Matrix Theory Solutions

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These are solutions to the exercises from the book *Matrix Theory* by *Joel Franklin*. If you find any mistakes, please send an email to kristof at resonata dot be. This website is also a github repository, to which you can send pull requests.

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Chapter 1

1.2

1.
2. minus
3. $\Delta = 3, \Delta_1 = 2, \Delta_2 = 10, \Delta_3 = 14, x_1 = \frac{2}{3}, x_2 = \frac{10}{3}, x_3 = \frac{14}{3}$

1.3

1. Row one plus row three is twice row two.
2. A term $s(j)a_{ij_1} \dots a_{nj_n}$ in the expansion of det A is zero if $j > 2$ when $i <= 2$. So for the remaining terms the j is a permutation of $(1, 2)$ when $i <= 2$, and consequently j is a permutation of $(3 \dots n)$ for $i > 2$. Then we can write the determinant as $(\sum_{i,j=1}^{2} s(j)a_{ij})(\sum_{i,j=3}^{n} s(j)a_{ij}) = detA_1 \cdot detA_2$
3. Proof by induction. The case $p = 2$ follows from Problem 2. For $p > 2$, consider the matrix $A_{2\dots p}$ containing only matrices $A_2 \dots A_p$. By the induction hypothesis this matrix has $detA_{2\dots p} = detA_2 \dots detA_p$. Then using the result of Problem 2 with A_1 and $A_{2\dots p}$ we find $detA = detA_1 \cdot detA_2 \dots detA_p$.
4. No. We can *reverse* $a, b, \dots z$ using 13 interchanges $a \leftrightarrow z, b \leftrightarrow y, c \leftrightarrow x, \dots$, so $s(z, y, \dots, a) = (-1)^{13} = -1$. But after 100 interchanges we would have $s(z, y, \dots, a) = (-1)^{100} = 1$, which is a contradiction.

1.4

1.

Chapter 2

2.4

1.

2.

3.

4.

5.

6.

7.
   - $x^*A^* = \overline{x^TA^T} = \overline{(Ax)^T} = (Ax)^*$
   - $A^*Ax = 0$ implies $x^*A^*Ax = (Ax)^*Ax = ||Ax||^2 = 0$ which implies $Ax = 0$
   - Since a solution vector x is a solution of all equations, A has the same null space as A^*A.

Chapter 4

4.5

1.

2.

3. Substituting in (1):

$$
\begin{aligned}
(x,y) &= \sum_{k=1}^{n} \xi_k \overline{\eta_k} = \overline{\sum_{k=1}^{n} \overline{\xi_k} \eta_k} = \overline{(y,x)} \\
(\lambda x, y) &= \sum_{k=1}^{n} \lambda \xi_k \overline{\eta_k} = \sum_{k=1}^{n} \xi_k \overline{\overline{\lambda} \eta_k} = (x, \overline{\lambda} y) \\
&= \lambda \sum_{k=1}^{n} \xi_k \overline{\eta_k} = \lambda (x,y) \\
(x+y, z) &= \sum_{k=1}^{n} (\xi_k + \eta_k) \overline{\zeta_k} = \sum_{k=1}^{n} \xi_k \overline{\zeta_k} + \sum_{k=1}^{n} \eta_k \overline{\zeta_k} = (x,z) + (y,z) \\
(x,x) &= \sum_{k=1}^{n} ||\xi_k||^2 \geq 0 \text{ (0 only when x is zero)}
\end{aligned}
$$

The vectors $u_i = \frac{b^{(i)}}{\|b^{(i)}\|}$ are orthogonal unit vectors.

4. Suppose the equation $Ax = 0$ has a solution $x \neq 0$. Then for all rows i:

$$
\sum_j \alpha_{ij} x_j = \sum_j x_j (a^j, a^i) = (\sum_j x_j a^j, a^i) = 0
$$

The vector $(\sum_i x_i a^i)$ is perpendicular to all a^i, so it is the zero vector. *proof,* multiply each row by $\overline{x_i}$ and sum all rows:

$$
\begin{aligned}
\overline{x_i}(\sum_j x_j a^j, a^i) &= (\sum_j x_j a^j, x_i a^i) = 0 \\
(\sum_j x_j a^j, \sum_i x_i a^i) &= 0 \\
\sum_j x_j a^j &= 0
\end{aligned}
$$

However since the a^i are linearly indenpendent, all x_i are zero. This is in contradiction with our assumption, so $detA \neq 0$.