MTH26001 - Elementary Number Theory.

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ASSIGNMENT 1.

Sec 2.2

Problem 8.

From the hint, we know that $111\ldots 11 = 111\ldots 108 + 3 = 4k + 3$.

Hence, any terms in the sequence has the form $4k+3$. By Division algorithm, for any terms, k is unique.

Suppose that $111\ldots 11 = x^2$, that is, there exists perfect square in the sequence. There are 4 cases.

Case 1. $x = 4q \Rightarrow x^2 = (4q)^2 = 4(4q^2)$, which is not in the form $4k+3$.

Case 2. $x = 4q+1 \Rightarrow x^2 = (4q+1)^2 = 16q^2 + 8q + 1 = 4(4q^2+2q)+1$, which is not in the form $4k+3$.

Case 3. $x = 4q+2 \Rightarrow x^2 = (4q+2)^2 = 16q^2 + 16q + 4 = 4(4q^2+4q+1)$, which is not in the form $4k+3$.

Case 4. $x = 4q+3 \Rightarrow x^2 = (4q+3)^2 = 16q^2 + 24q + 9 = 4(q^2+6q+2)+1$, which is not in the form $4k+3$.

Thus, there cannot exist x such that $111\ldots 11 = x^2$ $(x \in \mathbb{N})$.

Hence, there ~~is~~ are no perfect square in the sequence.

Problem 9.

Let the interger that is simultaneously a square and a cube be A.

$\Rightarrow A = x^3 = y^2$ $(x, y \in \mathbb{Z})$.

Let $x = 7q + r$, $0 \le r < 7$.

For $r = 0$, $x^3 = (7q)^3 = 7(49q^3)$, which has the form $7k$.

For $r = 1$, $x^3 = (7q+1)^3 = (7q)^3 + 3(7q)^2 + 3(7q) + 1$

$= 7[(7^2q^3) + 3.7q^2 + 3q] + 1$, which has the form $7k+1$.

For $r = 2$, $x^3 = (7q+2)^3 = (7q)^3 + 3(7q)^2.2 + 3(7q).2^2 + 2^3$

$= 7[7^2q^3 + 6.7q^2 + 12q] + 7 + 1$

$= 7[7^2q^3 + 6.7q^2 + 12q + 1] + 1$, which has the form $7k+1$.

For $r = 3$, $x^3 = (7q+3)^3 = (7q)^3 + 3(7q)^2.3 + 3(7q).3^2 + 3^3$

$= 7[7^2q^3 + 9.7q^2 + 27q] + 21 + 6$

$= 7[7^2q^2 + 9.7q^2 + 27q + 3] + 6$, which has the form $7k+6$.

For $r = 4$, $x^3 = (7q+4)^3 = (7q)^3 + 3(7q)^2.4 + 3(7q).4^2 + 4^3$

$= 7[7^2q^3 + 12.7q^2 + 3.4^2q] + 63 + 1$

$= 7[7^2q^3 + 12.7q^2 + 3.4^2q + 9] + 1$, which has the form $7k+1$.

For $r = 5$, $x^3 = (7q+5)^3 = (7q)^3 + 3(7q)^2.5 + 3(7q).5^2 + 5^3$

$= 7[7^2q^3 + 15.7q^2 + 3.5^2q] + 119 + 6$

$= 7[7^2q^3 + 15.7q^2 + 3.5^2q + 17] + 6$, which has the form $7k+6$.

For $r=6$, $x^3=(7q+6)^3=(7q)^3+3(7q)^2.6+3(7q).6^2+6^3$

$=7[7^2q^3+\cancel{18}\,18.7q^2+3.6^2q]+210+6$

$=7[7^2q^3+18.7q^2+3.6^2q+30]+6$, which has the form $7k+6$.

Thus, any cube has the form of either $7k$, $7k+1$ or $7k+6$. — (1).

* Let $y=7q+r$, $0\leq r<7$.

For $r=0$, $y^2=(7q)^2=7(7q^2)$, which has the form $7k$.

For $r=1$, $y^2=(7q+1)^2=(7q)^2+2(7q)+1$

$=7[7q^2+2q]+1$, which has the form $7k+1$.

For $r=2$, $y^2=(7q+2)^2=(7q)^2+2(7q).2+2^2$

$=7[7q^2+4q]+4$, which has the form $7k+4$.

For $r=3$, y^2 has the last term $3^2=9=7+2 \Rightarrow y^2=7k+2$.

For $r=4$, $y^2=(7q+4)^2$ has the last term $4^2=16=14+2=7.2+2 \Rightarrow y^2=7k+2$.

For $r=5$, $y^2=(7q+5)^2$ has the last term $5^2=25=7.3+4 \Rightarrow y^2=7k+4$.

For $r=6$, $y^2=(7q+6)^2$ has the last term $6^2=36=7.5+1 \Rightarrow y^2=7k+1$.

Thus, any square has the form of either $7k$, $7k+1$, $7k+2$ or $7k+4$. — (2).

* Since $A=x^3=y^2$, from (1), (2) and the uniqueness of Division algorithm, A must be in the form $7k$ or $7k+1$.

$\boxed{\text{Sec 2.3}}$

Problem 4 (d) $21 \mid 4^{n+1}+5^{2n-1}$

* Base case: $n=1$, $4^{n+1}+5^{2n-1}=4^{1+1}+5^{2.1-1}=21$

$\Rightarrow$ statement is true for $n=1$.

* Suppose that the statement is true for $n=k$, i.e. $21 \mid 4^{k+1}+5^{2k-1}$.

$\Rightarrow \exists x$ such that $21x=4^{k+1}+5^{2k-1}$

* Consider $4^{k+2}+5^{2(k+1)-1}=4.4^{k+1}+5^{2k+1}$

$=4.4^{k+1}+5^2.5^{2k-1}+4.5^{2k-1}-4.5^{2k-1}$

$=4(4^{k+1}+5^{2k-1})+(5^2-4).5^{2k-1}$

$=4(21x)+21.5^{2k-1}$

$=21(4x+5^{2k-1})$.

$\Rightarrow 21 \mid 4^{k+2}+5^{2(k+1)-1}$

$\Rightarrow$ The statement is also true for $n=k+1$.

$\Rightarrow$ The statement is true for $n\geq 1$.

Thus, $21 \mid 4^{n+1}+5^{2n-1}$

Problem 8 (~~d~~b)

Let the four consecutive integers be $x, x+1, x+2$ and $x+3$, $(x \in \mathbb{Z})$

The product of them is: $x(x+1)(x+2)(x+3) = [x(x+3)][(x+1)(x+2)]$

$$= (x^2+3x)(x^2+3x+2).$$
$$= (x^2+3x)^2 + 2(x^2+3x)$$
$$= (x^2+3x)^2 + 2(x^2+3x) + 1 - 1.$$
$$= (x^2+3x+1)^2 - 1.$$

$\Rightarrow$ ~~x(x+1~~ The product is one less than a ~~pefect~~ perfect square.

Problem 15.

~~Let $d = \gcd(2a-3b, 4a-5b) \Rightarrow d$ is a multiple of~~

~~$\Rightarrow \exists n$ such that~~

Let $d = \gcd(2a-3b, 4a-5b) \Rightarrow$ for all x, y, $x(2a-3b) + y(4a-5b)$ is a multiple of d.

$\Rightarrow \exists n$ such that $dn = x(2a-3b) + y(4a-5b)$ $\forall x, y \in \mathbb{Z}$.

Choose $(x,y) = (-2, 1)$, we obtain:

$$dn = (-2)(2a-3b) + 1(4a-5b)$$
$$= -4a + 6b + 4a - 5b$$
$$= b.$$

$\Rightarrow d \mid b \Rightarrow \gcd(2a-3b, 4a-5b) \mid b.$

Let $b = -1$, then $\gcd(2a+3, 4a+5) \mid (-1)$.

$\Rightarrow \gcd(2a+3, 4a+5) = 1.$

$\boxed{\text{Sec 2.4.}}$

Problem 4 (c).

Let $d = \gcd(a+b, a^2+b^2)$.

$$\Rightarrow \begin{cases} d \mid a+b \\ d \mid a^2+b \end{cases} \Rightarrow d \mid (a+b)(a-b) + 2b^2 \Rightarrow \begin{cases} dk_1 = a+b \\ dk_2 = (a+b)(a-b) + 2b^2 \end{cases} (k_1, k_2 \in \mathbb{Z})$$

$$\Rightarrow dk_2 = dk_1(a-b) + 2b^2$$
$$\Rightarrow 2b^2 = d[k_2 - k_1(a-b)]$$
$$\Rightarrow d \mid 2b^2. \quad (1)$$

Let $e = \gcd(b, d) \Rightarrow \begin{cases} e \mid b \\ e \mid d \end{cases} \Rightarrow \begin{cases} e \mid b \\ e \mid a+b \end{cases} \Rightarrow e \mid (a+b) - b \Rightarrow e \mid a.$

Since $\gcd(a,b) = 1 \Rightarrow \exists x, y : ax + by = 1.$
$d \mid a+b \Rightarrow \exists k : dk = a+b \Rightarrow dk - b = a$

$$\Rightarrow (dk-b)x + by = 1.$$
$$\Rightarrow dkx + b(y-x) = 1.$$
$$\Rightarrow \gcd(d, b) = 1. \quad (2).$$

From (1) and (2), we have $d \mid 2 \Rightarrow d \in \{1, 2\}$.

Thus, $\gcd(a+b, a^2+b^2) = 1$ or 2.

Problem 6.

* Let $c = \gcd(a+b, ab) \Longrightarrow \begin{cases} c \mid a+b \\ c \mid ab \end{cases}$.

* Since $\gcd(a,b) = 1$, $\exists x, y : ax + by = 1$.
  $c \mid a+b \Rightarrow \exists k : ck = a+b \Rightarrow ck - b = a$

  $\Rightarrow (ck-b)x + by = 1$.
  $\Rightarrow ckx + b(y-x) = 1$.
  $\Rightarrow \gcd(c,b) = 1$.

  Similarly, we can prove that $\gcd(c,a) = 1$.

* We have $c \mid ab$ and $\gcd(c,b) = 1$, then $c \mid a$.
  However, $c \mid ab$ and $\gcd(c,a) = 1$, then $c \mid b$.
  Thus, $c \le \gcd(a,b) = 1 \Rightarrow c = 1$.
  Therefore, $\gcd(a+b, ab) = 1$.

Sec 2.5

Problem 3(b) $54x + 21y = 906$.

$$54 = 2 \times 21 + 12$$
$$21 = 12 + 9$$
$$12 = 9 + 3$$
$$9 = 3 \times 3 + 0.$$
$$\Rightarrow \gcd(54, 21) = 3.$$

$\Rightarrow$

$$\begin{aligned} 3 &= 12 - 9 \\ &= 12 - (21 - 12) \\ &= 2 \times 12 - 21 \\ &= 2(54 - 2 \times 21) - 21 \\ &= 2 \times 54 - 5 \times 21. \end{aligned}$$

Thus, $3 = 2 \times 54 - 5 \times 21$

$$\begin{aligned} \Rightarrow 906 &= 302(2 \times 54 - 5 \times 21) \\ &= 604 \times 54 - 1510 \times 21. \end{aligned}$$

$\Rightarrow (604, -1510)$ is a solution.

$\Rightarrow$ The general solution is: $\begin{cases} x = 604 + 7t \\ y = -1510 - 18t \end{cases}$, $t \in \mathbb{Z}$.

For positive solution: $\begin{cases} x > 0 \\ y > 0 \end{cases} \Rightarrow \begin{cases} 604 + 7t > 0 \\ -1510 - 18t > 0 \end{cases} \Rightarrow \begin{cases} t > -86.3 \\ t < -83.9 \end{cases} \Rightarrow t \in \{-84, -85, -86\}$.

Therefore, the positive solutions are: $(x,y) \in \{(16, 2), (9, 20), (2, 38)\}$.

Problem 4. $ax - by = c$.

- Since $\gcd(a,b) = 1$ and $\gcd(a,b) \mid c$, the solution exists.
  Let x_0, y_0 be a particular solution, i.e. $ax_0 - by_0 = c$.
- The general solution is: $\begin{cases} x = x_0 - bt \\ y = y_0 - at \end{cases}$, $t \in \mathbb{Z}$.
- For $x, y > 0 \Rightarrow \begin{cases} x_0 - bt > 0 \\ y_0 - at > 0 \end{cases} \Rightarrow \begin{cases} \cancel{t < b/x_0}\ t < x_0/b \\ t < y_0/a \end{cases}$.
- If $t < \min\left(\frac{x_0}{b}, \frac{y_0}{a}\right)$ then the inequalities are always satisfied.

Therefore, there are infinitely many solutions as long as $t < \min\left(\frac{x_0}{b}, \frac{y_0}{a}\right)$.

Problem 5 (c) We have the equations: $\begin{cases} 6x + 9y = 126. \\ 6y + 9x = 114. \end{cases}$

$$\Leftrightarrow \begin{cases} 36x + 54y = 756 \\ 36x + 24y = 456. \end{cases}$$

$$\Leftrightarrow \begin{cases} 6x + 9y = 126 \\ 30y = 300 \end{cases}$$

$$\Leftrightarrow \begin{cases} 6x + 9y = 126 \\ y = 10. \end{cases}$$

$$\Leftrightarrow \begin{cases} 6x + 9(10) = 126 \\ y = 10 \end{cases}$$

$$\Leftrightarrow \begin{cases} 6x = 36 \\ y = 10 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 6 \\ y = 10 \end{cases}.$$

Thus, there are 6 sixes and 10 nines.