

# A NOTE ON MULTIPLICATIVE AUTOMATIC SEQUENCES

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RÉSUMÉ. Nous montrons qu'une fonction complètement multiplicative  $f : \mathbb{N} \rightarrow \mathbb{C}$  qui est également  $q$ -automatique est ultimement périodique, coïncidant donc avec un caractère de Dirichlet pour tout nombre premier suffisamment grand. Ceci résout un problème de J.-P. Allouche et L. Goldmakher, et confirme une conjecture de J. Bell, N. Bruin et M. Coons pour les fonctions complètement multiplicatives. De plus, sous l'hypothèse de Riemann généralisée, notre démonstration peut être adaptée pour démontrer cette conjecture pour toute fonction simplement supposée multiplicative.

ABSTRACT. We prove that any  $q$ -automatic completely multiplicative function  $f : \mathbb{N} \rightarrow \mathbb{C}$  essentially coincides with a Dirichlet character. This answers a question of J.-P. Allouche and L. Goldmakher and confirms a conjecture of J. Bell, N. Bruin and M. Coons for completely multiplicative functions. Further, assuming GRH, the methods allow us to replace completely multiplicative functions with multiplicative functions.

## 1. INTRODUCTION

Automatic sequences play important role in computer science and number theory. For a detailed account of the theory and applications we refer the reader to the classical monograph [2]. One of the applications of such sequences in number theory stems from a celebrated theorem of Christol [5] (also cf. [6]), which asserts that in order to show the transcendence of the power series  $\sum_{n \geq 1} f(n)z^n$  it is enough to establish that the function  $f : \mathbb{N} \rightarrow \mathbb{C}$  is *not* automatic. In this note, rather than working within the general set up, we confine ourselves to functions taking their values in  $\mathbb{C}$ . There are several equivalent definitions of automatic (or more precisely,  $q$ -automatic) sequences. It will be convenient for us to use the following one.

**Definition 1.1.** The sequence  $f : \mathbb{N} \rightarrow \mathbb{C}$  is called  $q$ -automatic if its  $q$ -kernel defined as a set of subsequences

$$K_q(f) = \left\{ \{f(q^i n + r)\}_{n \geq 0} \mid i \geq 1, 0 \leq r \leq q^i - 1 \right\}$$

is finite.

We remark that any  $q$ -automatic sequence takes only finitely many values, since it is a function on the states of a finite automation. A function  $f : \mathbb{N} \rightarrow \mathbb{C}$  is called completely multiplicative if  $f(mn) = f(m)f(n)$  for all  $m, n \in \mathbb{N}$ . The question of which multiplicative functions are  $q$ -automatic attracted considerable attention of several authors including [15], [14], [3], [13], [12] and [1]. In particular, the following conjecture was made in [3].

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**Conjecture 1.2** (Bell-Bruin-Coons). For any multiplicative  $q$ -automatic function  $f : \mathbb{N} \rightarrow \mathbb{C}$  there exists an eventually periodic function  $g : \mathbb{N} \rightarrow \mathbb{C}$ , such that  $f(p) = g(p)$  for all primes  $p$ .

This conjecture is still open in general, although some progress has been made when  $f$  is assumed to be completely multiplicative. In particular, Schlage-Puchta [14] showed that a completely multiplicative  $q$ -automatic sequence which does not vanish is almost periodic. Hu [9] improved on that result by showing that the same conclusion holds under a slightly weaker hypothesis. Our first result confirms a strong form of Conjecture 1.2 when  $f$  is additionally assumed to be *completely* multiplicative function.

**Theorem 1.3.** *Let  $q \geq 2$  and let  $f : \mathbb{N} \rightarrow \mathbb{C}$  be a completely multiplicative  $q$ -automatic sequence. Then there exists a Dirichlet character  $\chi$  of conductor  $Q$  such that either  $f(n) = \chi(n)$ , for all  $n \in \mathbb{N}$  such that  $(n, Q) = 1$ , or  $f(p) = 0$  for all sufficiently large  $p$ .*

Theorem 1.3 also confirms the first part of Conjecture 4.2 in [1]. We remark that a similar result has been very recently obtained independently by Li [12] using combinatorial methods relying on the techniques developed in the theory of automatic sequences. Our proof is shorter and builds upon two deep number theoretic results. Further, assuming the generalized Riemann hypothesis (which in particular implies a strong form of the Artin primitive root conjecture for primes in progressions) our method can be adapted to show the full conjecture (i.e., the assumption on *complete* multiplicativity can be removed).

*Note added in July 2019.* While the present article was under submission the authors [10], and independently J. Konieczny [11], found rather different ways to settle the Conjecture 1.2 unconditionally. Both approaches use a modification of our Proposition 2.1 to deal with the case where  $f$  vanishes on finitely many primes, but then deviate significantly from the approach taken here. In [11] the author proceeds by using the structural theory of automatic sequences, in particular the notion of “arid sets”, to deal with the remaining cases. In our work [10], instead of using the connection to the Artin conjecture (as in the present paper), we establish certain properties of the divisors of the sequences  $\{aq^n + b\}_{n \geq 1}$ , for  $a, b, q \in \mathbb{N}$ , to finish up the proof.

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## 2. PROOF OF THE MAIN RESULT

We begin with a simple albeit important remark. Since  $f$  is  $q$ -automatic and completely multiplicative the image of  $f : \mathbb{N} \rightarrow \mathbb{C}$  is finite and therefore for any prime  $p$ ,  $f(p) = 0$  or  $f(p)$  is a root of unity.

**Proposition 2.1.** *Let  $f : \mathbb{N} \rightarrow \mathbb{C}$  be a  $q$ -automatic completely multiplicative function and let  $\mathcal{M}_0 = \{p \mid f(p) = 0\}$ . If  $|\mathcal{M}_0| < \infty$ , then there exists a Dirichlet character  $\chi : \mathbb{N} \rightarrow \mathbb{C}$  such that  $f(p) = \chi(p)$  for all  $p \notin \mathcal{M}_0$ .*

*Proof.* Since  $f$  is  $q$ -automatic there exist positive integers  $i_1 \neq i_2$ , such that  $f(q^{i_1}n + 1) = f(q^{i_2}n + 1)$  for all  $n \geq 1$ . If  $n = m \prod_{p \in \mathcal{M}_0} p$ , then

$$\frac{f(q^{i_1}m \prod_{p \in \mathcal{M}_0} p + 1)}{f(q^{i_2}m \prod_{p \in \mathcal{M}_0} p + 1)} = 1 \neq 0,$$

for all  $m \geq 1$ . The conclusion now immediately follows from Theorem 2 of [7].  $\square$

Suppose that  $|\mathcal{M}_0| = \infty$ . We are going to show that in this case  $f(p) = 0$  for all sufficiently large primes  $p$ . Replacing  $f$  by  $|f|$ , which is also  $q$ -automatic, it is enough to prove the claim for the binary valued  $f : \mathbb{N} \rightarrow \{0, 1\}$ .

Let  $\overline{1, n} = [1, n] \cap \mathbb{Z}$ . Since  $f$  is  $q$ -automatic, the  $q$ -kernel of  $f$  is finite and therefore there exists  $k_0 = k_0(f)$ , such that for all  $i \geq 1$  and  $0 \leq r \leq q^i - 1$ , the equalities  $f(q^i n + r) = 0$  for  $n \in \overline{1, k_0}$  imply  $f(q^i n + r) = 0$  for all  $n \geq 1$ .

**Lemma 2.2.** *Suppose that  $|\mathcal{M}_0| = \infty$ . For  $k_0$  as above and for any  $p_1, p_2, \dots, p_{k_0} \in \mathcal{M}_0$ , such that  $p_j > \max\{q, k_0\}$  for all  $j \in \overline{1, k_0}$ , there exists  $r = r(q, p_1, \dots, p_{k_0})$  such that  $(r, qp_i) = 1$  for all  $i \in \overline{1, k_0}$  and  $f(n \prod_{i \leq k_0} p_i + r) = 0$  for all  $n \geq 1$ . We may further assume that  $r \equiv 3 \pmod{16}$ , and  $(r - 1, \prod_{i \leq k_0} p_i) = 1$ .*

*Proof.* For an integer parameter  $A \geq \log_q p_{k_0}$ , which we shall choose later, by the Chinese remainder theorem there exists  $r_A$  such that  $(r_A, q) = 1$  and  $r_A \equiv -sq^{2A} \pmod{p_s}$  for all  $s \in \overline{1, k_0}$ . Since  $p_s | q^{2A}s + r_A$  we have  $f(q^{2A}n + r_A) = 0$  for all  $n \in \overline{1, k_0}$ . The latter implies that  $f(q^{2A}n + r_A) = 0$  for all  $n \geq 1$ . We claim that  $f(r_A) = 0$ . Indeed, if this is not the case we choose a prime  $p$ , such that  $f(p) = 1$  and consider  $m = p^{\varphi(q^{2A})}r_A$ , where  $\varphi(n)$  denotes the usual Euler's totient function. Clearly  $m \equiv r_A \pmod{q^{2A}}$  and consequently  $0 = f(m) = (f(p))^{\varphi(q^{2A})}f(r_A) = 1$ , a contradiction. Note, that the same argument works for  $n \prod_{i \leq k_0} p_i + r_A$  in place of  $r_A$  and therefore we conclude that  $f(n \prod_{i \leq k_0} p_i + r_A) = 0$  for all  $n \geq 1$ . Setting  $r = r_A$  finishes the proof.  $\square$

Next, without loss of generality we may assume that there exist three sufficiently large primes  $t, t', t'' > \max(q, k_0)$  such that  $f(t) = f(t') = f(t'') = 1$ , since otherwise we are done. We write  $\prod_{a < p \leq b} p$  to denote the product of all primes  $a < p \leq b$ . We will require the following consequence of a result due to Heath-Brown [8].

**Lemma 2.3.** *Given distinct primes  $t, t', t'' > \max(q, k_0)$  and  $r = r(q, p_1, \dots, p_{k_0})$  as in Lemma 2.2, there exists infinitely many primes  $q_i \equiv r \pmod{16 \prod_{i \leq k_0} p_i}$  such that at least one of  $t, t', t''$  (say  $t$ ) is a primitive root modulo  $q_i$ . Moreover, by passing to a subsequence we may assume that for such primes  $(q_i - 1, q_j - 1) = 2$  for  $i \neq j$ , and for each  $l \in \overline{1, k_0}$  we have  $(l/q_i) = 1$  for all  $i \in \overline{1, k_0}$ .*

*Proof.* Let  $v = 16(\prod_{i \leq k_0} p_i) \prod_{2 < p \leq k_0} p$  and chose  $u$  such that  $u \equiv 3 \pmod{16}$  and  $u \equiv r \pmod{\prod_{i \leq k_0} p_i}$ , with  $r$  as in Lemma 2.2. Moreover, by quadratic reciprocity we may further select  $u \pmod{\prod_{2 < p \leq k_0} p}$  such that  $(u/p) = 1$  for all primes  $p \leq k_0$ , and  $(u-1, \prod_{2 < p \leq k_0} p) = 1$ . In particular, we have  $(-3/p) = -1$  for any prime  $p \equiv u \pmod{v}$ . Applying Lemma 3 of [8], with  $u, v$  as above and  $k = 1$  (and  $K = 2^k = 2$ ) there exists  $\alpha \in (1/4, 1/2]$  and  $\delta > 0$  such that

$$|\{p \leq x : p \equiv u \pmod{v}, (p-1)/K = P_2(\alpha, \delta)\}| \gg x/(\log x)^2,$$

with the implied constant possibly depending on  $\alpha$ , with  $P_2(\alpha, \delta)$  denoting the union of the set of primes, together with the set of almost primes  $n = t_1 t_2$  with  $t_1 < t_2$  both primes, and  $t_1 \in [n^\alpha, n^{1/2-\delta}]$ . Heath-Brown's argument then shows that at least one of  $t, t', t''$  is a primitive root for infinitely many primes  $p \equiv u \pmod{v}$ . Whether the primes  $q_i$  produced have the properties that  $(q_i - 1)/2$  is prime, or that  $(q_i - 1)/2 = t_1 t_2$ , we may pass to an infinite

subsequence of primes  $q_1 < q_2 < \dots$  (satisfying  $q_1 > q$ ) so that  $(q_i - 1, q_j - 1) = 2$  for  $i \neq j$  (for the latter case of almost primes, note that both  $t_1$  and  $t_2$  are growing.)  $\square$

**Proposition 2.4.** *Suppose that  $|\mathcal{M}_0| = \infty$ . Then  $f(p) = 0$  for all sufficiently large primes  $p$ .*

*Proof.* By Lemma 2.3, we may select prime  $t$  with  $f(t) = 1$ , which is a primitive root modulo infinitely primes  $q_1 < q_2 < \dots < q_{k_0}$  (satisfying  $q_1 > \max(k_0, q)$ ) such that  $q_i \equiv r \pmod{16 \prod_{j \leq k_0} p_{k_0}}$  and consequently  $f(q_i) = 0$ . From the proof of Lemma 2.2 it follows that there exists  $r_A$ , such that  $f(n \prod_{i \leq k_0} q_i + r_A) = 0$  for all  $n \geq 1$ . Since  $t$  is a primitive root modulo  $q_j$  for  $j \in \overline{1, k_0}$ , there exists  $\gamma_j$  such that  $t^{\gamma_j} \equiv r_A \pmod{q_j}$  for  $j \in \overline{1, k_0}$ . By the construction and Lemma 2.3 we have  $(r_A/q_i) = (-iq^{2A}/q_i) = -1$  and thus all  $\gamma_i$  have the same parity. Consequently, by the Chinese remainder theorem we can choose  $\gamma \in \mathbb{N}$ , such that  $\gamma \equiv \gamma_j \pmod{q_j - 1}$  for all  $j \in \overline{1, k_0}$ . For  $\gamma$  defined this way we have  $t^\gamma \equiv r_A \pmod{\prod_{j \leq k_0} q_j}$ . Hence,  $f(t^\gamma)$  must be zero. On the other hand  $f(t^\gamma) = f(t)^\gamma = 1$ , and this contradiction finishes the proof.  $\square$

Combining Proposition 2.1 and Proposition 2.4 yields the conclusion of Theorem 1.3

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