

ON THE ORDER OF UNIMODULAR MATRICES MODULO INTEGERS

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PRELIMINARY VERSION

1. INTRODUCTION

Given an integer b and a prime p such that $p \nmid b$, let $\text{ord}_p(b)$ be the multiplicative order of b modulo p . In other words, $\text{ord}_p(b)$ is the smallest non negative integer k such that $b^k \equiv 1 \pmod{p}$. Clearly $\text{ord}_p(b) \leq p-1$, and if the order is maximal, b is said to be a primitive root modulo p . Artin conjectured (see the preface in [1]) that if $b \in \mathbf{Z}$ is not a square, then b is a primitive root for a positive proportion¹ of the primes.

What about the “typical” behaviour of $\text{ord}_p(b)$? For instance, are there good lower bounds on $\text{ord}_p(b)$ that hold for a full density subset of the primes? In [3], Erdős and Murty proved that if $b \neq 0, \pm 1$, then there exists a $\delta > 0$ so that $\text{ord}_p(b)$ is at least $p^{1/2} \exp((\log p)^\delta)$ for a full density subset of the primes. However, we expect the typical order to be much larger. In [5] Hooley proved that the Generalized Riemann Hypothesis (GRH) implies Artin’s conjecture. Moreover, if $f : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ is an increasing function tending to infinity, it is possible to modify Hooley’s argument to show that GRH implies that the order of b modulo p is greater than $p/f(p)$ for full density subset of the primes.

In this paper we will investigate a related question, namely lower bounds on the order of unimodular matrices modulo $N \in \mathbf{Z}$. That is, if $A \in SL_2(\mathbf{Z})$, what can be said about lower bounds for $\text{ord}_N(A)$, the order of A modulo N , that hold for most N ? It is a natural generalization of the previous questions, but our main motivation comes from mathematical physics (quantum chaos): In [6] Rudnick and I proved that if A is hyperbolic², then quantum ergodicity for toral automorphisms follows from $\text{ord}_N(A)$ being slightly larger than $N^{1/2}$, and we then showed that this condition does hold for a full density subset of the integers.

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¹The constant is given by an Euler product that depends on b .

² A is hyperbolic if $|\text{tr}(A)| > 2$.

Again, we expect that the typical order is much larger. In order to give lower bounds on $\text{ord}_N(A)$, it is essential to have good lower bounds on $\text{ord}_p(A)$ for p prime, and this is our first goal.

Theorem 1. *Let $A \in SL_2(\mathbf{Z})$ be hyperbolic, and let $f : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ be an increasing function tending to infinity slower than $\log x$. Assuming GRH, there are at most $O(\frac{x}{\log x f(x)^{1-\epsilon}})$ primes $p \leq x$ such that $\text{ord}_p(A) < p/f(p)$. In particular, the set of primes p such that $\text{ord}_p(A) \geq p/f(p)$ has density one.*

Using this we obtain an improved lower bound on $\text{ord}_N(A)$ that is valid for most integers.

Theorem 2. *Let $A \in SL_2(\mathbf{Z})$ be hyperbolic. Assuming GRH, the number of $N \leq x$ such that $\text{ord}_N(A) \ll N^{1-\epsilon}$ is $o(x)$. That is, the set of integers N such that $\text{ord}_N(A) \gg N^{1-\epsilon}$ has density one.*

Remarks: If A is elliptic ($|\text{tr}(A)| < 2$) then A has finite order (in fact, at most 6). If A is parabolic ($|\text{tr}(A)| = 2$), then $\text{ord}_p(A) = p$ unless A is congruent to the identity matrix modulo p , and hence there exists a constant $c_A > 0$ so that $\text{ord}_N(A) > c_A N$. Apart from the application in mind, it is thus natural to only treat the hyperbolic case.

As far as unconditional results for primes go, we note that the proof in [3] relies entirely on analyzing the divisor structure of $p - 1$, and we expect that their method should give a similar lower bound on the order of A modulo p . An unconditional lower bound of the form

$$(1) \quad \text{ord}_p(b) \gg p^\eta$$

for a full proportion of the primes and $\eta > 1/2$ would be quite interesting. In this direction, Goldfeld proved [4] that if $\eta < 3/5$, then (1) holds for a positive, but not full, proportion of the primes.

Clearly $\text{ord}_p(A)$ is related to $\text{ord}_p(\epsilon)$, where ϵ is one of the eigenvalues of A . Since A is assumed to be hyperbolic, ϵ is a power of a fundamental unit in a real quadratic field. The question of densities of primes p such that $\text{ord}_p(\lambda)$ is maximal, for λ a fundamental unit in a real quadratic field, does not seem to have received much attention until quite recently; in [8] Roskam proved that GRH implies that the set of primes p for which $\text{ord}_p(\lambda)$ is maximal has positive density. (The work of Weinberger [2], Cooke and Weinberger [10] and Lenstra [7] does treat the case $\text{ord}_p(\lambda) = p - 1$, but not the case $\text{ord}_p(\lambda) = p + 1$.)

2. PRELIMINARIES

2.1. Notation. If \mathfrak{O}_F is the ring of integers in a number field F , we let $\zeta_F(s) = \sum_{\mathfrak{a} \in \mathfrak{O}_F} N(\mathfrak{a})^{-s}$ denote the zeta function of F . By GRH we

mean that all nontrivial zeroes of $\zeta_F(s)$ lie on the line $\operatorname{Re}(s) = 1/2$ for all number fields F .

Let ϵ be an eigenvalue of A , satisfying the equation

$$(2) \quad \epsilon^2 - \operatorname{tr}(A)\epsilon + \det(A).$$

Since A is hyperbolic, $K = \mathbf{Q}(\epsilon)$ is a real quadratic field. Let \mathfrak{O}_K be the integers in K , and let D_K be the discriminant of K . Since A has determinant one, ϵ is a unit in \mathfrak{O}_K . For $n \in \mathbf{Z}^+$ we let $\zeta_n = e^{2\pi i/n}$ be a primitive n -th root of unity, and $\alpha_n = \epsilon^{1/n}$ be an n -th root of ϵ . Further, with $Z_n = K(\zeta_n)$, $K_n = K(\zeta_n, \alpha_n)$, and $L_n = K(\alpha_n)$, we let σ_p denote the Frobenius element in $\operatorname{Gal}(K_n/\mathbf{Q})$ associated with p . We let F_{p^k} denote the finite field with p^k elements, and we let $F_{p^2}^1 \subset F_{p^2}^\times$ be the norm one elements in F_{p^2} , i.e., the kernel of the norm map from $F_{p^2}^\times$ to F_p^\times . Let $\langle A \rangle_p$ be the group generated by A in $SL_2(F_p)$. $\langle A \rangle_p$ is contained in a maximal torus (of order $p-1$ or $p+1$), and we let i_p be the index of $\langle A \rangle_p$ in this torus. Finally, let $\pi(x) = |\{p \leq x : p \text{ is prime}\}|$ be the number of primes up to x .

2.2. Kummer extensions and Frobenius elements. We want to characterize primes p such that $n|i_p$, and we can relate this to primes splitting in certain Galois extensions as follows:

Reduce equation (2) modulo p and let $\bar{\epsilon}$ denote a solution to equation (2) in F_p or F_{p^2} . (Note that if p does not ramify in K then the order of A modulo p equals the order of ϵ modulo p .) If p splits in K then $\bar{\epsilon} \in F_p$, and if p is inert, then $\bar{\epsilon} \in F_{p^2} \setminus F_p$. In the latter case, $\bar{\epsilon} \in F_{p^2}^1$ since the norm one property is preserved when reducing modulo p . Now, F_p^\times and $F_{p^2}^1$ are cyclic groups of order $p-1$ and $p+1$ respectively. Thus, if p splits in K then $\operatorname{ord}_p(\epsilon)|p-1$, whereas if p is inert in K then $\operatorname{ord}_p(\epsilon)|p+1$.

Lemma 3. *Let p be unramified in K_n , and let $C_n = \{1, \gamma\} \subset \operatorname{Gal}(K_n/\mathbf{Q})$, where γ is given by $\gamma(\zeta_n) = \zeta_n^{-1}$ and $\gamma(\alpha_n) = \alpha_n^{-1}$. Then the condition that $n|i_p$ is equivalent to $\sigma_p \in C_n$. Moreover, C_n is invariant under conjugation.*

Proof. The split case: Since $n|i_p$ and $i_p|p-1$ we have $\zeta_n \in F_p$, i.e. F_p contains all n -th roots of unity. Moreover, $\bar{\epsilon}$ is an n -th power of some element in F_p , and thus the equation $x^n - \epsilon$ splits completely in F_p . In other words, p splits completely in K_n and σ_p is trivial.

The inert case: Since n divides i_p , $\bar{\epsilon}$ is an n -th power of some element in $F_{p^2}^1$ and hence $\alpha_n \in F_{p^2}$. Moreover, $n|p^2-1$ implies that $\zeta_n \in F_{p^2}$.

Now, $N_{F_p}^{F_{p^2}}(\alpha_n) = 1$ and $N_{F_p}^{F_{p^2}}(\zeta_n) = \zeta_n^{p+1} = 1$ implies that

$$\sigma_p(\zeta_n) \equiv \zeta_n^{-1} \pmod{p}, \quad \sigma_p(\alpha_n) \equiv \alpha_n^{-1} \pmod{p}.$$

For p that does not ramify in K_n we thus have

$$(3) \quad \sigma_p(\zeta_n) = \zeta_n^{-1}, \quad \sigma_p(\alpha_n) = \alpha_n^{-1}$$

Now, an element $\tau \in \text{Gal}(K_n/\mathbf{Q})$ is of the form

$$\tau: \begin{cases} \zeta_n \rightarrow \zeta_n^t & t \in \mathbf{Z} \\ \alpha_n \rightarrow \alpha_n^u \zeta_n^s & s \in \mathbf{Z}, \quad u \in \{1, -1\} \end{cases}$$

Composing γ and τ then gives

$$\tau \circ \gamma: \begin{cases} \zeta_n \rightarrow \zeta_n^{-1} \rightarrow \zeta_n^{-t} \\ \alpha_n \rightarrow \alpha_n^{-1} \rightarrow \alpha_n^{-u} \zeta_n^{-s} \end{cases}$$

and

$$\gamma \circ \tau: \begin{cases} \zeta_n \rightarrow \zeta_n^t \rightarrow \zeta_n^{-t} \\ \alpha_n \rightarrow \alpha_n^u \zeta_n^s \rightarrow \alpha_n^{-u} \zeta_n^{-s} \end{cases}$$

which shows that γ is invariant under conjugation. \square

2.3. The Chebotarev density Theorem. In [9] Serre proved that the Generalized Riemann Hypothesis (GRH) implies the following version of the Chebotarev density Theorem:

Theorem 4. *Let E/\mathbf{Q} be a finite Galois extension of degree $[E : \mathbf{Q}]$ and discriminant D_E . For p a prime let $\sigma_p \in G = \text{Gal}(E/\mathbf{Q})$ denote the Frobenius conjugacy class, and let $C \subset G$ be a union of conjugacy classes. If the nontrivial zeroes of $\zeta_E(s)$ lie on the line $\text{Re}(s) = 1/2$, then for $x \geq 2$,*

$$|\{p \leq x : \sigma_p \in C\}| = \frac{|C|}{|G|} \pi(x) + O\left(\frac{|C|}{|G|} x^{1/2} (\log D_E + [E : \mathbf{Q}] \log x)\right)$$

Now, primes that ramify in K_n divides nD_K (see Lemma 9), so as far as densities are concerned, ramified primes can be ignored. The bounds on the size of D_{K_n} (see Lemma 9) and Lemma 3 then gives the following:

Corollary 5. *If GRH is true then*

$$(4) \quad |\{p \leq x : n|i_p\}| = \frac{2}{[K_n : \mathbf{Q}]} \times \pi(x) + O(x^{1/2}(\log(xn)))$$

Remark: For theorems 1 and 2 to be true, it is enough to assume that the Riemann hypothesis holds for all ζ_{K_n} , $n > 1$.

2.3.1. *Bounds on degrees.* In order to apply the Chebotarev density Theorem we need bounds on the degree $[K_n : \mathbf{Q}]$. We will first assume that ϵ is a fundamental unit.

Lemma 6. *If ϵ is a fundamental unit in K and if $n = 4$ or $n = q$, for q an odd prime, then $\text{Gal}(K_n/K)$ is nonabelian.*

Proof. We start by showing that $[K_n : Z_n] = n$. Consider first the case $n = q$. If $\alpha_q \in Z_q$ then $\beta = N_{K^q}^{Z_q}(\alpha_q) = \alpha_q^{[Z_q:K]} \zeta_q^t \in K \subset \mathbf{R}$ for some integer t . Since q is odd we may assume that $\alpha_q \in \mathbf{R}$, and this forces $\zeta_q^t = 1$, which in turn implies that $\alpha_q^{[Z_q:K]} \in K$. Because ϵ is a fundamental unit this means that $q \nmid [Z_q : K]$. On the other hand, $[Z_q : K] \mid \phi(q)$, a contradiction. Thus $\alpha_q \notin Z_q$, and hence K_q/Z_q is a Kummer extension of degree q .

For $n = 4$ we note that $i \in Z_4 = K(i)$. Thus $\alpha_2 = \sqrt{\epsilon} \in Z_4$ implies that $\sqrt{-\epsilon} \in Z_4$. However, either $\sqrt{\epsilon}$ or $\sqrt{-\epsilon}$ is real and generates a *real* degree two extension of K , whereas $K(i)$ is a non-real quadratic extension of K , and hence $\alpha_2 \notin Z_4$. Now, if $\alpha_4 \in Z_4(\alpha_2)$ then $N_{Z_4}^{Z_4(\alpha_2)}(\alpha_4) = \alpha_4^{2t} \in Z_4$ for some $t \in \mathbf{Z}$, and thus $\alpha_4^2 = \alpha_2 \in Z_4$ which contradicts $\alpha_2 \notin Z_4$. Therefore,

$$[Z_4(\alpha_4) : Z_4] = [Z_4(\alpha_4) : Z_4(\alpha_2)][Z_4(\alpha_2) : Z_4] = 4.$$

Finally we note that the commutator of any nontrivial element $\sigma_1 \in \text{Gal}(K_n/Z_n)$ with any nontrivial element $\sigma_2 \in \text{Gal}(K_n/L_n)$ is nontrivial (we may regard $\text{Gal}(K_n/Z_n)$ and $\text{Gal}(K_n/L_n)$ as subgroups of $\text{Gal}(K_n/K)$). Hence $\text{Gal}(K_n/K)$ is nonabelian. \square

Lemma 7. *If ϵ is a fundamental unit then*

$$[K_n : Z_n] \geq n/2.$$

Proof. Clearly $Z_n(\alpha_{q^k}) \subset K_n$, and since field extensions of relative prime degrees are disjoint, it is enough to show that if $q^k \nmid n$ is a prime power then $q^k \nmid [Z_n(\alpha_{q^k}) : Z_n]$ if q is odd, and $q^{k-1} \nmid [Z_n(\alpha_{q^k}) : Z_n]$ if $q = 2$.

If q is odd then Lemma 6 implies that $\alpha_q \notin Z_n$ since $\text{Gal}(Z_n/K)$ is abelian. Hence, if $m \in \mathbf{Z}$ and $\alpha_{q^k}^m \in Z_n$, we must have $q^k \mid m$. Now, if $\sigma \in \text{Gal}(Z_n(\alpha_{q^k})/Z_n)$ then $\sigma(\alpha_{q^k}) = \alpha_{q^k} \zeta_{q^k}^{t_\sigma}$ for some integer t_σ . Thus there exists an integer t such that

$$\beta = N_{Z_n}^{Z_n(\alpha_{q^k})}(\alpha_{q^k}) = \alpha_{q^k}^{[Z_n(\alpha_{q^k}):Z_n]} \zeta_{q^k}^t \in Z_n$$

Multiplying β by $\zeta_q^{-t} \in Z_n$ we find that $\alpha_{q^k}^{[Z_n(\alpha_{q^k}):Z_n]} \in Z_n$, and hence $q^k \nmid [Z_n(\alpha_{q^k}) : Z_n]$.

For $q = 2$ the proof is similar, except that a factor of two is lost if $\alpha_2 \in Z_n$. \square

Remark: K_2/Q is a Galois extension of degree four, hence abelian and therefore contained in some cyclotomic extension by the Kronecker-Weber Theorem, and it is thus possible that $\alpha_2 \in Z_n$ for some values of n .

Lemma 8. *We have*

$$n\phi(n) \ll_K [K_n : \mathbf{Q}] \leq 2n\phi(n)$$

Proof. We first note that $[Z_n : K]$ equals $\phi(n)$ or $\phi(n)/2$ depending on whether $K \subset \mathbf{Q}(\zeta_n)$ or not. We also have the trivial upper bound $[K_n : Z_n] \leq n$.

For a lower bound of $[K_n : Z_n]$ we argue as follows: Let $\gamma \in K$ be a fundamental unit. Since the norm of ϵ is one we may write $\epsilon = \gamma^k$ for some $k \in \mathbf{Z}$. (Note that k does not depend on n .) As $[Z_n(\gamma^{1/n}) : Z_n(\epsilon^{1/n})] \leq k$, Lemma 7 gives that $[Z_n(\epsilon^{1/n}) : Z_n] \geq n/k$. The upper and lower bounds now follows from

$$[K_n : \mathbf{Q}] = [K_n : Z_n][Z_n : K][K : \mathbf{Q}]$$

\square

2.3.2. Bounds on discriminants.

Lemma 9. *If p ramifies in K_n then $p \mid nD_K$. Moreover,*

$$\log(\text{disc}(K_n/\mathbf{Q})) \ll_K [K_n : K] \log(n)$$

Proof. First note that

$$\text{disc}(K_n/\mathbf{Q}) = N_{\mathbf{Q}}^K(\text{disc}(K_n/K)) \times \text{disc}(K/\mathbf{Q})^{[K_n:K]}.$$

From the multiplicativity of the different we get

$$\text{disc}(K_n/K) = \text{disc}(Z_n/K)^{[K_n:Z_n]} \times N_K^{Z_n}(\text{disc}(K_n/Z_n)),$$

Since ϵ is a unit, so is $\epsilon^{1/n}$. Thus, if we let $f(x) = x^n - \epsilon$ then $f'(x) = nx^{n-1}$, and therefore the principal ideal $f'(\epsilon^{1/n})\mathfrak{D}_{K_n}$ equals $n\mathfrak{D}_{K_n}$. In terms of discriminants this means that

$$\text{disc}(K_n/Z_n) \mid N_{Z_n}^{K_n}(n\mathfrak{D}_{K_n})$$

and similarly it can be shown that

$$\text{disc}(Z_n/K) \mid N_K^{Z_n}(n\mathfrak{D}_{Z_n}).$$

Thus $\text{disc}(K_n/\mathbf{Q})$ divides

$$N_{\mathbf{Q}}^K \left(N_K^{K_n}(n\mathfrak{D}_{K_n}) \times N_K^{Z_n}(n\mathfrak{D}_{Z_n})^{[K_n:Z_n]} \right) \times \text{disc}(K/\mathbf{Q})^{[K_n:K]}.$$

$$= n^{4[K_n:K]} \times \text{disc}(K/\mathbf{Q})^{[K_n:K]}$$

which proves the two assertions. \square

3. PROOF OF THEOREM 1

In order to bound the number of primes $p < x$ for which $i_p > x^{1/2}$ we will need the following Lemma:

Lemma 10. *The number of primes p such that $\text{ord}_p(A) \leq y$ is $O(y^2)$.*

Proof. Given A there exists a constant C_A such that $\det(A^n - I) = O(C_A^n)$. Now, if the order of $A \bmod p$ is n , then certainly p divides $\det(A^n - I) \neq 0$. Putting $M = \prod_{n=1}^y \det(A^n - I)$ we see that any prime p for which A has order $n \leq y$ must divide M . Finally, the number of prime divisors of M is bounded by

$$\log(M) \ll \sum_{n=1}^y n \log(C_A) \ll y^2.$$

\square

First step: We consider primes p such that $i_p \in (x^{1/2} \log x, x)$. By Lemma 10 the number of such primes is

$$(5) \quad O\left(\left(\frac{x}{x^{1/2} \log x}\right)^2\right) = O\left(\frac{x}{\log^2 x}\right).$$

Second step: Consider p such that $q|i_p$ for some prime $q \in (\frac{x^{1/2}}{\log^3 x}, x^{1/2} \log x)$. We may bound this by considering primes $p \leq x$ such that $p \equiv \pm 1 \pmod q$ for $q \in (\frac{x^{1/2}}{\log^3 x}, x^{1/2} \log x)$. Since $q \leq x^{1/2} \log x$, Brun's sieve gives (up to an absolute constant) the bound

$$\frac{x}{\phi(q) \log(x)}$$

and the total contribution from these primes is at most

$$(6) \quad \sum_{q \in (\frac{x^{1/2}}{\log^3 x}, x^{1/2} \log x)} \frac{x}{\phi(q) \log(x/q)} \ll \frac{x}{\log x} \sum_{q \in (\frac{x^{1/2}}{\log^3 x}, x^{1/2} \log x)} \frac{1}{q}.$$

Now, summing reciprocals of primes in a dyadic interval, we get

$$\sum_{q \in [M, 2M]} \frac{1}{q} \ll \frac{\pi(2M)}{M} \leq \frac{1}{\log M}$$

Hence

$$\sum_{q \in (\frac{x^{1/2}}{\log^3 x}, x^{1/2} \log x)} \frac{1}{q} \ll \frac{1}{\log x} \log_2 \left(\frac{x^{1/2} \log x}{x^{1/2} / \log^3 x} \right) \ll \frac{\log \log x}{\log x}.$$

and equation (6) is $O(\frac{x \log \log x}{\log^2 x})$.

Third step: Now consider p such that $q|i_p$ for some prime $q \in (f(x)^2, \frac{x^{1/2}}{\log^3 x})$. We are now in the range where GRH is applicable; by Corollary 5 and Lemma 8 we have

$$|\{p \leq x : q|i_p\}| \ll \frac{x}{q\phi(q) \log x} + O(x^{1/2} \log(xq^2))$$

Summing over $q \in (f(x)^2, \frac{x^{1/2}}{\log^3 x})$ we find that the number of such $p \leq x$ is bounded by

$$(7) \quad \sum_{q \in (f(x)^2, \frac{x^{1/2}}{\log^3 x})} \left(\frac{x}{q^2 \log x} + O(x^{1/2} \log(xq^2)) \right)$$

Now,

$$\sum_{q \in (f(x)^2, \frac{x^{1/2}}{\log^3 x})} \frac{1}{q^2} \ll \frac{1}{f(x)}$$

and thus equation (7) is

$$\ll \frac{x}{f(x) \log x} + \frac{x}{\log^2 x}.$$

Fourth step: For the remaining primes p , any prime divisor $q|i_p$ is smaller than $f(x)^2$. Hence i_p must be divisible by some integer $d \in (f(x), f(x)^3)$. Again Lemmas 5 and 8 give

$$|\{p \leq x : d|i_p\}| \ll \frac{x}{d\phi(d) \log x} + O(x^{1/2} \log(xd^2))$$

Noting that $\phi(d) \gg d^{1-\epsilon}$ and summing over $d \in (f(x), f(x)^3)$ we find that the number of such $p \leq x$ is bounded by

$$(8) \quad \sum_{d \in (f(x), f(x)^3)} \left(\frac{x}{d^{2-\epsilon} \log x} + O(x^{1/2} \log(xd^2)) \right)$$

Now,

$$\sum_{d \in (f(x), f(x)^3)} \frac{1}{d^{2-\epsilon}} \ll \frac{1}{f(x)^{1-\epsilon}}$$

and

$$\sum_{d \in (f(x), f(x)^3)} x^{1/2} \log(xd^2) \ll f(x)^3 x^{1/2} \log(x^2)$$

therefore equation (8) is

$$\ll \frac{x}{f(x)^{1-\epsilon} \log x}$$

4. PROOF OF THEOREM 2

With the results from the previous section we may now deduce Theorem 2. If p is prime such that $\text{ord}_p(A) \leq p/\log(p)$, or p ramifies in K , we say that p is “bad”. We let P_B denote the set of all bad primes, and we let $P_B(z)$ be the set of primes $p \in P_B$ such that $p \geq z$. Since only finitely many primes ramify in K , Theorem 1 gives that the number of bad primes $p \leq x$ is $O(\frac{x}{\log^{2-\epsilon} x})$. A key observation is the following:

Lemma 11. *We have*

$$(9) \quad \sum_{p \in P_B} \frac{1}{p} < \infty$$

In particular, if we let

$$\beta(z) = \sum_{p \in P_B(z)} 1/p,$$

then $\beta(z)$ tends to zero as z tends to infinity.

Proof. Immediate from partial summation and the $O(\frac{x}{\log^{2-\epsilon} x})$ estimate in Theorem 1. \square

Given $N \in \mathbb{Z}$, write $N = s^2 N_G N_B$ where $N_G N_B$ is square free and N_B is the product of “bad” primes dividing N . By the following Lemma, we find that few integers have a large square factor:

Lemma 12.

$$|\{N \leq x : s^2 | N, s \geq y\}| = O\left(\frac{x}{y}\right)$$

Proof. The number of $N \leq x$ such that $s^2 | N$ for $s \geq y$ is bounded by

$$\sum_{s \geq y} \frac{x}{s^2} \ll \frac{x}{y}$$

\square

Next we show that there are few N for which N_B is divisible by $p \in P_B(z)$. In other words, for most N , N_B is a product of small “bad” primes.

Lemma 13. *The number of $N \leq x$ such that $p \in P_B(z)$ divides N_B is $O(x\beta(z))$.*

Proof. Let $p \in P_B(z)$. The number of $N \leq x$ such that $p|N$ is less than x/p . Thus, the total number of $N \leq x$ such that some $p \in P_B(z)$ divides N , is bounded by

$$\sum_{p \in P_B(z)} \frac{x}{p} = x \sum_{p \in P_B(z)} \frac{1}{p} = x\beta(z).$$

□

Combining the previous results we get that the number of $N = s^2 N_G N_B \leq x$ such that N_B is z -smooth and $s \leq y$ is

$$x(1 + O(\beta(z) + 1/y)).$$

For such N we have $N_B \leq \prod_{p \leq z} p \ll e^z$. Letting $z = \log \log x$ and $y = \log x$ we get that

$$N_G = \frac{N}{s^2 N_B} \geq \frac{N}{\log^3 x}$$

for $N \leq x$ with at most $O(x(\beta(\log \log x) + (\log x)^{-1})) = o(x)$ exceptions. Now, the following Proposition gives that, for most N , $\text{ord}_N(A)$ is essentially given by $\prod_{p|N} \text{ord}_p(A)$.

Proposition ([6], Proposition 11). *Let $D_A = 4(\text{tr}(A)^2 - 4)$. For almost all ³ $N \leq x$,*

$$\text{ord}_N(A) \geq \frac{\prod_{p|d_0} \text{ord}_p(A)}{\exp(3(\log \log x)^4)}$$

where d_0 is given by writing $N = ds^2$, with $d = d_0 \gcd(d, D_A)$ square-free.

Finally, since $\text{ord}_p(A) \geq \frac{p}{\log p} \geq p^{1-\epsilon}$ for $p|N_G$ and p sufficiently large, we thus find that

$$\text{ord}_N(A) \gg \frac{\prod_{p|N_G} \text{ord}_p(A)}{\exp(3(\log \log x)^4)} \gg \frac{N_G^{1-\epsilon}}{\exp(3(\log \log x)^4)} \gg N^{1-2\epsilon}$$

for most N .

³By “for almost all $N \leq x$ ” we mean that there are $o(x)$ exceptional integers N that are smaller than x .

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