Reachability II
(Hamilton-Jacobi)

DD2415 – Lecture 5 – Nov 2, 2024

Last lectures

- Reachability on Graphs

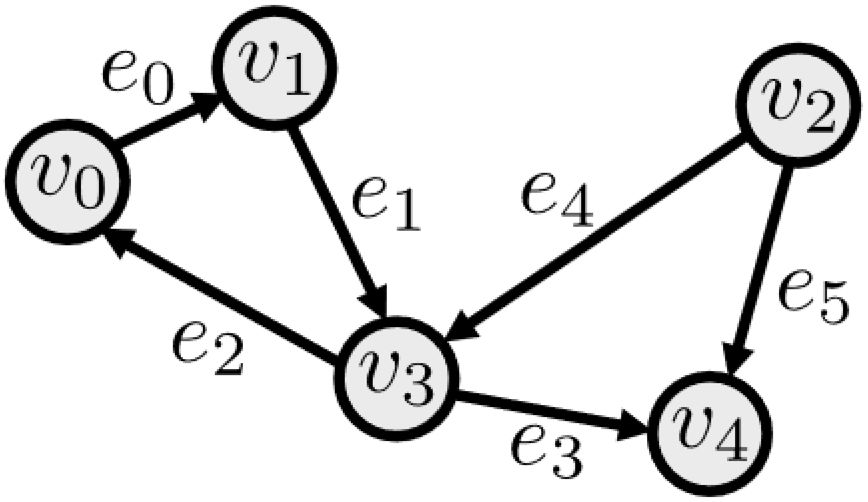


Fig: A Simple Digraph [Lec 3]

- Undisturbed linear systems

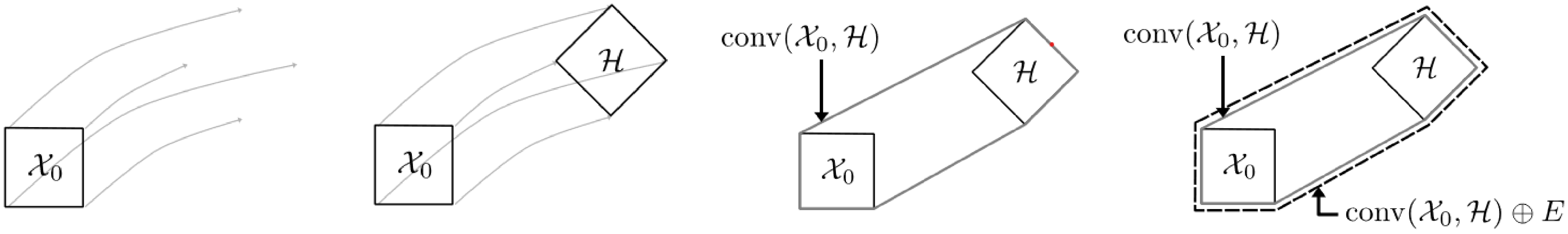


Fig: Algorithmic approach to FRS computation [Lec 3]

Last lectures

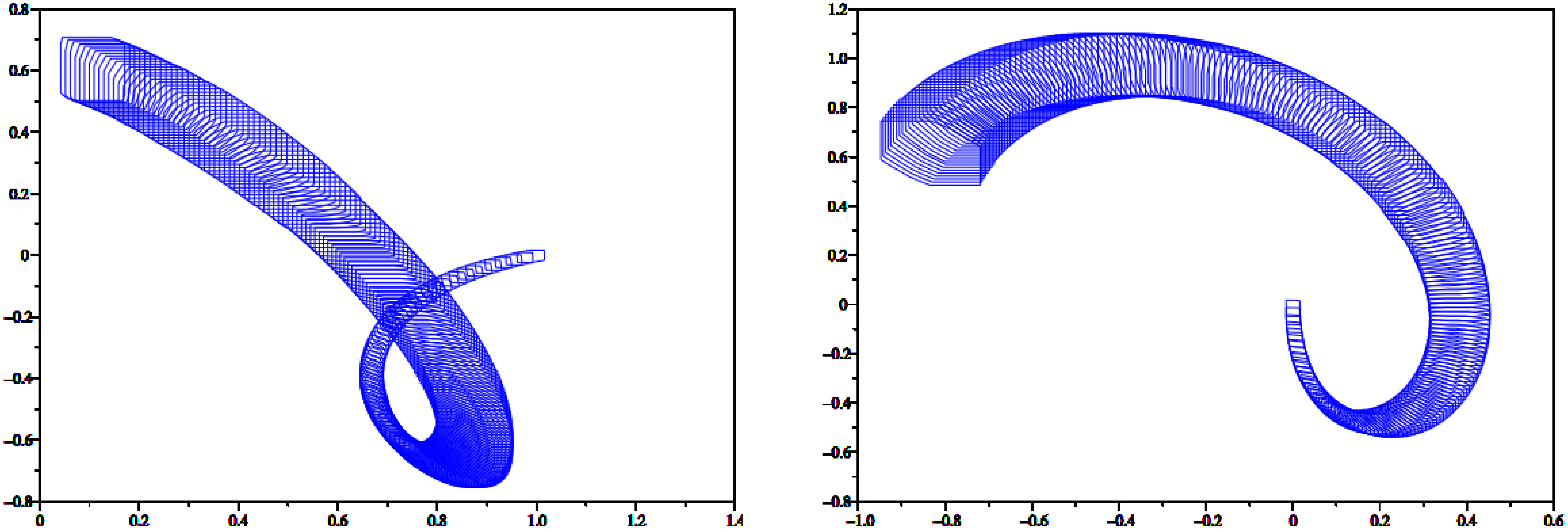


Fig: Projections of the reachable set [Lec 3]

Slight Extensions on Last lectures

- What if we add a disturbance?
  - $\dot{x} = Ax + Bu + Cd$
  - The disturbance 'shrinks' the reachable set after every step
- The Minkowski difference
  - $A \ominus B = \{a - b \mid a \in A, b \in B\}$,
    where $A \ominus B$ can be summed with B to recover A
  - Zonotopes are not closed under this operator!
  - Polytope <-> Zonotope conversions throughout!

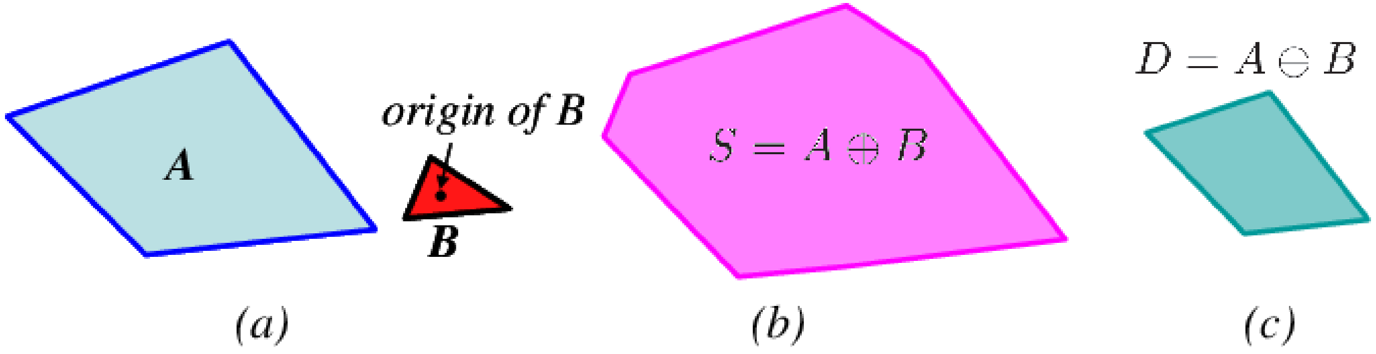


Fig: Minkowski difference of 2 polytopes [Bariki2010New]

Today

1. Introduction
   - Reachability
   - Our Problem formulation
2. A Brief Intro into Optimal Control
   - Dynamic Programming
   - Hamilton-Jacobi-Bellman
3. Differential Control Problems
   - Zero-sum Differential Games
   - Hamilton-Jacobi-Isaacs
4. Solving the HJI equation
   - The Levelset method
5. Research in HJ Reachability

Learning outcomes

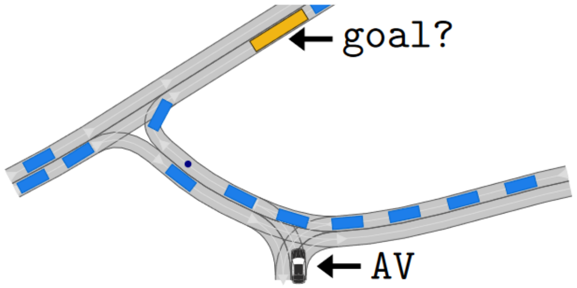
- Explain different flavours of reachability (BRS, BRT, Reach, Avoid) and how they are reflected in the final-value PDE
- Demonstrate the derivation of the HJB Equation and explain how this is used in optimal control
- Recall the relationship between HJB and HJI
- Demonstrate the ability to go from set-based to tube-based backwards reachability
- Demonstrate an intuitive understanding of the Levelset method for reachability computations

Introduction

- Reachability
- Our Problem Formulation

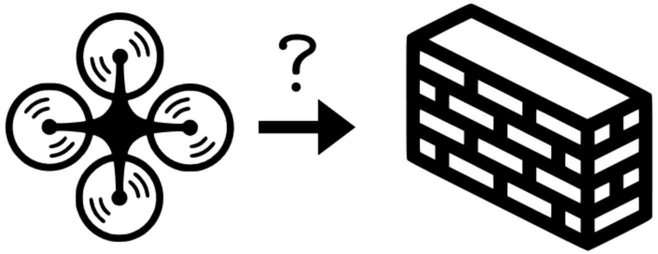
Reachability in General Terms

- We want to:
  - Reach a goal
  - Avoid an unsafe region
  - both
- Achieve this despite:
  - Disturbances
  - "opponents"



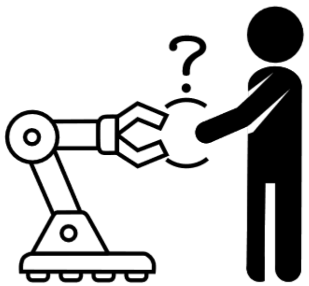
Can the AV reach the goal?

Despite other drivers?



Will the drone hit the wall?

Despite the wind?



Can the human reach the robot?

Despite uncertainty?

Our Reachability Problem

- Some nonlinear dynamics model

$$\dot{x} = f(x, u, d)$$

- Leading to a system trajectory

$$\Phi(t; x_0, u(\cdot), d(\cdot))$$

- A goal set

$$G(T)$$

- A finite time horizon

$$t \in [0, T]$$

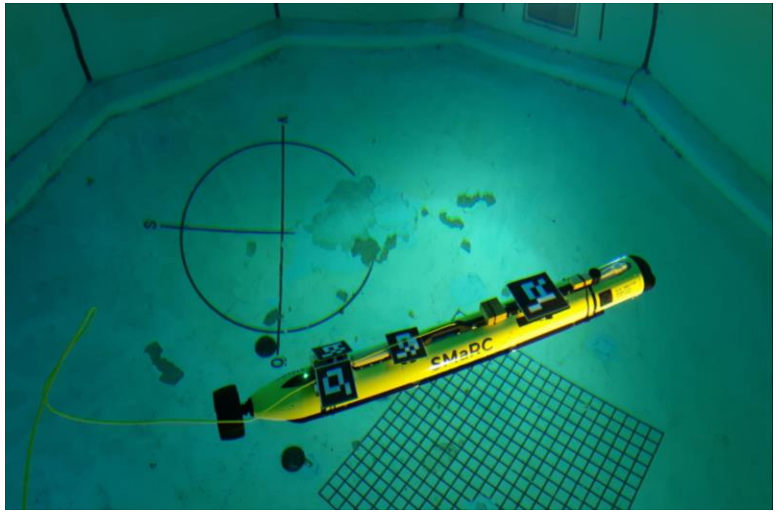
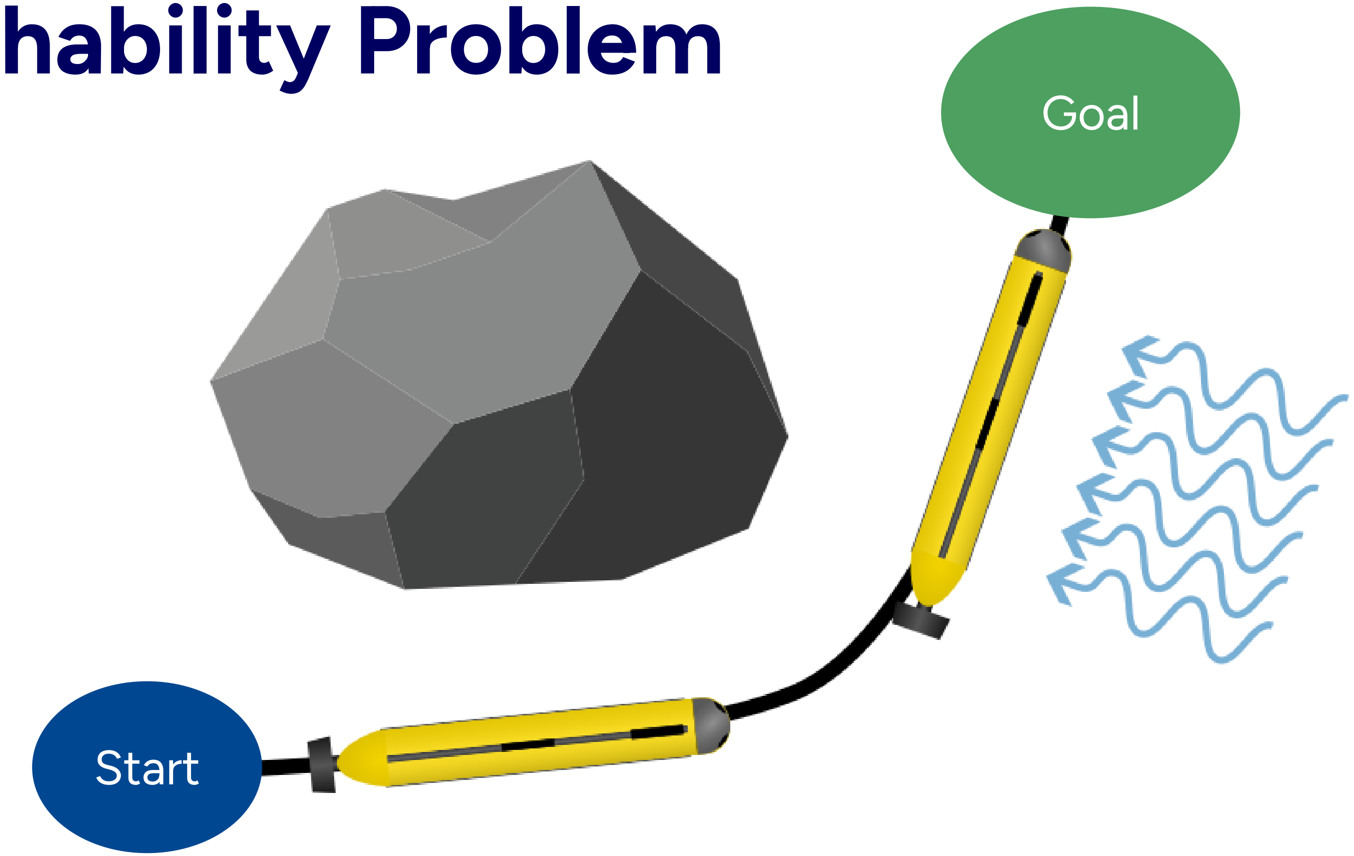


Fig: SAM (SMaRC, KTH)

Now we want to make statements about "reaching" states for a control action, for all disturbance actions

$$R_b(t) = \{x: \exists u(\cdot), \forall d(\cdot), \qquad s.t. \, \Phi\big(T; x, u(\cdot), d(\cdot)\big) \in G(T)\}$$

Our Reachability Problem



Now we want to make statements about ”reaching” states for a control action, for all disturbance actions

$$R_b(t) = \{x : \exists u(\cdot), \forall d(\cdot), \qquad s.t.\ \Phi\big(T; x, u(\cdot), d(\cdot)\big) \in G(T)\}$$

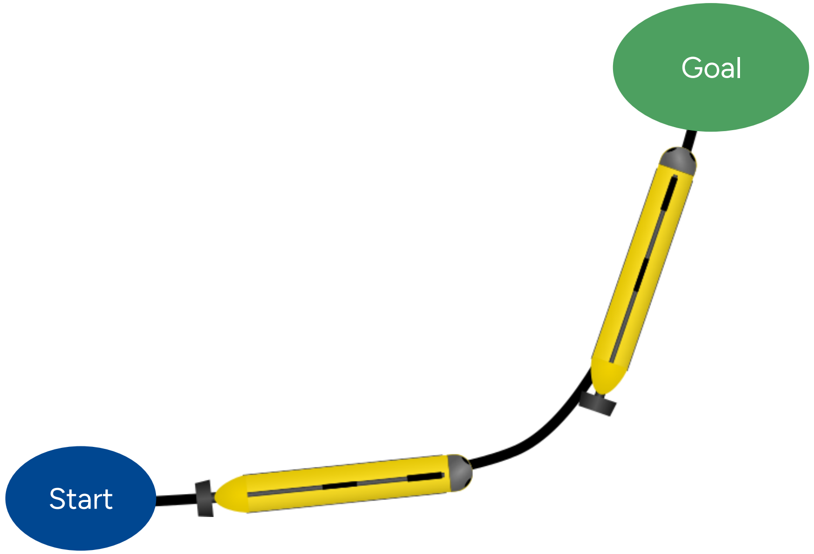
To reiterate

- We want to look at:
  - Disturbed nonlinear systems: $\dot{x} = f(x, u, d)$
    for example, nonlinear control affine systems: $\dot{x} = f(x) + g(x)u + h(x)d$
  - Backwards reachability!

A brief intro to optimal control

- Dynamic Programming
- Hamilton-Jacobi-Bellman

The Optimal Control Problem



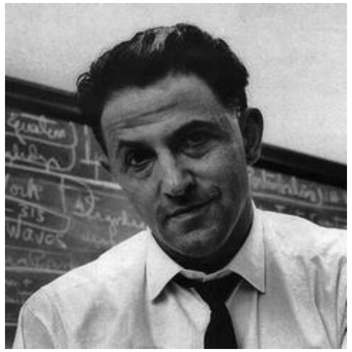
- Let's take a step back: undisturbed nonlinear system: $\dot{x} = f(x, u)$
- The standard, fundamental (deterministic) problem of control:

$$\min_{u} \int_0^T \overbrace{L\big(x(t), u(t)\big)ds}^{\textit{Running cost}} + \overbrace{l(x(T))}^{\textit{Terminal (goal) cost}}$$

$$\begin{array}{ll} s.t. & \left.\dot{x} = f(x, u)\right\} \textit{Dynamics constraints} \\ & \left.x(0) = x_0\right\} \textit{Initial state constraint} \end{array}$$

- What are we trying to get the system to do?
- How do we solve something like this?
  - remember tutorial 0

Dynamic Programming

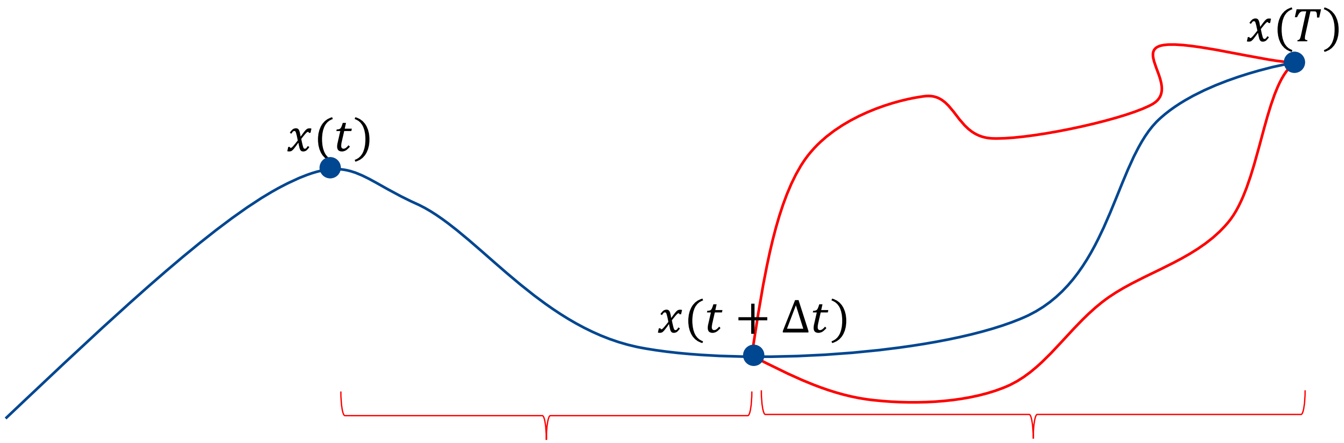


Richard Bellman

- Bellman’s optimality principle:
  - The optimal state trajectory remains optimal at intermediate points in time!
  - Or in Bellman’s own words:

    “An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.”

- So we can iteratively solve the OCP, starting backwards in time

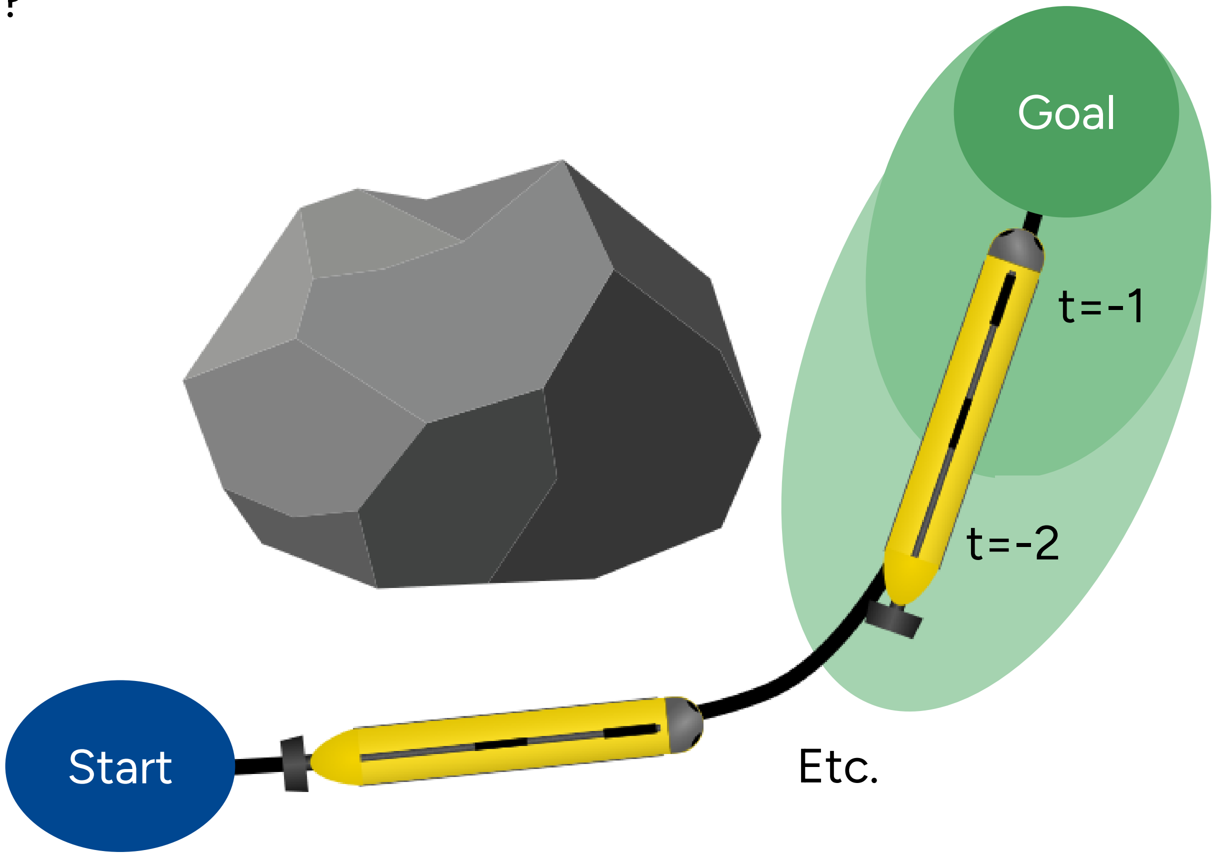


Captured by $\int_t^{t+\Delta t} L(x,u)dt$

Captured by $V(t+\Delta t, x(t+\Delta t))$

Dynamic Programming

- So we can iteratively solve the OCP, starting backwards in time
- Why is this so useful?



Blackboard time

Let's see what we can do with this

Hamilton-Jacobi-Bellman (HJB) PDE

$$\left.\begin{aligned} V(t,x) &= \min_u \int_0^T L\big(x(t),u(t)\big)ds + l(x(T)) \\ s.t.\quad \dot{x} &= f(x,u) \end{aligned}\right\} \text{A continuous opt. problem}$$

$$\left.\begin{aligned} &\frac{\partial V(t,x)}{\partial t} + \min_u \left\{\frac{\partial V(t,x)}{\partial x} f(x,u) + L(x,u)\right\} = 0 \\ &s.t.\quad V(T,x(T)) = l(x(T)) \end{aligned}\right\} \text{A final-value PDE}$$

- So what is $V(t,x)$?
  - The "cost" of starting at (t,x), under optimal control until T
- Now how do we obtain a control signal?
  - Generally, we have to solve a (high-dimensional) PDE
  - But there are some cases for which this is trivial: Linear Dynamics, Quadratic Cost!

Blackboard time

Let’s derive a solveable representation!

Differential Control Problems

- Zero-sum Differential Games
- Hamilton-Jacobi-Isaacs

Games of Degree vs. Games of Kind

- Fundamentally, our problem is a yes/no problem....
  - Starting at this state (t, x), will we reach the target within T?

$$R(t) = \{x: \exists u(\cdot), \forall d(\cdot), \qquad s.t.\ \Phi\big(T; x, u(\cdot), d(\cdot)\big) \in Goal\}$$

- We want to change our problem to a ”how much” problem...
  - Starting at this state, what is the value of reaching the target?
  - Why do we want this?
- If we can formulate some 'game', then the control $u(\cdot)$ and disturbance $d(\cdot)$ can play against eachother!
  - Zero-sum
  - Differential game

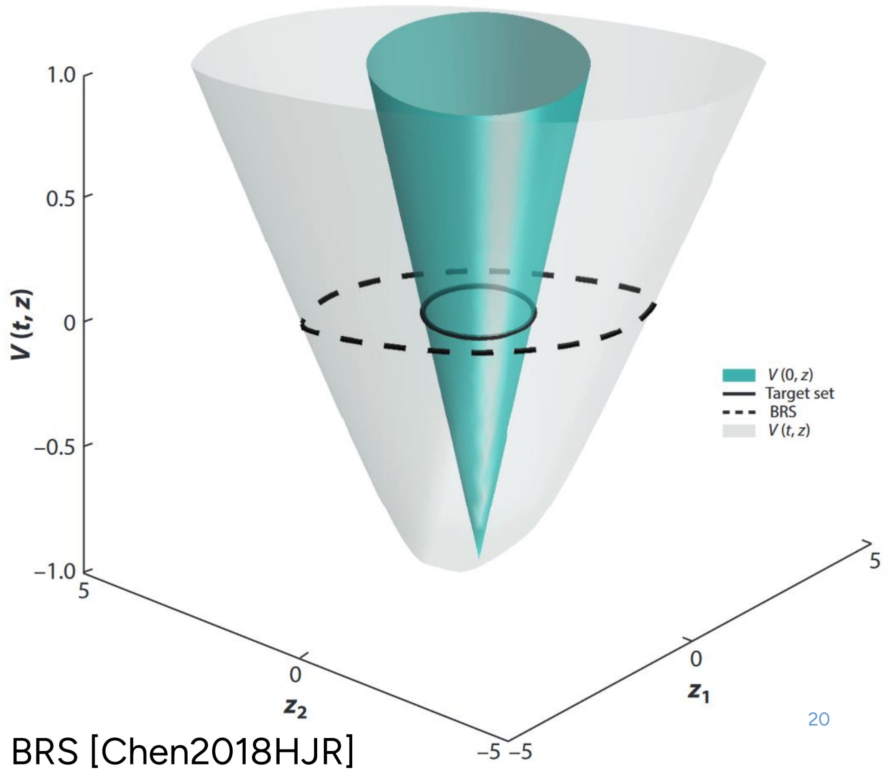


Fig: Zero-sublevel target set and BRS [Chen2018HJR]

From Optimal Control to Reachability

- Remember our Optimal Control problem:

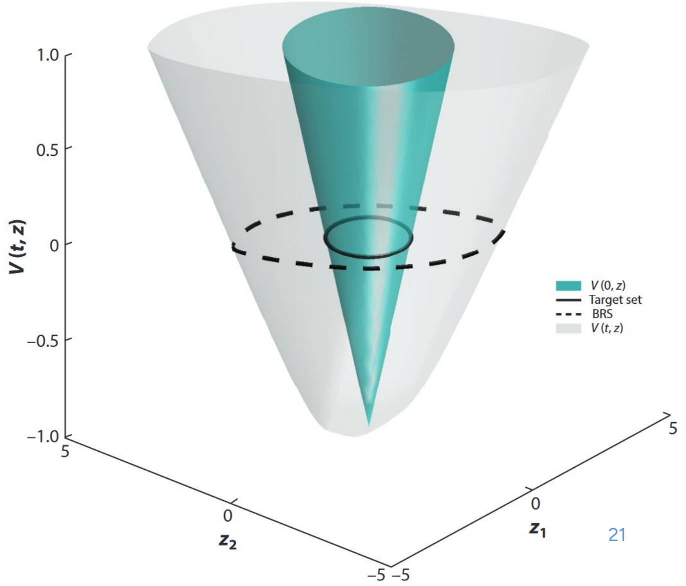
$$V(t,x) = \min_u \int_0^T L\big(x(t),u(t)\big)ds + l(x(T)) \quad s.t. \quad \dot{x} = f(x,u)$$

$$\frac{\partial V(t,x)}{\partial t} + \min_u \left\{ \frac{\partial V(t,x)}{\partial x} \cdot f(x,u) + L(x,u) \right\} = 0 \quad s.t. \quad V(T,x(T)) = l(x(T))$$

where we have a running cost $L(x,u)$, influencing what $V(t,x)$ represents

- But for reachability, we are interested the yes/no question (translated to a value problem)
  - Can we reach the set? No matter the cost!

$$\frac{\partial V(t,x)}{\partial t} + \min_u \left\{ \frac{\partial V(t,x)}{\partial x} \cdot f(x,u) \right\} = 0 \quad s.t. \quad V(T,x(T)) = l(x(T))$$



The Hamilton-Jacobi Isaacs Equation

Rufus Isaacs

$$\left.\begin{aligned} V(t,x) = \min_u \int_0^T L(x(t),u(t),d(t))ds + l(x(T)) \\ s.t. \quad \dot{x} = f(x,u,d) \end{aligned}\right\}$$

A continuous opt. problem

- How do we take care of the disturbance?
  - Assume the worst-case!

$$\left.\begin{aligned} V(t,x) = \overbrace{\min_u \max_d}^{\text{Lower-value}} \int_0^T L(x(t),u(t),d(t))ds + l(x(T)) \\ s.t. \quad \dot{x} = f(x,u,d) \end{aligned}\right\}$$

A continuous opt. problem

- So what is $V(t,x)$?
  - The "cost" of starting at (t,x) under optimal control and worst-case disturbance until T

The Hamilton-Jacobi Isaacs Equation

$$V(t,x) = \min_u \max_d \int_0^T L(x(t),u(t),d(t))ds + l(x(T))$$
$$s.t. \quad \dot{x} = f(x,u,d)$$

A continuous opt. problem

$$\frac{\partial V(t,x)}{\partial t} + \min_u \max_d [\nabla V(t,x) f(x,u,d)] = 0$$
$$s.t. \quad V\big(T, x(T)\big) = l(x(T))$$

A final-value PDE
Strict reachability

- So what is $V(t,x)$?
  - The "cost" of starting at (t,x) under optimal control and worst-case disturbance until T
  - We are interested in the sign of $V(t,x)$ to answer reachability questions
- Now how do we obtain a control signal?
  - $u^* = argmin_u \max_d [\nabla V(t,x) f(x,u,d)]$

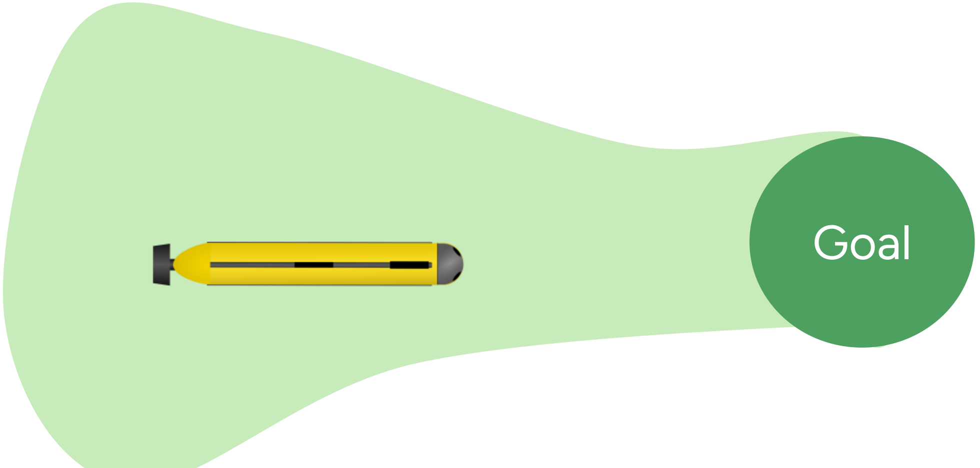
Comparisons/equivalences between OCP and HJI

Hamilton-Jacobi-Bellman

$$\frac{\partial V(t,x)}{\partial t} + \min_u [\nabla V(t,x) f(x,u)] = 0$$
$$s.t. \quad V\big(T, x(T)\big) = l(x(T))$$

"Standard" optimal control/reachability problem

"One-player" optimality

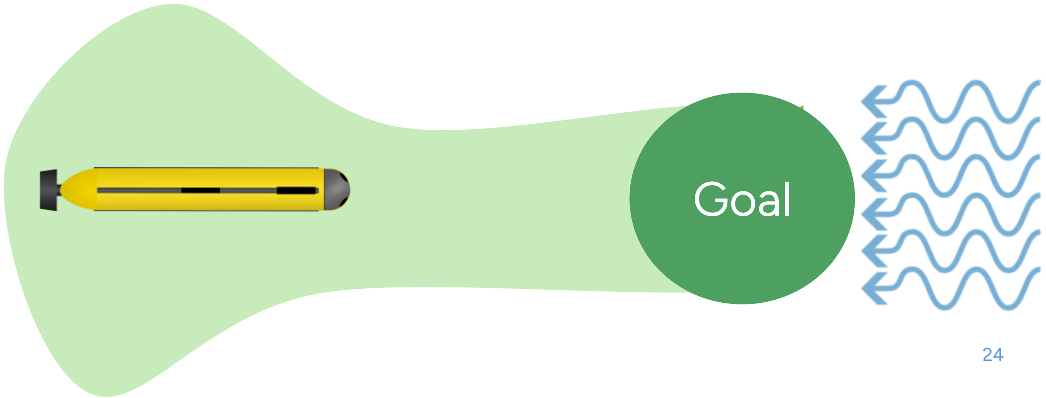


Hamilton-Jacobi-Isaacs

$$\frac{\partial V(t,x)}{\partial t} + \min_u \max_d [\nabla V(t,x) f(x,u,d)] = 0$$
$$s.t. \quad V\big(T, x(T)\big) = l(x(T))$$

Worst-case optimal control/reachability problem

"Two-player" optimality



The Hamilton-Jacobi Isaacs Equation

$$\frac{\partial V(t,x)}{\partial t} + \min_u \max_d [\nabla V(t,x) f(x,u,d)] = 0$$
$$s.t. \quad V\big(T, x(T)\big) = l(x(T))$$

A final-value PDE
Strict reachability

- *So what is $V(t,x)$?*
  - *The "cost" of starting at (t,x) under optimal control and worst-case disturbance until T*
- What if we swap $\min_u \max_d \ldots$ to $\max_u \min_d \ldots$
  - Avoid the target set for all disturbances $d(\cdot)$
- What if we swap $\min_u \max_d \ldots$ to $\min_u \min_d \ldots$
  - Reach the target set for some disturbance $d(\cdot)$

Blackboard time

Different flavours of HJI Reachability

Different Perspectives

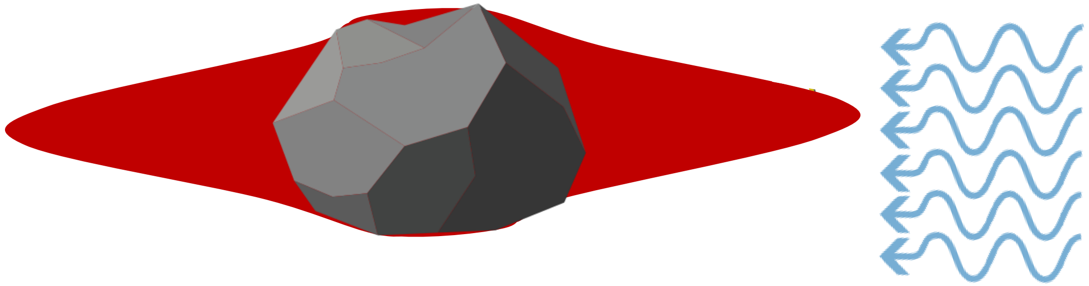
Avoid Set

$$R(t) = \{x: \exists d, \forall u(\cdot): \Phi\big(T; x, u(\cdot), d[u](\cdot)\big) \in Obs\}$$

$$\frac{\partial V(t,x)}{\partial t} + \max_u \min_d [\nabla V(t,x) f(x,u,d)] = 0$$

$$s.t. \quad V\big(T, x(T)\big) = l(x(T))$$

Enter G(t): You will collide at T



Reach Set

$$R(t) = \{x: \forall d, \exists u(\cdot): \Phi\big(T; x, u(\cdot), d[u](\cdot)\big) \in Goal\}$$

$$\frac{\partial V(t,x)}{\partial t} + \min_u \max_d [\nabla V(t,x) f(x,u,d)] = 0$$

$$s.t. \quad V\big(T, x(T)\big) = l(x(T))$$

Enter G(t): You'll be there at T!

Solving all of this

- The Levelset method

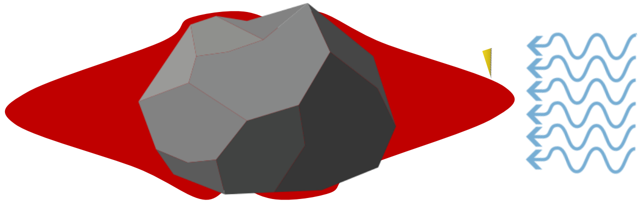
The Levelset Method

- What do we need to solve the HJI Equation?
  - We need a value-function $V(t, x)$ that, for any $x \in X$ and $t \in [0, T]$ captures whether we reach the goal/obstacle

- What do we have?
  - (high-dimensional) nonlinear dynamics, $\dot{x} = f(x, u, d)$
  - A control player and a disturbance player, $\min_u$ … and $\max_d$ …

- At the end, regardless of reach/avoid, helper/opposer
  - We need to solve a PDE…
  - But these are awful PDEs to analytically solve ☹

$$\frac{\partial V(t,x)}{\partial t} + \max_u \min_d [\nabla V(t,x) f(x,u,d)] = 0$$
$$s.t. \quad V\big(T, x(T)\big) = l(x(T))$$



Blackboard time

Different Perspectives

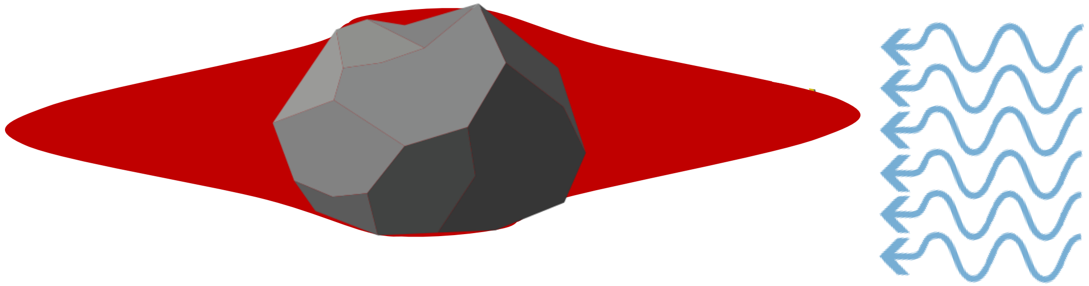
Avoid Set

$$R(t) = \{x: \exists d, \forall u(\cdot): \\ \Phi\big(T; x, u(\cdot), d[u](\cdot)\big) \in Obs\}$$

$$\frac{\partial V(t,x)}{\partial t} + \max_{u} \min_{d} [\nabla V(t,x) f(x,u,d)] = 0$$

$$s.t. \quad V\big(T, x(T)\big) = l(x(T))$$

Enter G(t): you'll collide at t=T



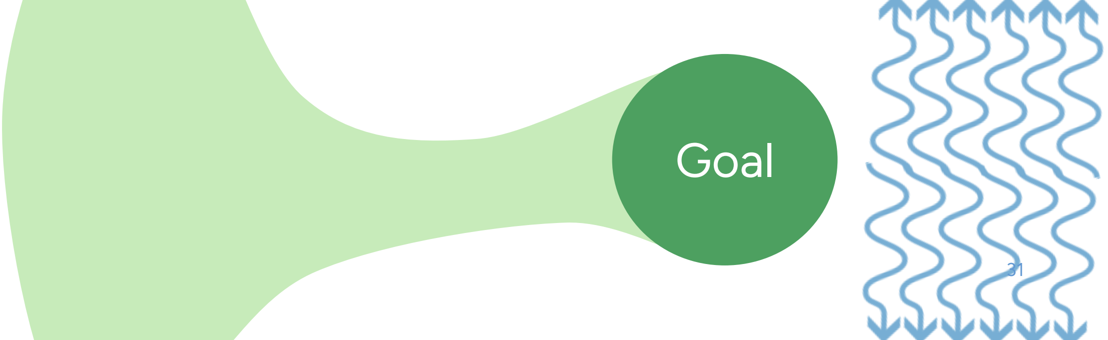
Reach Set

$$R(t) = \{x: \forall d, \exists u(\cdot): \\ \Phi\big(T; x, u(\cdot), d[u](\cdot)\big) \in Goal\}$$

$$\frac{\partial V(t,x)}{\partial t} + \min_{u} \max_{d} [\nabla V(t,x) f(x,u,d)] = 0$$

$$s.t. \quad V\big(T, x(T)\big) = l(x(T))$$

Enter G(t): You'll get there at t=T!



Different Perspectives

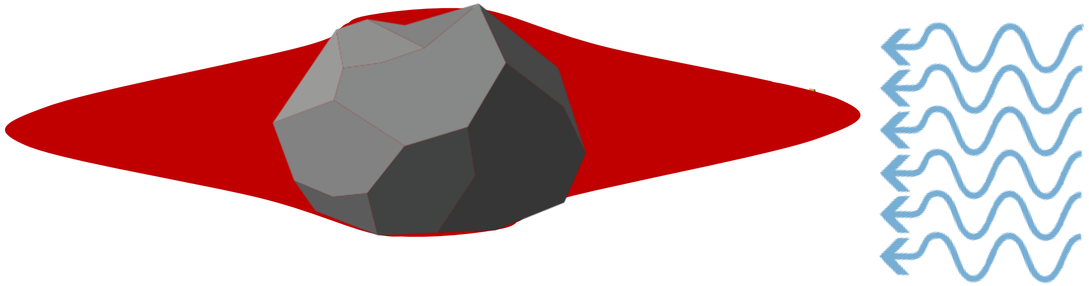
Avoid Tube

$$R(t) = \{x : \exists d, \forall u(\cdot), \exists \tau \in [0,T] : \\ \Phi\big(\tau; x, u(\cdot), d[u](\cdot)\big) \in Obs\}$$

$$\frac{\partial V(t,x)}{\partial t} + \min\{0, \max_u \min_d [\nabla V(t,x) f(x,u,d)]\} = 0$$

$$s.t. \quad V\big(T, x(T)\big) = l(x(T))$$

Enter G(t): you'll collide at some $t \in [0,T]$



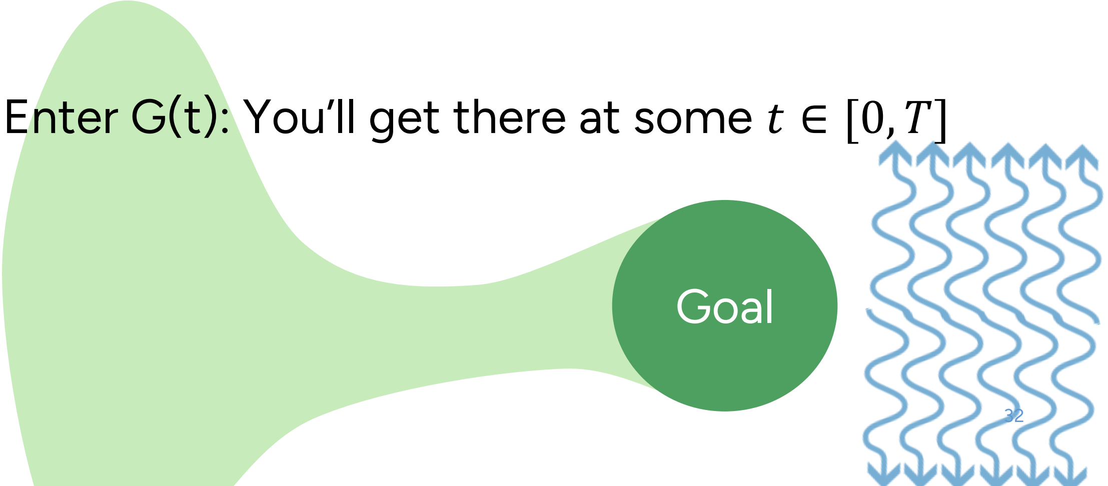
Reach Tube

$$R(t) = \{x : \forall d, \exists u(\cdot), \exists \tau \in [0,T] : \\ \Phi\big(\tau; x, u(\cdot), d[u](\cdot)\big) \in Goal\}$$

$$\frac{\partial V(t,x)}{\partial t} + \min\{0, \min_u \max_d [\nabla V(t,x) f(x,u,d)]\} = 0$$

$$s.t. \quad V\big(T, x(T)\big) = l(x(T))$$

Enter G(t): You'll get there at some $t \in [0,T]$



Different Perspectives

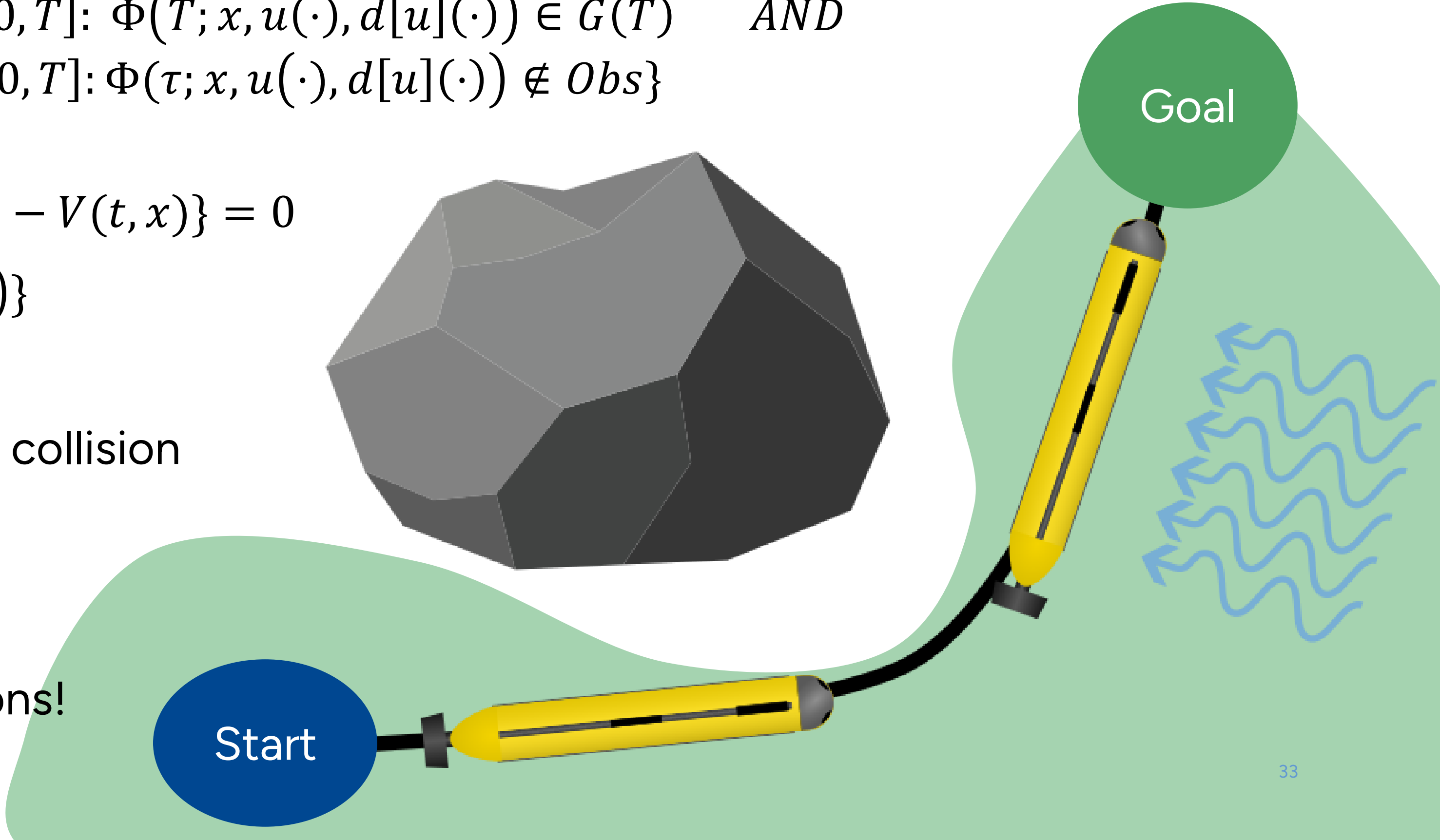
Reach-Avoid Tube

$$R(t) = \{x: \exists u, \forall d,$$
$$\exists t \in [0, T]: \; \Phi\big(T; x, u(\cdot), d[u](\cdot)\big) \in G(T) \qquad AND$$
$$\forall \tau \in [0, T]: \Phi(\tau; x, u(\cdot), d[u](\cdot)) \notin Obs\}$$

$$\max\{\frac{\partial V}{\partial t} + \min\{0, \min_u \max_d [\nabla V f(x, u, d)]\}, g(x) - V(t, x)\} = 0$$
$$s.t. \qquad V\big(T, x(T)\big) = \max\{l\big(x(T)\big), g\big(x(T)\big)\}$$

$g(x)$ = a function s.t. $g(x) \geq 0$ if the state is in collision

Enter $R(t)$: You'll get there without collisions!



The Levelset Method

- A set-based perspective on solving a PDE
  - Evolution of the set backwards in time
- We have to define a grid of the state-space
  - What does this allow us to do?
    - Arbitrary shapes of goals and obstacles
  - What does this mean for high-dimensional systems?
    - Curse of dimensionality!
  - What does this mean for the accuracy of our solution?
    - Only under- or over-approximations
- We solve for discrete points in time
  - What does this mean for the accuracy of our solution?
    - Again, only under- or over-approximations
- It is "trivial" to consider time-varying obstacles

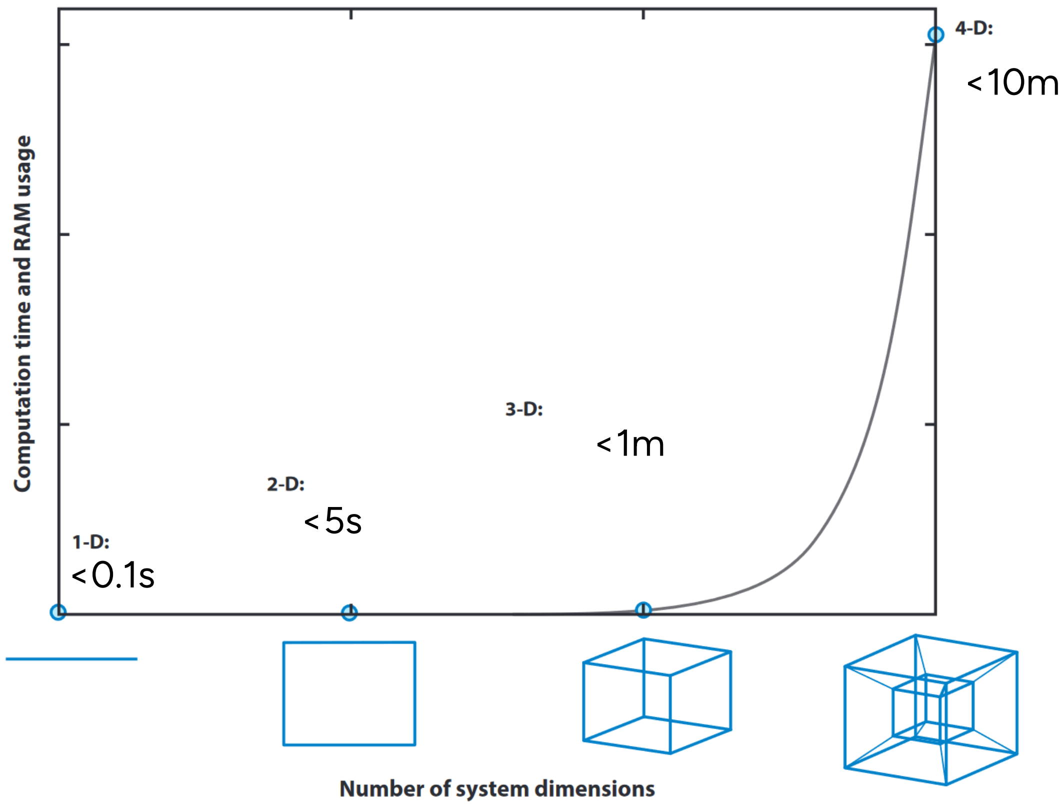


Fig: Computational complexity of HJ Reachability adapted from [Chen2018HJR]

Research <u>on</u> HJ Reachability

- Fundamentally, we need to solve a PDE
    - So lots of research on speeding up (approximate) methods

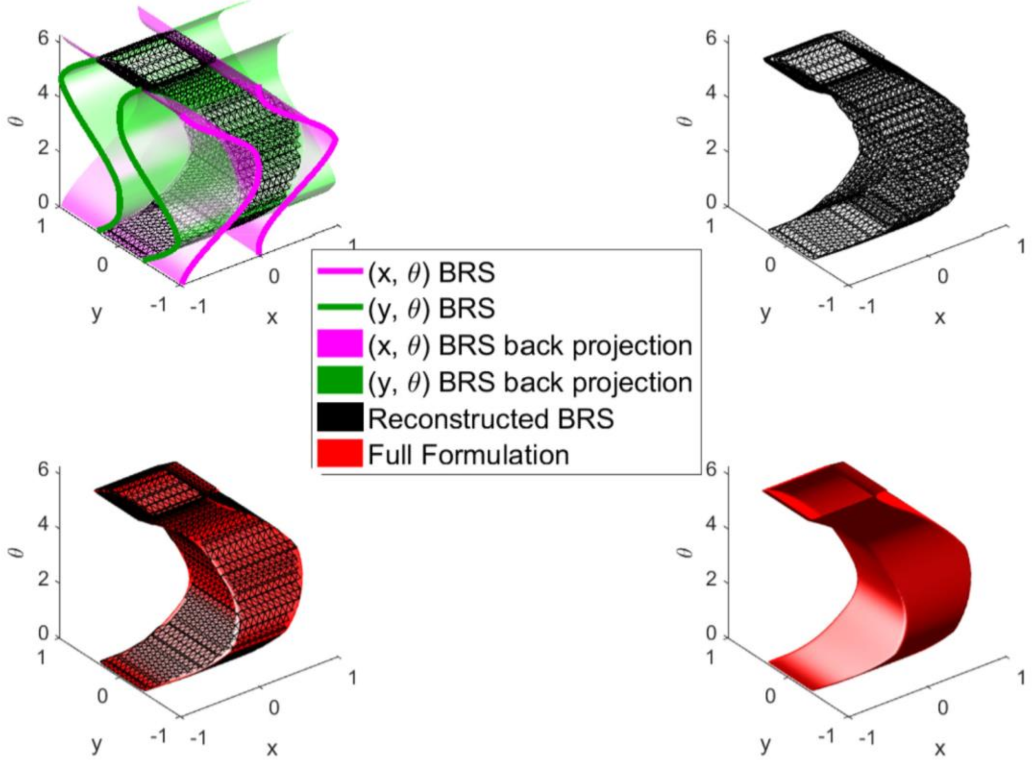
Research on HJ Reachability

- "Decomposition of reachable sets and tubes for a class of nonlinear systems" (Chen, Herbert, Vashishtha, Bansal & Tomlin, 2018)
- System decomposition:
  - Decouple high-order dynamics as different subsystems with shared control or disturbances
  - Trivial example: 2D double integrator
  - Non-trivial example: Dubin's car

$$\begin{bmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} v\cos\theta + d_x \\ v\sin\theta + d_y \\ \omega + d_\theta \end{bmatrix}$$

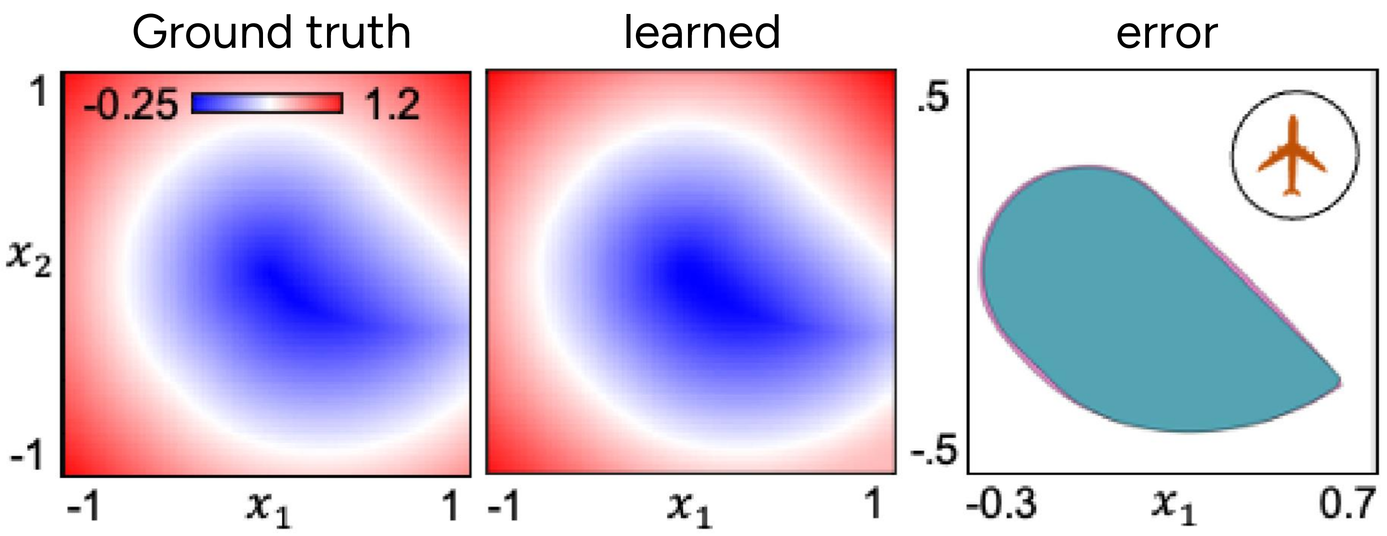
$$\omega \in \mathcal{U}, \quad (d_x, d_y, d_\theta) \in \mathcal{D}$$

Two self-contained subsystems!



Research on HJ Reachability

- "DeepReach: A Deep Learning Approach to High-Dimensional Reachability" (Bansil & Tomlin, 2021)
- Conditioned on a goal/obstacle state, can we learn
    - $V(t,x) = NN(t,x)$, in a self-supervised way (supervised by HJI value iteration)!
    - $Loss(x_i, t_i, \theta) = ||V_\theta(t_i, x_i) - l(x_i)||1(t_i = T) \quad + \quad \lambda|| \min\{\frac{\partial V_\theta(t_i,x_i)}{\partial t} + \max_u \min_d\{\nabla V(t_i, x_i) f(x,u,d)\}, l(x_i) - V_\theta(t_i, x_i)\}||$
- Extensions for statistical guarantees with conformal prediction



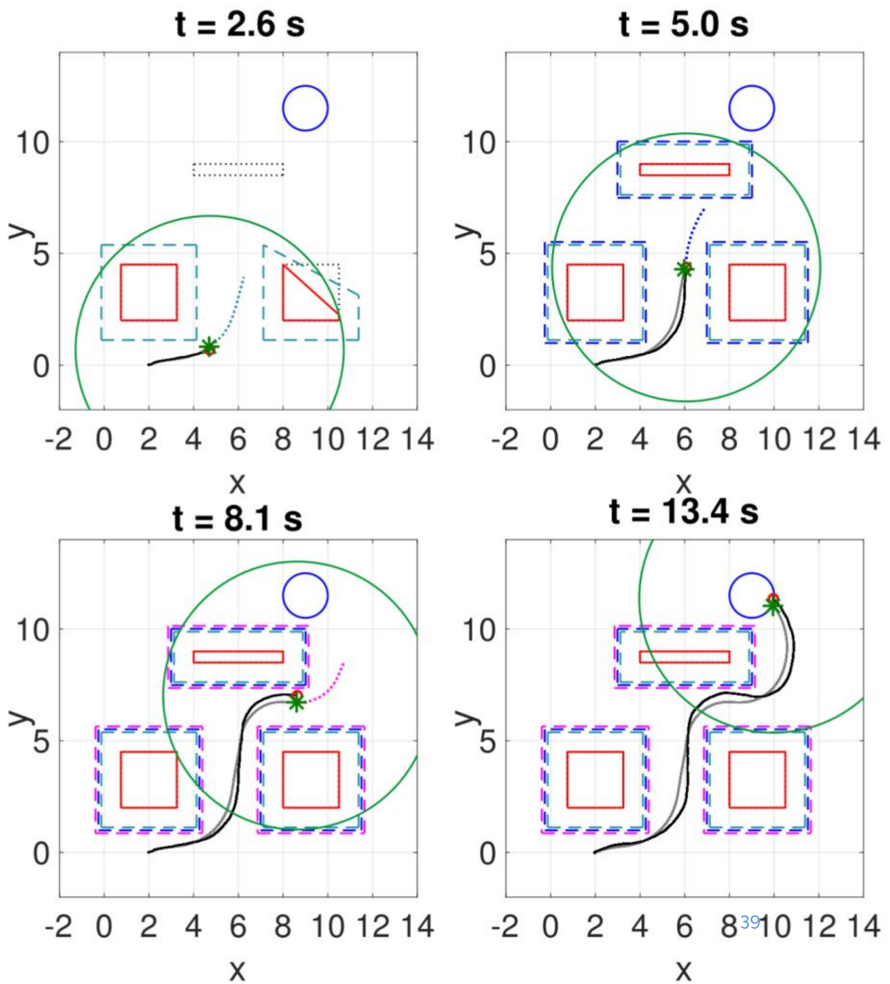
Research <u>with</u> HJ Reachability

- We can use HJ Reachability for
  - computing safe sets and plan with them!
  - Computing value functions and use them as backup controllers

Research with HJ Reachability

- FaSTrack: a Modular Framework for Fast and Guaranteed Safe Motion Planning (Herbert, Chen, Han, Bansal, Fisac & Tomlin, 2017)

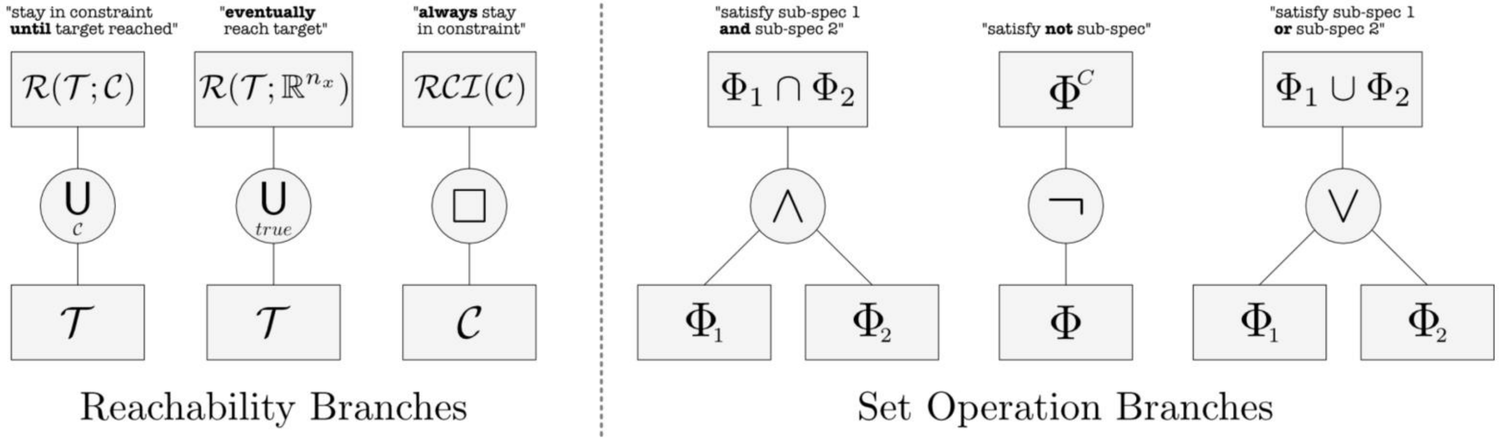
1. Define a simplified model of a complex dynamical model
2. Define tracking dynamics of the simplified and complex model
3. Offline reachability computation on these tracking dynamics
    1. Pursuit-evasion game between planner and tracker
    2. Tracker is pursuing the planner: what is the max tracking error?
4. Bloating obstacles for planning and safe controller look-up tables

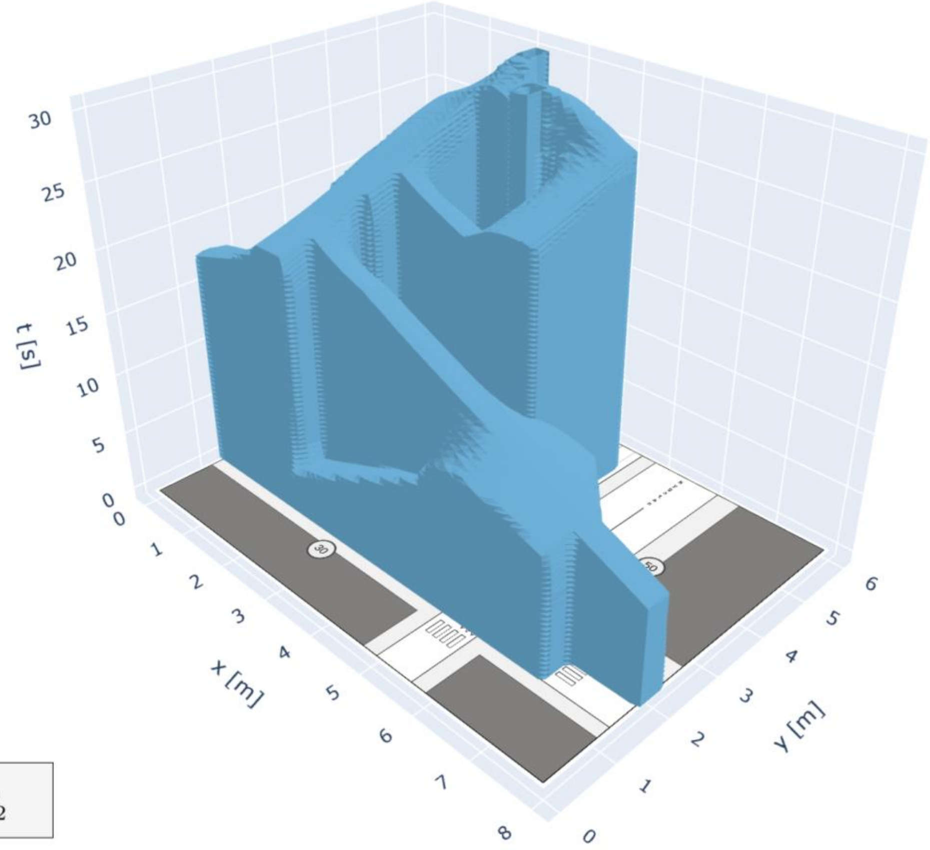


Research <u>with</u> HJ Reachability

- "Guaranteed Completion of Complex Tasks via Temporal Logic Trees and Hamilton-Jacobi Reachability" (Jiang, Arfvidsson, He, Chen, Johansson, 2024)

- Convert a high-level specification (e.g. do this and this or that and that) to a tree of reachable sets
- If you start in a valid region, you are guaranteed to satisfy the overall specification





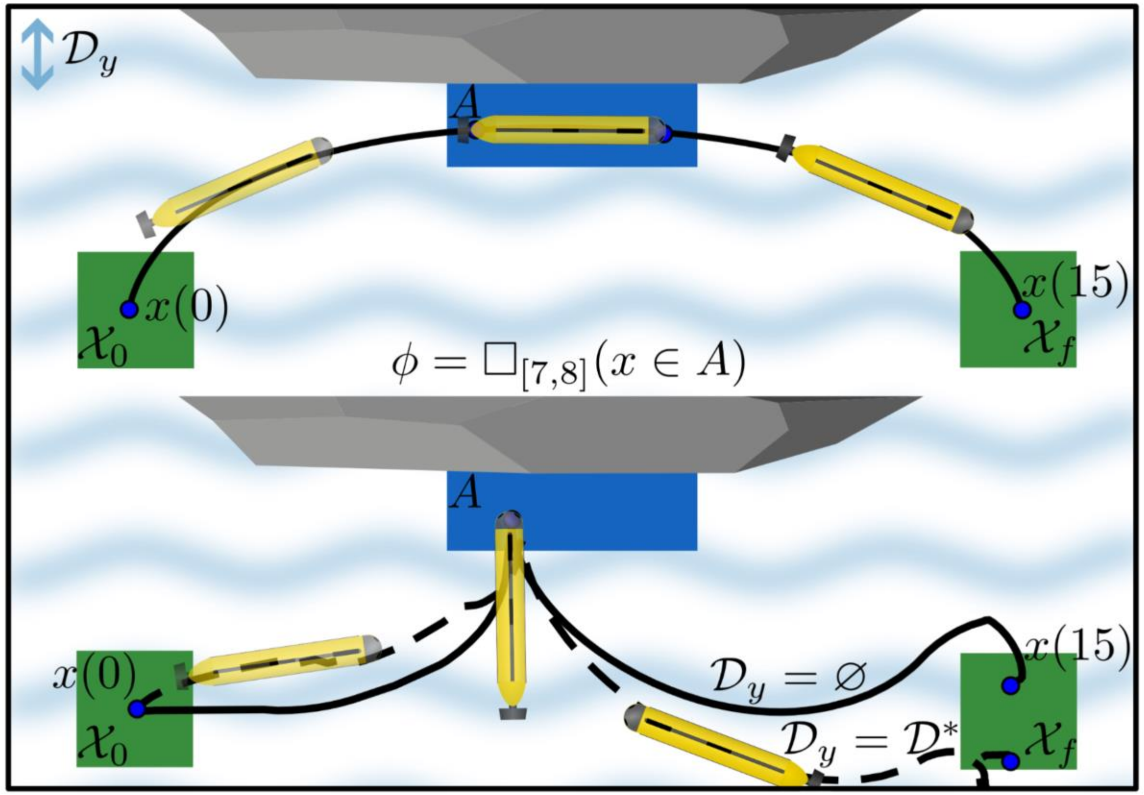
Research <u>with</u> HJ Reachability

- "Robust STL Control Synthesis under Maximal Disturbance Sets" (Verhagen, Lindemann, Tumova, 2024)

- We have some high-level specification ϕ
  - E.g. Always, between 7 and 8 seconds, be at location A
  - Always, Eventually, AND, OR
- Which "trajectory" permits the largest disturbance set $D, d \in D$

$$\begin{aligned} & \max\ D \\ s.t.\ \ & \Phi(t_0, X_0, u(\cdot) \in U, d(\cdot) \in D) \vDash \phi \end{aligned}$$

- Generally, no closed-form approach
- Iterative HJI computations (specific algs. per operator)



Key take-aways

- Reachability can be formulated as a dynamic programming problem
  - Yes/no to values: games of kind to games of degree
- Optimal control can be formulated as the solution to a PDE

$$V(t,x) = \min_u \int_0^T L\big(x(t),u(t)\big)ds + l(x(T)) \qquad s.t. \quad \dot{x} = f(x,u)$$

OCP

$$\frac{\partial V(t,x)}{\partial t} + \min_u \left\{ \frac{\partial V(t,x)}{\partial x} f(x,u) + L(x,u) \right\} = 0 \qquad s.t. \quad V(T,x(T)) = l(x(T))$$

PDE formulation

- Reachability can be formulated as the solution to a modified PDE

$$\frac{\partial V(t,x)}{\partial t} + \min_u \left\{ \frac{\partial V(t,x)}{\partial x} f(x,u) \right\} = 0 \qquad s.t. \quad V(T,x(T)) = l(x(T))$$

HJB

$$\frac{\partial V(t,x)}{\partial t} + \min_u \max_d [\nabla V(t,x) f(x,u,d)] = 0 \qquad s.t. \quad V\big(T,x(T)\big) = l(x(T))$$

HJI

- There exists different flavors (reach/avoid, opponent/helper, set/tube)

$$\frac{\partial V(t,x)}{\partial t} + \min\{0, \max_u \min_d [\nabla V(t,x) f(x,u,d)]\} = 0 \qquad s.t. \quad V\big(T,x(T)\big) = l(x(T))$$

Avoid tube

$$\frac{\partial V(t,x)}{\partial t} + \min\{0, \min_u \max_d [\nabla V(t,x) f(x,u,d)]\} = 0 \qquad s.t. \quad V\big(T,x(T)\big) = l(x(T))$$

Reach tube

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