

# Matrix Theory Solutions

Kristof Bastiaensen

These are solutions to the exercises from the book *Matrix Theory* by Joel Franklin. If you find any mistakes, please send an email to kristof at resonata dot be. This website is also a github repository, to which you can send pull requests.

**disclaimer:** no guarantee is made to the correctness of these solutions. I am not in any way affiliated with the author or the publisher. Use them at your own risk.

## Chapter 1

### 1.2

- 1.
2. minus
3.  $\Delta = 3, \Delta_1 = 2, \Delta_2 = 10, \Delta_3 = 14, x_1 = \frac{2}{3}, x_2 = \frac{10}{3}, x_3 = \frac{14}{3}$

### 1.3

1. Row one plus row three is twice row two.
2. A term  $s(j)a_{ij_1} \dots a_{nj_n}$  in the expansion of  $\det A$  is zero if  $j > 2$  when  $i \leq 2$ . So for the remaining terms the  $j$  is a permutation of  $(1, 2)$  when  $i \leq 2$ , and consequently  $j$  is a permutation of  $(3 \dots n)$  for  $i > 2$ . Then we can write the determinant as  $(\sum_{i,j=1}^2 s(j)a_{ij})(\sum_{i,j=3}^n s(j)a_{ij}) = \det A_1 \cdot \det A_2$
3. Proof by induction. The case  $p = 2$  follows from Problem 2. For  $p > 2$ , consider the matrix  $A_{2 \dots p}$  containing only matrices  $A_2 \dots A_p$ . By the induction hypothesis this matrix has  $\det A_{2 \dots p} = \det A_2 \dots \det A_p$ . Then using the result of Problem 2 with  $A_1$  and  $A_{2 \dots p}$  we find  $\det A = \det A_1 \cdot \det A_2 \dots \det A_p$ .
4. No. We can reverse  $a, b, \dots, z$  using 13 interchanges  $a \leftrightarrow z, b \leftrightarrow y, c \leftrightarrow x, \dots$ , so  $s(z, y, \dots, a) = (-1)^{13} = -1$ . But after 100 interchanges we would have  $s(z, y, \dots, a) = (-1)^{100} = 1$ .

#### 1.4

1.

### Chapter 4

#### 4.5

1.

2.

3. Substituting in (1):

$$\begin{aligned}
 (x, y) &= \sum_{k=1}^n \xi_k \overline{\eta_k} = \overline{\sum_{k=1}^n \overline{\xi_k} \eta_k} = \overline{(y, x)} \\
 (\lambda x, y) &= \sum_{k=1}^n \lambda \xi_k \overline{\eta_k} = \sum_{k=1}^n \xi_k \overline{\lambda \eta_k} = (x, \overline{\lambda} y) = \lambda \sum_{k=1}^n \xi_k \overline{\eta_k} = \lambda (x, y) \\
 (x + y, z) &= \sum_{k=1}^n (\xi_k + \eta_k) \overline{\zeta_k} = \sum_{k=1}^n \xi_k \overline{\zeta_k} + \sum_{k=1}^n \eta_k \overline{\zeta_k} = (x, z) + (y, z) \\
 (x, x) &= \sum_{k=1}^n \|\xi_k\|^2 \geq 0 \text{ (0 only when x is zero)}
 \end{aligned}$$

The vectors  $u_i = \frac{b^{(i)}}{\|b^{(i)}\|}$  are orthogonal unit vectors.

4. Suppose the equation  $Ax = 0$  has a solution  $x \neq 0$ . Then for all rows  $i$ :

$$\sum_j \alpha_{ij} x_j = \sum_j x_j (a^j, a^i) = \left( \sum_j x_j a^j, a^i \right) = 0$$

The vector  $(\sum_i x_i a^i)$  is perpendicular to all  $a^i$ , so it is the zero vector. However since the  $a^i$  are linearly independent, all  $x_i$  are zero. This is in contradiction with our assumption, so  $\det A \neq 0$ .