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HOMEWORK ASSIGNMENT 3

CSE 331 - Intro to Algorithm

Problem 1.

- For $n=0$, $T(0)=1$ ($\because$ definition). For $n=1$, $T(1)=1+T(0)=2T(0)$.
- For $n\geq 1$, $T(n) = 1+\sum_{j=0}^{n-1} T(j) = 1+T(n-1)+\sum_{j=0}^{n-2} T(j)$

$$= T(n-1) + 1 + \sum_{j=0}^{n-2} T(j)$$

$$= T(n-1) + T(n-1) \ (\because \text{definition})$$

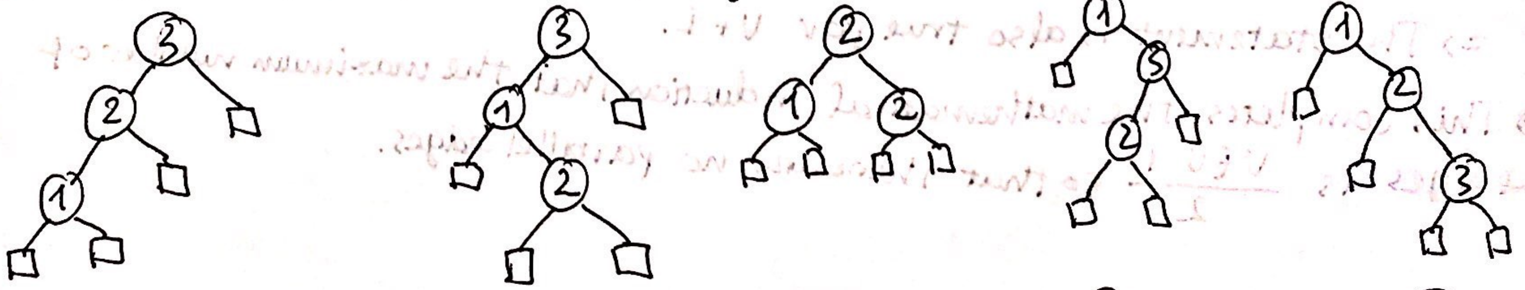
$$= 2T(n-1).$$

- Thus, we can rewrite the formula as $\begin{cases} T(n) = 1 & ,\text{if } n=0 \\ T(n) = 2T(n-1), & \text{if } n\geq 1 \end{cases}$.

- Let $P(n)$ be the statement "$T(n)=2^n$", i.e. $P(n): T(n)=2^n$.
- Base case: For $n=0$, $T(0)=1=2^0$.
  $\Rightarrow P(0)$ is true.
- Inductive case: Suppose that $P(n)$ is true, i.e. $T(n)=2^n$.
  Consider ~~$P(n+1)$~~ $T(n+1) = 2T(n)$
  $\Rightarrow T(n+1) = 2\times 2^n$ ($\because$ Induction hypothesis)
  $\Rightarrow T(n+1) = 2^{n+1}$.
  $\Rightarrow P(n+1)$ is also true.
- This completes the mathematical induction that $T(n)=2^n$.

Problem 2.

- We have 3 keys $k_1 < k_2 < k_3$. From the hint, there are 5 possible binary search tree, each of them is equally likely, i.e. probability of $\frac{1}{5}$.



1 2 3 4 5

Node (key)	The number of comparisions				
	Tree 1	Tree 2	Tree 3	Tree 4	Tree 5
1	3	2	2	1	1
2	2	3	1	3	2
3	1	1	2	2	3

- The Average numbers of comparisions of each key are:

$E(\cancel{no}k_1) = \frac{1}{5}(3+2+2+1+1) = 1.8$

$E(k_2) = \frac{1}{5}(2+3+1+3+2) = 2.2$

$E(k_3) = \frac{1}{5}(1+1+2+2+3) = 1.8$

- Thus, $E(\#\text{comparisions}) = 0.2 \times 1.8 + 0.5 \times 2.2 + 0.3 \times 1.8 = 2$.

Problem 3.

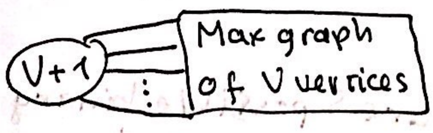
(a) Answer: $\frac{V(V-1)}{2}$ edges. This can be proven by induction.

- Base case: For $V = 1$, there's only 1 vertex $\Rightarrow$ There ~~is~~ should be no edges. $\Rightarrow$ The number of edge is $0 = \frac{1 \times 0}{2} = \frac{1 \times (1-1)}{2}$.

$\Rightarrow$ The statement is true for $V = 1$.

- Inductive ~~Suppose~~ case: Suppose the statement is true for V, i.e. the maximum number of edges is $\frac{V(V-1)}{2}$. Let's consider that we add one more vertex to the graph, i.e. there are $V+1$ vertices. If there are $V+1$ new edges, by the Pigeonhole Principle, there will be parallel ~~eg~~ edges ($\because$ there are only V choices for the new vertex to connect to). Thus, the maximum number of edges must be V.

$\Rightarrow$ The total maximum edges is: $V + (\#\text{edges in the original graph})$



$= V + \frac{V(V-1)}{2}$ ($\because$ Induction hypothesis)

$= \frac{2V + V(V-1)}{2}$

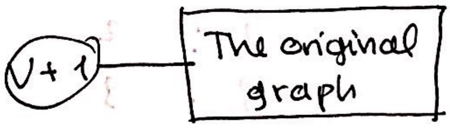
$= \frac{V(V+1)}{2}$

$\Rightarrow$ The statement is also true for $V+1$.

- This completes the mathematical induction that the maximum number of edges is $\frac{V(V-1)}{2}$ so that there are no parallel edges.

(b) Answer: $V-1$ edges. This can also be proven by induction.

- Base case: For $V = 1$, there's only 1 vertex. The minimum number of edge is 0, and since there's only 1 vertex, it can not be isolated.
  - $\Rightarrow$ The minimum number of edges is $0 = 1-1$.
  - $\Rightarrow$ The statement is true for $V = 1$.
- Inductive case: Suppose that the statement is true for V, i.e. the minimum number of edges is $V-1$. Let's consider that we add one more vertex to the original graph. If there's no new edge added, the new vertex will be isolated $\Rightarrow$ At least 1 new edge has to be added.
  - $\Rightarrow$ The minimum number of edge : $1 + (\#\text{edges in the original graph})$



$$
\begin{aligned}
&= 1 + V - 1 \\
&= V \\
&= (V+1) - 1.
\end{aligned}
$$

  - $\Rightarrow$ The statement is also true for $V+1$.
- This completes the mathematical induction that the minimum number of edges is $V-1$ so that none of which are isolated.