Homework Report

AE305 - Numerical Methods

Aerospace Engineering Department

Homework #1

Team #8

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1 Introduction

In this homework, falling of a parachute jumper chosen as physical process.Governing equation for this problem is $F = m * a$. To modeling drag force we choose $F_{DRAG} = \frac{1}{2}\rho A C_D V^2$ where C_D is drag coefficient, A is reference area of parachute and ρ is air density. The other force acting on the jumper is gravitational force which is given by $F_G = g * m$ where g is gravitational acceleration and m is mass. Resulting governing equation for this problem is following:

$$F = m * a \tag{1}$$

$$F_G - F_{DRAG} = m * a \tag{2}$$

$$m * g - \frac{1}{2}\rho A C_D V^2 = m * a \tag{3}$$

$$a = g - \frac{\rho A C_D V^2}{2m} \tag{4}$$

Since we know $a = \frac{dV}{dt}$ equation (4) can be written as follows:

$$\frac{dV}{dt} = g - \frac{\rho A C_D V^2}{2m} \tag{5}$$

Chosen values for the constants are as following $g = 9.81m/s^2$, $C_D = 1.15$, $\rho = 1.229kg/m^3$, $A = 3.14m^2$ and $m = 100kg$.The Initial condition is $v(0) = 0$.
Then, obtained analytic solution is as following

$$v(t) = \frac{21.0262e^{0.933124t} - 21.0262}{e^{0.933124t} + 1} \tag{6}$$

2 Method

2.1 Euler's Method

Euler's method is a first order numerical method to integrate functions. It is indicial form is following:

$$y_{i+1} = y_i + s\Delta x \tag{7}$$

where s is slope of function y in other words s is equal to y'

2.2 Heun's Method

https://www.overleaf.com/5472213733kpzpznnpqjvy Indicial notation of Heun's Method as following:

$$x_{i+1} = x_i + \frac{1}{2}(s_1 + s_2)h \tag{8}$$

where s_1 slope of beginning of interval and s_2 is slope of end of the interval which is founded by using Euler's method or in indicial equation form:

$$s_1 = f(x_i, y_i) \tag{9}$$
$$s_2 = f(x_i + \Delta x, y_i + k_1 \Delta x) \tag{10}$$

In those equations Δx is step size. For iteration the predictor step given by:

$$y_{i+1}^0 = y_i + s_1 \Delta x \tag{11}$$

Iterative step given by:

$$y_{i+1}^{n+1} = y_i + \frac{1}{2}(s_1 + s_2^n \Delta x) \tag{12}$$

2.3 Runge-Kutta 2 Method

Runge-Kutta method's are family of methods such that n^{th} order RK method gives some order of accuracy of the TSE. Incidial representation of RK2 method can be shown as:

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)\Delta x \tag{13}$$

where Δx is step size, k_1 and k_2 are functions

$$k_1 = f(x_i, y_i) \tag{14}$$
$$k_2 = f(x_i + p_1 \Delta x, y_i + k_1 p_1 \Delta x) \tag{15}$$
$$\tag{16}$$

From Taylor Series Expansion the following relations can be obtained:

$$a_1 * p_1 = \frac{1}{2} \tag{17}$$
$$a_1 + a_2 = 11 \tag{18}$$

In this homework p_1 is chosen as 0.37. For the iterations in RK2 method our team use the following corrector step:

$$k_2^{n+1} = f(x_i + p_1 \Delta x, y_i + k_2^n p_1 \Delta x) \tag{19}$$

3 Results and Discussion

3.1 Euler's method with different step size

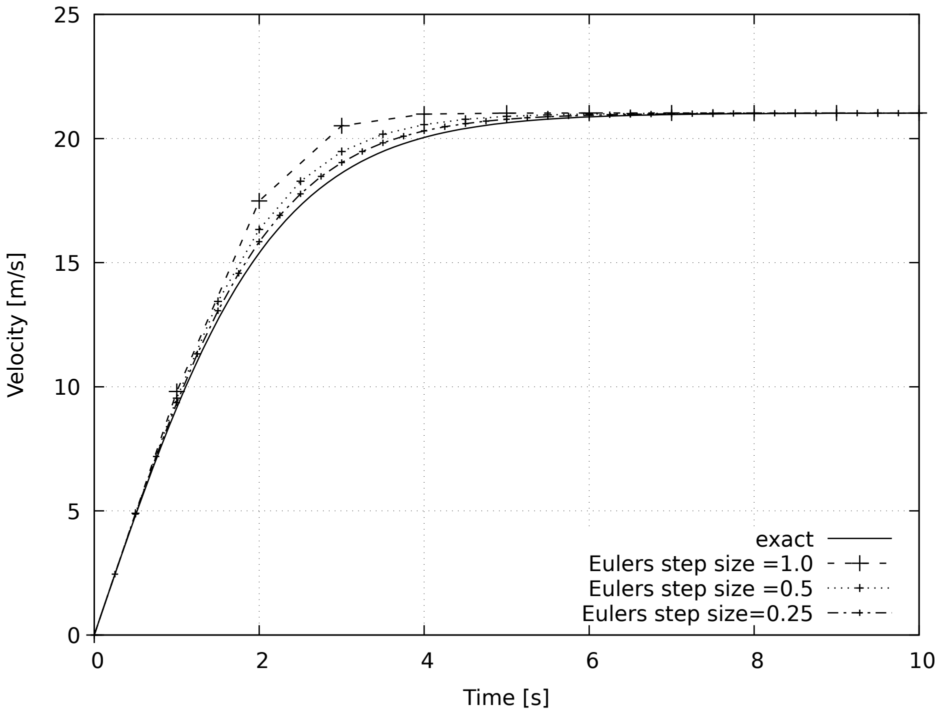

Figure 1: Velocity versus Time Graph of Falling Parachute Jumper

It can be seen from figure 1 lower step size value leads more accurate result for Euler's method.

3.2 Relative error distributions for the various step sizes

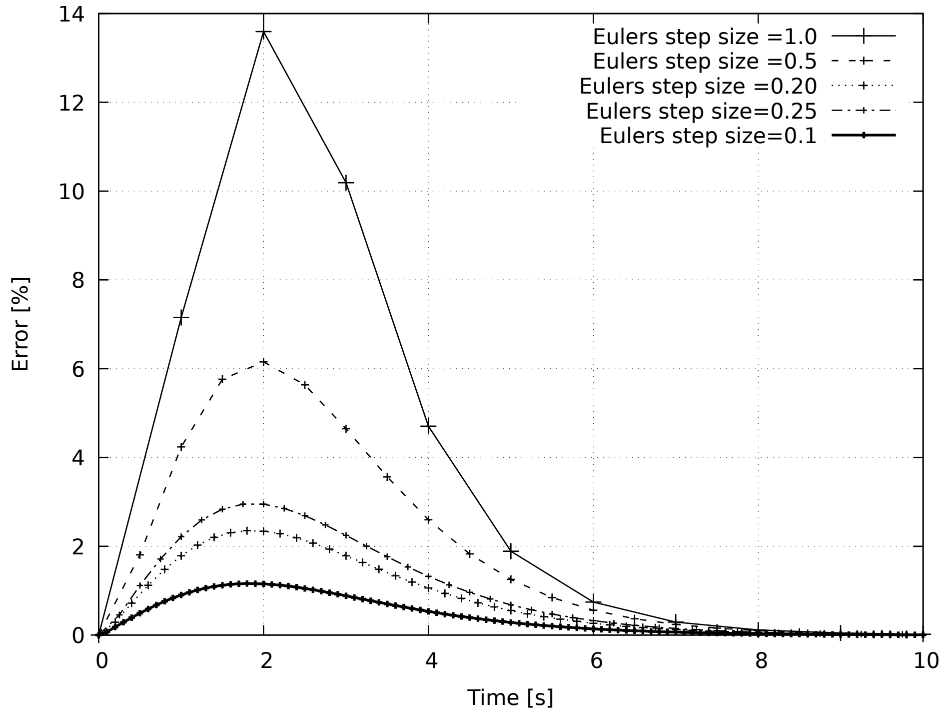

Figure 2: Relative Error Distributions of Euler's Method

It can be observable from figure 2 that small step sizes end up to a smaller relative error in Euler's method.

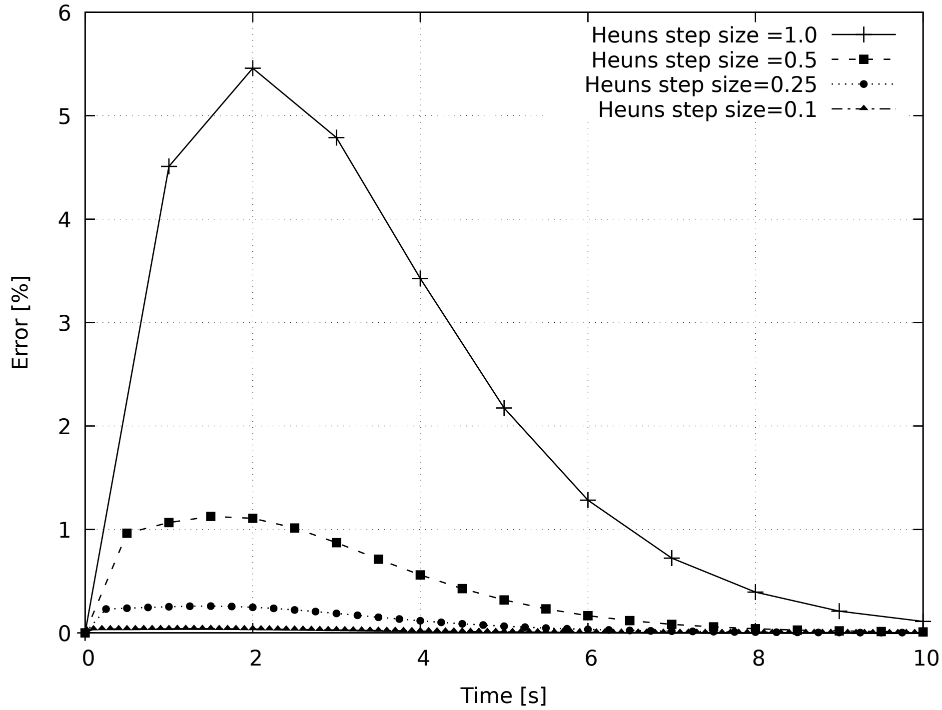

Figure 3: Relative Error Distribution of Heuns Method

It's observable from figure 3 that when we select small step sizes, relative errors have decreased in Heun's method. Also, for every step sizes relative errors first increase then it decrease as time pass. Around 2 second relative error is maximum for each solution solved by different step sizes. That relative error distribution depends on ODE which is solved.

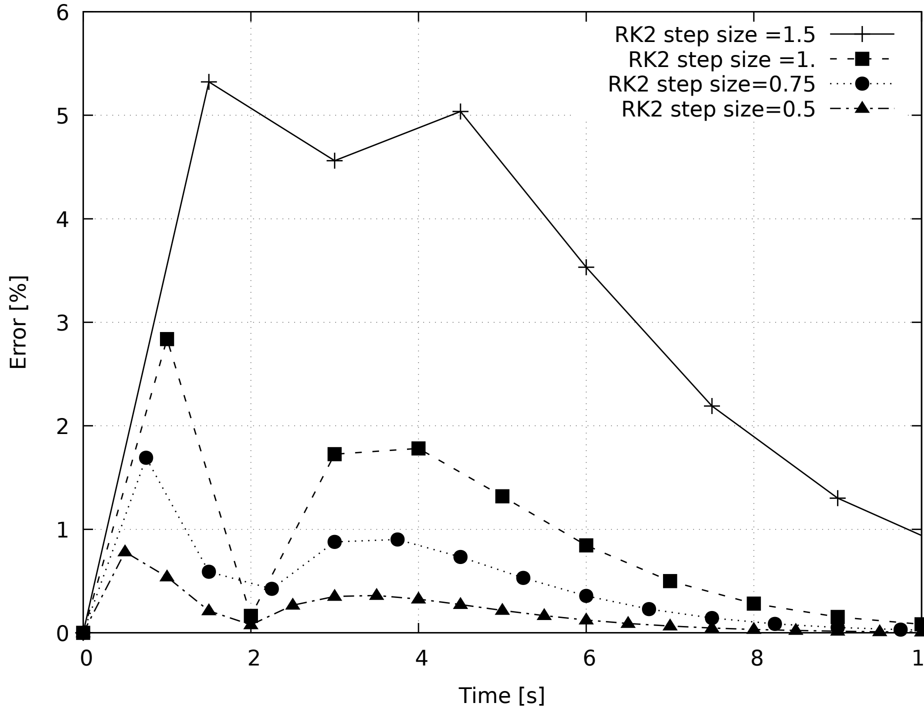

Figure 4: Relative Error Distribution of RK2 Method

It's observable from from figure 4 that lower step sizes result with lower relative error for RK2 Method. Also, relative error distrubiton is irregular until 4 second than regulary decrease for each step size.

3.3 Heun's method versus Euler's method

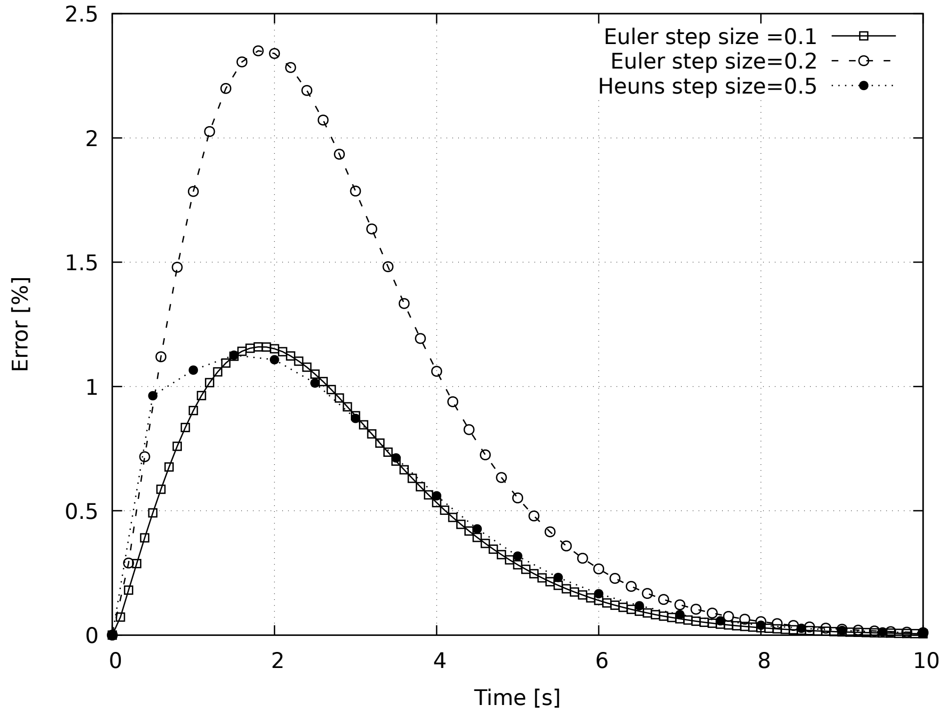

Figure 5: Relative Error Distribution of Euler's and Heun's Methods

This graph shows us 3 different approximations. They are Heun's Method with 0.5 step size, Euler's Method with 0.2 and 0.1 step sizes. It's observable from figure 5 that Heun's Method with step size 0.5 and Euler's Method with step size 0.1 have similar relative error distributions after t is equal to 2 second. Also, the worst distribution in the graph is Euler Method with 0.2 step size.

3.4 Maximum step size for a convergent solution

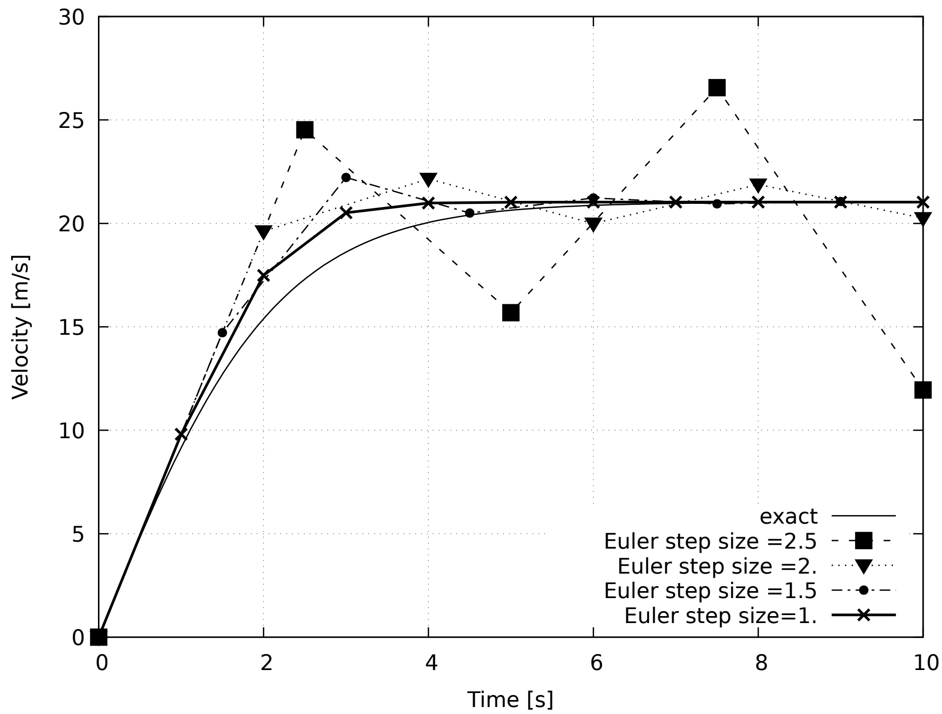

Figure 6: Velocity versus Time Graph of Falling Parachute Jumper

From figure 6, it can be observed that Euler's method over-predict the result even for small step size. Also it starts to fluctuate near step size 1.5 but it might be negligible. The fluctuation more visible when chosen step size equal to 2. and bigger for 2.5. The result for step size bigger than 2. is not acceptable for that problem.

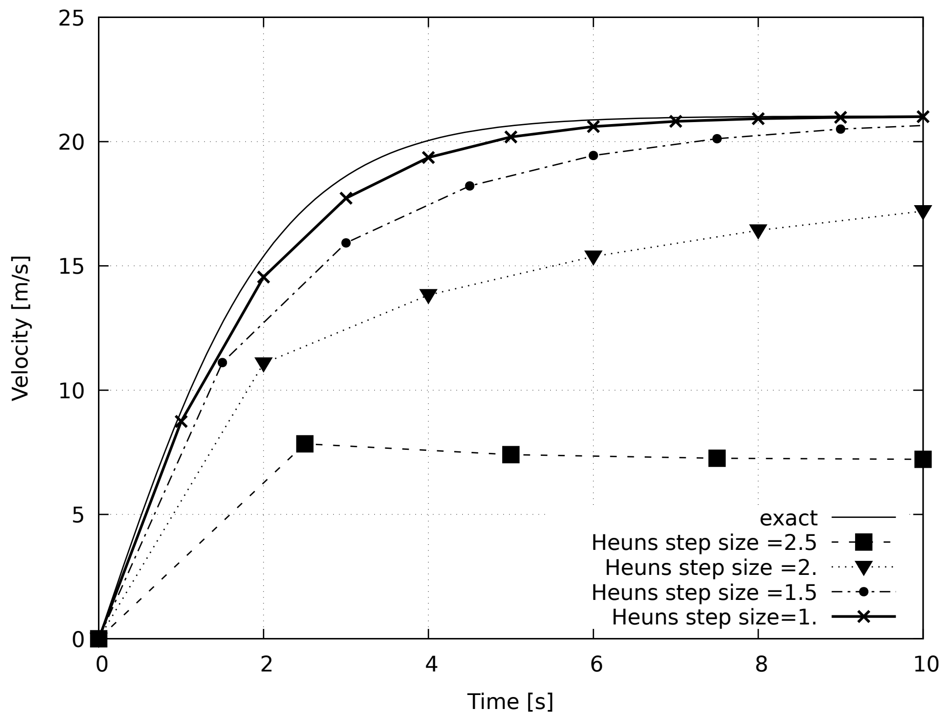

Figure 7: Velocity versus Time Graph of Falling Parachute Jumper

In this graph, there are 4 different approximations with Heun's Method and all of them are under-predict solutions. It can be seen from figure 7 that solutions with lower step sizes are more accurate. When the step size equal to 2.5 it diverges.

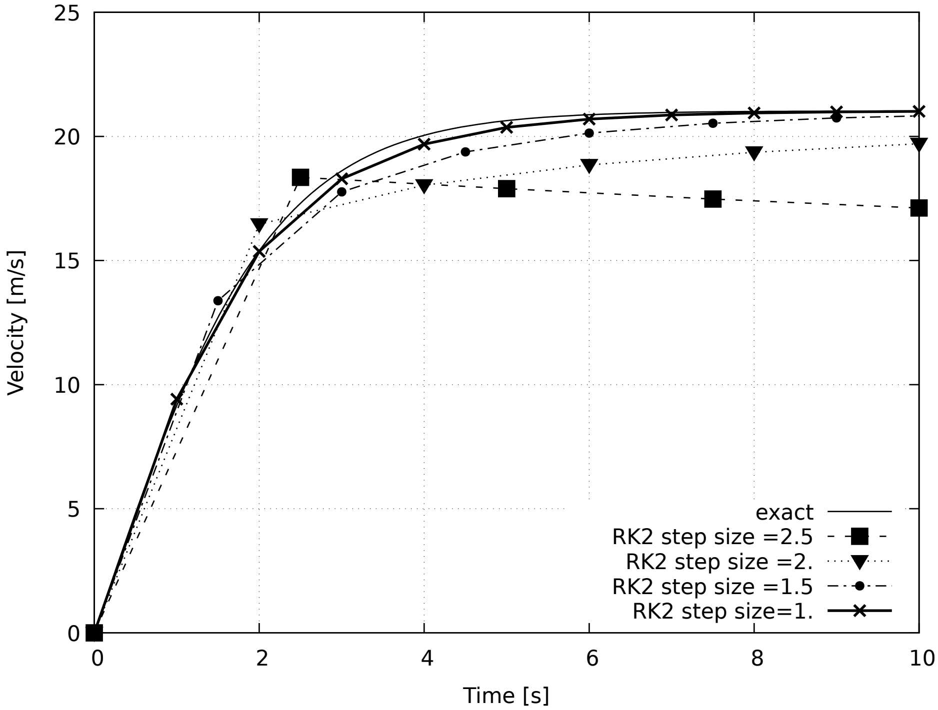

Figure 8: Velocity versus Time Graph of Falling Parachute Jumper

In figure 8, 4 different approaches for 4 different step sizes in RK2 method.If small step size values are used, more accurate results are obtained in RK2 methods.Also, if the step size value is increased more than 2, the solution does not converge. In this graph step size equal to 2.5 is diverge.

3.5 Comparison of Heun's and RK2 methods

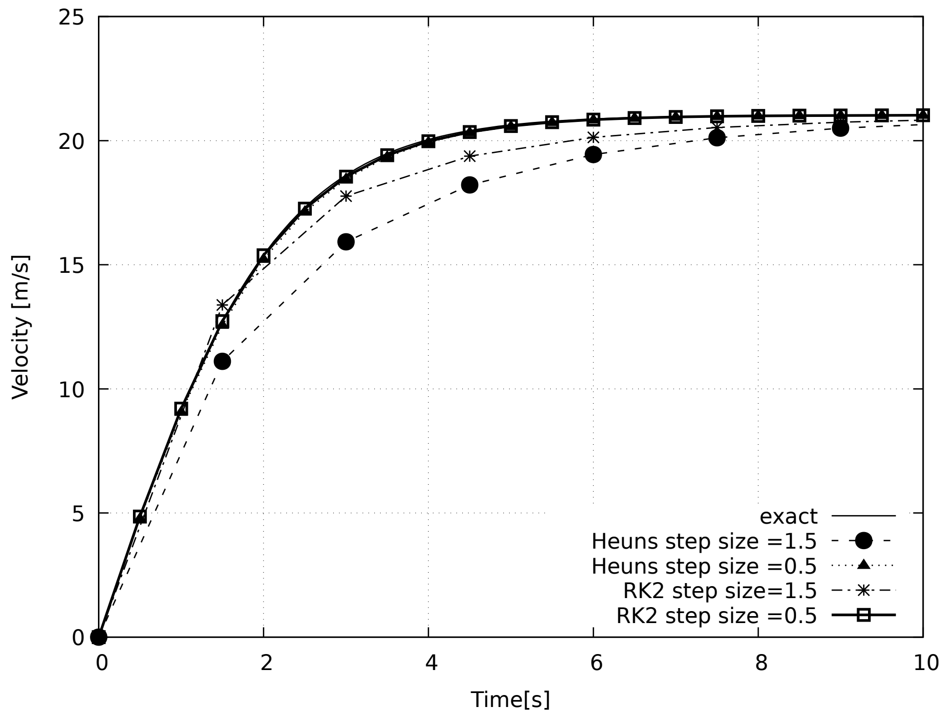

Figure 9: Velocity versus Time Graph of Falling Parachute Jumper

Figure 9 shows that RK2 method is more accurate than Heun's method at the same step size values.Also, when step size values are reduced in both methods, we get a more accurate solution.

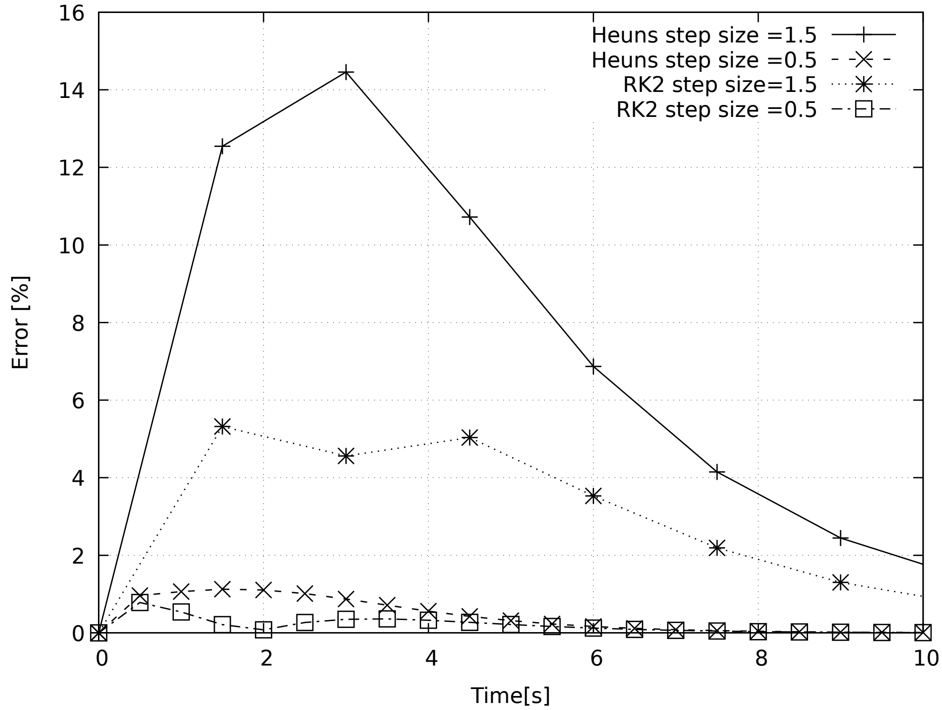

Figure 10: Relative Error Distributions of Heun's Method and RK2 Method

From figure 10, it can be said that RK2 method has less relative error than Heun's Method if both use same step size. Also it can be seen that using smaller step size gives more accurate for each methods.

3.6 Iterative RK2 method

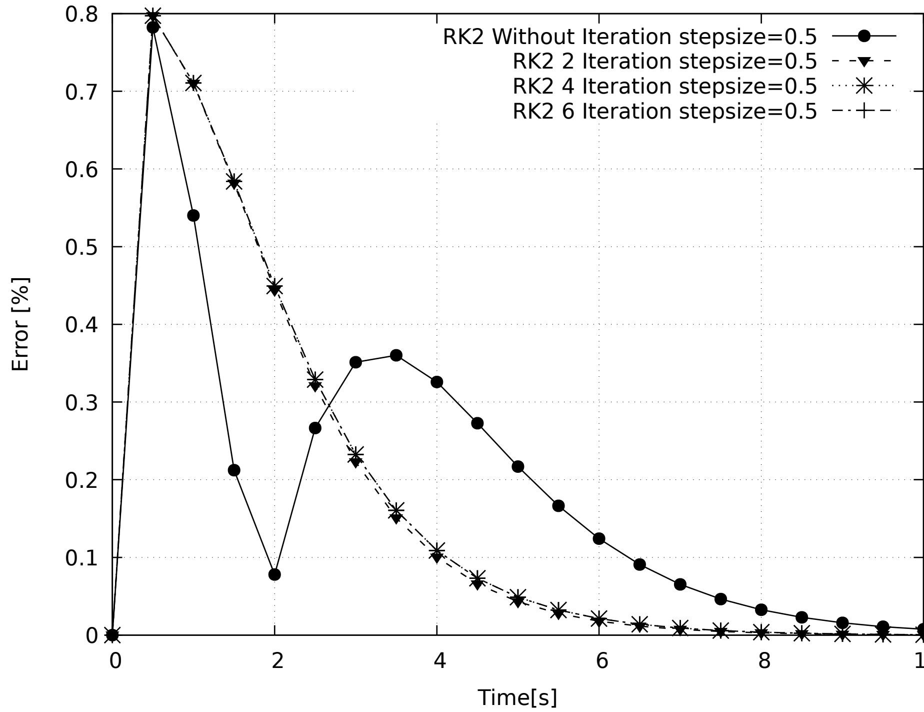

Figure 11: Relative Error Distribution of RK2 Method

It can be seen from figure 11 RK2 methods without iteration is less relative error than RK2 methods with iteration between 0-2 seconds.However, after 2 second has passed, the relative error with iteration is less. It can be said that more than 2 iteration is does not make much sense in terms of relative error.

3.7 Bonus

Leading truncation term is given by:

$$E = |A|\Delta x^n \tag{20}$$

whenever we take logarithm of both side the equation becames:

$$log_{10}E = log_{10}|A| + nlog_{10}\Delta x \tag{21}$$

Whenever $log_{10}E$, $log_{10}|A|$, n,$log_{10}\Delta x$ written as y,a,b and x respectively. Equation turn into linear equation as following:

$$y = a + bx \tag{22}$$

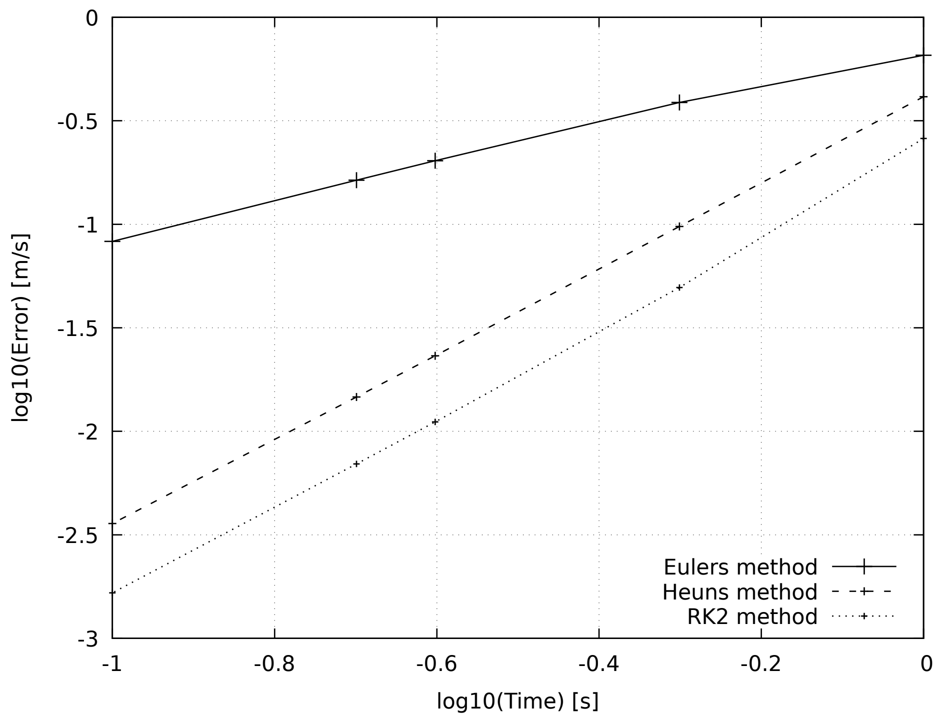

Figure 12: Error versus Time graph

From figure 12 it can be said that n value is equal to 1,2,2 for Euler's, Heun's and RK2 method respectively. Since we know that n value is slope of lines and slope of line indicates order of accuracy. It is obvious that RK2 and Heun's method have an order of accuracy of 2 and Euler's method has an order of accuracy 1 as expected.

4 Conclusion

In this homework, three method is investigated, namely Euler's method, Heun's method and Runge-Kutta 2 (RK2) method. All 3 methods can be used to solve ordinary differential equations (ODE) in a numerical manner. Firstly, all three methods have a limit step size to converge,, when that limit exceed all three will diverge. Secondly, for ODE which solved in this homework RK2 method gives the more accurate result whenever same step size chosen for all methods. Also it can be said that to achieve same accuracy in result RK2 method gives maximum step size which reduces computation time. Euler's method gives less accurate results for same step size, this is a result of it's order of accuracy. Euler's method has an order of accuracy of 1. On the other hand RK2 and Heun's method has an order of accuracy of 2. Lastly, iteration on RK2 methods help to get a more accurate result up to a point after some point accuracy gain from iteration can be negligible. To sum up, RK2 method is better in term of accuracy and step size. To sum up, in terms of accuracy all three method can be sorted as following: RK2 with Itreation > RK2 > Heun's > Euler.