Constructive theory of Jacobson rings

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Abstract

We give a constructive proof of the general Nullstellensatz: a univariate polynomial ring over a commutative Jacobson ring is Jacobson. This theorem implies that every finitely generated algebra over a zero-dimensional ring or the ring of integers is Jacobson, which has been an open problem in constructive algebra.

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1 Introduction

In this paper, all rings are considered to be commutative with identity.

Constructive algebra is the study of algebra without using non-constructive principles, such as Zorn's lemma or the law of excluded middle. See [17, 14, 29, 3] for recent developments in constructive algebra. It is closely related to computer algebra since constructive proofs have computational content, which can be extracted using proof assistants, such as Agda, Coq, and Lean.

Despite recent developments in constructive algebra, little work has been done on the constructive theory of Jacobson rings. In classical mathematics, a Jacobson ring is a ring A such that every prime ideal of A is an intersection of maximal ideals. One of the most important theorems about Jacobson rings is the general Nullstellensatz: a univariate polynomial ring over a Jacobson ring is Jacobson. This theorem has been proved independently by Goldman [9, Theorem 3] and Krull [10, Satz 1].

Recently, Wessel [26, Section 2.4.1] has proposed the following constructive definition of a Jacobson ring: a ring A is called Jacobson if $\operatorname{Nil} I = \operatorname{Jac} I$ for all ideals I of A, where $\operatorname{Nil} I$ and $\operatorname{Jac} I$ denote the nilradical and Jacobson radical of I, respectively. In this paper, based on this proposal by Wessel and the idea of the classical proof by Emerton [8, Theorem 8], we give a constructive proof of the general Nullstellensatz (Theorem 3.1). Our constructive proof provides a solution to two questions [25, 1] on MathOverflow and two open problems [12, 1.1, 1.2] in Lombardi's list.

We use the entailment relation of prime ideal and the entailment relation of maximal ideal [20] to obtain a constructive proof, although familiarity with entailment relations is not necessary to understand our proof. A notion similar to the entailment relation has been considered by Lorenzen [15], and Scott [21, 22] has later introduced the terminology. We mention that Neuwirth [16] has recently translated [15] into English. The use of entailment relations in algebra dates back to Cederquist and Coquand [2] and Coquand and Persson [5]. Their work is closely related to the dynamical method by Coste, Lombardi, and Roy [6], which is a generic method to turn a classical proof into a constructive proof. Classical arguments involving prime ideals and maximal ideals often use Zorn's lemma. The dynamical method simulates classical proofs in a simple deductive system, such as entailment relations, and extracts a constructive argument without prime ideals or maximal ideals.

The most important step in our constructive proof of the general Nullstellensatz is the generalization (Lemma 3.2) of Emerton's key lemma [8, Lemma 6]. Emerton's lemma is a lemma about integral domains, and we need some non-constructive principle to obtain the general Nullstellensatz from it. Our generalized lemma does not involve integral domains and is more useful in constructive mathematics.

The definition of a Jacobson ring used in this paper contains a quantification over all ideals of a ring. This definition works in some predicative foundations of constructive mathematics, such as Martin-Löf type theory with universes using setoids, although we have to pay attention to the universe levels. On the other hand, this definition does not work in other predicative foundations that cannot handle power sets at all, such as Martin-Löf type theory without universes. Such foundations do not accept constructing a proposition by quantifying over all subsets of a set. A definition of a Jacobson ring that does not use this problematic quantification is still not known.

2 Jacobson rings

In this section, we present the constructive definition of Jacobson rings and provide some examples. We first recall some basic definitions.

Definition 2.1. Let $\langle U \rangle_A$ denote the ideal of a ring A generated by a subset $U \subseteq A$. We define two ideals $\operatorname{Nil}_A U$, $\operatorname{Jac}_A U$ of A as follows:

$$\operatorname{Nil}_A U := \{a \in A : \exists n \geq 0.\ a^n \in \langle U \rangle\},$$
$$\operatorname{Jac}_A U := \{a \in A : \forall b \in A.\ 1 \in \langle U, 1 - ab \rangle\}.$$

Here $\langle U, 1-ab \rangle$ means $\langle U \cup \{1 - ab\} \rangle$. When the context is clear, we write $\langle U \rangle$, $\operatorname{Nil} U$, and $\operatorname{Jac} U$ for $\langle U \rangle_A$, $\operatorname{Nil}_A U$, and $\operatorname{Jac}_A U$, respectively. Note that $\operatorname{Nil} U = \operatorname{Nil} \langle U \rangle$ and $\operatorname{Jac} U = \operatorname{Jac} \langle U \rangle$ hold for all $U \subseteq A$.

The ideals $\operatorname{Nil}_A 0$ and $\operatorname{Jac}_A 0$ are called the *nilradical* and the *Jacobson radical* of A, respectively. Classically, it is well known that the ideal $\operatorname{Nil} U$ (resp. $\operatorname{Jac} U$) is equal to the intersection of all prime (resp. maximal) ideals containing U.

The above definition does not use prime ideals or maximal ideals. Therefore, it is reasonable to use the following definition of a Jacobson ring in a constructive setting.

Definition 2.2 ([26, Section 2.4.1])**.** We call a ring A *Jacobson* if every ideal I of A satisfies $\operatorname{Jac} I \subseteq \operatorname{Nil} I$.

In classical mathematics, the above definition is equivalent to the ordinary one. Note that every subset U of a ring A satisfies $\operatorname{Nil} U \subseteq \operatorname{Jac} U$. If we constructively prove that a ring A is Jacobson, then we can extract an algorithm such that

- its input is an ideal I of A, an element a of A, and a function $f : A \to I \times A$ such that for any $b \in A$, if $(i, c) = f(b)$, then $1 = i + (1 - ab)c$, and
- its output is a natural number $n \geq 0$ such that $a^n \in I$.

The following proposition easily follows from the definition.

Proposition 2.1. *Let I be an ideal of a ring A. Let $\pi : A \to A/I$ be the canonical projection. For every subset U of A, the following equalities hold:*

$$\pi^{-1}(\operatorname{Nil}_{A/I} \pi(U)) = \operatorname{Nil}_A(I \cup U), \quad \pi^{-1}(\operatorname{Jac}_{A/I} \pi(U)) = \operatorname{Jac}_A(I \cup U).$$

Two corollaries follow from the above proposition. They are useful for constructing some examples of Jacobson rings.

Corollary 2.1. *If I is an ideal of a Jacobson ring A, then A/I is Jacobson.*

Corollary 2.2. *A ring A is Jacobson if and only if $\operatorname{Jac}_{A/I} 0 \subseteq \operatorname{Nil}_{A/I} 0$ holds for all ideals I of A.*

We next prove a fundamental lemma that can be used to avoid the use of prime ideals and maximal ideals.

Lemma 2.1. *Let A be a ring, U be a subset of A, and $x, y \in A$. The following statements hold:*

1. *If $xy \in \operatorname{Nil} U$ and $x \in \operatorname{Nil}(U, y)$, then $x \in \operatorname{Nil} U$. Here $\operatorname{Nil}(U, y)$ means $\operatorname{Nil}(U \cup \{y\})$.*

2. *If $x \in \operatorname{Jac}(U, 1 - yz)$ for all $z \in A$, then $xy \in \operatorname{Jac} U$.*

Proof. 1. There exists $n \geq 0$ such that $x^n \in \langle U \rangle + \langle y \rangle$. Hence $x^{n+1} \in \langle U \rangle + \langle xy \rangle \subseteq \operatorname{Nil} U$.

2. For all $w \in A$, we have $1 \in \langle U, 1 - y(xw), 1 - x(yw) \rangle = \langle U, 1 - (xy)w \rangle$. Hence $xy \in \operatorname{Jac} U$. □

Remark 2.1. This remark is not needed to understand the constructive proof of our main theorem, but it explains the importance of Lemma 2.1.

In terms of the entailment relation $\vdash_{\mathrm{p}}$ and the geometric entailment relation $\vdash_{\mathrm{m}}$, which are defined in [20], item 1 of the above lemma corresponds to the fact that $U \vdash_{\mathrm{p}} x, y$ and $U, y \vdash_{\mathrm{p}} x$ together imply that $U \vdash_{\mathrm{p}} x$. Item 2 corresponds to the fact that if $U, 1 - yz \vdash_{\mathrm{m}} x$ holds for all $z \in A$, then $U \vdash_{\mathrm{m}} x, y$. They are related to the cut rule of $\vdash_{\mathrm{p}}$ and the infinitary cut rule of $\vdash_{\mathrm{m}}$, respectively.

In classical proofs, we can use results about integral domains to prove something about arbitrary rings. We do this by taking the quotient by a prime ideal, and this method does not work constructively, because we do not have enough prime ideals without non-constructive principles such as Zorn's lemma. Thus, in constructive algebra, we use the entailment relation $\vdash_{\mathrm{p}}$ to simulate an argument about integral domains, and then extract some results about arbitrary rings. We similarly use $\vdash_{\mathrm{m}}$ to extract some general result from an argument about fields.

Corollary 2.3. *Let A be a ring and $U \subseteq A$. Then $\operatorname{Jac}(\operatorname{Jac} U) \subseteq \operatorname{Jac} U$ holds.*

Proof. Let $a \in \operatorname{Jac}(\operatorname{Jac} U)$. Then we have $1 \in \langle \operatorname{Jac} U, 1 - ab \rangle \subseteq \operatorname{Jac}(U, 1 - ab)$ for every $b \in A$. Thus $a \in \operatorname{Jac} U$ by Lemma 2.1-(2) with $(A, U, x, y) = (A, U, 1, a)$. □

Before we provide some examples of Jacobson rings, we review the basic constructive theory of Krull dimension. Lombardi [11, Définition 5.1] has introduced the following constructively acceptable definition of Krull dimension.

Definition 2.3. Let $n \geq -1$. A ring A is of *Krull dimension at most n* if for every $x_0, \ldots, x_n \in A$, there exists $e_0, \ldots, e_n \geq 0$ such that

$$x_0^{e_0} \cdots x_n^{e_n} \in \langle x_0^{e_0+1}, x_0^{e_0} x_1^{e_1+1}, \ldots, x_0^{e_0} \cdots x_{n-1}^{e_{n-1}} x_n^{e_n+1} \rangle.$$

Let $\mathsf{Kdim}\, A \leq n$ denote the statement that A is of Krull dimension at most n. A ring A is called *n-dimensional* if $\mathsf{Kdim}\, A \leq n$.

Example 2.1. Every discrete field is zero-dimensional, where a discrete field means a ring K such that every element of K is null or invertible. We also have $\mathsf{Kdim}\, \mathbb{Z} \leq 1$ [14, Examples below Lemma XIII-2.4], [29, Examples 86].

Proposition 2.2 ([14, Proposition XIII-3.1, Lemma IX-1.2])**.** *Let A be a ring with $\mathsf{Kdim}\, A \leq n$. The following statements hold:*

1. *If I is an ideal of A, then $\mathsf{Kdim}\, A/I \leq n$.*

2. *If $n \geq 0$ and I is an ideal of A containing a regular element x, then $\mathsf{Kdim}\, A/I \leq n - 1$.*

3. *If $n = 0$, then $\operatorname{Jac}_A 0 \subseteq \operatorname{Nil}_A 0$.*

Now, we can show that every zero-dimensional ring is Jacobson, which is essentially contained in [14, Lemma IX-1.2].

Example 2.2. Let A be a zero-dimensional ring. By Proposition 2.2, we have $\mathrm{Jac}_{A/I}\, 0 \subseteq \mathrm{Nil}_{A/I}\, 0$ for all ideals I of A. Thus, A is Jacobson by Corollary 2.2. In particular, every discrete field is Jacobson.

We next prove that the ring of integers $\mathbb{Z}$ is Jacobson. Lombardi [13] has pointed out that the following lemma generalizes the essential part of our original proof that $\mathbb{Z}$ is Jacobson.

Lemma 2.2. *Let A be a 1-dimensional ring. If an ideal I of A contains some regular element, then $\mathrm{Jac}\, I \subseteq \mathrm{Nil}\, I$.*

Proof. We have $\mathsf{Kdim}\, A/I \leq 0$ by Proposition 2.2. Hence we have $\mathrm{Jac}_{A/I}\, 0 \subseteq \mathrm{Nil}_{A/I}\, 0$ by Example 2.2. Hence, the assertion follows from Proposition 2.1. □

The following proposition gives a class of Jacobson rings that contains $\mathbb{Z}$.

Proposition 2.3. *If A is a ring satisfying all of the following conditions, then A is Jacobson:*

1. *Every element of A is nilpotent or regular.*
2. *The Krull dimension of A is at most 1.*
3. *For all ideals I of A, if $\mathrm{Jac}\, I$ contains some regular element, then I also contains some regular element.*

In particular, $\mathbb{Z}$ is Jacobson.

Proof. Let $a \in \mathrm{Jac}\, I$.

1. If a is nilpotent, then $a \in \mathrm{Nil}\, I$.
2. If a is regular, then I contains some regular element. Hence $a \in \mathrm{Nil}\, I$ by Lemma 2.2. □

Remark 2.2. Assuming the law of excluded middle, a ring satisfies the first condition of Proposition 2.3 if and only if its zero ideal is primary.

Assuming the law of excluded middle, we can replace the third condition of the above proposition with $\mathrm{Jac}\, 0 \subseteq \mathrm{Nil}\, 0$ by the following proposition:

Proposition 2.4. *Let A be a ring satisfying the following two conditions:*

1. *Every element of A is nilpotent or regular.*
2. $\mathrm{Jac}\, 0 \subseteq \mathrm{Nil}\, 0$.

Then, for all ideals I of A, if $\mathrm{Jac}\, I$ contains some regular element, then the double negation of "I contains a regular element" holds.

Proof. Let I be an ideal of A such that $\mathrm{Jac}\, I$ contains some regular element.

- Assume that I does not contain a regular element. Then $I \subseteq \mathrm{Nil}\, 0 \subseteq \mathrm{Jac}\, 0$. Hence $\mathrm{Jac}\, I \subseteq \mathrm{Jac}(\mathrm{Jac}\, 0) \subseteq \mathrm{Jac}\, 0 \subseteq \mathrm{Nil}\, 0$ by Corollary 2.3. Since $\mathrm{Jac}\, I$ contains some regular element, A has a regular and nilpotent element. Hence $0 \in I$ is regular, and this contradicts the assumption that I does not contain a regular element.

Thus, the double negation of "I contains a regular element" holds. □

In Remark 2.3, we use the following lemma to present the computational content of $\mathbb{Z}$ being a Jacobson ring. The proof of the lemma is similar to the proof of 1-dimensionality of $\mathbb{Z}$ in [29, Examples 86].

Lemma 2.3. *Let $x \in \mathbb{Z} - \{0\}$ and $a \in \mathbb{Z}$. Then, there exist $d, e \in \mathbb{Z}$ such that $x = de$, $a \in \operatorname{Nil} d$, and $1 \in \langle a, e \rangle$.*

Proof. Computing successively

$$d_1 = \gcd(x, a),\ d_2 = \gcd(\frac{x}{d_1}, a),\ d_3 = \gcd(\frac{x}{d_1 d_2}, a), \dots,$$

we obtain a finite sequence $(d_1, \dots, d_n)$ of positive integers such that $d_i \mid a$ and $d_n = 1$. By letting $d := d_1 \cdots d_{n-1}$ and $e := x/d$, we have $x = de$, $a \in \operatorname{Nil} d$, and $1 \in \langle a, e \rangle$. $\square$

Remark 2.3. The ring of integers $\mathbb{Z}$ is Jacobson by Proposition 2.3. We can unfold the constructive proof and reveal the computational content as follows. Let I be an ideal of $\mathbb{Z}$ and $a \in \operatorname{Jac} I$.

1. If $a = 0$, then $a \in \operatorname{Nil} I$.
2. If $a \geq 1$, then there exists $c_{-1} \in \mathbb{Z}$ such that $1 - (1 + a)c_{-1} \in I$ by $a \in \operatorname{Jac} I$. Since $1 - (1 + a)c_{-1} \neq 0$, we deduce from the Lemma 2.3 that there exist $d, e \in \mathbb{Z}$ such that $1 - (1 + a)c_{-1} = de$, $a \in \operatorname{Nil} d$, and $1 \in \langle a, e \rangle$ all hold. By $1 \in \langle a, e \rangle$, there exists $b \in \mathbb{Z}$ such that $1 - ab \in \langle e \rangle$. By $a \in \operatorname{Jac} I$, there exists $c_b \in \mathbb{Z}$ such that $1 - (1 - ab)c_b \in I$. Since $d \in \langle 1 - (1 + a)c_{-1}, 1 - (1 - ab)c_b \rangle \subseteq I$, we have $a \in \operatorname{Nil} d \subseteq \operatorname{Nil} I$.
3. If $a \leq -1$, then $a \in \operatorname{Nil} I$ by a similar argument.

This solves a problem that Werner has mentioned in his comment to his question [25] on MathOverflow. Note that in the above argument, the ideal $I \subseteq \mathbb{Z}$ is not assumed to be finitely generated or detachable, where a detachable ideal of a ring A means an ideal $I \subseteq A$ such that $x \in I$ or $x \notin I$ holds for all $x \in A$.

3 The general Nullstellensatz

In this section, we provide a constructive proof of the general Nullstellensatz. We first recall three basic constructive results.

Proposition 3.1 ([17, Corollary VI-1.3])**.** *Let B be an A-algebra. If $b \in B$ is integral over A, then the algebra $A[b] \subseteq B$ is integral over A.*

Proposition 3.2 ([14, Theorem IX-1.7])**.** *If B is an integral extension of a ring A, then $B^\times \cap A \subseteq A^\times$.*

Proposition 3.3 ([14, Lemma II-2.6])**.** *For all rings A, $A[X]^\times \subseteq A^\times + (\operatorname{Nil}_A 0)[X]$.*

Corollary 3.1 (Snapper's theorem [24, Corollary 8.1])**.** *For all rings A, $\operatorname{Jac}_{A[X]} 0 \subseteq \operatorname{Nil}_{A[X]} 0$.*

Proof. Let $f \in \operatorname{Jac}_{A[X]} 0$. Then we have $1 - Xf \in A[X]^\times$. Hence $f \in (\operatorname{Nil}_A 0)[X]$ by Proposition 3.3, and thus $f \in \operatorname{Nil}_{A[X]} 0$. $\square$

For $a \in A$, let A_a denote the localization $A[1/a]$. Quitté [18] has pointed out that the following lemma is essentially contained in our original proof of Lemma 3.2.

Lemma 3.1. *Let A be a ring and $a \in A$. Let B be a ring extension of A such that B_a is integral over A_a. Then $a((\operatorname{Jac}_B 0) \cap A) \subseteq \operatorname{Jac}_A 0$.*

Proof. Let $a' \in (\operatorname{Jac}_B 0) \cap A$. For all $x \in A$, we have $1 - a'x \in B^\times$. Hence $1 - a'x \in B_a^\times \cap A_a \subseteq A_a^\times$ by Proposition 3.2. Hence $a \in \operatorname{Nil}_A(1 - a'x) \subseteq \operatorname{Jac}_A(1 - a'x)$. Thus, $aa' \in \operatorname{Jac}_A 0$ by Lemma 2.1-(2) with $(A, U, x, y) = (A, 0, a, a')$. $\square$

The following lemma is a constructive counterpart of the key lemma given by Emerton [8, Lemma 6]. Emerton's lemma assumes that A and B are integral domains. As explained in Remark 2.1, we need to remove this assumption to eliminate the use of prime ideals and maximal ideals and to obtain a constructive proof of the general Nullstellensatz. Generalizing Emerton's key lemma is the most crucial process in the proof of our main theorem.

Lemma 3.2. *Let A be a Jacobson ring, $a \in A$, and B be an A-algebra such that B_a is integral over A_a. Then $a \operatorname{Jac}_B J \subseteq \operatorname{Nil}_B J$ holds for all ideals J of B.*

Proof. Let $B' := B/J$ and $\varphi : A \to B'$ be the canonical homomorphism. Let $A' := A/\ker\varphi$. The ring B' is a ring extension of A'. Let $f \in \operatorname{Jac}_B J$. Since B'_a is integral over A'_a, there exist $n, l \geq 0$ and $a_0, \ldots, a_{n-1} \in A'$ such that $a^l f^n + a_{n-1} f^{n-1} + \cdots + a_0 =_{B'} 0$. For $k \in \{0, \ldots, n\}$, let $g_k := a^l f^k + a_{n-1} f^{k-1} + \cdots + a_{n-k}$, $J_k := \langle g_k, \ldots, g_{n-1} \rangle_{B'}$, and $(A_k, B_k) := (A'/(J_k \cap A'), B'/J_k)$. Note that B_k is a ring extension of A_k such that $(B_k)_a$ is integral over $(A_k)_a$. We prove that $af \in \operatorname{Nil}_{B_k} 0$ for all k by induction.

1. Since $a^l =_{A_0} 0$, we have $af \in \operatorname{Nil}_{B_0} 0$.

2. Let $k \geq 1$. We have $a_{n-k} \in \operatorname{Jac}_{B_k} 0$ by $f, g_k \in \operatorname{Jac}_{B_k} 0$. Hence $a a_{n-k} \in \operatorname{Jac}_{A_k} 0 \subseteq \operatorname{Nil}_{A_k} 0 \subseteq \operatorname{Nil}_{B_k} 0$ by Lemma 3.1 and the Jacobsonness of A_k. Hence $afg_{k-1} \in \operatorname{Nil}_{B_k} 0$ by $a_{n-k} = g_k - fg_{k-1} =_{B_k} -fg_{k-1}$. We have $af \in \operatorname{Nil}_{B_{k-1}} 0$ by the inductive hypothesis. Hence $af \in \operatorname{Nil}_{B_k} g_{k-1}$ by Proposition 2.1. Thus, $af \in \operatorname{Nil}_{B_k} 0$ by Lemma 2.1-(1) with $(A, U, x, y) = (B_k, 0, af, g_{k-1})$.

Thus, $af \in \operatorname{Nil}_{B_n} 0 = \operatorname{Nil}_{B'} 0$, and hence $af \in \operatorname{Nil}_B J$. □

Remark 3.1. The statement of Emerton's lemma [8, Lemma 6] is as follows:

> *If $A \to B$ is an injection of domains such that A is Jacobson, and for some $a \in A - \{0\}$, the induced morphism $A_a \to B_a$ is integral, then* $\operatorname{Jac}_B 0 = 0$.

In classical mathematics, we can use this lemma to prove Lemma 3.2 as follows:

> Define A' and B' as in the proof of Lemma 3.2.
>
> - Let $\mathfrak{p}$ be a prime ideal of B'. Let $B'' := B'/\mathfrak{p}$ and $A'' := A'/(\mathfrak{p} \cap A')$.
>   - If $a \in \mathfrak{p}$, then $a \operatorname{Jac}_{B'} 0 \subseteq \mathfrak{p}$.
>   - If $a \notin \mathfrak{p}$, then $a \in A'' - \{0\}$. Hence $\operatorname{Jac}_{B''} 0 = 0$ by Emerton's lemma. Hence $a \operatorname{Jac}_{B'} 0 \subseteq \operatorname{Jac}_{B'} 0 \subseteq \mathfrak{p}$.
>
> Hence $a \operatorname{Jac}_{B'} 0 \subseteq \bigcap_{\mathfrak{p} \in \operatorname{Spec} B'} \mathfrak{p} = \operatorname{Nil}_{B'} 0$. Thus $a \operatorname{Jac}_B J \subseteq \operatorname{Nil}_B J$. □

Lemma 3.2 implies the following two corollaries, which are useful for constructing examples of Jacobson rings.

Corollary 3.2 (pointed out by Quitté [18])**.** *If A is a Jacobson ring and $a \in A$, then A_a is Jacobson.*

Proof. Let $B = A_a$ in Lemma 3.2. □

Corollary 3.3. *If B is an algebra integral over a Jacobson ring A, then B is Jacobson.*

Proof. Let $a = 1$ in Lemma 3.2. □

We are now ready to prove our main theorem. Quitté [18] has simplified our original proof by removing redundant inductive hypotheses.

Theorem 3.1 (The General Nullstellensatz). *If A is a Jacobson ring, then so is $A[X]$.*

Proof. Let J be an ideal of $A[X]$ and $f \in \mathrm{Jac}_{A[X]} J$. Since $1 \in \langle J, 1 - fX \rangle_{A[X]}$, there exists $g \in J$ such that $1 \in \langle g, 1 - fX \rangle_{A[X]}$. There exist $n \geq 0$ and $a_0, \ldots, a_n \in A$ such that $g = a_n X^n + \cdots + a_0$.

Let $C_k := A[X]/\langle J, a_{k+1}, \ldots, a_n \rangle$ for $k \in \{-1, \ldots, n\}$. We prove that $f \in \mathrm{Nil}_{C_k} 0$ for all k by induction.

1. Let $A' := A/\langle a_0, \ldots, a_n \rangle$. We have $f \in \mathrm{Nil}_{A'[X]} 0$ by $1 \in \langle 1 - fX \rangle_{A'[X]}$ and Proposition 3.3. Hence $f \in \mathrm{Nil}_{C_{-1}} 0$.

2. Let $k \geq 0$. Then $X \in (C_k)_{a_k}$ is integral over A_{a_k} since $g =_{(C_k)_{a_k}} 0$. Hence, $(C_k)_{a_k}$ is integral over A_{a_k} by Proposition 3.1. Since $f \in \mathrm{Jac}_{C_k} 0$, we have $a_k f \in \mathrm{Nil}_{C_k} 0$ by Lemma 3.2. We have $f \in \mathrm{Nil}_{C_{k-1}} 0$ by the inductive hypothesis. Hence $f \in \mathrm{Nil}_{C_k} a_k$. Thus, $f \in \mathrm{Nil}_{C_k} 0$ by Lemma 2.1-(1) with $(A, U, x, y) = (C_k, 0, f, a_k)$.

Thus, $f \in \mathrm{Nil}_{C_n} 0 = \mathrm{Nil}_{A[X]/J} 0$, and hence $f \in \mathrm{Nil}_{A[X]} J$. □

The main theorem and Corollary 2.1 together imply the following corollary.

Corollary 3.4. *Every finitely generated algebra over a Jacobson ring is Jacobson.*

We obtain a solution to two problems [12, 1.1, 1.2] in Lombardi's list by Example 2.2, Remark 2.3, and the above corollary.

Corollary 3.5. *Let A be a zero-dimensional ring or the ring of integers $\mathbb{Z}$. Then every finitely generated algebra over A is Jacobson.*

Future research could use a similar method to provide a constructive version of the following Nullstellensatz [12, 1.4]:

Classical Theorem ([9, Theorem 5 and its corollary], [10, Satz 1], [7, Theorem 4.19]). *Let A be a Jacobson ring, B be a finitely generated A-algebra, and $\mathfrak{n}$ be a maximal ideal of B. Then $\mathfrak{m} := \mathfrak{n} \cap A$ is a maximal ideal of A, and $B/\mathfrak{n}$ is a finite extension field of $A/\mathfrak{m}$.*

We note that Wiesnet [27, Theorem 1], [28, Theorem 2] has obtained partial results in this direction.

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