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MTH20001 - Elementary Number Theory

Quiz 2.

Problem 1.

$F(1) = \sigma(1) = 1.$

Since $\sigma(n)$ is multiplicative, $F(n)$ is also multiplicative.

Consider $n = p^k$, p is prime, $k > 0$.

$$F(p^k) = \sum_{d \mid p^k} \sigma(d) = \sigma(1) + \sigma(p) + \dots + \sigma(p^k)$$
$$= 1 + \frac{p^2-1}{p-1} + \frac{p^3-1}{p-1} + \dots + \frac{p^{k+1}-1}{p-1}.$$

Thus,
$$\sum_{k=1}^{7} F(k) = F(1) + F(2) + F(3) + F(4) + F(5) + F(6) + F(7)$$
$$= F(1) + F(2) + F(3) + F(2^2) + F(5) + F(2)F(3) + F(7)$$
$$= 1 + 1 + 2 + 1 + 3 + 1 + 2 + 1 + 2 + 4 + 1 + 5 + (1+2)(1+3) + 1 + 7$$
$$= 44.$$

Problem 2.

- This can be proven by induction on n.
- Basis case: For $n = 2$, $\sum_{k=1}^{n} \sigma(k) = \sigma(1) + \sigma(2) = 1 + 1 + 2 = 4 \geq \frac{2^2+2+2}{2} = 4$

  $\Rightarrow$ It is true for $n = 2$.
- Inductive case: Suppose it is true for $n = s$, i.e. $\sum_{k=1}^{s} \sigma(k) \geq \frac{s^2+s+2}{2}$.

  Consider $n = s+1$:
  $$\sum_{k=1}^{s+1} \sigma(k) = \sigma(s+1) + \sum_{k=1}^{s} \sigma(k) \geq \sigma(s+1) + \frac{s^2+s+2}{2} \quad (\because \text{Induction hypothesis})$$

  Moreover, $\sigma(a) \geq 1 + a \ \forall a \in \mathbb{N}$ ($\because$ 1 and a always divide a).

  Hence,
  $$\sum_{k=1}^{s+1} \sigma(k) \geq 1 + (s+1) + \frac{s^2+s+2}{2} = s+2 + \frac{s^2+s+2}{2} = \frac{s^2+3s+6}{2}$$
  $$= \frac{s^2+2s+1 + s+1 + 2}{2}$$
  $$= \frac{(s+1)^2 + (s+1) + 2}{2}$$

  $$\Rightarrow \sum_{k=1}^{s+1} \sigma(k) \geq \frac{(s+1)^2+(s+1)+2}{2}.$$

$\Rightarrow$ It is also true for $n = s+1$.

This completes the mathematics induction that $\sum_{k=1}^{n} \sigma(k) \geq \frac{n^2+n+2}{2}$.

Problem 3. $\begin{cases} x \equiv 2 \pmod{15} \\ x \equiv 3 \pmod{8}. \end{cases}$

$\gcd(15, 8) = \cancel{1}\ \gcd(3\times5, 2^3) = 1 \Rightarrow$ solution exists.

~~Let M = 2×15 + 3×8~~

Let $M = 15 \times 8 = 120$

$M_1 = \frac{M}{15} = 8$

$M_2 = \frac{M}{8} = 15.$

$8x_1 \equiv 1 \pmod{15}$

$\Rightarrow 16x_1 \equiv 2 \pmod{15}$

$\Rightarrow x_1 \equiv 2 \pmod{15}$

$15x_2 \equiv 1 \pmod 8$

$\Rightarrow (15-8)x_2 \equiv 1 \pmod 8$

$\Rightarrow 7x_2 \equiv 1 \pmod 8$

$\Rightarrow -x_2 \equiv 1 \pmod 8$

$\Rightarrow x_2 \equiv -1 \pmod 8$

~~$\Rightarrow x_2 \equiv 7 \pmod 8$~~

Thus, $(x_1, x_2) = \cancel{(2, 7)}\ (2, -1)$

~~Solution is: $2\times8\times2 + 3\times15\times(-1)$~~

Solution: $x \equiv 2\times8\times2 + 3\times15\times(-1) \pmod{120}$

$\Rightarrow x \equiv -13 \pmod{120}$

$\Rightarrow x \equiv 107 \pmod{120}.$