Circuits and Transforms

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***Abstract*—This manual provides a simple introduction to Transforms**

1 Definitions

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \tag{1.1}$$

2. The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st}\, dt \tag{1.2}$$

2 Laplace Transform

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes $q_1\,\mu C$. Then S is switched to position Q. After a long time, the charge on the capacitor is $q_2\,\mu C$.

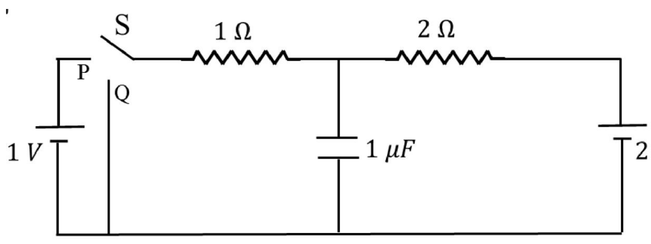

Fig. 2.1

2. Find q_1 and Draw the circuit using latex-tikz.
Solution: Let circuit be grounded at G then let relative potential at point X to be V.

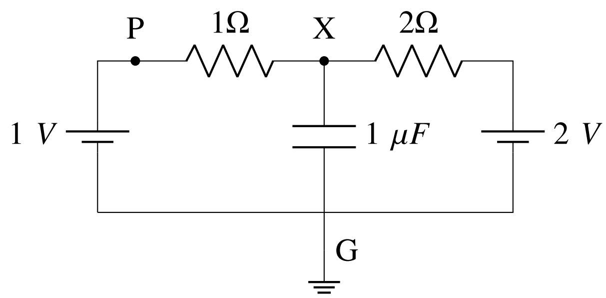

Fig. 2.2

From KCL,

$$\frac{V-1}{1} + \frac{V-2}{2} = 0$$
$$V = \frac{4}{3}$$

3. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC.
Solution: We have,

$$L[u(t)] = \int_{0}^{\infty} u(t)e^{-st}dt \tag{2.1}$$
$$= \int_{0}^{\infty} e^{-st}dt \tag{2.2}$$
$$= \frac{1}{s}, \quad \Re(s) > 0 \tag{2.3}$$

4. Show that

$$e^{-at}u(t) \overset{\mathcal{H}}{\longleftrightarrow} L\frac{1}{s+a}, \quad a > 0 \tag{2.4}$$

and find the ROC.
Solution:

$$L[e^{-at}u(t)] = \int_{0}^{\infty} u(t)e^{-(s+a)t}dt \tag{2.5}$$
$$= \int_{0}^{\infty} e^{-(s+a)t}dt \tag{2.6}$$
$$= \frac{1}{s+a}, \quad \Re(s) > -a \tag{2.7}$$

5. Now consider the following resistive circuit transformed from Fig. 2.1 where

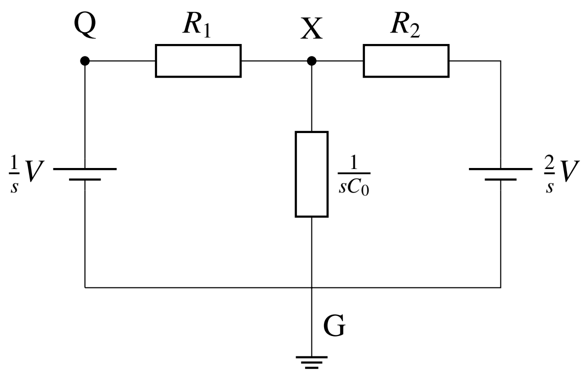

Fig. 2.3

$$L[u(t)] = V_1(s) \tag{2.8}$$
$$L[2u(t)] = V_2(s) \tag{2.9}$$

Find the voltage across the capacitor $V_{C_0}(s)$.
Solution:

$$V_1(s) = \frac{1}{s} \tag{2.10}$$
$$V_2(s) = \frac{2}{s} \tag{2.11}$$

Let circuit be grounded at G then let relative potential at point X to be V.
From KCL at X,

$$\frac{V - \frac{1}{s}}{R_1} + \frac{V - \frac{2}{s}}{R_2} + sC_0V = 0 \tag{2.12}$$
$$V\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right) = \frac{1}{s}\left(\frac{1}{R_1} + \frac{2}{R_2}\right) \tag{2.13}$$
$$V(s) = \frac{\frac{1}{R_1} + \frac{2}{R_2}}{s\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right)} \tag{2.14}$$
$$= \frac{\frac{1}{R_1} + \frac{2}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2}}\left(\frac{1}{s} - \frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + s}\right) \tag{2.15}$$

6. Find $v_{C_0}(t)$. Plot using python.
Solution:

$$L^{-1}[V(s)] = L^{-1}[\frac{\frac{1}{R_1} + \frac{2}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2}}\left(\frac{1}{s} - \frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + s}\right)] \tag{2.16}$$
$$= \frac{2R_1 + R_2}{R_1 + R_2}u(t)\left(1 - e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{t}{C_0}}\right) \tag{2.17}$$
$$= \frac{4}{3}\left(1 - e^{-(1.5\times10^6)t}\right)u(t) \tag{2.18}$$

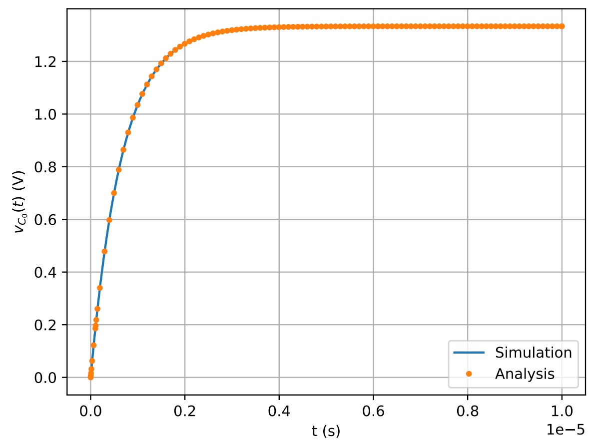

Fig. 2.4: $v_{C_0}(t)$ before the switch is flipped

```
$ wget https://github.com/kurugodukarthik11/
    EE3900/blob/main/Assignments/
    Assignment_5/codes/2.6.py
```

7. Verify your result using ngspice. **Solution:**

```
$ wget https://github.com/kurugodukarthik11/
    EE3900/blob/main/Assignments/
    Assignment_5/codes/2_7.cir
```

3 Initial Conditions

1. Find q_2 in Fig. 2.1.
Solution:
The equivalent circuit at steady state when the switch is at Q is shown below. Since capacitor

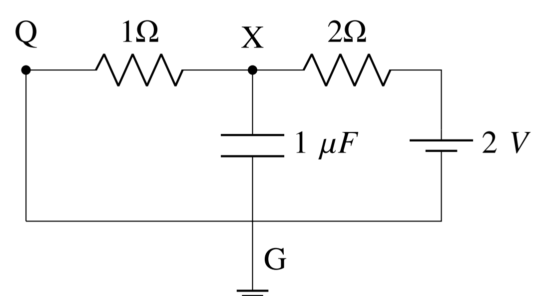

Fig. 3.1

behaves as an open circuit, we use KCL at X.

$$\frac{V-0}{1} + \frac{V-2}{2} = 0 \implies V = \frac{2}{3} \tag{3.1}$$

and hence, $q_2 = \frac{2}{3}\mu C$.

2. Draw the equivalent s-domain resistive circuit when S is switched to position Q. Use variables R_1, R_2, C_0 for the passive elements. Use latex-tikz.

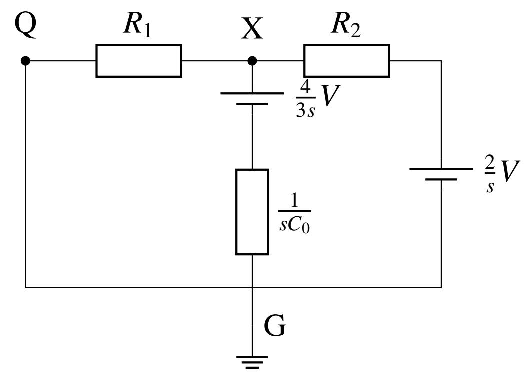

Fig. 3.2

3. $V_{C_0}(s) = ?$
Solution: Using KCL at node X in Fig. 3.2

$$\frac{V-0}{R_1} + \frac{V-\frac{2}{s}}{R_2} + sC_0\left(V - \frac{4}{3s}\right) = 0 \tag{3.2}$$

$$\implies V_{C_0}(s) = \frac{\frac{2}{sR_2} + \frac{4C_0}{3}}{\frac{1}{R_1} + \frac{2}{R_2} + sC_0} \tag{3.3}$$

4. $v_{C_0}(t) = ?$ Plot using python.
Solution:
From (3.3),

$$V_{C_0}(s) = \frac{4}{3}\left(\frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+s}\right) + \frac{2}{R_2\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}\left(\frac{1}{s} - \frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+s}\right) \tag{3.4}$$

Applying Inverse Laplace Transform on both sides

$$L^{-1}[V_{C_0}(t)] = v_{C_0}(t) = \frac{4}{3}e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}u(t) + \frac{2}{R_2\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}\left(1 - e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}\right)u(t) \tag{3.5}$$

Substituting R_1, R_2, C_0 in (3.5)

$$L^{-1}[V_{C_0}(t)] = v_{C_0}(t) = \frac{2}{3}\left(1 + e^{-(1.5\times10^6)t}\right)u(t) \tag{3.6}$$

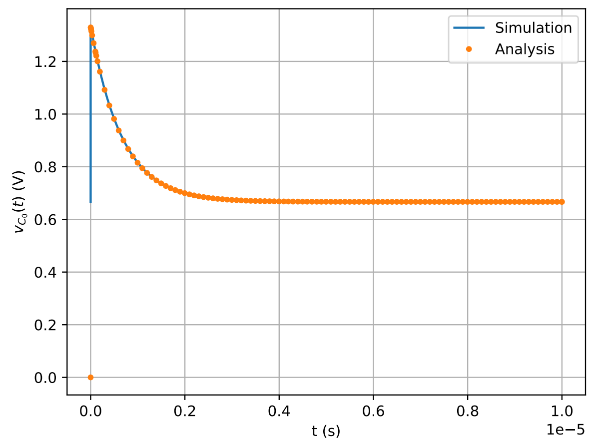

Fig. 3.3: $v_{C_0}(t)$ after the switch is flipped

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$ wget https://github.com/kurugodukarthik11/
    EE3900/blob/main/Assignments/
    Assignment_5/codes/3.4.py
```

5. Verify your result using ngspice.
Solution:

```
$ wget https://github.com/kurugodukarthik11/
    EE3900/blob/main/Assignments/
    Assignment_5/codes/3_5.cir
```

6. Find $v_{C_0}(0-), v_{C_0}(0+)$ and $v_{C_0}(\infty)$.
Solution: From the initial conditions,

$$v_{C_0}(0-) = \frac{q_1}{C} = \frac{4}{3} \tag{3.7}$$

From (3.6),

$$v_{C_0}(0+) = \lim_{t\to0+} v_{C_0}(t) = \frac{4}{3} \tag{3.8}$$

$$v_{C_0}(\infty) = \lim_{t\to\infty} v_{C_0}(t) = \frac{2}{3} \tag{3.9}$$

7. Obtain the Fig. in problem 3.2 using the equivalent differential equation.
Solution: The equivalent circuit in the t-domain is shown below. From KCL and KVL,

$$i_1 = i_2 + i_3 \tag{3.10}$$

$$i_1R_1 + \frac{4}{3} + \frac{1}{C_0}\int_0^t i_2dt = 0 \tag{3.11}$$

$$\frac{4}{3} + \frac{1}{C_0}\int_0^t i_2dt - i_3R_2 - 2 = 0 \tag{3.12}$$

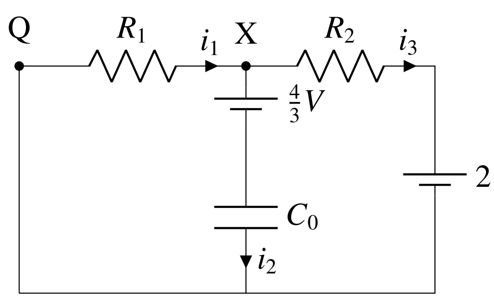

Fig. 3.4

Taking Laplace Transforms on both sides and using the properties of Laplace Transforms,

$$I_1 = I_2 + I_3 \tag{3.13}$$

$$I_1R_1 + \frac{4}{3} + \frac{1}{sC_0}I_2 = 0 \tag{3.14}$$

$$\frac{4}{3} + \frac{1}{sC_0}I_2 - I_3R_2 - 2 = 0 \tag{3.15}$$

$$\tag{3.16}$$

where $i(t) \overset{\mathcal{H}}{\longleftrightarrow} LI(s)$. Note that the capacitor is equivalent to a resistive element of resistance $R_C = \frac{1}{sC_0}$ in the s-domain. Equations (3.13) - (3.15) precisely describe Fig. 3.2.

4 Bilinear Transform

1. In Fig. 2.1,consider the case when S is switched to Q right in the beginning. Formulate the differential equation.
   Solution:
   The equivalent circuit in the t-domain is shown below.

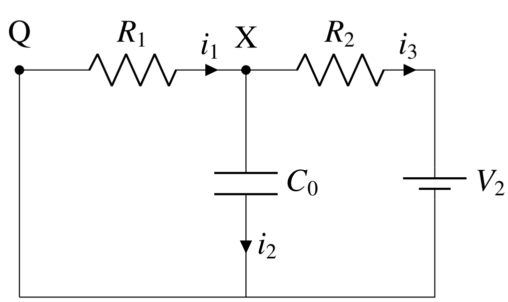

Fig. 4.1

Applying KCL and KVL,

$$i_1 = i_2 + i_3 \tag{4.1}$$

$$i_1R_1 = -\frac{1}{C_0}\int_0^t i_2\,dt \tag{4.2}$$

$$i_3R_2 = \frac{1}{C_0}\int_0^t i_2\,dt - 2 \tag{4.3}$$

Differentiating (4.1), (4.2), (4.3),

$$\frac{di_1}{dt} = \frac{di_2}{dt} + \frac{di_3}{dt} \tag{4.4}$$

$$R_1\frac{di_1}{dt} = -\frac{i_2}{C_0} \tag{4.5}$$

$$R_2\frac{di_3}{dt} = \frac{i_2}{C_0} \tag{4.6}$$

From (4.4),(4.6), (4.5),

$$\frac{di_1}{dt} = \left(\frac{-R_2}{R_1}\right)\frac{di_3}{dt} \tag{4.7}$$

$$\therefore \left(\frac{-R_2}{R_1}\right)\frac{di_3}{dt} = \frac{di_2}{dt} + \frac{di_3}{dt} \tag{4.8}$$

$$\frac{di_2}{dt} + \left(1 + \frac{R_2}{R_1}\right)\frac{di_3}{dt} = 0 \tag{4.9}$$

$$\frac{di_2}{dt} + \left(1 + \frac{R_2}{R_1}\right)\frac{i_2}{R_2C_0} = 0 \tag{4.10}$$

$$\frac{di_2}{dt} + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{i_2}{C_0} = 0 \tag{4.11}$$

$$\frac{di_2}{dt} + \frac{i_2}{\tau} = 0 \tag{4.12}$$

where $\tau = \frac{C_0R_1R_2}{R_1+R_2}$ Integrating (4.12) w.r.t t and at $t = 0$, $i_2 = \frac{V_2}{R_2}$

$$C_0\frac{dV}{dt} - \frac{V_2}{R_2} + \frac{C_0V}{\tau} = 0 \tag{4.13}$$

$$\frac{dV}{dt} + \frac{V}{\tau} = \frac{V_2}{C_0R_2} \tag{4.14}$$

2. Find $H(s)$ considering the ouput voltage at the capacitor.
   Solution: $H(s) = \frac{V(s)}{V_2(s)}$,

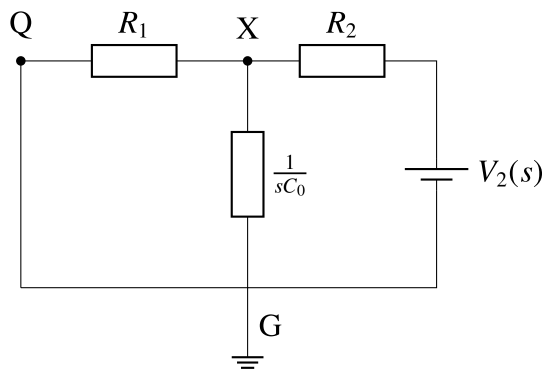

Fig. 4.2: Transforming Fig. 4.1 to s-domain

$$\frac{V}{R_1} + \frac{V}{\frac{1}{sC_0}} + \frac{V - V_2}{R_2} = 0 \tag{4.15}$$

$$H(s)\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right) = \frac{1}{R_2} \tag{4.16}$$

$$H(s) = \frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} \tag{4.17}$$

3. Plot $H(s)$. What kind of filter is it?
   Solution: $H(s)$ is a low-pass filter.

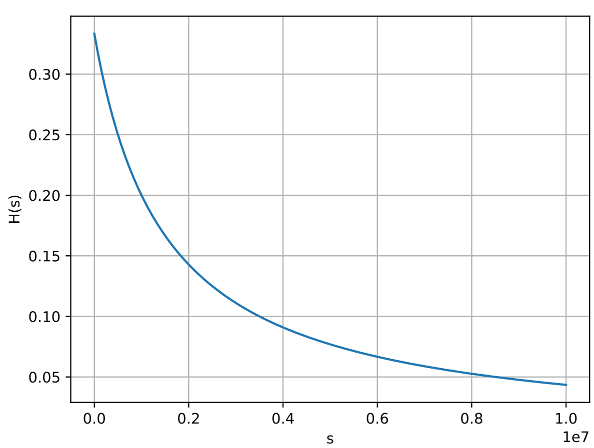

Fig. 4.3: Plot of $H(s)$.

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$ wget https://github.com/kurugodukarthik11/
    EE3900/blob/main/Assignments/
    Assignment_5/codes/4.3.py
```

4. Using trapezoidal rule for integration, formulate the difference equation by considering

$$y(n) = y(t)|_{t=n} \tag{4.18}$$

   Solution: Integrating (4.14) between limits n to $n+1$ and applying the trapezoidal formula,

$$v(n+1) - v(n) + \frac{v(n) + v(n+1)}{2\tau} = \frac{V_2\left(u(n) + u(n+1)\right)}{C_0 R_2} \tag{4.19}$$

$$v(n)(2\tau + 1) + v(n-1)(2\tau - 1) = \frac{V_2 \tau\left(u(n) + u(n-1)\right)}{C_0 R_2} \tag{4.20}$$

   for $n > 0$, where $v(0) = 0$.

5. Find $H(z)$.
   Solution: Note that for the input voltage, $v_i(n) = 2u(n)$ and so, $V_i(z) = \frac{2}{1-z^{-1}}$. Applying the Z-transform on both sides of (4.20),

$$V(z)\left[(2\tau+1) - z^{-1}(2\tau - 1)\right] = \frac{\tau\left(1 + z^{-1}\right)V_i(z)}{C_0 R_2} \tag{4.21}$$

   Hence,

$$H(z) = \frac{\tau\left(1 + z^{-1}\right)}{C_0 R_2\left((2\tau + 1) - (2\tau - 1)z^{-1}\right)} \tag{4.22}$$

   since $\left|\frac{2\tau-1}{2\tau+1}\right| < 1$, the ROC is $|z| > 1$.

6. How can you obtain $H(z)$ from $H(s)$?
   Solution: We use the bilinear transformation. Setting

$$s := \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}} \tag{4.23}$$

   we get

$$H(z) = \frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{2C_0}{T}\frac{1-z^{-1}}{1+z^{-1}}} \tag{4.24}$$

$$= \frac{T\tau\left(1 + z^{-1}\right)}{C_0 R_2\left((2\tau + T) - (2\tau - T)z^{-1}\right)} \tag{4.25}$$

   Setting $T = 1$ gives (4.22).