Homework Report

AE305 - Numerical Methods

Aerospace Engineering Department

Homework #5
Team #8

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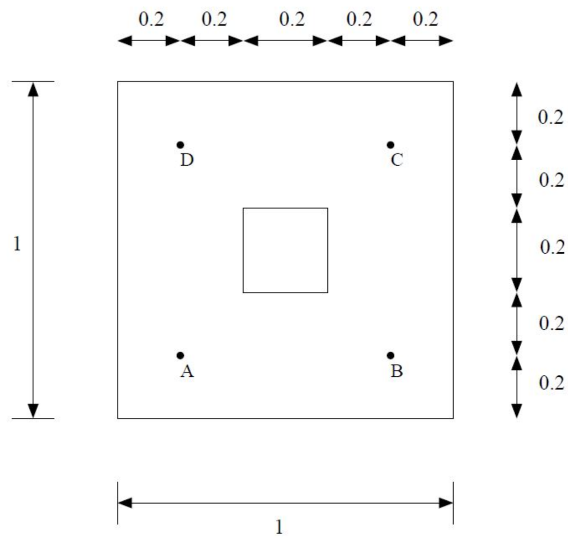
1 Introduction

In this homework, there is a solid elastic membrane. The membrane has a centered square hole as shown below.



The governing equation for this problem is Poisson equation which is elliptical partial differantial equation and can be seen below. Where u is non-dimensional transverse displacement of an elastic membrane and f is transverse load distribution.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f \tag{1}$$

While outer boundaries have zero transverse displacement, inner boundaries have constant upward transverse displacement with magnitude 0.01. In addition, there are four identical downward transverse loads applied on the points A, B, C and D.

Iterative methods like Point Jacobi, Gauss-Seidel and Successive Over Relaxation(SOR) will be implement on the problem by the guide of given incomplete Fortran. Finally, it will be examined which method works well and yield fast convergence.

2 Method

The non-dimensional transverse displacement of an elastic membrane, u, under a transverse load distribution, f, is governed by the Poisson equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f \tag{2}$$

the Poisson equation for the displacement is solved by using iterative solution methods. Those methods are as follows.

2.1 Point Jacobi Method

Poisson equation is written as center finite difference method so this equation is as follows:

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2} = f \tag{3}$$

And also;

$$\beta^2 = \frac{\Delta x^2}{\Delta y^2} \tag{4}$$

Substitute to β^2 in the equation 2.

$$u_{i,j} = \frac{u_{i+1,j} + u_{i-1,j} + \beta^2(u_{i,j+1} + u_{i,j-1}) - \Delta x^2 f}{2(1+\beta^2)} \tag{5}$$

2.2 Gauss-Seidel

With the Jacobi method, the values of $x_{i,j}^k$ obtained in the k^{th} iteration remain unchanged until the entire $(k+1)^{th}$ iteration has been calculated. With the Gauss-Seidel method, we use the new values $x_{i,j}^{k+1}$ as soon as they are known. Thus, $i-1$ and $j-1$ terms are takes in calculated values.

$$u_{i,j}^{k+1} = \frac{u_{i+1,j}^k + u_{i-1,j}^{k+1} + \beta^2(u_{i,j+1}^k + u_{i,j-1}^{k+1}) - \Delta x^2 f}{2(1+\beta^2)} \tag{6}$$

2.3 Successive Over Relaxation(SOR)

$$\Delta u^{k+1} \mid_{GS} = u_{i,j}^{k+1} \mid_{GS} - u_{i,j}^k \tag{7}$$

And then:

$$u^{k+1} \mid_{SOR} = u_{i,j}^k + w \Delta u_{i,j}^{k+1} \mid_{GS} \tag{8}$$

Thus,this equation is as follows:

$$u_{i,j}^{k+1} = \frac{w u_{i+1,j}^k + w u_{i-1,j}^{k+1} + w\beta^2(u_{i,j+1}^k + u_{i,j-1}^{k+1}) - w\Delta x^2 f}{2(1+\beta^2)} + (1-w)u_{i,j}^k \tag{9}$$

2.4 Alternating Direction Implicit (ADI)

In the ADI Method, at first we use to fixed j row and all iteration are made in this j row.And for each row, we use tridiagonal matrix.This equation is as follows:

$$u_{i-1,j}^{k+1} - 2(1+\beta^2)u_{i,j}^{k+1} + u_{i+1,j}^{k+1} = -\beta^2 u_{i,j-1}^{k} - \beta^2 u_{i,j+1}^{k} + f\Delta x^2 \tag{10}$$

And matrix form which is used for each j row is as follows:

$$\underbrace{\begin{bmatrix} -2(1+\beta^2) & 1 & 0 & \dots & 0 & 0 \\ 1 & -2(1+\beta^2) & 1 & \dots & 0 & 0 \\ 0 & 1 & -2(1+\beta^2) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -2(1+\beta^2) & 1 \\ 0 & 0 & 0 & \dots & 1 & -2(1+\beta^2) \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} u_{2,j}^{n+1} \\ u_{3,j}^{n+1} \\ u_{4,j}^{n+1} \\ \vdots \\ u_{imax-2,j}^{n+1} \\ u_{imax-1,j}^{n+1} \end{bmatrix}}_{\mathbf{u}} \underbrace{\begin{bmatrix} RHS(2) + u_{1,j}^{n+1} \\ RHS(3) \\ RHS(4) \\ \vdots \\ RHS(imax-2) \\ RHS(imax-1) + u_{imax,j}^{n+1} \end{bmatrix}}_{\mathbf{RHS}}$$

where $RHS(i) = -\beta^2 u_{i,j-1}^{k} - \beta^2 u_{i,j+1}^{k+1} + f\Delta x^2$

After this step, we used to alternative direction(fixed i row).This time, for each i row, iteration is calculated.This equation is as follows:

$$\beta^2 u_{i,j-1}^{k+1} - 2(1+\beta^2)u_{i,j}^{k+1} + \beta^2 u_{i,j+1}^{k+1} = -u_{i-1,j}^{k} - u_{i+1,j}^{k} + f\Delta x^2 \tag{11}$$

And matrix form which is used for each i row is as follows:

$$\underbrace{\begin{bmatrix} -2(1+\beta^2) & \beta^2 & 0 & \dots & 0 & 0 \\ \beta^2 & -2(1+\beta^2) & \beta^2 & \dots & 0 & 0 \\ 0 & \beta^2 & -2(1+\beta^2) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -2(1+\beta^2) & \beta^2 \\ 0 & 0 & 0 & \dots & \beta^2 & -2(1+\beta^2) \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} u_{i,2}^{n+1} \\ u_{i,3}^{n+1} \\ u_{i,4}^{n+1} \\ \vdots \\ u_{i,jmax-2}^{n+1} \\ u_{i,jmax-1}^{n+1} \end{bmatrix}}_{\mathbf{u}} \underbrace{\begin{bmatrix} RHS(2) + u_{i,1}^{n+1} \\ RHS(3) \\ RHS(4) \\ \vdots \\ RHS(imax-2) \\ RHS(imax-1) + u_{i,jmax}^{n+1} \end{bmatrix}}_{\mathbf{RHS}}$$

where $RHS(i) = -u_{i-1,j}^{k} - u_{i+1,j}^{k} + f\Delta x^2$

3 Results and Discussion

3.1 Successive Over Relaxation, Gauss-Seidel and Point Jacobi Methods

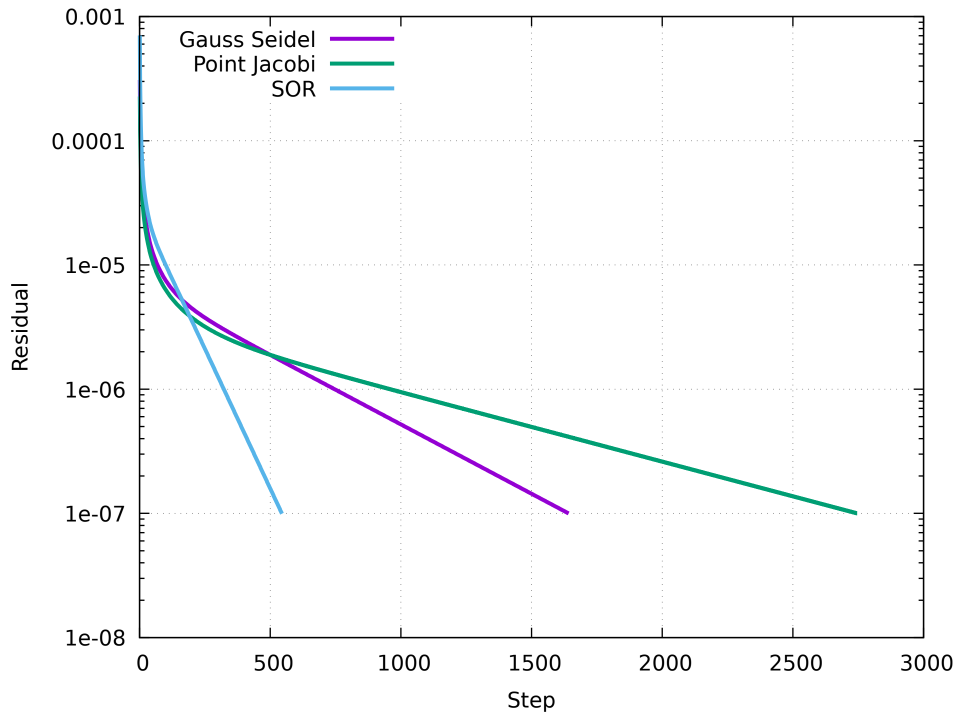
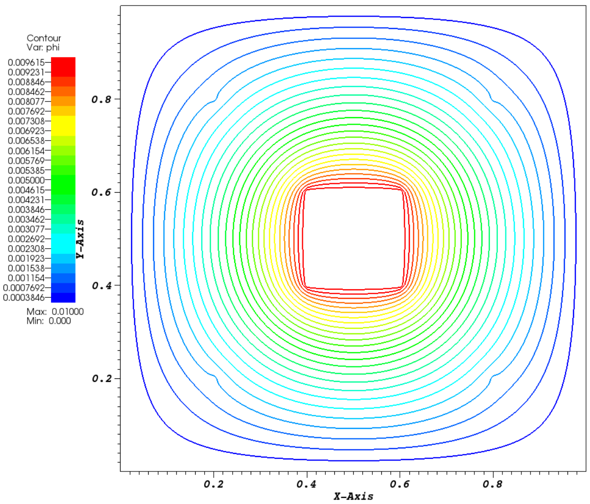
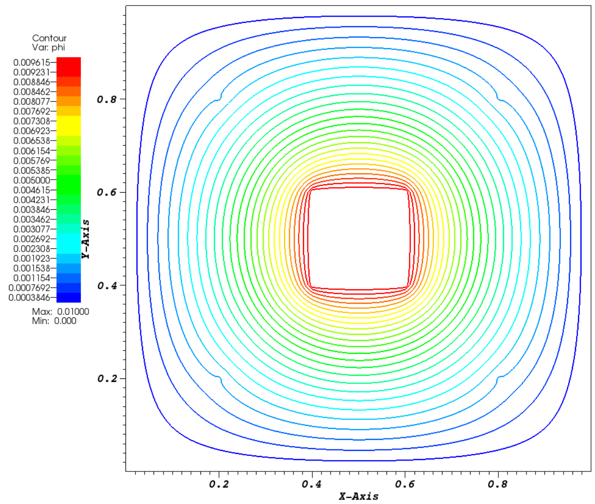


Figure 1: Convergence History of SOR, PJ, and GS methods

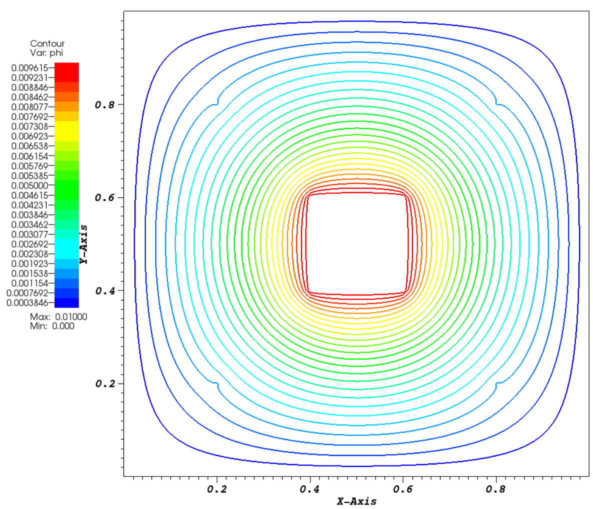
In the Figure 1, SOR method is almost three times faster than Gauss-Seidel method and also almost five times faster than Point Jacobi method.



(a)



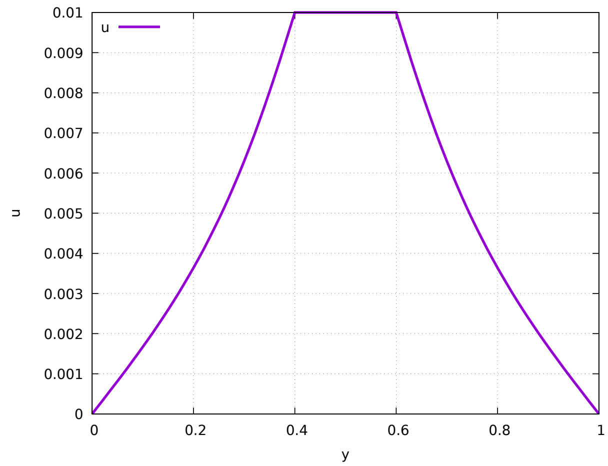
(b)



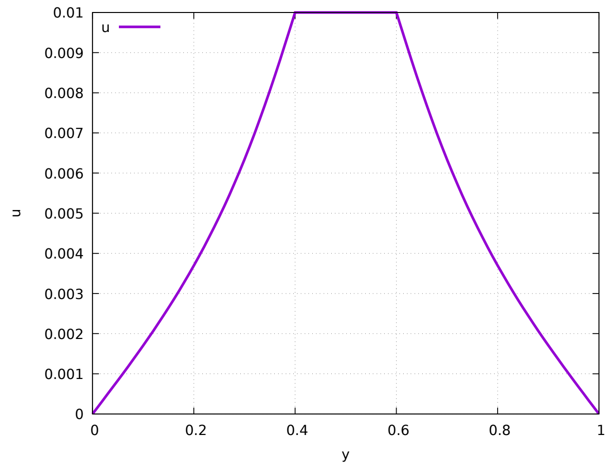
(c)

Figure 2: Contour plot of different methods
a) Point Jacobi b) Gauss-Seidel c) Successive Over Relaxation

It can be seen from figures 2a,2b and 2c that solutions are almost same in the SOR, GS and PJ methods at last. Because of small force distributions, there is a small deflections at force locations.



(a)



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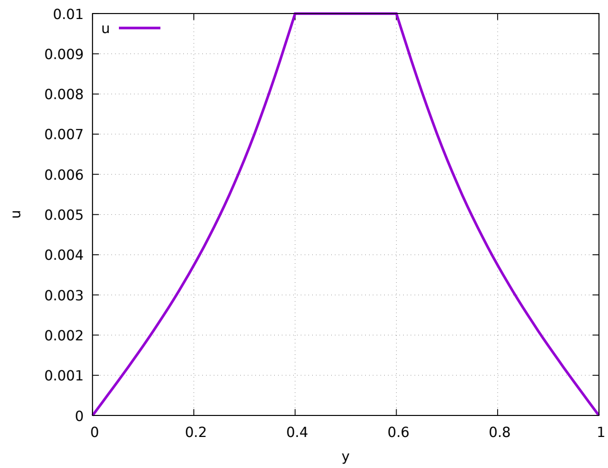
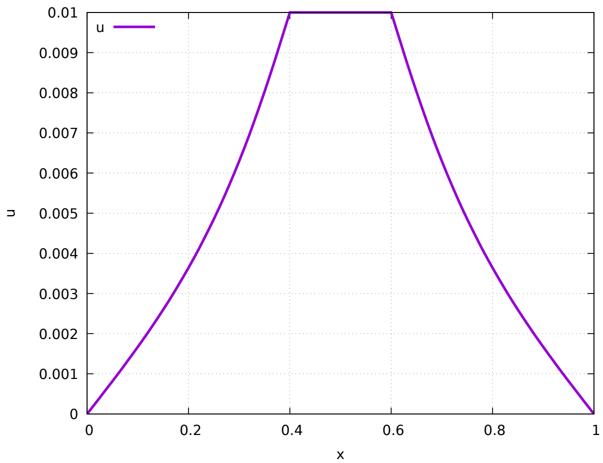


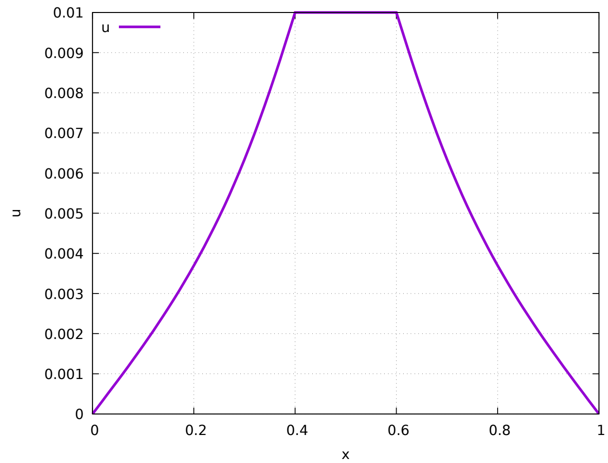
(c)

Figure 3: Vertical midsection u distribution
a) Point Jacobi b) Gauss-Seidel c) Successive Over Relaxation

It is observable that in figure 3a, 3b and 3c there are not differences between those figures in vertical midsection.



(a)



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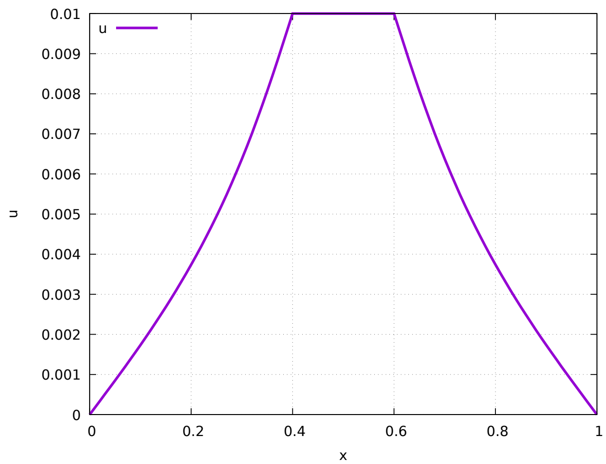
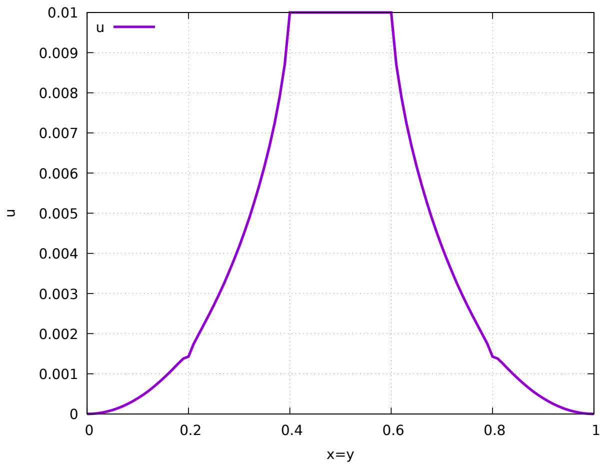


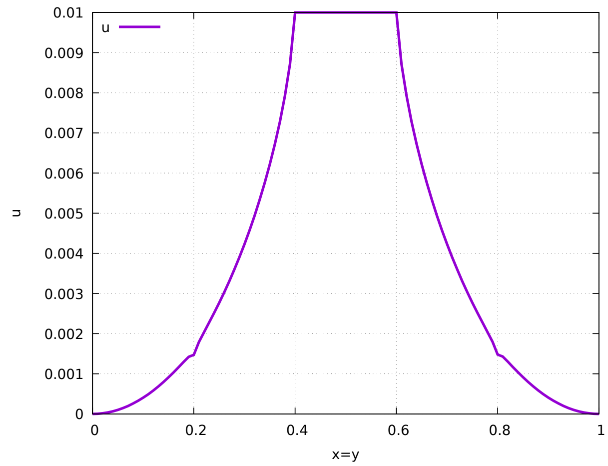
(c)

Figure 4: Horizontal midsection u distribution
a) Point Jacobi b) Gauss-Seidel c) Successive Over Relaxation

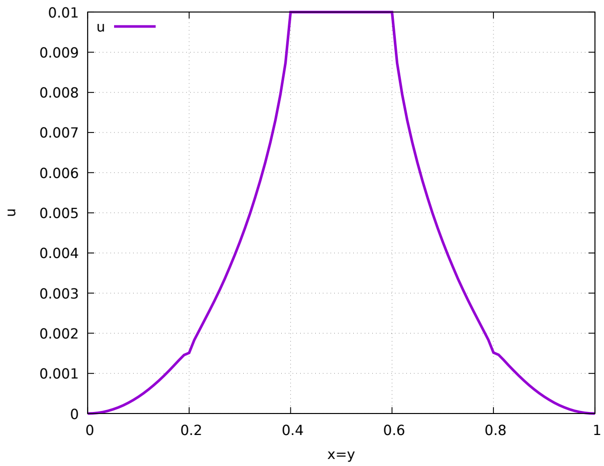
In Figure 4a,4b and 4c,it can be seen that the same solutions are given by using each methods in horizontal midsection.



(a)



(b)



(c)

Figure 5: Diagonal midsection u distribution
a) Point Jacobi b) Gauss-Seidel c) Successive Over Relaxation

In Figures 5a, 5b and 5c show that when varies iteration methods are used, same solutions are taken and there is a deformation in force locations in diagonal midsection.

3.2 Different Force Distributions

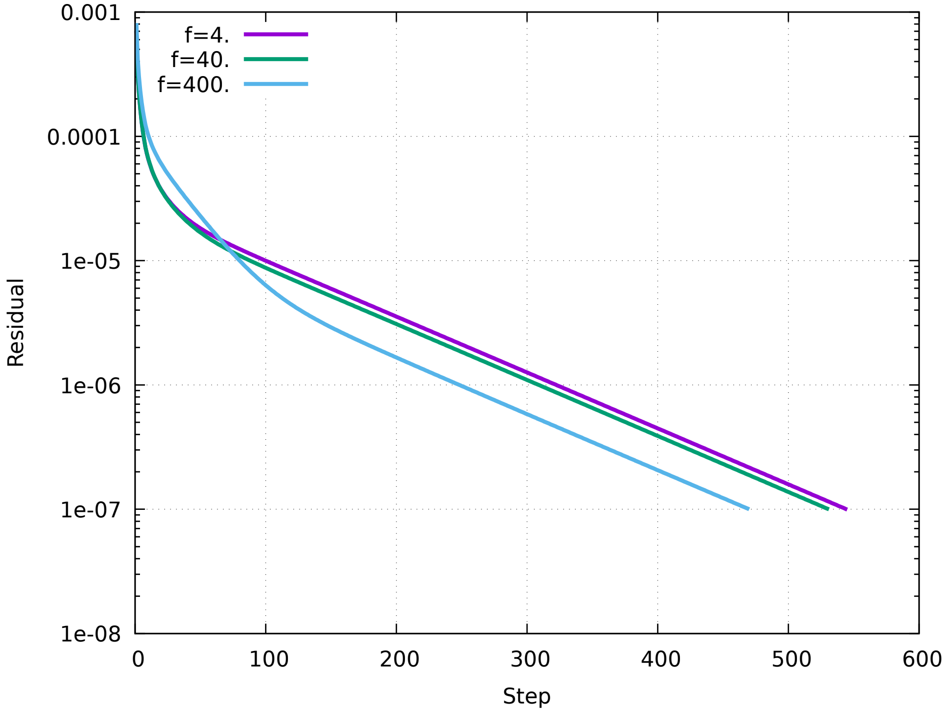
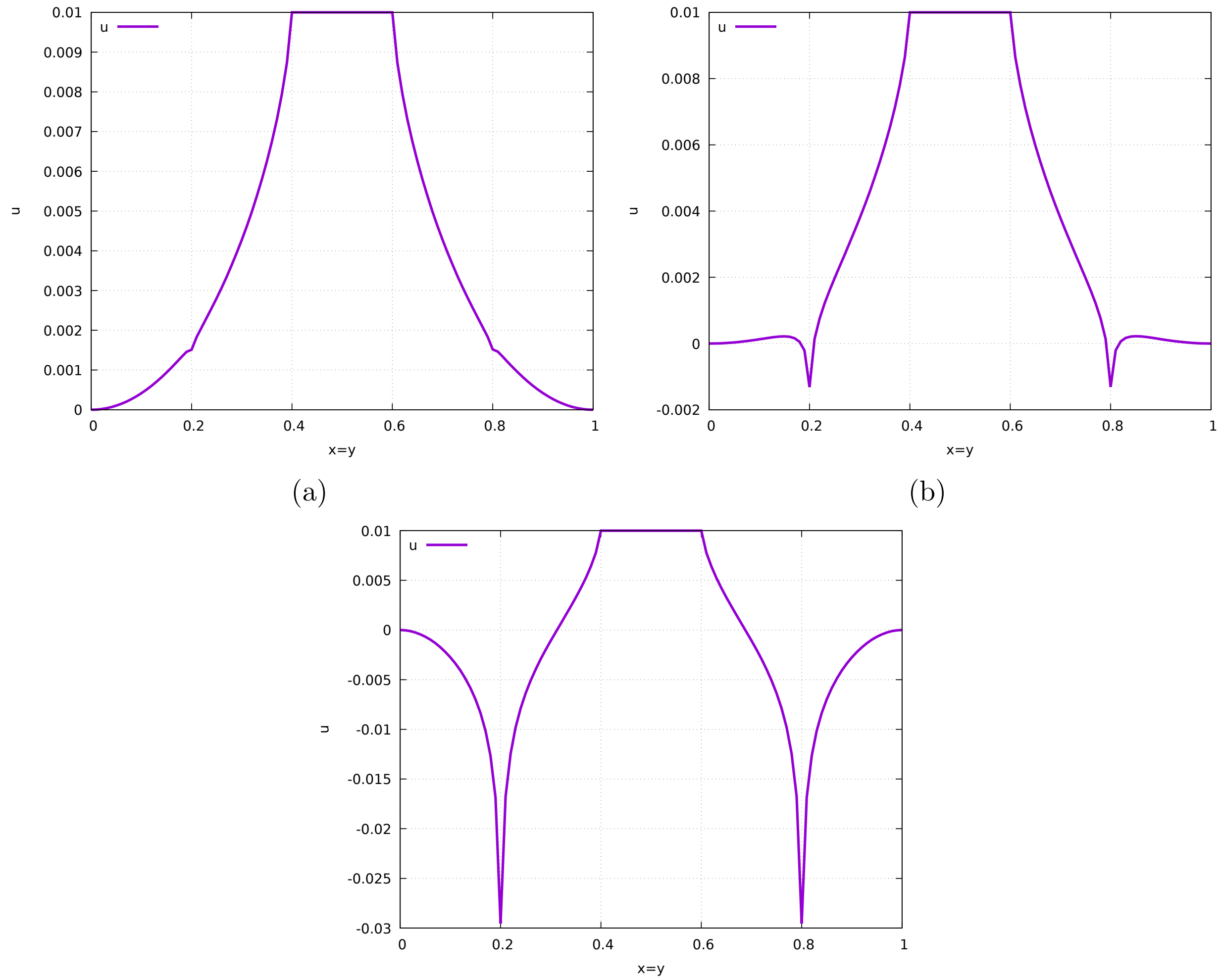


Figure 6: Convergence history of different force distributions

It can be clearly seen from Figure 6, when we change f distribution by chancing increasing f, the solution looks like converges slowly at first but after around 100 steps, the final solution converge faster. We calculate those solution by using SOR method.



(a) (b)

(c)

Figure 7: Diagonal midsection u distribution
a) f=4. b) f=40. c) f=400.

In Figure 7, there is same boundary all figures but their forces are different. When force is increased, deformation of force location increases.

3.3 Different dx,dy

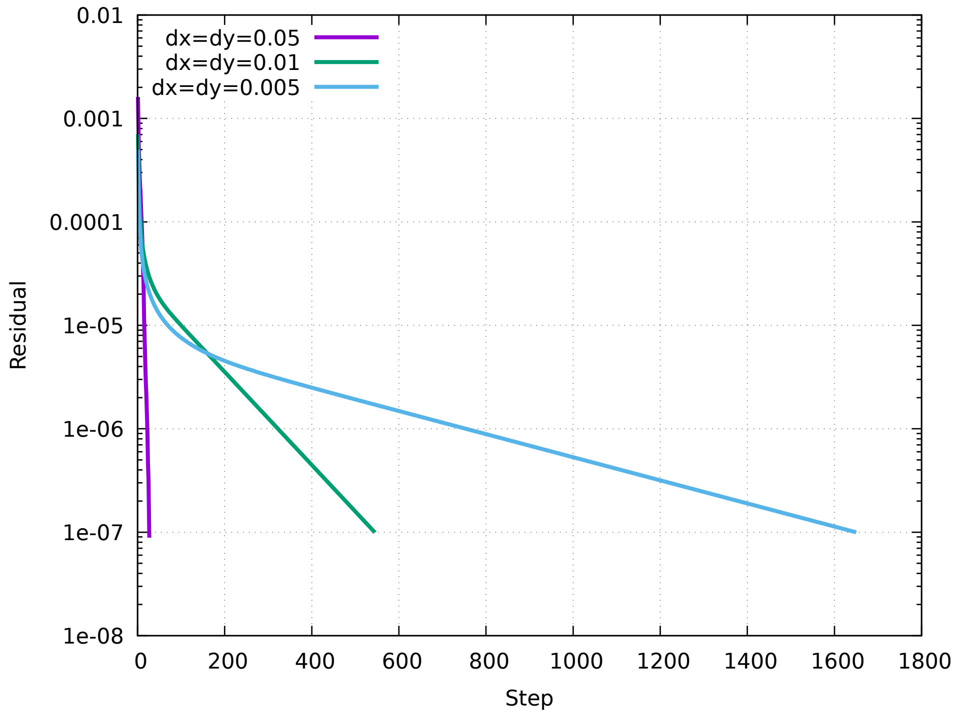
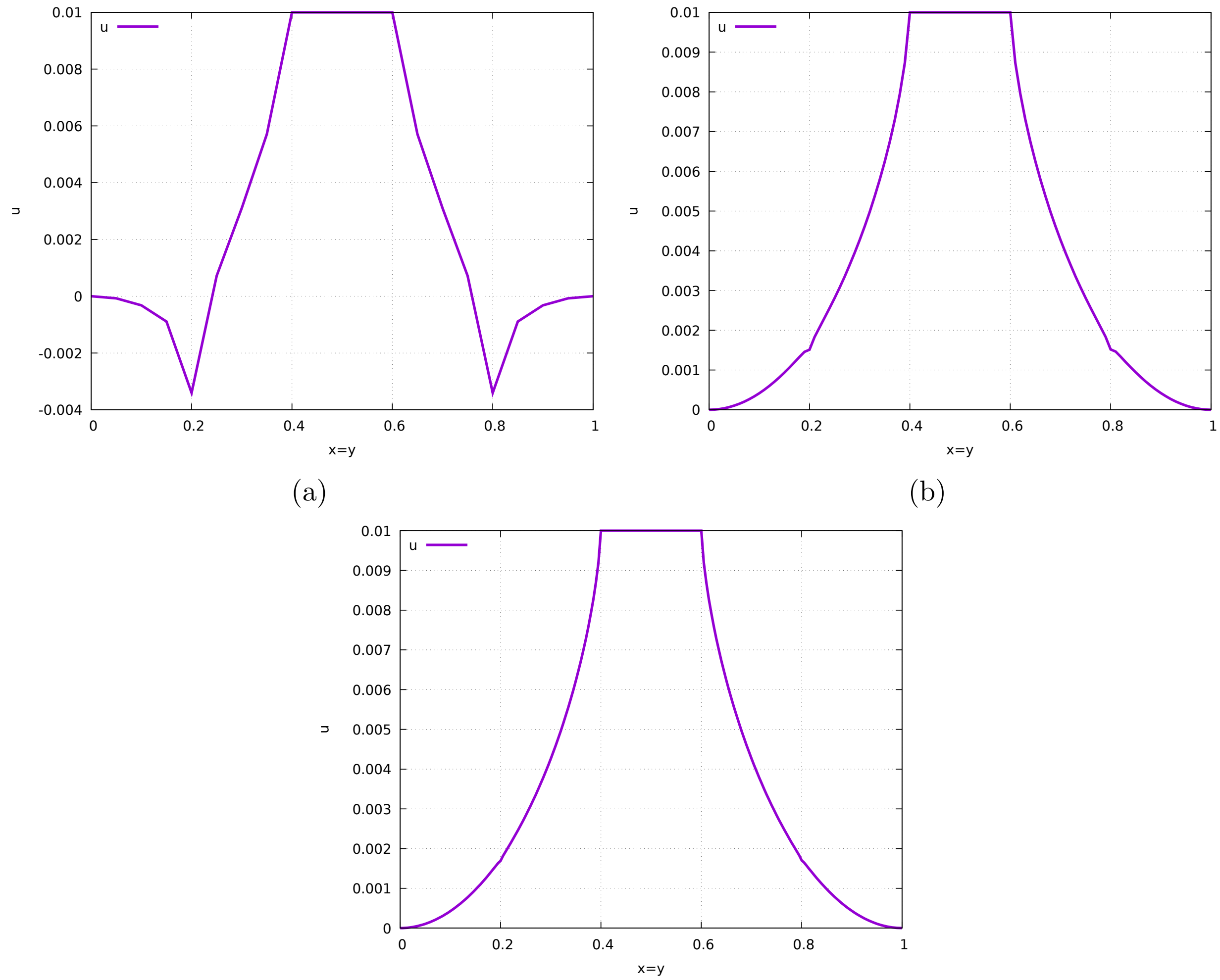


Figure 8: Converge history of different dx,dy value cases

In Figure 8, when dx and dy are equal to 0.05, the approach in almost one step. 0.05 value of dx and dy is almost ten times faster than 0.01 values of dx and dy. And also, 0.01 values of dx and dy is almost three times faster than 0.005 values of dx and dy.



(a) (b) (c)

Figure 9: Diagonal midsection u distribution
a) dx=dy=0.05 b) dx=dy=0.01 c) dx=dy=0.005

Since the boundary is a square, it is choosen that dx=dy. From figure 9 it can be said that changing grid size affect both effect area and magnitude of effect of the force applied on the membrane. Higher values of dx,dy gives less resolution which result with a larger effect of force on larger area. Lower values of dx,dy increases resolution which results with a smaller effect of force on smaller area.

3.4 Alternating Direction Implicit

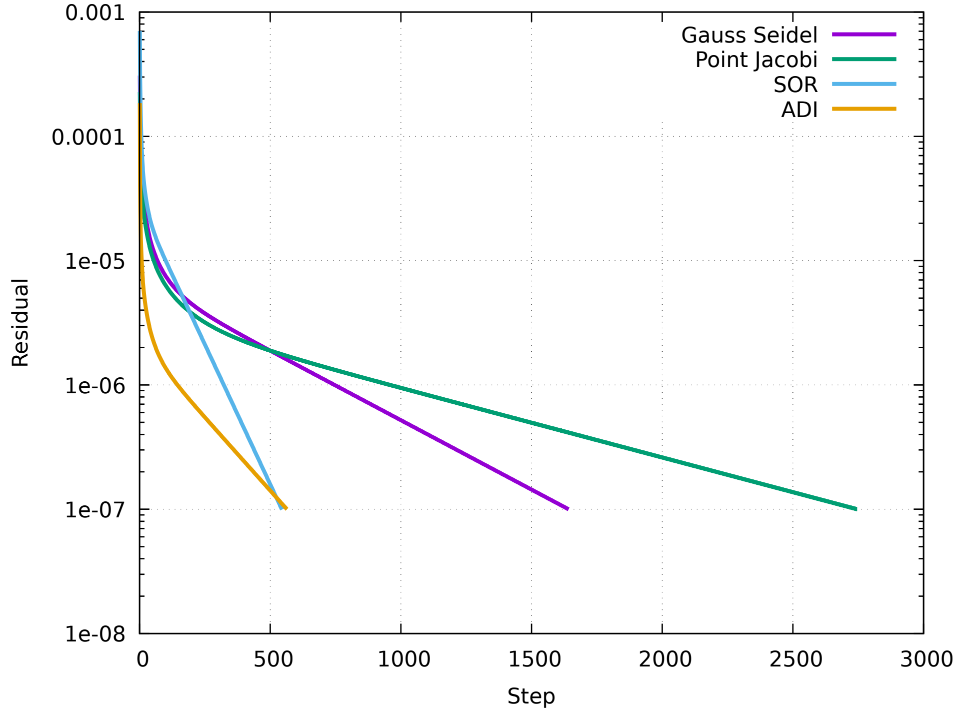
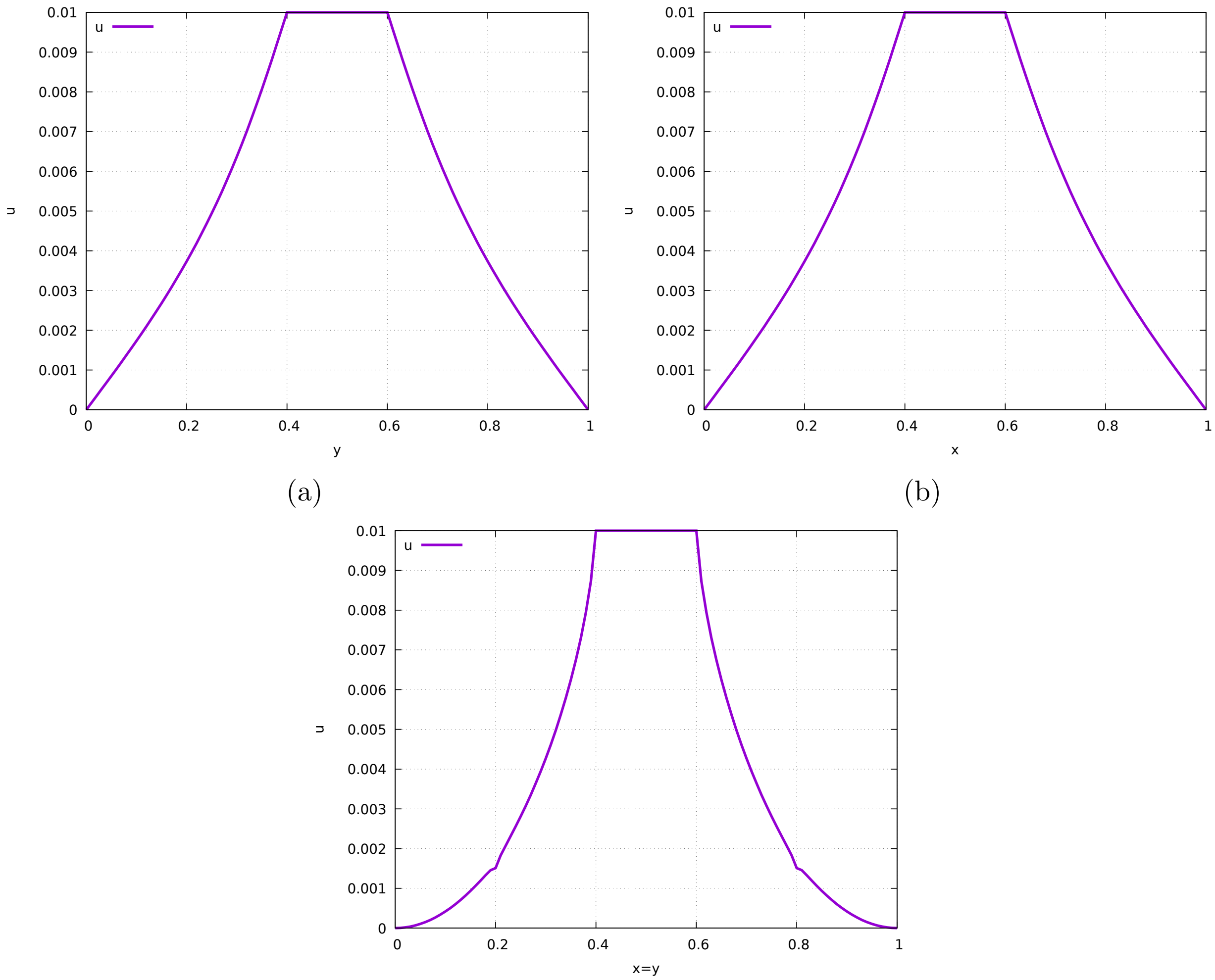


Figure 10: Convergence history of ADI method with SOR,PJ and GS

Even though the convergence profile is not same with the SOR method, ADI methods approximately have same iteration to converge solution. Since it is harder to code ADI method for this problem use of SOR is much more suitable for this homework.



(c)

Figure 11: U distributions along cross sections a) Mid-vertical b) Mid-horizontal c) Diagonal

Figure 11a and figure 11b are same because there is not force distribution on mid-vertical and mid-horizontal cross-section.However, we can see that deformation occurs because of force distribution.

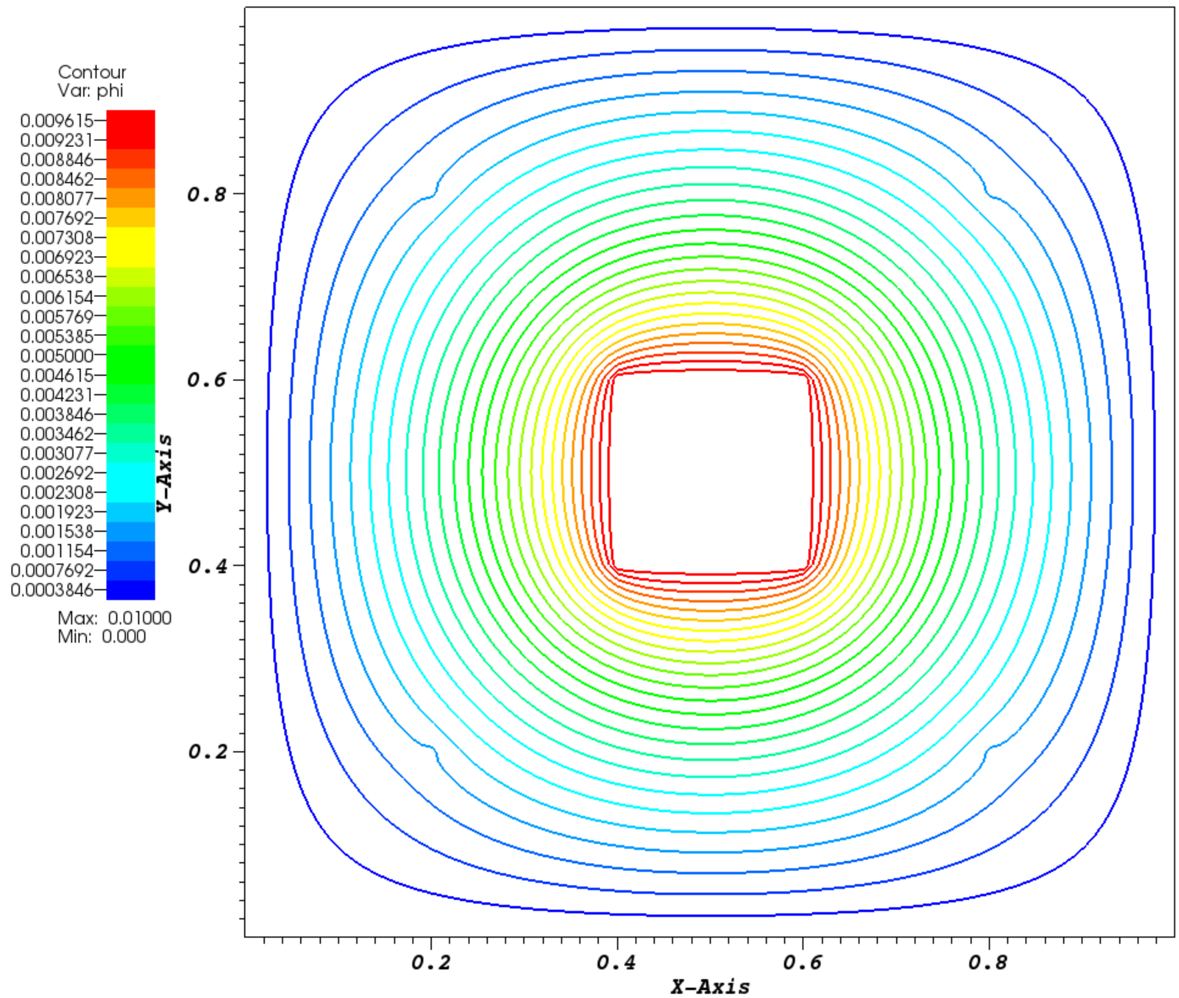


Figure 12: Contour plot of u distribution of ADI method

As seen from figure 12 contour plot of u using ADI method is perfectly same as the ones defined from SOR, GS, and PJ methods which can be seen from figure 5. An alteration can be seen at the point where the force is applied.

4 Conclusion

In conclusion, non-dimensional transverse displacement of an elastic membrane problem which is governed by elliptical Poisson equation is solved with Point Jacobi, Gauss-Seidel, Successive Over Relaxation(SOR) and Alternating Direction Implicit (ADI) methods. It understood that for convergence history(converge speed) the Successive Over Relaxation(SOR) and ADI were the fastest ones by far, the Gauss-Seidel had an average performance and Point Jacobi was quite inefficient. Therefore, it can be concluded that if the convergence speed is important for a problem, SOR, ADI and Gauss-Seidel methods are practical solutions of problems governed by elliptical equations. From the contour plots of different methods it can be observed that solutions are exactly same for each method. Maximum displacement seen around the inner boundary and it is steadily decreasing towards the edges. Also, effects of the loads on displacements are not very visible due to small value of load but of course there are certain small effects at the points where the loads applied. Therefore, effect of constant displacement on inner boundary is much stronger than effects of load. Another issue is when horizontal and vertical midsection displacements are examined, again there is no difference between different solution methods and it can be seen that displacements values are high near the hole and lower near the outer boundary and there is no visible effects of loads. This is reasonable because loads are applied at the points that are on the diagonal of membrane so effect of load cannot reach the midsections. However when diagonal sections of membrane are examined load effects can be seen. In this homework, change in force distribution has been also viewed. It shows that increase in load yields to recognizable change in convergence speed but it does not change so much. Also, as the load increases, displacement at point which exposed to load also increases significantly. This also leads to change in displacements of other sections of membrane. This problem has lots of parameters which can effect the solution and one of the these parameters is size of dx and dy values. Size of them has a significant importance on convergence history if their size is decreased, problem converge slower but it results with better resolution and smooth solution.