{bmatrix}$

Steps to find closed solutions For $\vec{u}_{K+1} = A\vec{u}_K$ Problems:

1. Write $\vec{u}_0 = c_1\vec{X}_1 + c_2\vec{X}_2 + \cdots + c_n\vec{X}_n$
2. Multiply each $\vec{X}_i$ by $(\lambda_i)^K$
3. Add up the pieces $c_i(\lambda_i)^K\vec{X}_i$

Example: Each week 1/10 in section A and 1/20 in section B drop Calculus. Also, each week 1/10 of the students in each section switch to the other section.

(a.) Cook up the 3×3 matrix that governs this system:

$$A = \begin{array}{c|ccc} \text{End} \backslash \text{Start} & A & B & \text{Drop} \\ \hline A & 8/10 & 1/10 & 0 \\ B & 1/10 & 17/20 & 0 \\ \text{Drop} & 1/10 & 1/20 & 1 \end{array}$$

(b.) We start with 20 in each section. What are the enrollments at the end of 14 weeks?

$$A^k \vec{u}_0 = A^{14} \begin{bmatrix} 20 \\ 20 \\ 0 \end{bmatrix} \approx \begin{bmatrix} \sim 6 & \text{Section A} \\ \sim 8 & \text{Section B} \\ \sim 26 & \text{Dropped} \end{bmatrix}$$

(c.) Diagonalize the matrix A.

Want $A = X \Lambda X^{-1}$

Technology gives:

$\lambda_1 \approx 0.7219$, $\vec{x}_1 \approx (-4.5616, 3.5616, 1)$

$\lambda_2 \approx 0.9281$, $\vec{x}_2 \approx (-0.4384, -0.5616, 1)$

$\lambda_3 = 1$, $\vec{x}_3 = (0, 0, 1)$

$$A^k = X \Lambda^k X^{-1}$$

$$A^k = \begin{bmatrix} -4.5616 & -0.4384 & 0 \\ 3.5616 & -0.5616 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0.7219 & 0 & 0 \\ 0 & 0.9281 & 0 \\ 0 & 0 & 1 \end{bmatrix}^k \begin{bmatrix} -0.1362 & 0.1063 & 0 \\ -0.8638 & -1.1063 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Push k inside Λ

(d.) In theory, what happens to calculus enrollment as $k \to \infty$?

200

$$\lim_{k\to\infty} A^k = X \lim_{k\to\infty} \begin{bmatrix} 0.7219^k & 0 & 0 \\ 0 & 0.9281^k & 0 \\ 0 & 0 & 1^k \end{bmatrix} X^{-1}$$

$$\lim_{k\to\infty} A^k = X \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} X^{-1}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Now, with starting enrollments of $\begin{bmatrix} 20 \\ 20 \\ 0 \end{bmatrix}$,

$$\lim_{k\to\infty} A^k \vec{u}_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 20 \\ 20 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 40 \end{bmatrix}.$$

Everyone drops, given enough time!

Example: NW Mexico is populated by two competing species, coyotes and roadrunners.

We wish to model the populations $c(t)$ and $r(t)$ t years from now if the current populations are c_0 and r_0.

We have the model:

$$c(t+1) = 0.86\,c(t) + 0.08\,r(t)$$
$$r(t+1) = -0.12\,c(t) + 1.14\,r(t)$$

As a matrix equation

$$\underset{(2\times 1)}{\begin{bmatrix} c(t+1) \\ r(t+1) \end{bmatrix}} = \underset{(2\times 2)}{\begin{bmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{bmatrix}} \underset{(2\times 1)}{\begin{bmatrix} c(t) \\ r(t) \end{bmatrix}}$$

The vector $x(t) = \begin{bmatrix} c(t) \\ r(t) \end{bmatrix}$ is the state vector at time t

$$\vec{x}(t+1) = A\vec{x}(t)$$

If we know $\vec{x}(0) = \vec{x}_0 = \begin{bmatrix} c_0 \\ r_0 \end{bmatrix}$ then

$$\vec{x}(t) = A^t \vec{x}_0$$

e.g. $\vec{x}(10) = A^{10}\vec{x}_0$ and long-term behavior depends on initial populations.

- We want closed-form solutions for $c(t)$ and $r(t)$ as a function of t.

<u>Case 1</u>: (Specially chosen) Say we have $c_0 = 100$ and $r_0 = 300$ at $t = 0$.

$$\vec{x}_0 = \begin{bmatrix} 100 \\ 300 \end{bmatrix}$$

$$\vec{x}_1 = A\vec{x}_0 = \begin{bmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{bmatrix} \begin{bmatrix} 100 \\ 300 \end{bmatrix} = \begin{bmatrix} 110 \\ 330 \end{bmatrix}$$

- Each population grows by 10%, and at time t,

$$\vec{x}_t = A^t \vec{x}_0 \quad \text{with } \lambda_1 = 1.1$$

and $A\vec{x}_0 = \lambda_1 \vec{x}_0$ for eigenvector $\vec{x}_0 = \begin{bmatrix} 100 \\ 300 \end{bmatrix}$.

Both populations grow without bound

($X_{100} = \begin{bmatrix} 1,378,061 \\ 4,134,183 \end{bmatrix}$ by the way)

$$\vec{x}(t) = \begin{bmatrix} c(t) \\ r(t) \end{bmatrix} = (1.1)^t \vec{x}_0 = (1.1)^t \begin{bmatrix} 100 \\ 300 \end{bmatrix}$$

with $c(t) = 100(1.1)^t$
$r(t) = 300(1.1)^t$

Case 2: Suppose $\vec{x}_0 = \begin{bmatrix} 200 \\ 100 \end{bmatrix}$ (Carefully chosen)

$$\vec{X}(1) = A\vec{X}_0 = \begin{bmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{bmatrix} \begin{bmatrix} 200 \\ 100 \end{bmatrix} = \begin{bmatrix} 180 \\ 90 \end{bmatrix}$$

$$= 0.9\,\vec{X}_0$$

Here, $\lambda_2 = 0.9$ and $\vec{x}_0 = \begin{bmatrix} 200 \\ 100 \end{bmatrix}$ is the eigenvector.

Both populations decline 10% each year.

Too many coyotes compared to roadrunners.

Case 3: $\vec{X}_0 = \begin{bmatrix} 1000 \\ 1000 \end{bmatrix}$ gives

$$\vec{X}_1 = A\vec{X}_0 = \begin{bmatrix} 940 \\ 1020 \end{bmatrix}, \quad \vec{X}_2 = A\vec{X}_1 = \begin{bmatrix} 890 \\ 1050 \end{bmatrix}$$

- not a scalar multiple of x_0!
- Cannot detect the trend.
- We must work with the two eigenvectors $\vec{V}_1 = \begin{bmatrix} 100 \\ 300 \end{bmatrix}$ and

$$\vec{V}_2 = \begin{bmatrix} 200 \\ 100 \end{bmatrix}$$

Fact: Since $\vec{v}_1$ and $\vec{v}_2$ form a basis for $\mathbb{R}^2$, any vector in $\mathbb{R}^2$ can be written as $\vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2$

In fact, $\vec{x}_0 = \begin{bmatrix} 1000 \\ 1000 \end{bmatrix} = 2\vec{v}_1 + 4\vec{v}_2$

Recall: $A^t\vec{v}_1 = (1.1)^t\vec{v}_1$ and

$A^t\vec{v}_2 = (0.9)^t\vec{v}_2$

$$\vec{x}(t) = A^t\vec{x}_0 = A^t(2\vec{v}_1 + 4\vec{v}_2) =$$

$$2A^t\vec{v}_1 + 4A^t\vec{v}_2 =$$

$$2(1.1)^t\vec{v}_1 + 4(0.9)^t\vec{v}_2 =$$

$$2(1.1)^t\begin{bmatrix} 100 \\ 300 \end{bmatrix} + 4(0.9)^t\begin{bmatrix} 200 \\ 100 \end{bmatrix}$$

or

$$c(t) = 200(1.1)^t + 800(0.9)^t$$
$$r(t) = 600(1.1)^t + 400(0.9)^t$$

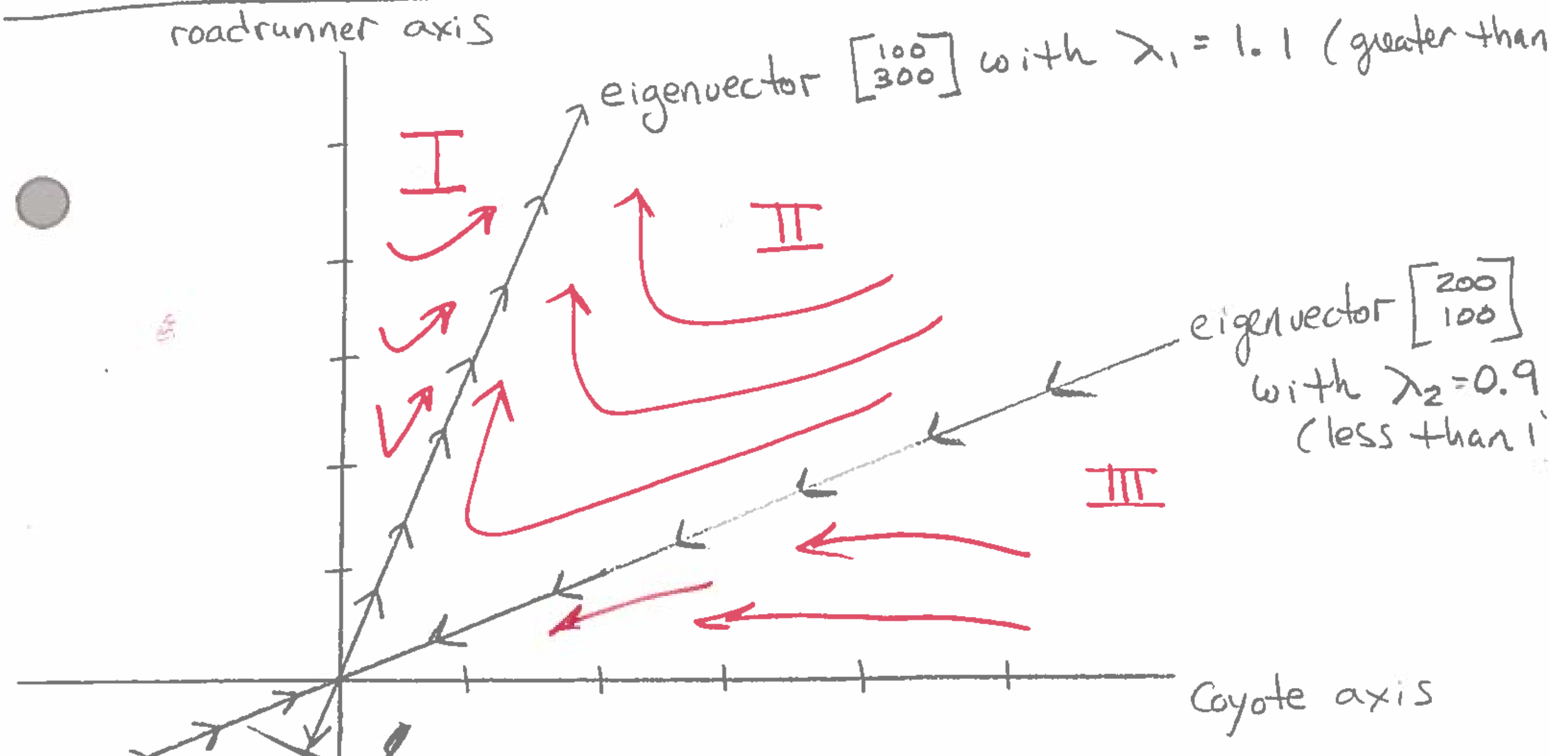
200

As time progresses, the $(0.9)^t$ term goes to 0, both populations (recall $\vec{x}_0 = \begin{bmatrix} 1000 \\ 1000 \end{bmatrix}$) grow at 10% per year.

The ratio of roadrunners to coyotes goes to

$$\frac{r(t)}{c(t)} = \frac{600(1.1)^t}{200(1.1)^t} = 3$$

The 3 is the slope of the 'steady state vector, $\vec{x}(t)$.



* Any starting population vectors X_0 in regions I or II will eventually go to the $\frac{r(t)}{c(t)} = 3$ ratio.

* Any starting in region III will go extinct.

Eigenvectors govern the system!

With $\vec{x}_0 = \begin{bmatrix} 1000 \\ 1000 \end{bmatrix}$ we get

t	$c(t)$	$r(t)$	$\frac{r(t)}{c(t)}$
0	1000	1000	1
1	940	1020	1.09
5	794	1203	1.51
10	798	1696	2.13
20	1443	4085	2.83
50	23482	70437	2.99
500	1.E+23	3.E+23	3

Another starting at $\vec{X}_0 = \begin{bmatrix} 400 & \text{Coyotes} \\ 100 & \text{Roadrunners} \end{bmatrix}$

Find closed-form solutions for $c(t), r(t)$.

Matrix A still same λ and $\vec{v}_1, \vec{v}_2$!

Write $\begin{bmatrix} 400 \\ 100 \end{bmatrix} = c_1 \vec{v}_1 + c_2 \vec{v}_2 = c_1 \begin{bmatrix} 100 \\ 300 \end{bmatrix} + c_2 \begin{bmatrix} 200 \\ 100 \end{bmatrix}$

Using TI: $\text{rref} \left[\begin{array}{cc|c} 100 & 200 & 400 \\ 300 & 100 & 100 \end{array}\right] =$

$$\left[\begin{array}{cc|c} 1 & 0 & -0.4 \\ 0 & 1 & 2.2 \end{array}\right]$$

So $c_1 = -0.4$ and $c_2 = 2.2$

$$\vec{X}_0 = \begin{bmatrix} 400 \\ 100 \end{bmatrix} = -0.4\begin{bmatrix} 100 \\ 300 \end{bmatrix} + 2.2\begin{bmatrix} 200 \\ 100 \end{bmatrix}$$

And $\vec{x}(t) = A^t(-0.4\vec{v}_1 + 2.2\vec{v}_2)$

$$= -0.4A^t\vec{v}_1 + 2.2A^t\vec{v}_2$$

$$= -0.4(1.1)^t\begin{bmatrix} 100 \\ 300 \end{bmatrix} + 2.2(0.9)^t\begin{bmatrix} 200 \\ 100 \end{bmatrix}$$

And $c(t) = -40(1.1)^t + 440(0.9)^t$

$r(t) = -120(1.1)^t + 220(0.9)^t$

t	$c(t)$	$r(t)$	$\frac{r(t)}{c(t)}$
0	400	100	0.25
1	352	66	0.19
2	308	33	0.11
3	268	1	0.002
4	230	-31	-0.14
500	-2E+22	-6E+22	3?

Mathmatically, populations gravitate to eigenvector $\begin{bmatrix} 100 \\ 300 \end{bmatrix}$ in negative direction.

Strang 6.4 Symmetric Matrices

Let $S = S^T$ and $S\vec{x} = \lambda\vec{x}$

- All λ_i will be real
- $\vec{x}_i$ can be chosen to be orthonormal
- Spectral Theorem

$$S = Q\Lambda Q^T, \quad Q^T = Q^{-1}$$

$$\begin{aligned} S = {} & \lambda_1 \vec{q}_1 \vec{q}_1^{\,T} + \\ & \lambda_2 \vec{q}_2 \vec{q}_2^{\,T} + \cdots + \\ & \lambda_n \vec{q}_n \vec{q}_n^{\,T} \end{aligned}$$

If A is real, but not symmetric, the eigenvalues & eigenvectors come in conjugate pairs.

- If $A\vec{x} = \lambda\vec{x}$, then $A\overline{\vec{x}} = \overline{\lambda}\,\overline{\vec{x}}$

with $\lambda = a + ib$
$\lambda = a - ib$

Similar Matrices (Back to 6.2)

We have $A = X \Lambda X^{-1}$ OR
$$S = Q \Lambda Q^T$$

If we change X OR Q, but keep Λ fixed, that family of matrices is called "similar".

This extends to non-diagonalizable matrices

Fix matrix C, allow all invertible B.

$A = BCB^{-1}$ will have

A similar to C

A and C share the same λ.

Example: We know $\lambda_1 = 4$, $\lambda_2 = 4$, $\lambda_3 = 7$.

Matrix A is ....

Invertible? Yes. No $\lambda = 0$.

Diagonalizable? MAYBE. Repeated λ may or may not have a full set of eigenvectors.

Example: We know all eigenvectors of A are multiples of $\vec{x} = (4, 7)$.

Is A invertible? No way to tell. $\lambda = 0$ is possible.

Is λ repeated? Yes. Missing eigenvector, so yes, repeated λ.

Is $X \Lambda X^{-1}$ possible? No. Need a second indep. eigenvector.

Example: Complete matrices A, B, C such that $\det(A) = \det(B) = \det(C) = 25$

$$A = \begin{bmatrix} 8 & 3 \\ -3 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 9 & 4 \\ -4 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 10 & 5 \\ -5 & 0 \end{bmatrix}$$

$\det(A) = 16 - bc = 25$ $\quad$ $\det(B) = 9 - 4c = 25$ $\quad$ $\det(C) = 10d + 25 = 25$

For all 3 matrices, $\lambda_1 = \lambda_2 = 5$

$$A - 5I = \begin{bmatrix} 3 & 3 \\ -3 & -3 \end{bmatrix}$$

A, B, C only have one line of eigenvectors.

$$B - 5I = \begin{bmatrix} 4 & 4 \\ -4 & -4 \end{bmatrix}$$

$$C - 5I = \begin{bmatrix} 5 & 5 \\ -5 & -5 \end{bmatrix}$$

A, B, C are not diagonalizable.

Example: Let $A = X \Lambda X^{-1}$.

Show $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$.

Proof:
$$\begin{aligned} \det(A) &= \det(X \Lambda X^{-1}) \\ &= \det(X)\det(\Lambda)\det(X^{-1}) \\ &= \det(\Lambda) = \lambda_1 \lambda_2 \cdots \lambda_n \end{aligned}$$

Example: Let $A\vec{x} = \lambda \vec{x}$.

If $\lambda = 0$, $\vec{x} \in N(A)$. Recall, A is $n \times n$.

If $\lambda \neq 0$, $\vec{x} \in C(A)$.

$$\dim(C(A)) + \dim(N(A)) = r + (n-r) = n$$

Explain: Why doesn't every square matrix A have n independent eigenvectors?

Problem 1: $N(A)$ and $C(A)$ can overlap.

Problem 2: We might not have r indep. eigenvectors in $C(A)$.

Example: If all $|\lambda_i| < 1$, $A^k \to 0$

If any $|\lambda_i| > 1$, $A^k \to$ Blow Up.

What about $A = \begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix}$?

Find A^{1024}.

$$\det(A-\lambda I) = \begin{bmatrix} 3-\lambda & 2 \\ -5 & -3-\lambda \end{bmatrix} =$$

$$-9-3\lambda+3\lambda+\lambda^2+10 = \lambda^2+1 \overset{\text{set}}{=} 0$$

$$\lambda_1 = i \quad \lambda_2 = -i$$

with

$$\vec{X}_1 = \begin{bmatrix} -3-i \\ 5 \end{bmatrix}, \quad \vec{X}_2 = \begin{bmatrix} -3+i \\ 5 \end{bmatrix}$$

$$A^K = \begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix}^K = X \Lambda^k X^{-1}$$

$$A^K = \begin{bmatrix} -3-i & -3+i \\ 5 & 5 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}^K \begin{bmatrix} X \end{bmatrix}^{-1}$$

$$A^K = \begin{bmatrix} -3-i & -3+i \\ 5 & 5 \end{bmatrix} \begin{bmatrix} i^K & 0 \\ 0 & -i^K \end{bmatrix} \begin{bmatrix} X \end{bmatrix}^{-1}$$

$i = \sqrt{-1}$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$ and also $(-i)^4 = 1$

So $A^{1024} = X I X^{-1} = I$

$1024 \div 4 = 256$ even

Example: Find the 2017^{th} power of

$$A = \begin{bmatrix} 110 & 55 & -164 \\ 42 & 21 & -62 \\ 88 & 44 & -131 \end{bmatrix}$$

Note: $\text{trace}(A) = 0$ So $\lambda_1 + \lambda_2 + \lambda_3 = 0$

Note: $2 \times$ Column 2 $=$ Column 1 so $\det(A) = 0$

$\lambda_1 = 0$

Note: $A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ So $\lambda_2 = 1$

$\lambda_3 = -1$

$\lambda_1 = 0$ has $\vec{x}_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$

$\lambda_2 = 1$ has $\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\lambda_3 = -1$ has $\vec{x}_3 = \begin{bmatrix} 33 \\ -1 \\ 22 \end{bmatrix}$

$$A = X \Lambda X^{-1} =$$

$$A = \begin{bmatrix} -1 & 1 & 33 \\ 2 & 1 & -1 \\ 0 & 1 & 22 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -23 & -11 & 34 \\ 44 & 22 & -65 \\ -2 & -1 & 3 \end{bmatrix}$$

$$A^{2017} = \begin{bmatrix} -1 & 1 & 33 \\ 2 & 1 & -1 \\ 0 & 1 & 22 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1^{2017} & 0 \\ 0 & 0 & -1^{2017} \end{bmatrix} \begin{bmatrix} -23 & -11 & 34 \\ 44 & 22 & -65 \\ -2 & -1 & 3 \end{bmatrix}$$

$$A^{2017} = \begin{bmatrix} 0 & 1 & -33 \\ 0 & 1 & 1 \\ 0 & 1 & -22 \end{bmatrix} \begin{bmatrix} -23 & -11 & 34 \\ 44 & 22 & -65 \\ -2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 110 & 55 & -164 \\ 42 & 21 & -62 \\ 88 & 44 & -131 \end{bmatrix} = A$$

Example: Find Q to diagonalize $A = \begin{bmatrix} -2 & 6 \\ 6 & 7 \end{bmatrix}$.

$\lambda_1 = -5$ has $\vec{x}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ $\qquad$ $\lambda_2 = 10$ has $\vec{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\|\vec{x}_1\| = \sqrt{5}$ $\qquad$ $\|\vec{x}_2\| = \sqrt{5}$

$$A = Q \Lambda Q^T = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -5 & 0 \\ 0 & 10 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$$

Example: We have $S = \begin{bmatrix} 1 & b \\ b & 1 \end{bmatrix}$

(a.) How do we know S does not have two negative eigenvalues? $\text{trace}(S) = 2$

and $\lambda_1 + \lambda_2 = 2$ can't both be negative

(b.) Give the two pivots:

$S \sim \begin{bmatrix} 1 & b \\ 0 & 1-b^2 \end{bmatrix}$ has pivots $1, 1-b^2$

(c.) When will S have two positive eigenvalues?

For symmetric S, pivots & λ_i have same signs.

So, if $1-b^2 > 0$ or $1 > b^2$ or $b^2 < 1$

We get two positive λ_i

Then if $b^2 > 1$, we will get $\lambda_1 > 0$
$\lambda_2 < 0$

Example: Why does a 3×3 real matrix always have at least one real eigenvalue?

If $\lambda_1 = a+bi$, then $\lambda_2 = a-bi$

$\lambda_1 + \lambda_2 = 2a$

trace(A) = real, so λ_3 must be real.

Example: Write $S = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$ as

$\lambda_1 \vec{x}_1 \vec{x}_1^T + \lambda_2 \vec{x}_2 \vec{x}_2^T$ (Spectral Theorem)

$S = Q \Lambda Q^T$

For S, we have:

$\lambda_1 = 0$, $\vec{x}_1 = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ or $\vec{x}_1 = \frac{1}{5}\begin{bmatrix} -4 \\ 3 \end{bmatrix}$ (unit)

$\lambda_2 = 25$, $\vec{x}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ or $\vec{x}_2 = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ (unit)

$$S = 0 \cdot \frac{1}{25}\begin{bmatrix} -4 \\ 3 \end{bmatrix}\begin{bmatrix} -4 & 3 \end{bmatrix} + 25 \cdot \frac{1}{25}\begin{bmatrix} 3 \\ 4 \end{bmatrix}\begin{bmatrix} 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} \checkmark$$

6.5 Positive Definite Matrices

- Matrix S is symmetric, all $\lambda_i > 0$

To test for positive definiteness, all are equivalent:

1. All n pivots of S are all positive
2. All n upper-left determinants are pos.
3. All n eigenvalues are positive
4. $\vec{x}^T S \vec{x}$ = a polynomial > 0 for all $\vec{x} \neq 0$ (energy)
5. $S = A^T A$ with A having indep. columns.

Any verification of 1-5 gives all other properties.

Why are positive definite matrices important?

See handout —

Positive Definite Matrices – Why Important

From Anonymous on math.stackexchange

There are many uses for definite and semi-definite matrices. I can give just a few examples although undoubtedly I will be missing many.

1. Positive-definite matrices are the matrix analogues to positive *numbers*. It is generally not possible to define a consistent notion of "positive" for matrices other than symmetric matrices. As a consequence, positive definite matrices are a special class of symmetric matrices (which themselves are another very important, special class of matrices). It turns out that many useful matrices fall under this class such the covariance matrix, overlap matrices used in quantum chemistry and dynamical matrices used in calculation of molecular vibrations (which is positive semi-definite).

2. Definiteness is a useful measure for optimization. Quadratic forms on positive definite matrices are always positive for non-zero x and are convex. Analogous results hold for negative-definite matrices. This is a very desirable property for optimization since it guarantees the existences of maxima and minima. It is properties like these for example, that allow you to use the Hessian matrix to optimize multivariate functions.

3. Perhaps equally (or more) important, especially to a mathematician, is the fact that the theory of (semi)definite matrices is an incredibly rich and beautiful field. There are chains of elegant results concerning these matrices, especially for positive-definite matrices. That is motivation enough.

From AlephZero on www.physicsforum.com

Numerical algorithms on positive definite matrices are usually well behaved. The underlying reason is that all the eigenvalues are positive, so the sort of operations that occur in numerical methods don't lose precision when positive and negative quantities are added and cancel out. (Of course individual elements of a positive definite matrix can be negative, but in a sense they can't be "negative enough" to cause numerical problems.) This means there are usually faster and simpler numerical algorithms for positive definite matrices than for general matrices. In physics, matrices are often Hermitian (which includes real symmetric matrices) as well as positive definite, and the product x^t A x represents some kind of work or energy.

Example: Positive definite or not?

$S_1 = \begin{bmatrix} 5 & 6 \\ 6 & 7 \end{bmatrix}$ $\det[5] = 5 > 0$ ✓

$\det[S_1] = 35 - 36 = -1$ ×

No. $\lambda_1 \approx 12.083$, $\lambda_2 \approx -0.083$

$S_2 = \begin{bmatrix} -1 & -2 \\ -2 & -5 \end{bmatrix}$, No! First pivot is -1

$S_3 = \begin{bmatrix} 1 & 10 \\ 10 & 100 \end{bmatrix}$, $\det[S_3] = 0$ Positive Semi-Definite

$$\vec{x}^T S_3 \vec{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 10 \\ 10 & 100 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_1 + 10x_2 \\ 10x_1 + 100x_2 \end{bmatrix} = x_1^2 + 10x_1x_2 + 10x_1x_2 + 100x_2^2$$

$$= x_1^2 + 20x_1x_2 + 100x_2^2$$

$$= ax_1^2 + 2bx_1x_2 + cx_2^2$$

$S_4 = \begin{bmatrix} 1 & 10 \\ 10 & 101 \end{bmatrix}$ has $\text{pivot}_1 = 1$, $\text{pivot}_2 = 1$ So yes.

$\lambda_1 \approx 0.0098$

$\lambda_2 \approx 101.99$

$x_1^2 + 20x_1x_2 + 101x_2^2 > 0 \quad \forall\, x_1, x_2$ ≠

Example: Find b,c so that $S_1 = \begin{bmatrix} 1 & b \\ b & 9 \end{bmatrix}$ is positive definite.

Strategy 1: $\det(S_1) = 9 - b^2 > 0$

$$9 > b^2$$
$$b^2 < 9$$
$$-3 < b < 3$$

$S_2 = \begin{bmatrix} 2 & 4 \\ 4 & c \end{bmatrix}$ Strategy 2 → Write $S_2 = L \underset{\substack{\uparrow \\ \text{pivots}}}{D} L^T$

$\begin{bmatrix} 2 & 4 \\ 4 & c \end{bmatrix} \sim \begin{bmatrix} 2 & 4 \\ 0 & c-8 \end{bmatrix}$ has $l_{21} = 2$

$$S_2 = LDL^T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & c-8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

2^{nd} pivot is positive if $c - 8 > 0$ or $c > 8$

$$S_3 = \begin{bmatrix} c & b \\ b & c \end{bmatrix} \sim \begin{bmatrix} c & b \\ 0 & c - \frac{b^2}{c} \end{bmatrix} = \begin{bmatrix} c & b \\ 0 & \frac{c^2 - b^2}{c} \end{bmatrix} = U$$

$l_{21} = \frac{b}{c}$

$$S_3 = LDL^T = \begin{bmatrix} 1 & 0 \\ b/c & 1 \end{bmatrix} \begin{bmatrix} c & 0 \\ 0 & \frac{c^2 - b^2}{c} \end{bmatrix} \begin{bmatrix} 1 & b/c \\ 0 & 1 \end{bmatrix}$$

Need
$c > 0$
$c^2 > b^2$
$|c| > |b|$

Example: $f(x,y) = 2xy$ has a saddle point and not a minimum at $(0,0)$.

What matrix S produced this $f(x,y)$? Give λ_1, λ_2.

$$S = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \Rightarrow f(x,y) = ax^2 + 2bxy + cy^2$$

$$S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow f(x,y) = [x \ y] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$f(x,y) = 2xy$$

$$\det(S - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - 1 \overset{\text{set}}{=} 0$$

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

Example:

Test that $S = A^T A$ is positive definite for $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix}$ (independent columns!)

$$S = A^T A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}$$

$S = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}$ has $\det[6] = 6$

$\det[S] = 36 - 25 = 11$

✓ Pos. Def.

FACT: If S is positive def, then S^{-1} is positive def. Why?

Eigenvalues of S^{-1} are $1/\lambda_i$

FACT: A pos. def. matrix cannot have 0 on the diagonal. Show it for $S = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 5 \end{bmatrix}$ using $\vec{x}^T S \vec{x}$.

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 4 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 4x_1 + x_2 + x_3 \\ x_1 + 2x_3 \\ x_1 + 2x_2 + 5x_3 \end{bmatrix}$$

$= \underline{4x_1^2} + \underline{x_1 x_2} + \underline{x_1 x_3} + \underline{x_1 x_2} + \underline{2x_2 x_3} + \underline{x_1 x_3} + \underline{2x_2 x_3} + \underline{5x_3^2}$

$= 4x_1^2 + 2x_1 x_2 + 2x_1 x_3 + 4x_2 x_3 + 5x_3^2$ can easily be negative if $x_2 < 0$

(Notice 2nd pivot is negative too!)