Theory of Discrete Fourier Transform and Cross-correlation function

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January 20, 2019

1 Introduction

In this note, we describe the definition of Fourier transform, Inverse Fourier transform, Discrete Fourier transform (DFT), Inverse Discrete Fourier transform (IDFT) and Cross-correlation function for time series analysis.

2 Definition

Let $f(t)$ be a time series (*e.g.* acceleration [m/s^2], velocity [m/s], displacement [m]). The unit of t is time [s].

Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{1}$$

where i is imaginary unit.

Fourier Transform

$$F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}dt = \mathcal{F}[f(t)], \tag{2}$$

where ω is frequency [Hz]. $\mathcal{F}$ indicates the Fourier transform operation.

Inverse Fourier Transform

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)e^{i\omega t}d\omega = \mathcal{F}^{-1}[F(\omega)]. \tag{3}$$

Thus

$$\mathcal{F}^{-1}[\mathcal{F}[f(t)]] = f(t). \tag{4}$$

Cross-correlation function

Let $u_1(t)$ and $u_2(t)$ be time series.

$$\psi_{12}(\tau) = \int_{-\infty}^{+\infty} u_1(t)u_2(t+\tau)dt. \tag{5}$$

When $u_1(t)$ and $u_2(t)$ are finite duration time series ($t_1 \leq t \leq t_2$), the cross-correlation function is defined as

$$\psi_{12}(\tau) = \int_{t_1}^{t_2} u_1(t)u_2(t+\tau)dt, \tag{6}$$

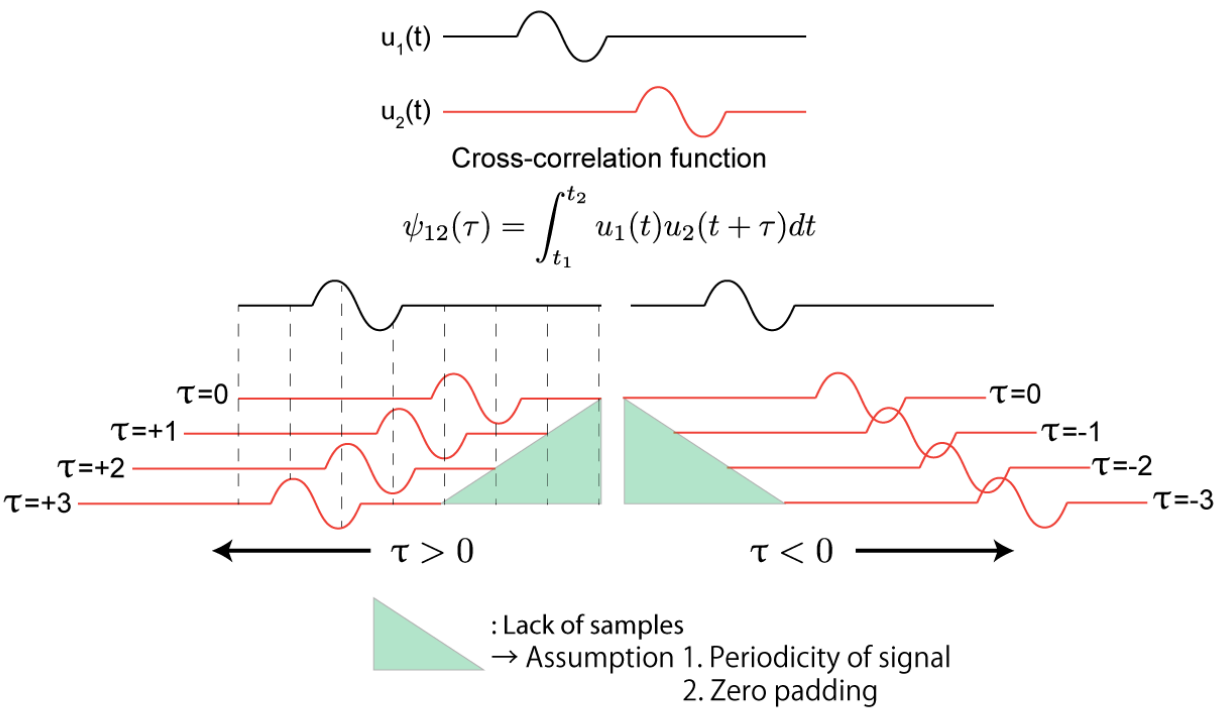

Figure 1: Schematic of Cross-correlation function.

where

$$-(t_2 - t_1) \leq \tau \leq (t_2 - t_1). \tag{7}$$

Cross spectrum

$$\Psi_{12}(\omega) = U_1^*(\omega)U_2(\omega), \tag{8}$$

where U_1^* is complex conjugate of $U_1(\omega)$. It is noteworthy that we assume the periodicity of finite duration time series $[t_1, t_2]$ to regard it as infinite duration time series. Thus $-\infty < \omega < \infty$.

Then

$$\psi_{12}(\tau) = \mathcal{F}^{-1}[\Psi_{12}(\omega)]. \tag{9}$$

Discrete Fourier Transform (DFT)

Let $u[n]$ be a discrete time series. $n = 0, 1, 2, \cdots, N-1$, where N is data length. Then $t = n \times dt$.

$$U[k] = \sum_{m=0}^{N-1} u[m]e^{-\frac{i2\pi km}{N}}. \quad k = 0, 1, 2, \cdots, N-1 \tag{10}$$

Inverse Discrete Fourier Transform(IDFT)

$$u[n] = \frac{1}{N}\sum_{l=0}^{N-1} U[l]e^{\frac{i2\pi ln}{N}}. \quad n = 0, 1, 2, \cdots, N-1 \tag{11}$$

Then

$$u[n] = IDFT[DFT[u[n]]]. \tag{12}$$

Discrete Cross-correlation function

$$\psi_{12}[n] = \sum_{k=0}^{N-1} u_1[k]u_2[k+n]. \quad n = 0, \pm 1, \pm 2, \cdot, \pm (N-1) \tag{13}$$

Thus the size of $\psi_{12}[n]$ is $2N-1$. $t = n \times dt$. For the Discrete Cross spectrum, we can also define the Discrete Cross-correlation function as following:

$$\psi_{12}[n] = \begin{cases} \sum_{k=0}^{N-1} u_1[k]u_2[k+n] & n = 0, 1, 2, \cdots, N-1 \\ \sum_{k=0}^{N-1} u_1[k-n]u_2[k] & n = -1, -2, \cdots, -(N-1) \end{cases}. \tag{14}$$

Note

For negative n, shifting u_2 towards right-hand side is identical with shifting u_1 towards left-hand side (See Figure 1).

Discrete Cross spectrum

$$\Psi_{12}^{+}[k] = U_1^*[k]U_2[k] \quad (k \geq 0) \tag{15}$$

$$\Psi_{12}^{-}[-k] = U_1[-k]U_2^*[-k] \quad (k < 0). \tag{16}$$

where $-(N-1) \leq k \leq (N-1)$.

Note

Since the definition of DFT allows the index to be only positive, we have to consider the conditions $k \geq 0$ or $k < 0$.

Then

$$\psi_{12}[n] = \begin{cases} IDFT[\Psi_{12}^{+}[n]] & n = 0, 1, 2, \cdots, N-1 \\ IDFT[\Psi_{12}^{-}[-n]] & n = -1, -2, \cdots, -(N-1) \end{cases}. \tag{17}$$

Fast Fourier Transform (FFT)

FFT is the algorithm to obtain the components of DFT effectively by using periodicity to reduce the calculation.

3 Derivation

Lemma 1.

$$\mathcal{F}^{-1}[\mathcal{F}[f(t)]] = f(t). \tag{18}$$

Proof.

$$\mathcal{F}^{-1}[\mathcal{F}[f(t)]] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} f(t') e^{-i\omega t'} dt' \right] e^{i\omega t} d\omega \tag{19}$$

$$= \int_{-\infty}^{+\infty} f(t') \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\omega(t'-t)} d\omega \right] dt'. \tag{20}$$

Since

$$
\begin{aligned}
\int_{-\infty}^{+\infty} e^{-i\omega(t'-t)} d\omega &= \lim_{a\to\infty} \left(\int_0^a - \int_0^{-a} \right) e^{-i\omega(t'-t)} d\omega &(21)\\
&= \lim_{a\to\infty} \frac{2\pi \sin a(t'-t)}{\pi(t'-t)} &(22)\\
&= 2\pi\delta(t'-t), &(23)
\end{aligned}
$$

where $\delta(t'-t)$ is Dirac delta function.

Note

$$
\lim_{a\to\infty} \frac{\sin ax}{\pi x} = \delta(x) \tag{24}
$$

Thus equation (20) is rewritten as following

$$
\begin{aligned}
\int_{-\infty}^{+\infty} f(t') \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\omega(t'-t)} d\omega \right] dt' &= \int_{-\infty}^{+\infty} f(t')\delta(t'-t) dt' &(25)\\
&= f(t) &(26)
\end{aligned}
$$

□

Lemma 2.

$$
\Psi_{12}(\omega) = U_1^*(\omega) U_2(\omega), \tag{27}
$$

Proof.

$$
\begin{aligned}
\Psi_{12}(\omega) &= \mathcal{F}[\psi_{12}(\tau)] &(28)\\
&= \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} u_1(t) u_2(t+\tau) dt \right] e^{-i\omega(\tau)} d\tau &(29)\\
&= \int_{-\infty}^{+\infty} u_1(t) e^{+i\omega t} \left[\int_{-\infty}^{+\infty} u_2(t+\tau) e^{-i\omega(t+\tau)} d\tau \right] dt &(30)\\
&= \int_{-\infty}^{+\infty} u_1(t) e^{+i\omega t} \left[\int_{-\infty}^{+\infty} u_2(\nu) e^{-i\omega(\nu)} d\nu \right] dt &(31)\\
&= U_2(\omega) \int_{-\infty}^{+\infty} u_1(t) e^{+i\omega t} dt &(32)\\
&= U_1^*(\omega) U_2(\omega). &(33)
\end{aligned}
$$

□

Lemma 3.

$$
\Psi_{12}[k] = U_1^*[k] U_2[k] \quad (k = 0, 1, 2, \cdots, N-1) \tag{34}
$$

Proof.

$$\begin{aligned}
\Psi_{12}[k] &= \sum_{m=0}^{N-1} \psi[m] e^{-\dfrac{i2\pi km}{N}} && (35)\\
&= \sum_{m=0}^{N-1}\sum_{l=0}^{N-1} u_1[l]u_2[l+m] e^{-\dfrac{i2\pi km}{N}} && (36)\\
&= \sum_{m=0}^{N-1} u_1[l] e^{\dfrac{i2\pi km}{N}} \sum_{l=0}^{N-1} u_2[l+m] e^{-\dfrac{i2\pi k(l+m)}{N}} && (37)\\
&= \sum_{m=0}^{N-1} u_1[l] e^{\dfrac{i2\pi km}{N}} \sum_{\nu=m}^{N+m-1} u_2[\nu] e^{-\dfrac{i2\pi k\nu}{N}}. && (38)
\end{aligned}$$

Considering periodicity of $e^{-\dfrac{i2\pi k\nu}{N}}$ and time series,

$$\begin{aligned}
e^{-\dfrac{i2\pi k(N+\alpha)}{N}} &= e^{-i2\pi k} e^{-\dfrac{i2\pi k\alpha}{N}} && (39)\\
&= e^{-\dfrac{i2\pi k\alpha}{N}}. && (40)
\end{aligned}$$

$$u[N+\alpha] = u[N]. \tag{41}$$

Note

Equation (41) infers the assumption that the time series is periodic with [u[0], u[N-1]]

Thus

$$\sum_{\nu=m}^{N+m-1} u_2[\nu] e^{-\dfrac{i2\pi k\nu}{N}} = \sum_{\nu=0}^{N-1} u_2[\nu] e^{-\dfrac{i2\pi k\nu}{N}}. \tag{42}$$

Then the equation (38) is written as

$$\begin{aligned}
\Psi_{12}[k] &= \sum_{m=0}^{N-1} u_1[l] e^{\dfrac{i2\pi km}{N}} \sum_{\nu=0}^{N-1} u_2[\nu] e^{-\dfrac{i2\pi k\nu}{N}} && (43)\\
&= U_1^*[k]U_2[k]. && (44)
\end{aligned}$$

□