

Circuits and Transforms

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Abstract—This manual provides a simple introduction to Transforms

1 DEFINITIONS

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \quad (1.1)$$

2. The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st} dt \quad (1.2)$$

2 LAPLACE TRANSFORM

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes $q_1 \mu C$. Then S is switched to position Q. After a long time, the charge on the capacitor is $q_2 \mu C$.

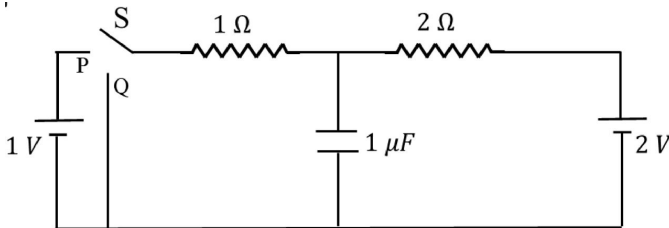


Fig. 2.1

2. Find q_1 and Draw the circuit using latex-tikz.
Solution: Let circuit be grounded at G then let

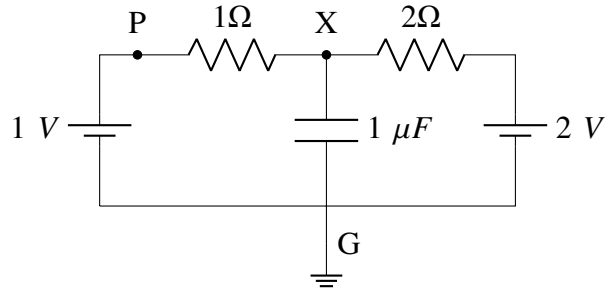


Fig. 2.2

relative potential at point X to be V .
From KCL,

$$\frac{V-1}{1} + \frac{V-2}{2} = 0$$

$$V = \frac{4}{3}$$

3. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC.

Solution: We have,

$$L[u(t)] = \int_0^{\infty} u(t)e^{-st} dt \quad (2.1)$$

$$= \int_0^{\infty} e^{-st} dt \quad (2.2)$$

$$= \frac{1}{s}, \quad \Re(s) > 0 \quad (2.3)$$

4. Show that

$$e^{-at}u(t) \xleftrightarrow{\mathcal{H}} L \frac{1}{s+a}, \quad a > 0 \quad (2.4)$$

and find the ROC.

Solution:

$$L[e^{-at}u(t)] = \int_0^{\infty} u(t)e^{-(s+a)t} dt \quad (2.5)$$

$$= \int_0^{\infty} e^{-(s+a)t} dt \quad (2.6)$$

$$= \frac{1}{s+a}, \quad \Re(s) > -a \quad (2.7)$$

5. Now consider the following resistive circuit transformed from Fig. 2.1 where

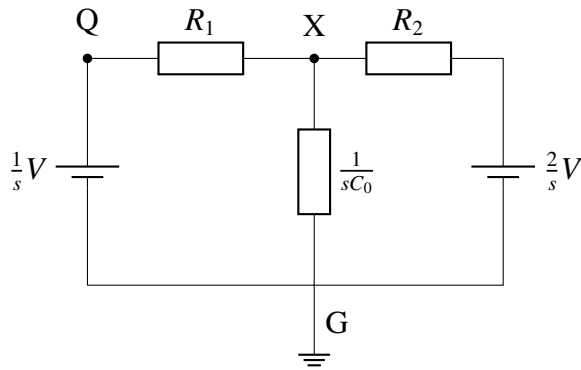


Fig. 2.3

$$L[u(t)] = V_1(s) \quad (2.8)$$

$$L[2u(t)] = V_2(s) \quad (2.9)$$

Find the voltage across the capacitor $V_{C_0}(s)$.

Solution:

$$V_1(s) = \frac{1}{s} \quad (2.10)$$

$$V_2(s) = \frac{2}{s} \quad (2.11)$$

Let circuit be grounded at G then let relative potential at point X to be V.

From KCL at X,

$$\frac{V - \frac{1}{s}}{R_1} + \frac{V - \frac{2}{s}}{R_2} + sC_0 V = 0 \quad (2.12)$$

$$V \left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0 \right) = \frac{1}{s} \left(\frac{1}{R_1} + \frac{2}{R_2} \right) \quad (2.13)$$

$$V(s) = \frac{\frac{1}{R_1} + \frac{2}{R_2}}{s \left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0 \right)} \quad (2.14)$$

$$= \frac{\frac{1}{R_1} + \frac{2}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2}} \left(\frac{1}{s} - \frac{1}{\frac{1}{C_0} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) + s} \right) \quad (2.15)$$

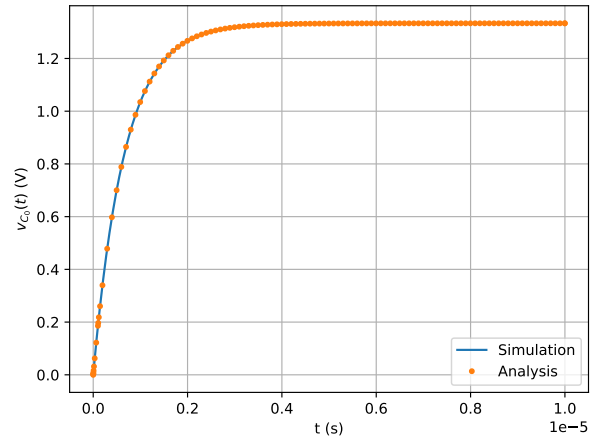
6. Find $v_{C_0}(t)$. Plot using python.

Solution:

$$L^{-1}[V(s)] = L^{-1} \left[\frac{\frac{1}{R_1} + \frac{2}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2}} \left(\frac{1}{s} - \frac{1}{\frac{1}{C_0} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) + s} \right) \right] \quad (2.16)$$

$$= \frac{2R_1 + R_2}{R_1 + R_2} u(t) \left(1 - e^{-\left(\frac{1}{R_1} + \frac{1}{R_2} \right) \frac{t}{C_0}} \right) \quad (2.17)$$

$$= \frac{4}{3} \left(1 - e^{-(1.5 \times 10^6)t} \right) u(t) \quad (2.18)$$

Fig. 2.4: $v_{C_0}(t)$ before the switch is flipped

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$ wget https://github.com/kurugodukarthik11/EE3900/blob/main/Assignments/Assignment_1/codes/7_11.py
```

7. Verify your result using ngspice. **Solution:**

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$ wget https://github.com/kurugodukarthik11/EE3900/blob/main/Assignments/Assignment_1/codes/7_11.py
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3 INITIAL CONDITIONS

1. Find q_2 in Fig. 2.1.

Solution:

The equivalent circuit at steady state when the switch is at Q is shown below. Since capacitor

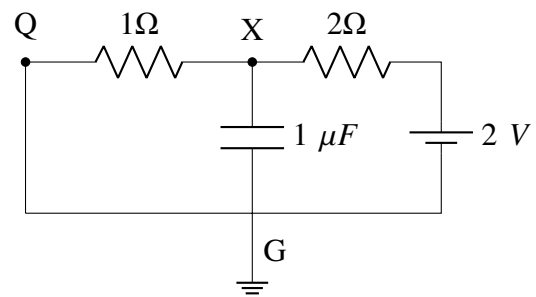


Fig. 3.1

behaves as an open circuit, we use KCL at X.

$$\frac{V - 0}{1} + \frac{V - 2}{2} = 0 \implies V = \frac{2}{3} \quad (3.1)$$

and hence, $q_2 = \frac{2}{3} \mu C$.

2. Draw the equivalent s -domain resistive circuit when S is switched to position Q. Use variables R_1, R_2, C_0 for the passive elements. Use latex-tikz.

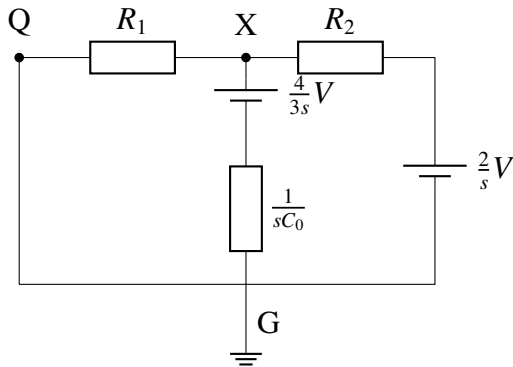


Fig. 3.2

3. $V_{C_0}(s) = ?$

Solution: Using KCL at node X in Fig. 3.2

$$\frac{V - 0}{R_1} + \frac{V - \frac{2}{s}}{R_2} + sC_0 \left(V - \frac{4}{3s} \right) = 0 \quad (3.2)$$

$$\Rightarrow V_{C_0}(s) = \frac{\frac{2}{sR_2} + \frac{4C_0}{3}}{\frac{1}{R_1} + \frac{2}{R_2} + sC_0} \quad (3.3)$$

4. $v_{C_0}(t) = ?$ Plot using python.

Solution:

From (3.3),

$$V_{C_0}(s) = \frac{4}{3} \left(\frac{1}{\frac{1}{C_0} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) + s} \right) + \frac{2}{R_2 \left(\frac{1}{R_1} + \frac{1}{R_2} \right)} \left(\frac{1}{s} - \frac{1}{\frac{1}{C_0} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) + s} \right) \quad (3.4)$$

Applying Inverse Laplace Transform on both sides

$$L^{-1}[V_{C_0}(t)] = v_{C_0}(t) = \frac{4}{3} e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{t}{C_0}} u(t) + \frac{2}{R_2 \left(\frac{1}{R_1} + \frac{1}{R_2} \right)} \left(1 - e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{t}{C_0}} \right) u(t) \quad (3.5)$$

Substituting R_1, R_2, C_0 in (3.5)

$$L^{-1}[V_{C_0}(t)] = v_{C_0}(t) = \frac{2}{3} \left(1 + e^{-(1.5 \times 10^6)t} \right) u(t) \quad (3.6)$$

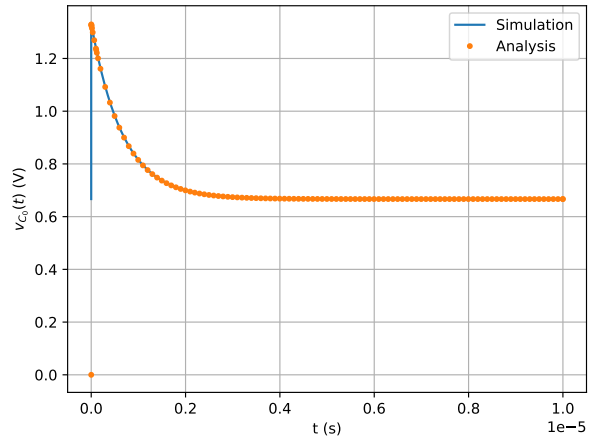


Fig. 3.3: $v_{C_0}(t)$ after the switch is flipped

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$ wget https://github.com/kurugodukarthik11/EE3900/blob/main/Assignments/Assignment_1/codes/7_11.py
```

5. Verify your result using ngspice.

Solution:

```
$ wget https://github.com/kurugodukarthik11/EE3900/blob/main/Assignments/Assignment_1/codes/7_11.py
```

6. Find $v_{C_0}(0-)$, $v_{C_0}(0+)$ and $v_{C_0}(\infty)$.

Solution: From the initial conditions,

$$v_{C_0}(0-) = \frac{q_1}{C} = \frac{4}{3} \quad (3.7)$$

From (3.6),

$$v_{C_0}(0+) = \lim_{t \rightarrow 0+} v_{C_0}(t) = \frac{4}{3} \quad (3.8)$$

$$v_{C_0}(\infty) = \lim_{t \rightarrow \infty} v_{C_0}(t) = \frac{2}{3} \quad (3.9)$$

7. Obtain the Fig. in problem 3.2 using the equivalent differential equation.

Solution: The equivalent circuit in the t -domain is shown below. From KCL and KVL,

$$i_1 = i_2 + i_3 \quad (3.10)$$

$$i_1 R_1 + \frac{4}{3} + \frac{1}{C_0} \int_0^t i_2 dt = 0 \quad (3.11)$$

$$\frac{4}{3} + \frac{1}{C_0} \int_0^t i_2 dt - i_3 R_2 - 2 = 0 \quad (3.12)$$

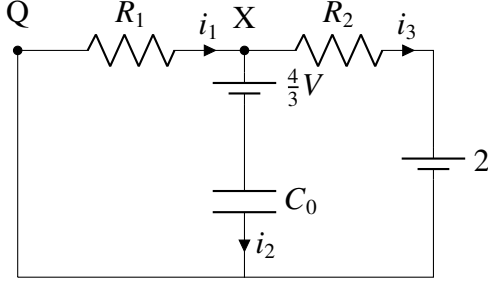


Fig. 3.4

Taking Laplace Transforms on both sides and using the properties of Laplace Transforms,

$$I_1 = I_2 + I_3 \quad (3.13)$$

$$I_1 R_1 + \frac{4}{3} + \frac{1}{sC_0} I_2 = 0 \quad (3.14)$$

$$\frac{4}{3} + \frac{1}{sC_0} I_2 - I_3 R_2 - 2 = 0 \quad (3.15)$$

$$(3.16)$$

where $i(t) \xleftrightarrow{\mathcal{H}} LI(s)$. Note that the capacitor is equivalent to a resistive element of resistance $R_C = \frac{1}{sC_0}$ in the s -domain. Equations (3.13) - (3.15) precisely describe Fig. 3.2.

4 BILINEAR TRANSFORM

1. In Fig. 2.1, consider the case when S is switched to Q right in the beginning. Formulate the differential equation.

Solution:

The equivalent circuit in the t -domain is shown below.

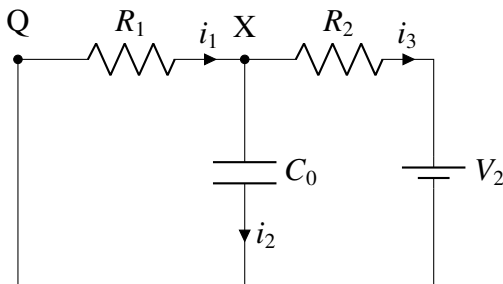


Fig. 4.1

Applying KCL and KVL,

$$i_1 = i_2 + i_3 \quad (4.1)$$

$$i_1 R_1 = -\frac{1}{C_0} \int_0^t i_2 dt \quad (4.2)$$

$$i_3 R_2 = \frac{1}{C_0} \int_0^t i_2 dt - 2 \quad (4.3)$$

Differentiating (4.1), (4.2), (4.3),

$$\frac{di_1}{dt} = \frac{di_2}{dt} + \frac{di_3}{dt} \quad (4.4)$$

$$R_1 \frac{di_1}{dt} = -\frac{i_2}{C_0} \quad (4.5)$$

$$R_2 \frac{di_3}{dt} = \frac{i_2}{C_0} \quad (4.6)$$

From (4.4), (4.6), (4.5),

$$\frac{di_1}{dt} = \left(\frac{-R_2}{R_1} \right) \frac{di_3}{dt} \quad (4.7)$$

$$\therefore \left(\frac{-R_2}{R_1} \right) \frac{di_3}{dt} = \frac{di_2}{dt} + \frac{di_3}{dt} \quad (4.8)$$

$$\frac{di_2}{dt} + \left(1 + \frac{R_2}{R_1} \right) \frac{di_3}{dt} = 0 \quad (4.9)$$

$$\frac{di_2}{dt} + \left(1 + \frac{R_2}{R_1} \right) \frac{i_2}{R_2 C_0} = 0 \quad (4.10)$$

$$\frac{di_2}{dt} + \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \frac{i_2}{C_0} = 0 \quad (4.11)$$

$$\frac{di_2}{dt} + \frac{i_2}{\tau} = 0 \quad (4.12)$$

where $\tau = \frac{C_0 R_1 R_2}{R_1 + R_2}$ Integrating (4.12) w.r.t t and at $t = 0$, $i_2 = \frac{V_2}{R_2}$

$$C_0 \frac{dV}{dt} - \frac{V_2}{R_2} + \frac{C_0 V}{\tau} = 0 \quad (4.13)$$

$$\frac{dV}{dt} + \frac{V}{\tau} = \frac{V_2}{C_0 R_2} \quad (4.14)$$

2. Find $H(s)$ considering the output voltage at the capacitor.

Solution: $H(s) = \frac{V(s)}{V_2(s)}$,

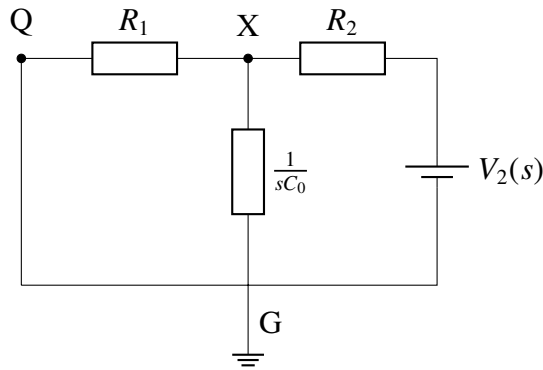


Fig. 4.2: Transforming Fig. 4.1 to s -domain

$$\frac{V}{R_1} + \frac{V}{\frac{1}{sC_0}} + \frac{V - V_2}{R_2} = 0 \quad (4.15)$$

$$H(s) \left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0 \right) = \frac{1}{R_2} \quad (4.16)$$

$$H(s) = \frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} \quad (4.17)$$

3. Plot $H(s)$. What kind of filter is it?

Solution: $H(s)$ is a low-pass filter.

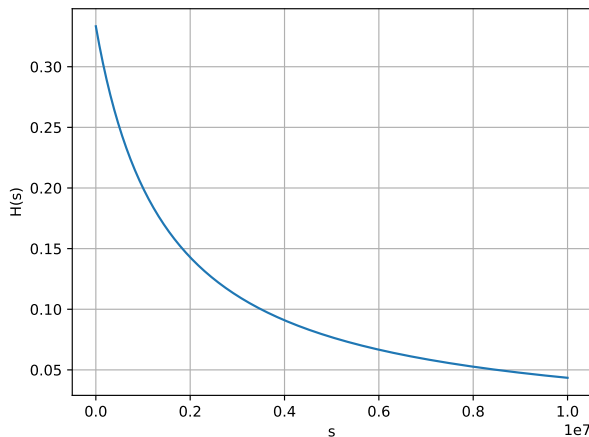


Fig. 4.3: Plot of $H(s)$.

\$ wget https://github.com/kurugodukarthik11/EE3900/blob/main/Assignments/Assignment_1/codes/7_11.py

4. Using trapezoidal rule for integration, formulate the difference equation by considering

$$y(n) = y(t)|_{t=n} \quad (4.18)$$

Solution: Integrating (4.14) between limits n

to $n + 1$ and applying the trapezoidal formula,

$$v(n+1) - v(n) + \frac{v(n) + v(n+1)}{2\tau} = \frac{V_2(u(n) + u(n+1))}{C_0 R_2} \quad (4.19)$$

$$v(n)(2\tau + 1) + v(n-1)(2\tau - 1) = \frac{V_2\tau(u(n) + u(n-1))}{C_0 R_2} \quad (4.20)$$

for $n > 0$, where $v(0) = 0$.

5. Find $H(z)$.

Solution: Note that for the input voltage, $v_i(n) = 2u(n)$ and so, $V_i(z) = \frac{2}{1-z^{-1}}$. Applying the Z-transform on both sides of (4.20),

$$\begin{aligned} V(z) \left[(2\tau + 1) - z^{-1}(2\tau - 1) \right] \\ = \frac{\tau(1 + z^{-1}) V_i(z)}{C_0 R_2} \end{aligned} \quad (4.21)$$

Hence,

$$H(z) = \frac{\tau(1 + z^{-1})}{C_0 R_2 ((2\tau + 1) - (2\tau - 1)z^{-1})} \quad (4.22)$$

since $\left| \frac{2\tau-1}{2\tau+1} \right| < 1$, the ROC is $|z| > 1$.

6. How can you obtain $H(z)$ from $H(s)$?

Solution: We use the bilinear transformation. Setting

$$s := \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}} \quad (4.23)$$

we get

$$H(z) = \frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{2C_0}{T} \frac{1-z^{-1}}{1+z^{-1}}} \quad (4.24)$$

$$= \frac{T\tau(1 + z^{-1})}{C_0 R_2 ((2\tau + T) - (2\tau - T)z^{-1})} \quad (4.25)$$

Setting $T = 1$ gives (4.22).