Adaptive Sparse Grids

Seminar High Dimensional Methods in Scientific Computing

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***Abstract*—**

***Index Terms*—Sparse Grid,Interpolation,Multivariate, Multidimension,Adaptivity**

I. Introduction

Ever increasing internet speed and data production and collection come with some new challenges to computational science. Beside the classical high dimensional problem like finance, some new player are also jumped to the train like machine learning with an immense amount of dimension. High dimensional problems are suffer from curse of dimensionality, i.e. they have an exponential dependency on dimension. This is a barrier in numerical treatment of the high-dimensonal problems. This exponential dependency makes harder to use classical mesh based approaches to solve this kind of problems. One could also use mesh-free methods like Monte-Carlo quadratures.

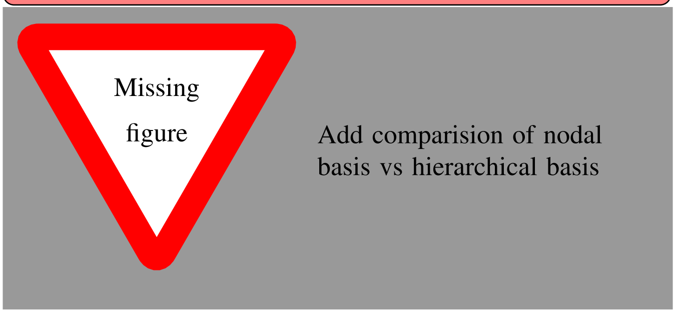
In order to overcome such a problem, the sparse grid method gains more and more popularity. The sparse grid method is a general numerical discretization technique which is first introduced by the Russian mathematician Smolyak in 1963 [1].

Sparse grids offers a new way to reduce the required number of grid points by the order of magnitude $O(2^{nd})$ to just only $O(2^n n^{d-1})$ while preseving a similar error as using the full grid. In order to achieve these bounds, the mixed second derivatives have to be bounded

Find reference for this.

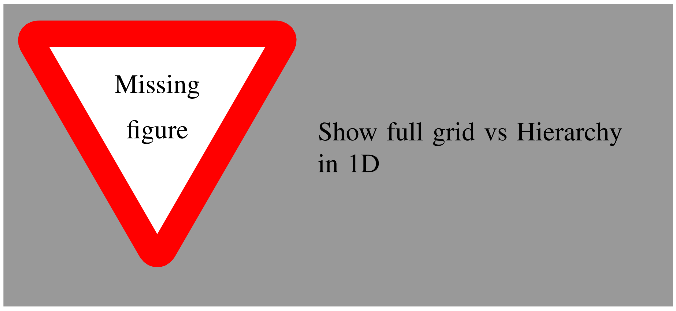
. For the problems which are do not satisfy smoothness criteria or required further reduce in meshs size, one can use advantage of adaptivity. The hierarchical basis is a direct indicator of areas where further refinement required.

Sparse grids are based on a hierarchical (and thus inherently incremental and adaptive) formulation of the one-dimensional basis which is then extended to the d-dimensional setting via a tensor product approach.



	Storage Requirement	L2 Norm of Interpolation Error
Full Grid	$O(2^{nd})$	$O(2^{-2n})$
Sparse Grid	$O(2^n n^{d-1})$	$O(2^{-2n} n^{d-1})$

A. Comparision to Full Grids



II. Adaptivity

A regular sparse grid is constructed in a way that takes a cut in diagonal hyperplane, which means it treats all the dimensions equally. However, there can be an importance difference between the dimensions, i.e. one dimension might be more important than others. This can be solved by so called dimensional adaptivity.

The most straightforward approach for this type of refinement is adding some new subspaces in the dimension which function changes rapidly. In order to add a new subspace $W_{\vec{l}}$ one should include all the backward neighbours in the current set of subspaces. This refiment treads all the grid points in one dimension as a uniform way,and called as dimensionally adaptive refinement. Leads more point in one dimension than other one.

May be add an example of dimensionally adaptive sparse grid to here.

Morever, there are some cases exist, where dimensional adaptivity is not to be enough to solve problem. For instance take a function that is mostly flat, have peaks at certain regions in the domain. Franke's function is a good example for this [2].

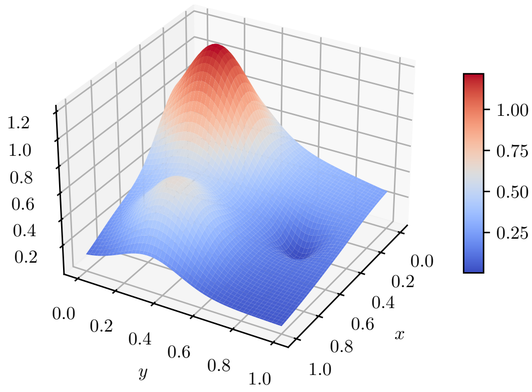


Fig. 1. Surface plot of Franke's Function.

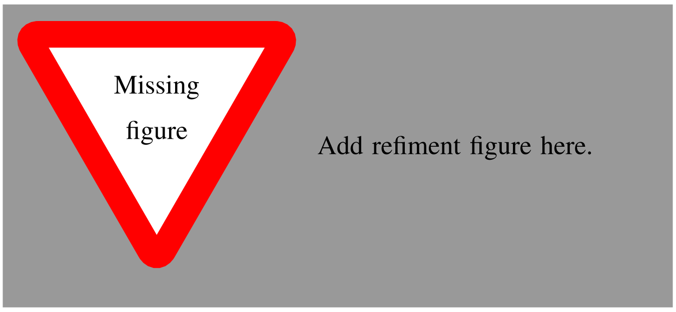


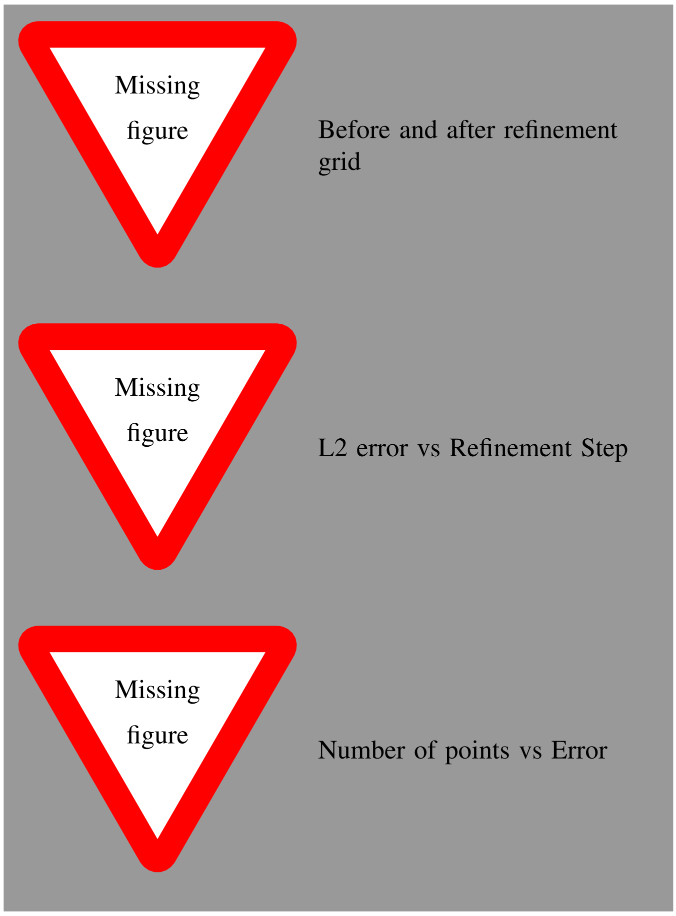
Fig. 2. Spatial refinement of a selected node in two dimension.

$$\begin{aligned} f(\mathbf{x}) = &\frac{3}{4}\exp\left(-\frac{(9x_1-2)^2}{4}-\frac{(9x_2-2)^2}{4}\right) \\ &+\frac{3}{4}\exp\left(-\frac{(9x_1+1)}{49}-\frac{9x_2+1}{10}\right) \\ &+\frac{1}{2}\exp\left(-\frac{(9x_1-7)^2}{4}-\frac{(9x_2-3)^2}{4}\right) \\ &-\frac{1}{5}\exp\left(-(9x_1-4)^2-(9x_2-7)\right) \end{aligned} \tag{1}$$

A dimensionally adaptive grid would also add more point to regions where the function is mostly flat. A spatially adaptive grid would overcome this problem. Instead of adding whole incremental grids, spatial adaption would only include a subset of those points near the region of interest and saves points.

The general approach for spatially adaptive grids is doing the refinement process iteratively. One could start and initial coarse grid or if knows a grid which is tailored to problem. Using an iterative process one could simply adds new points (neighboring grid point in the next higher level) to region of interest. To enable usage of sparse grids algorithms there is also a consistency constraint exist, the grid should contain all the hierarchical ancestors of all grid points. Figure 2 shows how this process is done in two-dimensional regular sparse grid.

III. Examples



References

[1] S. A. Smolyak, "Quadrature and interpolation formulas for tensor products of certain classes of functions," in *Doklady Akademii Nauk*, vol. 148, no. 5. Russian Academy of Sciences, 1963, pp. 1042–1045.

[2] R. Franke, "A critical comparison of some methods for interpolation of scattered data," Tech. Rep., Dec. 1979. [Online]. Available: https://doi.org/10.21236/ada081688