BOUNDS ON SUPREMUM NORMS FOR HECKE EIGENFUNCTIONS OF QUANTIZED CAT MAPS

PÄR KURLBERG

ABSTRACT. We study extreme values of desymmetrized eigenfunctions (so called Hecke eigenfunctions) for the quantized cat map, a quantization of a hyperbolic linear map of the torus. In a previous paper it was shown that for prime values of the inverse Planck's constant $N = 1/h$, such that the map is diagonalizable (but not upper triangular) modulo N, the Hecke eigenfunctions are uniformly bounded. The purpose of this paper is to show that the same holds for any prime N provided that the map is not upper triangular modulo N. We also find that the supremum norms of Hecke eigenfunctions are $\ll_\epsilon N^\epsilon$ for all $\epsilon > 0$ in the case of N square free.

1. INTRODUCTION

The behavior of eigenfunctions, such as their value distribution and extreme values, of classically chaotic quantum systems has received considerable attention in the past few years [4, 10, 23, 2, 11, 1] (see section 1.1 for a discussion of these and related results.) The aim of this note is to investigate supremum norms for eigenfunctions in the context of "quantized cat maps". The classical maps, known as "cat maps", are given by the action of hyperbolic linear maps $A \in SL_2(\mathbf{Z})$ on the two dimensional torus. A procedure to quantize such maps was first introduced by Berry and Hannay in [9], and was further developed in [14, 5, 6, 13, 30, 15, 8, 18]. The quantization procedure restricts Planck's constant to be of the form $h = 1/N$ where N is a positive integer. For each integer $N \geq 1$, let $U_N(A)$ denote the quantization of A as a unitary operator on the Hilbert space of states H_N. (For more details, see section 2 and references mentioned therein.) For certain values of N, $U_N(A)$ can have very large spectral degeneracies, and in [15] it was observed that these degeneracies are connected to quantum

symmetries of $U_N(A)$, namely a commutative group of unitary opera-
tors which contains $U_N(A)$. These operators are called Hecke operators
in analogy with the classical theory of modular forms. The eigenspaces
of $U_N(A)$ admit an orthonormal basis consisting of eigenfunctions of all
the Hecke operators, which are called *Hecke eigenfunctions*. In [16] it
was shown that for prime values of N for which the map is diagonaliz-
able (but not upper triangular) modulo N (the so called "split primes"),
normalized Hecke eigenfunctions are uniformly bounded, and, by us-
ing results of N. Katz [12], that their value distribution is given by
the semi-circle law. The aim of this paper is to show that the same
bound holds for all odd prime N, provided that the map is not upper
triangular modulo N:

Theorem 1. *If N is an odd inert prime, $\psi \in H_N$ is a Hecke eigen-
function normalized so that $\|\psi\|_2 = 1$, and A is not upper triangular
modulo N, then*

$$\|\psi\|_\infty \le \frac{2}{\sqrt{1 + 1/N}}$$

Remark: By a different method (namely expressing the Hecke eigen-
functions in terms of perverse sheaves and then bounding the dimen-
sions of certain cohomology groups) the same result has independently
been obtained by Gurevich and Hadani in [7].

Theorem 1 together with Theorem 2 of [16], and a short treatment
of the case of ramified primes gives that Hecke eigenfunctions are uni-
formly bounded for all prime values of N as long as A is not upper
triangular modulo N.

Theorem 2. *If N is an odd prime, $\psi \in H_N$ is an L^2-normalized Hecke
eigenfunction, and A is not upper triangular modulo N, then*

$$\|\psi\|_\infty \le 2$$

*On the other hand, if A is upper triangular modulo N, there exists
Hecke eigenfunctions ψ such that*

$$\|\psi\|_\infty \ge \sqrt{N/2}$$

Remark: Since A is assumed to be hyperbolic, it can only be upper
triangular modulo N for finitely many N.

For composite N, it was shown in [16] that $\|\phi\|_\infty \ll_\epsilon N^{3/8+\epsilon}$ for all
$\epsilon > 0$ if $\phi \in H_N$ is a Hecke eigenfunction. However, for N square free,
this bound can be improved considerably by using Theorem 2.

Theorem 3. *If N is square free and $\phi \in H_N$ is an L^2-normalized Hecke eigenfunction, then*

$$\|\phi\|_\infty \ll_\epsilon N^\epsilon$$

for all $\epsilon > 0$.

However, it should be noted that this bound does not hold for general integers N — Olofsson has shown [21] that there exists a subsequence $\{N_i\}_{i=1}^\infty$ for which there exists Hecke eigenfunctions $\phi_i \in H_{N_i}$ satisfying $\|\phi_i\|_\infty \geq N_i^{1/4}$.

1.1. **Comparison with eigenfunctions of the Laplacian.** A rich class of examples of chaotic classical dynamics are given by the geodesic flow on negatively curved compact Riemannian manifolds. In this setting, the quantization of the dynamics is essentially given by the Laplace-Beltrami operator Δ acting on smooth functions on the manifold, and the eigenfunctions of the quantized cat map can be viewed as an analogue of eigenfunctions of $-\hbar^2\Delta$, i.e., functions ψ_λ (always assumed to be normalized to have L^2-norm one) satisfying $-\hbar^2\Delta\psi_\lambda = \lambda\psi_\lambda$ where $\hbar$ is Planck's constant and λ is the energy associated with the eigenstate ψ_λ. Keeping the energy fixed and letting $\hbar \to 0$ (the semiclassical limit) is equivalent to setting $\hbar = 1$ and letting $\lambda \to \infty$ (the large energy limit), and when making comparisons with the cat map we should think of N, the inverse Planck's constant, to be of size $\sqrt{\lambda}$.

As a model for eigenfunctions in the case of classical chaotic dynamics, Berry proposed [4] a superposition of random plane waves. Consequently, eigenfunctions should have a Gaussian value distribution, and this prediction is matched very well by numerics for certain arithmetic surfaces [10, 2]. As for extreme values, the random wave model predicts (see [10, 23], and also [1] for numerics specifically investigating large values of eigenfunctions) that $\|\psi_\lambda\|_\infty$ should be of order $\sqrt{\log \lambda}$. As for rigorous results, a well known bound (e.g., see [25]) valid for any compact Riemannian manifold of dimension n is that $\|\psi_\lambda\|_\infty \ll \lambda^{\frac{n-1}{4}}$, and the bound is sharp as can be seen by considering zonal harmonics on the sphere. (For cat maps, the corresponding bound $\|\psi\|_\infty \ll \sqrt{N}$ is trivial.) It is of considerable interest to improve this bound using various dynamical properties of the geodesic flow. For real analytic *surfaces*, Sogge and Zelditch proved [26] that if the flow is ergodic, then $\|\psi_\lambda\|_\infty = o(\lambda^{\frac{n-1}{4}})$. Moreover, in the case of negative curvature, the slightly stronger bound $\|\psi_\lambda\|_\infty \ll \lambda^{\frac{n-1}{4}}/\log \lambda$ follows from Berard

[3], but this is probably quite far from the truth, especially in dimension two. In fact, Iwaniec and Sarnak conjectured [11] that for surfaces of constant negative curvature, $\|\psi_\lambda\|_\infty \ll_\epsilon \lambda^\epsilon$ holds for all $\epsilon > 0$. In this direction, they also proved that for certain arithmetic quotients of the upper half plane, $\|\psi_\lambda\|_\infty \ll_\epsilon \lambda^{5/24+\epsilon}$, for all $\epsilon > 0$, as well as that the lower bound $\|\psi_\lambda\|_\infty \gg \sqrt{\log\log\lambda}$ holds for infinitely many eigenvalues. (To be precise, they assume that $\{\psi_\lambda\}$ is a basis of Hecke eigenfunctions.) Note that their results, as well as their conjecture, are consistent with the random wave model prediction for extreme values of eigenfunctions. However, it should also be noted that in higher dimensions, eigenfunctions can have rather large supremum norms even though the curvature is negative — in [22], Rudnick and Sarnak showed that for certain arithmetic manifolds of dimension three, $\|\psi_\lambda\|_\infty \gg \lambda^{\frac{1}{4}}$ for infinitely many eigenvalues.

In the integrable case eigenfunctions are better understood, and it is sometimes possible to infer fairly refined information about the geometry of the manifold from the growth of eigenfunctions. For irrational flat tori eigenfunctions are uniformly bounded, and in [27, 28] Toth and Zelditch proved a partial converse: under certain assumptions (bounded eigenvalue multiplicity, "complete quantum integrability" and "bounded complexity") it turns out that uniformly bounded eigenfunctions implies that the metric is flat. (However, note that rational flat tori have unbounded multiplicities and hence unbounded eigenfunctions.) Moreover, Sogge and Zelditch has shown [26] (also see [28]) that for manifolds with completely integrable flow satisfying a certain non-degeneracy condititon, non-flatness is equivalent with large growth rates of the L^∞ and L^p-norms for a subsequence of eigenfunctions. Further, they also showed that if M is manifold with an infinite subsequence of maximal growth eigenfunctions (i.e., $\|\phi_\lambda\|_\infty \gg \lambda^{(n-1)/4}$) then there exists a point $x \in M$ for which the set of directions of geodesic loops at x has positive measure. In particular, if M is a real analytic surface with maximal eigenfunction growth, then M is topologically either a 2-sphere or the real projective plane.

For further reading, see [23, 20, 24, 29] for some nice surveys.

2. Background

2.1. **Notation.** Since we only will be concerned with Planck's constant h taking values among inverse primes, we will use the notation $h = 1/p$ where p is a prime (rather than $h = 1/N$.) $\mathbb{F}_p$ will denote the finite

field with p elements, $\mathbb{F}_{p^2}$ the the finite field with p^2 elements, and

$$\mathbb{F}_{p^2}^1 := \{x \in \mathbb{F}_{p^2} : N_{\mathbb{F}_p}^{\mathbb{F}_{p^2}}(x) = 1\}$$

is the kernel of the norm map. Further, $\psi : \mathbb{F}_p \to \mathbf{C}^\times$ will denote a nontrivial additive character of $\mathbb{F}_p$.

2.2. **Classical dynamics.** The classical dynamics are given by a hyperbolic linear map $A \in SL_2(\mathbf{Z})$ acting on the phase space $T^2 = \mathbf{R}^2/\mathbf{Z}^2$, and the time evolution on a classical observable $f \in C^\infty(T^2)$ is given by $f \to f \circ A$. Since A is assumed to be hyperbolic, the eigenvalues of A are powers of a fundemental unit in a real quadratic field K. For simplicity, we will assume that $A \equiv I \mod 2$, but we will outline how this restriction can be avoided in Remark 2.2.

2.3. **The Hilbert space of states.** The Hilbert space of states may be identified with $H_p = L^2(\mathbb{F}_p)$, where the inner product is given by

$$(1) \qquad \langle f, g \rangle := \frac{1}{p} \sum_{x=1}^{p} f(x)\overline{g(x)}.$$

2.4. **The quantum propagator and the Weil representation.** We will use the quantization procedure introduced in [15]. For the convenience of the reader we briefly recall some of its key properties. For a classical map $A \equiv I \mod 2$ and Planck's constant of the form $h = 1/p$, the quantization associates a unitary operator $U_p(A)$ acting on H_p in such a way that the two following important properties hold: $U_p(A)$ only depends on the value of A modulo $2p$, and the quantization is multiplicative in the sense that $U_p(AB) = U_p(A)U_p(B)$ if A, B are two different classical maps (both congruent to I modulo 2.) In fact, $U_p(A)$ can be defined via the Weil representation of $SL_2(\mathbb{F}_p)$ acting on $H_p = L^2(\mathbb{F}_p)$. (We abuse notation and also let A denote the image of A in $SL_2(\mathbb{F}_p)$ under the reduction modulo p map.) Since $SL_2(\mathbb{F}_p)$ is generated by matrices of the form

$$(2) \qquad \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

it suffices to specify U_p on such matrices. Let $e(x) = e^{2\pi i x}$, and let r be an integer such that $2r \equiv 1 \mod p$. Further, with

$$S_r(a, p) := \frac{1}{\sqrt{p}} \sum_{x \bmod p} e(\frac{-rax^2}{p}),$$

and Λ being the nontrivial quadratic character on $\mathbb{F}_p^\times$, the action on a state $\phi \in H_p$ is given by

$$(3) \qquad \left(U_p \left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right) \phi \right)(x) = e(\frac{rbx^2}{p})\phi(x)$$

$$(4) \qquad \left(U_p \left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) \phi \right)(x) = \Lambda(t)\phi(tx)$$

$$(5) \qquad \left(U_p \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \phi \right)(x) = S_r(-1,p)\frac{1}{\sqrt{p}} \sum_{y \bmod p} \phi(y)e(\frac{2rxy}{p}).$$

To simplify the notation, we let $\psi(x) = e(rx/p)$; ψ is then a nontrivial additive character on $\mathbb{F}_p$.

As an immediate consequence, we obtain the following description of how the Weil representation acts on delta functions:

Lemma 4. *Let δ_i be a delta function supported at i, i.e. $\delta_i(x) = 1$ if $x \equiv i \mod p$, and $\delta_i(x) = 0$ otherwise. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element in $SL_2(\mathbb{F}_p)$ such that $c \not\equiv 0 \mod p$. Then*

$$(6) \qquad \left(U_p(M)\delta_i \right)(x) = \frac{S_r(-1,p)}{\sqrt{p}}\Lambda(-c)\psi\left(\frac{ax^2 + di^2 - 2xi}{c} \right)$$

Proof. Since $c \not\equiv 0 \mod p$, we can write M as the following product of generators

$$(7) \qquad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & b_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & b_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$$

where $t = -c, b_1 = a/c$ and $b_2 = cd$. Hence

$$(8) \qquad (U_p(M)\delta_i)(x) = \frac{S_r(-1,p)}{\sqrt{p}}\Lambda(t)\sum_y \psi(b_1 x^2 + b_2 y^2 + 2xy)\delta_i(ty)$$

and since the only nonvanishing term in the sum over y is when $ty = -cy = i$, we find that

$$(U_p(M)\delta_i)(x) = \frac{S_r(-1,p)}{\sqrt{p}}\Lambda(-c)\psi\left(\frac{a}{c}x^2 + cd(\frac{i}{c})^2 + 2x\frac{i}{-c} \right)$$

$$= \frac{S_r(-1,p)}{\sqrt{p}}\Lambda(-c)\psi\left(\frac{ax^2 + di^2 - 2xi}{c} \right)$$

$\square$

Remark 2.1. As an immediate consequence of the Lemma we can, up to a phase, determine the trace of $U_p(M)$ in many cases:

$$\left| \text{tr}\left(U_p\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \right) \right| = 1$$

if $c \neq 0$ and $a + d \neq 2$ since the absolute value of the Gauss sum $\sum_{x \mod p} \psi(\alpha x^2)$ equals $\sqrt{p}$ if $\alpha \neq 0$. Moreover, from (4), we see that $|\text{tr}(U_p(-I))| = 1$, and trivially, $\text{tr}(U_p(I)) = p$.

Remark 2.2. We may relax the condition $A \equiv I \mod 2$ by defining $\mathcal{C}_A$ in a different way and using a slightly different quantization procedure: the image of A in $SL_2(\mathbf{Z}/2p\mathbf{Z})$ is contained in some cyclic group of maximal order, say generated by some element B. Define $\mathcal{C}_A$ to be this cyclic subgroup. The Weil representation then gives a quantization $U_p(B)$ which is well defined up to a choice of scalar, and this scalar can be chosen so that multiplicativity holds on $\mathcal{C}_A$.

2.5. **Hecke operators and eigenfunctions.** Let p be a fixed inert prime (i.e., the characteristic polynomial of A remains irreducible modulo p.) With

$$\mathcal{C}_A := \{B \in SL_2(\mathbb{F}_p) : AB \equiv BA \mod p\},$$

the Hecke operators are then given by $\{U_p(B) : B \in \mathcal{C}_A\}$. (In [15], the Hecke operators were defined in a somewhat different way: a subgroup of the centralizer of A modulo $2N$ was identified with the norm one elements of $\mathfrak{O}/(2N\mathfrak{O})$ (with certain parity conditions) where $\mathfrak{O}$ is an order contained in $\mathbf{Q}(\epsilon)$ and ϵ is an eigenvalue of A. However, in the case of $N = p$ an inert prime it is straightforward to verify that these notions are the same.)

A Hecke eigenfunction is then a state $\phi_\nu \in H_p$ such that

$$U_p(B)\phi_\nu = \nu(B)\phi_\nu \quad \forall B \in \mathcal{C}_A$$

where $\nu : \mathcal{C}_A \to \mathbf{C}^\times$ is a character of $\mathcal{C}_A$. (Note that the eigenspaces of the Hecke operators are parametrized by characters of $\mathcal{C}_A$.) Unless otherwise noted, all Hecke eigenfunctions will be normalized so that $\|\phi_\nu\|_2 = 1$.

Our goal is to express Hecke eigenfunctions in terms of exponential sums in one variable, and in order to do this we need a geometric parametrization of $\mathcal{C}_A$. Since we assume that p is inert, there exist some $M = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \in SL_2(\mathbb{F}_p)$ so that

(9) $$\mathcal{C}_A = M \left\{ \begin{pmatrix} a & bD \\ b & a \end{pmatrix} : a^2 - Db^2 \equiv 1 \mod p \right\} M^{-1}$$

where D is not a square in $\mathbb{F}_p$. We note that the solutions to $a^2 - Db^2 = 1$ can be parametrized by letting

$$(a, b) = \left(\frac{1 + Dt^2}{1 - Dt^2}, \frac{2t}{1 - Dt^2} \right)$$

where t takes values in $P^1(\mathbb{F}_p)$. We next determine how Hecke operators act on delta functions.

Lemma 5. *Let* $M = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$ *and* $B = \begin{pmatrix} a & bD \\ b & a \end{pmatrix}$ *be elements in* $SL_2(\mathbb{F}_p)$. *If* $b \neq 0$, *then*

(10)
$$\left(U_p(MBM^{-1}) \delta_i \right)(x) = \Lambda(W^2 - DZ^2) \psi \left(\frac{(YW - DXZ)(x^2 - i^2)}{(W^2 - DZ^2)} \right) \times$$
$$\times S_r(-1, p) \Lambda(-b) \psi \left(\frac{a(x^2 + i^2) - 2xi}{b(W^2 - DZ^2)} \right)$$

Proof. Since

$$MBM^{-1} = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \begin{pmatrix} a & bD \\ b & a \end{pmatrix} \begin{pmatrix} W & -Y \\ -Z & X \end{pmatrix}$$
$$= \begin{pmatrix} a(XW - YZ) + b(YW - DXZ) & * \\ b(W^2 - DZ^2) & a(XW - YZ) + b(DXZ - YW) \end{pmatrix}$$
$$= \begin{pmatrix} a + b(YW - DXZ) & * \\ b(W^2 - DZ^2) & a + b(DXZ - YW) \end{pmatrix},$$

equation (6) gives that[1]

$$\left(U_p(MBM^{-1}) \delta_i \right)(x) = S_r(-1, p) \Lambda(-b(W^2 - DZ^2)) \times$$
$$\times \psi \left(\frac{(a + b(YW - DXZ))x^2 + (a + b(DXZ - YW))i^2 - 2xi}{b(W^2 - DZ^2)} \right) =$$
$$= S_r(-1, p) \Lambda(W^2 - DZ^2) \Lambda(-b) \psi \left(\frac{(YW - DXZ)(x^2 - i^2)}{(W^2 - DZ^2)} \right) \times$$
$$\times \psi \left(\frac{a(x^2 + i^2) - 2xi}{b(W^2 - DZ^2)} \right)$$

$\square$

---

[1] Note that $b(W^2 - DZ^2) \neq 0$ since $b \neq 0$ and $W^2 - DZ^2 \neq 0$ since D is not a square in $\mathbb{F}_p$.

3. Proof of Theorem 1

3.1. Hecke eigenfunctions via projections.

Given a character ν on $\mathcal{C}_A$, let

$$V_\nu := \{\phi \in L^2(\mathbb{F}_p) : U_p(B)\phi = \nu(B)\phi \ \ \forall B \in \mathcal{C}_A\}.$$

Since p is odd and inert in K, the Weil representation restricted to $\mathcal{C}_A$ is multiplicity free and hence $\dim(V_\nu) \leq 1$. For a proof of this, see Proposition 3 in [19]. Alternatively, we might argue as follows: let d_ν be the dimension of V_ν. Then $\sum_\nu d_\nu = \dim(L^2(\mathbb{F}_p)) = p$, and by the character formula for group representations, $\sum_\nu d_\nu^2 = |\mathcal{C}_A|^{-1}\sum_{B \in \mathcal{C}_A}|\operatorname{tr}(U_p(B))|^2$, which by Remark 2.1 equals $|\mathcal{C}_A|^{-1}(1 \cdot p^2 + (p-1) \cdot 1 + 1 \cdot 1) = p(p+1)/|\mathcal{C}_A|$. Since $|\mathcal{C}_A| = p + 1$ in the inert case, we find that $\sum_\nu d_\nu = p = \sum_\nu d_\nu^2$ which implies that $d_\nu \leq 1$. In fact, since $\dim(L^2(\mathbb{F}_p)) = p$ has dimension p, we note that $\dim(V_\nu) = 1$ for all but one character ν of $\mathcal{C}_A$. In what follows we let ν be a fixed character of $\mathcal{C}_A$ such that $\dim(V_\nu) = 1$.

To construct the Hecke eigenfunction corresponding to ν we define a projection operator $P_\nu : L^2(\mathbb{F}_p) \to V_\nu$ by letting

$$P_\nu f := \frac{1}{|\mathcal{C}_A|} \sum_{B \in \mathcal{C}_A} \overline{\nu}(B)U_p(B)f$$

Clearly, $P_\nu f$ is a Hecke eigenfunction; the main difficulty is to control the L^2-norm of $P_\nu f$. However, if f is a delta function, the L^2-norm of the projection can be expressed in a simple manner.

Lemma 6. *We have*

$$\|P_\nu \delta_i\|_2^2 = \frac{(P_\nu \delta_i)(i)}{p}$$

Proof. Since P_ν is an orthogonal projection[2], $P_\nu^2 = P_\nu$ and P_ν is self adjoint. Thus

$$\|P_\nu \delta_i\|_2^2 = \langle P_\nu \delta_i, P_\nu \delta_i \rangle = \langle P_\nu^2 \delta_i, \delta_i \rangle =$$

$$= \langle P_\nu \delta_i, \delta_i \rangle = \frac{1}{p} \sum_{x \in \mathbb{F}_p} (P_\nu \delta_i)(x)\overline{\delta_i(x)} = \frac{(P_\nu \delta_i)(i)}{p}$$

$\square$

---

[2]The inner product defined by (1) is invariant by the action of $\mathcal{C}_A$, hence the Hilbert space of states decomposes into an orthogonal sum of $\mathcal{C}_A$-invariant subspaces.

3.2. **Hecke eigenfunctions and exponential sums.** We can now express the Hecke eigenfunctions in terms of exponential sums. Since we may regard $\mathcal{C}_A$ as $\mathbb{F}^1_{p^2}$, the group of norm one elements in $\mathbb{F}^\times_{p^2}$, and $\mathbb{F}_{p^2} = \mathbb{F}_p[\sqrt{D}]$, Hilbert's theorem 90 gives us a parametrization $P^1(\mathbb{F}_p) \to \mathbb{F}^1_{p^2}$ via the map $t \to \frac{1+t\sqrt{D}}{1-t\sqrt{D}}$.

Proposition 7. *Given a character $\nu : \mathbb{F}^1_{p^2} \to \mathbf{C}^\times$ on the group of norm one elements in $\mathbb{F}_{p^2}$, define an exponential sum*

$$E_\nu(i,x) := \sum_{t \neq 0} \overline{\nu}\left(\frac{1+t\sqrt{D}}{1-t\sqrt{D}}\right) \Lambda\left(\frac{-2t}{1-Dt^2}\right) \psi_F\left(\frac{(x-i)^2}{2t} + Dt(x+i)^2\right)$$

where $F = (W^2 - DZ^2)^{-1}$ and $\psi_F : \mathbb{F}_p \to \mathbf{C}^\times$ is a nontrivial additive character defined by $\psi_F(x) = \psi(Fx)$. Putting

$$\alpha(i,x) = \Lambda(W^2 - DZ^2)\psi\left(\frac{(YW - DXZ)(x^2 - i^2)}{W^2 - DZ^2}\right),$$

we then have

(11) $(P_\nu \delta_i)(x) =$
$$= \frac{1}{p+1}\left(\delta_i(x) + \Lambda(-1)\delta_i(-x) + \alpha(i,x)S_r(-1,p)\frac{E_\nu(i,x)}{\sqrt{p}}\right)$$

Proof. We first note that $\pm I \in \mathcal{C}_A$ corresponds to $t = 0$ or $t = \infty$ in the parametrization $P^1(\mathbb{F}_p) \to \mathcal{C}_A$. Since $U_p(I)\delta_i = \delta_i$ and $(U_p(-I)\delta_i)(x) = \Lambda(-1)\delta_i(-x)$ we find that

$$(P_\nu \delta_i)(x) = \frac{1}{p+1}\left(\delta_i(x) + \Lambda(-1)\delta_i(-x) + \sum_{B \in \mathcal{C}_A, B \neq \pm I} \overline{\nu}(B)\left(U_p(B)\delta_i\right)(x)\right)$$

If $B \in \mathcal{C}_A$ and $B \neq \pm I$, then

$$B = M\begin{pmatrix} a & bD \\ b & a \end{pmatrix}M^{-1}$$

where $a^2 - Db^2 \equiv 1 \mod p$ and $b \not\equiv 0 \mod p$. These solutions are all of the form $(a,b) = \left(\frac{1+Dt^2}{1-Dt^2}, \frac{2t}{1-Dt^2}\right)$ for $t \in \mathbb{F}^\times_p$. From (10) we obtain

$\overline{\nu}(B)\left(U_p(B)\delta_i\right)(x)$
$$= \overline{\nu}\left(\frac{1+t\sqrt{D}}{1-t\sqrt{D}}\right)\alpha(i,x)\Lambda(-b)S_r(-1,p)\psi\left(\frac{a(x^2 + i^2) - 2xi}{b(W^2 - DZ^2)}\right)$$

and since

$$\frac{a(x^2 + i^2) - 2xi}{b} = \frac{(1 + Dt^2)(x^2 + i^2) - 2xi(1 - Dt^2)}{2t}$$

$$= \frac{Dt(x + i)^2}{2} + \frac{(x - i)^2}{2t}.$$

the result follows. $\qquad\qquad\square$

Remark: If $x = i$ then $\alpha(i, x) = \Lambda(W^2 - DZ^2) = \pm 1$, and it can be shown that $S_r(-1, p)E_\nu(i, i)$ is in fact real valued.

Using the Riemann hypothesis for curves, we will now bound the exponential sums $E_\nu(i, x)$.

Proposition 8. *If $x \neq \pm i$, then*

$$|E_\nu(i, x)| \leq 4\sqrt{p},$$

and if $x = \pm i$, then

$$|E_\nu(i, x)| \leq 3\sqrt{p}.$$

Moreover,

$$|E_\nu(0, 0)| \leq 2\sqrt{p}.$$

Proof. Let w be the place corresponding to the field extension $\mathbb{F}_{p^2}/\mathbb{F}_p$. By chapter 6 of [17] (see Theorems 4, 6, and exercise 3) there exists idele class characters $\tilde{\nu}, \tilde{\psi}, \tilde{\Lambda}$ such that

$$\tilde{\nu}(\pi_v) = \bar{\nu}\left(\frac{1 + t\sqrt{D}}{1 - t\sqrt{D}}\right)$$

$$\tilde{\psi}(\pi_v) = \psi\left(\frac{(x - i)^2}{2t} + Dt(x + i)^2\right)$$

$$\tilde{\Lambda}(\pi_v) = \Lambda\left(\frac{-2t}{1 - Dt^2}\right)$$

for all degree one places in $\mathbb{F}_p[X]$ of the form $\pi_v = (X + t)$, $t \in \mathbb{F}_p^\times$. Moreover, the conductors are as follows:

$$\text{cond}(\tilde{\nu}) = (w),$$

$$\text{cond}(\tilde{\Lambda}) = (0) + (w) + (\infty),$$

and

$$\text{cond}(\tilde{\psi}) = \begin{cases} 2(0) + 2(\infty) & \text{if } x \neq \pm i \\ 2(\infty) & \text{if } x = i, \\ 2(0) & \text{if } x = -i, \\ 0 & \text{if } x = i = 0. \text{ (I.e., } \tilde{\psi} \text{ is trivial.)} \end{cases}$$

Letting $\chi = \tilde{\nu} \cdot \tilde{\psi} \cdot \tilde{\Lambda}$ we have

$$E_\nu(i, x) = \sum_{\substack{v : \deg(v) = 1 \\ v \text{ unramified}}} \chi(\pi_v)$$

and Corollary 3 in 6.1 of [17] then gives

$$\left| \sum_{\substack{v : \deg(v) = 1 \\ v \text{ unramified}}} \chi(\pi_v) \right| \le \sqrt{p}(m - 2)$$

where m is the degree of the conductor of χ. Now, if $i \ne \pm x$ then $\mathrm{cond}(\chi) = (w) + 2(0) + 2(\infty)$, which has degree $2 + 2 + 2 = 6$, and hence

$$|E_\nu(i, x)| \le 4\sqrt{p}$$

If $x = i$, then $\mathrm{cond}(\chi) = (w) + (0) + 2(\infty)$, and if $x = -i$, then $\mathrm{cond}(\chi) = (w) + 2(0) + (\infty)$. In either case, the degree is 5, hence

$$|E_\nu(i, \pm i)| \le 3\sqrt{p}.$$

Finally, a similar argument gives that $|E_\nu(0, 0)| \le 2\sqrt{p}$. $\square$

3.3. Conclusion.
Let $f_{\nu,i} = P_\nu \delta_i$. Since $f_{\nu,i}$ is always a scalar multiple (possibly zero) of an L^2-normalized eigenfunction ϕ_ν, we may find functions g_ν, h_ν such that

$$f_{\nu,i}(x) = g_\nu(i) h_\nu(x)$$

where h_ν is a scalar multiple of ϕ_v normalized so that $\sum_x |h_\nu(x)|^2 = 1$. Since

$$f_{\nu,i}(x) = p\langle P_\nu \delta_i, \delta_x \rangle = p\langle \delta_i, P_\nu \delta_x \rangle = p\overline{\langle P_\nu \delta_x, \delta_i \rangle} = \overline{f_{\nu,x}(i)}$$

we find that

$$g_\nu(i) h_\nu(x) = \overline{g_\nu(x) h_\nu(i)}$$

for all i, x. Now, if $P_\nu \ne 0$ then $P_\nu \delta_i \ne 0$ for some i, hence $g(i) \ne 0$ for some i and we find that

$$h_\nu(x) = \overline{g_\nu(x)} \cdot \frac{\overline{h_\nu(i)}}{g_\nu(i)}$$

for all x. In order to determine $\overline{h_\nu(i)}/g_\nu(i)$ we argue as follows: since we have chosen h_ν so that $\sum_x |h_\nu(x)|^2 = 1$,

$$\|f_{\nu,i}\|_2^2 = \frac{1}{p} \sum_x |f_{\nu,i}(x)|^2 = \frac{|g_\nu(i)|^2}{p} \sum_x |h_\nu(x)|^2 = \frac{|g_\nu(i)|^2}{p},$$

which, using Lemma 6, also can be written as

$$\|f_{\nu,i}\|_2^2 = \|P_\nu \delta_i\|_2^2 = \frac{(P_\nu \delta_i)(i)}{p} = \frac{f_{\nu,i}(i)}{p} = \frac{g_\nu(i) h_\nu(i)}{p}.$$

Thus $h_\nu(i) = \overline{g_\nu(i)}$ and we find that $h_\nu(x) = \overline{g_\nu(x)}$ for *all* x. Hence

$$|h_\nu(x)|^2 = |h_\nu(x) g_\nu(x)| = |f_{\nu,x}(x)| = |(P_\nu \delta_x)(x)|,$$

and by Proposition 7 we find that

$$|h_\nu(x)|^2 = \begin{cases} \left| \frac{1}{p+1} \left(1 + \alpha(x,x) S_r(-1,p) \frac{E_\nu(x,x)}{\sqrt{p}} \right) \right| & \text{if } x \neq 0, \\ \left| \frac{1}{p+1} \left(1 + \Lambda(-1) + \alpha(0,0) S_r(-1,p) \frac{E_\nu(0,0)}{\sqrt{p}} \right) \right| & \text{if } x = 0. \end{cases}$$

By Proposition 8, $|E_\nu(x,x)| \leq 3\sqrt{p}$ if $x \neq 0$, and $|E_\nu(0,0)| \leq 2\sqrt{p}$, hence

$$|\phi_\nu(x)| = p^{1/2} |h_\nu(x)| \leq 2 \sqrt{\frac{p}{p+1}}$$

since $|S_r(-1,p)| = 1$, and $|\alpha(x,x)| = 1$ for all x.

4. Proof of Theorem 2

There are three different types of primes that needs to be considered: inert, split and ramified primes. Theorem 1 gives the inert case (note that A cannot be upper triangular modulo p if p is inert), and the case of A split and not upper triangular is Theorem 2 of [16]. What remains to be done is to treat the ramified case, and the case of p split and A upper triangular modulo p.

4.1. **The ramified case.** If the characteristic polynomial of A has a double root modulo p then p is said to be ramified. In this case,

$$A = M \begin{pmatrix} \pm 1 & s \\ 0 & \pm 1 \end{pmatrix} M^{-1}$$

for some $M \in SL_2(\mathbb{F}_p)$ and $s \in \mathbb{F}_p$. Moreover, the norm one elements used to define the Hecke operators correspond to matrices conjugate to upper triangular matrices with ± 1 on the diagonal, i.e.,

$$\mathcal{C}_A = M \left\{ \begin{pmatrix} \pm 1 & t \\ 0 & \pm 1 \end{pmatrix} : t \in \mathbb{F}_p \right\} M^{-1}.$$

The normalized Hecke eigenfunctions are then given by

$$\phi_i^\pm = \sqrt{p/2} \cdot U_p(M)(\delta_i \pm \delta_{-i})$$

for $0 \leq i \leq (p-1)/2$ and

$$\phi_0 = \sqrt{p} \cdot U_p(M) \delta_0$$

We note that A is upper triangular if and only if M is upper triangular. In the upper triangular case, $U_p(M)$ acts via multiplication by scalars (of absolute value 1), and by permuting the arguments (see equations (3) and (4)), hence

$$\left\|\phi_i^{\pm}\right\|_{\infty} = \sqrt{p/2} \left\|\delta_i \pm \delta_{-i}\right\|_{\infty} = \sqrt{p/2}$$

and

$$\left\|\phi_0\right\|_{\infty} = \sqrt{p} \left\|\delta_0\right\|_{\infty} = \sqrt{p}$$

If A is not upper triangular, then $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for some $c \not\equiv 0$ mod p, and equation 6 then gives that

$$\left|(U_p(M)(\delta_i \pm \delta_{-i})(x)\right| = \frac{1}{\sqrt{p}} |\psi(\frac{ax^2 + di^2 - 2xi}{c}) + \psi(\frac{ax^2 + di^2 + 2xi}{c})|$$

and thus

$$\left\|\phi_i^{\pm}\right\|_{\infty} = \sqrt{2},$$

(for instance, take $x = 0$) and a similar calculation gives that

$$\left\|\phi_0\right\|_{\infty} = 1.$$

4.2. **The upper triangular split case.** We assume that A is upper triangular and conjugate to a diagonal matrix, i.e.,

$$A = M \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} M^{-1}$$

where M is upper triangular. Arguing as in section 4.1 of [16], we find that a basis of normalized Hecke eigenfunctions are given by

$$\phi_{\chi} = \sqrt{\frac{p}{p-1}} U_p(M)\chi$$

where $\chi : \mathbb{F}_p \to \mathbf{C}^{\times}$ ranges over all multiplicative characters of $\mathbb{F}_p^{\times}$ (extended to $\mathbb{F}_p$ by letting $\chi(0) = 0$) and

$$\phi_0 = \sqrt{p} \cdot U_p(M)\delta_0.$$

Since M is upper triangular, $U_p(M)$ acts via multiplication by scalars of absolute value 1 and by permuting the arguments, hence

$$\left\|\phi_{\chi}\right\|_{\infty} = \sqrt{\frac{p}{p-1}}$$

and

$$\left\|\phi_0\right\|_{\infty} = \sqrt{p}$$

Remark: Since one of the eigenspaces have dimension two, we can find a character χ and choose scalars α, β so that $\phi = \alpha\phi_0 + \beta\phi_{\chi}$

is a normalized Hecke eigenfunction with supremum norm equal to $\sqrt{p^2/(p-1)}$.

5. Proof of Theorem 3

If N is square free, say with prime factorization $N = \prod_{i=1}^{k} p_i$, then $U_N(A)$ can be expressed as a tensor product (cf. [15], section 4.1) of the form $U_N(A) = U_{p_1}(A) \otimes U_{p_2}(A) \otimes \cdots \otimes U_{p_k}(A)$. Thus any Hecke eigenfunction $\phi \in H_N$ can be written as a product of Hecke functions $\phi_i \in H_{p_i}$, i.e., $\phi(x) = \prod_{i=1}^{k} \phi_i(x_i)$ where each x_i is the image of x under the projection from $\mathbf{Z}/N\mathbf{Z}$ to $\mathbf{Z}/p_i\mathbf{Z}$. Since A cannot be upper triangular modulo p for more than a finite number of primes, Theorem 2 gives that $\|\phi\|_\infty = \prod_{i=1}^{k} \|\phi_i\|_\infty \ll 2^k \ll_\epsilon N^\epsilon$.

References

[1] R. Aurich, A. Bäcker, R. Schubert, and M. Taglieber. Maximum norms of chaotic quantum eigenstates and random waves. *Phys. D*, 129(1-2):1–14, 1999.

[2] R. Aurich and F. Steiner. Statistical properties of highly excited quantum eigenstates of a strongly chaotic system. *Phys. D*, 64(1-3):185–214, 1993.

[3] P. H. Bérard. On the wave equation on a compact Riemannian manifold without conjugate points. *Math. Z.*, 155(3):249–276, 1977.

[4] M. V. Berry. Regular and irregular semiclassical wavefunctions. *J. Phys. A*, 10(12):2083–2091, 1977.

[5] M. Degli Esposti. Quantization of the orientation preserving automorphisms of the torus. *Ann. Inst. H. Poincaré Phys. Théor.*, 58(3):323–341, 1993.

[6] M. Degli Esposti, S. Graffi, and S. Isola. Classical limit of the quantized hyperbolic toral automorphisms. *Comm. Math. Phys.*, 167(3):471–507, 1995.

[7] S. Gurevich and R. Hadani. Heisenberg Realizations, Eigenfunction and Proof of the Kurlberg-Rudnick Supremum Conjecture. *Preprint (preliminary version)*.

[8] S. Gurevich and R. Hadani. On Hannay-Berry Equivariant Quantization of the Torus. *Preprint*, 2002.

[9] J. H. Hannay and M. V. Berry. Quantization of linear maps on a torus-Fresnel diffraction by a periodic grating. *Phys. D*, 1(3):267–290, 1980.

[10] D. A. Hejhal and B. N. Rackner. On the topography of Maass waveforms for $PSL(2, \mathbf{Z})$. *Experiment. Math.*, 1(4):275–305, 1992.

[11] H. Iwaniec and P. Sarnak. L^∞ norms of eigenfunctions of arithmetic surfaces. *Ann. of Math. (2)*, 141(2):301–320, 1995.

[12] N. M. Katz. Sato-Tate equidistribution of Kurlberg-Rudnick sums. *Internat. Math. Res. Notices*, (14):711–728, 2001.

[13] S. Klimek, A. Leśniewski, N. Maitra, and R. Rubin. Ergodic properties of quantized toral automorphisms. *J. Math. Phys.*, 38(1):67–83, 1997.

[14] S. Knabe. On the quantisation of Arnold's cat. *J. Phys. A*, 23(11):2013–2025, 1990.

[15] P. Kurlberg and Z. Rudnick. Hecke theory and equidistribution for the quantization of linear maps of the torus. *Duke Math. J.*, 103(1):47–77, 2000.

[16] P. Kurlberg and Z. Rudnick. Value distribution for eigenfunctions of desymmetrized quantum maps. *Int. Math. Res. Not.*, 2001(18):985–1002, 2001.

[17] W. C. W. Li. *Number theory with applications.* World Scientific Publishing Co. Inc., River Edge, NJ, 1996.

[18] F. Mezzadri. On the multiplicativity of quantum cat maps. *Nonlinearity*, 15(3):905–922, 2002.

[19] C. Moen. The dual pair $(U(1), U(1))$ over a p-adic field. *Pacific J. Math.*, 158(2):365–386, 1993.

[20] N. Nadirashvili, D. Tot, and D. Yakobson. Geometric properties of eigenfunctions. *Uspekhi Mat. Nauk*, 56(6(342)):67–88, 2001.

[21] R. Olofsson. Large supremum norms for hecke eigenfunctions of quantized cat maps. *Preprint.*

[22] Z. Rudnick and P. Sarnak. The behaviour of eigenstates of arithmetic hyperbolic manifolds. *Comm. Math. Phys.*, 161(1):195–213, 1994.

[23] P. Sarnak. Arithmetic quantum chaos. In *The Schur lectures (1992) (Tel Aviv)*, pages 183–236. Bar-Ilan Univ., Ramat Gan, 1995.

[24] P. Sarnak. Spectra of hyperbolic surfaces. *Bull. Amer. Math. Soc. (N.S.)*, 40(4):441–478 (electronic), 2003.

[25] C. D. Sogge. Concerning the L^p norm of spectral clusters for second-order elliptic operators on compact manifolds. *J. Funct. Anal.*, 77(1):123–138, 1988.

[26] C. D. Sogge and S. Zelditch. Riemannian manifolds with maximal eigenfunction growth. *Duke Math. J.*, 114(3):387–437, 2002.

[27] J. A. Toth and S. Zelditch. Riemannian manifolds with uniformly bounded eigenfunctions. *Duke Math. J.*, 111(1):97–132, 2002.

[28] J. A. Toth and S. Zelditch. Norms of modes and quasi-modes revisited. In *Harmonic analysis at Mount Holyoke (South Hadley, MA, 2001)*, volume 320 of *Contemp. Math.*, pages 435–458. Amer. Math. Soc., Providence, RI, 2003.

[29] S. Zelditch. Quantum ergodicity and mixing, to appear in Encyclopedia of Mathematical Physics.

[30] S. Zelditch. Index and dynamics of quantized contact transformations. *Ann. Inst. Fourier (Grenoble)*, 47(1):305–363, 1997.

E-mail address: kurlberg@math.kth.se

Department of Mathematics, Royal Institute of Technology, SE-100 44 Stockholm, Sweden