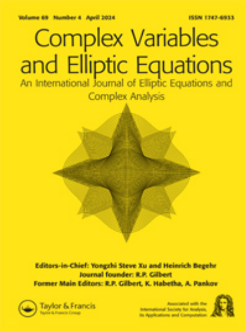

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On a fractional (p,q)-Laplacian equation with critical nonlinearities

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ABSTRACT

In this paper, we consider the following fractional (p,q)-Laplacian equations with critical Hardy-Sobolev exponents

$$\begin{cases} (-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u = \lambda |u|^{r-2}u + \mu \frac{|u|^{p_{s_1}^*(\alpha)-2}u}{|x|^\alpha} & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases}$$

where $0 < s_2 < s_1 < 1 < q < r \leq p < \frac{N}{s_1}$, $\lambda, \mu > 0$ are two parameters, $0 \leq \alpha < ps_1$ and $p_{s_1}^*(\alpha) = \frac{p(N-\alpha)}{N-ps_1}$ is the fractional Hardy-Sobolev critical exponent, $\Omega \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary. By using variational methods, we show that the problem has a nontrivial nonnegative weak solution.

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1. Introduction and main results

In the present paper, we study the existence of a nontrivial nonnegative solution for the following fractional (p,q)-Laplacian equations involving critical Hardy-Sobolev exponents:

$$\begin{cases} (-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u = \lambda |u|^{r-2}u + \mu \frac{|u|^{p_{s_1}^*(\alpha)-2}u}{|x|^\alpha} & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain containing the origin, $0 < s_2 < s_1 < 1$, $\lambda, \mu > 0$, $1 < q < r \leq p < \frac{N}{s_1}$, $0 \leq \alpha < ps_1$ and $p_{s_1}^*(\alpha) = \frac{p(N-\alpha)}{N-ps_1}$. Up to a normalization factor, the fractional a-Laplacian $(-\Delta)_a^s$ $(a > 1)$ is defined by

$$(-\Delta)_a^s u(x) = 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \backslash B_\varepsilon(x)} \frac{|u(x) - u(y)|^{a-2}(u(x) - u(y))}{|x-y|^{N+as}} \mathrm{d}y, \quad x \in \mathbb{R}^N.$$

Equation (1) descends from the study of the (p,q)-Laplacian problem, which has been widely investigated in the past decade. As is well known, when $s_1 = s_2 = 1$, the (p,q)-Laplacian problem appears in general reaction-diffusion system

$$u_t = \mathrm{div}(a(u)\nabla u) + c(x,u), \tag{2}$$

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where $a(u) = |\nabla u|^{p-2} + |\nabla u|^{q-2}$. This system has a wide range of applications in biophysics, plasma physics and chemical reactions etc. In this regard, there is a vast amount of literature concerning on the existence and multiplicity of nontrivial solutions for various (p,q)-Laplacian questions related to (2), we refer to e.g. [1–5] and the references therein.

Regarding the fractional (p,q)-Laplacian problems, we mention that in [6], Bhakta and Mukherjee investigated the following equations in smooth bounded domains

$$\begin{cases} (-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u = \theta V(x)|u|^{r-2}u + |u|^{p_{s_1}^*-2}u + \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3}$$

with $\lambda, \theta > 0, 0 < s_2 < s_1 < 1$. By imposing certain assumptions on functions f and V, the authors first established the existence of infinitely many nontrivial solutions to problem (3) in the case $1 < r < q < p < \frac{N}{s_1}$. Further, via variational argument, they proved that (3) has a nontrivial nonnegative weak solution for $V(x) = 1$, $\lambda = 0$ and $2 \le q < p < r < p_{s_1}^*(0)$. After the work [6], problem (3) has been generalized in several ways. In a recent paper, Fan [7] studied a fractional (p,q)-Laplacian equation with subcritical or critical Hardy-Sobolev exponents

$$\begin{cases} (-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u = f(x)|u|^{r-2}u + \frac{|u|^{m-2}u}{|x|^\alpha} & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{4}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $0 < s_2 < s_1 < 1$, $0 \le \alpha < ps_1 < N$, $1 < q < p < r < m \le p_{s_1}^*(\alpha)$ and f is a weight function satisfying certain assumptions. By using the Nehari manifold together with variational methods, the author showed that problem (4) admits at least one nontrivial weak solution under two different cases.

Concerning the critical fractional (p,q)-Laplacian systems related to (3), Chen [8] proved the following problem

$$\begin{cases} (-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u = \lambda Q(x)|u|^{r-2}u + \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^\beta & \text{in } \Omega, \\ (-\Delta)_p^{s_1} v + (-\Delta)_q^{s_2} v = \mu Q(x)|v|^{r-2}v + \frac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{5}$$

has a nontrivial weak solution for $1 < q < p < r < p_{s_1}^*(0)$, where $0 < s_2 < s_1 < 1, ps_1 < N$ and $\alpha + \beta = p_{s_1}^*(0)$. In the case $1 < r < q < p$, the existence of infinitely many solutions to system (5) was obtained in [9] by using Lusternik-Schnirelmann's theory for $s_1 = s_2$ and $Q(x) = 1$. Now, an interesting question about problems (3)–(5) is whether or not nontrivial solutions exist when $1 < q < r < p$. As is well known, the nonlinear term grows as r with $q < r < p$ is called the sandwich-type growth. For this problem, recently Ding and Yang [10] gave an investigation and they obtained the existence of a nontrivial solution for fractional p&q-Laplacian system involving critical sandwich-type nonlinearities. It should be mentioned that in [11], Ho and Sim first considered the existence result for this kind of problems. For more results about the existence of solutions for fractional (p,q)-Laplacian problems, the readers can refer to the papers [12–21] and the references therein.

Motivated by above-mentioned results, in this paper we study the existence of nontrivial nonnegative solutions for critical fractional (p,q)-Laplacian equation (1) under two different scenarios: the sandwich case, that is, $q < r < p$ and the linear case, with $r = p$.

Before stating our main results, we introduce some definitions. For $0 < s_2 < s_1 < 1$ and $1 < q < r \leq p$, define

$$E := \left\{ u \in W_0^{s_1,p}(\Omega) : \int_\Omega |u|^r \mathrm{d}x > 0 \right\},$$

and let

$$\lambda_* := \inf_{u \in E} C_0(p,q,r) \left(\frac{[u]_{s_1,p}^p}{p} \right)^{\frac{r-q}{p-q}} \left(\frac{[u]_{s_2,q}^q}{q} \right)^{\frac{p-r}{p-q}} \frac{r}{\int_\Omega |u|^r \mathrm{d}x}, \tag{6}$$

where $C_0(p,q,r) = (\frac{p-q}{p-r})^{\frac{p-r}{p-q}} (\frac{p-q}{r-q})^{\frac{r-q}{p-q}}$ and the symbol $[\cdot]_{s_1,p}$ ($[\cdot]_{s_2,q}$) stands for the Gagliardo seminorm (see Sect.2 below for details). With above definitions, we have $E \neq \phi$ and $\lambda_* \in [0, \infty)$ exists.

The first result of this paper is the following:

Theorem 1.1: *Let $0 < s_2 < s_1 < 1 < q < r < p < \frac{N}{s_1}$. Then for $\lambda > \lambda_*$ with λ_* given by* (6)*, there exists $\mu_* > 0$ such that for any $\mu \in (0, \mu_*)$, problem* (1) *has a nontrivial nonnegative weak solution.*

Remark 1.2: Theorem 1.1 complements the results in [6], where the situations $1 < r < q$ and $p < r < p_{s_1}^*$ (for simplicity, from now on, we write $p_{s_1}^* = p_{s_1}^*(0)$) have been considered, and generalizes the result in [11] to the fractional case. Furthermore, we would like to point out that our result seems to correct the mistake of paper [10], where the authors studied the vectorial situation of (1). Indeed, Ding et al. [10] wanted to prove the existence of a mountain pass solution with positive energy, by studying the asymptotic behaviour of mountain pass level $\lim_{\lambda \to \infty} m_\lambda = 0$. However, from Lemma 3.1 in the present paper, we can see that the threshold for the Palais-Smale condition $c^* = c^*(\lambda)$ strongly depends on λ and in particular, $c^* \to \infty$ as $\lambda \to \infty$. As a result, it is not possible that $m_\lambda < c^*$ when λ is sufficiently large. For further details, please see [10, Lemmas 2.3 and 2.4]. In their proof of lower bound estimate for $J_{\lambda,\mu}(u,v)$ ((u,v) denotes the weak limit of a $(PS)_c$ sequence), since $p_{s_1}^* > q_{s_2}^*$, the embedding $X_{0,s_2,q} \hookrightarrow L^{p_{s_1}^*}(\Omega)$ may not continuous, hence we could not check the validity of the second inequality in the proof of Lemma 2.3 in [10]. For this reason, we choose to exploit a different variational approach and study the existence of a nontrivial nonnegative solution with negative energy.

Our second goal is to study problem (1) with $r = p$ and $\mu = 1$, namely, we investigate the existence of solutions to the following equations:

$$\begin{cases} (-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u = \lambda |u|^{p-2} u + \frac{|u|^{p_{s_1}^*(\alpha)-2} u}{|x|^\alpha} & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega. \end{cases} \tag{7}$$

Our result can be stated as follows:

Theorem 1.3: *Let $0 < s_2 < s_1 < 1 < q < p < \min\{\frac{N}{s_1}, q_{s_2}^*\}$. Then for $\lambda > 0$ sufficiently large, problem* (7) *admits a nontrivial nonnegative weak solution.*

The proofs of Theorems 1.1 and 1.3 are based on the use of the variational methods. In order to prove the main results, we will encounter serious difficulties because of the lack of compactness due to the presence of critical Hardy-Sobolev exponent. To overcome the challenge, we use the concentration-compactness principle [22–24] and follow some ideas developed in [6, 11, 14]. Then by the Ekeland variational principle and the Mountain Pass theorem, we prove that problem (1) admits a nontrivial nonnegative weak solution for $r < p$ and $r = p$, respectively.

The rest of this paper is organized in the following way: In Section 2, we gather some preliminary results. Section 3 is devoted to proving Theorem 1.1 and in Section 4, we give the proof of Theorem 1.3.

Notation Throughout the paper, we denote by C a positive constant that may change from line to line.

For any $x \in \mathbb{R}^N$, we denote the ball of radius $r > 0$ centred at x by $B_r(x)$.

For smooth bounded domain $\Omega \subset \mathbb{R}^N$ containing the origin, we denote $d_\Omega := \max\{|x| : x \in \overline{\Omega}\}$.

We shall use $o_n(1)$ to denote quantities that tend to 0 as n tends to ∞.

2. Preliminaries

Let $0 < s < 1$, $1 < p < \infty$ be such that $ps < N$, the fractional Sobolev space is defined by

$$W^{s,p}(\mathbb{R}^N) = \left\{u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty\right\}$$

endowed with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} = \left(\|u\|^p_{L^p(\mathbb{R}^N)} + [u]^p_{s,p}\right)^{\frac{1}{p}},$$

where

$$[u]_{s,p} = \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}}$$

is the Gagliardo seminorm of a measurable function $u : \mathbb{R}^N \to \mathbb{R}$.

Let now $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain, we consider the closed subspace

$$W_0^{s,p}(\Omega) := \left\{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \backslash \Omega\right\}.$$

It is well known that $W_0^{s,p}(\Omega)$ is a uniformly convex Banach space with the norm $[\cdot]_{s,p}$. The embedding $W_0^{s,p}(\Omega) \hookrightarrow L^r(\Omega)$ is continuous for any $r \in [1, p_s^*]$ and compact for $r \in [1, p_s^*)$. For more details on the fractional Sobolev spaces, see [25] and the references therein.

For $0 < s_2 < s_1 < 1$ and $1 < q \le p$, we have the following result.

Lemma 2.1 ([6, Lemma 2.2]): *Let $0 < s_2 < s_1 < 1$, $1 < q \le p$ and Ω be a smooth bounded domain in $\mathbb{R}^N$, where $s_1 p < N$. Then $W_0^{s_1,p}(\Omega) \subset W_0^{s_2,q}(\Omega)$ and there exists $C(|\Omega|, N, s_1, s_2, p, q) > 0$ such that*

$$[u]_{s_2,q} \le C(|\Omega|, N, s_1, s_2, p, q)[u]_{s_1,p}, \quad \forall\, u \in W_0^{s_1,p}(\Omega). \tag{8}$$

In this paper, we work in the space $W_0^{s_1,p}(\Omega)$ and use the variational method to find critical points of the energy functional associated with (1), which is given by

$$J_{\lambda,\mu}(u) = \frac{1}{p}[u]_{s_1,p}^p + \frac{1}{q}[u]_{s_2,q}^q - \frac{\lambda}{r}\int_\Omega |u|^r \mathrm{d}x - \frac{\mu}{p_{s_1}^*(\alpha)}\int_\Omega \frac{|u|^{p_{s_1}^*(\alpha)}}{|x|^\alpha}\mathrm{d}x, \quad u \in W_0^{s_1,p}(\Omega).$$

Then $J_{\lambda,\mu} \in \mathcal{C}^1(W_0^{s_1,p}(\Omega), \mathbb{R})$ and its critical points correspond to solutions of (1).

Definition 2.2: We say that $u \in W_0^{s_1,p}(\Omega)$ is a weak solution of (1) if for every $\varphi \in W_0^{s_1,p}(\Omega)$, one has

$$\langle(-\Delta)_p^{s_1}u, \varphi\rangle_{s_1,p} + \langle(-\Delta)_q^{s_2}u, \varphi\rangle_{s_2,q} = \lambda\int_\Omega |u|^{r-2}u\varphi \mathrm{d}x + \mu\int_\Omega \frac{|u|^{p_{s_1}^*(\alpha)-2}u\varphi}{|x|^\alpha}\mathrm{d}x,$$

where

$$\langle(-\Delta)_p^{s_1}u, \varphi\rangle_{s_1,p} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps_1}}\mathrm{d}x\mathrm{d}y,$$
$$\langle(-\Delta)_q^{s_2}u, \varphi\rangle_{s_2,q} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+qs_2}}\mathrm{d}x\mathrm{d}y.$$

In our context, the fractional Hardy-Sobolev constant S_α is given by

$$S_\alpha := \inf_{u\in W_0^{s_1,p}(\Omega)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps_1}}\mathrm{d}x\mathrm{d}y}{\left(\int_\Omega \frac{|u|^{p_{s_1}^*(\alpha)}}{|x|^\alpha}\mathrm{d}x\right)^{\frac{p}{p_{s_1}^*(\alpha)}}}, \quad 0 \le \alpha < ps_1. \tag{9}$$

By results of [26–28], we know that $S_\alpha > 0$. Furthermore, when $\Omega \subset \mathbb{R}^N$ is bounded (we assume here that $0 \in \Omega$), the embedding $W_0^{s_1,p}(\Omega) \hookrightarrow L^r(|x|^{-\alpha}, \Omega)$ is continuous for $r \in [1, p_{s_1}^*(\alpha)]$ and compact for $r \in [1, p_{s_1}^*(\alpha))$.

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1 by using the Ekeland variational principle together with the concentration-compactness principle. To this aim, we define the fractional gradient of a function $u \in W_0^{s_1,p}(\Omega)$ as

$$|D^{s_1}u(x)|^p = \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps_1}}\mathrm{d}y.$$

In what follows, we shall use the notation $|D^{s_1}u|$ denotes the fractional gradient of a function u, then $|D^{s_1}u| \in L^p(\mathbb{R}^N)$ and it is well defined a.e. in $\mathbb{R}^N$.

Lemma 3.1: *Assume that $\lambda \in \mathbb{R}$ and $\mu > 0$. Let $\{u_n\} \subset W_0^{s_1,p}(\Omega)$ be a sequence such that $\|J'_{\lambda,\mu}(u_n)\|_{(W_0^{s_1,p}(\Omega))^*} \to 0$ and $J_{\lambda,\mu}(u_n) \to c$ with $c < c^*$, then there exists a strongly* *convergent subsequence. Here*

$$c^* = \left(\frac{1}{p} - \frac{1}{p_{s_1}^*(\alpha)}\right)\mu^{-\frac{p}{p_{s_1}^*(\alpha)-p}} S_\alpha^{\frac{p_{s_1}^*(\alpha)}{p_{s_1}^*(\alpha)-p}} - C(s_1,\alpha,p,r,|\Omega|)|\lambda|^{\frac{p_{s_1}^*(\alpha)}{p_{s_1}^*(\alpha)-r}}\mu^{-\frac{r}{p_{s_1}^*(\alpha)-r}}, \tag{10}$$

where

$$\begin{aligned} C(s_1,\alpha,p,r,|\Omega|) =& \frac{p_{s_1}^*(\alpha)-r}{p_{s_1}^*(\alpha)}|\Omega| d_\Omega^{\frac{\alpha r}{p_{s_1}^*(\alpha)-r}}\left(\frac{1}{r}-\frac{1}{p}\right)^{\frac{p_{s_1}^*(\alpha)}{p_{s_1}^*(\alpha)-r}} \\ &\cdot\left(\frac{1}{p}-\frac{1}{p_{s_1}^*(\alpha)}\right)^{-\frac{r}{p_{s_1}^*(\alpha)-r}}\left(\frac{r}{p_{s_1}^*(\alpha)}\right)^{\frac{r}{p_{s_1}^*(\alpha)-r}}. \end{aligned}$$

Proof: For $\lambda \in \mathbb{R}$ and $\mu > 0$, since $\{u_n\} \subset W_0^{s_1,p}(\Omega)$ is a sequence such that $\|J'_{\lambda,\mu}(u_n)\|_{(W_0^{s_1,p}(\Omega))^*} \to 0$ and $J_{\lambda,\mu}(u_n) \to c$ as $n \to \infty$. By (9) and using the Hölder inequality, we obtain

$$\begin{aligned} & c + o_n(1) + o_n(1)[u_n]_{s_1,p} \\ &= J_{\lambda,\mu}(u_n) - \frac{1}{p_{s_1}^*(\alpha)}\langle J'_{\lambda,\mu}(u_n), u_n\rangle \\ &= \left(\frac{1}{p}-\frac{1}{p_{s_1}^*(\alpha)}\right)[u_n]_{s_1,p}^p + \left(\frac{1}{q}-\frac{1}{p_{s_1}^*(\alpha)}\right)[u_n]_{s_2,q}^q \\ &\quad - \lambda\left(\frac{1}{r}-\frac{1}{p_{s_1}^*(\alpha)}\right)\int_\Omega |u_n|^r \mathrm{d}x \\ &\geq \left(\frac{1}{p}-\frac{1}{p_{s_1}^*(\alpha)}\right)[u_n]_{s_1,p}^p - |\lambda|\left(\frac{1}{r}-\frac{1}{p_{s_1}^*(\alpha)}\right)|\Omega|^{\frac{p_{s_1}^*-r}{p_{s_1}^*}}\left(\int_\Omega |u_n|^{p_{s_1}^*}\mathrm{d}x\right)^{\frac{r}{p_{s_1}^*}} \\ &\geq \left(\frac{1}{p}-\frac{1}{p_{s_1}^*(\alpha)}\right)[u_n]_{s_1,p}^p - |\lambda|\left(\frac{1}{r}-\frac{1}{p_{s_1}^*(\alpha)}\right)|\Omega|^{\frac{p_{s_1}^*-r}{p_{s_1}^*}} S_0^{-\frac{r}{p}}[u_n]_{s_1,p}^r. \end{aligned} \tag{11}$$

Recalling that $1 < q < r < p$, (11) implies that $\{u_n\} \subset W_0^{s_1,p}(\Omega)$ is bounded. Thus, up to a subsequence, still denoted by $\{u_n\}$, there exists $u \in W_0^{s_1,p}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{s_1,p}(\Omega)$. By the concentration-compactness principle [22–24], there exists $\sigma, \nu \in \mathcal{M}(\mathbb{R}^N)$, an at most enumerable set $\mathcal{J}$, a set of points $\{x_k\} \subset \overline{\Omega}$, and two sequences of nonnegative real numbers $\{\sigma_k\}, \{\nu_k\}$ such that, up to a subsequence,

$$|D^{s_1}u_n|^p \rightharpoonup \sigma \geq |D^{s_1}u|^p + \sum_{k\in\mathcal{J}} \sigma_k\delta_{x_k}, \tag{12}$$

$$\frac{|u_n|^{p_{s_1}^*(\alpha)}}{|x|^\alpha} \rightharpoonup \nu = \frac{|u|^{p_{s_1}^*(\alpha)}}{|x|^\alpha} + \sum_{k\in\mathcal{J}} \nu_k\delta_{x_k}, \tag{13}$$

in the sense of measures, with

$$S_\alpha \nu_k^{\frac{p}{p_{s_1}^*(\alpha)}} \leq \sigma_k, \quad \text{for all } k \in \mathcal{J}, \tag{14}$$

here δ_{x_k} denotes the Dirac Delta at x_k.

We aim to prove that the set $\mathcal{J}$ is finite. For this purpose, let x_k be a singular point of the measures σ and ν, and we define a cut-off function $\phi_{\varepsilon,k} = \phi(\frac{x-x_k}{\varepsilon})$, where $\phi \in C_0^\infty(\mathbb{R}^N)$ is such that $0 \leq \phi \leq 1$, $\phi(x) = 1$ in $B_1(0)$, $\phi(x) = 0$ in $\mathbb{R}^N \backslash B_2(0)$. Taking into account that $\{\phi_{\varepsilon,k} u_n\}$ is bounded in $W_0^{s_1,p}(\Omega)$, we deduce that $\langle J'_{\lambda,\mu}(u_n), \phi_{\varepsilon,k} u_n\rangle \to 0$ as $n \to \infty$, which together with (12) and (13) yields

$$\begin{aligned}
&\int_{\mathbb{R}^N} \phi_{\varepsilon,k} d\sigma - \mu \int_\Omega \phi_{\varepsilon,k} d\nu - \lambda \int_\Omega |u|^r \phi_{\varepsilon,k} \mathrm{d}x \\
&\leq \lim_{n\to\infty} \Big\{ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p \phi_{\varepsilon,k}(x)}{|x-y|^{N+ps_1}} \mathrm{d}x\mathrm{d}y - \mu \int_\Omega \frac{|u_n|^{p_{s_1}^*(\alpha)}}{|x|^\alpha} \phi_{\varepsilon,k}(x)\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^q \phi_{\varepsilon,k}(x)}{|x-y|^{N+qs_2}} \mathrm{d}x\mathrm{d}y - \lambda \int_\Omega |u_n|^r \phi_{\varepsilon,k}(x)\mathrm{d}x \Big\} \\
&= -\lim_{n\to\infty} \Big\{ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\phi_{\varepsilon,k}(x)-\phi_{\varepsilon,k}(y))u_n(y)}{|x-y|^{N+ps_1}} \mathrm{d}x\mathrm{d}y \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^{q-2}(u_n(x)-u_n(y))(\phi_{\varepsilon,k}(x)-\phi_{\varepsilon,k}(y))u_n(y)}{|x-y|^{N+qs_2}} \mathrm{d}x\mathrm{d}y \Big\} \\
&=: -\lim_{n\to\infty} (I_{1,n} + I_{2,n}).
\end{aligned}$$

We now estimate $I_{1,n}$ and $I_{2,n}$. First of all, by the Hölder inequality together with the boundedness of $\{u_n\}$ in $W_0^{s_1,p}(\Omega)$, we derive that

$$\begin{aligned}
|I_{1,n}| &= \Big| \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\phi_{\varepsilon,k}(x)-\phi_{\varepsilon,k}(y))u_n(y)}{|x-y|^{N+ps_1}} \mathrm{d}x\mathrm{d}y \Big| \\
&\leq [u_n]_{s_1,p}^{p-1} \Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\phi_{\varepsilon,k}(x)-\phi_{\varepsilon,k}(x)|^p |u_n(y)|^p}{|x-y|^{N+ps_1}} \mathrm{d}x\mathrm{d}y \Big)^{\frac{1}{p}} \\
&\lesssim \Big(\int_{\mathbb{R}^N} |D^{s_1}\phi_{\varepsilon,k}(y)|^p |u_n(y)|^p \mathrm{d}y \Big)^{\frac{1}{p}}.
\end{aligned}$$

Applying [22, Lemma 2.4] and arguing as in the proof of (2.4) in [22], we have

$$\begin{aligned}
\lim_{\varepsilon\to 0}\lim_{n\to\infty} |I_{1,n}| &\lesssim \lim_{\varepsilon\to 0} \Big(\int_{\mathbb{R}^N} |D^{s_1}\phi_{\varepsilon,k}(y)|^p |u(y)|^p \mathrm{d}y \Big)^{\frac{1}{p}} \\
&= 0.
\end{aligned}$$

By a similar argument, we get from Lemma 2.1 that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}|I_{2,n}|=0.$$

Gathering all these information, it follows that

$$\mu\nu_k\ge\sigma_k,$$

which combined with (14) yields

$$\nu_k=0\text{ or }\nu_k\ge(\mu^{-1}S_\alpha)^{\frac{p^*_{s_1}(\alpha)}{p^*_{s_1}(\alpha)-p}}. \tag{15}$$

Thus, $\mathcal{J}$ is finite. In what follows, we show that $\mathcal{J}=\phi$. To this end, we argue by contradiction and assume that $\nu_k\neq 0$ for some $k\in\mathcal{J}$, then by the boundedness of $\{u_n\}$ in $W_0^{s_1,p}(\Omega)$, one gets that

$$\begin{aligned}
c+o_n(1)=&J_{\lambda,\mu}(u_n)-\frac{1}{p}\langle J'_{\lambda,\mu}(u_n),u_n\rangle\\
=&\left(\frac{1}{q}-\frac{1}{p}\right)[u_n]^q_{s_2,q}+\lambda\left(\frac{1}{p}-\frac{1}{r}\right)\int_\Omega|u_n|^r\mathrm{d}x+\mu\left(\frac{1}{p}-\frac{1}{p^*_{s_1}(\alpha)}\right)\int_\Omega\frac{|u_n|^{p^*_{s_1}(\alpha)}}{|x|^\alpha}\mathrm{d}x\\
\ge&-|\lambda|\left(\frac{1}{r}-\frac{1}{p}\right)\int_\Omega|u_n|^r\mathrm{d}x+\mu\left(\frac{1}{p}-\frac{1}{p^*_{s_1}(\alpha)}\right)\int_\Omega\frac{|u_n|^{p^*_{s_1}(\alpha)}}{|x|^\alpha}\mathrm{d}x
\end{aligned}$$

Taking the limit $n\to\infty$ in the above inequality and using (13), we derive that

$$c\ge-|\lambda|\left(\frac{1}{r}-\frac{1}{p}\right)\int_\Omega|u|^r\mathrm{d}x+\mu\left(\frac{1}{p}-\frac{1}{p^*_{s_1}(\alpha)}\right)\left(\int_\Omega\frac{|u|^{p^*_{s_1}(\alpha)}}{|x|^\alpha}\mathrm{d}x+\nu_k\right),$$

which together with (15) yields

$$\begin{aligned}
c\ge&-|\lambda|\left(\frac{1}{r}-\frac{1}{p}\right)\int_\Omega|u|^r\mathrm{d}x+\mu\left(\frac{1}{p}-\frac{1}{p^*_{s_1}(\alpha)}\right)\int_\Omega\frac{|u|^{p^*_{s_1}(\alpha)}}{|x|^\alpha}\mathrm{d}x\\
&+\mu\left(\frac{1}{p}-\frac{1}{p^*_{s_1}(\alpha)}\right)(\mu^{-1}S_\alpha)^{\frac{p^*_{s_1}(\alpha)}{p^*_{s_1}(\alpha)-p}}\\
\ge&-|\lambda|\left(\frac{1}{r}-\frac{1}{p}\right)|\Omega|^{\frac{p^*_{s_1}(\alpha)-r}{p^*_{s_1}(\alpha)}}d_\Omega^{\frac{\alpha r}{p^*_{s_1}(\alpha)}}\left(\int_\Omega\frac{|u|^{p^*_{s_1}(\alpha)}}{|x|^\alpha}\mathrm{d}x\right)^{\frac{r}{p^*_{s_1}(\alpha)}}\\
&+\mu\left(\frac{1}{p}-\frac{1}{p^*_{s_1}(\alpha)}\right)\int_\Omega\frac{|u|^{p^*_{s_1}(\alpha)}}{|x|^\alpha}\mathrm{d}x+\mu\left(\frac{1}{p}-\frac{1}{p^*_{s_1}(\alpha)}\right)(\mu^{-1}S_\alpha)^{\frac{p^*_{s_1}(\alpha)}{p^*_{s_1}(\alpha)-p}}\\
=&f(\tau)+\left(\frac{1}{p}-\frac{1}{p^*_{s_1}(\alpha)}\right)\mu^{\frac{-p}{p^*_{s_1}(\alpha)-p}}S_\alpha^{\frac{p^*_{s_1}(\alpha)}{p^*_{s_1}(\alpha)-p}},
\end{aligned} \tag{16}$$

where

$$f(\tau) = -m\tau^{\frac{r}{p^*_{s_1}(\alpha)}} + l\tau, \quad \tau \geq 0,$$

$m = |\lambda|(\frac{1}{r} - \frac{1}{p})|\Omega|^{\frac{p^*_{s_1}(\alpha)-r}{p^*_{s_1}(\alpha)}} d_{\Omega}^{\frac{\alpha r}{p^*_{s_1}(\alpha)}}, l = \mu(\frac{1}{p} - \frac{1}{p^*_{s_1}(\alpha)})$.

By a straightforward computation, one has

$$\begin{aligned} \min_{\tau \geq 0} f(\tau) =& f\left(\left(\frac{mr}{lp^*_{s_1}(\alpha)}\right)^{\frac{p^*_{s_1}(\alpha)}{p^*_{s_1}(\alpha)-r}}\right) \\ =& -\frac{p^*_{s_1}(\alpha) - r}{p^*_{s_1}(\alpha)} m^{\frac{p^*_{s_1}(\alpha)}{p^*_{s_1}(\alpha)-r}} l^{-\frac{r}{p^*_{s_1}(\alpha)-r}} \left(\frac{r}{p^*_{s_1}(\alpha)}\right)^{\frac{r}{p^*_{s_1}(\alpha)-r}}. \end{aligned} \tag{17}$$

In view of (16) and (17), we infer that

$$c \geq \left(\frac{1}{p} - \frac{1}{p^*_{s_1}(\alpha)}\right)\mu^{-\frac{p}{p^*_{s_1}(\alpha)-p}} S_\alpha^{\frac{p^*_{s_1}(\alpha)}{p^*_{s_1}(\alpha)-p}} - C(s_1, \alpha, p, r, \Omega)|\lambda|^{\frac{p^*_{s_1}(\alpha)}{p^*_{s_1}(\alpha)-r}} \mu^{-\frac{r}{p^*_{s_1}(\alpha)-r}},$$

which contradicts the assumption $c < c^*$. Thus, $\mathcal{J} = \phi$ and $u_n \to u$ in $L^{p^*_{s_1}(\alpha)}(|x|^{-\alpha}, \Omega)$.

Next, we prove that $u_n \to u$ in $W_0^{s_1,p}(\Omega)$. For this purpose, recall that $\{u_n\} \subset W_0^{s_1,p}(\Omega)$ is bounded, taking into account that $u_n \to u$ in $L^{p^*_{s_1}(\alpha)}(|x|^{-\alpha}, \Omega)$, we get from (9) that

$$\begin{aligned} &\left|\lim_{n\to\infty} \int_\Omega \frac{|u_n|^{p^*_{s_1}(\alpha)-2} u_n(u_n - u)}{|x|^\alpha} \mathrm{d}x\right| \\ &\leq \lim_{n\to\infty} \left(\int_\Omega \frac{|u_n|^{p^*_{s_1}(\alpha)}}{|x|^\alpha}\mathrm{d}x\right)^{\frac{p^*_{s_1}(\alpha)-1}{p^*_{s_1}(\alpha)}} \left(\int_\Omega \frac{|u_n - u|^{p^*_{s_1}(\alpha)}}{|x|^\alpha}\mathrm{d}x\right)^{\frac{1}{p^*_{s_1}(\alpha)}} \\ &= 0. \end{aligned} \tag{18}$$

Furthermore, having in mind that Ω is a bounded domain of $\mathbb{R}^N$, then for $1 < q < r < p$, the compact embedding $L^r(\Omega) \hookrightarrow W_0^{s_1,p}(\Omega)$ implies that

$$\lim_{n\to\infty} \int_\Omega |u_n|^{r-2} u_n(u_n - u)\mathrm{d}x = 0. \tag{19}$$

As a consequence, using [27, Lemma 2.2], (18) and (19) yield

$$\begin{aligned} &[u_n]^p_{s_1,p} - [u]^p_{s_1,p} + [u_n]^q_{s_2,q} - [u]^q_{s_2,q} \\ &= \langle(-\Delta)^{s_1}_p u_n, u_n - u\rangle_{s_1,p} + \langle(-\Delta)^{s_2}_q u_n, u_n - u\rangle_{s_2,q} + o_n(1) \\ &= \langle J'_{\lambda,\mu}(u_n), u_n - u\rangle + \lambda \int_\Omega |u_n|^{r-2} u_n(u_n - u)\mathrm{d}x \end{aligned}$$

$$+\mu\int_\Omega \frac{|u_n|^{p_{s_1}^*(\alpha)-2}u_n(u_n-u)}{|x|^\alpha}\mathrm{d}x+o_n(1)$$
$$=o_n(1).$$

Applying the Brezis-Lieb lemma [29], we have

$$o_n(1)=[u_n-u]_{s_1,p}^p+[u_n-u]_{s_2,q}^q\geq [u_n-u]_{s_1,p}^p,$$

which implies that $u_n\to u$ in $W_0^{s_1,p}(\Omega)$. This ends the proof of Lemma 3.1. ■

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1: The proof goes through the three following claims.

Claim 3.1: Let λ_* be given by (6) and $\lambda>\lambda_*$ be arbitrary and fixed, there exist $\rho_\lambda,\delta>0$ and $\mu_1>0$ such that for any $\mu\in(-\mu_1,\mu_1)$, there holds

$$\inf_{u\in B_{\rho_\lambda}(0)}J_{\lambda,\mu}(u)<0<\delta\leq\inf_{u\in\partial B_{\rho_\lambda}(0)}J_{\lambda,\mu}(u).$$

To prove the claim, observe that from (6), we can find $\phi\in E$ such that

$$\lambda>\left(\frac{p-q}{p-r}\right)^{\frac{p-r}{p-q}}\left(\frac{p-q}{r-q}\right)^{\frac{r-q}{p-q}}\left(\frac{[\phi]_{s_1,p}^p}{p}\right)^{\frac{r-q}{p-q}}\left(\frac{[\phi]_{s_2,q}^q}{q}\right)^{\frac{p-r}{p-q}}\frac{r}{\int_\Omega|\phi|^r\mathrm{d}x}. \tag{20}$$

Then for $t>0$, one has

$$\begin{aligned}J_{\lambda,\mu}(t\phi)=&\frac{t^p}{p}[\phi]_{s_1,p}^p+\frac{t^q}{q}[\phi]_{s_2,q}^q-\frac{\lambda t^r}{r}\int_\Omega|\phi|^r\mathrm{d}x-\frac{\mu t^{p_{s_1}^*(\alpha)}}{p_{s_1}^*(\alpha)}\int_\Omega\frac{|\phi|^{p_{s_1}^*(\alpha)}}{|x|^\alpha}\mathrm{d}x\\ =&g_\lambda(t)-\frac{\mu t^{p_{s_1}^*(\alpha)}}{p_{s_1}^*(\alpha)}\int_\Omega\frac{|\phi|^{p_{s_1}^*(\alpha)}}{|x|^\alpha}\mathrm{d}x,\end{aligned} \tag{21}$$

where

$$g_\lambda(t)=c_1t^p+c_2t^q-c_3\lambda t^r,$$

here $c_1=\frac{[\phi]_{s_1,p}^p}{p}$, $c_2=\frac{[\phi]_{s_2,q}^q}{q}$, $c_3=\frac{1}{r}\int_\Omega|\phi|^r\mathrm{d}x$.

For the convenience of our argument, we write

$$g_\lambda(t)=t^q\tilde{g}_\lambda(t),$$

where $\tilde{g}_\lambda(t)=c_1t^{p-q}-c_3\lambda t^{r-q}+c_2$.

By a straightforward computation, we derive that

$$\min_{t>0}\tilde{g}_\lambda(t)=\tilde{g}_\lambda\left[\left(\frac{r-q}{p-q}c_3c_1^{-1}\lambda\right)^{\frac{1}{p-r}}\right]$$

$$=c_2 - \frac{p-r}{p-q} c_1^{-\frac{r-q}{p-r}} c_3^{\frac{p-q}{p-r}} \left(\frac{r-q}{p-q}\right)^{\frac{r-q}{p-r}} \lambda^{\frac{p-q}{p-r}}.$$

In view of (20), we get $\min_{t>0} \tilde{g}_\lambda(t) < 0$. Therefore, we can find $t_{1,\lambda}$ and $t_{2,\lambda}$ satisfying

$$0 < t_{1,\lambda} < t_\lambda := \left(\frac{r-q}{p-q} c_3 c_1^{-1} \lambda\right)^{\frac{1}{p-r}} < t_{2,\lambda},$$

and

$$\tilde{g}_\lambda(t) \begin{cases} > 0, & t \in (0, t_{1,\lambda}) \cup (t_{2,\lambda}, \infty), \\ < 0, & t \in (t_{1,\lambda}, t_{2,\lambda}). \end{cases}$$

Consequently, $g_\lambda(t_\lambda) < 0$, which together with (21) gives that

$$J_{\lambda,\mu}(t_\lambda \phi) = g_\lambda(t_\lambda) - \frac{\mu t_\lambda^{p_{s_1}^*(\alpha)}}{p_{s_1}^*(\alpha)} \int_\Omega \frac{|\phi|^{p_{s_1}^*(\alpha)}}{|x|^\alpha} \mathrm{d}x < 0, \tag{22}$$

provided $\mu > -\tilde{\mu}$, where

$$\tilde{\mu} = -\frac{g_\lambda(t_\lambda) p_{s_1}^*(\alpha)}{t_\lambda^{p_{s_1}^*(\alpha)} \int_\Omega \frac{|\phi|^{p_{s_1}^*(\alpha)}}{|x|^\alpha} \mathrm{d}x} > 0.$$

Now, set

$$\rho_\lambda := \max\left\{ (1+t_{2,\lambda})[\phi]_{s_1,p}, \left(\frac{1}{2p} \frac{r S_0^{\frac{r}{p}}}{\lambda |\Omega|^{\frac{p_{s_1}^*-r}{p_{s_1}^*}}}\right)^{\frac{1}{r-p}} \right\},$$

and

$$\mu_1 := \min\left\{ \tilde{\mu}, \frac{p_{s_1}^*(\alpha) \rho_\lambda^{p-p_{s_1}^*(\alpha)}}{2p} S_\alpha^{\frac{p_{s_1}^*(\alpha)}{p}} \right\}.$$

Then for $\mu \in (-\mu_1, \mu_1)$ and $u \in W_0^{s_1,p}(\Omega)$ with $[u]_{s_1,p} = \rho_\lambda$, it follows that

$$\begin{aligned} J_{\lambda,\mu}(u) =& \frac{1}{p}[u]_{s_1,p}^p + \frac{1}{q}[u]_{s_2,q}^q - \frac{\lambda}{r}\int_\Omega |u|^r \mathrm{d}x - \frac{\mu}{p_{s_1}^*(\alpha)} \int_\Omega \frac{|u|^{p_{s_1}^*(\alpha)}}{|x|^\alpha} \mathrm{d}x \\ \geq& \frac{1}{p}[u]_{s_1,p}^p - \frac{\lambda}{r}|\Omega|^{\frac{p_{s_1}^*-r}{p_{s_1}^*}} \left(\int_\Omega |u|^{p_{s_1}^*} \mathrm{d}x\right)^{\frac{r}{p_{s_1}^*}} - \frac{|\mu|}{p_{s_1}^*(\alpha)} \int_\Omega \frac{|u|^{p_{s_1}^*(\alpha)}}{|x|^\alpha} \mathrm{d}x \\ \geq& \frac{1}{p}[u]_{s_1,p}^p - \frac{\lambda}{r}|\Omega|^{\frac{p_{s_1}^*-r}{p_{s_1}^*}} S_0^{-\frac{r}{p}} [u]_{s_1,p}^r - \frac{|\mu|}{p_{s_1}^*(\alpha)} S_\alpha^{-\frac{p_{s_1}^*(\alpha)}{p}} [u]_{s_1,p}^{p_{s_1}^*(\alpha)} \\ =& \frac{1}{p}[u]_{s_1,p}^p - \frac{\lambda}{r}|\Omega|^{\frac{p_{s_1}^*-r}{p_{s_1}^*}} S_0^{-\frac{r}{p}} \rho_\lambda^{r-p} [u]_{s_1,p}^p - \frac{|\mu|}{p_{s_1}^*(\alpha)} S_\alpha^{-\frac{p_{s_1}^*(\alpha)}{p}} [u]_{s_1,p}^{p_{s_1}^*(\alpha)}. \end{aligned}$$

Having in mind that $1 < q < r < p$, by the definition of ρ_λ and μ_1, we infer that

$$\begin{aligned} J_{\lambda,\mu}(u) &\geq \frac{1}{2p}\rho_\lambda^p - \frac{|\mu|}{p_{s_1}^*(\alpha)} S_\alpha^{-\frac{p_{s_1}^*(\alpha)}{p}} \rho_\lambda^{p_{s_1}^*(\alpha)} \\ &= \left(\frac{1}{2p}\rho_\lambda^{p-p_{s_1}^*(\alpha)} - \frac{|\mu|}{p_{s_1}^*(\alpha)} S_\alpha^{-\frac{p_{s_1}^*(\alpha)}{p}} \right) \rho_\lambda^{p_{s_1}^*(\alpha)} \\ &=: \delta > 0. \end{aligned} \tag{23}$$

On the other hand, by the choice of ρ_λ, we have $t_\lambda \phi \in \overline{B_{\rho_\lambda}(0)}$. Thus, (22) and (23) yield

$$\inf_{u \in \overline{B_{\rho_\lambda}(0)}} J_{\lambda,\mu}(u) < 0 < \delta \leq \inf_{u \in \partial \overline{B_{\rho_\lambda}(0)}} J_{\lambda,\mu}(u),$$

this finishes the proof of claim 3.1.

Claim 3.2: There exists $\mu_* \in (0, \mu_1)$ such that for $\mu \in (0, \mu_*)$, $c^* > 0$.

Actually, by (10), we can write that

$$c^* = \left[\tilde{c}_1 - \tilde{c}_2(\lambda) \mu^{\frac{(p-r)p_{s_1}^*(\alpha)}{(p_{s_1}^*(\alpha)-p)(p_{s_1}^*(\alpha)-r)}} \right] \mu^{-\frac{p}{p_{s_1}^*(\alpha)-p}},$$

where $\tilde{c}_1, \tilde{c}_2(\lambda)$ are positive constants independent of μ. Taking into account that $1 < q < r < p$, we deduce that there exists $\mu_* \in (0, \mu_1)$ such that

$$c^* > 0, \text{ for any } \mu \in (0, \mu_*). \tag{24}$$

So claim 3.2 holds.

Claim 3.3: For $\mu \in (0, \mu_*)$ and $\lambda > \lambda_*$, define

$$c_{\lambda,\mu} := \inf_{u \in \overline{B_{\rho_\lambda}(0)}} J_{\lambda,\mu}(u),$$

where $\rho_\lambda > 0$ is given in Claim 1, then there exists a sequence $\{u_n\} \subset W_0^{s_1,p}(\Omega)$ such that

$$\|J'_{\lambda,\mu}(u_n)\|_{(W_0^{s_1,p}(\Omega))^*} \to 0 \quad \text{and} \quad J_{\lambda,\mu}(u_n) \to c_{\lambda,\mu}, \text{ as } n \to \infty. \tag{25}$$

Indeed, from Claim 3.1, we get

$$-\infty < c_{\lambda,\mu} = \inf_{u \in \overline{B_{\rho_\lambda}(0)}} J_{\lambda,\mu}(u) < 0 < \delta \leq \inf_{u \in \partial \overline{B_{\rho_\lambda}(0)}} J_{\lambda,\mu}(u).$$

For $n \in \mathbb{N}$, set $\frac{1}{n} \in (0, \inf_{u \in \partial \overline{B_{\rho_\lambda}(0)}} J_{\lambda,\mu}(u) - c_{\lambda,\mu})$, by the Ekeland variational principle, we obtain that there is a $\{u_n\} \subset \overline{B_{\rho_\lambda}(0)}$ such that

$$\begin{cases} J_{\lambda,\mu}(u_n) \leq \inf_{u \in \overline{B_{\rho_\lambda}(0)}} J_{\lambda,\mu}(u) + \frac{1}{n}, \\ J_{\lambda,\mu}(u_n) \leq J_{\lambda,\mu}(u) + \frac{1}{n}[u_n - u]_{s_1,p}, \end{cases} \tag{26}$$

for all $u \in \overline{B_{\rho_\lambda}(0)}$.

With (26), for each $n \in \mathbb{N}$, one gets that

$$J_{\lambda,\mu}(u_n) \leq \inf_{u \in \overline{B_{\rho_\lambda}(0)}} J_{\lambda,\mu}(u) + \frac{1}{n} < \inf_{u \in \partial \overline{B_{\rho_\lambda}(0)}} J_{\lambda,\mu}(u),$$

this implies that $u_n \in B_{\rho_\lambda}(0)$. Now, we define $\varphi_n : W_0^{s_1,p}(\Omega) \to \mathbb{R}$ by

$$\varphi_n(u) = J_{\lambda,\mu}(u) + \frac{1}{n}[u - u_n]_{s_1,p}.$$

According to (26), we have $\{u_n\} \subset B_{\rho_\lambda}(0)$ minimizes of $\varphi_n(u)$ on $\overline{B_{\rho_\lambda}(0)}$. Hence, for all $v \in W_0^{s_1,p}(\Omega)$ with $[v]_{s_1,p} = 1$, take $h > 0$ sufficiently small such that $u_n + hv \in \overline{B_{\rho_\lambda}(0)}$, one has

$$\begin{aligned} 0 \leq & \frac{\varphi_n(u_n + hv) - \varphi_n(u_n)}{h} \\ = & \frac{J_{\lambda,\mu}(u_n + hv) - J_{\lambda,\mu}(u_n)}{h} + \frac{1}{n}. \end{aligned}$$

Letting $h \to 0^+$, we get

$$\langle J'_{\lambda,\mu}(u_n), v \rangle \geq -\frac{1}{n}.$$

Replacing v by $-v$ in the argument above, one can deduce that

$$\langle J'_{\lambda,\mu}(u_n), v \rangle \leq \frac{1}{n}.$$

Putting the latest two inequalities together, it follows that

$$\|J'_{\lambda,\mu}(u_n)\|_{(W_0^{s_1,p}(\Omega))^*} \leq \frac{1}{n}. \tag{27}$$

Combining (26) with (27), we obtain that (25) holds, this proves the Claim.

Now, let us complete the proof of Theorem 1.1. For $\lambda > \lambda_*$ and $\mu \in (0, \mu_*)$, we deduce from Claims 3.1 and 3.3 that there exists a sequence $\{u_n\} \subset \overline{B_{\rho_\lambda}(0)} \subset W_0^{s_1,p}(\Omega)$ such that $J_{\lambda,\mu}(u_n) \to c_{\lambda,\mu} < 0$ and $\|J'_{\lambda,\mu}(u_n)\|_{(W_0^{s_1,p}(\Omega))^*} \to 0$ as $n \to \infty$. Further, by Claim 3.2 and using Lemma 3.1, we get that $\{u_n\}$ has a strongly convergent subsequence. Without loss of generality, we assume that $u_n \to u_\lambda$ in $W_0^{s_1,p}(\Omega)$ as $n \to \infty$. Thus,

$$J'_{\lambda,\mu}(u_\lambda) = 0 \quad \text{and} \quad J_{\lambda,\mu}(u_\lambda) = c_{\lambda,\mu} < 0.$$

That is, u_λ is a nontrivial solution to problem (1). By [27, Lemma 2.5], we have $u_\lambda \in W_0^{s_1,p}(\Omega)$ is nonnegative, this completes the proof of Theorem 1.1. ∎

4. Proof of Theorem 1.3

In this section, we apply the Mountain Pass theorem to study the existence of nontrivial nonnegative weak solutions to problem (1) in linear case, that is, $r = p$ and $\mu = 1$. For this purpose, we first prove that the functional

$$J_\lambda(u) = \frac{1}{p}[u]_{s_1,p}^p + \frac{1}{q}[u]_{s_2,q}^q - \frac{\lambda}{p}\int_\Omega |u|^p \mathrm{d}x - \frac{1}{p_{s_1}^*(\alpha)}\int_\Omega \frac{|u|^{p_{s_1}^*(\alpha)}}{|x|^\alpha}\mathrm{d}x,$$

possesses the mountain pass geometry in $W_0^{s_1,p}(\Omega)$.

Theorem 4.1 ([30]): *Let $(E, \|\cdot\|)$ be a Banach space and $I : E \to \mathbb{R}$ a C^1-functional satisfying the following conditions:*

(a) $I(0) = 0$;
(b) *There exists $\rho, \alpha > 0$ such that $I(u) \geq \alpha$ for all $u \in E$, with $\|u\| = \rho$;*
(c) *There exists $e \in E$ such that $\limsup_{t\to\infty} I(te) < 0$.*

Let $t_0 > 0$ be such that $\|t_0 e\| > \rho$ and $I(t_0 e) < 0$. Define $c_\lambda := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t))$, where

$$\Gamma := \{\gamma \in C([0,1], E) : \gamma(0) = 0 \quad \text{and} \quad \gamma(1) = t_0 e\}.$$

Then I possesses a Palais-Smale sequence at level c_λ, that is, there exists a sequence $\{u_n\} \subset E$ such that

$$I(u_n) \to c_\lambda, \quad I'(u_n) \to 0 \quad \text{strongly in } E'.$$

Lemma 4.2: *For any $\lambda \in \mathbb{R}$, the functional J_λ verifies hypotheses of Theorem* 4.1.

Proof: Clearly, $J_\lambda(0) = 0$. Under the assumption $1 < q < p < q_{s_2}^*$, we deduce that $W_0^{s_2,q}(\Omega) \hookrightarrow L^p(\Omega)$. Moreover, with Lemma 2.1, we know that $W_0^{s_1,p}(\Omega) \subset W_0^{s_2,q}(\Omega)$, therefore

$$T := \inf_{u\in W_0^{s_1,p}(\Omega)\setminus\{0\}} \frac{[u]_{s_2,q}^q}{\|u\|_{L^p(\Omega)}^q} \geq \inf_{u\in W_0^{s_2,q}(\Omega)\setminus\{0\}} \frac{[u]_{s_2,q}^q}{\|u\|_{L^p(\Omega)}^q} > 0. \tag{28}$$

Next, we verify ***(b)*** holds. By using (9) and (28), it follows that

$$\begin{aligned} J_\lambda(u) =& \frac{1}{p}[u]_{s_1,p}^p + \frac{1}{q}[u]_{s_2,q}^q - \frac{\lambda}{p}\int_\Omega |u|^p \mathrm{d}x - \frac{1}{p_{s_1}^*(\alpha)}\int_\Omega \frac{|u|^{p_{s_1}^*(\alpha)}}{|x|^\alpha}\mathrm{d}x \\ \geq& \frac{1}{p}[u]_{s_1,p}^p - \frac{1}{p_{s_1}^*(\alpha)} S_\alpha^{-\frac{p_{s_1}^*(\alpha)}{p}} [u]_{s_1,p}^{p_{s_1}^*(\alpha)} + \frac{1}{q}[u]_{s_2,q}^q - \frac{\lambda}{p} T^{-\frac{p}{q}}[u]_{s_2,q}^p \\ =& \left(\frac{1}{p} - \frac{1}{p_{s_1}^*(\alpha)} S_\alpha^{-\frac{p_{s_1}^*(\alpha)}{p}} [u]_{s_1,p}^{p_{s_1}^*(\alpha)-p}\right)[u]_{s_1,p}^p + \left(\frac{1}{q} - \frac{\lambda}{p}T^{-\frac{p}{q}}[u]_{s_2,q}^{p-q}\right)[u]_{s_2,q}^q. \end{aligned}$$

Let now $\rho > 0$ sufficiently small satisfying

$$\rho < \rho^* := \min\left\{\left(\frac{pT^{\frac{p}{q}}}{q\lambda}\right)^{\frac{1}{p-q}} \frac{1}{C(|\Omega|, N, s_1, s_2, p, q)}, \left(\frac{p^*_{s_1}(\alpha)}{2p} S_\alpha^{\frac{p^*_{s_1}(\alpha)}{p}}\right)^{\frac{1}{p^*_{s_1}(\alpha)-p}}\right\}.$$

Then for $u \in W_0^{s_1,p}(\Omega)$ with $[u]_{s_1,p} = \rho < \rho^*$, it follows from Lemma 2.1 that

$$\frac{1}{q} - \frac{\lambda}{p} T^{-\frac{p}{q}} [u]_{s_2,q}^{p-q} > 0.$$

Consequently, we get

$$\begin{aligned} J_\lambda(u) &\geq \left(\frac{1}{p} - \frac{1}{p^*_{s_1}(\alpha)} S_\alpha^{-\frac{p^*_{s_1}(\alpha)}{p}} (\rho^*)^{p^*_{s_1}(\alpha)-p}\right)\rho^p \\ &\geq \frac{1}{2p}\rho^p =: \alpha > 0, \end{aligned}$$

for all $[u]_{s_1,p} = \rho$.

(c) To prove **(c)**, we let $\phi \in W_0^{s_1,p}(\Omega)\backslash\{0\}$, then

$$J_\lambda(t\phi) = \frac{t^p}{p}[\phi]_{s_1,p}^p + \frac{t^q}{q}[\phi]_{s_2,q}^q - \frac{\lambda t^p}{p}\int_\Omega |\phi|^p \mathrm{d}x - \frac{t^{p^*_{s_1}(\alpha)}}{p^*_{s_1}(\alpha)}\int_\Omega \frac{|\phi|^{p^*_{s_1}(\alpha)}}{|x|^\alpha}\mathrm{d}x.$$

Since $1 < q < p < p^*_{s_1}(\alpha)$, we derive that

$$\lim_{t\to+\infty} J_\lambda(t\phi) = -\infty,$$

this ends the proof. ■

From Theorem 4.1 and Lemma 4.2, we can obtain a Palais-Smale sequence for the functional J_λ at level c_λ. In the following lemma, we prove the convergence result for the $(PS)_c$ sequence.

Lemma 4.3: *Suppose that $1 < q < p < \frac{N}{s_1}$ holds. Then for $\lambda > 0$, every $(PS)_c$ sequence of J_λ has a strongly convergent subsequence for*

$$c < c^{**} := \frac{ps_1 - \alpha}{p(N-\alpha)} S_\alpha^{\frac{N-\alpha}{ps_1-\alpha}}.$$

Proof: Let $\{u_n\} \subset W_0^{s_1,p}(\Omega)$ be a $(PS)_c$ sequence of J_λ, that is,

$$J_\lambda(u_n) \to c, \quad J'_\lambda(u_n) \to 0, \quad \text{as } n \to \infty,$$

from which we deduce that

$$c + o_n(1) = J_\lambda(u_n) = \frac{1}{p}[u_n]_{s_1,p}^p + \frac{1}{q}[u_n]_{s_2,q}^q - \frac{\lambda}{p}\int_\Omega |u_n|^p \mathrm{d}x$$

$$-\frac{1}{p_{s_1}^*(\alpha)}\int_\Omega \frac{|u_n|^{p_{s_1}^*(\alpha)}}{|x|^\alpha}\mathrm{d}x, \tag{29}$$

and

$$\left|\langle J_\lambda'(u_n), u_n\rangle\right| \le \|J_\lambda'(u_n)\|_{(W_0^{s_1,p}(\Omega))^*}[u_n]_{s_1,p} = o_n(1)[u_n]_{s_1,p}, \tag{30}$$

where

$$\begin{aligned}\langle J_\lambda'(u_n), u_n\rangle = [u_n]_{s_1,p}^p + [u_n]_{s_2,q}^q - \lambda\int_\Omega |u_n|^p\mathrm{d}x \\ - \int_\Omega \frac{|u_n|^{p_{s_1}^*(\alpha)}}{|x|^\alpha}\mathrm{d}x.\end{aligned} \tag{31}$$

Combining (29), (30) and (31), it follows that

$$\begin{aligned}c + o_n(1) + o_n(1)[u_n]_{s_1,p} &\ge J_\lambda(u_n) - \frac{1}{p}\langle J_\lambda'(u_n), u_n\rangle \\ &= \left(\frac{1}{q}-\frac{1}{p}\right)[u_n]_{s_2,q}^q + \left(\frac{1}{p}-\frac{1}{p_{s_1}^*(\alpha)}\right)\int_\Omega \frac{|u_n|^{p_{s_1}^*(\alpha)}}{|x|^\alpha}\mathrm{d}x \\ &\ge \frac{ps_1-\alpha}{p(N-\alpha)}\int_\Omega \frac{|u_n|^{p_{s_1}^*(\alpha)}}{|x|^\alpha}\mathrm{d}x,\end{aligned}$$

which yields

$$\int_\Omega \frac{|u_n|^{p_{s_1}^*(\alpha)}}{|x|^\alpha}\mathrm{d}x \le C + o_n(1)[u_n]_{s_1,p}. \tag{32}$$

Plugging (32) into (29), we get by Hölder's inequality that

$$\begin{aligned}\frac{1}{p}[u_n]_{s_1,p}^p &\le C + \frac{\lambda}{p}\int_\Omega |u_n|^p\mathrm{d}x + o_n(1)[u_n]_{s_1,p} \\ &\le C + \frac{\lambda}{p}\left(\int_\Omega |u_n|^{p_{s_1}^*(\alpha)}\mathrm{d}x\right)^{\frac{p}{p_{s_1}^*(\alpha)}} |\Omega|^{\frac{p_{s_1}^*(\alpha)-p}{p_{s_1}^*(\alpha)}} + o_n(1)[u_n]_{s_1,p} \\ &\le C + \frac{\lambda}{p} d_\Omega^{\frac{\alpha p}{p_{s_1}^*(\alpha)}} |\Omega|^{\frac{p_{s_1}^*(\alpha)-p}{p_{s_1}^*(\alpha)}} \left(\int_\Omega \frac{|u_n|^{p_{s_1}^*(\alpha)}}{|x|^\alpha}\mathrm{d}x\right)^{\frac{p}{p_{s_1}^*(\alpha)}} + o_n(1)[u_n]_{s_1,p} \\ &\le C + C + o_n(1)[u_n]_{s_1,p}^{\frac{p}{p_{s_1}^*(\alpha)}} + o_n(1)[u_n]_{s_1,p} \\ &\le C + o_n(1)[u_n]_{s_1,p} + o_n(1)[u_n]_{s_1,p}^{\frac{p}{p_{s_1}^*(\alpha)}}.\end{aligned}$$

Since $1 < p < p_{s_1}^*(\alpha)$, the last inequality implies that $\{u_n\} \subset W_0^{s_1,p}(\Omega)$ is bounded. Thus, up to a subsequence, denote by itself, there exists $u \subset W_0^{s_1,p}(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{in } W_0^{s_1,p}(\Omega), \\ u_n \rightharpoonup u & \text{in } L^{p_{s_1}^*(\alpha)}(|x|^{-\alpha}, \Omega), \\ u_n \to u & \text{in } L^r(\Omega), \quad 1 \le r < p_{s_1}^*, \\ u_n \to u & \text{a.e. in } \mathbb{R}^N. \end{cases}$$

With the help of [27, Lemma 2.2], we have that u is a weak solution of problem (7), which gives that

$$[u]_{s_1,p}^p + [u]_{s_2,q}^q = \lambda \int_\Omega |u|^p \mathrm{d}x + \int_\Omega \frac{|u|^{p_{s_1}^*(\alpha)}}{|x|^\alpha} \mathrm{d}x. \tag{33}$$

Let now $v_n = u_n - u$. Using the Brezis-Lieb lemma [29], we derive that

$$[u_n]_{s_1,p}^p = [v_n]_{s_1,p}^p + [u]_{s_1,p}^p + o_n(1), \tag{34}$$

$$[u_n]_{s_2,q}^q = [v_n]_{s_2,q}^q + [u]_{s_2,q}^q + o_n(1), \tag{35}$$

$$\int_\Omega \frac{|u_n|^{p_{s_1}^*(\alpha)}}{|x|^\alpha} \mathrm{d}x = \int_\Omega \frac{|v_n|^{p_{s_1}^*(\alpha)}}{|x|^\alpha} \mathrm{d}x + \int_\Omega \frac{|u|^{p_{s_1}^*(\alpha)}}{|x|^\alpha} \mathrm{d}x + o_n(1). \tag{36}$$

By (30), (31), (33)–(36) together with the boundedness of $\{u_n\}$ in $W_0^{s_1,p}(\Omega)$, one gets that

$$[v_n]_{s_1,p}^p + [v_n]_{s_2,q}^q - \int_\Omega \frac{|v_n|^{p_{s_1}^*(\alpha)}}{|x|^\alpha} \mathrm{d}x = o_n(1). \tag{37}$$

Taking into account that $\{v_n\} \subset W_0^{s_1,p}(\Omega)$ is bounded, up to a subsequence if necessary (still denoted by $\{v_n\}$), we may assume that

$$[v_n]_{s_1,p}^p = a + o_n(1), \quad [v_n]_{s_2,q}^q = b + o_n(1),$$
$$\int_\Omega \frac{|v_n|^{p_{s_1}^*(\alpha)}}{|x|^\alpha} \mathrm{d}x = d + o_n(1).$$

Then $a, b, d \ge 0$ and (37) implies that

$$a + b = d + o_n(1). \tag{38}$$

In view of (9) and (38), it follows that

$$a \ge S_\alpha (a+b)^{\frac{p}{p_{s_1}^*(\alpha)}} \ge S_\alpha a^{\frac{p}{p_{s_1}^*(\alpha)}}, \tag{39}$$

thus $a = 0$ or $a \ge S_\alpha^{\frac{N-\alpha}{ps_1-\alpha}}$. If $a = 0$, we complete the proof. In what follows we consider the case $a \ge S_\alpha^{\frac{N-\alpha}{ps_1-\alpha}}$. In this case, by (29), (33)–(36) and (38), we infer that

$$c + o_n(1) = J_\lambda(u_n) = \frac{a}{p} + \frac{b}{q} - \frac{d}{p_{s_1}^*(\alpha)} + \frac{1}{p}[u]_{s_1,p}^p + \frac{1}{q}[u]_{s_2,q}^q$$

$$
\begin{aligned}
&- \frac{1}{p_{s_1}^*(\alpha)} \int_\Omega \frac{|u|^{p_{s_1}^*(\alpha)}}{|x|^\alpha} \mathrm{d}x - \frac{\lambda}{p} \int_\Omega |u|^p \mathrm{d}x \\
&= \left(\frac{1}{p} - \frac{1}{p_{s_1}^*(\alpha)} \right) a + \left(\frac{1}{q} - \frac{1}{p_{s_1}^*(\alpha)} \right) b + \left(\frac{1}{q} - \frac{1}{p} \right) [u]_{s_2,q}^q \\
&+ \left(\frac{1}{p} - \frac{1}{p_{s_1}^*(\alpha)} \right) \int_\Omega \frac{|u|^{p_{s_1}^*(\alpha)}}{|x|^\alpha} \mathrm{d}x.
\end{aligned}
$$

Letting $n \to \infty$ in the above inequality, we get

$$
c \geq \left(\frac{1}{p} - \frac{1}{p_{s_1}^*(\alpha)} \right) a \geq \frac{ps_1 - \alpha}{p(N-\alpha)} S_\alpha^{\frac{N-\alpha}{ps_1-\alpha}},
$$

which contradicts the assumption $c < c^{**}$. As a consequence, $u_n \to u$ in $W_0^{s_1,p}(\Omega)$, this concludes the proof. ■

Next, we estimate the upper bound on c_λ.

Proposition 4.4: *Let c^{**} be defined as in Lemma 4.3, then $c_\lambda < c^{**}$ for $\lambda > 0$ sufficiently large.*

Proof: Let $\phi \in W_0^{s_1,p}(\Omega)\backslash\{0\}$ be arbitrary and fixed. From the proof of Lemma 4.2, we are able to find $t_0 > 0$ such that $[t_0\phi]_{s_1,p} > \rho$ and $J_\lambda(t_0\phi) < 0$. Thus, the path $\gamma(t) = tt_0\phi \in \Gamma$, from where it follows that

$$
\begin{aligned}
0 < c_\lambda &= \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_\lambda(\gamma(t)) \leq \max_{t \geq 0} J_\lambda(t\phi) \\
&\leq \max_{t \geq 0} \left\{ \frac{t^p}{p} \left([\phi]_{s_1,p}^p - \frac{\lambda}{2} \int_\Omega |\phi|^p \mathrm{d}x \right) - \frac{t^{p_{s_1}^*(\alpha)}}{p_{s_1}^*(\alpha)} \int_\Omega \frac{|\phi|^{p_{s_1}^*(\alpha)}}{|x|^\alpha} \mathrm{d}x \right\} \\
&+ \max_{t \geq 0} \left\{ \frac{t^q}{q} [\phi]_{s_2,q}^q - \frac{\lambda}{2} \frac{t^p}{p} \int_\Omega |\phi|^p \mathrm{d}x \right\}.
\end{aligned}
$$

Note that $\int_\Omega |\phi|^p \mathrm{d}x > 0$. Define $\lambda_1 = \frac{2[\phi]_{s_1,p}^p}{\int_\Omega |\phi|^p \mathrm{d}x}$, then for $\lambda \geq \lambda_1$, it holds $[\phi]_{s_1,p}^p - \frac{\lambda}{2} \int_\Omega |\phi|^p \mathrm{d}x \leq 0$. This gives

$$
\max_{t \geq 0} \left\{ \frac{t^p}{p} \left([\phi]_{s_1,p}^p - \frac{\lambda}{2} \int_\Omega |\phi|^p \mathrm{d}x \right) - \frac{t^{p_{s_1}^*(\alpha)}}{p_{s_1}^*(\alpha)} \int_\Omega \frac{|\phi|^{p_{s_1}^*(\alpha)}}{|x|^\alpha} \mathrm{d}x \right\} = 0.
$$

On the other hand, a straightforward computation yields

$$
\begin{aligned}
\varphi(\lambda) &:= \max_{t \geq 0} \left\{ \frac{t^q}{q} [\phi]_{s_2,q}^q - \frac{\lambda}{2} \frac{t^p}{p} \int_\Omega |\phi|^p \mathrm{d}x \right\} \\
&= \left(\frac{1}{q} - \frac{1}{p} \right) \frac{[\phi]_{s_2,q}^{\frac{pq}{p-q}}}{\left(\frac{\lambda}{2} \int_\Omega |\phi|^p \mathrm{d}x \right)^{\frac{q}{p-q}}}.
\end{aligned}
$$

Observe that $\varphi(\lambda)$ is monotonically decreasing with respect to $\lambda > 0$ and $\lim_{\lambda\to 0^+}\varphi(\lambda) = +\infty$, $\lim_{\lambda\to+\infty}\varphi(\lambda) = 0$. Hence, there exists $\lambda_2 > 0$ such that $\varphi(\lambda) < c^{**}$ for all $\lambda > \lambda_2$. Consequently, for any $\lambda > \max\{\lambda_1, \lambda_2\}$, there holds $0 < c_\lambda < c^{**}$, this proves the Proposition 4.4. ■

We are now ready to complete the proof of Theorem 1.3.

Proof of Theorem 1.3: From Lemma 4.2, we know that the functional J_λ satisfies the mountain pass geometry, then Theorem 4.1 shows that there exists a $(PS)_{c_\lambda}$ sequence $\{u_n\} \subset W_0^{s_1,p}(\Omega)$ of J_λ. By Proposition 4.4, we infer that $0 < c_\lambda < c^{**}$ when $\lambda > 0$ is sufficiently large. With the help of Lemma 4.3, we know that $\{u_n\}$ has a strongly convergent subsequence. Without loss of generality, we assume that $u_n \to u_\lambda$ in $W_0^{s_1,p}(\Omega)$. Then $J_\lambda(u_\lambda) = c_\lambda$ and u_λ is a nontrivial weak solution to problem (7).

Next, we prove (7) possesses a nontrivial nonnegative weak solution. To this end, we set $u^+ = \max\{u, 0\}$, $u^- = \min\{u, 0\}$ and define

$$J_\lambda^+(u) = \frac{1}{p}[u]_{s_1,p}^p + \frac{1}{q}[u]_{s_2,q}^q - \frac{\lambda}{p}\int_\Omega (u^+)^p \mathrm{d}x - \frac{1}{p_{s_1}^*(\alpha)}\int_\Omega \frac{(u^+)^{p_{s_1}^*(\alpha)}}{|x|^\alpha}\mathrm{d}x.$$

By using a similar argument, we can obtain a nontrivial weak solution $u \in W_0^{s_1,p}(\Omega)$ for the following equation

$$\begin{cases} (-\Delta)_p^{s_1} + (-\Delta)_q^{s_2} = \lambda(u^+)^{p-1} + \frac{(u^+)^{p_{s_1}^*(\alpha)-1}}{|x|^\alpha} & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N\backslash\Omega. \end{cases} \tag{40}$$

Testing (40) with u^-, [27, Lemma 2.5] gives that

$$[u^-]_{s_1,p}^p + [u^-]_{s_2,q}^q \le \langle(-\Delta)_p^{s_1}u, u^-\rangle_{s_1,p} + \langle(-\Delta)_q^{s_2}u, u^-\rangle_{s_2,q} = 0,$$

from which we get that $u^- = 0$. Hence, $u \ge 0$ is a nontrivial nonnegative solution to (7), this concludes the proof of Theorem 1.3. ■

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