Digital Signal Processing

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Contents

***Abstract*—This manual provides a simple introduction to digital signal processing.**

1 Software Installation

Run the following commands

```
sudo apt−get update
sudo apt−get install libffi−dev libsndfile1 python3
    −scipy python3−numpy python3−matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

```
wget https://raw.githubusercontent.com/
    gadepall/
EE1310/master/filter/codes/Sound_Noise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution:

```python
import soundfile as sf
from scipy import signal

#read .wav file
input_signal,fs = sf.read('Sound_Noise.wav'
    )

#sampling frequency of Input signal
sampl_freq=fs

#order of the filter
order=4

#cutoff frquency 4kHz
cutoff_freq=4000.0

#digital frequency
Wn=2*cutoff_freq/sampl_freq

# b and a are numerator and denominator
    polynomials respectively
b, a = signal.butter(order,Wn, 'low')

#filter the input signal with butterworth filter
output_signal = signal.filtfilt(b, a,
    input_signal)
#output_signal = signal.lfilter(b, a,
    input_signal)

#write the output signal into .wav file
sf.write('Sound_With_ReducedNoise.wav',
    output_signal, fs)
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectrogram in Problem 2.2. What do you observe?

Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

3 Difference Equation

3.1 Let

$$x(n) = \left\{\underset{\uparrow}{1}, 2, 3, 4, 2, 1\right\} \tag{3.1}$$

Sketch $x(n)$.

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$
$$y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$.

Solution: The following code yields Fig. 3.3.

```
wget https://github.com/gadepall/EE1310/raw/
    master/filter/codes/xnyn.py
```

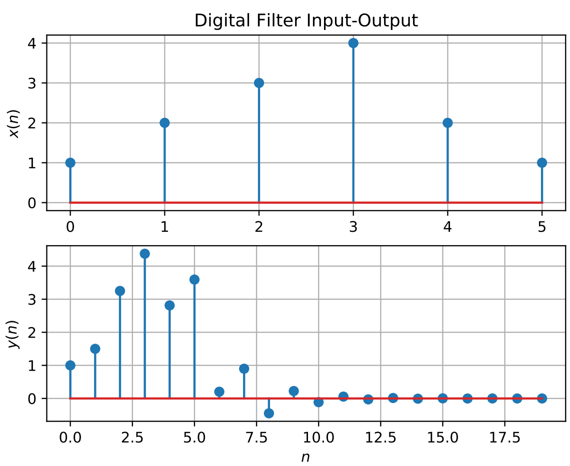

Fig. 3.2

3.3 Repeat the above exercise using a C code.

Solution: The following code yields Fig. 3.3.

```
wget https://github.com/kurugodukarthik11/
    EE3900/blob/main/Assignments/
    Assignment_1/codes/xny.cpp
```

4 Z-transform

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

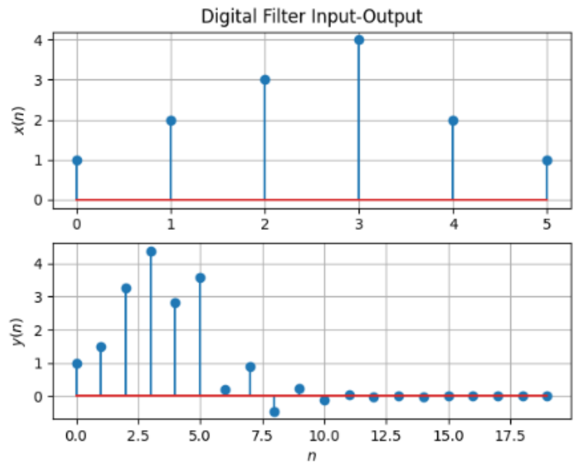

Fig. 3.3

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution: From (4.1),

$$\mathcal{Z}\{x(n-1)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \tag{4.4}$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.5}$$

Similarly,

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-k)z^{-n} \tag{4.6}$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-k} \tag{4.7}$$

$$= z^{-k}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.8}$$

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \tag{4.9}$$

4.2 Obtain $X(z)$ for $x(n)$ defined in problem 3.1.

Solution: We know from (4.1)

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.10}$$

Substitute values of $x(n)$ from (3.1) in (4.1)

$$X(z) = \mathcal{Z}\{x(n)\} \tag{4.11}$$
$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.12}$$
$$X(z) = \sum_{n=0}^{5} x(n)z^{-n} \tag{4.13}$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.14}$$

from (3.2) assuming that the Z-transform is a linear operation.
Solution: Applying (4.9) in (3.2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{4.15}$$
$$\implies \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{4.16}$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.17}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.18}$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.19}$$

Solution:

$$\mathcal{Z}\{\delta(n)\} = \sum_{n=-\infty}^{\infty} \delta(n)z^{-n} \tag{4.20}$$
$$= \delta(0)z^{0} \tag{4.21}$$
$$= 1 \tag{4.22}$$

Thus

$$\delta(n) \overset{\mathcal{Z}}{\rightleftharpoons} 1 \tag{4.23}$$

and from (4.18),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{4.24}$$
$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.25}$$

using the fomula for the sum of an infinite geometric progression.

4.5 Show that

$$a^{n}u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.26}$$

Solution: Infinite GP:

$$\sum_{n=0}^{\infty} r^{n} = \frac{1}{1 - r}, \quad |r| < 1 \tag{4.27}$$

$$U(z) = \sum_{n=0}^{\infty} a^{n}z^{-n} \tag{4.28}$$
$$= \frac{1}{1 - az^{-1}}, \quad |z| > |a| \tag{4.29}$$

4.6 Let

$$H\left(e^{\jmath\omega}\right) = H\left(z = e^{\jmath\omega}\right). \tag{4.30}$$

Plot $|H\left(e^{\jmath\omega}\right)|$. Is it periodic? If so, find the period. $H(e^{\jmath\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $x(n)$.
Solution:

$$H\left(e^{\jmath\omega}\right) = \frac{1 + e^{-2\jmath\omega}}{1 + \frac{1}{2}e^{-\jmath\omega}} \tag{4.31}$$
$$|H\left(e^{\jmath\omega}\right)| = \frac{\left|1 + \cos 2\omega - \jmath \sin 2\omega\right|}{\left|1 + \frac{1}{2}\cos\omega - \frac{1}{2}\sin\omega\right|} \tag{4.32}$$
$$= \sqrt{\frac{(1 + \cos 2\omega)^2 + (\sin 2\omega)^2}{(1 + \frac{1}{2}\cos\omega)^2 + (\frac{1}{2}\sin\omega)^2}} \tag{4.33}$$
$$= \sqrt{\frac{2 + 2\cos 2\omega}{\frac{5}{4} + \cos\omega}} \tag{4.34}$$
$$= \sqrt{\frac{2(2\cos^2\omega)4}{5 + 4\cos\omega}} \tag{4.35}$$
$$= \frac{4\left|\cos\omega\right|}{\sqrt{5 + 4\cos\omega}} \tag{4.36}$$

Therefore the period of above equation is 2π since lcm of period of numerator(π) and period of denominator(2π) is 2π.

The following code plots Fig4.6.

```
wget https://github.com/kurugodukarthik11/
    EE3900/blob/main/Assignments/
    Assignment_1/codes/dtft.py
```

4.7 Express $h(n)$ in terms of $H\left(e^{\jmath\omega}\right)$. **Solution:** We

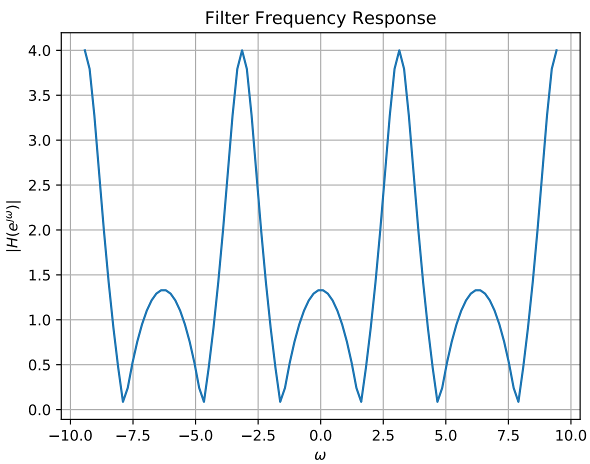

Fig. 4.6: $|H(e^{\jmath\omega})|$

have,

$$H(e^{\jmath\omega}) = \sum_{k=-\infty}^{\infty} h(k)e^{-\jmath\omega k} \tag{4.37}$$

However,

$$\int_{-\pi}^{\pi} e^{\jmath\omega(n-k)} d\omega = \begin{cases} 2\pi & n = k \\ 0 & \text{otherwise} \end{cases} \tag{4.38}$$

and so,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{\jmath\omega n} d\omega \tag{4.39}$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_{-\pi}^{\pi} h(k)e^{\jmath\omega(n-k)} d\omega \tag{4.40}$$

$$= \frac{1}{2\pi}2\pi h(n) = h(n) \tag{4.41}$$

which is known as the Inverse Discrete Fourier Transform. Thus,

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{\jmath\omega n} d\omega \tag{4.42}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 + e^{-2\jmath\omega}}{1 + \frac{1}{2}e^{-\jmath\omega}} e^{\jmath\omega n} d\omega \tag{4.43}$$

5 Impulse Response

5.1 Using long division, find

$$h(n), \quad n < 5 \tag{5.1}$$

for H(z) in (4.16).

Solution: For long division, substitute

$x := z^{-1}$

$$\begin{array}{r|l} & \quad 2x - 4 \\ \hline \frac{1}{2}x + 1) & \; x^2 \qquad + 1 \\ & - x^2 - 2x \\ \hline & \qquad - 2x + 1 \\ & \qquad\;\; 2x + 4 \\ \hline & \qquad\qquad 5 \end{array}$$

Therefore,

$$H(z) = -4 + 2z^{-1} + \frac{5}{1 + \frac{1}{2}z^{-1}} \tag{5.2}$$

$$= -4 + 2z^{-1} + 5\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n z^{-n} \tag{5.3}$$

$$= 1 - \frac{1}{2}z^{-1} + 5\sum_{n=2}^{\infty}\left(-\frac{1}{2}\right)^n z^{-n} \tag{5.4}$$

$$= \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n z^{-n} + 4\sum_{n=2}^{\infty}\left(-\frac{1}{2}\right)^n z^{-n} \tag{5.5}$$

$$= \sum_{n=-\infty}^{\infty} u(n)\left(-\frac{1}{2}\right)^n z^{-n} + \sum_{n=-\infty}^{\infty} u(n-2)\left(-\frac{1}{2}\right)^{n-2} z^{-n} \tag{5.6}$$

Now, from (4.1), we get

$$h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.7}$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.8}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).

Solution: From (4.16),

$$H(z) = \frac{1}{1 + \frac{1}{2}z^{-1}} + \frac{z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{5.9}$$

$$\implies h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.10}$$

ROC: (-∞,-1/2) ∪ (-1/2,∞)
using (4.26) and (4.9).

5.3 Sketch $h(n)$. Is it bounded? Justify theoretically.

Solution:
We know

$$a)\left|\left(\frac{-1}{2}\right)^n\right| \leq 1 \tag{5.11}$$

$$b)\, |u(n)| \le 1 \tag{5.12}$$

$$c)\, \left|\left(\frac{-1}{2}\right)^{n-2}\right| \le 1 \tag{5.13}$$

$$d)\, |u(n-2)| \le 1 \tag{5.14}$$

Therefore

$$\left|\left(\frac{-1}{2}\right)^{n} u(n)\right| \le 1 \tag{5.15}$$

$$\left|\left(\frac{-1}{2}\right)^{n-2} u(n-2)\right| \le 1 \tag{5.16}$$

Therefore

$$\left|\left(\frac{-1}{2}\right)^{n} u(n) + \left(\frac{-1}{2}\right)^{n-2} u(n-2)\right| \le 2 \tag{5.17}$$

Hence Bounded
The following code plots Fig. 5.3.

```
wget https://github.com/kurugodukarthik11/
    EE3900/blob/main/Assignments/
    Assignment_1/codes/hn.py
```

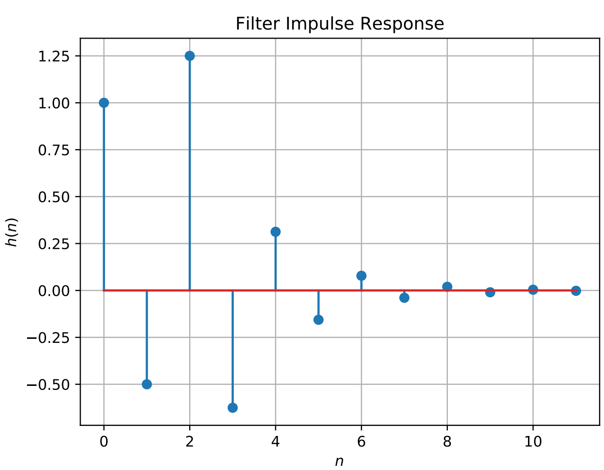

Fig. 5.3: $h(n)$ as the inverse of $H(z)$

5.4 Convergent? Justify using the ratio test.
Solution: Using the ratio test for convergence

$$\lim_{n\to\infty} \left|\frac{h(n+1)}{h(n)}\right| \tag{5.18}$$

$$= \lim_{n\to\infty} \left|\frac{\left(-\frac{1}{2}\right)^{n+1} u(n+1) + \left(-\frac{1}{2}\right)^{n-1} u(n-1)}{\left(-\frac{1}{2}\right)^{n} u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2)}\right| \tag{5.19}$$

$$\le \lim_{n\to\infty} \left|\frac{\left(-\frac{1}{2}\right)^{n-1} \left(\frac{1}{4}+1\right)}{\left(-\frac{1}{2}\right)^{n-2} \left(\frac{1}{4}+1\right)}\right| \tag{5.20}$$

$$= \lim_{n\to\infty} \left|-\frac{1}{2}\right| \tag{5.21}$$

$$= \frac{1}{2} < 1 \tag{5.22}$$

Hence by ratio test $h(n)$ is convergent

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.23}$$

Is the system defined by (3.2) stable for the impulse response in (5.8)?
Solution:

$$\tag{5.24}$$

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=-\infty}^{\infty} \left(-\frac{1}{2}\right)^{n} u(n) + \sum_{n=-\infty}^{\infty} \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.25}$$

$$= \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^{n} + \sum_{n=2}^{\infty} \left(-\frac{1}{2}\right)^{n-2} \tag{5.26}$$

From Infinite GP

$$= \frac{2}{3} + \frac{2}{3} \tag{5.27}$$

$$= \frac{4}{3} < \infty \tag{5.28}$$

Thus System is stable from (5.23)

5.6 Verify the above result using a python code.
Solution:

```
https://github.com/kurugodukarthik11/EE3900
    /blob/main/Assignments/Assignment_1/
    codes/5_6.py
```

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \tag{5.29}$$

This is the definition of $h(n)$.
Solution: The following code plots Fig. 5.7. Note that this is the same as Fig. 5.3.

```
wget https://github.com/kurugodukarthik11/
    EE3900/blob/main/Assignments/
```

Assignment_1/codes/hndef.py

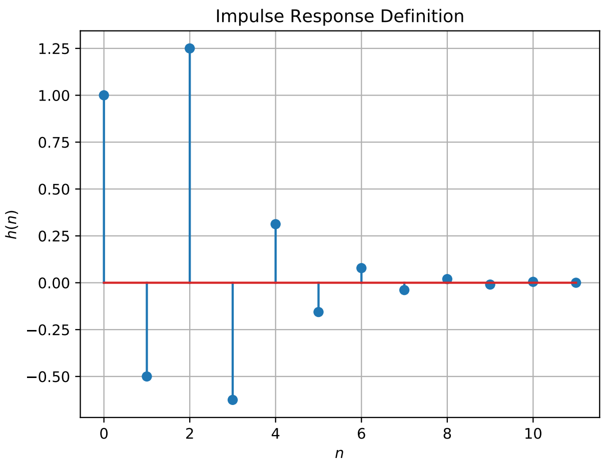

Fig. 5.7: $h(n)$ from the definition

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{n=-\infty}^{\infty} x(k)h(n-k) \quad (5.30)$$

Comment. The operation in (5.30) is known as *convolution*.
Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 3.3.

```
wget https://raw.githubusercontent.com/
    gadepall/EE1310/master/filter/codes/
    ynconv.py
```

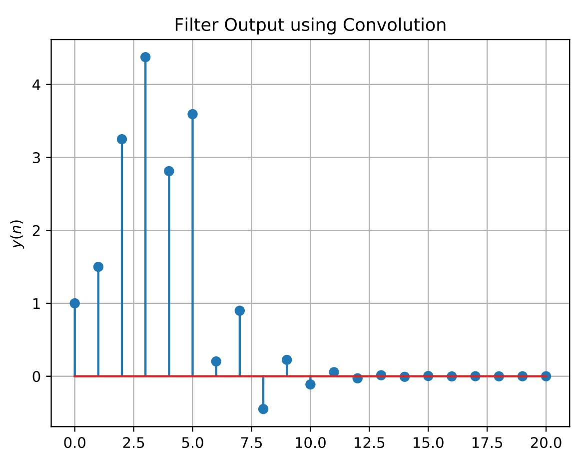

Fig. 5.8: $y(n)$ from the definition of convolution

5.9 Express the above convolution using a Teoplitz matrix.

Solution:
The Toeplitz matrices for convolution are,

$$\mathbf{y} = \mathbf{x} \circledast \mathbf{h} \quad (5.31)$$

$$\mathbf{y} = \begin{pmatrix} h_1 & 0 & . & . & . & 0 \\ h_2 & h_1 & . & . & . & 0 \\ h_3 & h_2 & h_1 & . & . & 0 \\ . & . & . & . & . & . \\ 1 & . & . & h_3 & h_2 & h_1 \\ 2 & . & . & . & h_2 & h_1 \\ 3 & . & . & . & 0 & h_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad (5.32)$$

5.10 Show that

$$y(n) = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \quad (5.33)$$

Solution: Solution: From eqn (5.30)

$$y(n) = x(n) * h(n) = \sum_{n=-\infty}^{\infty} x(k)h(n-k) \quad (5.34)$$

Replace 'k' with 'n-k'

$$y(n) = x(n) * h(n) \quad (5.35)$$

$$y(n) = \sum_{n=-\infty}^{\infty} x(n-k)h(n-(n-k)) \quad (5.36)$$

$$y(n) = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \quad (5.37)$$

Hence Proved

6 DFT and FFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-\jmath 2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \quad (6.1)$$

and $H(k)$ using $h(n)$.
Solution:

```
wget https://github.com/kurugodukarthik11/
    EE3900/blob/main/Assignments/
    Assignment_1/codes/6_1.py
```

6.2 Compute

$$Y(k) = X(k)H(k) \quad (6.2)$$

Solution:

```
wget https://github.com/kurugodukarthik11/
    EE3900/blob/main/Assignments/
    Assignment_1/codes/6_2.py
```

6.3 Compute

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k)\cdot e^{j2\pi kn/N}, \quad n = 0, 1, \ldots, N-1 \tag{6.3}$$

Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 3.3.

```
wget https://raw.githubusercontent.com/
    gadepall/EE1310/master/filter/codes/yndft.
    py
```

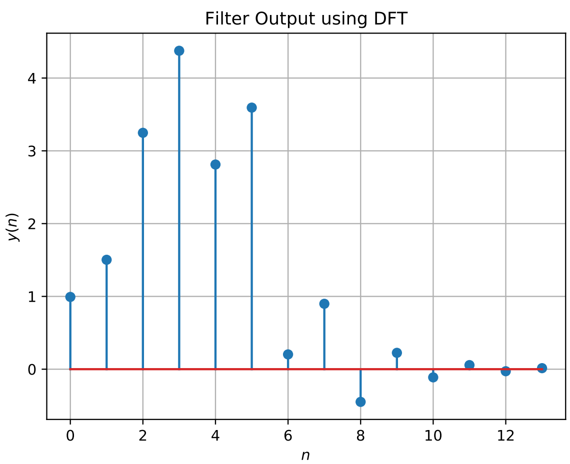

Fig. 6.3: $y(n)$ from the DFT

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.
Solution:

```
wget https://github.com/kurugodukarthik11/
    EE3900/blob/main/Assignments/
    Assignment_1/codes/6_4.py
```

7 FFT

1. The DFT of $x(n)$ is given by

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{7.1}$$

2. Let

$$W_N = e^{-j2\pi/N} \tag{7.2}$$

Then the N-point *DFT matrix* is defined as

$$\mathbf{F}_N = [W_N^{mn}], \quad 0 \le m, n \le N-1 \tag{7.3}$$

where W_N^{mn} are the elements of $\mathbf{F}_N$.

3. Let

$$\mathbf{I}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^2 & \mathbf{e}_4^3 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.4}$$

be the 4×4 identity matrix. Then the 4 point *DFT permutation matrix* is defined as

$$\mathbf{P}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^3 & \mathbf{e}_4^2 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.5}$$

4. The 4 point *DFT diagonal matrix* is defined as

$$\mathbf{D}_4 = diag\begin{pmatrix} W_8^0 & W_8^1 & W_8^2 & W_8^3 \end{pmatrix} \tag{7.6}$$

5. Show that

$$W_N^2 = W_{N/2} \tag{7.7}$$

Solution: We write

$$W_N^2 = \left(e^{-\frac{j2\pi}{N}}\right)^2 = e^{-\frac{j2\pi}{N/2}} = W_{N/2} \tag{7.8}$$

6. Show that

$$\mathbf{F}_4 = \begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \mathbf{P}_4 \tag{7.9}$$

Solution: Observe that for $n \in \mathbb{N}$, $W_4^{4n} = 1$ and $W_4^{4n+2} = -1$. Using (7.7),

$$\mathbf{D}_2\mathbf{F}_2 = \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{bmatrix} \tag{7.10}$$

$$= \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} W_4^0 & W_4^0 \\ W_4^0 & W_4^2 \end{bmatrix} \tag{7.11}$$

$$= \begin{bmatrix} W_4^0 & W_4^0 \\ W_4^1 & W_4^3 \end{bmatrix} \tag{7.12}$$

$$\implies -\mathbf{D}_2\mathbf{F}_2 = \begin{bmatrix} W_4^2 & W_4^6 \\ W_4^3 & W_4^9 \end{bmatrix} \tag{7.13}$$

and

$$\mathbf{F}_2 = \begin{pmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{pmatrix} \tag{7.14}$$

$$= \begin{pmatrix} W_4^0 & W_4^0 \\ W_4^0 & W_4^2 \end{pmatrix} \tag{7.15}$$

Hence,

$$\mathbf{W}_4 = \begin{pmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^2 & W_4^1 & W_4^3 \\ W_4^0 & W_4^4 & W_4^2 & W_4^6 \\ W_4^0 & W_4^6 & W_4^3 & W_4^9 \end{pmatrix} \tag{7.16}$$

$$= \begin{bmatrix} \mathbf{I}_2\mathbf{F}_2 & \mathbf{D}_2 F_2 \\ \mathbf{I}_2\mathbf{F}_2 & -\mathbf{D}_2 F_2 \end{bmatrix} \tag{7.17}$$

$$= \begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & \mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \tag{7.18}$$

Multiplying (7.18) by $\mathbf{P}_4$ on both sides, and noting that $\mathbf{W}_4\mathbf{P}_4 = \mathbf{F}_4$ gives us (7.9).

7. Show that

$$\mathbf{F}_N = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \mathbf{P}_N \tag{7.19}$$

Solution: Observe that for even N and letting $\mathbf{f}_N^i$ denote the i^{th} column of $\mathbf{F}_N$, from (7.12) and (7.13),

$$\begin{pmatrix} \mathbf{D}_{N/2}\mathbf{F}_{N/2} \\ -\mathbf{D}_{N/2}\mathbf{F}_{N/2} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_N^2 & \mathbf{f}_N^4 & \dots & \mathbf{f}_N^N \end{pmatrix} \tag{7.20}$$

and

$$\begin{pmatrix} \mathbf{I}_{N/2}\mathbf{F}_{N/2} \\ \mathbf{I}_{N/2}\mathbf{F}_{N/2} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_N^1 & \mathbf{f}_N^3 & \dots & \mathbf{f}_N^{N-1} \end{pmatrix} \tag{7.21}$$

Thus,

$$\begin{bmatrix} \mathbf{I}_2\mathbf{F}_2 & \mathbf{D}_2\mathbf{F}_2 \\ \mathbf{I}_2\mathbf{F}_2 & -\mathbf{D}_2\mathbf{F}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix}$$
$$= \begin{pmatrix} \mathbf{f}_N^1 & \dots & \mathbf{f}_N^{N-1} & \mathbf{f}_N^2 & \dots & \mathbf{f}_N^N \end{pmatrix} \tag{7.22}$$

and so,

$$\begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \mathbf{P}_N$$
$$= \begin{pmatrix} \mathbf{f}_N^1 & \mathbf{f}_N^2 & \dots & \mathbf{f}_N^N \end{pmatrix} = \mathbf{F}_N \tag{7.23}$$

8. Find

$$\mathbf{P}_4\mathbf{x} \tag{7.24}$$

Solution: We have,

$$\mathbf{P}_4\mathbf{x} = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^3 & \mathbf{e}_4^2 & \mathbf{e}_4^4 \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{pmatrix} = \begin{pmatrix} x(0) \\ x(2) \\ x(1) \\ x(3) \end{pmatrix} \tag{7.25}$$

9. Show that

$$\mathbf{X} = \mathbf{F}_N\mathbf{x} \tag{7.26}$$

where $\mathbf{x}, \mathbf{X}$ are the vector representations of $x(n), X(k)$ respectively. **Solution:** Writing the terms of X,

$$X(0) = x(0) + x(1) + \ldots + x(N-1) \tag{7.27}$$

$$X(1) = x(0) + x(1)e^{-\frac{\jmath 2\pi}{N}} + \ldots + \\ + x(N-1)e^{-\frac{\jmath 2(N-1)\pi}{N}} \tag{7.28}$$

$$\vdots$$

$$X(N-1) = x(0) + x(1)e^{-\frac{\jmath 2(N-1)\pi}{N}} + \ldots + \\ + x(N-1)e^{-\frac{\jmath 2(N-1)(N-1)\pi}{N}} \tag{7.29}$$

Clearly, the term in the m^{th} row and n^{th} column is given by ($0 \le m \le N-1$ and $0 \le n \le N-1$)

$$T_{mn} = x(n)e^{-\frac{\jmath 2mn\pi}{N}} \tag{7.30}$$

and so, we can represent each of these terms as a matrix product

$$\mathbf{X} = \mathbf{F}_N\mathbf{x} \tag{7.31}$$

where $\mathbf{F}_N = \left[e^{-\frac{-\jmath 2mn\pi}{N}}\right]_{mn}$ for $0 \le m \le N-1$ and $0 \le n \le N-1$.

10. Derive the following Step-by-step visualisation of 8-point FFTs into 4-point FFTs and so on

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.32}$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.33}$$

4-point FFTs into 2-point FFTs

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.34}$$

$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.35}$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.36}$$

$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.37}$$

$$P_8\begin{bmatrix}x(0)\\x(1)\\x(2)\\x(3)\\x(4)\\x(5)\\x(6)\\x(7)\end{bmatrix}=\begin{bmatrix}x(0)\\x(2)\\x(4)\\x(6)\\x(1)\\x(3)\\x(5)\\x(7)\end{bmatrix} \tag{7.38}$$

$$P_4\begin{bmatrix}x(0)\\x(2)\\x(4)\\x(6)\end{bmatrix}=\begin{bmatrix}x(0)\\x(4)\\x(2)\\x(6)\end{bmatrix} \tag{7.39}$$

$$P_4\begin{bmatrix}x(1)\\x(3)\\x(5)\\x(7)\end{bmatrix}=\begin{bmatrix}x(1)\\x(5)\\x(3)\\x(7)\end{bmatrix} \tag{7.40}$$

Therefore,

$$\begin{bmatrix}X_3(0)\\X_3(1)\end{bmatrix}=F_2\begin{bmatrix}x(0)\\x(4)\end{bmatrix} \tag{7.41}$$

$$\begin{bmatrix}X_4(0)\\X_4(1)\end{bmatrix}=F_2\begin{bmatrix}x(2)\\x(6)\end{bmatrix} \tag{7.42}$$

$$\begin{bmatrix}X_5(0)\\X_5(1)\end{bmatrix}=F_2\begin{bmatrix}x(1)\\x(5)\end{bmatrix} \tag{7.43}$$

$$\begin{bmatrix}X_6(0)\\X_6(1)\end{bmatrix}=F_2\begin{bmatrix}x(3)\\x(7)\end{bmatrix} \tag{7.44}$$

Solution: We write out the values of performing an 8-point FFT on $\mathbf{x}$ as follows.

$$X(k)=\sum_{n=0}^{7}x(n)e^{-\frac{\jmath 2kn\pi}{8}} \tag{7.45}$$

$$=\sum_{n=0}^{3}\left(x(2n)e^{-\frac{\jmath 2kn\pi}{4}}+e^{-\frac{\jmath 2k\pi}{8}}x(2n+1)e^{-\frac{\jmath 2kn\pi}{4}}\right) \tag{7.46}$$

$$=X_1(k)+e^{-\frac{\jmath 2k\pi}{4}}X_2(k) \tag{7.47}$$

where $\mathbf{X}_1$ is the 4-point FFT of the even-numbered terms and $\mathbf{X}_2$ is the 4-point FFT of the odd numbered terms. Noticing that for $k\geq 4$,

$$X_1(k)=X_1(k-4) \tag{7.48}$$

$$e^{-\frac{\jmath 2k\pi}{8}}=-e^{-\frac{\jmath 2(k-4)\pi}{8}} \tag{7.49}$$

we can now write out $X(k)$ in matrix form as in (7.32) and (7.33). We also need to solve the two 4-point FFT terms so formed.

$$X_1(k)=\sum_{n=0}^{3}x_1(n)e^{-\frac{\jmath 2kn\pi}{8}} \tag{7.50}$$

$$=\sum_{n=0}^{1}\left(x_1(2n)e^{-\frac{\jmath 2kn\pi}{4}}+e^{-\frac{\jmath 2k\pi}{8}}x_2(2n+1)e^{-\frac{\jmath 2kn\pi}{4}}\right) \tag{7.51}$$

$$=X_3(k)+e^{-\frac{\jmath 2k\pi}{4}}X_4(k) \tag{7.52}$$

using $x_1(n)=x(2n)$ and $x_2(n)=x(2n+1)$. Thus we can write the 2-point FFTs

$$\begin{bmatrix}X_3(0)\\X_3(1)\end{bmatrix}=F_2\begin{bmatrix}x(0)\\x(4)\end{bmatrix} \tag{7.53}$$

$$\begin{bmatrix}X_4(0)\\X_4(1)\end{bmatrix}=F_2\begin{bmatrix}x(2)\\x(6)\end{bmatrix} \tag{7.54}$$

Using a similar idea for the terms X_2,

$$\begin{bmatrix}X_5(0)\\X_5(1)\end{bmatrix}=F_2\begin{bmatrix}x(1)\\x(5)\end{bmatrix} \tag{7.55}$$

$$\begin{bmatrix}X_6(0)\\X_6(1)\end{bmatrix}=F_2\begin{bmatrix}x(3)\\x(7)\end{bmatrix} \tag{7.56}$$

But observe that from (7.25),

$$\mathbf{P}_8\mathbf{x}=\begin{pmatrix}\mathbf{x}_1\\\mathbf{x}_2\end{pmatrix} \tag{7.57}$$

$$\mathbf{P}_4\mathbf{x}_1=\begin{pmatrix}\mathbf{x}_3\\\mathbf{x}_4\end{pmatrix} \tag{7.58}$$

$$\mathbf{P}_4\mathbf{x}_2=\begin{pmatrix}\mathbf{x}_5\\\mathbf{x}_6\end{pmatrix} \tag{7.59}$$

where we define $x_3(k)=x(4k)$, $x_4(k)=x(4k+2)$, $x_5(k)=x(4k+1)$, and $x_6(k)=x(4k+3)$ for $k=0,1$.

11. For

$$\mathbf{x}=\begin{pmatrix}1\\2\\3\\4\\2\\1\end{pmatrix} \tag{7.60}$$

compte the DFT using (7.26) **Solution:** Download the Python code from

```
$ wget https://github.com/kurugodukarthik11/
    EE3900/blob/main/Assignments/
    Assignment_1/codes/7_11.py
```

12. Repeat the above exercise using the FFT after zero padding **x**.
13. Write a C program to compute the 8-point FFT.

```
$ wget https://github.com/kurugodukarthik11/
    EE3900/blob/main/Assignments/
    Assignment_1/codes/7_13.c
```

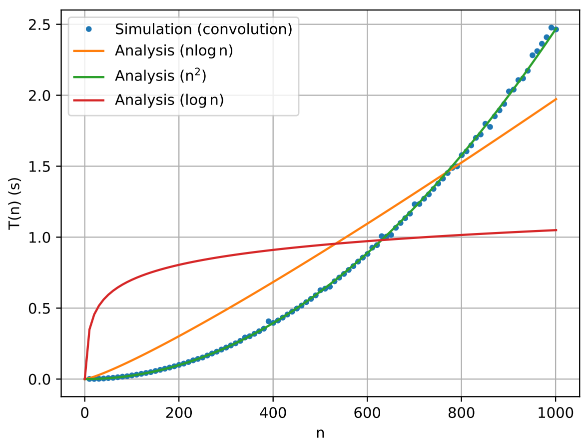

Fig. 7.13: Complexity Analysis of convolution

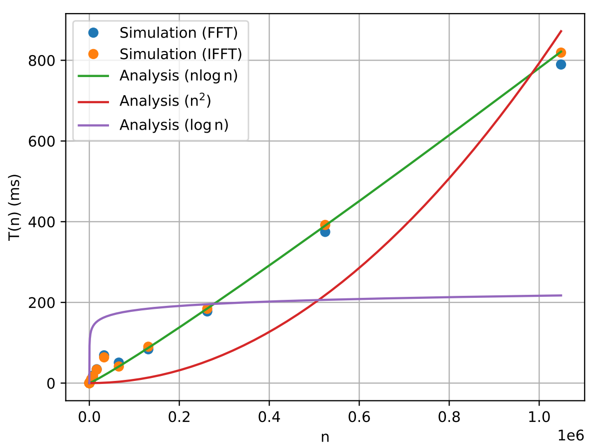

Fig. 7.13: Complexity Analysis of FFT and IFFT

8 Exercises

Answer the following questions by looking at the python code in Problem 2.3.

8.1 The command

```
output_signal = signal.lfilter(b, a,
    input_signal)
```

in Problem 2.3 is executed through the following difference equation

$$\sum_{m=0}^{M} a(m)\, y(n-m) = \sum_{k=0}^{N} b(k)\, x(n-k) \tag{8.1}$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.
Solution: On taking the Z-transform on both sides of the difference equation

$$\sum_{m=0}^{M} a(m)\, z^{-m} Y(z) = \sum_{k=0}^{N} b(k)\, z^{-k} X(z) \tag{8.2}$$

$$\implies H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N} b(k)\, z^{-k}}{\sum_{m=0}^{M} a(m)\, z^{-m}} \tag{8.3}$$

For obtaining the discrete Fourier transform, put $z = \jmath^{\frac{2\pi i}{I}}$ where I is the length of the input signal and $i = 0, 1, \dots, I-1$ Run the following code

```
wget https://github.com/kurugodukarthik11/
    EE3900/blob/main/Assignments/
    Assignment_1/codes/8_1.py
```

8.2 Repeat all the exercises in the previous sections for the above a and b.
Solution: The polynomial coefficients obtained are

$$\mathbf{a} = \begin{pmatrix} 1.000 \\ -2.519 \\ 2.561 \\ -1.206 \\ 0.220 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 0.003 \\ 0.014 \\ 0.021 \\ 0.014 \\ 0.003 \end{pmatrix} \tag{8.4}$$

The difference equation is then given by

$$\begin{aligned} &\mathbf{a}^\top \mathbf{y} = \mathbf{b}^\top \mathbf{x} \\ &y(n) - (2.519)y(n-1) + (2.561)y(n-2) \\ &-(1.206)y(n-3) + (0.220)y(n-4) = (0.003)x(n) \\ &+(0.014)x(n-1) + (0.021)x(n-2) + (0.014)x(n-3) \\ &+ (0.003)x(n-4) \end{aligned} \tag{8.5}$$

where

$$\mathbf{y} = \begin{pmatrix} y(n) \\ y(n-1) \\ y(n-2) \\ y(n-3) \\ y(n-4) \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} x(n) \\ x(n-1) \\ x(n-2) \\ x(n-3) \\ x(n-4) \end{pmatrix} \tag{8.6}$$

Run the following code

```
wget https://github.com/kurugodukarthik11/
    EE3900/blob/main/Assignments/
    Assignment_1/codes/8_2a.py
```

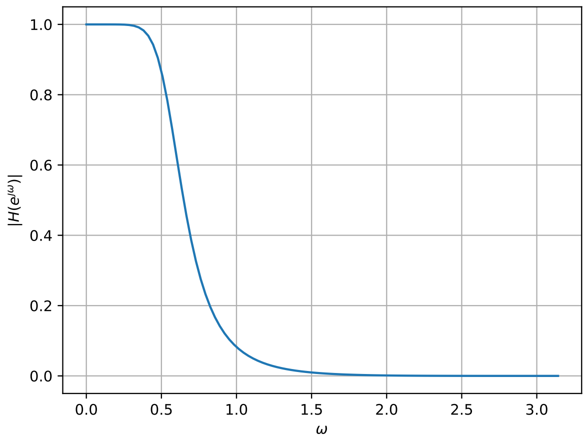

Fig. 8.2: $|H(e^{j\omega})|$

Run the following code

```
wget https://github.com/kurugodukarthik11/
    EE3900/blob/main/Assignments/
    Assignment_1/codes/8_2b.py
```

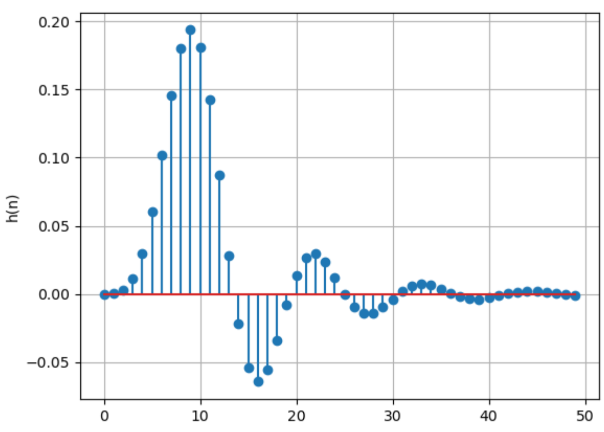

Fig. 8.2: h(n)

8.3 What is the sampling frequency of the input signal?
Solution: Sampling frequency(fs)=44.1kHZ.

8.4 What is type, order and cutoff-frequency of the above butterworth filter
Solution: The given butterworth filter is low pass with order=4 and cutoff-frequency=4kHz.

8.5 Modifying the code with different input parameters and to get the best possible output.
Solution: Order: 10
Cutoff frequency: 3000 Hz