Eidgenössische Technische Hochschule Zürich

Analysis II

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Chapter 1

Metric spaces

1.1 Euclidean Space $\mathbb{R}^n$

We all know the set $\mathbb{R}^n = \{x = (x_1, \dots, x_n) \mid x_i \in \mathbb{R}\}$, all n-tuples of real numbers. We want a vector space structure on this

$$x + y = (x_1 + y_1, \dots, x_n + y_n), \quad \lambda x = (\lambda x_1, \dots, \lambda x_n),$$

but with a notion on distance.

DEFINITION 1.1: EUCLIDEAN STRUCTURE OF $\mathbb{R}^n$

Given $x, y \in \mathbb{R}^n$, we define the *scalar product*

$$x \cdot y = \langle x, y \rangle := \sum_{i=1}^{n} x_i y_i,$$

the *Euclidean norm*

$$\|x\| := \sqrt{x_1^2 + \cdots + x_n^2}$$

and *Euclidean distance*

$$d_{\text{Euclidean}}(x, y) := \|x - y\|.$$

LEMMA 1.2: CAUCHY-SCHWARTZ

For all $x, y \in \mathbb{R}^n$, $x \cdot y \leq \|x\|\|y\|$.

Proof. If either $x = 0$ or $y = 0$, both sides are zero and the inequality is true. So assume wlog both $x, y \neq 0$. Let $\lambda > 0$ and using $2ab \leq a^2 + b^2$

$$2x \cdot y = 2 \sum_{i=1}^{n} \lambda x_i \frac{y_i}{\lambda} \leq \sum_{i=1}^{n} \lambda^2 x_i^2 + \frac{y_i^2}{\lambda} = \lambda^2 \|x\|^2 + \frac{1}{\lambda^2} \|y\|^2.$$

Take $\lambda = \dfrac{\|y\|}{\|x\|}$ and the equation becomes

$$\lambda^2 \|x\|^2 + \frac{1}{\lambda^2} \|y\|^2 = 2\|x\|\|y\|,$$

which proves the inequality. $\square$

Lemma 1.3: Triangle inequality

For all $x, y, z \in \mathbb{R}^n$,

$$\|x - z\| \leq \|x - y\| + \|y - z\|.$$

Proof. We want to prove

$$\forall x, y \in \mathbb{R}^n : \|x + y\| \leq \|x\| + \|y\|.$$

This is equivalent by substitution. We square this to get

$$\|x + y\|^2 \leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\|.$$

But the LHS computes to

$$\sum_{i=1}^{n} (x_i + y_i)^2 = \sum_{i=1}^{n} x_i^2 + y_i^2 + 2x_i y_i = \|x\|^2 + \|y\|^2 + 2x \cdot y.$$

And the statement is equivalent to $x \cdot y \leq \|x\|\|y\|$, which holds by the Cauchy-Schwartz, Lemma 1.2. □

1.2 Definition of metric space

We will work with just three axioms for distance, staying general until we later apply what we learned to $\mathbb{R}^n$ specifically.

Definition 1.4: Metric space

A metric space is a pair (X, d), where X is a set and the *distance* $d : X \times X \to [0, \infty)$ satisfies

(1) $\forall x, y \in X : d(x, y) = 0 \iff x = y$.
(2) $\forall x, y \in X : d(x, y) = d(y, x)$.
(3) $\forall x, y, z \in X : d(x, z) \leq d(x, y) + d(y, z)$.

Example 1.5

$(\mathbb{R}^n, d_{\text{Euclidean}})$ is a metric space.
But for $\mathbb{R}^2$, there is also the distance $d(x, y) = d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$.
For a metric (X, d) and $Y \subset X$, (Y, d) is also a metric space.
On the sphere $X = \left\{x \in \mathbb{R}^3 \mid \|x\| = 1\right\}$, you can also either define the distance as the Euclidean distance or by going on a circular arc from one point to another. This also demonstrates that the most common metric isn't always the most natural/useful pick.
Let $X = C_0([a, b])$, the space of all real-valued continuos functions on $[a, b]$. Then

$$d_1(f, g) = \max_{[a,b]} |f - g|, \quad d_2(f, g) = \left(\int_a^b (f - g)^2 \, dx\right)^{\frac{1}{2}}$$

are both metrics.

1.3 Sequences in metric spaces

Like in the real numbers, we can define sequences in all metric spaces. And in fact, we need to understand sequences in real numbers to define sequences in metric spaces.

DEFINITION 1.6: SEQUENCE

For a set X, we call a sequence in X a map $x : \mathbb{N} \to X$ and write $x_n = x(n)$. For the entire sequence, we write $(x_n)_n^\infty \subset X$.

DEFINITION 1.7: CONVERGENCE AND LIMIT

Let $(x_n)_n^\infty$ be a sequence in a metric space (X, d). We say $(x_n)_n^\infty$ converges to $x \in X$ if $d(x_n, x) \to 0$ as a sequence of real numbers.
Equivalently, $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ s.t. $\forall n \geq N : d(x_n, x) < \varepsilon$.
We write $\lim\limits_{n\to\infty} x_n = x$, or $x_n \to x$.

LEMMA 1.8: LIMIT IS UNIQUE

Let (X, d) be a metric space and $(x_n)_n^\infty \subset X$ a sequence with $x_n \to x, x_n \to y$. Then $x = y$.

Proof. Trivial. $\square$

DEFINITION 1.9: SUBSEQUENCE

Let $(x_n)_n^\infty$ be a sequence in X. We define a subsequence as any sequence of the form $\left(x_{f(k)}\right)_k^\infty$ where $f : \mathbb{N} \to \mathbb{N}$ is increasing. We write $f(k) = n_k$, so the subsequence is denoted $(x_{n_k})_k^\infty$.

DEFINITION 1.10: ACCUMULATION POINT

Let (X, d) be a metric space.
(1) For a subset $Y \subset X$, we say that $y \in X$ is an accumulation point of Y if $\exists (y_n)_n^\infty \subset Y$ such that $y_n \to y$.
(2) For a sequence $(x_n)_n^\infty \subset X$, we say that $x \in X$ is an accumulation point if $\exists$ a subsequence $x_{n_k} \to x$.

LEMMA 1.11: SUBSEQUENCES AND LIMIT

A sequence $(x_n)_n \subset X$, for (X, d) metric space, converges to x if and only if every subsequence $(x_{n_k})_k^\infty$ converges to x.

Proof. $\Longleftarrow$: If every subsequence converges to x, since $(x_n)_n^\infty$ is also a subsequence, the limit is x.

$\Longrightarrow$: Let $(x_n)_n^\infty \to x$ and $(x_{n_k})_k^\infty$ any subsequence. For $\varepsilon > 0$ find $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for $n \geq N$, then since $n_k \geq k$, we get $d(x_{n_k}, 0) < \varepsilon$, so $x_{n_k} \to x$ as well. $\square$

LEMMA 1.12: SUBSEQUENCE AND LIMIT 2

Under the same assumptions, $x_n \to x$ if and only if every subsequence $(x_{n_k})_k^\infty$ has a sub-subsequence $x_{n_k} \to x$.

Proof. Exercise. $\square$

Definition 1.13: Cauchy sequence

In a metric space, $(x_n)_n^\infty$ is Cauchy if

$$\forall \varepsilon > 0\, \exists N \in \mathbb{N} \text{ s.t. } \forall n, m \geq N : d(x_n, x_m) < \varepsilon.$$

Definition 1.14: Completeness

(X, d) is complete if every Cauchy sequence in it converges.

Example 1.15

Consider $(\mathbb{R}, d_{\text{Euclidean}})$. The space $A = (0, \infty)$ is *not* complete, but $A = [a, b]$ is. $\mathbb{Q} \subset \mathbb{R}$ is not complete, because you can have a sequence going to $\sqrt{2}$, which is not in $\mathbb{Q}$.

Remark 1.16

There is a conflict of notation in the following proofs, because for a sequence $(x_n)_n^\infty \subset \mathbb{R}^n$, x_n could either mean $x(n)$ or the n-th component of the vector x. So in the following two proofs, m will be the index in the sequence and i, j the coordinate index. And then $x_{m,i}$ means the i-th component of $x(m)$.

Lemma 1.17: Coordinate wise convergence

Let $(x_m)_m^\infty$ be a sequence. Then $x_m \to x$ if and only if $x_{m,i} \to x_i$ for every $1 \leq i \leq n$.

Proof. $\implies$: Assume $x_m \to x$. This means for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|x_m - x\| < \varepsilon$ if $m \geq N$. Given $1 \leq i \leq n$, notice

$$|x_{m,i} - x_i| \leq \sqrt{\sum_{j=1}^{n} |x_{m,j} - x_j|^2} = \|x_m - x\| < \varepsilon.$$

This way, $x_{m,i} \to x_i$.

$\impliedby$: Let $\varepsilon > 0$. By assumption for all $1 \leq i \leq n$, $\exists N_i \in \mathbb{N}$ s.t. $\forall m \geq N_i : |x_{m,i} - x_i| < \frac{\varepsilon}{\sqrt{n}}$. If we define $N = \max_{1 \leq i \leq n} N_i$, then for $m \geq N$,

$$\|x_m - x\| = \sqrt{\sum_{j=1}^{n} |x_{m,i} - x_i|^2} \leq \sqrt{n \frac{\varepsilon^2}{n}} = \varepsilon.$$

This way, $x_m \to x$ as a whole. □

Theorem 1.18: Euclidean $\mathbb{R}^n$ is complete

The metric space $(\mathbb{R}^n, d_{\text{Euclidean}})$ with the Euclidean metric 1.1 is complete.

Proof. So let $(x_n)_n^\infty$ be a Cauchy sequence in $\mathbb{R}^n$. Because $|x_i - y_i| \leq \|x - y\|$ for $x, y \in \mathbb{R}^n$, we get that all the component-wise sequence are Cauchy. But those are in $\mathbb{R}$, so they converge. By 1.17, $x_n \to x$. □

1.4 Open and closed sets

Continuing our quest to generalise everything, we extend the definition of open and closed sets in $\mathbb{R}$ to any metric space (X, d).

Definition 1.19: Open ball

Let (X, d) be a metric space. Define the open ball of radius $r > 0$ centered at $x \in X$ as

$$B_r(x) = B(x, r) = \{y \in X \mid d(y, x) < r\}.$$

Definition 1.20: Open and closed sets

For (X, d) a metric space, we say

- $U \subset X$ is *open* if $\forall x \in U\ \exists r > 0$ s.t. $B(x, r) \subset U$.
- $A \subset X$ is *closed* if $X \setminus A$ is open.

The collection of all open sets $\mathcal{T}_d = \{U \subset X \mid U \text{ is open}\}$ is called the *topology*.

Lemma 1.21: Union and intersection of open sets is open

Let (X, d) be a metric space and $\{U_i\}_{i \in I}$ a family of open sets in X.

- Arbitrary unions of open sets are open. So $\bigcup_{i \in I} U_i$ is open.
- Finite intersections of open sets are open. So if I is finite, $\bigcap_{i \in I} U_i$ is open.

Proof. Set

$$U = \bigcup_{i \in I} U_i.$$

If $x \in U$, Then there exists $i \in I$ with $x \in U_i$ and since U_i is open, there exists an $r > 0$ such that $B(x, r) \subset U_i$, so also contained in U. Thus, U is open.

For the second one, consider that if x is in all the open sets U_i, then for each i, there exists a small enough radius r_i such that $B(r_i, x) \subset U_i$. Then, out of all r_i's, pick the smallest one, call it r and $B(r, x)$ is contained in all the U_i, so in U. $\square$

Lemma 1.22: Union and intersection of closed sets

Let (X, d) be a metric space and $\{A_i\}_{i \in I}$ a family of closed sets.

- Arbitrary intersections of closed sets are closed. So $\bigcup_{i \in I} A_i$ is closed.
- Finite unions of closed sets are closed. So if I is finite, $\bigcap_{i \in I} A_i$ is closed.

Proof. Consider that

$$X \setminus \bigcup_{i \in I} A_i = \bigcap_{i \in I} (X \setminus A_i),$$
$$X \setminus \bigcap_{i \in I} A_i = \bigcup_{i \in I} (X \setminus A_i).$$

Using this, one applies Lemma 1.21 to get the statement. $\square$

Example 1.23

The intersection of infinitely many open sets might not be open. In particular, in $(\mathbb{R}, d_{|\cdot|})$, one gets for $U_k = \left(-\frac{1}{k}, \frac{1}{k}\right)$ that the intersection over all $k \in \mathbb{N}$ is $\{0\}$, which is not open. Passing this into the complement also gives a counterexample to "infinitely many unions of closed sets are closed.

Definition 1.24: Interior, closure and boundary

Given $\Omega \subset X$, for (X,d) a metric space, we define:

- The interior $\Omega^\circ = \bigcup\{U \subset \Omega \mid U \text{ is open}\}$.
- The closure $\overline{\Omega} = \{x \in X \mid \exists (x_n)_n^\infty \subset \Omega \text{ s.t. } x_n \to x\} = \bigcap\{A \supset \Omega \mid A \text{ is closed}\}^a$.
- The (topological) boundary $\partial\Omega := \overline{\Omega} \setminus \Omega^\circ$.

One shows $\Omega^\circ, \partial\Omega$ are closed, while $\overline{\Omega}$ is open using 1.22 and 1.21.

[a] Provable with 1.27. Exercise.

Example 1.25

The set $[0,1) \times [0,1) \subset \mathbb{R}^2$ will have for instance $(0,0)$ in the closure, but also $(1,1)$. It will not contain $(0,0)$ in the interior.

Remark 1.26

We say a proposition $\mathcal{P}(x_n)$ holds "eventually" for a sequence $(x_n)_n^\infty$, when there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $\mathcal{P}(x_n)$ holds. We will use this terminology a lot, so it's good to get accustomed to it. Sometimes, we also say "for n large enough".

Lemma 1.27: Open and closed sets through sequences

Let (X,d) be a metric space.

- A subset $U \subset X$ is open if and only if for every sequence $(x_n)_n^\infty \subset X$ with $x_n \to x \in U$, x_n lies eventually in U.
- A subset $A \subset X$ is closed if and only if for every sequence $(x_n)_n^\infty \subset A : x_n \to x \in X \implies x \in A$.

Proof. First, we prove the statement about open sets.

$\implies$: Take $x \in U$. Since U is open, there exists $r > 0$ such that $B(x,r) \subset U$. Since $x_n \to x$, $\exists N$ s.t. $d(x_n,x) < r$, so $x_n \in B(r,x)$ for all $n \geq N$.

$\impliedby$: We will prove this by contraposition. Assume that U is not open. This means $\exists x \in U$ such that $\forall r > 0$, $B(x,r) \not\subset U$. This means there exists $x_r \in B(x,r) \cap (X \setminus U)$ for any $r > 0$. Taking $r = \frac{1}{n}$, we can produce a sequence of points $x_n \to x$, with none of the x_n in U.

Now we prove the one about closed sets.

$\implies$: Assume A is closed, so $X \setminus A$ is open. Assume that $x_n \to x$, but $x \in X \setminus A$. By what we just proved, x_n must eventually lie in $X \setminus A$, a contradiction to $(x_n)_n^\infty \subset A$.

$\impliedby$: By contraposition. Assume A is not closed. This means $X \setminus A$ is not open. By the statement about open sets we just proved, there exists a sequence $(x_n)_n^\infty \subset X$ with $x_n \to x \in X \setminus A$ that never lies in $X \setminus A$, so it lies entirely in A. $\square$

Exercise. Prove that if (X,d) is complete and $A \subset X$ is closed, then (A,d) is complete.

1.5 Continuity

This time we generalise for continuity. So we will consider maps $f : X \to Y$ for metric spaces (X, d_X) and (Y, d_Y), which might not be the same. This is a significant difference to the old continuity, which was defined between two sets that are both the reals, hence share the same metric.

DEFINITION 1.28: CONTINUITY

Let (X, d) be a metric space. Consider a function $f : X \to Y$. We say f is continuous if one of the following 3 equivalent properties hold.

(1) $\forall x \in X\ \forall \varepsilon > 0\ \exists \delta > 0$ s.t. $f(B_\delta(x)) \subset B_\varepsilon(f(x))$.

(2) For $(x_n)_n^\infty \subset X$ with $x_n \to x$, $(f(x_n))_n^\infty$ is a convergent sequence in Y with limit $f(x_n)$.

(3) $\forall V \subset Y$ open, $f^{-1}(V) \subset X$ is open.

We call them $\varepsilon - \delta$-continuity (1), sequential continuity (2) and topological continuity (3).

PROPOSITION 1.29: EQUIVALENT DEFINITIONS OF CONTINUITY

The three definitions of continuity are equivalent.

Proof. We will prove an implication cycle.

(1) $\implies$ (2): Assume $f : X \to Y$ is $\varepsilon - \delta$ continuous and let $x_n \to x$. Let $\varepsilon > 0$. I want to prove $d(f(x_n), f(x)) < \varepsilon$ eventually, or in other words $f(x_n) \in B_\varepsilon(f(x))$ eventually. By $\varepsilon - \delta$ continuity, there exists $\delta > 0$ such that $f(B_\delta(x)) \subset B_\varepsilon(f(x))$. But $x_n \in B_\delta(x)$ eventually, so $f(x_n) \in B_\varepsilon(f(x))$ eventually.

$\neg$(3) $\implies$ $\neg$(2): By $\neg$(3), we have $\exists V \subset Y$ open such that $f^{-1}(V)$ not open. By Lemma 1.27, $\exists x \in f^{-1}(V)$ such that there is a sequence $x_n \to x$ with $x_n \in X \setminus f^{-1}(V)$ for all n. In particular, $f(x_n) \in Y \setminus V$, but $f(x) \in V$. Because V is open, $f(x_n) \not\to f(x)$, by Lemma 1.27.

(3) $\implies$ (1): Let $x \in X$ and $\varepsilon > 0$. The set $V = B_\varepsilon(f(x))$ is open, so $f^{-1}(V)$ is also open. We have of course $x \in f^{-1}(V)$. In particular, there exists $\delta > 0$ such that $B_\delta(x) \subset f^{-1}(V)$. But then, for $x' \in B_\delta(x)$, we have $f(x') \in V$, so $f(x') \in B_\varepsilon(f(x))$. This proves $f(B_\delta(x)) \subset B_\varepsilon(f(x))$. $\square$

DEFINITION 1.30: UNIFORM / LIPSCHITZ CONTINUITY

Let $f : X \to Y$ be a map between metric spaces.

(1) f is uniformly continuous if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall x \in X : f(B_\delta(x)) \subset B_\varepsilon(f(x))$.

(2) f is L-Lipschitz for $L > 0$ if $\forall x, y \in X : d_Y(f(x), f(y)) \le L \cdot d_X(x, y)$.

EXERCISE. Prove that any Lipschitz function is also uniformly continuous, hence also continuous.

EXAMPLE 1.31

The distance is a continuous map: Let (X, d) be a metric space and $x_0 \in X$. Define $f(x) = d(x, x_0)$, mapping to the space $Y = [0, \infty)$ with the Euclidean distance (from now on implied, unless specified otherwise). Then f is 1-Lipschitz. Indeed,

$$d_Y(f(x), f(y)) = |f(x) - f(y)| = |d(x, x_0) - d(y, x_0)| \le d(x, y),$$

using triangle inequality.

There is this very useful and fundamental theorem about Lipschitz maps in complete spaces.

THEOREM 1.32: BANACH FIXED POINT THEOREM

Let (X, d) be a complete metric space and $T : X \to X$ be a λ-Lipschitz map for $\lambda \in (0, 1)$. Then T has a unique fixed point $x \in X$ such that $T(x) = x$.

Proof. Fix $x_0 \in X$. Define $x_{n+1} = T(x_n)$. Then,

$$d(x_{n+1}, x_n) = d(T(x_n), T(x_{n-1})) \leq \lambda d(x_n, x_{n-1}),$$

using that T is λ-Lipschitz. We claim that $(x_n)_n^\infty$ is Cauchy. We have

$$d(x_{n+1}, x_n) \leq \lambda d(x_n, x_{n-1}) \leq \lambda^2 d(x_{n-1}, x_{n-2}) \leq \cdots \leq \lambda^n d(x_1, x_0).$$

Hence for wlog $m < n$,

$$d(x_n, x_m) \leq \sum_{k=m}^{n-1} d(x_{k+1}, x_k) \leq \sum_{k=m}^{n-1} \lambda^k d(x_1, x_0) = d(x_1, x_0)\lambda^m \frac{1}{1-\lambda}.$$

As $\lambda^m \to 0$ with $m \to \infty$, the sequence is Cauchy and converges to $x = \lim_{n\to\infty} x_n$, because X is complete. For this point, we have, using sequential continuity,

$$T(x) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = x.$$

If x, y are two fixed points, then

$$d(x, y) = d(T(x), T(y)) \leq \lambda d(x, y)),$$

which is possible only if $d(x, y) = 0$, so $x = y$. □

1.6 Compactness

Compact is *not* just bounded and closed, like in $\mathbb{R}$.

Definition 1.33: Cover and subcover

Given a set X and $E \subset X$, we say $\mathcal{U} = \{U_i\}_{i\in I}$, a family of subsets of X, is a cover of E if

$$E \subset \bigcup \mathcal{U} = \bigcup_{i\in I} U_i.$$

If $\mathcal{V} \subset \mathcal{U}$ is still a cover, we call $\mathcal{V}$ a subcover. If $\mathcal{U}$ is a collection of open sets, we call it an open cover.

Definition 1.34: Compactness

Let (X, d) be a metric space. A set $K \subset X$ is called

(1) sequentially compact if $\forall (x_n)_n^\infty \subset K$, $\exists$ a subsequence s.t. $\lim_{k\to\infty} x_{n_k} \in K$.

(2) topologically compact if $\forall$ open covers $\mathcal{U}$ of K, $\exists$ a finite subcover.

Example 1.35

In $\mathbb{R}$, the intervals $[a, b]$ are compact by Bolzano-Weierstrass, while $\mathbb{Q} \cap [0, 1]$ is for instance not compact.

Consider $X = (0, 1) \cap \mathbb{Q}$. Enumerate $\mathbb{Q} \subset \{x_n \mid n \geq 0\}$. If we let the cover $\mathcal{U} = \left\{B_{2^{-n-10^3}}(x_n)\right\}$. The total "length" of the intervals in this cover will be much less than 1. But this cover by definition includes all rationals in $(0, 1)$, but it doesn't even cover the length of the interval $(0, 1)$.

Proposition 1.36: Compactness

The two definitions of compactness are equivalent.

Proof. (1) $\implies$ (2): Assume $K \subset X$ is sequentially compact. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover. This means $\forall x \in K\ \exists U_i$ open s.t. $x \in U_i$. For $x \in K$, we let

$$r(x) = \min\left\{\sup\{r > 0 \mid B_r(x) \subset U_j \text{ for some } U_j \in \mathcal{U}\}, 1\right\}.$$

Using this, given $x \in K$, select $U_{i(x)} \in \mathcal{U}$ s.t.

$$B_{r(x)/2}(x) \subset U_{i(x)}.$$

In a way, this is an open set containing a (half) maximally big neighborhood of x. With this in mind, we can construct a finite subcover for K.

Pick any $x_0 \in K$. Define

$$\mathcal{V} = \{U_0, U_1, \dots\},$$

by setting $U_0 = U_{i(x_0)}$ and always picking x_n so that it's in none of the previous open sets $U_0, \dots, U_{n-1}$, and letting $U_n = U_{i(x_n)}$. If this is not possible, then we're done. We will prove that assuming this goes on forever leads to a contradiction.

Formally, unless this stops, we'll produce a sequence $(x_n)_n^\infty$ with

$$x_n \in K \setminus \bigcup_{k=0}^{n-1} U_{i(x_k)}.$$

By sequential compactness, $(x_n)_n^\infty$ has a converging subsequence $(x_{n_\ell})_\ell^\infty$ with $x = \lim_{\ell \to \infty} x_{n_\ell} \in K$. Note that $x \notin U_{n_\ell}$ for all ℓ, because the complement of the union of all the open sets U that came before U_{n_ℓ} is closed (see Lemma 1.22 and 1.27) and the sequence elements that come afterward and the limit must hence remain in the complement. In particular, $r(x_{n_\ell}) \to 0$, otherwise x would have to lie in one of the U_{n_ℓ} at some point.

Since $B_{r(x)/2}(x)$ is open, x_{n_ℓ} must eventually lie inside it as $x_{n_\ell} \to x$. And since $r(x_{n_\ell}) \to 0$, it must eventually contain

$$B_{2r\left(x_{n_\ell}\right)}(x_{n_\ell}) \subset B_{r(x)/2}(x) \subset U_{i(x)},$$

which is impossible, because this would imply $2r(x_{n_\ell})(x) \leq r(x_{n_\ell})$, as $r(x_{n_\ell})$ was supposed to be the supremum of all the radii such that $B_r(x)$ is contained in any open set in the cover.

(2) $\implies$ (1): Let $(x_n)_n^\infty \subset K$ be a sequence. We need to show that there exists a convergent subsequence. So assume by contradiction that $\forall x \in K$, x is not an accumulation point of x_n. But this means for any $x \in K$ there exists $\varepsilon(x) > 0$, such that eventually, $x_n \in K \setminus B_{\varepsilon(x)}(x)$ (quick contraposition).

Define $\mathcal{U} = \left\{B_{\varepsilon(x)}(x) \mid x \in K\right\}$. This is an open cover of K, so it admits a finite subcover, such that

$$K \subset \bigcup_{j=1}^{n} B_{\varepsilon(y_j)}(y_j).$$

But then $(x_n)_n^\infty$ can only have finitely many terms, by the definition of $\varepsilon(x)$, a contradiction. $\square$

Corollary 1.37

The last proposition gives us the following corollaries:

(1) $K \subset X$ compact $\implies$ K closed.

(2) $K \subset X$ compact $\implies$ K complete.

(3) $K \subset X$, $A \subset X$ closed $\implies$ $K \cap A$ compact.

Proof. (1) is simple, we can use sequential closedness 1.27. If a sequence converges, then it must converge to the limit of any of its subsequences. (2) also follows since if a Cauchy sequence has a convergent subsequence, then it also converges as a whole, and converges into K, because it is closed by (1). (3) also follows from sequential compactness and the fact that the limit of the subsequence must be in A. $\square$

Proposition 1.38: Continuous image of a compact set is compact

Let (X, d_X) and (Y, d_Y) be metric spaces and $K \subset X$ be compact. If $f : X \to Y$ is continuous, then $f(K)$ is compact in Y.

Proof. The goal is to show that $f(K)$ is topologically compact, so for $\mathcal{V}$ an open cover of $f(K)$, I want to find a finite subcover. So let $\mathcal{U} = \left\{ f^{-1}(V) \mid V \in \mathcal{V} \right\}$, which is an open (by continuity) cover of K. Extract a finite subcover of K and map it back to the codomain. $\square$

Theorem 1.39: Extreme value theorem

Let (X, d) be a metric space and $f : K \to \mathbb{R}$ be continuous and $K \subset X$ compact. Then both $\sup \{f(x) \mid x \in K\}$ and $\inf \{f(x) \mid x \in K\}$ are attained.

Proof. We will prove it for the supremum only. By definition of the supremum $s = \sup_{x \in K} \{f(x)\}$, there exists a sequence $(x_n)_n^\infty \subset K$ such that $f(x_n) \to s$. By compactness, this has a convergent subsequence $x_{n_k} \to \bar{x}$. By sequential continuity, $f(\bar{x}) = s$. $\square$

Remark 1.40

We take a subset $K \subset \mathbb{R}^n$ to be bounded w.r.t. the Euclidean distance if there exists $N \in \mathbb{N}$ such that $\|x\| < N$ for all $x \in K$, equivalently $K \subset B_N(0)$.

Theorem 1.41: Heine-Borel

For the metric space $\mathbb{R}^n$ with the Euclidean distance, $K \subset \mathbb{R}^n$ is compact if and only if it is bounded and closed.

Proof. $\implies$: K is closed by Corollary 1.37. If K is unbounded, $\exists (x_n)_n^\infty \subset K$ such that $\|x_n\| \geq N$ for all $N \in \mathbb{N}$. Any subsequence will hence diverge. Indeed, $x_{n_k} \to x$ would imply $\|x_{n_k} - x\| \to 0$, and by triangle inequality, 1.3

$$\|x_{n_k}\| \leq \|x\| + \|x_{n_k} - x\|,$$

a contradiction to x_{n_k} being arbitrarily large.

$\impliedby$: It suffices to show that for $N \in \mathbb{N}$, $[-N, N]^n \subset \mathbb{R}^n$ is compact, assuming that $K \subset \mathbb{R}^n$ is closed and bounded by $B_N(0)$. This is because $K \subset B_N(0) \subset [-N, N]^n$, and a closed subset of a compact set is compact, so K would be compact, by Corollary 1.37.

The proof will have to use the Bolzano-Weierstrass Theorem in $\mathbb{R}$. Given a sequence $(x_k)_k^\infty$, we can write this component-wise

$$x_k = (x_{k,1}, \ldots, x_{k,n}).$$

Consider the sequences $(x_{k,i})_{k=1}^\infty \subset [-N, N]$ for $1 \le i \le n$. By Bolzano–Weierstrass, there exists a strictly increasing sequence of indices $(k_m^{(1)})_{m=1}^\infty$ such that

$$x_{k_m^{(1)},1} \to \ell_1 \in [-N, N] \quad \text{as } m \to \infty.$$

Now consider the sequence $(x_{k_m^{(1)},2})_{m=1}^\infty \subset [-N, N]$. Again by Bolzano–Weierstrass, there exists a strictly increasing sequence $(k_m^{(2)})_{m=1}^\infty$ which is a subsequence of $(k_m^{(1)})_{m=1}^\infty$ such that

$$x_{k_m^{(2)},2} \to \ell_2 \in [-N, N] \quad \text{as } m \to \infty.$$

Since $(k_m^{(2)})$ is a subsequence of $(k_m^{(1)})$, we still have

$$x_{k_m^{(2)},1} \to \ell_1.$$

Proceeding inductively, for each $j = 1, \ldots, n$ we obtain a strictly increasing sequence $(k_m^{(j)})_{m=1}^\infty$ which is a subsequence of $(k_m^{(j-1)})_{m=1}^\infty$ and such that

$$x_{k_m^{(j)},j} \to \ell_j \in [-N, N] \quad \text{as } m \to \infty,$$

and for all $i \le j$,

$$x_{k_m^{(j)},i} \to \ell_i.$$

After n steps we obtain $(k_m^{(n)})_{m=1}^\infty$ such that

$$x_{k_m^{(n)},i} \to \ell_i \quad \text{for all } i = 1, \ldots, n.$$

Define $x := (\ell_1, \ldots, \ell_n) \in [-N, N]^n$. Then $x_{k_m^{(n)}} \to x$ componentwise, hence $x_{k_m^{(n)}} \to x$ in $\mathbb{R}^n$ by Lemma 1.17. $\square$

EXAMPLE 1.42

Consider $X = \mathbb{R}$ and $A = (0, 1]$. Why is this not compact? Because

$$A \subset \bigcup_{n=1}^{\infty} \left(\frac{1}{n}, 2\right)$$

gives us a cover of A. But any finite cover of this will definitely miss some numbers of A.

PROPOSITION 1.43: UNIFORM CONTINUITY

Let $f : X \to Y$ be a continuous map between metric spaces. If $K \subset X$ is compact, then $f|_K$ is uniformly continuous.

Proof. The goal is to show that given $\varepsilon > 0$, there exists $\delta > 0$ such that $\forall x \in K : f(B_\delta(x)) \subset B_\varepsilon(f(x))$. Using continuity, $\forall x \in K$, there exists $\delta(x) > 0$ such that $f(B_{\delta(x)}(x)) \subset B_{\frac{\varepsilon}{2}}(f(x))$. Define

$$\mathcal{U} = \left\{ B_{\frac{\delta(x)}{2}}(x) \mid x \in K \right\}.$$

This is an open cover of K. So it has a finite subcover. So

$$K \subset \bigcup_{i=1}^{n} B_{\frac{\delta(x_i)}{2}}(x_i)$$

and we define $\delta = \min_{1 \le i \le n} \left\{ \frac{\delta(x_i)}{2} \right\}$.

If $x \in K$ and $y \in B_\delta(x)$, then $x \in B_{\delta(x_i)/2}(x_i)$ for some i. Consider that

$$d(x, y) < \delta \le \frac{\delta(x_i)}{2}, \quad d(x_i, y) < \frac{\delta(x_i)}{2} + \frac{\delta(x_i)}{2} < \delta(x_i).$$

That means

$$f(B_\delta(x)) \subset f(B_{\delta(x_i)}(x_i)) \subset B_{\varepsilon/2}(f(x_i)) \subset B_\varepsilon(f(x)).$$

□

1.7 Connectedness

In any metric space, $X, \emptyset$ are always both open and closed (clopen). Are there other clopen sets and how many? Or, consider the following question. For the sphere $X = \mathbb{S}^2 = \left\{ v \in \mathbb{R}^3 \mid \|v\| = 1 \right\}$ and the Euclidean distance, is it possible to write $\mathbb{S} = U \cup V$ with $U \cap V = \emptyset$ such that U, V are both open and non-empty? Intuitively, it seems quite improbable. One can ask the same question for $(0, 1) \subset \mathbb{R}$. We want to explore the concept of connected space in this section and end up answering these seemingly unrelated questions.

Definition 1.44: Connectedness

Let (X, d) be a metric space and $A \subset X$ a nonempty subset.
X is *disconnected* if $\exists U, V \subset A$ open, disjoint and nonempty such that

$$A \subset U \cup V \text{ and } A \cap U \neq \emptyset, A \cap V \neq \emptyset.$$

Any subset A that is not disconnected is called *connected*.

Example 1.45

In $\mathbb{R}$, the set $A = (0, 1) \cup (2, 3)$ is immediately seen to be disconnected, as is $A = \{0, 1\}$.

Remark 1.46

A subset I of $\mathbb{R}$ is an interval if and only if $\forall x_1, x_2 \in I$ with $x_1 \le x_2$, $[x_1, x_2] \subset I$.

Proposition 1.47: Connected subsets of $\mathbb{R}$

For the metric space $(\mathbb{R}, d_{\text{Eucl}})$, $E \subset \mathbb{R}$ is connected if and only if E is an interval.

Proof. $\Longrightarrow$: By contraposition. Assume E is not an interval. That means there exist points $x_1, x_2 \in E$ and $y \in \mathbb{R} \setminus E$ such that $x_1 < y < x_2$. This gives us the disjoint open cover $(-\infty, y)$ and (y, ∞). Hence, E is disconnected.

$\Longrightarrow$ Again, by contraposition. Assume E is disconnected. This means $\exists U_1, U_2$ open s.t. $E \subset U_1 \cup U_2$ and $U_1 \cap U_2 = \emptyset$. There exist $x_1 \in E \cap U_1$ and $x_2 \in E \cap U_2$. Assume wlog that $x_1 < x_2$. We define

$$y = \sup \{ t \ge x_1 \mid [x_1, t) \subset U_1 \} \in \mathbb{R}.$$

This exists because the set is clearly bounded by x_2. y is the supremum, hence the limit of points in $\mathbb{R} \setminus U_2$, which is closed, so itself in $\mathbb{R} \setminus U_2$. Furthermore, $y \in \mathbb{R} \setminus U_1$. Because if $y \in U_1$, there exists a ball $(y - \varepsilon, y + \varepsilon) \subset U_1$ for $\varepsilon > 0$, but then that would give us $[x_1, y + \varepsilon) \subset U_1$, a contradiction. As a consequence, $y \in \mathbb{R} \setminus (U_1 \cup U_2) \subset \mathbb{R} \setminus E$. This gives us x_1, x_2 and $y \in (x_1, x_2)$ with $y \notin E$. $\square$

PROPOSITION 1.48: CONNECTEDNESS UNDER CONTINUOUS MAPS

Let $f : X \to Y$ be a continuous map between metric spaces and $E \subset X$ connected. Then $f(E)$ is connected.

Proof. By contraposition. Assume that $f(E)$ is disconnected. This means

$$\exists V_1, V_2 \subset Y \text{ open disjoint, with } f(E) \subset V_1 \cup V_2 \text{ and } V_i \cap f(E) \neq \emptyset.$$

We define $U_1 = f^{-1}(V_1)$ and $U_2 = f^{-1}(V_2)$. Then $E \subset f^{-1}(V_1) \cup f^{-1}(V_2) = U_1 \cup U_2$. Since the images were disjoint, we have

$$U_1 \cap U_2 = f^{-1}(V_1) \cap f^{-1}(V_2) = f^{-1}(V_1 \cap V_2) = f^{-1}(\emptyset) = \emptyset.$$

U_1, U_2 are also open since f is continuous and the intersection with E is also non-empty, so U_1, U_2 split E, so it is disconnected. $\square$

COROLLARY 1.49: INTERMEDIATE VALUE THEOREM

Let (X, d) be a connected metric space and $f : X \to \mathbb{R}$ continuous, with $f(x) = a, f(y) = b$ and $a \leq b$ for some $x, y \in X$. Then $\exists z \in X$ s.t. $f(z) = c$ for all $c \in [a, b]$.

Proof. Because X is connected, $f(X)$ is connected. But in $\mathbb{R}$, by Proposition 1.48, the only connected sets are the intervals, immediately proving the corollary. $\square$

DEFINITION 1.50: CURVE

Let (X, d) be a metric space. A curve or path in X is a map $\gamma : [0, 1] \to X$ that is continuous. We call $\gamma(0)$ the starting point and $\gamma(1)$ the end point.
A path with $\gamma(0) = \gamma(1)$ is called a closed curve, or a loop.

DEFINITION 1.51: PATH-CONNECTED

Let (X, d) be a metric space. A set $E \subset X$ is called path connected if $\forall x, y \in E$, there exists a path $\gamma : [0, 1] \to E$ joining x and y, i.e. $\gamma(0) = x$ and $\gamma(1) = y$.

PROPOSITION 1.52: PATH CONNECTED IMPLIES CONNECTED

Let (X, d) be a metric space and $E \subset X$. If E is path connected, then E is connected.

Proof. By contraposition. Assume E is disconnected, so there exist U_1, U_2 open, disjoint s.t. $E \subset U_1 \cup U_2$ and $\exists x_i \in U_i \cap E$. Assume further that there exists a path $\gamma : [0, 1] \to E$ joining x_1 and x_2. Define $V_i = \gamma^{-1}(U_i) \subset [0, 1]$. These are open disjoint nonempty ($0 \in V_1$ and $1 \in V_2$) and covering $[0, 1]$. Hence $[0, 1]$ is disconnected, which is impossible by 1.47. $\square$

EXAMPLE 1.53: TOPOLIGSTS' CURVE

Consider $(\mathbb{R}^2, d_{\text{Eucl}})$. Define $E = (\{0\} \times [0,1]) \cup \{(t, \sin(1/t)) \mid t > 0\}$. One can prove this is connected as an exercise, but not path connected, creating sequences that converge to some point in $[0, y]$, which would make a path connection from $[0, y]$ to any point on the sine curve impossible. So connected does not imply path connected, at least not in general.

DEFINITION 1.54: COMPOSED AND REVERSED PATHS

Let $\gamma_1 : [0,1] \to X$ with $\gamma_1(0) = x$ and $\gamma_1(1) = y$ and $\gamma_2 : [0,1] \to X$ with $\gamma_2(0) = y$ and $\gamma_2(1) = z$. We define $\gamma_1^*(t) = \gamma_1(1-t)$, the reverse path. Since $1 - t$ is continuous, γ_1^* is a continuous map as well and represents the reversion of γ_1.
We also define the composition

$$\gamma_3(t) = \begin{cases} \gamma_1(2t) & t \in \left[0, \frac{1}{2}\right] \\ \gamma_2(2t-1) & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

This is also a curve (check that it's continuous, best with $\varepsilon - \delta$).

Finally, we are ready to show that in Euclidean space, path connected and connected mean the same thing, also revealing for instance that $\mathbb{S}^2$ is connected (because it is path connected), so can not be decomposed into open sets. And I'll throw in a remark about clopen sets. With this and the work leading up to this, all of the questions posted at the beginning of the section should be answered.

REMARK 1.55

It turns out that the answer to the question about clopen sets was right in front of us the whole time. Assume X is connected and let $A \subset X$ be clopen. Then $X \setminus A$ and A give a separation of X, so one of them must be empty. Conversely, if X is disconnected, then $X = U \cup V$ for $U, V \subset X$ a separation. But then $U \cap V = \emptyset$, so we're left with $V = X \setminus U$. But these sets are by assumption non-empty, giving a non-trivial clopen set V.

THEOREM 1.56: CONNECTED IFF PATH CONNECTED IN $\mathbb{R}^n$

In Euclidean $\mathbb{R}^n$, an open subset $U \subset \mathbb{R}^n$ is connected if and only if it is path connected.

Proof. One direction we already proved in Proposition 1.52. The goal is to fix an $x_0 \in U$, and then prove we can join it with any other point $x \in U$ by a path. Then, any two points $x, y \in U$ can be joined by composing the path from x to x_0 and the path from x_0 to y.

Define the set $G \subset U$, with

$$G = \{x \in U \mid \exists \text{ path } \gamma : [0,1] \to U \text{ with } \gamma(0) = x_0, \gamma(1) = x\}.$$

We will prove that (1) G is open and (2) $U \setminus G$ is open. But $x_0 \in G$, and since U is connected, $U \setminus G$ will have to be empty, leaving us with $G = U$.

The key observation is that, using openness of U, $\forall x \in U\ \exists B_r(x) \subset U$, and that for $y \in B_r(x)$, $y \in G \iff x \in G$. Why? Because the map $t \in [0,1] \mapsto (1-t)x + ty \in \mathbb{R}^n$ is continuous and lives only inside $B_r(x) \subset U$, as

$$\|(1-t)x + ty - x\| = \|t(y-x)\| = t\|y-x\| < r,$$

connecting y and x, hence if $y \in G$, then x can be connected to x_0 via y. And conversely, if $x \in G$, then $y \in G$ by the same argument.

But this just shows that if $x \in G$, then there exists a small enough ball $B_r(x) \subset G$, showing G is open, but also that if $x \notin G$, that there exists another small ball $B_r(x)$ that has *no* elements in G, so $B_r(x) \subset X \setminus G$. But this proves $X \setminus G$ is open and we're done. $\square$