Chair of Scientific Computing
Department of Informatics
Technical University of Munich

Adaptive Sparse Grids

Seminar: High Dimensional Methods in Scientific Computing

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Outline

Introduction

What is a sparse grid?

A **numerical discretization technique** that can be used in a wide range of computatitonal problems:

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However, in some cases, the dimensionality of the problem is high ($d \geq 4$) and the solution on a *full grid* is not feasible.
This dilemma is called *curse of dimensionality* and is the reason why sparse grids are used in many scientific applications.

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A brief comparision

The required grid points and the Euclidian norm of the interpolation error on a regular sparse grid and on a full grid is shown in the following table below for d dimensional space and grid level of n.

	Number of grid points	L2 Norm of Interpolation Error
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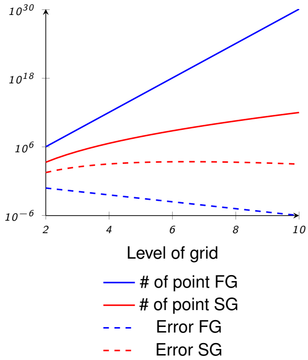
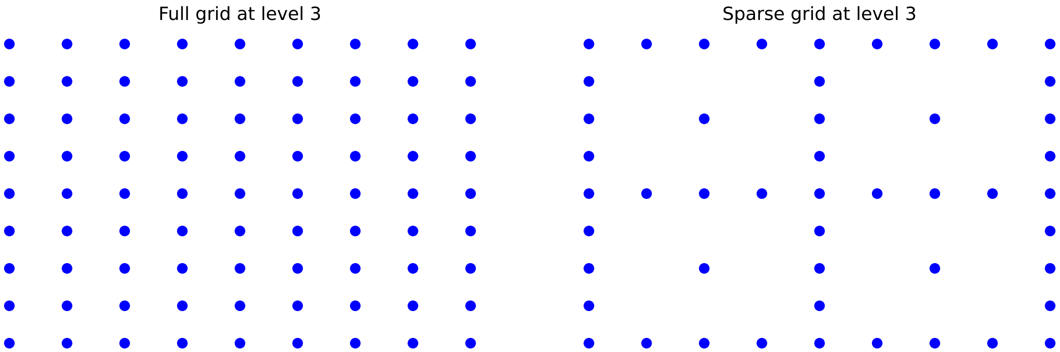
Comparision for $d = 10$



Introduction

How they look?



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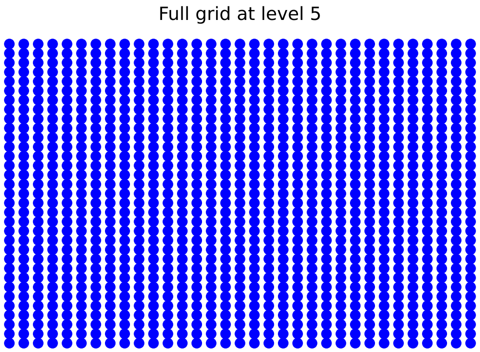

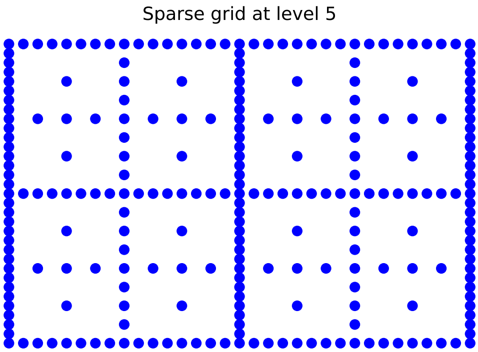

Sparse Grids

Requirements for a sparse grid

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- f can be computable in any point in hyper-cube.
- It is assumed that the function is *computatinally expensive*. So that we need to change f with another function which is *cheaper* and approximate original function well.

Sparse Grids

Hierachical Basis Functions

Using a standard hat function as a basis function, we can construct a sparse grid.

$$\phi(x) = \begin{cases} 1 - |x| & \text{if } x \in [-1, 1], \\ 0 & \text{otherwise} \end{cases}$$

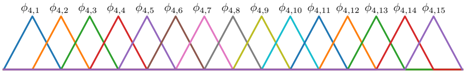

Figure: Nodal Basis Functions

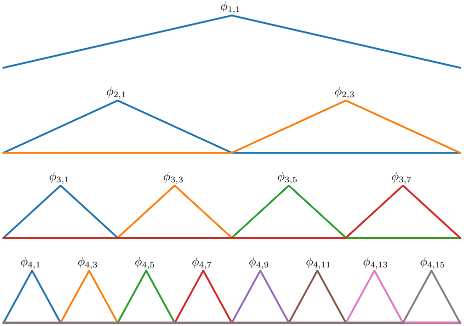

Figure: Hierarchical Basis Functions

Sparse Grids

Tensorial product and construction on d-dimensional space

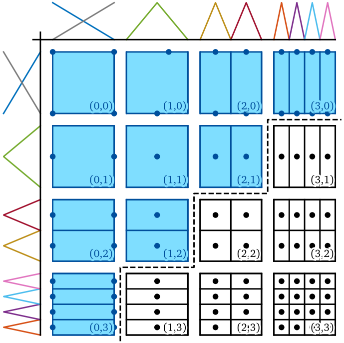

Figure: Construction of the regular sparse grid of level in 2D, by *Julian Valentin*

Adaptivity on Sparse Grids

Types of Adaptivity

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The point where f is mostly flat are wasted.

Spatial adaptivity allows to use more grid points locally.

Adaptivity on Sparse Grids

Adaptivity Criterion

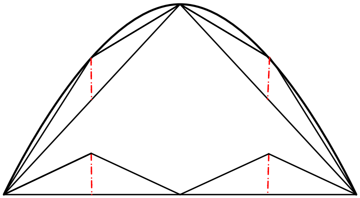

Figure: Interpolation of a parabola using 2 level hierarchical basis and surpluses, surpluses are shown in red lines.

- One of the popular criterion is so called *surplus-based criterion*.

Adaptivity on Sparse Grids

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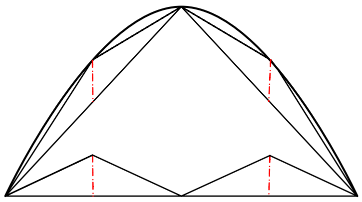

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- It uses the absolute values of of hierarchical surpluses α, to estimate second derivative of the function f. In general, a larger absolute surplus means a larger second derivative.

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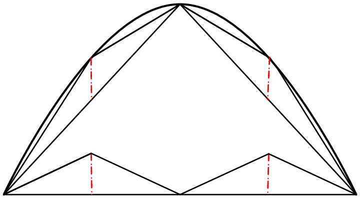

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- It uses the absolute values of of hierarchical surpluses α, to estimate second derivative of the function f. In general, a larger absolute surplus means a larger second derivative.
- More grid points are inserted to vicinity of larger surplus values.

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Spatial Adaptivity Requirements

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- Iteratively choose region of interests.
- Add new points to region of interests.
- The grid should contain all the hierarchical ancestors of all grid points.
- Due to the consistency constraint, the number of points added can be larger than $2 \cdot d$.

Adaptivity on Sparse Grids

Spatial Adaptivity in Action

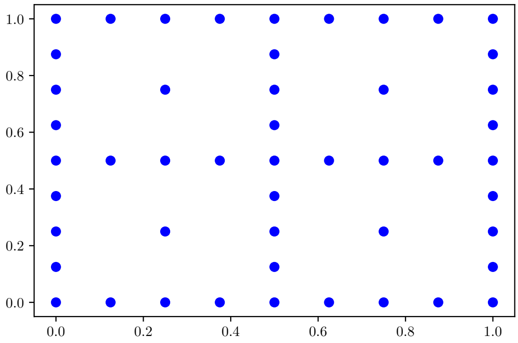


Figure: Level 3 Regular grid before adaptation.

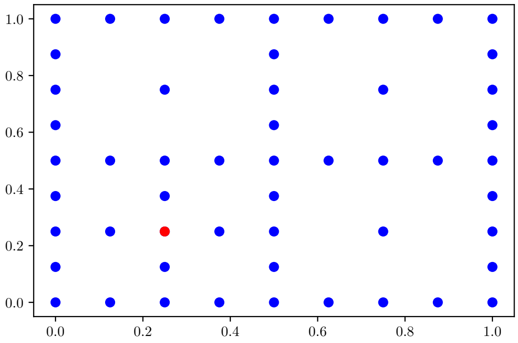


Figure: Level 3 Regular grid after adaptation.

Sparse Grid in Action: Interpolation

Franke's Function

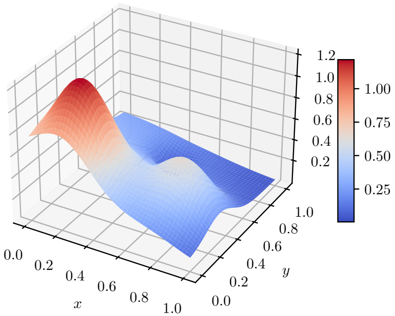
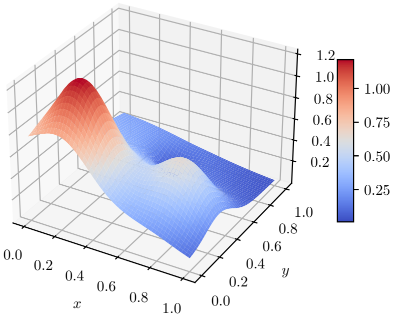


Figure: Calculated surface of Franke's function



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Sparse Grid in Action: Interpolation

Franke's Function

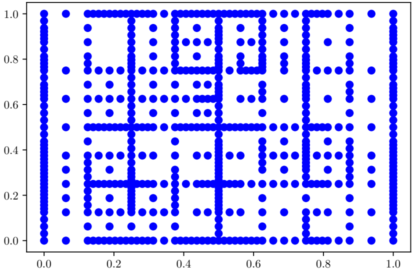

Figure: The final sparse grid with 570 grid points.

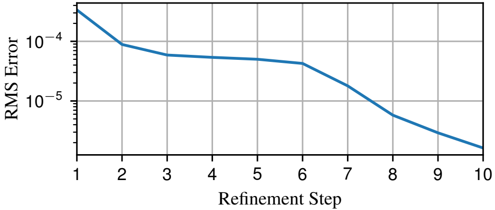

Figure: Error reduction plot of Franke's function w.r.t. refinement steps.

Sparse Grid in Action: Interpolation

Genz Test Functions: Product Peak

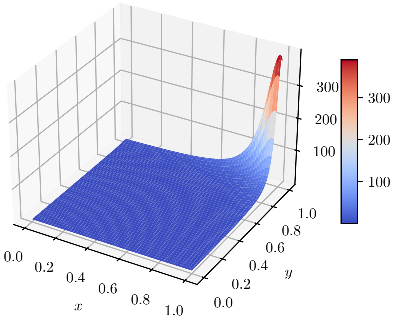

Figure: Calculated surface of Product Peak function

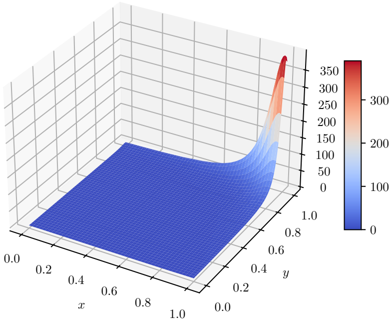

Figure: Interpolated surface of Product Peak function

Sparse Grid in Action: Interpolation

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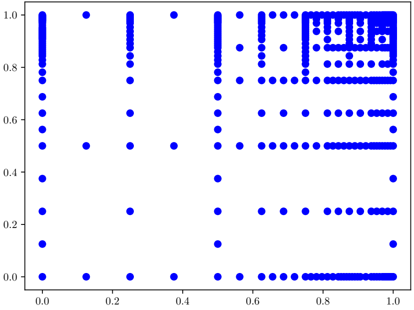

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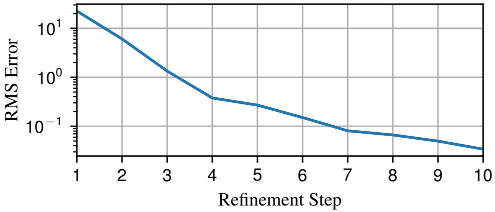


Figure: Error reduction plot of Product Peak function w.r.t. refinement steps.

Sparse Grid in Action: Interpolation

Genz Test Functions: Discontinuous

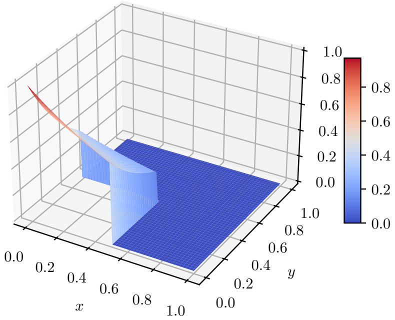
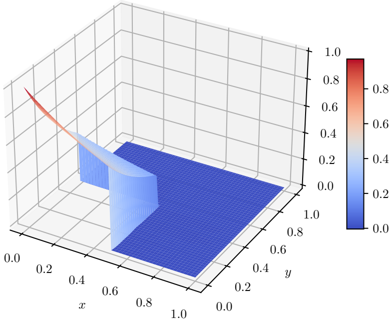


Figure: Calculated surface of Discontinuous function



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Sparse Grid in Action: Interpolation

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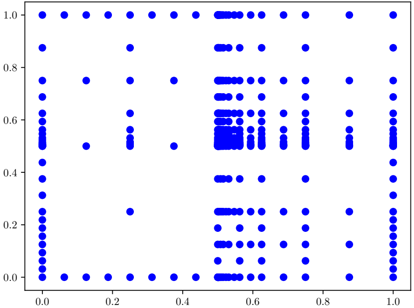

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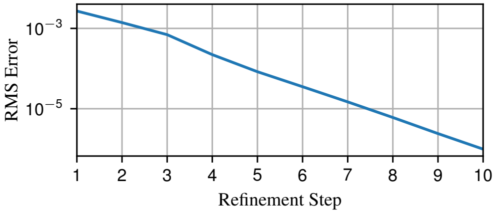

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Thank you for your attention!