

# Entailment relations for the constructive theory of free modules

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# About me

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# What is constructive algebra?

Constructive algebra: Algebra without nonconstructive principles (e.g., excluded middle, Zorn's lemma, ...).

Constructive proofs have computational content. They can be regarded as programs for proof assistants.

Proof of  $\exists n \in \mathbb{N}. \varphi(n) \rightsquigarrow$  Algorithm to compute  $n$  s.t.  $\varphi(n)$

## 1 Classical proof

## 2 Entailment relations

## 3 Conservative extension

## 4 Summary

$(e_s = e_t) \Rightarrow (s = t) \vee (1 = 0)$ , classically

### Theorem 1

If  $e_s =_{A^{\oplus S}} e_t$ , then  $s =_S t$  or  $1 =_A 0$ .

#### Classical proof.

By LEM, we can define  $f : S \rightarrow A^S$  by

$$f(s)(t) := \begin{cases} 1 & \text{if } s = t, \\ 0 & \text{if } s \neq t. \end{cases}$$

This map induces a homomorphism  $\tilde{f} : A^{\oplus S} \rightarrow A^S$ . Then we have  $f(s)(t) = \tilde{f}(e_s)(t) = \tilde{f}(e_t)(t) = f(t)(t) = 1$ .

- If  $s = t$ , we have  $s = t$ .
- If  $s \neq t$ , we have  $0 = f(s)(t) = 1$ .

By LEM, we have  $s = t$  or  $1 = 0$ .



# Constructivizing the proof

Some constructive proofs are collected in the following MathOverflow discussion:

- *Constructively, is the unit of the “free abelian group” monad on sets injective?* <https://mathoverflow.net/q/302516>.

According to Blechschmidt, we can apply the baby version of Barr's theorem (Friedman's translation).

In this talk, I will show a simple constructivization method using entailment relations. (I believe that this method is essentially the same as baby Barr.)

# Entailment relations

Definition 2 (Lorenzen [1951], Scott [1971])

A binary relation  $\vdash$  on the set of finite subsets of  $P$  is called an entailment relation on  $P$  if  $\vdash$  satisfies the following conditions:

$$\frac{}{\varphi \vdash \varphi}, \quad \frac{U \vdash V}{U, U' \vdash V, V'}, \quad \frac{U, \varphi \vdash V \quad U' \vdash \varphi, V'}{U, U' \vdash V, V'}.$$

The application of entailment relations to constructive algebra dates back to Cederquist and Coquand [2000], Coquand and Persson [2001] (The application of distributive lattices dates back to Joyal [1975]).

## Examples: $\vdash_S, \vdash_A$

We define an entailment relation  $\vdash_S$  on  $\{\text{Eq}_S(s, t) : s, t \in S\}$ .

Axioms of  $\vdash_S$ :

$$\begin{aligned} & \vdash \text{Eq}_S(s, s), \quad \text{Eq}_S(s, t), \text{Eq}_S(t, u) \vdash \text{Eq}_S(s, u), \\ & \text{Eq}_S(s, t) \vdash \text{Eq}_S(t, s). \end{aligned}$$

### Proposition 1

- $U \vdash_S \varphi_0, \dots, \varphi_{n-1} \iff \exists k. U \vdash_S \varphi_k.$
- $U \vdash_S \text{Eq}_S(s, t) \iff s \sim_U t.$

### Corollary 1

$$\vdash_S \text{Eq}_S(s, t) \iff s = t.$$

We define  $\vdash_A$  similarly.

# Example: $\vdash_{S,A}$

We define an entailment relation  $\vdash_{S,A}$  on  $\{\text{Eq}_S(s,t) : s,t \in S\} \cup \{\text{Eq}_A(a,b) : a,b \in A\}$ .

Axioms of  $\vdash_{S,A}$ : axioms of  $\vdash_S$  and  $\vdash_A$ .

## Proposition 2

$$U_S, U_A \vdash_{S,A} V_S, V_A \iff (U_S \vdash_S V_S \text{ or } U_A \vdash_A V_A).$$

## Corollary 2

- $\vdash_{S,A} \text{Eq}_S(s,t) \iff s = t$ .
- $\vdash_{S,A} \text{Eq}_A(a,b) \iff a = b$ .
- $\vdash_{S,A} \text{Eq}_S(s,t), \text{Eq}_A(1,0) \iff (s = t \text{ or } 1 =_A 0)$ .

We will extend  $\vdash_{S,A}$  conservatively.

# Adjoining $\neg$

Theorem 3 (Lorenzen [1951], Cederquist and Coquand [2000])

We can conservatively adjoin  $\{\neg\varphi : \varphi \in P\}$  to an entailment relation with the following axioms:

$$\varphi, \neg\varphi \vdash, \quad \vdash \varphi, \neg\varphi.$$

Proof sketch.

By induction,

$$U_0, \neg U_1 \vdash_{\neg} V_0, \neg V_1 \implies U_0, V_1 \vdash V_0, U_1 \quad (U_k, V_k: \text{negation-free}).$$



# Key point: Adjoining $f$

## Proposition 3

We can conservatively adjoin a function symbol  $f$  to  $\vdash_{S,A,\neg}$  with the following axioms:

$$\begin{aligned} \text{Eq}_S(s,t) &\vdash \text{Eq}_A(f(s,t), 1), \quad \neg \text{Eq}_S(s,t) \vdash \text{Eq}_A(f(s,t), 0), \\ &\vdash \text{Eq}_A(\alpha, \alpha), \quad \text{Eq}_A(\alpha, \beta), \text{Eq}_A(\beta, \gamma) \vdash \text{Eq}_A(\alpha, \gamma), \dots \\ &(\alpha, \beta, \gamma \text{ may contain } f). \end{aligned}$$

We use the fundamental theorem of entailment relation to prove this.

# Fundamental theorem of entailment relation

Theorem 4 (Lorenzen [1951], Cederquist and Coquand [2000])

We can conservatively adjoin connectives  $\top, \wedge, \perp, \vee$  with the following axioms:

$$\begin{aligned} & \vdash \top, \quad \varphi, \psi \vdash \varphi \wedge \psi, \quad \varphi \wedge \psi \vdash \varphi, \quad \varphi \wedge \psi \vdash \psi, \\ & \perp \vdash, \quad \varphi \vdash \varphi \vee \psi, \quad \psi \vdash \varphi \vee \psi, \quad \varphi \vee \psi \vdash \varphi, \psi. \end{aligned}$$

Proof sketch.

$$[\![U; V]\!] := U \vdash V \quad (U, V: (\top, \wedge, \perp, \vee)\text{-free}),$$

$$[\![U, \top; V]\!] := [\![U; V]\!],$$

$$[\![U, \varphi \wedge \psi; V]\!] := [\![U, \varphi, \psi; V]\!],$$

$$[\![U, \perp; V]\!] := \text{true},$$

$$[\![U, \varphi \vee \psi; V]\!] := [\![U, \varphi; V]\!] \text{ and } [\![U, \psi; V]\!], \dots$$

Then, prove  $U \vdash_{\text{DLat}} V \Rightarrow [\![U; V]\!]$  by induction. □

# Adjoining $f$ (proof)

## Proposition 3

We can conservatively adjoin a function symbol  $f$  to  $\vdash_{S,A,\neg}$  with the following axioms:

$$\begin{aligned} \text{Eq}_S(s,t) &\vdash \text{Eq}_A(f(s,t),1), \quad \neg \text{Eq}_S(s,t) \vdash \text{Eq}_A(f(s,t),0), \\ &\vdash \text{Eq}_A(a,a), \quad \text{Eq}_A(a,b), \text{Eq}_A(b,c) \vdash \text{Eq}_A(a,c), \dots \\ &\quad (a,b,c \text{ may contain } f). \end{aligned}$$

## Proof sketch.

$$\begin{aligned} &[\![\varphi(f(s_0, t_0), \dots, f(s_{n-1}, t_{n-1}))]\!] \\ &:= \bigvee_{i_0, \dots, i_{n-1} \in \{0,1\}} (\neg^{i_0} \text{Eq}_S(s_0, t_0) \wedge \dots \wedge \neg^{i_{n-1}} \text{Eq}_S(s_{n-1}, t_{n-1}) \\ &\quad \wedge \varphi(1 - i_0, \dots, 1 - i_{n-1})). \end{aligned}$$

Then, prove  $U \vdash_{S,A,\neg,f} V \Rightarrow [\![U]\!] \vdash_{S,A,\neg,\text{DLat}} [\![V]\!]$  by induction. □

# Adjoining $+'_A, \cdot'_A$

## Proposition 5

We can conservatively adjoin a function symbol  $+', \cdot'$  to  $\vdash_{S,A,\neg,f}$  with the following axioms:

$$\vdash \text{Eq}_A(\alpha, \alpha), \quad \text{Eq}_A(\alpha, \beta), \text{Eq}_A(\beta, \gamma) \vdash \text{Eq}_A(\alpha, \gamma), \dots,$$

$$\text{Eq}_A(\alpha, \beta) \vdash \text{Eq}_A(\alpha +' \gamma, \beta +' \gamma), \dots,$$

$$\vdash \text{Eq}_A(0 +' \alpha, \alpha), \vdash \text{Eq}_A(1 \cdot' \alpha, \alpha), \dots \quad (\alpha, \beta, \gamma \text{ may contain } f, +', \cdot'),$$

$$\vdash \text{Eq}_A(a +' b, a + b), \quad \vdash \text{Eq}_A(a \cdot' b, ab) \quad (a, b \in A).$$

## Proof sketch.

$$\llbracket \varphi(f(s_0, t_0), \dots, f(s_{n-1}, t_{n-1})) \rrbracket$$

$$:= \bigvee_{i_0, \dots, i_{n-1} \in \{0,1\}} (\neg^{i_0} \text{Eq}_S(s_0, t_0) \wedge \dots \wedge \neg^{i_{n-1}} \text{Eq}_S(s_{n-1}, t_{n-1})$$

$$\wedge \varphi(1 - i_0, \dots, 1 - i_{n-1})). \quad (+', \cdot' \rightsquigarrow +, \cdot) \quad \square$$

# Adjoining $\tilde{f}$ (1/2)

Let  $A_S$  be the set generated by the following constructors:

$$\frac{s \in S}{e_s \in A_S}, \quad \frac{}{0 \in A_S}, \quad \frac{x, y \in A_S}{x + y \in A_S}, \quad \frac{a \in A, x \in A_S}{ax \in A_S}.$$

Note that  $A^{\oplus S} \cong A_S/\sim$ , where  $\sim$  is generated by  $0 + x \sim x, \dots$

We adjoin elements of the form  $\tilde{f}(x, t)$  ( $x \in A_S, t \in S$ ) and extend  $+', .'$  to elements containing  $\tilde{f}$  with the following axioms:

- $\vdash \text{Eq}_A(\tilde{f}(e_s, t), f(s, t)), \quad \vdash \text{Eq}_A(\tilde{f}(0, t), 0_A),$
- $\vdash \text{Eq}_A(\tilde{f}(x + y, t), \tilde{f}(x, t) +' \tilde{f}(y, t)), \quad \vdash \text{Eq}_A(\tilde{f}(ax, t), a .' \tilde{f}(x, t)),$
- $\vdash \text{Eq}_A(0 +' \alpha, \alpha), \dots \quad (\alpha \text{ may contain } \tilde{f})$

# Adjoining $\tilde{f}$ (2/2)

## Proposition 6

We can conservatively adjoin  $\tilde{f}$  to  $\vdash_{S, A, \neg, f, +', \cdot'}$  (and extend  $+'$ ,  $\cdot'$ ).

## Proof sketch.

$$\llbracket \tilde{f}(e_s, t) \rrbracket := f(s, t)$$

$$\llbracket \tilde{f}(0, t) \rrbracket := 0$$

$$\llbracket \tilde{f}(x + y, t) \rrbracket := \llbracket \tilde{f}(x, t) \rrbracket +' \llbracket \tilde{f}(y, t) \rrbracket$$

$$\llbracket \tilde{f}(ax, t) \rrbracket := a \cdot' \llbracket \tilde{f}(x, t) \rrbracket.$$



$(e_s = e_t) \Rightarrow (s = t) \vee (1 = 0)$ , constructively

### Proposition 7

If  $x =_{A^{\oplus S}} y$ , then  $\vdash_{S, A, \neg, f, +', .', \tilde{f}} \text{Eq}_A(\tilde{f}(x, t), \tilde{f}(y, t))$ .

### Proof Sketch.

Note that  $A^{\oplus S} \cong A_S / \sim$ , where  $\sim$  is the equivalence relation generated by  $0 + x \sim x, \dots$



### Constructive proof of Theorem 1.

Suppose  $e_s =_{A^{\oplus S}} e_t$ . Then  $\vdash_{S, A, \neg, f, +', .', \tilde{f}} \text{Eq}_A(\tilde{f}(e_s, t), \tilde{f}(e_t, t))$ . By simulating the classical proof on the entailment relation, we have

$$\vdash_{S, A, \neg, f, +', .', \tilde{f}} \text{Eq}_S(s, t), \text{Eq}_A(1, 0)$$

By the conservativity, we have  $\vdash_{S, A} \text{Eq}_S(s, t), \text{Eq}_A(1, 0)$ . Hence  $(s =_S t) \vee (1 =_A 0)$  holds.



## Summary and future work

Using entailment relations, we can deal with *ideal objects* such as  $f$  and  $\tilde{f}$ .

Future work: Combine this method with other applications of entailment relations.

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