Entailment relations for the constructive theory of free modules

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About me

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- Research interest: constructive algebra

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What is constructive algebra?

Constructive algebra: Algebra without nonconstructive principles (e.g., excluded middle, Zorn's lemma, ...).

Constructive proofs have computational content. They can be regarded as programs for proof assistants.

Proof of $\exists n \in \mathbb{N}.\ \varphi(n) \rightsquigarrow$ Algorithm to compute n s.t. $\varphi(n)$

$(e_s = e_t) \Rightarrow (s = t) \vee (1 = 0)$, classically

Theorem 1

If $e_s =_{A \oplus S} e_t$, *then* $s =_S t$ *or* $1 =_A 0$.

Classical proof.

By LEM, we can define $f : S \to A^S$ by

$$f(s)(t) := \begin{cases} 1 & \text{if } s = t, \\ 0 & \text{if } s \neq t. \end{cases}$$

This map induces a homomorphism $\tilde{f} : A^{\oplus S} \to A^S$. Then we have $f(s)(t) = \tilde{f}(e_s)(t) = \tilde{f}(e_t)(t) = f(t)(t) = 1$.

- If $s = t$, we have $s = t$.
- If $s \neq t$, we have $0 = f(s)(t) = 1$.

By LEM, we have $s = t$ or $1 = 0$. $\square$

Constructivizing the proof

Some constructive proofs are collected in the following MathOverflow discussion:

- *Constructively, is the unit of the "free abelian group" monad on sets injective?* `https://mathoverflow.net/q/302516`.

According to Blechschmidt, we can apply the baby version of Barr's theorem (Friedman's translation).

In this talk, I will show a simple constructivization method using entailment relations. (I believe that this method is essentially the same as baby Barr.)

Entailment relations

Definition 2 (Lorenzen [1951], Scott [1971])

A binary relation $\vdash$ on the set of finite subsets of P is called an entailment relation on P if $\vdash$ satisfies the following conditions:

$$\frac{}{\varphi \vdash \varphi}\,, \quad \frac{U \vdash V}{U, U' \vdash V, V'}\,, \quad \frac{U, \varphi \vdash V \qquad U' \vdash \varphi, V'}{U, U' \vdash V, V'}\,.$$

Application of entailment relations to constructive algebra dates back to Cederquist and Coquand [2000], Coquand and Persson [2001] (Application of distributive lattices dates back to Joyal [1975]).

Examples: $\vdash_S, \vdash_A$

We define an entailment relation $\vdash_S$ on $\{\mathrm{Eq}_S(s,t) : s,t \in S\}$.
Axioms of $\vdash_S$:

$$\vdash \mathrm{Eq}_S(s,s), \quad \mathrm{Eq}_S(s,t), \mathrm{Eq}_S(t,u) \vdash \mathrm{Eq}_S(s,u),$$
$$\mathrm{Eq}_S(s,t) \vdash \mathrm{Eq}_S(t,s).$$

Proposition 1

- $U \vdash_S \varphi_0, \ldots, \varphi_{n-1} \iff \exists k.\, U \vdash_S \varphi_k.$
- $U \vdash_S \mathrm{Eq}_S(s,t) \iff s \sim_U t.$

Corollary 1

$\vdash_S \mathrm{Eq}_S(s,t) \iff s = t.$

We define $\vdash_A$ similarly.

Example: $\vdash_{S,A}$

We define an entailment relation $\vdash_{S,A}$ on $\{\mathrm{Eq}_S(s,t) : s,t \in S\} \cup \{\mathrm{Eq}_A(a,b) : a,b \in A\}$.
Axioms of $\vdash_{S,A}$: axioms of $\vdash_S$ and $\vdash_A$.

Proposition 2

$U_S, U_A \vdash_{S,A} V_S, V_A \iff (U_S \vdash_S V_S \text{ or } U_A \vdash_A V_A)$.

Corollary 2

- $\vdash_{S,A} \mathrm{Eq}_S(s,t) \iff s = t$.
- $\vdash_{S,A} \mathrm{Eq}_A(a,b) \iff a = b$.
- $\vdash_{S,A} \mathrm{Eq}_S(s,t), \mathrm{Eq}_A(1,0) \iff (s = t \text{ or } 1 =_A 0)$.

We will extend $\vdash_{S,A}$ conservatively.

Adjoining $\neg$

Theorem 3 (Lorenzen [1951], Cederquist and Coquand [2000])

We can conservatively adjoin $\{\neg\varphi : \varphi \in P\}$ *to an entailment relation with the following axioms:*

$$\varphi, \neg\varphi \vdash, \quad \vdash \varphi, \neg\varphi.$$

Proof sketch.

By induction,

$$U_0, \neg U_1 \vdash_\neg V_0, \neg V_1 \implies U_0, V_1 \vdash V_0, U_1 \quad (U_k, V_k\text{: negation-free}).$$

$\square$

Key point: Adjoining f

Proposition 3

We can conservatively adjoin a function symbol f to $\vdash_{S,A,\neg}$ with the following axioms:

$$\mathrm{Eq}_S(s,t) \vdash \mathrm{Eq}_A(f(s,t),1), \quad \neg\,\mathrm{Eq}_S(s,t) \vdash \mathrm{Eq}_A(f(s,t),0),$$

$$\vdash \mathrm{Eq}_A(\alpha,\alpha), \quad \mathrm{Eq}_A(\alpha,\beta), \mathrm{Eq}_A(\beta,\gamma) \vdash \mathrm{Eq}_A(\alpha,\gamma), \ \ldots$$

$(\alpha, \beta, \gamma$ *may contain* $f)$.

We use the fundamental theorem of entailment relation to prove this.

Fundamental theorem of entailment relation

Theorem 4 (Lorenzen [1951], Cederquist and Coquand [2000])

We can conservatively adjoin connectives $\top, \wedge, \bot, \vee$ *with the following axioms:*

$$\vdash \top, \quad \varphi, \psi \vdash \varphi \wedge \psi, \quad \varphi \wedge \psi \vdash \varphi, \quad \varphi \wedge \psi \vdash \psi,$$
$$\bot \vdash, \quad \varphi \vdash \varphi \vee \psi, \quad \psi \vdash \varphi \vee \psi, \quad \varphi \vee \psi \vdash \varphi, \psi.$$

Proof sketch.

$$\begin{aligned}
[\![U;V]\!] &:= U \vdash V \quad (U, V\colon\ (\top, \wedge, \bot, \vee)\text{-free}),\\
[\![U,\top;V]\!] &:= [\![U;V]\!],\\
[\![U,\varphi \wedge \psi;V]\!] &:= [\![U,\varphi,\psi;V]\!],\\
[\![U,\bot;V]\!] &:= \text{true},\\
[\![U,\varphi \vee \psi;V]\!] &:= [\![U,\varphi;V]\!] \text{ and } [\![U,\psi;V]\!], \ \ldots
\end{aligned}$$

Then, prove $U \vdash_{\mathsf{DLat}} V \Rightarrow [\![U;V]\!]$ by induction. $\square$

Adjoining f (proof)

Proposition 3

We can conservatively adjoin a function symbol f to $\vdash_{S,A,\neg}$ with the following axioms:

$$\mathrm{Eq}_S(s,t) \vdash \mathrm{Eq}_A(f(s,t),1), \quad \neg\,\mathrm{Eq}_S(s,t) \vdash \mathrm{Eq}_A(f(s,t),0),$$
$$\vdash \mathrm{Eq}_A(a,a), \quad \mathrm{Eq}_A(a,b), \mathrm{Eq}_A(b,c) \vdash \mathrm{Eq}_A(a,c), \ \ldots$$
$$(a,b,c \textit{ may contain } f).$$

Proof sketch.

$$[\![\varphi(f(s_0,t_0),\ldots,f(s_{n-1},t_{n-1}))]\!]$$
$$:= \bigvee_{i_0,\ldots,i_{n-1}\in\{0,1\}} (\neg^{i_0}\,\mathrm{Eq}_S(s_0,t_0) \wedge \cdots \wedge \neg^{i_{n-1}}\,\mathrm{Eq}_S(s_{n-1},t_{n-1})$$
$$\wedge\, \varphi(1-i_0,\ldots,1-i_{n-1})).$$

Then, prove $U \vdash_{S,A,\neg,f} V \Rightarrow [\![U]\!] \vdash_{S,A,\neg,\mathsf{DLat}} [\![V]\!]$ by induction. $\square$

Adjoining $+'_A, \cdot'_A$

Proposition 5

We can conservatively adjoin a function symbol $+', \cdot'$ *to* $\vdash_{S,A,\neg,f}$ *with the following axioms:*

$$\vdash \mathrm{Eq}_A(\alpha,\alpha), \quad \mathrm{Eq}_A(\alpha,\beta), \mathrm{Eq}_A(\beta,\gamma) \vdash \mathrm{Eq}_A(\alpha,\gamma), \ \ldots,$$
$$\mathrm{Eq}_A(\alpha,\beta) \vdash \mathrm{Eq}_A(\alpha +' \gamma, \beta +' \gamma), \ \ldots,$$
$$\vdash \mathrm{Eq}_A(0 +' \alpha, \alpha), \vdash \mathrm{Eq}_A(1 \cdot' \alpha, \alpha), \ \ldots \ (\alpha,\beta,\gamma \textit{ may contain } f, +', \cdot'),$$
$$\vdash \mathrm{Eq}_A(a +' b, a+b), \quad \vdash \mathrm{Eq}_A(a \cdot' b, ab) \quad (a, b \in A).$$

Proof sketch.

$$[\![\varphi(f(s_0,t_0),\ldots,f(s_{n-1},t_{n-1}))]\!]$$
$$:= \bigvee_{i_0,\ldots,i_{n-1}\in\{0,1\}} (\neg^{i_0} \mathrm{Eq}_S(s_0,t_0) \wedge \cdots \wedge \neg^{i_{n-1}} \mathrm{Eq}_S(s_{n-1},t_{n-1})$$
$$\wedge\, \varphi(1-i_0,\ldots,1-i_{n-1})). \quad (+', \cdot' \rightsquigarrow +, \cdot) \ \square$$

Adjoining $\tilde{f}$ (1/2)

Let A_S be the set generated by the following constructors:

$$\frac{s \in S}{e_s \in A_S}, \quad \frac{}{0 \in A_S}, \quad \frac{x, y \in A_S}{x + y \in A_S}, \quad \frac{a \in A,\ x \in A_S}{ax \in A_S}.$$

Note that $A^{\oplus S} \cong A_S/{\sim}$, where $\sim$ is generated by $0 + x \sim x$,...

We adjoin elements of the form $\tilde{f}(x, t)$ $(x \in A_S,\ t \in S)$ and extend $+', \cdot'$ to elements containing $\tilde{f}$ with the following axioms:

$$\vdash \mathrm{Eq}_A(\tilde{f}(e_s, t), f(s, t)), \quad \vdash \mathrm{Eq}_A(\tilde{f}(0, t), 0_A),$$

$$\vdash \mathrm{Eq}_A(\tilde{f}(x + y, t), \tilde{f}(x, t) +' \tilde{f}(x, t)), \vdash \mathrm{Eq}_A(\tilde{f}(ax, t), a \cdot' \tilde{f}(x, t)),$$

$$\vdash \mathrm{Eq}_A(0 +' \alpha, \alpha), \ldots \quad (\alpha \text{ may contain } \tilde{f})$$

Adjoining $\tilde{f}$ (2/2)

Proposition 6

We can conservatively adjoin $\tilde{f}$ to $\vdash_{S,A,\neg,f,+',\cdot'}$ (and extend $+', \cdot'$).

Proof sketch.

$$\begin{aligned}
[\![\tilde{f}(e_s, t)]\!] &:= f(s,t) \\
[\![\tilde{f}(0, t)]\!] &:= 0 \\
[\![\tilde{f}(x + y, t)]\!] &:= [\![\tilde{f}(x,t)]\!] +' [\![\tilde{f}(y,t)]\!] \\
[\![\tilde{f}(ax, t)]\!] &:= a \cdot' [\![\tilde{f}(x,t)]\!].
\end{aligned}$$

$\square$

$(e_s = e_t) \Rightarrow (s = t) \vee (1 = 0)$, constructively

Proposition 7

If $x =_{A^{\oplus S}} y$, *then* $\vdash_{S,A,\neg,f,+',\cdot',\tilde{f}} \mathrm{Eq}_A(\tilde{f}(x,t), \tilde{f}(y,t))$.

Proof Sketch.

Note that $A^{\oplus S} \cong A_S/{\sim}$, where $\sim$ is the equivalence relation generated by $0 + x \sim x$,... $\square$

Constructive proof of Theorem 1.

Suppose $e_s =_{A^{\oplus S}} e_t$. Then $\vdash_{S,A,\neg,f,+',\cdot',\tilde{f}} \mathrm{Eq}_A(\tilde{f}(e_s,t), \tilde{f}(e_t,t))$. By simulating the classical proof on the entailment relation, we have

$$\vdash_{S,A,\neg,f,+',\cdot',\tilde{f}} \mathrm{Eq}_S(s,t), \mathrm{Eq}_A(1,0)$$

By the conservativity, we have $\vdash_{S,A} \mathrm{Eq}_S(s,t), \mathrm{Eq}_A(1,0)$. Hence $(s =_S t) \vee (1 =_A 0)$ holds. $\square$

Summary and future work

Using entailment relations, we can deal with *ideal objects* such as f and $\tilde{f}$.

Future work: Combine this method with other applications of entailment relations.

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