

ME 254 Computational Fluid Dynamics
Homework 3

Due on 02/19/19 at 11:59 pm (through Catcourses)
Maximum points: 100

1. (**55 points**) Write a code to solve the 1-D heat equation (given below) using an explicit forward-time centered-space finite difference scheme.

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \quad (1)$$

The computational domain extends from $x = 0$ to $x = 1$. The boundary conditions are given by $T(0, t) = 700 \text{ K}$ and $T(1, t) = 300 \text{ K}$. The initial condition can be assumed to be $T(x, 0) = 500 \text{ K}$. Assume $\alpha = 1$ and $\Delta x = 0.05$

- (a) Use $\alpha\Delta t/(\Delta x)^2 = 0.5$ and plot the temperature profiles (on the same plot to show evolution) at $t = \Delta t, 2\Delta t, 5\Delta t, 10\Delta t, 20\Delta t, 50\Delta t, 100\Delta t$, and $150\Delta t$.
 - (b) Does the temperature profile reach a steady state.
 - (c) Now increase $\alpha\Delta t/(\Delta x)^2$ to 1.5 and re-run your code. Compare your results with those obtained above.
2. (**15 points**) The DuFort-Frankel method for solving the heat equation requires solution of the difference equation

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} = \frac{\alpha}{(\Delta x)^2} (u_{j+1}^n - u_j^{n+1} - u_j^{n-1} + u_{j-1}^n) \quad (2)$$

Determine the stability restrictions for this scheme.

3. (**15 points**) An implicit scheme for solving the heat equation is given by

$$u_j^{n+1} = u_j^n + \frac{\alpha\Delta t}{(\Delta x)^2} \left[\frac{1}{3}(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}) + \frac{2}{3}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) \right] \quad (3)$$

Determine the stability restrictions for this scheme.

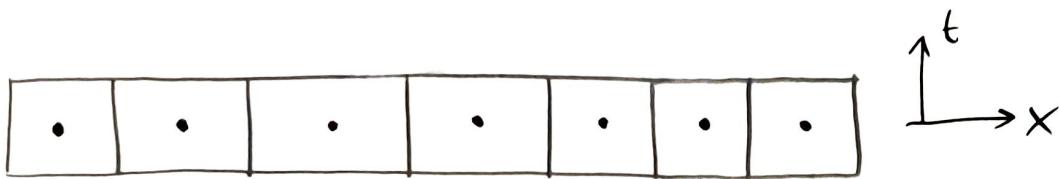
4. (**15 points**) The leap frog method for solving the 1-D wave equation is given by

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} + c \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0 \quad (4)$$

Determine the stability restrictions for this scheme.

Write a code to solve 1-D heat equation (given below)
 using an explicit / forward - time / centered space / finite difference scheme

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$



forward - time

$$\left. \frac{\partial T}{\partial t} \right|_{i,j} = \frac{T_x^{N+1} - T_x^N}{\Delta t}$$

centered scheme

$$\left. \frac{\partial^2 T}{\partial x^2} \right|_{i,j} = \frac{T_{x+1}^N - 2T_x^N + T_{x-1}^N}{(\Delta x)^2}$$

Total equation

$$\frac{T_x^{N+1} - T_x^N}{\Delta t} = \alpha \frac{[T_{x+1}^N - 2T_x^N + T_{x-1}^N]}{(\Delta x)^2}$$

B.C's

$$x=0 \rightarrow x=1 \quad \alpha = 0.1$$

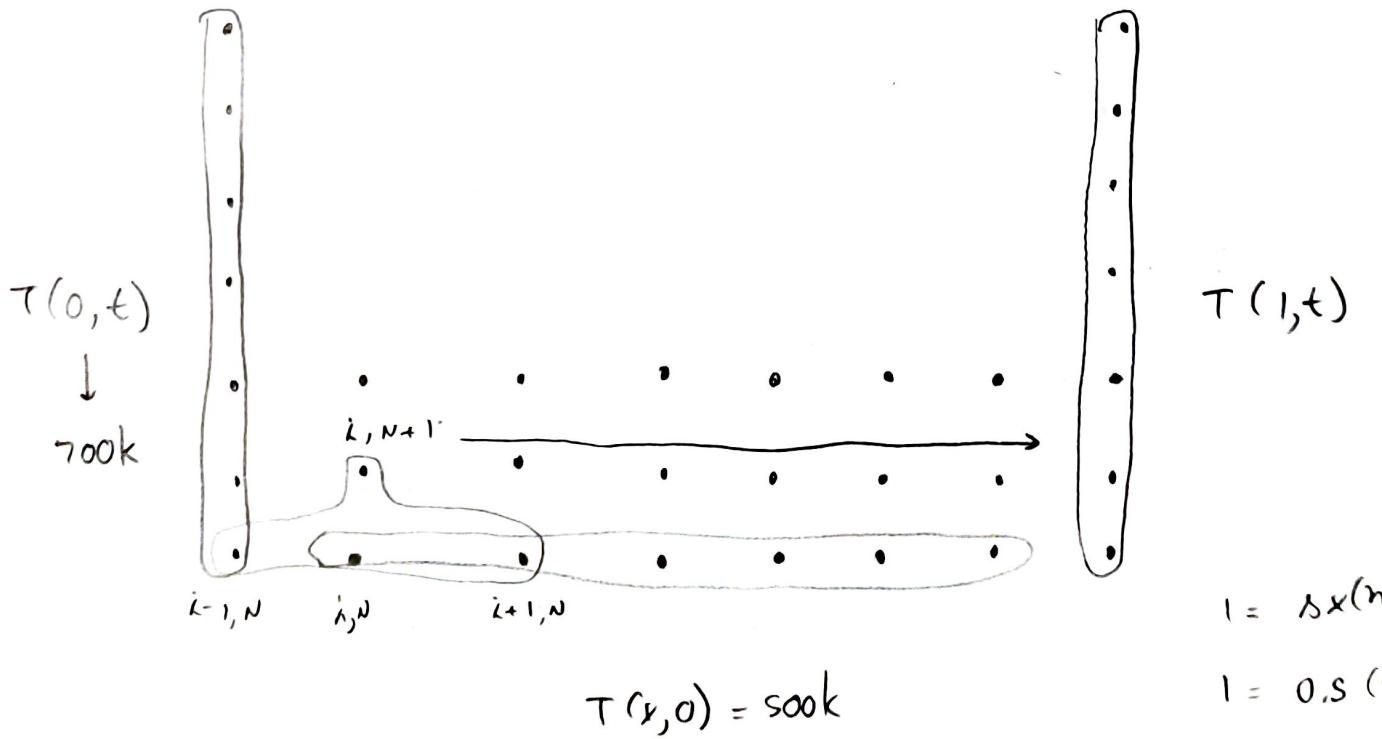
$$T(0,t) = 700k \quad \Delta x = 0.05$$

$$T(1,t) = 300k$$

$$T(x,0) = 500k$$

$$T_i^{N+1} - T_i^N = \left(\frac{\alpha \Delta t}{(\Delta x)^2} \right) [T_{i-1}^N - 2T_i^N + T_{i+1}^N]$$

$$T_i^{N+1} = \left(\frac{\alpha \Delta t}{(\Delta x)^2} \right) [T_{i+1}^N - 2T_i^N + T_{i-1}^N] + T_i^N$$



$$\textcircled{1} \quad \frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{1}{2} \gamma \left(u_{j+1}^n - u_{j+1}^{n+1} - u_{j-1}^{n+1} + u_{j-1}^n \right) \quad \textcircled{2}$$

$$\gamma = \frac{\alpha \Delta t}{(\Delta x)^2}$$

$$\textcircled{1} \quad \frac{b_m(t + \Delta t) e^{ik_m x}}{b_m(t) e^{ik_m x}} = G$$

$$\textcircled{2} \quad \frac{b_m(t - \Delta t) e^{ik_m x}}{b_m(t) e^{ik_m x}} = \frac{1}{G}$$

$$\frac{e^{at - a\Delta t}}{e^{at}} = e^{-a\Delta t} = \frac{1}{e^{a\Delta t}} = \frac{1}{G}$$

$$\textcircled{3} \quad \frac{b_m(t) e^{ik_m(x + \Delta x)}}{b_m(t) e^{ik_m x}} = e^{ik_m \Delta x}$$

$$\textcircled{4} \quad \frac{b_m(t + \Delta t) e^{ik_m x}}{b_m(t) e^{ik_m x}} = G$$

$$\textcircled{5} \quad \frac{1}{G}$$

$$\textcircled{6} \quad \frac{b_m(t) e^{ik_m(x - \Delta x)}}{b_m(t) e^{ik_m x}} = e^{-ik_m \Delta x}$$

$$G - \frac{1}{G} = 2\gamma \left(e^{ik_m \Delta x} - G - \frac{1}{G} - e^{-ik_m \Delta x} \right)$$

$$e^{ik_m \Delta x} + e^{-ik_m \Delta x} = e^{i\beta} + e^{-i\beta} = \cos \beta + i \sin \beta + \cos \beta - i \sin \beta = 2 \cos \beta$$

$$G - \frac{1}{G} = 2\gamma \left(2 \cos \beta - G - \frac{1}{G} \right)$$

Solving for G this gives...

$$G = \frac{2\gamma \cos \beta \pm \sqrt{1 - 2\gamma^2 + 2\gamma^2 \cos[2\beta]}}{1 + 2\gamma}$$

$$\rightarrow 1 - 2\gamma^2 (1 - \cos^2 \beta + \sin^2 \beta)$$

$$1 - 2\gamma^2 (2 \sin^2 \beta) \rightarrow 1 - 4\gamma^2 \sin^2 \beta$$

$$G = \frac{2\gamma \cos \beta \pm \sqrt{1 - 4\gamma^2 \sin^2 \beta}}{1 + 2\gamma}$$

$$\left| \frac{2\gamma \cos \beta + \sqrt{1 - 4\gamma^2 \sin^2 \beta}}{1 + 2\gamma} \right| \geq 1$$

$$\left(\left(\frac{2\gamma \cos \beta + \sqrt{1 - 4\gamma^2 \sin^2 \beta}}{1 + 2\gamma} \right)^2 \right)^{1/2} \geq 1$$

$$\frac{2\gamma \cos \beta + \sqrt{1 - 4\gamma^2 \sin^2 \beta}}{1 + 2\gamma} \geq 1$$

$$2\gamma \cos \beta + \sqrt{1 - 4\gamma^2 \sin^2 \beta} \geq 1 + 2\gamma$$

$$\gamma \leq \frac{-1 + \cos \beta - \sin \beta}{1 - 2\cos \beta + \cos^2 \beta}$$

$[1, -1]$

$$\gamma \leq \frac{-1 + \cos \beta - \sin \beta}{(-1 + \cos \beta)^2}$$

at $\cos(x) = 1$

$$\gamma \leq \frac{(-1 + 1 - 0)}{(-1 + 1)} \rightarrow \text{Nope}$$

at $\cos(\kappa) \rightarrow -1$

$$\gamma \leq \frac{-1 - 1 - 0}{-1 - 1}$$

$$\left[\gamma \leq \frac{-2}{-2} = 1 \right] \quad \text{unconditionally stable}$$

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{\omega}{(\Delta x)^2} \left[\frac{1}{3} (U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}) + \frac{2}{3} (U_{j+1}^n - 2U_j^n + U_{j-1}^n) \right]$$

LHS

$$\frac{b_m(t+\Delta t) e^{ik_m x} - b_m(t) e^{ik_m x}}{\Delta t}$$

$$\left[\frac{b_m(t+\Delta t) e^{ik_m x}}{b_m(t) e^{ik_m x}} - \frac{b_m(t) e^{ik_m x}}{b_m(t) e^{ik_m x}} \right] \frac{1}{\Delta t}$$

$$\frac{1}{\Delta t} [6 - 1]$$

RHS

(A)

$$\frac{2}{(\Delta x)^2} \left[\frac{1}{3} (b_m(t+\Delta t) e^{ik_m(x+\Delta x)} - 2b_m(t+\Delta t) e^{ik_m x} + b_m(t+\Delta t) e^{ik_m(x-\Delta x)}) \right]$$

$$+ \frac{2}{3} \left(b_m(t) e^{ik_m(x+\Delta x)} - 2b_m(t) e^{ik_m x} + b_m(t) e^{ik_m(x-\Delta x)} \right)$$

(B)

(A)

$$\frac{\alpha}{(\Delta x)^2} \left[\left(\frac{1}{3} \right) \left(\frac{6_m(t+\Delta t) e^{ik_m(x+\Delta x)}}{6_m(t) e^{ik_m x}} - \frac{26_m(t) e^{ik_m x}}{6_m(t) e^{ik_m x}} + \frac{6_m(t-\Delta t) e^{ik_m(x-\Delta x)}}{6_m(t) e^{ik_m x}} \right) \right]$$

$$\frac{\alpha}{(\Delta x)^2} \left[\left(\frac{1}{3} \right) \left(6 e^{ik_m \Delta x} - 26 + 6 e^{-ik_m \Delta x} \right) \right]$$

(B)

$$\frac{\alpha}{(\Delta x)^2} \left[\left(\frac{2}{3} \right) \left(\frac{6_m(t) e^{ik_m(x-\Delta x)}}{6_m(t) e^{ik_m x}} - \frac{26_m(t) e^{ik_m x}}{6_m(t) e^{ik_m x}} + \frac{6_m(t) e^{ik_m(x-\Delta x)}}{6_m(t) e^{ik_m(x)}} \right) \right]$$

$$\frac{\alpha}{(\Delta x)^2} \left[\left(\frac{2}{3} \right) \left(e^{ik_m \Delta x} - 2 + e^{-ik_m \Delta x} \right) \right]$$

$$G - 1 = \frac{\alpha \Delta t}{(\Delta x)^2} \left[\left(\frac{1}{3} \right) \left(6 e^{ik_m \Delta x} - 26 + 6 e^{-ik_m \Delta x} \right) + \left(\frac{2}{3} \right) \left(e^{ik_m \Delta x} - 2 + e^{-ik_m \Delta x} \right) \right]$$

(AA)

$$6(e^{i\beta} + e^{-i\beta}) - 26 = 6(\cos \beta + i \cancel{\sin \beta} + \cos \beta - i \cancel{\sin \beta}) - 26$$

$$= 6(2 \cos \beta) - 26$$

(BB)

$$(2 \cos \beta) - 2$$

$$\gamma = \frac{\alpha \Delta t}{(\Delta x)^2}$$

$$6-1 = \gamma \left[\left(\frac{1}{3}\right)(6(2\cos\beta) - 26) + \left(\frac{2}{3}\right)(2\cos\beta - 2) \right]$$

Solving for 6 gives...

$$6 \rightarrow \frac{3 - 4\gamma + 4\gamma \cos\beta}{3 + 2\gamma - 2\gamma \cos\beta}$$

$$\left| \frac{3 - 4\gamma + 4\gamma \cos\beta}{3 + 2\gamma - 2\gamma \cos\beta} \right| \leq 1$$

$$\begin{aligned} 3 - 4\gamma + 4\gamma \cos\beta &\leq 3 + 2\gamma - 2\gamma \cos\beta \\ -6\gamma &\leq -6\gamma \cos\beta \end{aligned}$$

$$1 \geq \cos\beta$$

$$\frac{3 - 4\gamma + 4\gamma \cos\beta}{3 + 2\gamma - 2\gamma \cos\beta} \geq -1$$

$$\frac{3 - 4\gamma + 4\gamma \cos\beta}{3 + 2\gamma - 2\gamma \cos\beta} \geq -1$$

$$3 - 4\gamma + 4\gamma \cos\beta \geq -3 - 2\gamma + 2\gamma \cos\beta$$

$$-6 \geq -2\gamma + 2\gamma \cos\beta + 4\gamma - 4\gamma \cos\beta$$

$$-6 \geq 2\gamma - 2\gamma \cos\beta$$

$$-6 \geq 2\gamma(1 - \cos\beta)$$

$$\frac{-6}{2(1 - \cos\beta)} \geq \gamma \quad \cos\beta \in [-1, 1]$$

$$\frac{-6}{2(1+1)} \geq \gamma$$

$$\frac{-6}{2} \rightarrow -3$$

$$\frac{-6}{2(1-0.5)}$$

$$\frac{-6}{2(-0.5)} = -5$$

$$\left[\frac{-6}{4} \geq \gamma \right]$$

$$\left[-\frac{3}{2} \geq \gamma \right]$$

$\left[\begin{array}{l} \text{conditionally} \\ \text{stable} \end{array} \right]$

$$\frac{u_j^{n+1} - u^{n-1}}{2\delta t} + \frac{c(u_{j+1}^n - u_{j-1}^n)}{2\delta x} = 0$$

$$\frac{u_j^{n+1} - u^{n-1}}{2\delta t} = \left(\frac{c}{2\delta x}\right)(u_{j-1}^n - u_{j+1}^n)$$

LHS

$$\left(\frac{b_m(t+\delta t) e^{ik_m x} - b_m(t-\delta t) e^{ik_m x}}{2\delta t} \right) \left(\frac{1}{b_m(t) e^{ik_m x}} \right)$$

$$\left(\frac{1}{2\delta t} \right) \left(\frac{b_m(t+\delta t) e^{ik_m x}}{b_m(t) e^{ik_m x}} - \frac{b_m(t-\delta t) e^{ik_m x}}{b_m(t) e^{ik_m x}} \right)$$

$$\left(\frac{1}{2\delta t} \right) \left(G - \frac{1}{G} \right)$$

RHS

$$\left(\frac{c}{2\delta x} \right) \left(\frac{b_m(t) e^{ik_m(x+\delta x)}}{b_m(t) e^{ik_m(x)}} - \frac{b_m(t) e^{ik_m(x-\delta x)}}{b_m(t) e^{ik_m(x)}} \right)$$

$$\left(\frac{c}{2\delta x}\right) (e^{ik_n \delta x} - e^{-ik_n \delta x})$$

$$\left(\frac{c}{2\delta x}\right) (e^{i\beta} - e^{-i\beta})$$

Ok... the equation is

$$\left(\frac{1}{2\delta t}\right) \left(G - \frac{1}{G}\right) = \frac{c}{2\delta x} (e^{i\beta} - e^{-i\beta})$$

$$G - \frac{1}{G} = \frac{c \delta t}{\delta x} (e^{i\beta} - e^{-i\beta})$$

$$G - \frac{1}{G} = \gamma (e^{i\beta} - e^{-i\beta})$$

$$\textcircled{A} \quad \cancel{\cos \beta + i \sin \beta} - (\cancel{\cos \beta} - i \sin \beta) \\ = 2i \sin \beta$$

$$G - \frac{1}{G} = \gamma (2i \sin \beta)$$

$$G = \frac{1}{2} \left[2i\gamma \sin \beta \pm \sqrt{4 - 4\gamma^2 \sin^2 \beta} \right]$$

$$G = \gamma \sin \beta i \pm \sqrt{1 - \gamma^2 \sin^2 \beta} \quad (\text{now, let's do } G)$$

$$\left| \gamma \sin \beta i + \sqrt{1 - \gamma^2 \sin^2 \beta} \right| \leq 1 \quad \textcircled{A}$$

$$\left| \gamma \sin \beta i - \sqrt{1 - \gamma^2 \sin^2 \beta} \right| \leq 1 \quad \textcircled{B}$$

(for A)

$$\left[|a+bi| = \sqrt{a^2+b^2} \right]$$

$$\sqrt{(\gamma \sin \beta)^2 + ((1 - \gamma^2 \sin^2 \beta)^{1/2})^2} \leq 1$$

$$\left(\gamma^2 \sin^2 \beta + (1 - \gamma^2 \sin^2 \beta) \right)^{1/2} \leq 1$$

$$(1)^{1/2} = 1$$

$$[1 \leq 1]$$

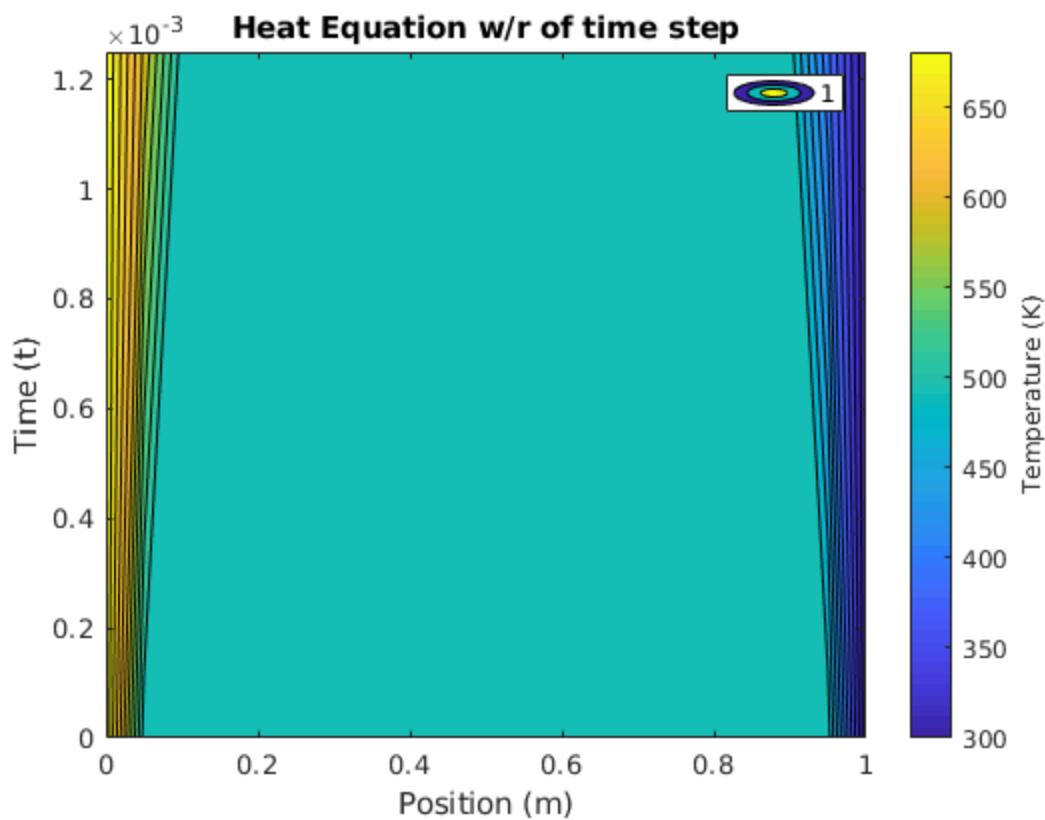
unconditionally stable

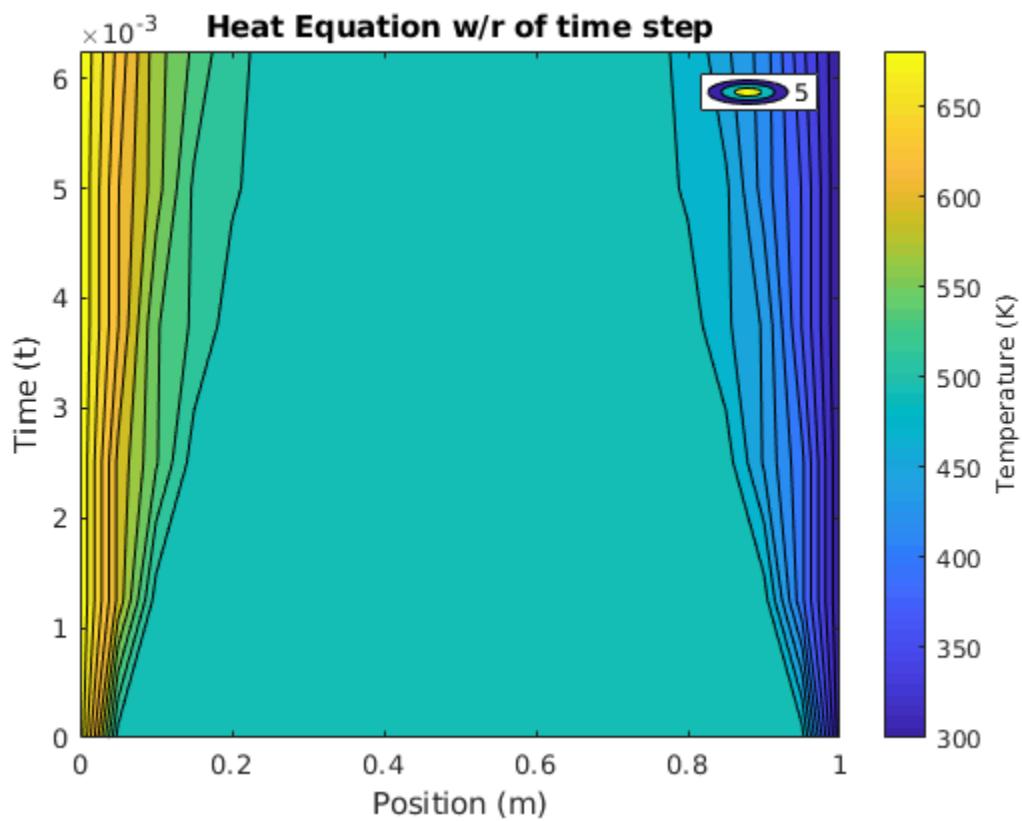
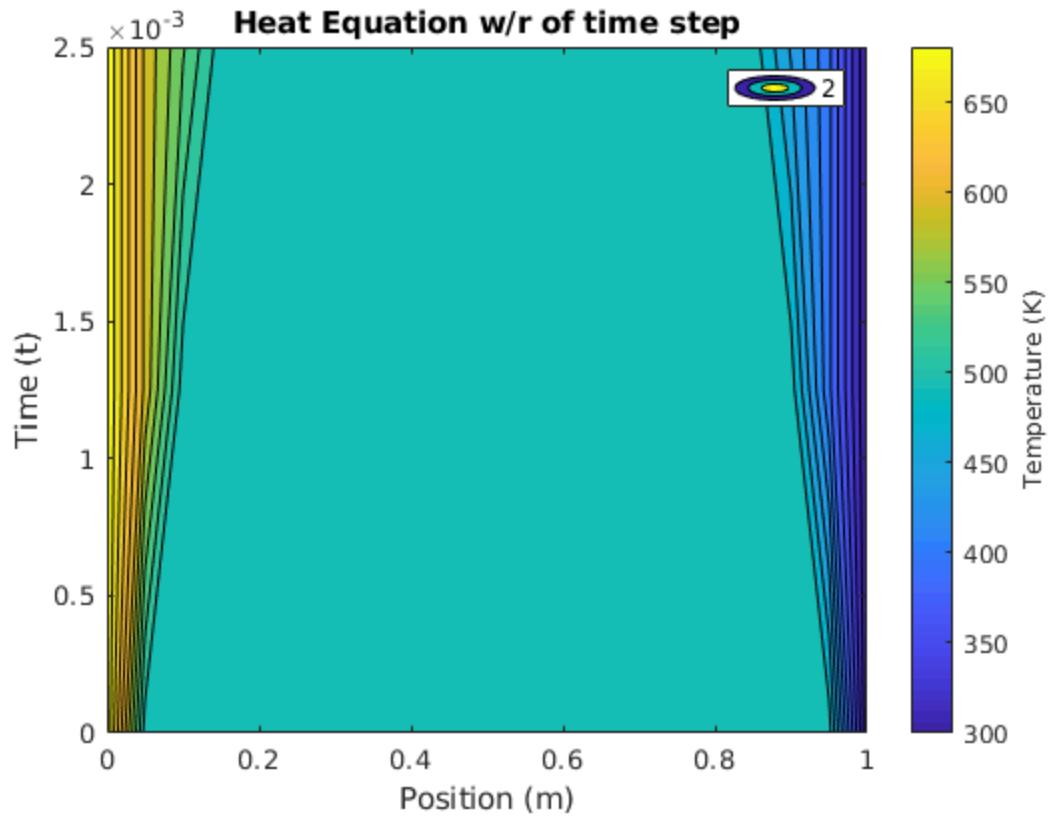
→ is going to be the same for B...

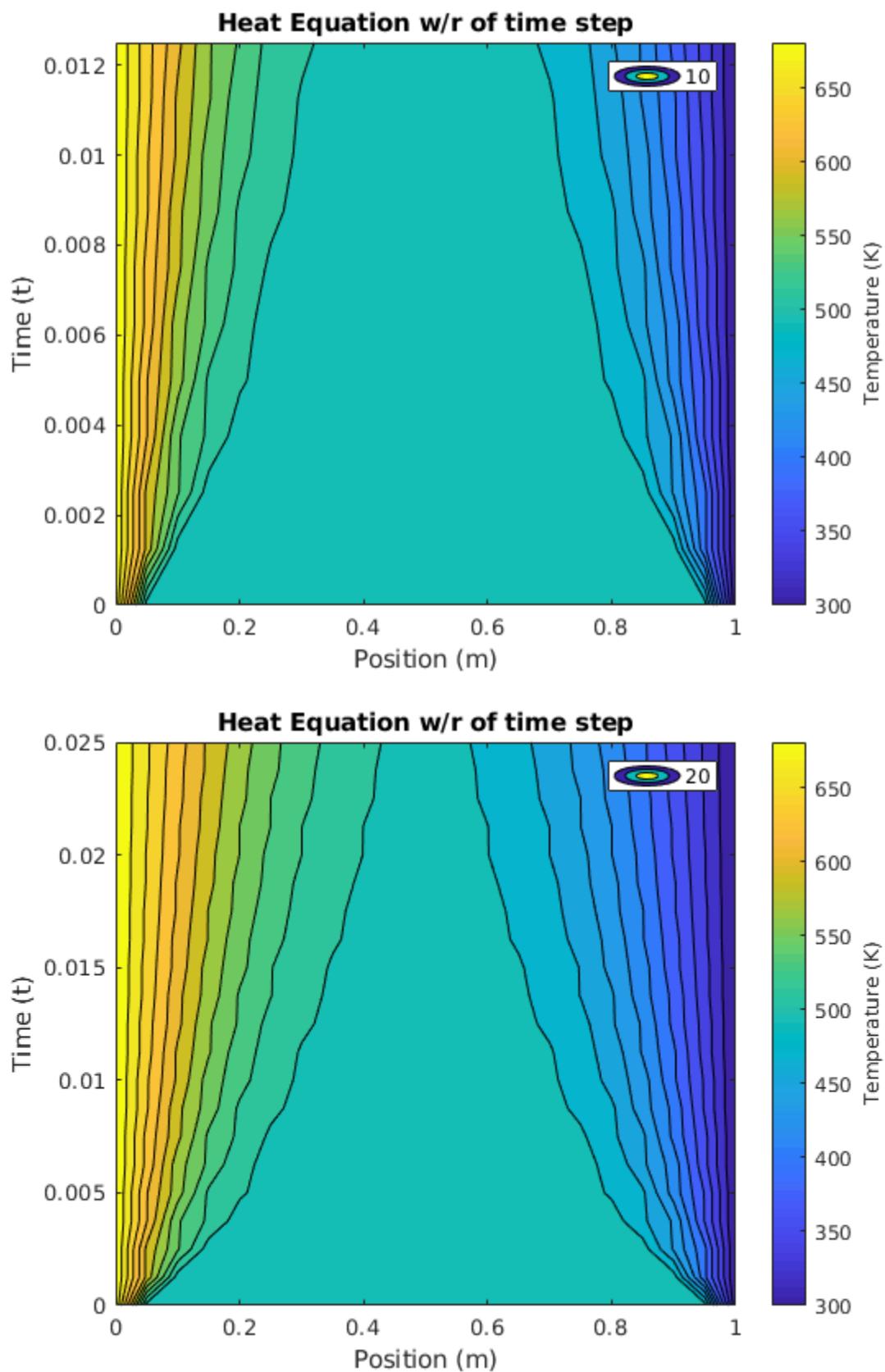
```
% The following script solves the 1-D heat equation using a
% explicit forward-time centered space finite difference scheme
% For this case I use a Y of 0.5
```

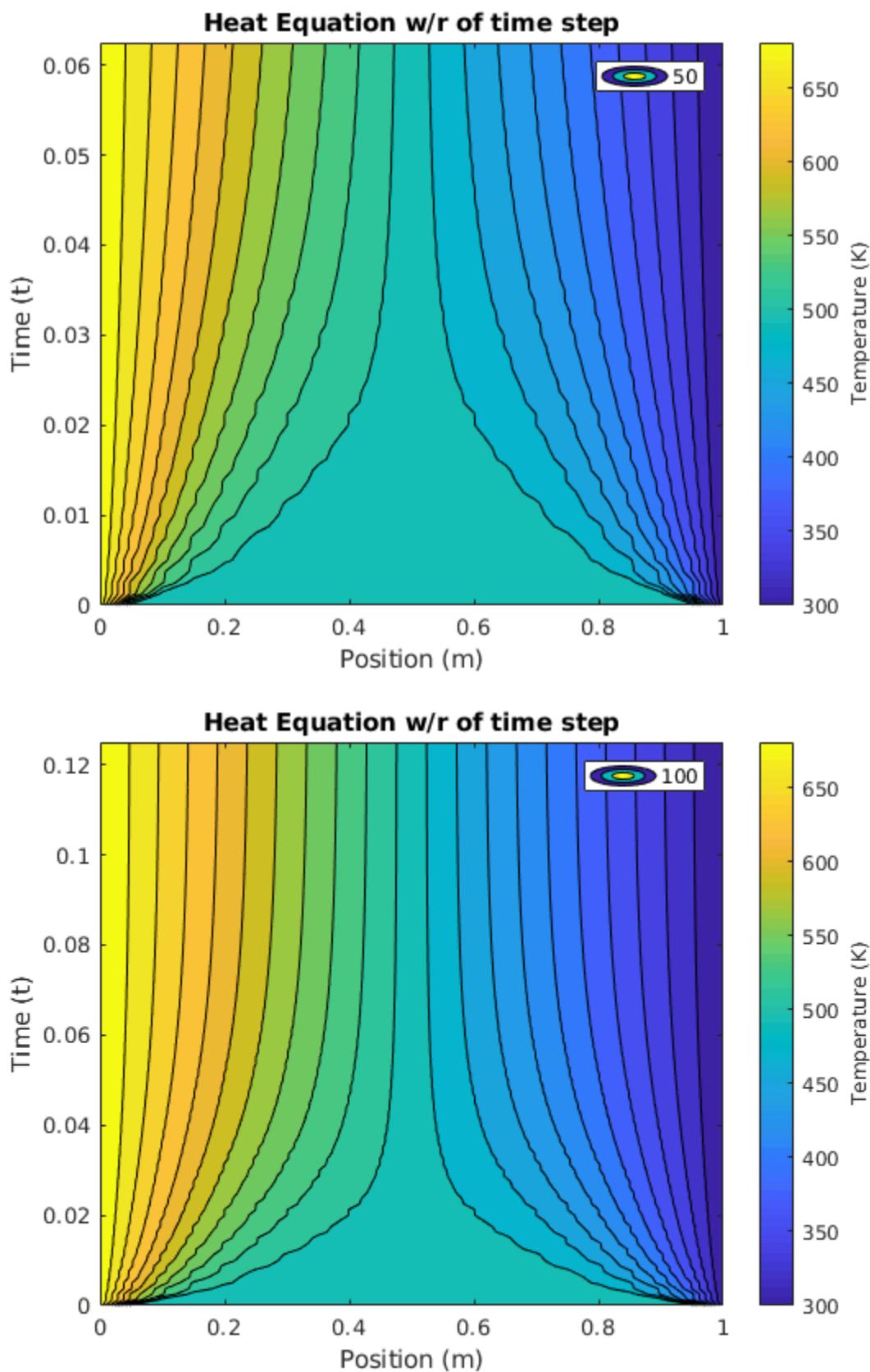
```
A = [1,2,5,10,20,50,100,150];
heatTransferPlots(0.5,A);
%temperaturePosition(0.5,A,length(A));
heatTransferPlots(1.5,A);
```

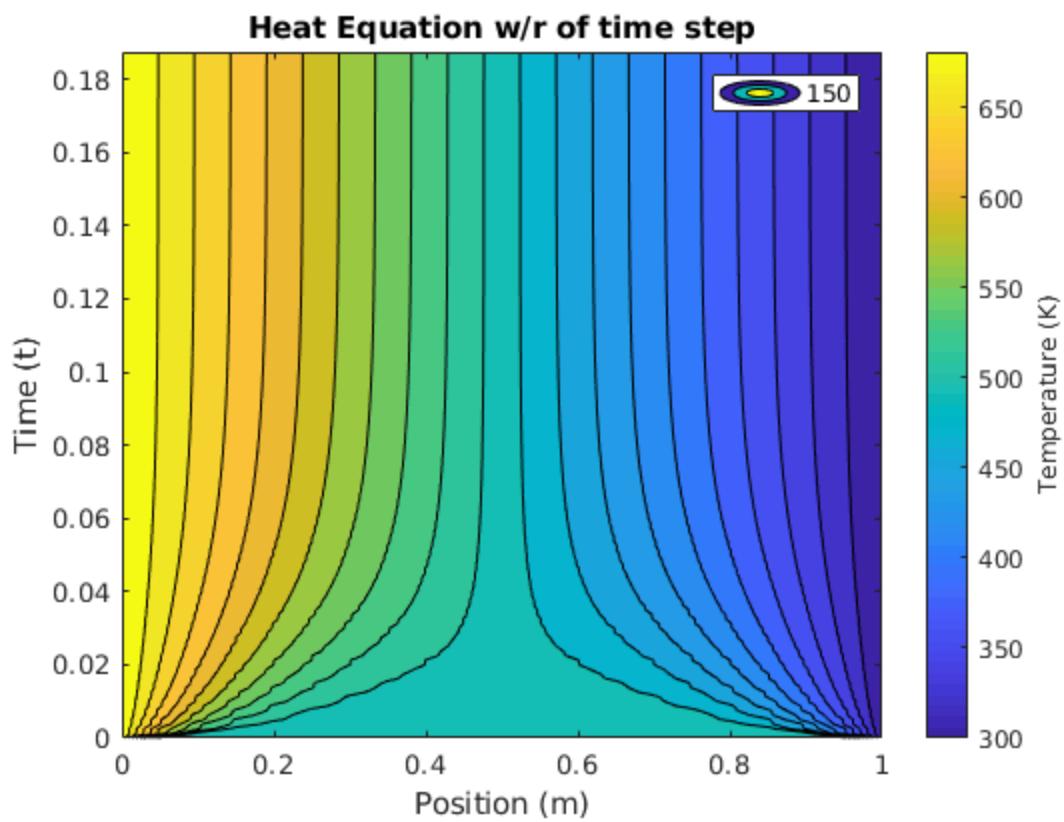
```
Warning: Ignoring extra legend entries.
```











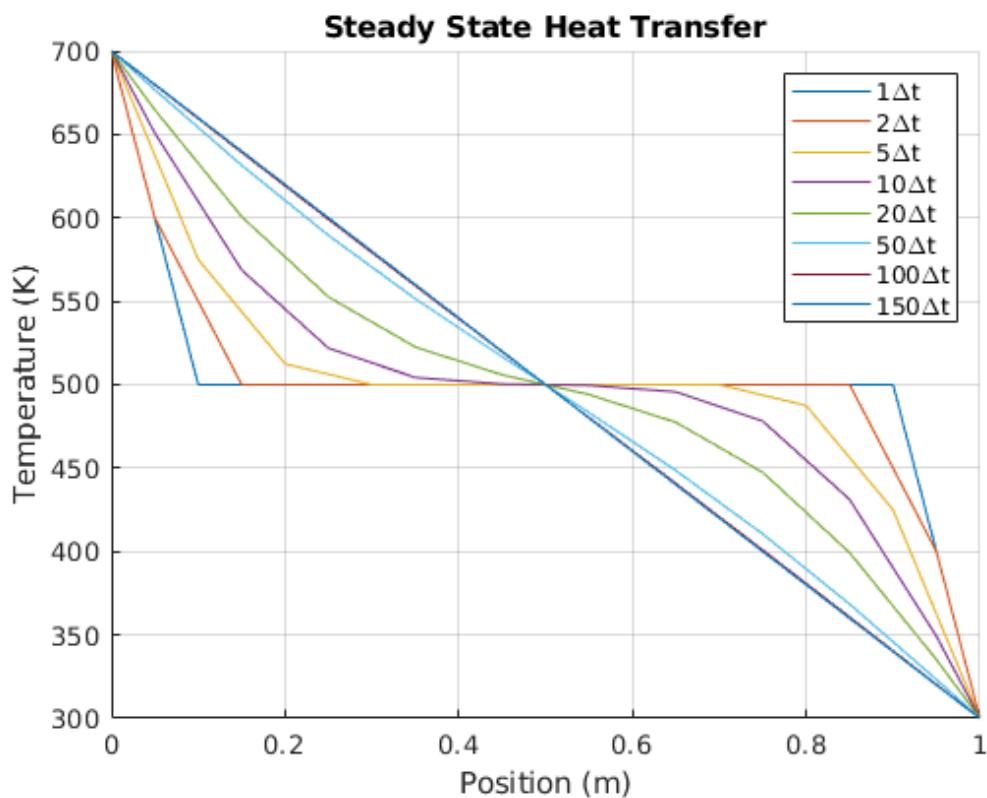
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```

% The following script solves the 1-D heat equation using a
% explicit foward-time centered space finite difference scheme
% For this case I use a Y of 0.5
% This time I only plot position and temperature
clc
clear all
close all

A = [1,2,5,10,20,50,100,150];
%heatTransferPlots(0.5,A);
temperaturePosition(0.5,A,length(A));
%heatTransferPlots(1.5,A);

```



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```

% The following script compare the value at a initial iteration
% with the following iteration in order to see whenever there is a
% convergence. In other words, if I see in the countour plot a value
% other than zero that means that more iterations are needed in order
% to reach convergence. Tested until 10 time steps in order to see
% results

clc;
clear all;

dx = 0.05;
Y = 1.5;
dt = Y*(dx)^2;

n_time = 10;

% this define columns needed
n_x = (1/dx)+1;
% this define rows needed
n_t = n_time+1;

T = zeros(n_t,n_x);
%this gives us our boundary conditions
boundaryx0 = 700*ones(n_t,1);
boundaryx1 = 300*ones(n_t,1);
boundaryt0 = 500*ones(1,n_x-2);

T(:,1) = boundaryx0;
T(:,n_x) = boundaryx1;
T(n_t,2:n_x-1) = boundaryt0;

if n_t>1
    for i=n_t-1:-1:1
        for w=2:1:n_x-1
            T(i,w) = (0.5)*(T(i+1,w+1)-2*T(i+1,w)+T(i+1,w-1))+T(i+1,w);
        end
    end
end

% for x values the following is done
valuedx = 0:dx:1;

% for time values the following is done
valuedt = n_time*dt:-dt:0;

matrix_x_direction = valuedx.*ones(n_t,n_x);
matrix_t_direction = (valuedt)' .*ones(n_t,n_x);

change_t_direction_d = matrix_t_direction(1:n_t-1,:);
matrix_x_direction_d = matrix_x_direction(1:n_t-1,:);
change_T_directionl = zeros((n_t-1),n_x);

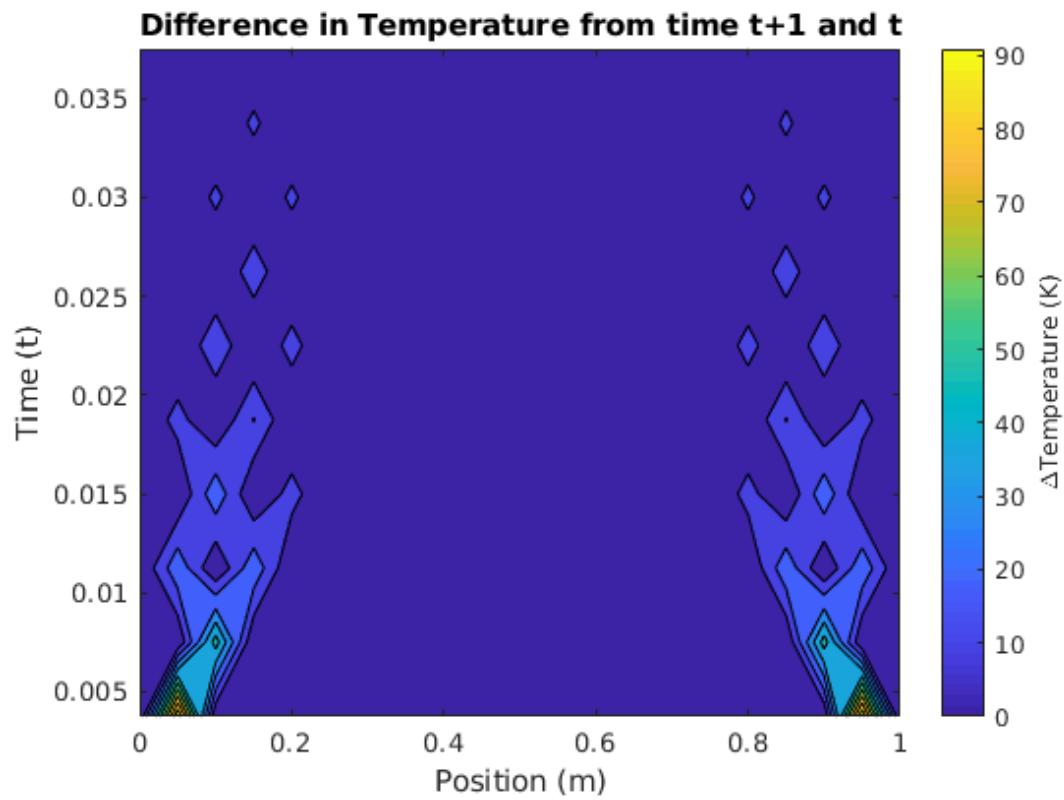
```

```

for i=1:l:n_t-1
    change_T_direction1(i,:) = abs(T(i,:)-T(i+1,:));
end

figure(1)
contourf(matrix_x_direction_d,change_t_direction_d,change_T_direction1,n_time)
c = contourbar;
c.Label.String = '\Delta Temperature (K)';
title('Difference in Temperature from time t+1 and t');
xlabel('Position (m)');
ylabel('Time (t)');

```



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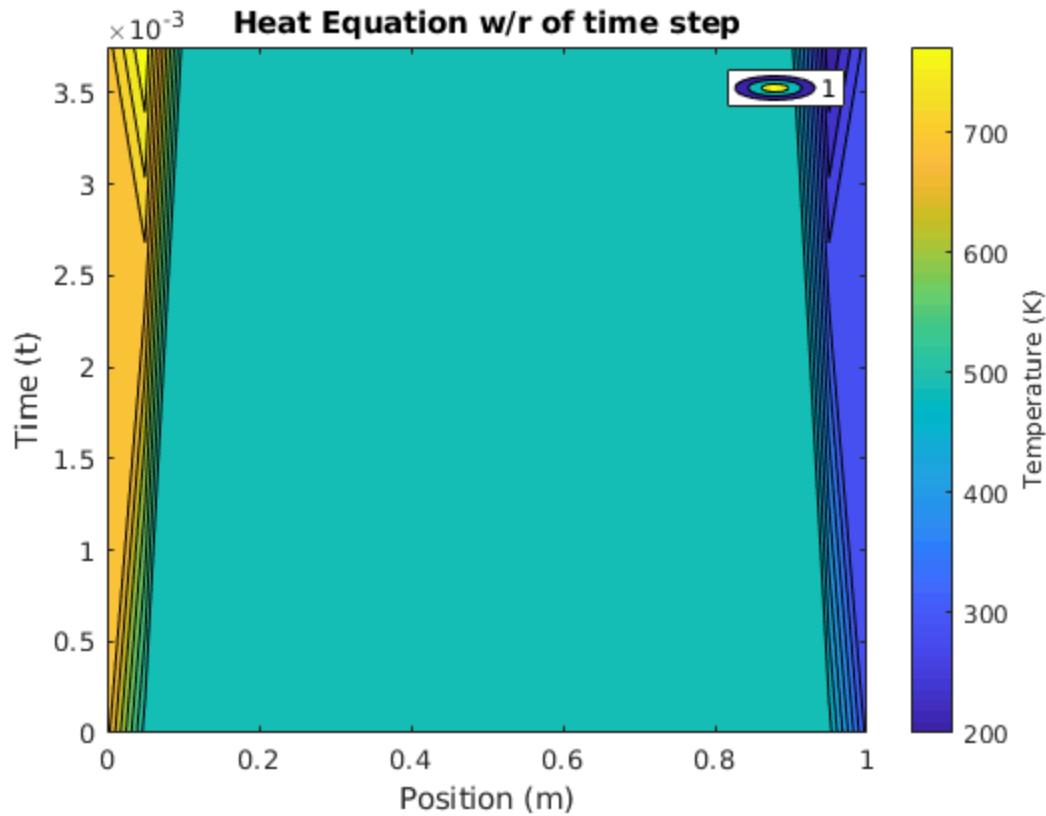
```

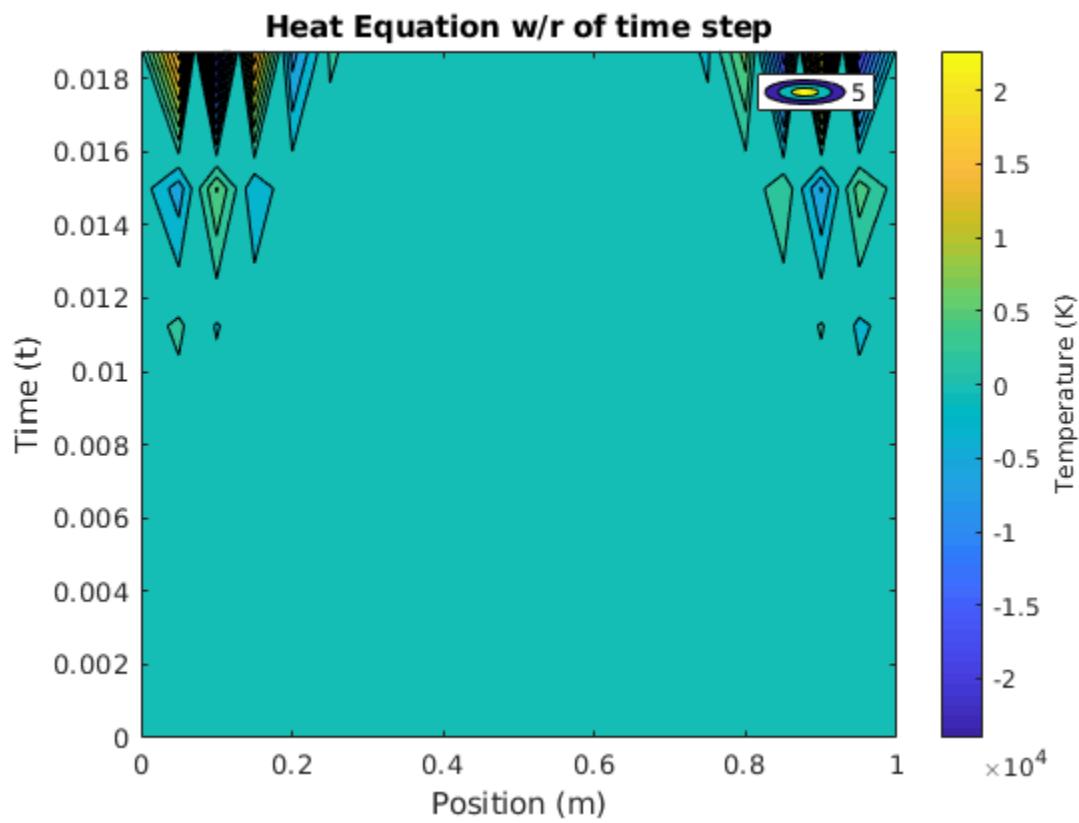
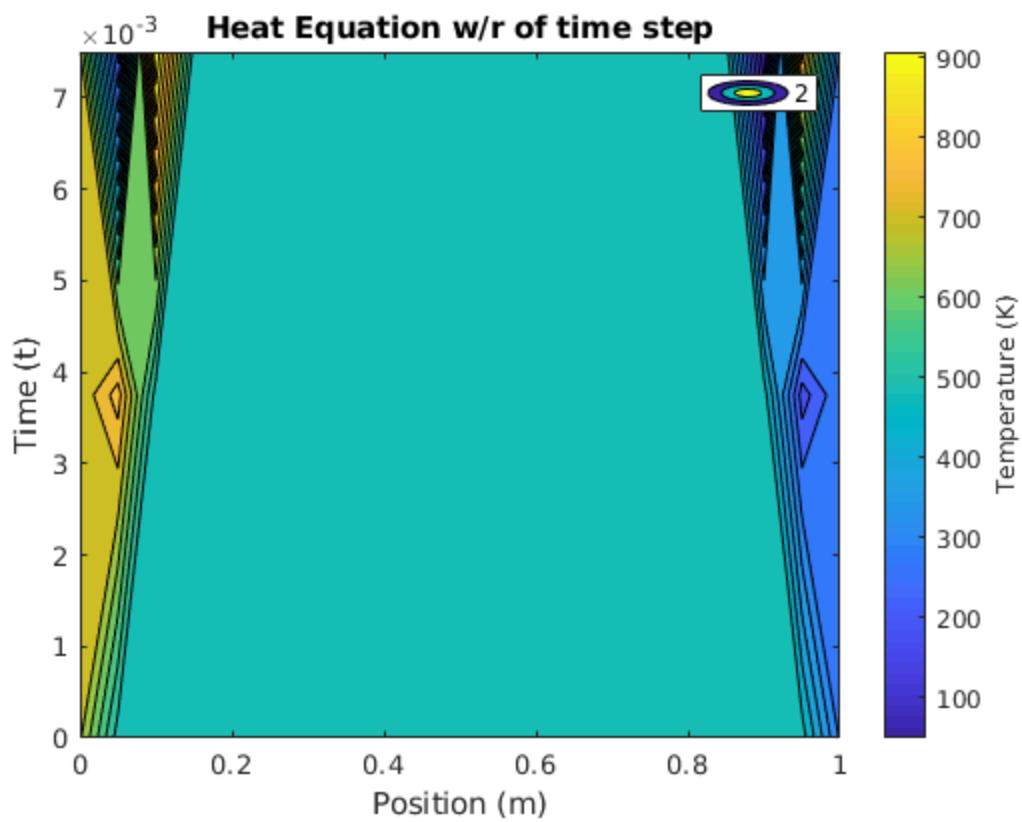
% The following script solves the 1-D heat equation using a
% explicit foward-time centered space finite difference scheme
% For this case I use a Y of 1.5
% This time I will plot the contour plot
clc
clear all
close all

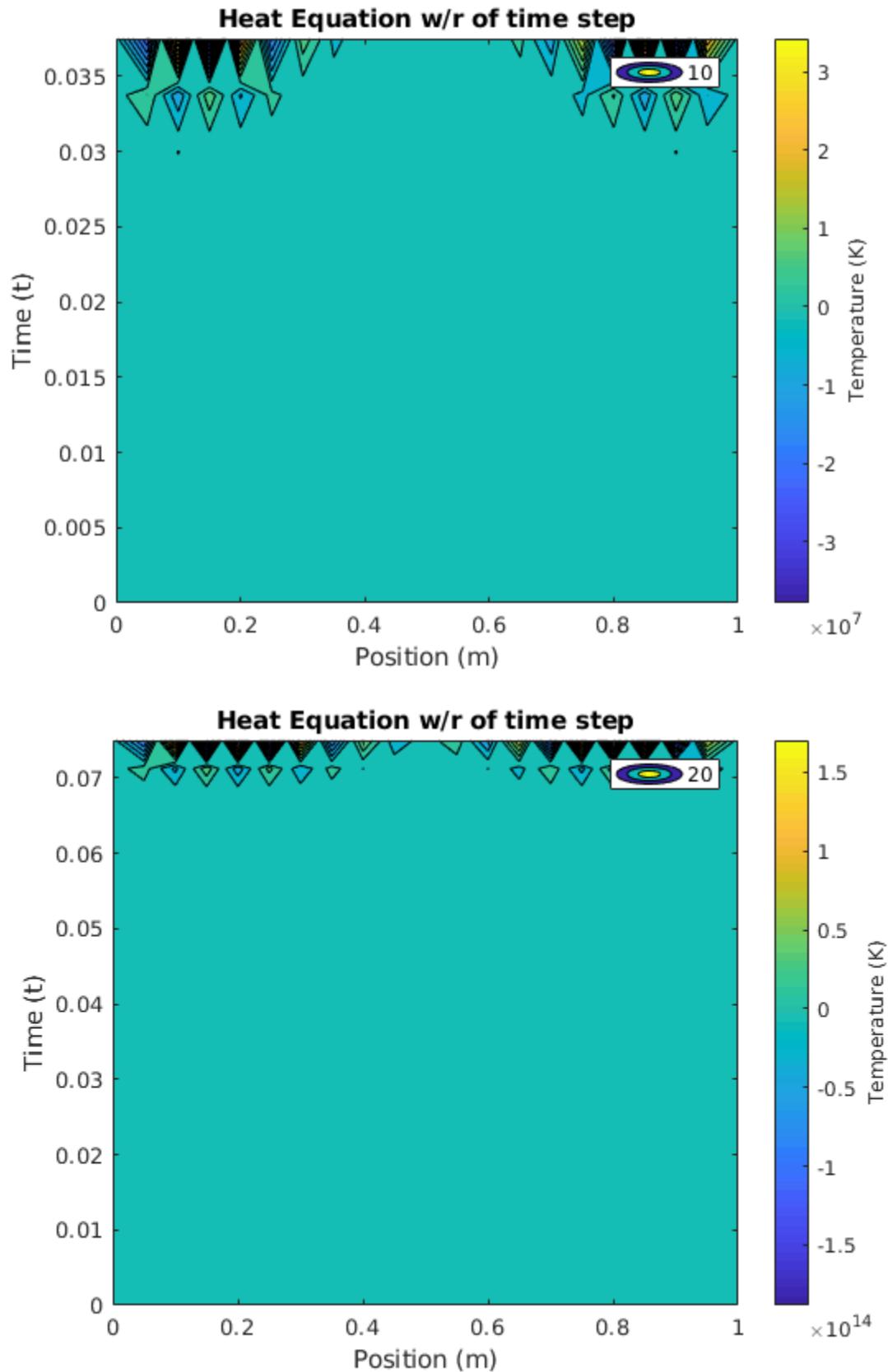
A = [1,2,5,10,20,50,100,150];
%heatTransferPlots(0.5,A);
%temperaturePosition(0.5,A,length(A));
heatTransferPlots(1.5,A);

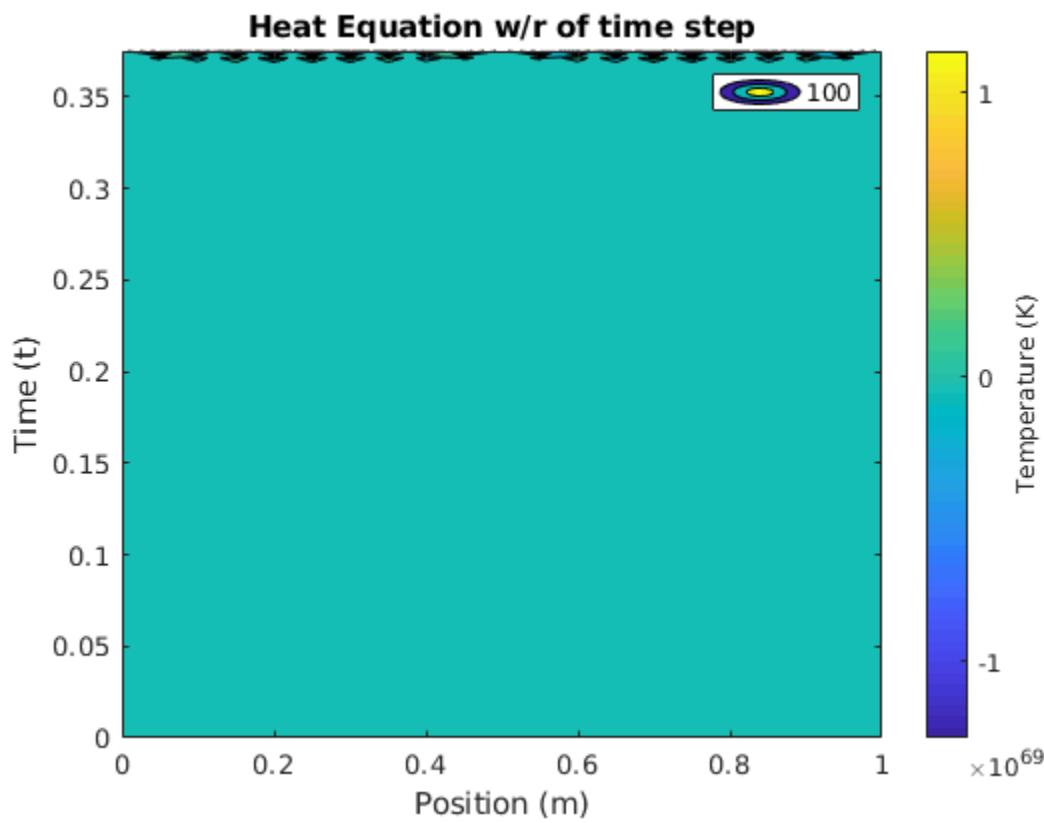
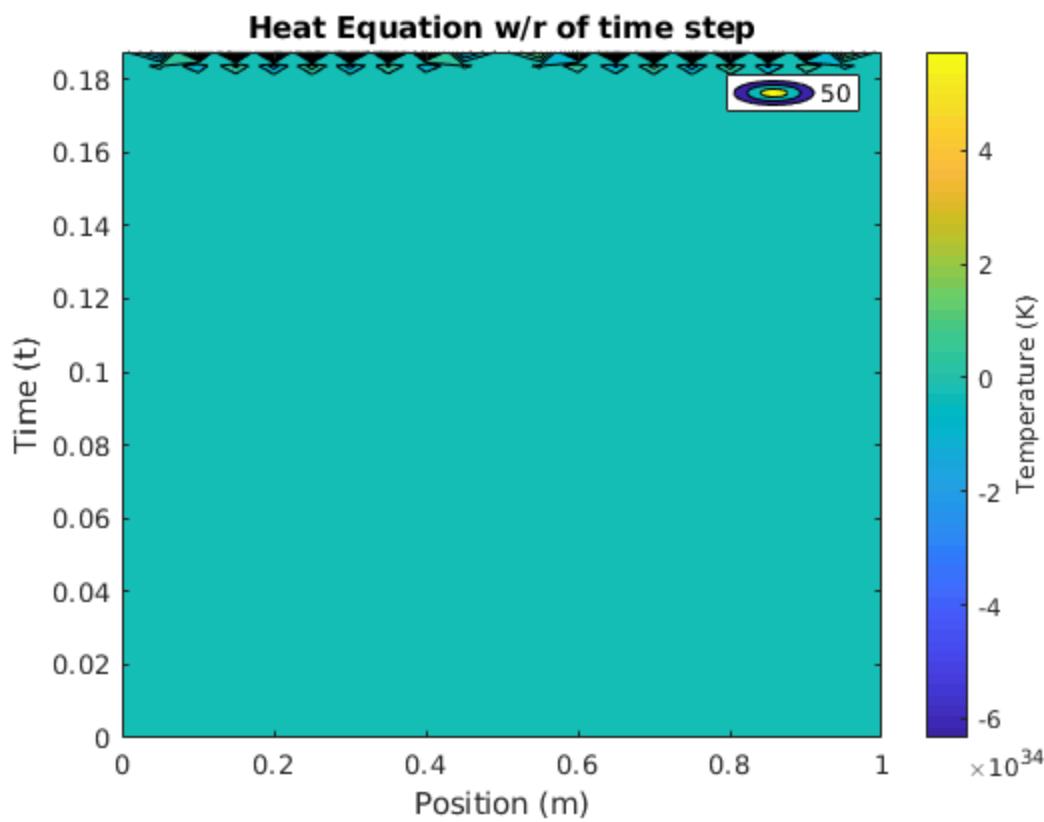
Warning: Ignoring extra legend entries.

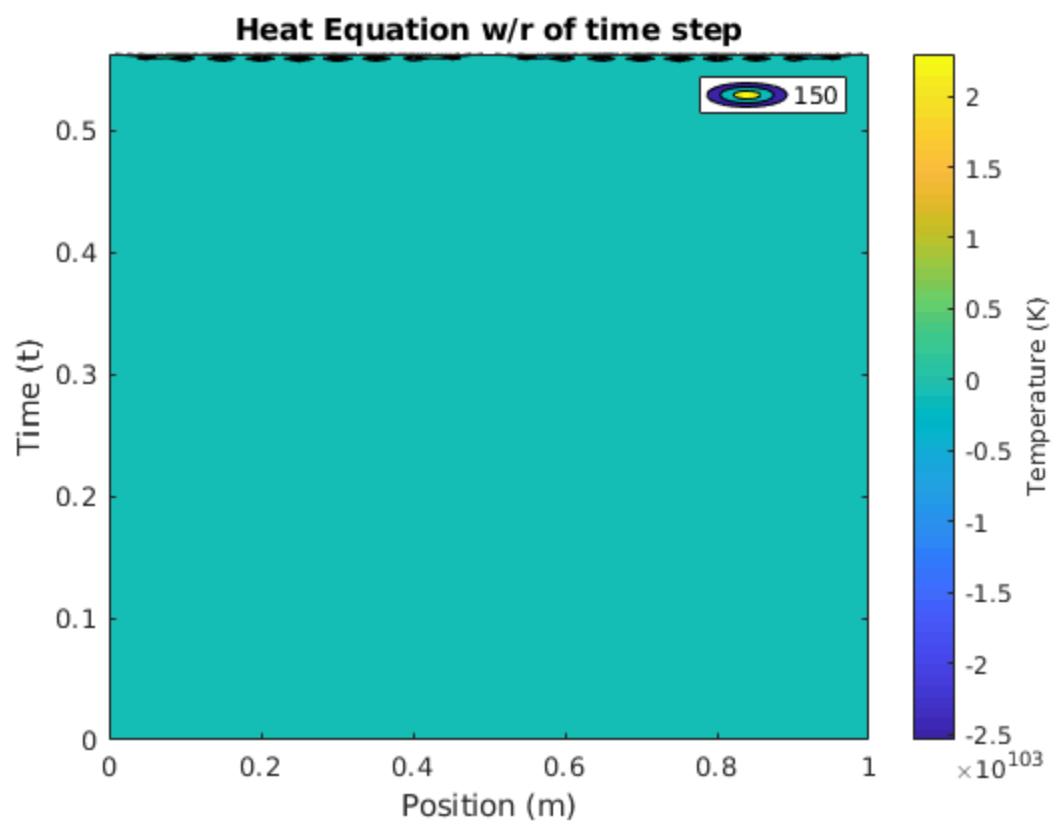
```











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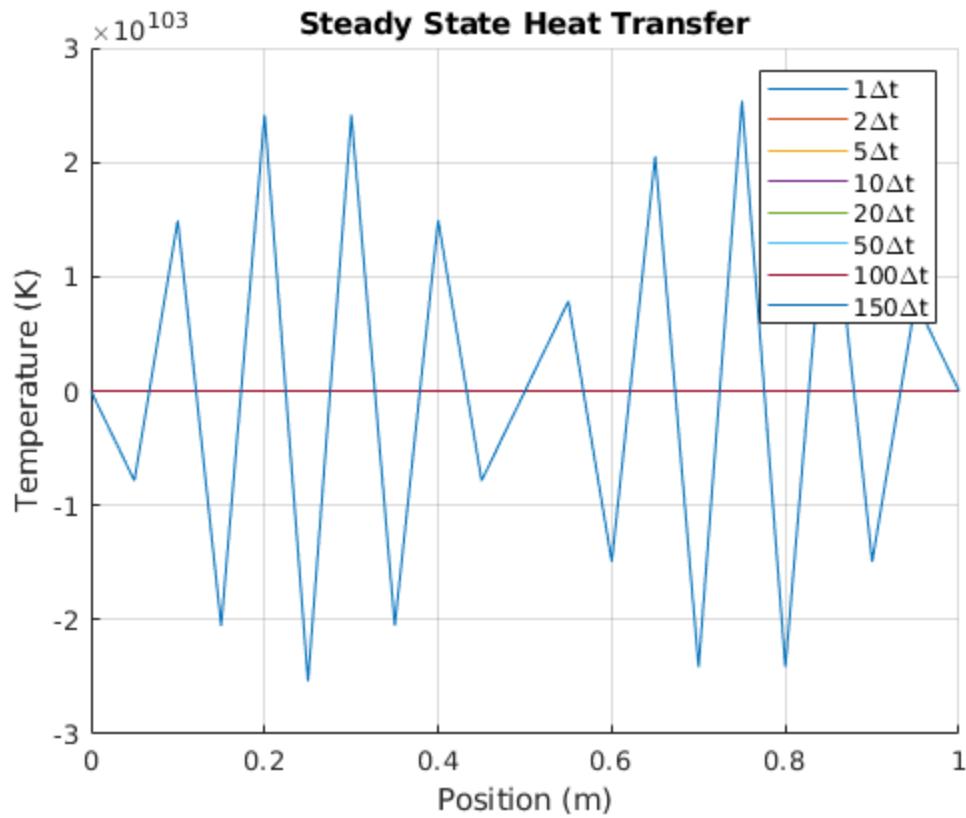
```

% The following script solves the 1-D heat equation using a
% explicit foward-time centered space finite difference scheme
% For this case I use a Y of 1.5
% This time I will plot temperature with respect of position
% clearly this is diverging and the answer makes no sence

clc
clear all
close all

A = [1,2,5,10,20,50,100,150];
%heatTransferPlots(0.5,A);
%temperaturePosition(0.5,A,length(A));
%heatTransferPlots(1.5,A);
temperaturePosition(1.5,A,length(A));

```



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```

function temperaturePosition(Y,n_time,figure_number)

for x=1:1:length(n_time)

dx = 0.05;
% this define columns needed
n_x = (1/dx)+1;
% this define rows needed
n_t = n_time(x)+1;

T = zeros(n_t,n_x);
%this gives us our boundary conditions
boundaryx0 = 700*ones(n_t,1);
boundaryx1 = 300*ones(n_t,1);
boundaryt0 = 500*ones(1,n_x-2);

T(:,1) = boundaryx0;
T(:,n_x) = boundaryx1;
T(n_t,2:n_x-1) = boundaryt0;

if n_t>1
    for i=n_t-1:-1:1
        for w=2:1:n_x-1
            T(i,w) = (Y)*(T(i+1,w+1)-2*T(i+1,w)+T(i+1,w-1))+T(i+1,w);
        end
    end
end

% for x values the following is done
valuedx = 0:dx:1;

figure(figure_number)
hold on
plot(valuedx,T(1,:),'DisplayName',[num2str(n_time(x)) '\Delta t']);
title('Steady State Heat Transfer');
xlabel('Position (m)');
ylabel('Temperature (K)');
legend(gca,'show')
%legend('\Delta',num2str(x),'t')
%legend(num2str(x))
grid on
end

end

Not enough input arguments.

Error in temperaturePosition (line 4)
for x=1:1:length(n_time)

```

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```

function heatTransferPlots(Y,n_time)

for x=1:1:length(n_time)
dx = 0.05;
dt = Y*(dx)^2;
% this define columns needed
n_x = (1/dx)+1;
% this define rows needed
n_t = n_time(x)+1;

T = zeros(n_t,n_x);
%this gives us our boundary conditions
boundaryx0 = 700*ones(n_t,1);
boundaryx1 = 300*ones(n_t,1);
boundaryt0 = 500*ones(1,n_x-2);

T(:,1) = boundaryx0;
T(:,n_x) = boundaryx1;
T(n_t,2:n_x-1) = boundaryt0;

if n_t>1
    for i=n_t-1:-1:1
        for w=2:1:n_x-1
            T(i,w) = (Y)*(T(i+1,w+1)-2*T(i+1,w)+T(i+1,w-1))+T(i+1,w);
        end
    end
end

% for x values the following is done
valuedx = 0:dx:1;

% for time values the following is done
valuedt = n_time(x)*dt:-dt:0;

matrix_x_direction = valuedx.*ones(n_t,n_x);
matrix_t_direction = (valuedt)' .*ones(n_t,n_x);

figure(x)
contourf(matrix_x_direction,matrix_t_direction,T,20)
c = contourbar;
c.Label.String = 'Temperature (K)';
title('Heat Equation w/r of time step');
xlabel('Position (m)');
ylabel('Time (t)');
legend(num2str(n_time(x)), '/Deltat')
end
end

```

Not enough input arguments.

Error in heatTransferPlots (line 3)
for x=1:1:length(n_time)
