# LECTURE 1: EE 381K - CONVEX OPTIMIZATION

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## 1. Class Policy Extras

Linear Programming Books:

(1) Bertsimas, Tsitsiklis "Introduction to Linear Optimization"
(2) R.J. Vanderbli "Linear Programming Foundation and Extension"

Convex Optimization Books:

(1) Ben Tal & Nemirovski "Lectures on Modern Convex Opt
(2) Bertsekas, Nedic, Oz

## 2. Minimization

General Formulation:

$$\begin{aligned}
&\text{minimize } f_0(x)\\
&\text{subject to } f_i(x) \leq b_i(x)\ i = 1, ..., m\\
&f_0 : \mathbb{R}^n \to \mathbb{R} \Rightarrow \text{Objective function}\\
&f_i : \mathbb{R}^n \to \mathbb{R}\ i = 1, ..., m \Rightarrow \text{Constraint function}\\
&x \in \mathbb{R}^n \Rightarrow \text{ optimization/decision variable}\\
&\hat{x} \text{ feasible if } f_i(\hat{x}) \leq b_i(\hat{x})\\
&x^* : \text{ optimal solution}\\
&f_0(x^*) \leq f_0(\hat{x}) \forall \text{ feasible } \hat{x}
\end{aligned}$$

## 3. Linear Programming

**Definition 3.1.** Linear programs are one type of convex optimization problems, and they consist of:

- Objective function $f_0$
- All the constraints $f_1, ..., f_m$

If these are all linear functions, then it is a linear program.

$$\min_{x_1,...,x_n} \sum_{j=1}^{n} c_j x_j \mid c_j \text{ cost of } x_j$$

$$s.t. \sum_{j=1}^{n} a_{ij} x_j \leq b_i \text{ for } i = 1, ..., m \Rightarrow m \text{ Inequality constraints}$$

$$\sum_{j=1}^{n} d_{ij}x_j = e_i \text{ for } i = 1, ..., P \Rightarrow P \text{ Equality constraints}$$

Example: Resource Allocation

**Parameters** i.e. given variables

- $n$: Number of activities $j = 1, ..., n$
- $m$: Number of Resources $i = 1, ..., m$
- $P_j$: Profit of activity $j$
- $b_i$: Amount of available resource $i$
- $a_{ij}$: amount of resource $i$ used by activity $j$

**Variables**

- $x_j$ amount of activity $j$ selected for resource $i$

**Goal**

$$\max \textstyle\sum_{j=1}^{n} P_j x_j$$
$$\text{s.t. } \textstyle\sum_{j=1}^{n} a_{ij}x_j \leq b_i \text{ for } i = 1, ...m$$
$$\text{s.t. } x_j \geq 0 \text{ for } j = 1, ..., m$$

Example: Matching Problem

We have $N$ people and $N$ tasks.
People indexed by $i = 1, ...N$
Tasks indexed by $j = 1, ..., N$
$a_{ij}$ cost of assigning task $j$ to person $i$

$$x_{ij} = \begin{cases} 1 & \text{Assign task } j \text{ to person } i \\ 0 & \text{otherwise} \end{cases}$$

**Goal**

$$\text{minimize } \textstyle\sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}x_{ij}$$
$$\text{s.t. } \textstyle\sum_{i=1}^{N} x_{ij} = 1 \; j = 1, ..., N$$
$$\text{s.t. } \textstyle\sum_{j=1}^{N} x_{ij} = 1 \; i = 1, ..., N$$
$$\text{AND } x_{ij} \in \{0, 1\}$$

**Problem:** Feasible set is only $\{0, 1\}$.
Thus we should **relax** constraints such that $0 \leq x_{ij} \leq 1$. We will come back to this later.

## 4. Vectorization

Now let's vectorize these formulations.

$$c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$$

$$a_i = \begin{bmatrix} a_{i1} \\ \vdots \\ a_{in} \end{bmatrix} i = 1, .., m \quad d_i = \begin{bmatrix} d_{i1} \\ \vdots \\ d_{in} \end{bmatrix} i = 1, .., P$$

$$\text{minimize } c^T x$$
$$\text{s.t. } a_i^T x \leq b_i \ i = 1, ..., m$$
$$d_i^T n = e_i \ i = 1, ..., P$$

Now let

$$A = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ \vdots & \vdots & ... & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix} \in \mathbb{R}^{mn} \quad D = \begin{bmatrix} d_{11} & d_{12} & ... & d_{1n} \\ \vdots & \vdots & ... & \vdots \\ d_{P1} & d_{P2} & ... & d_{Pn} \end{bmatrix} \in \mathbb{R}^{Pn}$$

Now we can rewrite the constraints to be:

$$Ax \leq b$$
$$Dx = e$$

*Remark* 4.1. Whenever using $\leq$ with vectors, this implies all elements in one vector are $\leq$ than their respective element (same idx) in the other vector.

## 5. Shit ton of Definitions

**Definition 5.1.** $x$ is feasible if it satisfies $Ax \leq b$ & $Dx = e$

**Definition 5.2.** feasible set $S = \{x \in \mathbb{R}^n | Ax \leq b \ \& \ Dx = e\}$

**Definition 5.3.** $x^*$ is optimal if $c^T x^* \leq c^T x$ for any $x \in S$

**Definition 5.4.** optimal value $p^* = c^T x^*$

**Definition 5.5.** Unbounded LP $p^* = c^T x^*$

**Definition 5.6.** Hyperplane: Solution set of one linear equation with nonzero coeff. vector $a$ ($a_i \neq 0$) s.t. $a^T x = b$

**Definition 5.7.** Half Space: Solution set of one linear inequality with nonzero coefficients. i.e. $a^T x \leq b$.

**Definition 5.8.** Subspace Intersection of a set of hyperplanes. Or a solution to a system of equality equations.

**Definition 5.9.** Polyhedron Intersection of a set of half-spaces. Or a solution to a finite number of linear inequalities.

**Definition 5.10.** Function set Set of points where some $f$ has value $\alpha$. i.e. $f(x) = \alpha$. e.g. hyperplanes

## 6. Polyhedrons

$$P = \{x | Ax \leq B, Cx = d\}$$

**Definition 6.1.** Lineality space: The lineality space of $P$ is defined as

$$L = \text{nullspace}( \begin{bmatrix} A \in \mathbb{R}^{mn} \\ C \in \mathbb{R}^{Pn} \end{bmatrix}$$

*Claim.* Let $x \in P, v \in L$. Then $x + v \in L$.

*Proof.* Trivial. $x$ in the solution space, and $Av = 0 = Cv$ ∎

**Definition 6.2.** Pointed polyhderon A polyhderon $P$ with Lineality space $L = \{0\}$. i.e. the null space of both $A$ and $C$ is trivial.

$\Rightarrow$ A polyhderon is pointed if it doesn't contain a *line*.
Example 1
a half space $\{xa^T x \leq b\}$.
Only when $x \in \mathbb{R}$ is this a pointed polyhedron.
Example 2
a half space $\{x - 1 \leq a^T x \leq 1\}$.
Only when $x \in \mathbb{R}$ is this a pointed polyhedron.
Example 3
a half space $\{x|x| \leq 1|y| \leq 1\}$.
$S = \{(0, 0, z)|z \in \mathbb{R}\}$
therefore always not pointed.
Example 4
$\{x|1^T x = 1, x \geq 0\}$.
YEET it is pointed