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ASSIGNMENT 1.

Problem 1.

⊛ Firstly, let's prove that $2^{n+1} = \Omega(2^n)$, i.e. $\exists\, c, n_0 > 0$ such that $2^{n+1} \geq c2^n$ for $n \geq n_0$

choose $c = 1$, then we have: $2^{n+1} \geq 2^n \Rightarrow 2 \times 2^n \geq 2^n \Rightarrow 2 \geq 1$ ($\because 2^n > 0\ \forall n$),

which is true for any values of n.

Thus, n_0 could be any values, for simplicity, let's take $n_0 = 1$.

Then, for $c = 1$ and $n_0 = 1$, we have: $2^{n+1} \geq 1 \times 2^n$, for $n \geq 1$,

which implies $2^{n+1} = \Omega(2^n)$ —— (1)

⊛ Secondly, let's prove that $2^{n+1} = O(2^n)$, i.e. $\exists\, c, n_0 > 0$ such that $2^{n+1} \leq c2^n$ for $n \geq n_0$.

choose $c = 3$, then we have: $2^{n+1} \leq 3 \times 2^n \Rightarrow 2 \times 2^n \leq 3 \times 2^n \Rightarrow 2 \leq 3$ ($\because 2^n > 0\ \forall n$),

which is true for any values of n.

Thus, n_0 could be any values, for simplicity, let's take $n_0 = 1$.

Then, for $c = 3$ and $n_0 = 1$, we have: $2^{n+1} \leq 3 \times 2^n$, for $n \geq 1$,

which implies $2^{n+1} = O(2^n)$ —— (2)

⊛ From (1) and (2), we obtain the fact that $2^{n+1} = \Theta(2^n)$.

Problem 2.

⊛ Firstly, let's simplify some functions:

$n^{\lg\lg n} = (2^{\lg n})^{\lg\lg n} = (2^{\lg\lg n})^{\lg n} = (\lg n)^{\lg n}$.

$4^{\lg n} = (2^2)^{\lg n} = (2^{\lg n})^2 = n^2$.

$2^{\lg n} = n$

$(\sqrt{2})^{\lg n} = (2^{\frac{1}{2}})^{\lg n} = (2^{\lg n})^{\frac{1}{2}} = n^{\frac{1}{2}} = \sqrt{n}$.

$n^{1/\lg n} = (2^{\lg n})^{1/\lg n} = 2^{\lg n \times \frac{1}{\lg n}} = 2^1 = 2$.

⊛ Rank the functions by order of growth.

$g_1 = 2^{2^{n+1}}$

$g_2 = 2^{2^n}$

$g_3 = (n+1)!$

$g_4 = n!$

$g_5 = e^n$

$g_6 = n2^n$

$g_7 = 2^n$

$g_8 = (\frac{3}{2})^n$

$g_9 = n^{\lg\lg n}$

$g_{10} = (\lg n)^{\lg n}$

$g_{11} = (\lg n)!$

$g_{12} = n^3$

$g_{13} = 4^{\lg n}$

$g_{14} = n^2$

$g_{15} = n\lg n$

$g_{16} = \lg(n!)$

$g_{17} = 2^{\lg n}$

$g_{18} = n$

$g_{19} = (\sqrt{2})^{\lg n}$

$g_{20} = 2^{\sqrt{2\lg n}}$

$g_{21} = \lg^2 n$

$g_{22} = \ln(n)$

$g_{23} = \sqrt{\lg n}$

$g_{24} = \ln(\ln(n))$

$g_{25} = n^{1/\lg n}$

$g_{26} = 1$.

⊛ Partition into equivalence classes

Class 1: $2^{2^{n+1}}$

Class 2: 2^{2^n}

Class 3: $(n+1)!$

Class 4: $n!$

Class 5: e^n

Class 6: $n2^n$

Class 7: 2^n

Class 8: $(\frac{3}{2})^n$

Class 9: $\begin{cases} n^{\lg\lg n} \\ (\lg n)^{\lg n} \end{cases}$

Class 10: $(\lg n)!$

Class 11: n^3

Class 12: $\begin{cases} 4^{\lg n} \\ n^2 \end{cases}$

Class 13: $\begin{cases} n\lg n \\ \lg(n!) \end{cases}$

Class 14: $\begin{cases} 2^{\lg n} \\ n \end{cases}$

Class 15: $(\sqrt{2})^{\lg n}$

Class 16: $2^{\sqrt{2\lg n}}$

Class 17: $\lg^2 n$

Class 18: $\ln(n)$

Class 19: $\sqrt{\lg n}$

Class 20: $\ln(\ln(n))$

Class 21: $\begin{cases} n^{1/\lg n} \\ 1 \end{cases}$

Problem 3.

⊛ Base case: For $n=1$, $S_1 = 1 \leq 1 = k^0 = k^{1-1}$

For $n=2$, $S_2 = 1 \leq \frac{3}{2} = \frac{1}{2} + \frac{2}{2} = \frac{1}{2} + \frac{\sqrt{4}}{2} < \frac{1}{2} + \frac{\sqrt{5}}{2} = \frac{1+\sqrt{5}}{2} = k^1 = k^{2-1}$

For $n=3$, $S_3 = 2 = \frac{8}{4} = \frac{6+2}{4} \leq \frac{6+2\sqrt{5}}{4} = \frac{1+2\sqrt{5}+5}{4} = (\frac{1+\sqrt{5}}{2})^2 = k^2 = k^{3-1}$

Thus, the equality holds for the 3 basis cases.

⊛ Induction case: let's assume that the equality holds for all values of $n \leq p$.

i.e. $S_n \leq k^{n-1}$, for $n \leq p$.

We need to prove that the equality also holds for $n = p+1$, i.e. $S_{p+1} \leq k^p$.

Note that $S_{p+1} = S_p + S_{p-1}$

By the Induction Hypothesis, $S_{p+1} = S_p + S_{p-1} \leq k^{p-1} + k^{p-2}$

$$\begin{aligned} &= k^{p-2}(k+1) \\ &= k^{p-2}(\tfrac{1+\sqrt{5}}{2}+1) \\ &= k^{p-2} \times \tfrac{3+\sqrt{5}}{2} \\ &= k^{p-2} \times \tfrac{6+2\sqrt{5}}{4} \\ &= k^{p-2} \times (\tfrac{1+\sqrt{5}}{2})^2 \\ &= k^{p-2} \times k^2 \\ &= k^p. \end{aligned}$$

$\Rightarrow S_{p+1} \leq k^p$,

which implies the equality also holds for $n = p+1$.

⊛ Therefore, this completes the mathematical induction that $S_n \leq k^{n-1}$ for all $n > 0$.

Problem 4.

⊛ Firstly, let's prove that the number of different doubles that can be chosen from n items is $\frac{n(n-1)}{2}$.

The first item can be chosen from n choices.

The second one has $n-1$ choices

Each double is counted twice, then the product should be divided by 2.

Since they are independent event, the total number of doubles is $\frac{n(n-1)}{2}$. □

⊛ Now, let's prove the main theorem using mathematical induction.

~~Base~~

It is obvious that 3 items or more are needed, hence, $n \geq 3$.

<u>Basis case:</u> For $n = 3$, there is only one triplet, which implies the formula is true ($1 = \frac{3 \times 2 \times 1}{6}$).

<u>Induction case:</u> let's assume that the ~~statem~~ statement is true for n items.

We need to prove the statement is also true for $n+1$ items.

I.e. the ~~ne~~ number of triplets is $\frac{(n+1)(n+1-1)(n+1-2)}{6} = \frac{(n+1)n(n-1)}{6}$.

Now, there are two cases:

Case 1. The $(n+1)$-th item is not chosen.

Then, the triplets have to come from the other n items, which yeilds the number of triplets to be $\frac{n(n-1)(n-2)}{6}$ ($\because$ From Induction Hypothesis)

Case 2. The $(n+1)$-th item is chosen.

Then, the other two in the triplet must come from the other n items, which yeilds the number of triplets to be $\frac{n(n-1)}{2}$ ($\because$ by the fact proven above)

Combining the two cases, we obtain the total number of triplets:

$$\frac{n(n-1)(n-2)}{6} + \frac{n(n-1)}{2} = \frac{n(n-1)(n-2) + 3n(n-1)}{6}$$

$$= \frac{n(n-1)(n-2+3)}{6}$$

$$= \frac{(n+1)n(n-1)}{6},$$

which implies the statement also holds for $n+1$ items.

This completes the mathematical induction that the number of different ~~items~~ triplets from n items is ~~per~~ precisely $\frac{n(n-1)(n+2)}{6}$.

Problem 5.

⊛ The pseudocode:

```
Find_Min_Moves(x, y):
    distance = y - x
    if distance == 0:
        return 0
    peak = ⌊√distance⌋
    if peak² == distance:
        return 2×peak - 1
    else if distance - peak² ≤ peak:
        return 2×peak
    else:
        return 2×peak + 1
```

⊛ Algorithm Explanation.

- From the condition, it follows that the optimal sequence should be ~~symmetrical~~ symmetrical, and of the form: $1+2+3+\dots+(q-1)+q+q\dots+3+2+1$.
- Let $\Delta = y-x$ be the distance between x and y.
- The number of moves depends on Δ, let $q = \lfloor\sqrt{\Delta}\rfloor$
- If Δ is a perfect square, it follows that:

$$\Delta = q^2 = \frac{2q^2}{2} = \frac{(q+1+q-1)q}{2} = \frac{(q+1)q}{2} + \frac{(q-1)q}{2}$$
$$= 1+2+\dots+q+(q-1)+(q-2)+\dots+2+1,$$

which yeilds the number of steps is $2q-1$.

- If the sequence $1+2+3+\dots+q+q+(q-1)+\dots+2+1 \geq \Delta$; ~~it can change that~~ we can change its element (decrease the middle ones) so that it could reach Δ and still maintain the conditions. This results in $2q$ moves
- Otherwise, the sequence $1+2+\dots+q+q+\dots+2+1 < \Delta$ needs to add one more element, which yeilds $2q+1$ moves.

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