CS388C: LECTURE 2

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1. Inclusion-Exclusion

$$|A \cup B| = |A| + |B| - |A \cap B| \tag{1}$$

Proposition 1.1.

$$|\bigcup_{i=1}^{n} A_i| \geq \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j|$$

Proof.

Fix $x \in \bigcup_{i=1}^{n} A_i$. Say x appears in **r** of the A_i.
Then x contributes $r - \binom{r}{2}$ elements to the RHS. Which is equivalent to:

$$\begin{aligned}
&= r - \frac{r(r-1)}{2} \\
&= r(1 - \frac{r-1}{2}) \\
&\leq 1
\end{aligned}$$

Therefore the LHS must be greater. (See $r \geq 3$) ∎

Example 1.2. Suppose we have n sets $A_1, ..., A_n$ s.t. $|A_i| = k$ and $|A_i \cap A_j| \leq 1$.
Using **Proposition 1.1**, we have:

$$| \cup A_i| \geq nk - \binom{n}{2}$$

However, if $k <<< n$, this can be a negative lower bound, so we can relax to only the **first k** sets...

$$| \cup^n A_i| \geq | \cup^k A_i| \geq k^2 - \binom{k}{2}$$

Proposition 1.3.

$$
\begin{aligned}
|\bigcup_{i=1}^{n} A_i| = \sum_{i=1}^{n} |A_i| &- \sum_{i<j} |A_i \cap A_j| \\
&+ \sum_{i<j<k} |A_i \cap A_j \cap A_k| \\
&- \sum_{i<j<k<l} |A_i \cap A_j \cap A_k \cap A_l| \\
&\vdots \\
&\pm |A_1 \cap A_2 \cap ... \cap A_n|
\end{aligned}
$$

Proof. Let use the same strategy as before. i.e. Suppose $x \in \bigcup A_i$.
Say x appears in **r** of the A_i. We just need to find its contribution to the RHS:

$$
\begin{aligned}
\sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| \\
&+ \sum_{i<j<k} |A_i \cap A_j \cap A_k| \\
&\vdots \\
&\pm |A_1 \cap A_2 \cap ... \cap A_n| \\
&= r - \binom{r}{2} + \binom{r}{3} - \binom{r}{4} + ... \pm 1 \\
&= \binom{r}{1} - \binom{r}{2} + \binom{r}{3} - \binom{r}{4} + ... \pm \binom{r}{r}
\end{aligned}
$$

By the binomial theorem, we know this is exactly equal to

$$\binom{r}{0} - (1-1)^r = 1$$

∎

Definition 1.4. <u>derangement</u> - permutation π $s.t.$ $\forall i$ $\pi(i) \neq i$

Example 1.5. Now let us try and find the # derangements of a sequence of size n.

We can try and frame this problem by breaking it up into smaller sets A_i as before...

Let $A_i = \{\pi : \pi(i) = i\}$. Then notice that:

$$n! - \text{\# derangements} = |\bigcup A_i|$$

Since there are $n!$ possible permutations. Now we can use **Proposition 1.3**. Notice that due to the definition of A_i, we can directly compute the value of each term – since each set defines permutations that hold $1, 2, ..., n$ elements constant:

$$
\begin{aligned}
|\bigcup A_i| &= n \cdot (n-1)! - \binom{n}{2} \cdot (n-2)! + \binom{n}{3} \cdot (n-3)! + ... \pm \binom{n}{n} 1! \\
&= n! - \frac{n!}{2!(n-2)!} \cdot (n-2)! + \frac{n!}{3!(n-3)!} \cdot (n-3)! + ... \pm 1 \\
&= n! \cdot (1 - \frac{1}{2!} + \frac{1}{3!} + ... \pm \frac{1}{n!}) \\
&= \frac{n!}{e}
\end{aligned}
$$

Recall that

$$exp(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + ...$$

Thus we have that the number of derangements given n is about

$$n!(1 - \frac{1}{e})$$

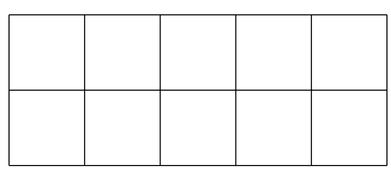
Remark. There exist **Bonferroni Inequalities** in the same form as **Proposition 1.1** that continuously add in longer intersection terms... e.g.:

$$|\bigcup_{i=1}^{n} A_i| \leq \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k|$$

$$|\bigcup_{i=1}^{n} A_i| \geq \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \sum_{i<j<k<l} |A_i \cap A_j \cap A_k \cap A_l|$$

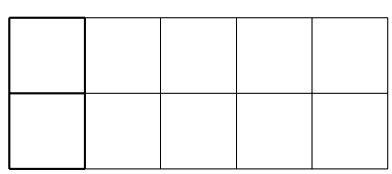
2. Counting with Recursion

Example 2.1. Consider a $2 \times n$ grid. $n = 5$ as shown:



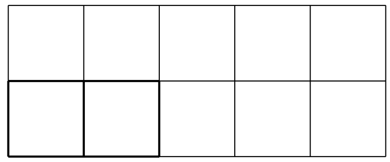
Question: How many ways can we tile 1×2 dominoes on this grid?

Solution: This can be done by first noticing the following: if we place a single domino all the way to the left of this grid, we get exactly **2 situations**. The first is:



Once this domino is set, notice that we now have to fill a 2×**(n-1)** grid. i.e. if we define $T(n) =$ # ways to tile an $2 \times n$ grid, then we would only have to find $T(n-1)$.

The second situation is:



When we place the domino like so, we are forced to place at least one domino to fill the space above the current one. We are then left with a 2×**(n-2)** grid to tile, i.e. $T(n-2)$ possibilities.

Therefore, we have $T(n) = T(n-1) + T(n-2)$ as our recurrence relation.
The base cases are $T(0) = 1$ and $T(1) = 1$. The answer then leads us to the Fibonacci numbers as they share the same relation, except we start with 0 indexing. Thus we have:

$$T(n) = F_{n+1} = (n+1)^{th} \text{ Fibonacci number}$$

3. Probabilistic Method

This section will now cover examples of how to use the *Probabilistic Method* to prove the existence of solutions. In general, we will be showing that there is a non-zero probability of obtaining a valid solution using some type of random assignment.

Example 3.1. Let S be a set of elements. Let $A_i \subseteq S$ such that $|A_i| = k$. (k-set)
Define family $\mathcal{F}$ as a set of k-sets:

$$\mathcal{F} = \{A_1, A_2, ..., A_n\}$$

Definition 3.2. A 2-coloring of $\mathcal{F}$ is a map $\mathcal{X} : S \to \{0, 1\}$ such that no A_i is **monochromatic**. i.e.:

$$\forall i \exists\, j, k \in A_i \ s.t.\ \mathcal{X}(j) \neq \mathcal{X}(k)$$

Theorem 3.3. *If $n < 2^{k-1}$, then $\exists$ 2-coloring of $\mathcal{F}$.*

Solution. We can employ the Probabilistic Method.

First choose a coloring uniformly at random. We want to show:

With probability > 0, we can get a valid 2-coloring of $\mathcal{F}$.

To do this, we can instead show its inverse:

$$\Pr[\exists \text{ monochromatic } A_i] < 1$$

Let E_i be the events (i.e. random colorings) such that A_i is monochromatic. Then we are trying to compute:

$$\Pr\left[\bigcup E_i\right] \leq \sum \Pr\left[E_i\right]$$

By **Union Bound** inequality. Clearly $\Pr[E_i] = 2(\frac{1}{2})^k$, since we have 2 colors and each element is assigned with $\frac{1}{2}$ probability. Now since we know $n < 2^{k-1}$, then:

$$\Pr\left[\bigcup E_i\right] \leq n \cdot 2(\frac{1}{2})^k < 1$$

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Definition 3.4. Universal Set. Let us define the following notation:

Set of strings of length n: $A \subseteq \{0,1\}^n$

Set of $k < n$ indeces: $S = \{i_1, i_2, ..., i_k\}$

Restriction of A onto S: $A_{|S} = \left\{(a_{i_1}, a_{i_2}, ..., a_{i_k}) : a = (a_1, ..., a_n) \in A\right\}$

Then A is a (n,k) - Universal Set if:

$\forall S$, $A_{|S}$ contains all 2^k possible binary strings of length k.

Example 3.5. Question: How large is a smallest (n,k)-universal set?

Theorem 3.6. *$\exists$ (n,k)-universal sets of size $\leq 2^k(kln(n)+1)$*

Proof. We will employ the Probabilistic Method again. Suppose A contains r random substrings. If we are trying to prove that these (n,k) - universal sets of this size exist, then we want to show:

$$\Pr\left[\exists\ A_{|S} \neq \{0,1\}^k\right] < 1$$

To do this, let us further break this event into smaller ones, by considering every possible $v \in \{0,1\}^k$:

$$\Pr\left[\bigcup_v \left\{v \notin A_{|S}\right\}\right] \leq 2^k \cdot (1 - (\frac{1}{2})^k)^r$$

This is because:

(1) Union Bound to get inequality.
(2) 2^k possible v exist
(3) Given a substring of length k, v, then the probability of $A_{|S}$ not containing it would be $(1-(\frac{1}{2})^k)^r$. Since A contains r random substrings and the probability of creating a random substring of length k is $(\frac{1}{2})^k$.

Further, we can extend the inequality:

$$\begin{aligned}
&\leq 2^k \cdot (1 - (\frac{1}{2})^k)^r \\
&\leq \binom{n}{k} 2^k \cdot (1 - 2^{-k})^r \\
&\leq \frac{n}{k!} 2^k \cdot e^{-2^{-k} \cdot r}
\end{aligned}$$

Now just set $r = 2^k(k\ln(n)+1)$. We get:

$$\begin{aligned}\Pr\left[\exists\, A_{|S} \neq \{0,1\}^k\right] &\leq \frac{n^k}{k!} 2^k e^{-k\ln(n)+1} \\ &= \frac{2^k}{k!} \cdot \frac{1}{e} \\ &< 1\end{aligned}$$

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