# Adaptive Sparse Grids

## Seminar: High Dimensional Methods in Scientific Computing

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# Outline

# Introduction

## What is a sparse grid?

A **numerical discretization technique** that can be used in a wide range of computaitonal problems:

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However, in some cases, the dimensionality of the problem is high ($d \geq 4$) and the solution on a *full grid* is not feasible.

This dilemma is called *curse of dimensionality* and is the reason why sparse grids are used in many scientific applications.

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## A brief comparision

The required grid points and the Euclidian norm of the interpolation error on a regular sparse grid and on a full grid is shown in the following table below for $d$ dimensional space and grid level of $n$.

| | Number of grid points | L2 Norm of Interpolation Error |
|---|---|---|
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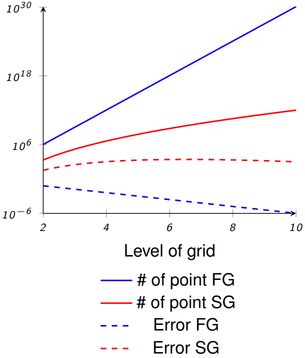
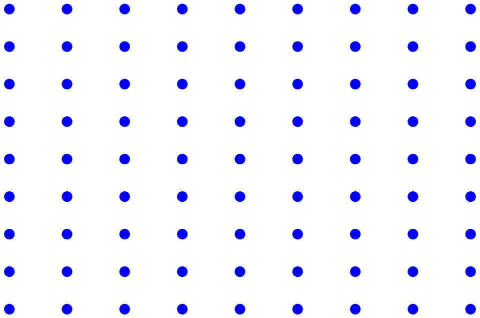
Comparision for $d = 10$



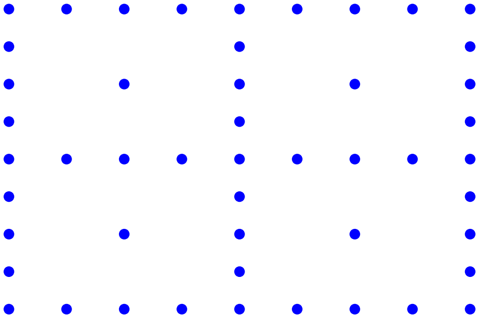
# Introduction

## How they look?

Full grid at level 3



Sparse grid at level 3



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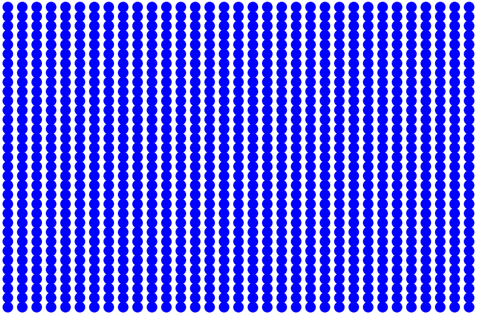
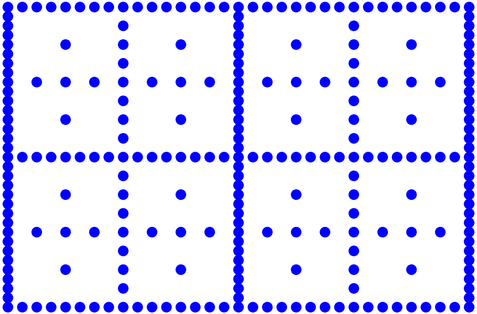
## How they look?

Full grid at level 5



Sparse grid at level 5



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- $f$ can be computable in any point in hyper-cube.
- It is assumed that the function is *computatinally expensive*. So that we need to change $f$ with another function which is *cheaper* and approximate original function well.

# Sparse Grids

## Hierachical Basis Functions

Using a standard hat function as a basis function, we can construct a sparse grid.

$$\phi(x) = \begin{cases} 1 - |x| & \text{if } x \in [-1, 1], \\ 0 & \text{otherwise} \end{cases}$$

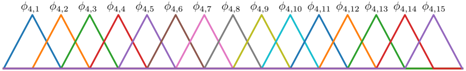


Figure: Nodal Basis Functions

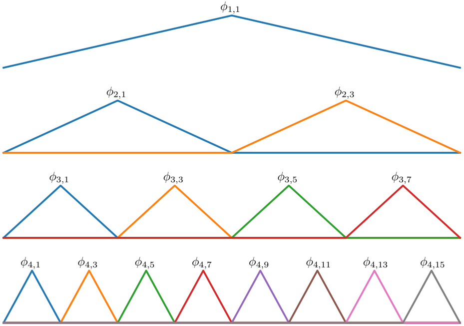


Figure: Hierarchical Basis Functions

# Sparse Grids

## Tensorial product and construction on d-dimensional space

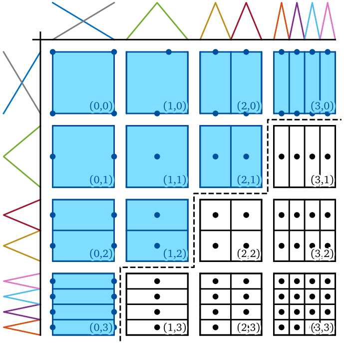


Figure: Construction of the regular sparse grid of level in 2D, by *Julian Valentin*

# Adaptivity on Sparse Grids

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**Spatial Adaptivity**

- Can be used when $f$ has a different characteristic in specific regions

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The point where $f$ is mostly flat are wasted.

Spatial adaptivity allows to use more grid points locally.

# Adaptivity on Sparse Grids

## Adaptivity Criterion

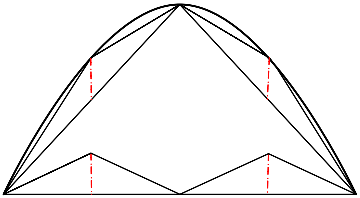

Figure: Interpolation of a parabola using 2 level hierarchical basis and surpluses, surpluses are shown in red lines.

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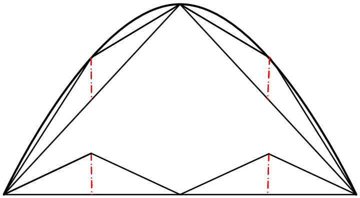

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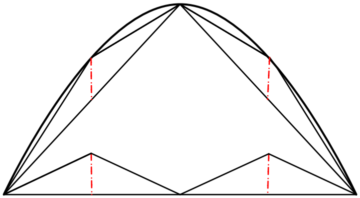

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- More grid points are inserted to vicinity of larger surplus values.

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- The grid should contain all the hierarchical ancestors of all grid points.
- Due to the consistency constraint, the number of points added can be larger than $2 \cdot d$.

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## Spatial Adaptivity in Action

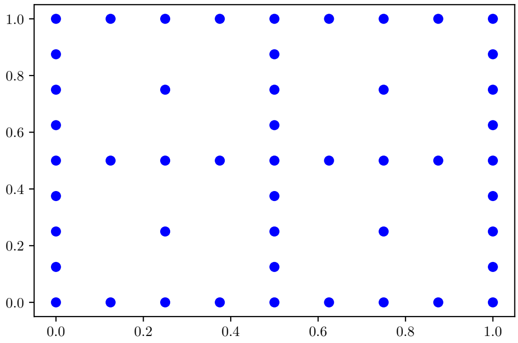
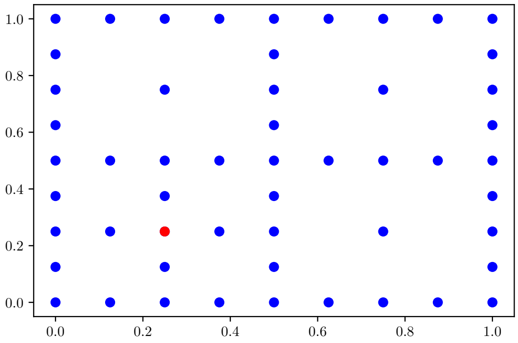



Figure: Level 3 Regular grid before adaptation.

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# Sparse Grid in Action: Interpolation

## Franke's Function

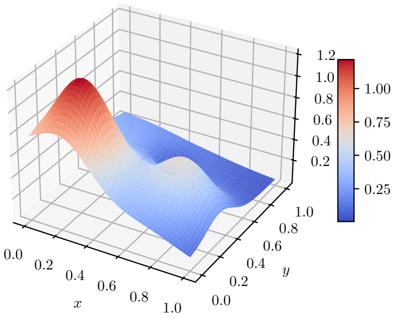
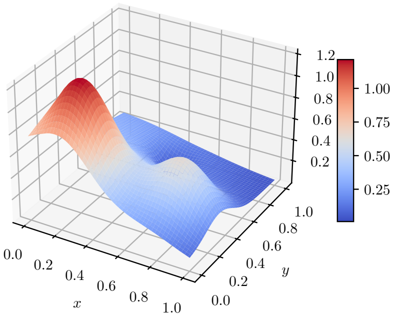



Figure: Calculated surface of Franke's function

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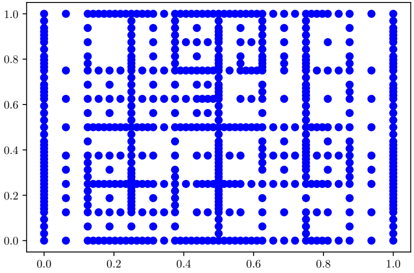



Figure: The final sparse grid with 570 grid points.

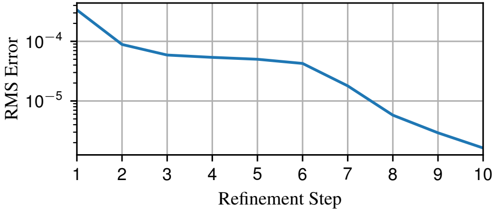
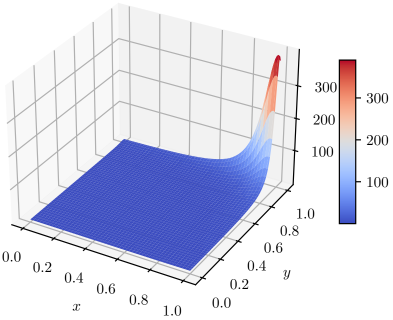



Figure: Error reduction plot of Franke's function w.r.t. refinement steps.

# Sparse Grid in Action: Interpolation

## Genz Test Functions: Product Peak

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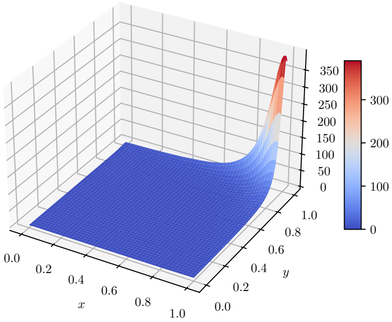


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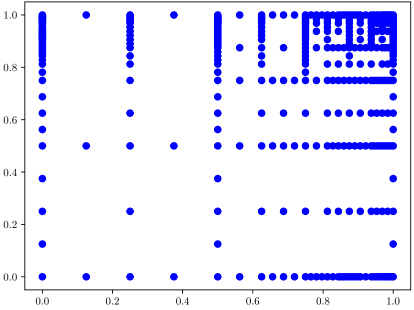


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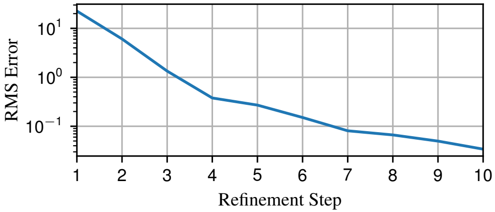


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# Sparse Grid in Action: Interpolation

## Genz Test Functions: Discontinuous

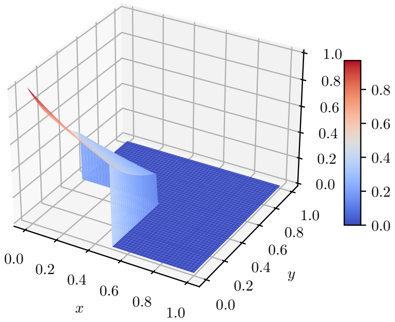
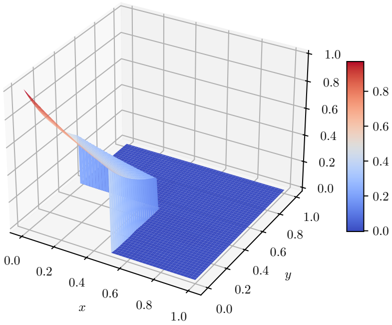



Figure: Calculated surface of Discontinuous function

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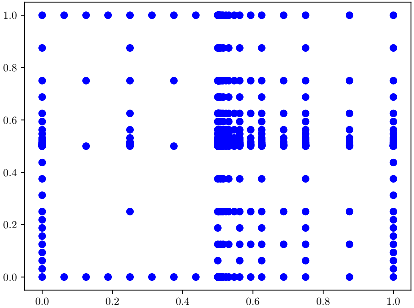


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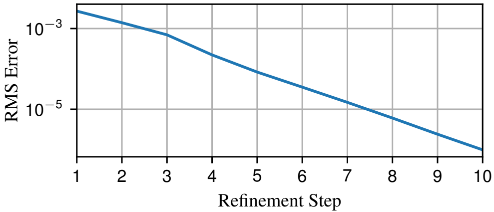


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Thank you for your attention!