# Conic Sections

# *Learning Outcomes*

At the end of this lesson, you should be able to:

Define conic sections;

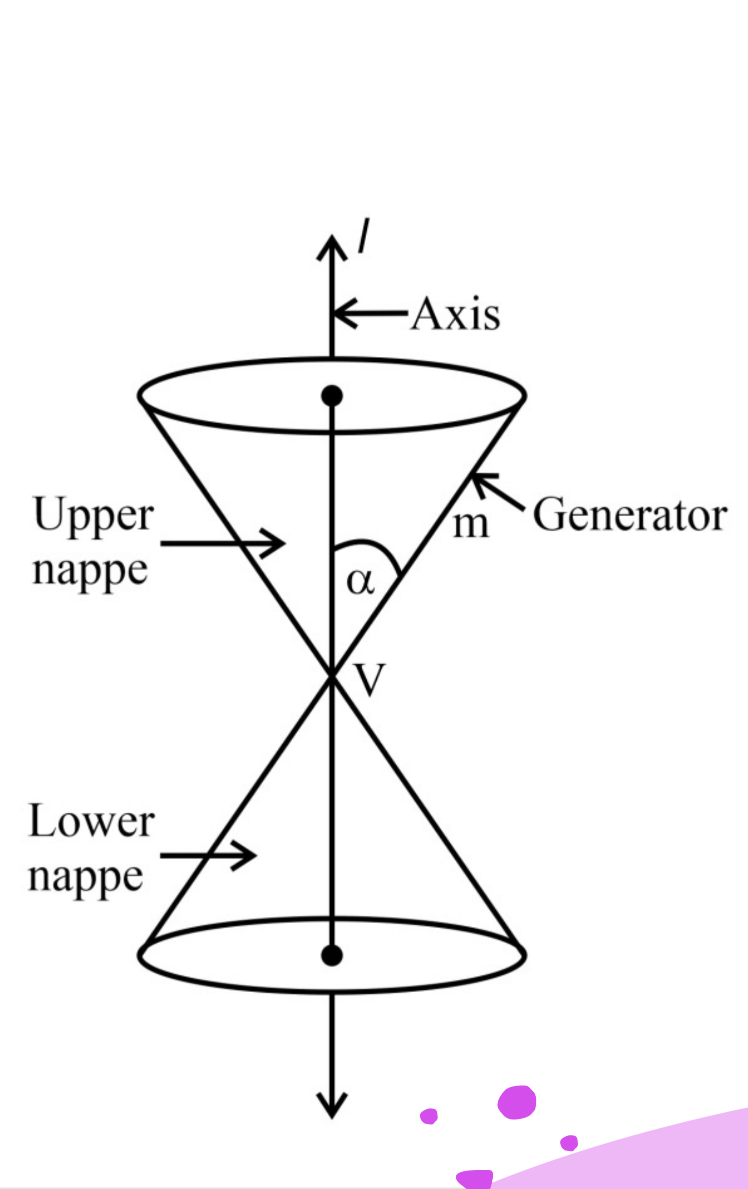
Illustrate the different types of conic sections: circle, parabola, ellipse, and hyperbola;

Illustrate the degenerate cases of conic sections; and

Identify a type of conic section given the general form of an equation.

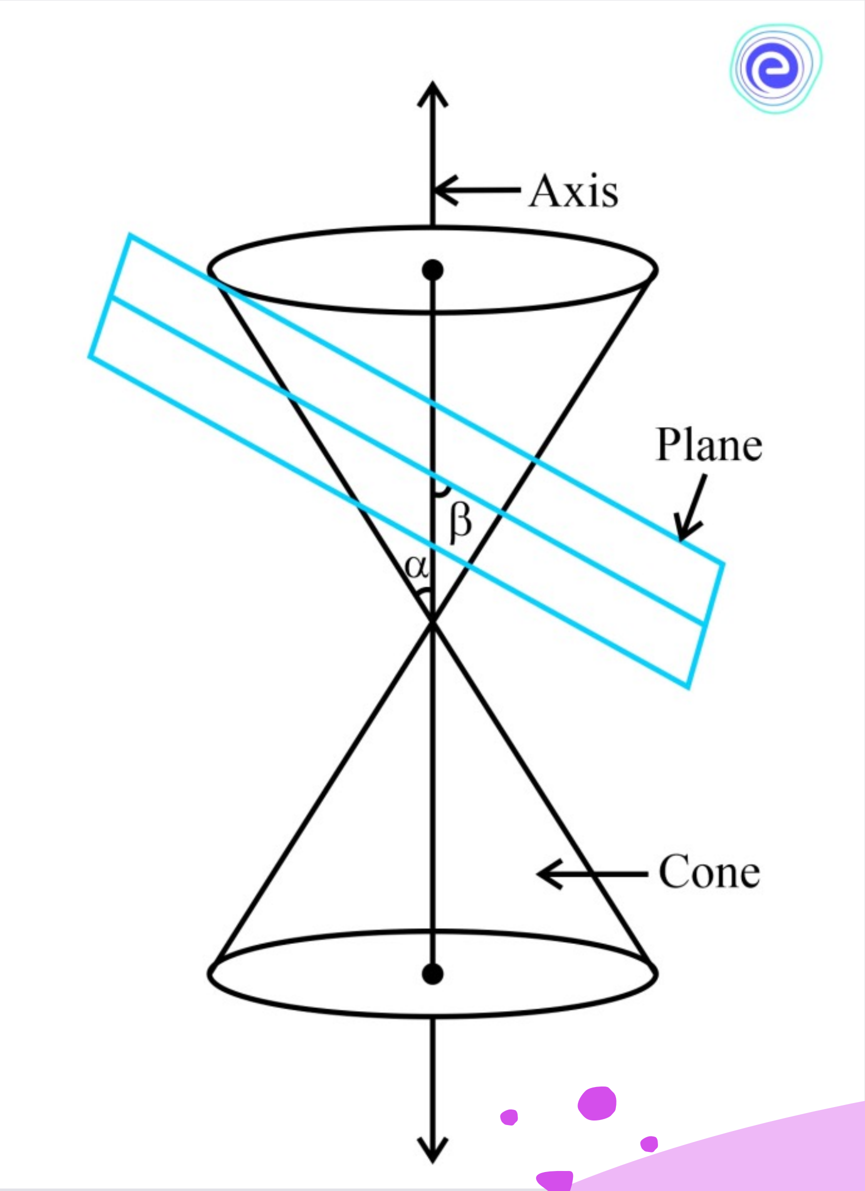
# *Definition*

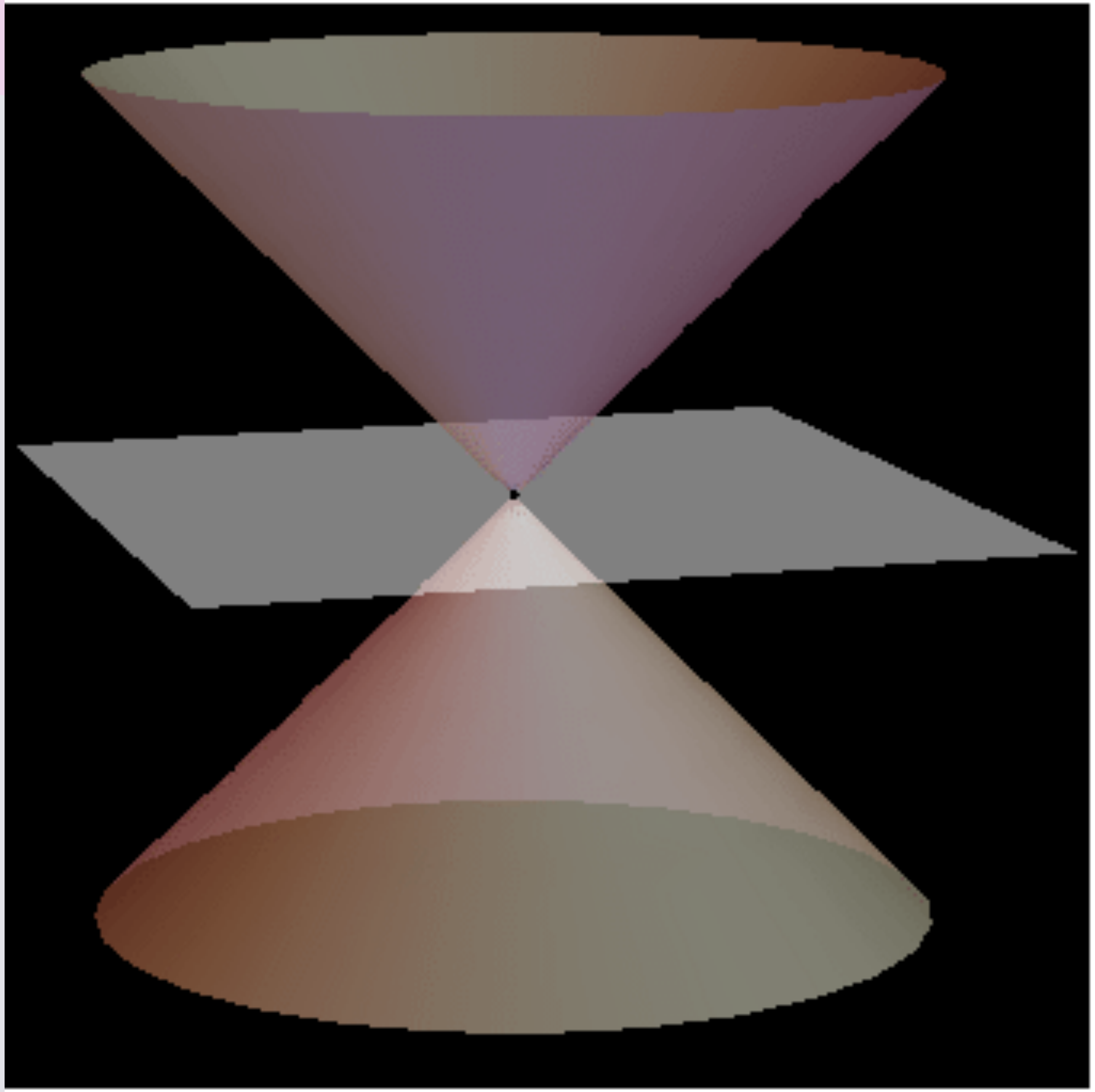
A **double-napped cone** is a pair of congruent opposite cones connected by a common vertex. It is formed when a line called the **generator** rotates about another line called the **axis** through their intersection point. This point of intersection becomes the **common vertex** of the opposite cones.



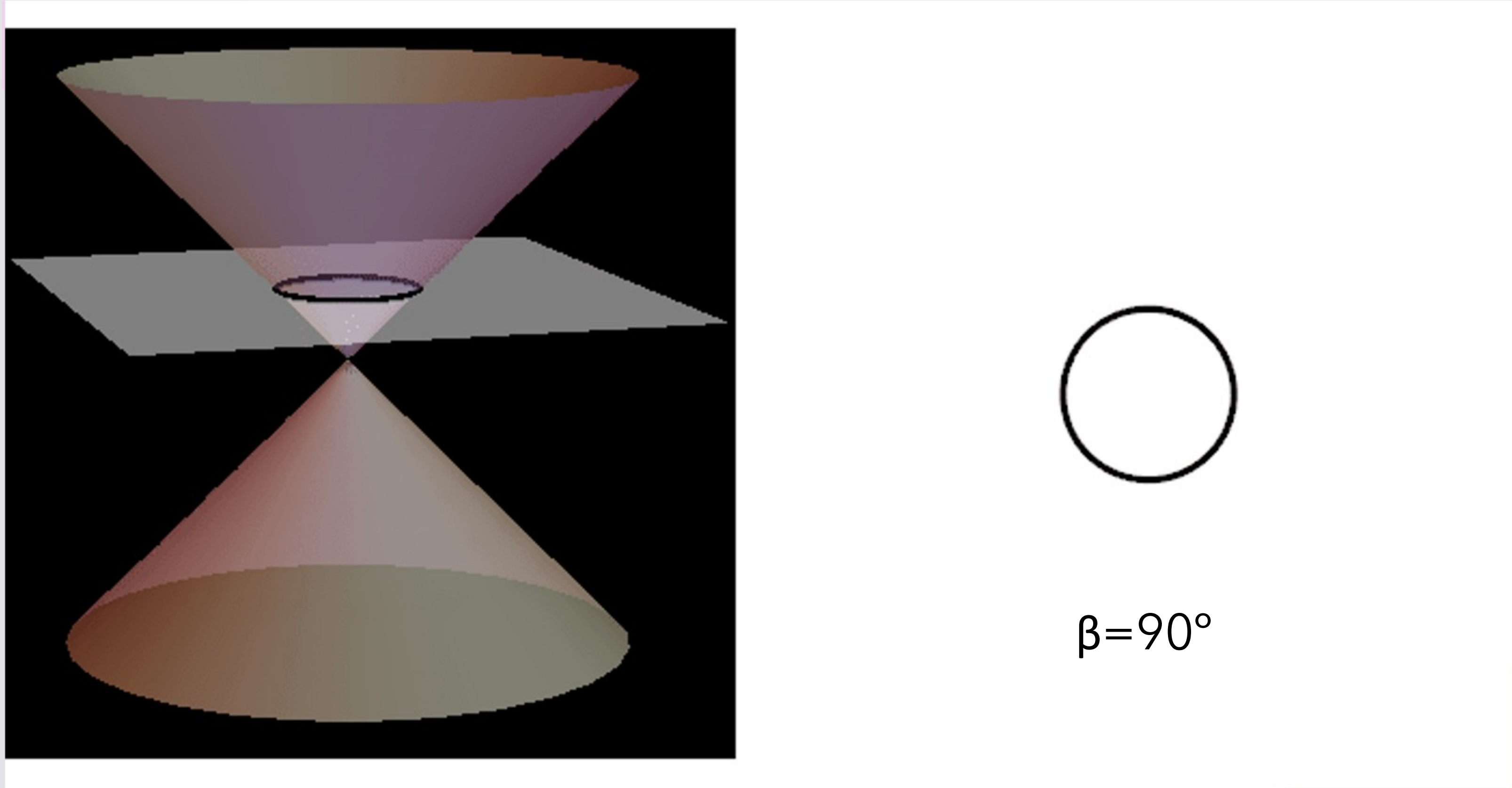
# *Definition*

**Conic sections** are two-dimensional curves that are formed when a plane intersects a double-napped cone. The shapes of these curves depend on the angle ($\beta$) at which the plane intersects the axis and the angle ($\alpha$) that the generator makes with the axis.

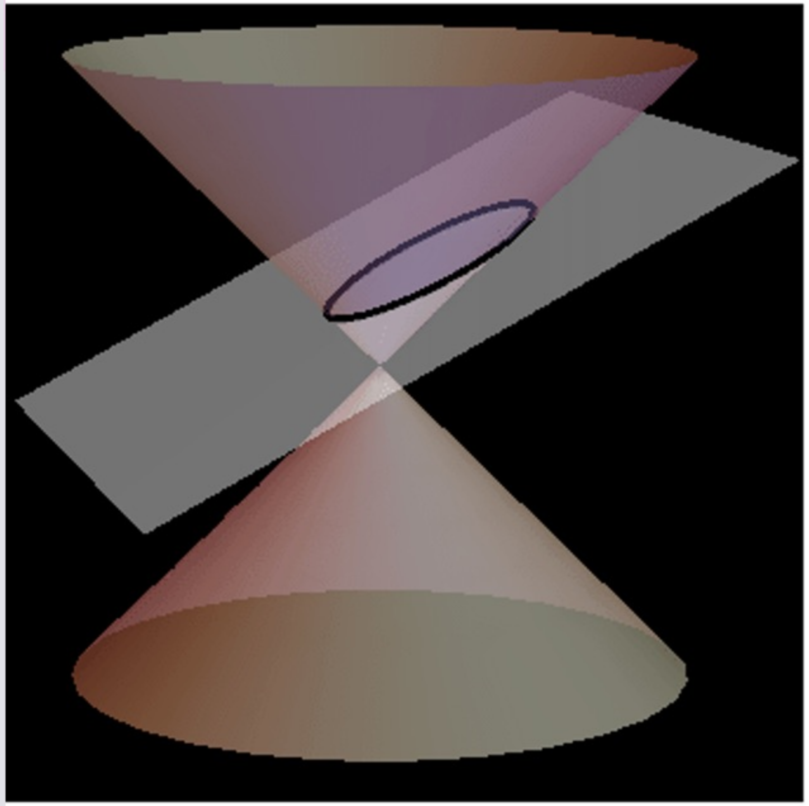


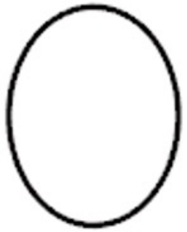


# Circle



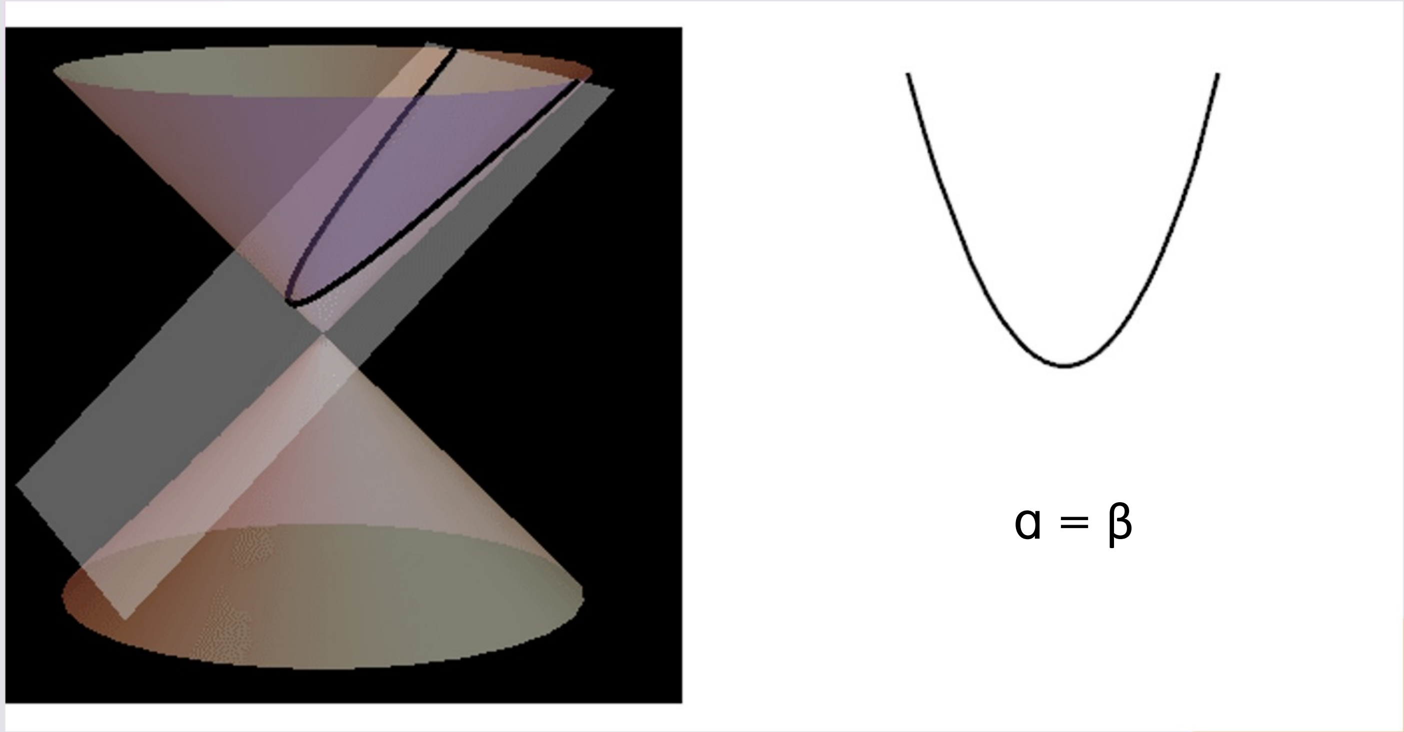
# *Ellipse*



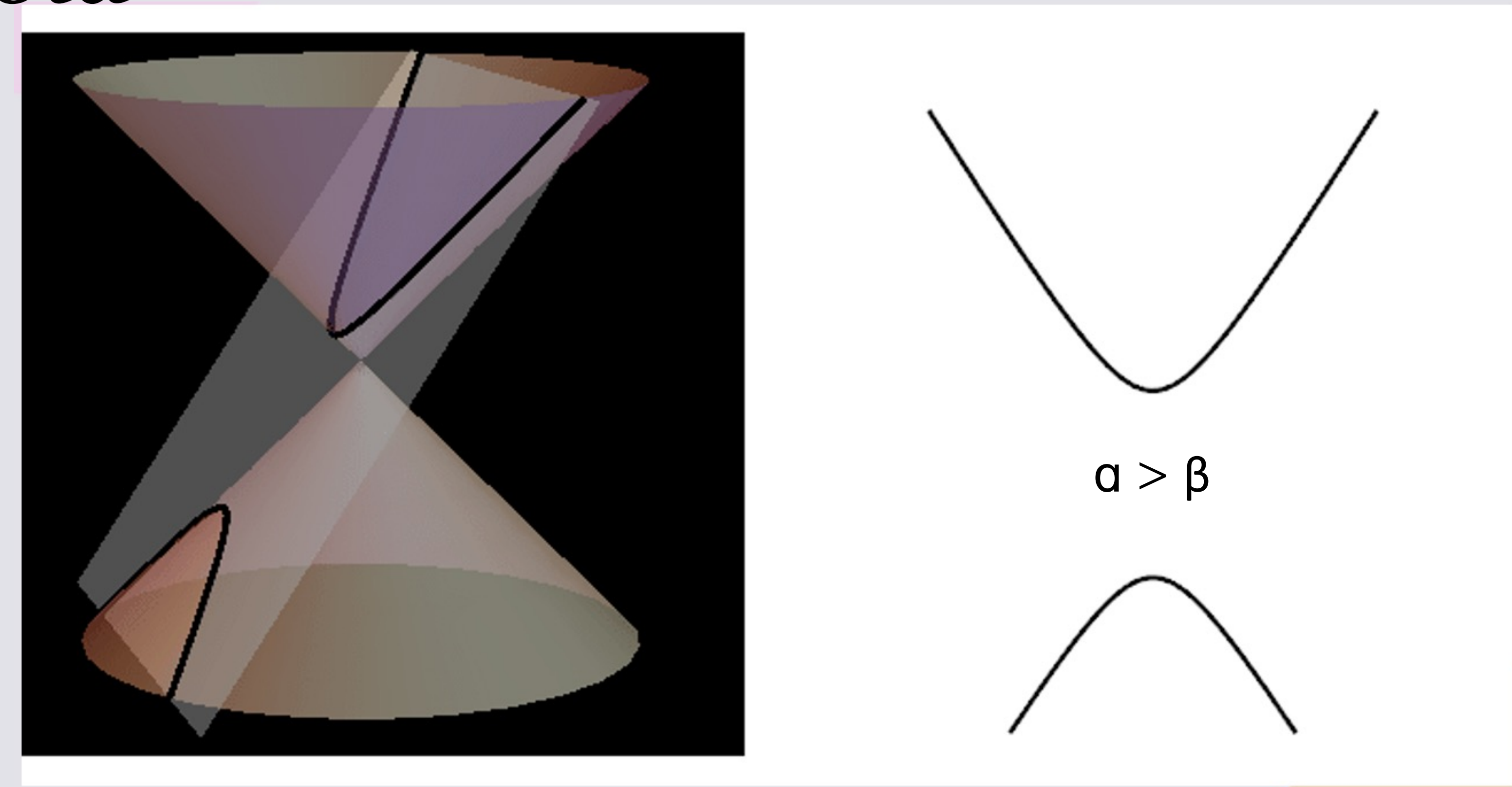


$\alpha < \beta < 90^\circ$

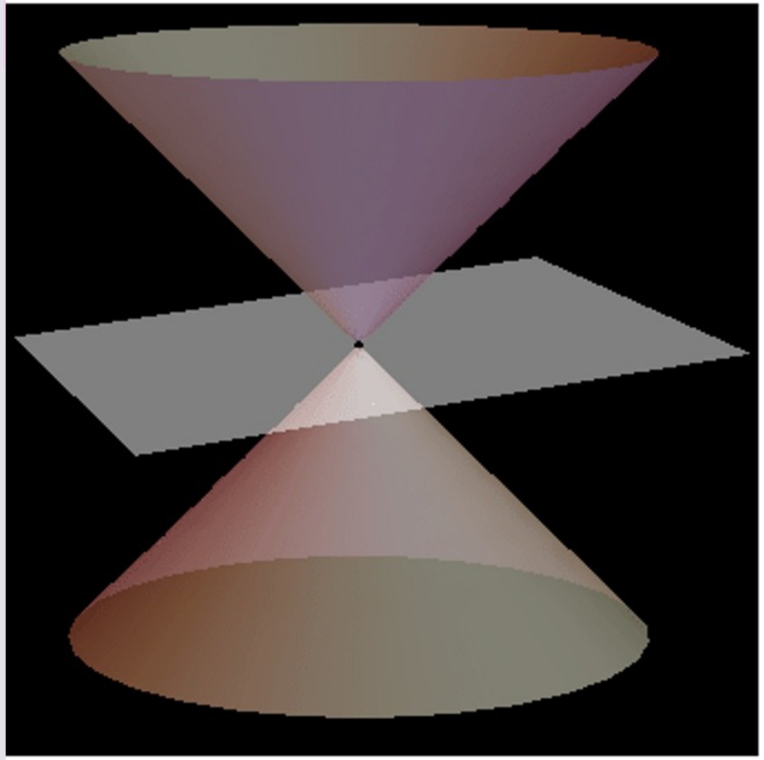
# *Parabola*



# *Hyperbola*



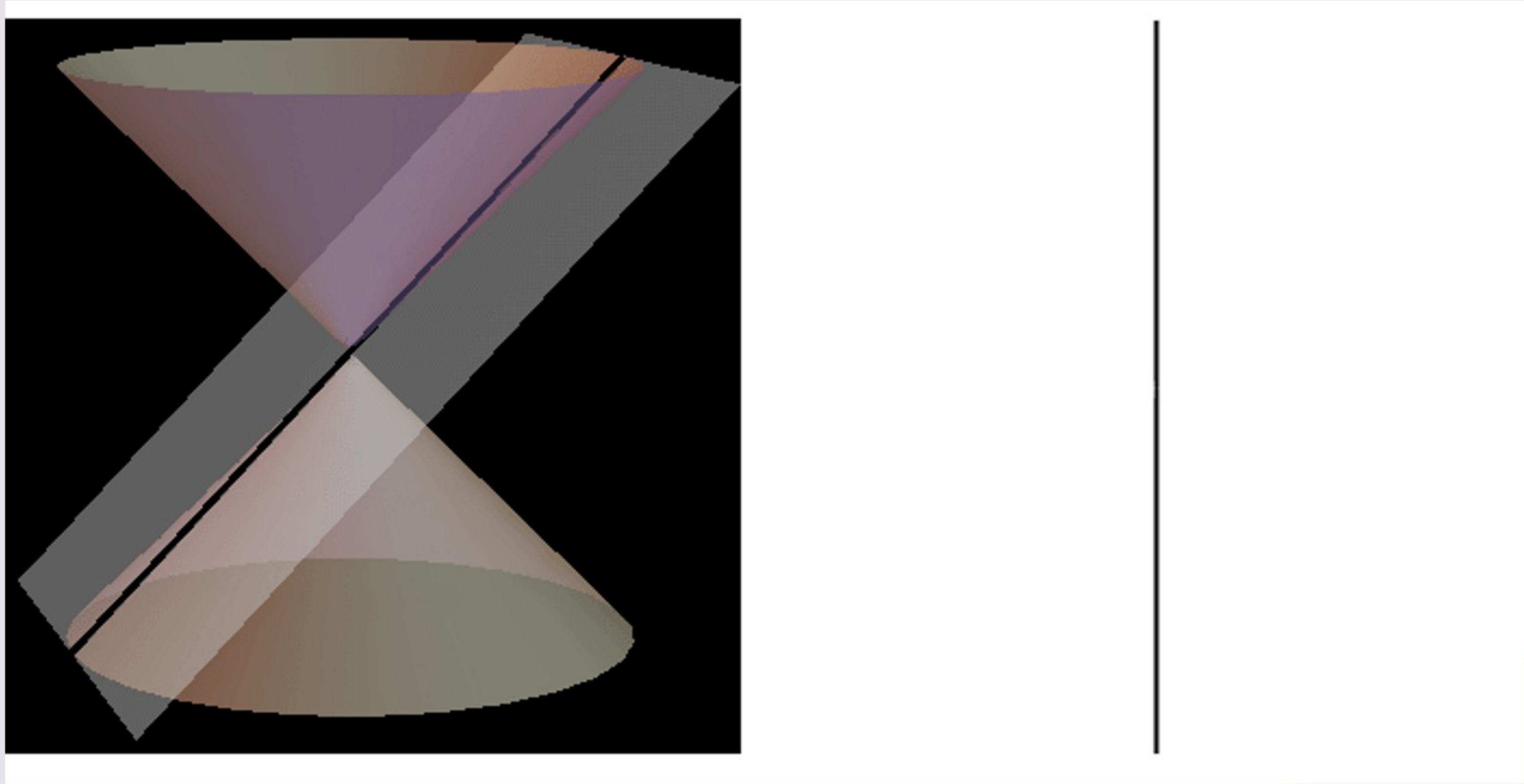
# *Point*





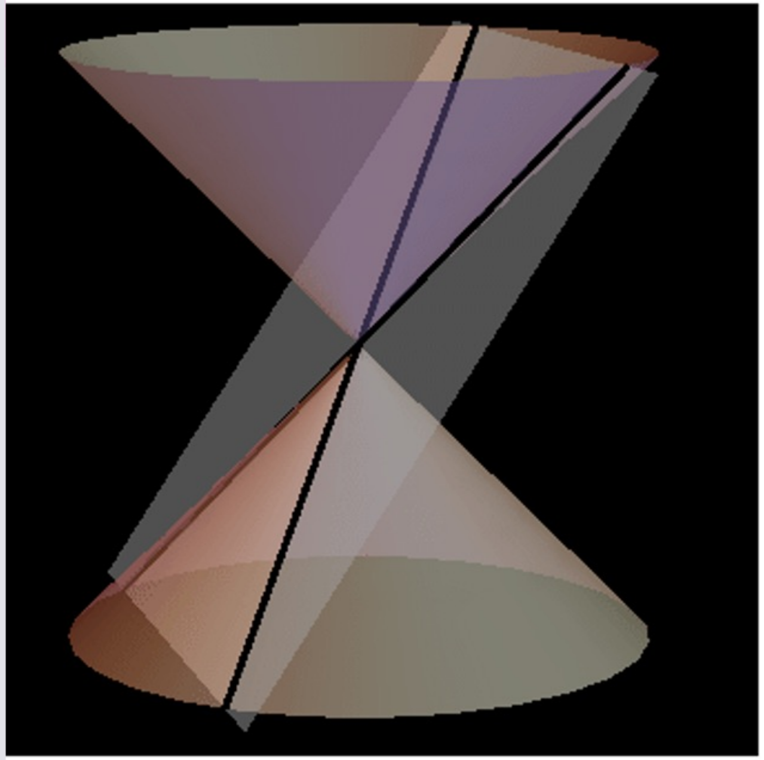
α < β < 90° or β = 90° and passes through vertex

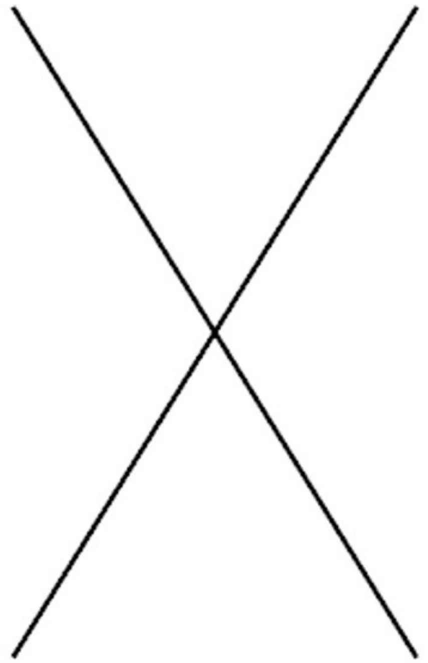
# *Line*



α = β and passes through vertex

# *Two Intersecting Lines*





α > β and passes through vertex

# *Definition*

The **general form of the equation** of conic sections is a quadratic equation in $x$ and $y$ given by $Ax^2 + By^2 + Cx + Dy + E = 0$, where $A$, $B$, $C$, $D$, and $E$ are all real numbers, with $A$ and $B$ not equal to zero.

Since there is only one general form for all conic sections, here are some useful hints to help us determine what type of conic section is being described by a given equation.

1. If $A$ and $B$ are equal, the equation may be that of a circle.

- $4x^2 + 4y^2 + 8x + 20y + 1 = 0$
- $3x^2 + 3y^2 + 12x - 6y - 11 = 0$

# *Definition*

2. If $A$ and $B$ are not equal but have the same signs, the equation may be that of an ellipse.

- $4x^2 + 5y^2 + 8x - 20y - 15 = 0$
- $x^2 + 2y^2 - x + 20y + 12 = 0$

3. If $A$ and $B$ have opposite signs, the equation may be that of a hyperbola.

- $5x^2 - 6y^2 + 8x + 2y - 44 = 0$
- $-4x^2 + y^2 + x + 15y - 21 = 0$

4. If $A = 0$ or $B = 0$ but not both, the equation may be that of a parabola.

- $13x^2 + 8x - 11y + 15 = 0$; here, $B = 0$.
- $-2y^2 - 8x + 5y + 19 = 0$; here, $A = 0$.

# *Definition*

However, degenerate conics also share the same general equation. The following are examples of equations of some degenerate conics that look exactly like the equations of normal conic sections.

1. $2x^2 + 5y^2 + 12x - 20y + 38 = 0$ is not an ellipse but a point.
2. $2x^2 - 3y^2 - 20x - 18y + 23 = 0$ is not a hyperbola but a pair of intersecting lines.
3. $x^2 + y^2 + 2x - 6y + 15 = 0$ will not form a circle because its graph is imaginary.

# *Definition*

The **standard form** of each conic section is given as follows:

1. Circle: $(x-h)^2+(y-k)^2=r^2$
2. Ellipse: $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ or $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$
3. Parabola: $4a(x-h)=(y-k)^2$ or $4a(y-k)=(x-h)^2$, $a\neq 0$
4. Hyperbola: $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ or $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$

# Definition

Degenerate conics are formed from these standard forms if a special situation is present.

5. Point – the constant side of the standard form of a circle or ellipse is zero
6. Pair of intersecting lines – the constant side of the standard form of a hyperbola is zero
7. Line – either side of the standard form of a parabola is zero
8. Imaginary graph – the constant side of the standard form of a circle or ellipse is a negative number

# *Examples*

Transform $6x^2 + 6y^2 + 12x - 24y + 12 = 0$ into its standard form. Then, identify what type of conic section it is.

Transform $2x^2 + y^2 - 4x - 2y - 1 = 0$ into its standard form. Then, identify what type of conic section it is.

Convert the hyperbola with equation $4x^2 - 3y^2 - 8x + 24y - 80 = 0$ into its standard form.

Convert the parabola with equation $x^2 + 8x - 4y + 36 = 0$ into its standard form.

Convert the hyperbola with equation $\frac{(x-3)^2}{9} - \frac{(y+4)^2}{4} = 1$ into its general form.

Convert the ellipse with equation $\frac{(x+5)^2}{8} + \frac{(y+2)^2}{8} = 1$ into its general form.