LEVEL REPULSION FOR ARITHMETIC TORAL POINT SCATTERERS IN DIMENSION 3

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ABSTRACT. We show that arithmetic toral point scatterers in dimension three ("Šeba billiards on $\mathbb{R}^3/\mathbb{Z}^3$") exhibit strong level repulsion between the set of "new" eigenvalues. More precisely, let $\Lambda := \{\lambda_1 < \lambda_2 < \ldots\}$ denote the ordered set of new eigenvalues. Then, given any $\gamma > 0$,

$$\frac{|\{i \leq N : \lambda_{i+1} - \lambda_i \leq \epsilon\}|}{N} = O_\gamma(\epsilon^{4-\gamma})$$

as $N \to \infty$ (and $\epsilon > 0$ small.)

1. INTRODUCTION

1.1. **Background.** The statistics of gaps between energy levels in the semiclassical limit is a central problem in the theory of spectral statistics [15, 7]. The Berry-Tabor conjecture [4] asserts that (typical) integral systems have Poisson spacing statistics, and the Bohigas-Giannoni-Schmit conjecture [8] asserts that (generic) chaotic systems should have spacing statistics given by random matrix theory; in particular small gaps are very unlikely.

More precisely, with $\{\lambda_1 \leq \lambda_2 \leq \ldots\}$ denoting the energy levels, suitably unfolded using the main term in Weyl's law so that $|\{i : \lambda_i < E\}| \sim E$ for E large, define consecutive gaps, or spacings, $s_i := \lambda_{i+1} - \lambda_i$. The level spacing distribution $P(s)$, if it exists, is defined by

$$\lim_{N\to\infty} \frac{|\{i \leq N : s_i < x\}|}{N} = \int_0^x P(s)\, ds$$

for all $x \geq 0$. For Poisson spacing statistics, $P(s) = e^{-s}$, whereas (time reversible) chaotic systems should have Gaussian Orthogonal Ensemble (GOE) spacings, where $P(s) \approx \pi s/2$ for s small, and $P(s) \approx (\pi s/2)\exp(-\pi s^2/4)$ for s large; in particular, there is linear vanishing at $s = 0$ ("level repulsion").

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1.1.1. *Systems with intermediate statistics.* There are also "pseudo integrable" systems that are neither integrable nor chaotic. Their spectral statistics do not fall into the models described above and are believed to exhibit "intermediate statistics", e.g. there is level repulsion as for random matrix theory systems, whereas $P(s)$ has exponential tail decay similar to Poisson statistics.

The point scatterer, or the Laplacian perturbed by a delta potential, for rectangular domains (i.e., in dimension $d = 2$, with Dirichlet boundary conditions) was introduced by Šeba [18] as a model for investigating the transition between integrability and chaos in quantum systems. For this model Šeba found evidence for level repulsion of GOE type for small gaps as well as "wave chaos", in particular Gaussian value distribution of eigenfunctions.

Shigehara [19] later pointed out that level repulsion in dimension two only occurs if "strength" of the perturbation grows logarithmically with the eigenvalue λ. (The perturbation is formally defined using von Neumann's theory of self adjoint extensions; in this setting there is a one-parameter family of extensions, but any fixed parameter choice turns out to result in a "weak coupling limit" with no level repulsion, cf. [21, Section 3].) On the other hand, for dimension $d = 3$, Cheon and Shigehara [20] found GOE type level repulsion for "fixed strength" perturbations for rectangular boxes, again with Dirichlet boundary conditions and the scatterer placed at the center of the box; placing the scatterer elsewhere appeared to weaken the repulsion.

Under certain randomness assumptions on the unperturbed spectrum, GOE-type level repulsion and tails of Poisson type has been shown to hold by Bogomolny, Gerland and Schmit [6, 5]. In particular, for Šeba billiards with periodic boundary conditions, $P(s) \sim (\pi\sqrt{3}/2)s$ (for s small), whereas $P(s) \sim (1/8\pi^3)s \log^4 s$ in case of Dirichlet boundary condition and generic scatterer position. Interestingly, Berkolaiko, Bogomolny and Keating [3] has shown that the pair correlation in the former case is *identical* to the pair correlation for quantum star graphs (for growing number of bonds with generic lengths.)

1.1.2. *Toral point scatterers.* With periodic boundary conditions the location of the scatterer is irrelevant and it is natural to consider tori. In dimensions two and three, Rudnick and Ueberschär [16] used trace formula techniques to investigate the spacing distribution for toral point scatterers. In dimension two, for a fixed self adjoint extension, resulting in a weak coupling limit, they showed that the spectral statistics of the new eigenvalues is the same for the old spectrum, possibly after removing multiplicities. (In the irrational aspect ratio case it is believed that the old spectrum is of Poisson type, for partial results in this direction cf. [17, 9]; for the square torus $\mathbb{R}^2/\mathbb{Z}^2$ the same was shown to hold assuming certain analogs of the Hardy-Littlewood prime k-tuple conjecture for sums of two integer squares [13].) For $d = 3$, a fixed self adjoint extension results in a strong coupling limit, and here Rudnick-Ueberschär gave evidence for level repulsion: the mean displacement between old and new eigenvalues was shown to equal half the mean spacing between the old eigenvalues. However, the method does not rule out the level spacing distribution having (say) positive mass at $s = 0$.

1.2. **Results.** The purpose of this paper is to show that there is strong level repulsion between the set of new eigenvalues for point scatters on arithmetic tori in dimension three. To state our main result we need to describe some basic properties of the model.

1.2.1. *Toral point scatterers for arithmetic tori.* Let $\mathbb{T} := \mathbb{R}^3/(2\pi\mathbb{Z}^3)$ denote the standard flat torus in dimension three. A point scatterer on $\mathbb{T}$ is formally given by the Hamiltonian

$$H = H_{x_0,\alpha} = -\Delta + \alpha\delta_{x_0}, \quad \alpha \in \mathbb{R}$$

where Δ is the Laplace operator acting on $L^2(\mathbb{T})$, $x_0 \in \mathbb{T}$ is the location of the point scatterer, and α is the "strength" of the perturbation; in the physics literature α is known as the coupling constant.

A mathematically rigorous definition of H can be made via von Neumann's theory of self adjoint extensions, below we will briefly summarize the most important properties but refer the reader to [1, 16, 21] for detailed discussions.

The spectrum of the unperturbed Laplacian on $\mathbb{T}$ is arithmetic in nature, and given by $\{m \in \mathbb{Z} : r_3(m) > 0\}$, where

$$r_3(m) := |\{v \in \mathbb{Z}^3 : |v|^2 = m\}|$$

denotes the number of ways $m \in \mathbb{Z}$ can be written as a sum of three integer squares, and the multiplicity of an eigenvalue m is given by $r_3(m)$. The addition of a δ-potential is a rank-1 perturbation of the Laplacian, and the spectrum of H consists of two kinds of eigenvalues: "old" and "new" eigenvalues. Namely, an eigenvalue m of the unperturbed Laplacian is also an eigenvalue of H, but with multiplicity $r_3(m) - 1$; the corresponding H-eigenspace is just Laplace eigenfunctions vanishing at x_0. For simpler notation we shall, without loss of generality, from here on assume that $x_0 = 0$.

Associated to m there is also a new eigenvalue λ_m (of multiplicity one) of H, and the set of new eigenvalues $\{\lambda_m : m \in \mathbb{Z}, r_3(m) > 0\}$ interlace between the old eigenvalues. More precisely, the corresponding new eigenfunction $\psi_{\lambda_m}(x)$ is given by the Green's function

$$\sum_{v\in\mathbb{Z}^3} \frac{e^{2\pi i v\cdot x}}{|v|^2 - \lambda_m}, \quad x \in \mathbb{T},$$

(in L^2 sense), with the new eigenvalue λ_m being a solution to the spectral equation

$$G(\lambda) = \frac{1}{\nu}; \quad G(\lambda) := \sum_n r_3(n) \left(\frac{1}{n-\lambda} - \frac{n}{n^2+1} \right), \tag{1}$$

where $\nu \neq 0$ is known as the "formal strength" of the perturbation in the physics literature (cf. [20, Eq. (4)]).

As mentioned above, the set of new eigenvalues interlace with the old spectrum $\{m \in \mathbb{Z} : r_3(m) > 0\}$, and it is convenient to abuse notation and use the following labeling of the new spectrum: given m such that $r_3(m) > 0$, let λ_m denote the largest solution to $G(\lambda) = 1/\nu$ such that $\lambda < m$. (In particular, λ_m *does not* denote the m-th new eigenvalue since $r_3(m) = 0$ for a positive proportion of integers.) For m such that $r_3(m) > 0$, let m_+ denote the smallest integer $n > m$ such that $r_3(n) > 0$, and define the consecutive spacing

$$s_m := \lambda_{m_+} - \lambda_m.$$

Note that $\{s_m\}_{m:r_3(m)>0}$ has mean $6/5$ (rather than one), but as we are concerned mainly with the frequency of very small spacings this shall not concern us.

1.2.2. *Statement of the main result.* We show that the cumulant of the nearest-neighbor distribution essentially has fourth order vanishing near the origin, and hence considably stronger repulsion than the quadratic order vanishing of the cumulant in the GOE-model.

Theorem 1. *Given any small $\gamma > 0$, we have*

$$\frac{|\{m \leq x : r_3(m) > 0, s_m < \epsilon\}|}{|\{m \leq x : r_3(m) > 0\}|} = O_\gamma(\epsilon^{4-\gamma})$$

as $x \to \infty$ (and $\epsilon > 0$ small.)

1.3. **Discussion.** As Figures 1 and 2 indicate, spectral gap statistics for $3d$ arithmetic point scatterers is clearly non-generic since there is no mass at all in the tail. The reason is the "old" spectrum being very rigid — all positive integers n except the ones ruled out by simple congruence conditions (i.e., n's of the form $n = 4^k \cdot m$ for $m \equiv 7 \mod 8$) satisfy $r_3(n) > 0$, hence the gaps between new eigenvalues is easily seen to be bounded above by 4.

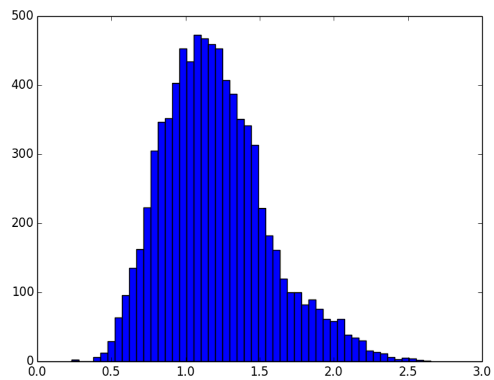

FIGURE 1. Histogram illustration of the distribution of s_m, for $m \leq 10000$ (and $r_3(m) > 0$.)

The main driving force in the fluctuations in gaps, say between two new eigenvalues λ_m and λ_{m+}, are arithmetic in nature and mainly due to fluctuations in $r_3(m)$. In particular, the small gap repulsion is not due to lack of time reversal symmetry, but rather due to $r_3(m)/\sqrt{m}$ being small extremely rarely *unless* $4^l|m$ for some high exponent l. In fact, as indicated by the plots below (as well as by the proof of Theorem 1), small gap occurences is mainly due to integers m that are divisible by large powers of 4.

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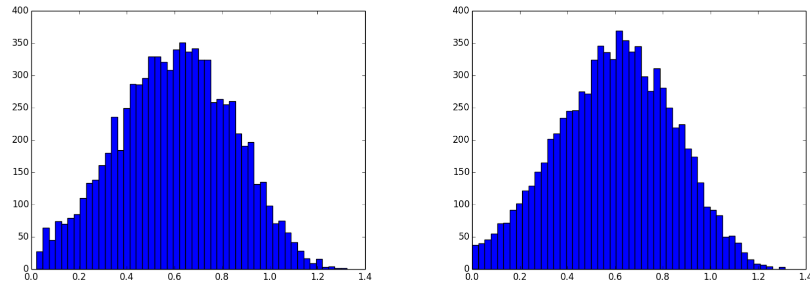

FIGURE 2. Histogram illustration of the distribution of s_m (for m such that $r_3(m) > 0$), along the progressions $\{m = 4^{10} \cdot k : k \leq 10000\}$ (left) and for $\{m = 4^{20} \cdot k : k \leq 10000\}$ (right).

2. NUMBER THEORETIC BACKGROUND

We begin by recalling some results on $r_3(n)$, the number of ways to write $n \geq 0$ as a sum of three integer squares. A classical result of Legendre (cf. [11, Chapter 3.1]) asserts that $r_3(n) \neq 0$ if and only if n is not of the form $4^a(8k+7)$ for $a, k \in \mathbb{Z}_{\geq 0}$, and we also have

$$r_3(4^a n) = r_3(n).$$

We also recall a result of Heath-Brown [12], namely that

$$\sum_{n \leq R^2} r_3(n) = \sum_{v \in \mathbb{Z}^3 : |v|^2 \leq R^2} 1 = \frac{4\pi}{3} R^3 + O(R^{21/16}) \tag{2}$$

We remark that for the main theorem any error term of the form $O(R^{3-\epsilon})$ and $\epsilon > 0$ would suffice, but the exponent $21/16$ is very helpful for the numerics behind Figures 1 and 2.

2.1. Sums of three squares and values of L-functions.

With $R_3(n)$ denoting the number of *primitive* representations of n, i.e., the number of ways to write $n = x^2 + y^2 + z^2$ for x, y, z coprime, we have the following basic identity:

$$r_3(n) = \sum_{d^2 | n} R_3(n/d^2).$$

The reason for introducing $R_3(n)$ is Gauss' marvelous identity (cf. [11, Ch. 4.8])

$$R_3(n) = \pi^{-1} \mu_n \sqrt{n} L(1, \chi_{-4n})$$

where $\mu_n = 0$ for $n \equiv 0,4,7 \mod 8$, $\mu_n = 16$ for $n \equiv 3 \mod 8$, and $\mu_n = 24$ for $n \equiv 1,2,5,6 \mod 8$;

$$L(1,\chi_{-4n}) = \sum_{m=1}^{\infty} \chi_{-4n}(m)/m$$

where χ_{-4n} is defined via the Kronecker symbol, namely

$$\chi_{-4n}(m) := \left(\frac{-4n}{m}\right).$$

Now, $-4n$ is not always a fundamental discriminant[1], but if $R_3(n) > 0$ and we write

$$n = c^2 d$$

where d is square free and $4 \nmid c^2 d$, then $L(1,\chi_{-4n})$ and $L(1,\chi_{-4d})$ have the same Euler factors, except at primes p dividing $4^l c^2$. Moreover, $-4d$ is a fundamental discriminant if $n \equiv 1,2,5,6 \mod 8$. If $n \equiv 3 \mod 8$ then $-4d$ is not a fundamental discrimant but $-d$ is, and the Euler factors of $L(1,\chi_{-4d})$ and $L(1,\chi_{-d})$ are the same at all odd primes. (Recall that $n \not\equiv 0,4,7 \mod 8$ since we assume that $R_3(n) > 0$.)

Thus, if $4 \nmid n_0$ and we write $n_0 = c^2 d$ with d square free, we note the following useful lower bound in terms of L-functions attached to primitive characters (associated with fundamental discriminants)

$$L(1,\chi_{-4n_0}) \gg \begin{cases} (\phi(c)/c)L(1,\chi_{-4d}) & \text{if } n_0 \equiv 1,2,5,6 \mod 8, \\ (\phi(c)/c)L(1,\chi_{-d}) & \text{if } n_0 \equiv 3 \mod 8, \end{cases} \tag{3}$$

where $f(x) \gg g(x)$ means that $f(x) > cg(x)$ for some absolute constant $c > 0$.

We next show that $L(1,\chi_{-4n})$ is small very rarely. With

$$FD := \{d \in \mathbb{Z} : d < 0 \text{ and } d \text{ is a fundamental discriminant}\}.$$

the following Proposition is an easy consequence of [10, Proposition 1].

Proposition 2. *There exists $c > 0$ such that for $T \geq 1$, we have*[2]

$$|\{d \leq x : -d \in FD, L(1,\chi_{-d}) < \frac{\pi^2}{6e^{\gamma}T}\}| \ll x \exp(-c \cdot e^T/T)$$

as $x \to \infty$.

For imprimitive quadratic characters we will use the following weaker bound.

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[1]A fundamental discriminant is an integer $d \equiv 0,1 \mod 4$ such that d is square-free if $d \equiv 1$, and $d = 4m$ for $m \equiv 2,3$ square free if $d \equiv 0 \mod 4$.

[2]We recall Vinogradov's "$\ll$ notation": $f \ll g$ is equivalent to $f = O(g)$. When allowing implicit constants to depend on a parameter (say γ), $f \ll_\gamma g$ is equivalent to $f = O_\gamma(g)$.

Proposition 3. *The number of integers $n \leq x$ of the form $n = 4^l n_0 = 4^l c^2 d$, where d is square free, $c \leq C$, $4 \nmid c^2 d$, and*

$$L(1, \chi_{-4n_0}) \leq 1/T$$

is

$$\ll \frac{x}{4^l} \exp(-(T/\log\log C)^4)$$

for all integer $l \geq 0$.

Proof. Using (3), together with the bound $c/\phi(c) \ll \log\log c \leq \log\log C$ we find that for c, l fixed, we have (note that either $-d$ or $-4d$ is a fundamental discriminant)

$$|\{n \leq x : n = 4^l n_0 = 4^l c^2 d, L(1, \chi_{-4n_0}) \leq 1/T\}|$$

$$\ll |\{d \leq 4x/(c^2 4^l) : -d \in FD, L(1, \chi_{-d}) \ll \frac{\log\log C}{T}\}|$$

which, by Proposition 2, is

$$\ll x/(c^2 4^l) \exp(-(T/\log\log C)^4)$$

as $x \to \infty$. Summing over $c \leq C$ the proof is concluded. □

2.2. **Estimates on moments of $r_3(n)$.** We recall the following bound by Barban [2].

Theorem 4. *For $k \in \mathbb{Z}^+$ we have*

$$\sum_{1 \leq d \leq x} L(1, \chi_{-d})^k \ll_k x.$$

We can now easily deduce bounds on the (normalized) moments of $r_3(n)$.

Proposition 5. *For $k \in \mathbb{Z}^+$ we have*

$$\sum_{n \leq x} (r_3(n)/\sqrt{n})^k \ll_k x$$

Proof. Writing $n = 4^l n_0$ so that $4 \nmid n_0$ we have

$$r_3(n) = r_3(n_0)$$

and we further recall the identities (cf. Section 2.1)

$$r_3(n) = \sum_{d^2 | n} R_3(n/d^2)$$

and

$$R_3(n) = \mu_n \sqrt{n} L(1, \chi_{-4n}),$$

where $\mu_n = 0$ if $n \equiv 0, 4, 7 \mod 8$; otherwise $16 \le \mu_n \le 24$. Thus

$$\sum_{n \le x} (r_3(n)/\sqrt{n})^k = \sum_{n \le x} \left(\sum_{\substack{d^2 | n, d \equiv 1 \mod 2 \\ 4^l || n}} \frac{R_3(n/(4^l d^2))}{\sqrt{n}} \right)^k$$

$$\ll \sum_{l : 4^l \le x} \sum_{\substack{d^2 \le x/4^l \\ d \text{ odd}}} \sum_{\substack{d_1, \ldots, d_k \\ [d_1, \ldots, d_k] = d}} \sum_{\substack{n \le x : d^2 | n \\ 4^l || n}} \prod_{i=1}^k \frac{R_3(n/(4^l d_i^2))}{\sqrt{n}} \ll$$

$$\sum_{l : 4^l \le x} \sum_{\substack{d^2 \le x/4^l \\ d \text{ odd}}} \sum_{\substack{d_1, \ldots, d_k \\ [d_1, \ldots, d_k] = d}} \sum_{n_0 \le x/d^2 4^l} \prod_{i=1}^k \frac{R_3(n_0 d^2/d_i^2)}{\sqrt{d^2 4^l n_0}} \ll$$

$$\sum_{l : 4^l \le x} \sum_{\substack{d^2 \le x/4^l \\ d \text{ odd}}} \sum_{\substack{d_1, \ldots, d_k \\ [d_1, \ldots, d_k] = d}} \sum_{n_0 \le x/d^2 4^l} \prod_{i=1}^k \frac{L(1, \chi_{-4n_0 d^2/d_i^2})}{d_i} \ll$$

which, on noting[3] that $L(1, \chi_{-4n_0 d^2/d_i^2}) \ll L(1, \chi_{-4n_0}) d/\phi(d)$ is

$$\ll \sum_{l : 4^l \le x} \sum_{\substack{d^2 \le x/4^l \\ d \text{ odd}}} \sum_{d_1, \ldots, d_k | d} \frac{(d/\phi(d))^k}{d_1 \cdots d_k} \sum_{n_0 \le x/d^2 4^l} L(1, \chi_{-4n_0})^k. \tag{4}$$

By Theorem 4, the inner sum over n_0 is $\ll_k x/(d^2 4^l)$, and hence (4) is

$$\ll \sum_{l : 4^l \le x} \sum_{\substack{d^2 \le x/4^l \\ d \text{ odd}}} \sum_{d_1, \ldots, d_k | d} \frac{(d/\phi(d))^k}{d_1 \cdots d_k} \frac{x}{d^2 4^l} \ll \sum_{l : 4^l \le x} \sum_{\substack{d^2 \le x/4^l \\ d \text{ odd}}} (d/\phi(d))^{2k} \frac{x}{d^2 4^l} \ll x$$

using that $\sum_{d_i | d} 1/d_i \ll d/\phi(d) \ll d^{o(1)}$.

□

By Chebychev's inequality we immediately deduce:

Corollary 6. *Given $k \in \mathbb{Z}^+$, we have*

$$|\{n \le x : r_3(n)/\sqrt{n} > T\}| \ll_k x/T^k$$

[3] The Euler products for $L(1, \chi_{-4n_0 d^2/d_i^2})$ and $L(1, \chi_{-4n_0})$ agree at all primes p such that $p \nmid d$.

3. Proof of Theorem 1

Let $\mathcal{N}_3 := \{n \in \mathbb{Z} : r_3(n) > 0\}$ denote the old spectrum. Given $n \in \mathcal{N}_3$, let n_+ denote its nearest right neighboor in $\mathcal{N}_3$, and recall our definition

$$s_n := \lambda_{n_+} - \lambda_n$$

of the nearest neighboor spacing between the two new eigenvalues λ_n, λ_{n_+}.

The spectral equation (cf. (1)) is then given by

$$\sum_n r_3(n) \left(\frac{1}{n-\lambda} - \frac{n}{n^2+1} \right) = 1/\nu$$

where ν is constant. For simpler notation we shall only treat the case of when the R.H.S. equals zero, corresponding to $\nu = \infty$, but we remark that the method of proof gives the same result when $1/\nu$ is allowed to change, sufficiently smoothly, with λ, as long as we are in the strong coupling region $1/\nu = O(\lambda^{1/2-\epsilon})$ for $\epsilon > 0$ (cf. [20, Section II]).

We first show that the sum may be truncated without significantly changing the the new eigenvalues (the proof is relegated to the appendix.)

Lemma 7. *There exists $\alpha, \delta \in (0,1)$ such that*

$$\sum_{n:|n-\lambda|>\lambda^\alpha} r_3(n) \left(\frac{1}{n-\lambda} - \frac{n}{n^2+1} \right) \ll \lambda^{1/2-\delta}$$

(in fact, using (2) we may take $\alpha = 1/6$.)

The following simple result was used in [14] (cf. Lemma 7) in order to study scarred eigenstates for arithmetic toral point scatterers in dimensions two and three.

Lemma 8. *Assume that $A, B > 0$. Let $f(x)$ be a smooth function with $f'(x) \geq \epsilon > 0$ for all $x \in [-1/2, 1/2]$, and assume that $f'(x) \leq B$ for $|x| \leq 1/2$. If the equation*

$$f(x) = A/x$$

has two roots $x_1 \in [-1/2, 0)$ and $x_2 \in (0, 1/2]$, then

$$|x_2 - x_1| \gg \sqrt{A/B}.$$

Given $m \in \mathcal{N}_3$, set $\lambda = m + \delta$, and define

$$G_m(\delta) := \sum_{n:0<|n-m|\leq\sqrt{m}} r_3(n) \left(\frac{1}{n-m-\delta} - \frac{n}{n^2+1} \right);$$

we can then rewrite the spectral equation as

$$G_m(\delta) - 1/\nu = \frac{r_3(m)}{\delta}$$

For $|\delta| < 1/2$, we find (note that all terms are positive)

$$0 < G_m'(\delta) \ll \sum_{n:0<|n-m|\leq\sqrt{m}} \frac{r_3(n)}{(n-m)^2} = \sum_{0<|h|<\sqrt{m}} \frac{r_3(m+h)}{h^2}$$

To apply Lemma 8, we will need to bound G_m' from above.

Lemma 9. *Given $k \in \mathbb{Z}^+$, we have*

$$\frac{1}{x}\sum_{n\leq x}\left(\sum_{0<|h|<n^\alpha} \frac{r_3(n+h)}{h^2\sqrt{n}}\right)^k \ll_k x$$

and consequently

$$|\{n \leq x : \sum_{0<|h|<n^\alpha} \frac{r_3(n+h)}{h^2\sqrt{n}} > T\}| \ll_k x/T^k$$

Proof. Expanding out the k-th power expression it is enough to show that for $0 < |h_1|, |h_2|, \ldots, |h_k| < x^\alpha$ we have

$$\frac{1}{x}\sum_{n\leq x} \frac{r_3(n+h_1)\cdots r_3(n+h_k)}{n^{k/2}} \ll_k x$$

and this follows from Hölder's inequality and the bound on $\sum_{n\leq x}(r_3(n)/\sqrt{n})^k$ given by Proposition 5. □

In particular, for $k \in \mathbb{Z}^+$, we have

$$|\{m \leq x : G_m'(\delta)/\sqrt{m} > T \text{ for } |\delta| \leq 1/2\}| \ll_k x/T^k \tag{5}$$

Now, if $s_m = \lambda_{m_+} - \lambda_m$ denotes the distances between two consecutive new eigenvalues near $m \in \mathcal{N}_3$, we have one of the following: either one or both new eigenvalues lies outside $[m-1/2, m+1/2]$ in which case $s_m \geq 1/2$. In case both eigenvalues lie in $[m-1/2, m+1/2]$, Lemma 8 gives that

$$s_m \gg \sqrt{r_3(m)/G_m'(0)} = \sqrt{A(m)/B(m)}. \tag{6}$$

where we define

$$A = A(m) := r_3(m)/\sqrt{m}, \quad B = B(m) := G_m'(0)/\sqrt{m}$$

Proposition 10. *For any $\gamma > 0$ we have, as $x \to \infty$,*

$$|\{n \leq x : 0 < r_3(n)/G_n'(0) \leq \epsilon^2\}| \ll_\gamma \epsilon^{4-\gamma} \cdot x$$

Proof. Given n such that $r_3(n) > 0$, write $n = 4^l n_0 = 4^l c^2 d$, where $4 \nmid n_0$ and d is squarefree. Recalling that $G'_n(0) = \sqrt{n}B(n)$ and

$$r_3(n) \geq R_3(n_0) \gg \frac{\sqrt{n}}{2^l} L(1, \chi_{-4n_0})$$

it is enough to estimate the number of $n = 4^l n_0 \leq x$, for which

$$\epsilon^2 \geq r_3(n)/G'_n(0) \gg \frac{L(1, \chi_{-4n_0})}{2^l B(4^l n_0)} \tag{7}$$

The number of $n \leq x$ such that $n = 4^l c^2 d$ for $c \geq \epsilon^{-4}$ is $\ll x/\epsilon^{-4} = \epsilon^4 x$. We may thus assume that $c \leq \epsilon^{-4}$ in any such decomposition.

Now, given n, ϵ, define $a = a(n), b = b(n)$ and $m = m(\epsilon)$ such that

$$L(1, \chi_{-4n_0}) = 2^{-a} \quad G'_n(0) = 2^b, \quad \epsilon = 2^{-m}.$$

Moreover, (7) is equivalent to $2^{-2m} \gg \frac{2^{-a}}{2^l 2^b}$ i.e.,

$$a + b + l \geq 2m + O(1).$$

First case: $a > \sqrt{m}$. Noting that $c \leq 1/\epsilon^4 = 2^{4m}$ implies that $\log\log c \leq \log 4m$, Proposition 3 gives that the number of $n \leq x$ such that $L(1, \chi_{-4_{n_0}}) = 2^{-a} \leq 2^{-\sqrt{m}}$ is

$$\ll \quad x \exp(-(2^{\sqrt{m}}/(\log 4m))^4) \quad \ll \quad x \exp(-1000m) \quad = \quad o(\epsilon^{10} x)$$

Second case: $a \leq \sqrt{m}$. Given a large positive integer $k \asymp 8/\gamma$, we first consider l such that $l < 2m(1-\gamma) - \sqrt{m}$. We then find that

$$b \geq 2m+O(1)-a-l \geq 2m+O(1)-\sqrt{m}-2m(1-\gamma)+\sqrt{m} \geq 2m\gamma+O(1)$$

for all sufficiently large m. Using the bound in (5) (recall that $k \asymp 8/\gamma$ and that $B(n) = G'_n(0)/\sqrt{n}$), the number of $n \leq x$ such that $B(n) \geq 2^{\gamma m}$ is

$$\ll_\gamma x/(2^{\gamma m})^{8/\gamma} \ll_\gamma x/(2^{8m}) = O_\gamma(\epsilon^8 x)$$

Finally, the number of $n \leq x$ such that $4^l | n$ for some $l \geq 2m(1-\gamma) - \sqrt{m}$ is

$$\ll x/2^{2(2m(1-\gamma)-\sqrt{m})} \ll x\epsilon^{4(1-\gamma)+o(1)}$$

as $m \to \infty$ (or equivalently, that $\epsilon = 2^{-m} \to 0$), thereby concluding the proof. $\square$

Appendix A. Spectral cutoff justification

Before proving Lemma 7 we record complete cancellation in a certain smooth approximation to the spectral counting function.

Lemma 11. *With P.V.* $\int_0^\infty \dots\, dt = \lim_{\epsilon\to 0}(\int_0^{\lambda-\epsilon} \dots + \int_{\lambda+\epsilon}^\infty \dots)\, dt$ *denoting the principal value of the integral (with respect to the singularity at* $t=\lambda$*), we have*

$$P.V. \int_0^\infty \sqrt{t}\left(\frac{1}{t-\lambda} - \frac{1}{t}\right) dt = 0.$$

Proof. Using the change of variables $t = \lambda s^2$ (taking apropriate principle values of the integrals, in particular for the integrals over the s-parameter, the principal value is taken with respect to the singularity at $s=1$) we find that

$$\int_0^\infty \left(\frac{1}{\lambda - t} + \frac{1}{t}\right)\sqrt{t}\, dt = \int_0^\infty \left(\frac{1}{\lambda - \lambda s^2} + \frac{1}{\lambda s^2}\right)\sqrt{\lambda s^2} 2\lambda s\, ds$$

$$= 2\sqrt{\lambda}\int_0^\infty \left(\frac{1}{1-s^2} + \frac{1}{s^2}\right) s^2\, ds = 2\sqrt{\lambda}\int_0^\infty \frac{2}{1-s^2}\, ds = 0.$$

∎

Proof of Lemma 7. As $n/(n^2+1) - 1/n = O(1/n^3)$, we find that

$$\sum_{n=1}^\infty r_3(n)\left(\frac{1}{n-\lambda} - \frac{n}{n^2+1}\right) = \sum_{n=1}^\infty r_3(n) g(n) + O(1)$$

if we define

$$g(n) := \frac{1}{n-\lambda} - \frac{1}{n}$$

We now study sums

$$\sum_{n\in I} r_3(n) g(n)$$

for various intervals $I = (A,B]$. More precisely, define $I_1 := (1, \lambda - \lambda^\alpha]$, $I_2 = (\lambda - \lambda^\alpha, \lambda + \lambda^\alpha]$, and let $I_3 := (\lambda+\lambda^\alpha, \infty)$, for $\alpha \in (0,1)$. It is our goal to show that for $k = 1, 3$, we have

$$\sum_{n\in I_k} r_3(n) g(n) \ll \lambda^{1/2-\delta}$$

for some small $\delta > 0$, if α is chosen appropriately.

With

$$R(t) := \sum_{n\le t} r_3(n)$$

we have (cf. (2))

$$R(t) = \frac{4\pi}{3}t^{3/2} + O(t^{21/32}) =: M(t) + E(t).$$

By Abel's summation formula,

$$\sum_{A<n\leq B} r_3(n)g(n) = R(B)g(b) - R(A)g(A) - \int_A^B R(t)g'(t)\,dt,$$

and the contribution from the smooth main term $M(t)$ is then

$$= M(B)g(b) - M(A)g(A) - \int_A^B M(t)g'(t)\,dt$$

which, on using partial integration, equals (after taking appropriate principal values)

$$M(B)g(b) - M(A)g(A) - \left(M(B)g(b) - M(A)g(A) + \int_A^B m(t)g(t)\,dt \right) = \int_A^B m(t)g(t)\,dt$$

where $m(t) = M'(t) = c\sqrt{t}$ for some $c > 0$.

Thus, taking principal values as appropriate, and using Lemma 11

$$\int_{I_1 \cup I_3} m(t)g(t)\,dt = -\int_0^1 m(t)g(t)\,dt - \int_{I_2} m(t)g(t)\,dt = O(1) - \int_{I_2} m(t)g(t)\,dt$$

and it is enough to bound

$$\int_{I_2} \sqrt{t}\left(\frac{1}{t-\lambda} - \frac{1}{t}\right)dt = \int_{-\lambda^\alpha}^{\lambda^\alpha} \sqrt{\lambda+s}\left(\frac{1}{s} - \frac{1}{\lambda+s}\right)ds. \tag{8}$$

Since $\sqrt{\lambda+s} = \sqrt{\lambda}\cdot(1+O(s/\lambda))$ we find that (8) equals

$$\int_{-\lambda^\alpha}^{\lambda^\alpha} \frac{\lambda^{1/2}}{s}\,ds + \int_{-\lambda^\alpha}^{\lambda^\alpha} \frac{\lambda^{1/2}s}{\lambda s}\,ds + O(\lambda^{\alpha-1/2}) = 0 + O(\lambda^{\alpha-1/2}) = O(\lambda^{\alpha-1/2}).$$

Further, the contribution from the error term $E(t)$ equals

$$\begin{aligned} &E(B)g(B) - E(A)g(A) - \int_A^B E(t)g'(t)\,dt \\ &\quad = B^{21/32}\left(\frac{1}{B-\lambda} - \frac{1}{B}\right) - \left(A^{21/32}\left(\frac{1}{A-\lambda} - \frac{1}{A}\right)\right) \\ &\qquad + O\left(\int_A^B t^{21/32}\left(-\frac{1}{(t-\lambda)^2} + \frac{1}{t^2}\right)dt\right). \end{aligned}$$

For the integral over the first interval I_1 we have $A = 1$, $B = \lambda - \lambda^\alpha$ and thus

$$\int_{I_1} \ll \lambda^{21/32-\alpha} + O(1) + \lambda^{21/32-\alpha} + O(1) \ll \lambda^{21/32-\alpha}$$

(to see that $\int_1^{\lambda-\lambda^\alpha} \frac{t^{21/32}}{(t-\lambda)^2} dt \ll \lambda^{21/32-\alpha}$, consider $\int_1^{\lambda/2}$ and $\int_{\lambda/2}^{\lambda-\lambda^\alpha}$ separately.) Similarly, the contribution to $\int_{I_3}$, from terms involving $E(t)$, is

$$\ll \lambda^{21/32-\alpha}$$

(to see that $\int_{\lambda+\lambda^\alpha}^{\infty} \frac{t^{21/32}}{(t-\lambda)^2} dt \ll \lambda^{21/32-\alpha}$, consider $\int_{\lambda+\lambda^\alpha}^{2\lambda}$ and $\int_{2\lambda}^{\infty}$ separately.)

In conclusion, we find that

$$\sum_{n \in I_1 \cup I_2} r_3(n) g(n) \ll \lambda^{21/32-\alpha} = \lambda^{1/2-\delta}$$

for $\delta > 0$ if we take $\alpha = 1/6$ (note that $21/32 < 2/3$.)

$\square$

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