

# Digital Signal Processing

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**Abstract**—This manual provides a simple introduction to digital signal processing.

## 1 SOFTWARE INSTALLATION

Run the following commands

```
sudo apt-get update
sudo apt-get install libffi-dev libsndfile1 python3
-sciPy python3-numpy python3-matplotlib
sudo pip install cffi pysoundfile
```

## 2 DIGITAL FILTER

2.1 Download the sound file from

```
wget https://raw.githubusercontent.com/
gadepall/
EE1310/master/filter/codes/Sound_Noise.wav
```

2.2 You will find a spectrogram at <https://academo.org/demos/spectrum-analyzer>. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?

**Solution:** There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.

**Solution:**

```
import soundfile as sf
from scipy import signal

#read .wav file
input_signal,fs = sf.read('Sound_Noise.wav')

#sampling frequency of Input signal
sampl_freq=fs

#order of the filter
order=4

#cutoff frequency 4kHz
cutoff_freq=4000.0

#digital frequency
Wn=2*cutoff_freq/sampl_freq

# b and a are numerator and denominator
polynomials respectively
b, a = signal.butter(order,Wn, 'low')

#filter the input signal with butterworth filter
output_signal = signal.filtfilt(b, a,
input_signal)
#output_signal = signal.lfilter(b, a,
input_signal)

#write the output signal into .wav file
sf.write('Sound_With_ReducedNoise.wav',
output_signal, fs)
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound\_With\_ReducedNoise.wav. Play the file in the spectrogram in Problem 2.2. What do you observe?

**Solution:** The key strokes as well as background noise is subdued in the audio.

Also, the signal is blank for frequencies above 5.1 kHz.

### 3 DIFFERENCE EQUATION

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \quad (3.1)$$

Sketch  $x(n)$ .

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$

$$y(n) = 0, n < 0 \quad (3.2)$$

Sketch  $y(n)$ .

**Solution:** The following code yields Fig. 3.3.

wget <https://github.com/gadepall/EE1310/raw/master/filter/codes/xnyn.py>

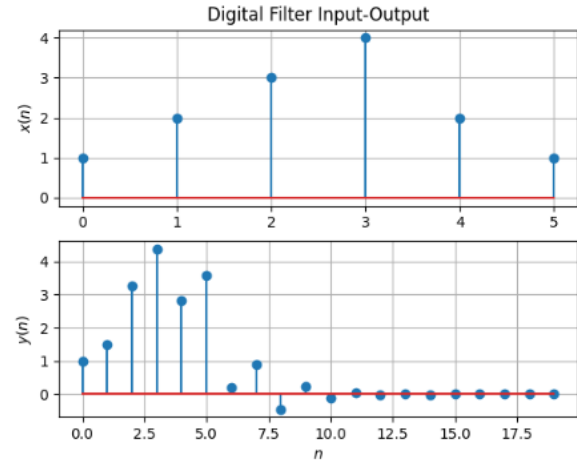


Fig. 3.3

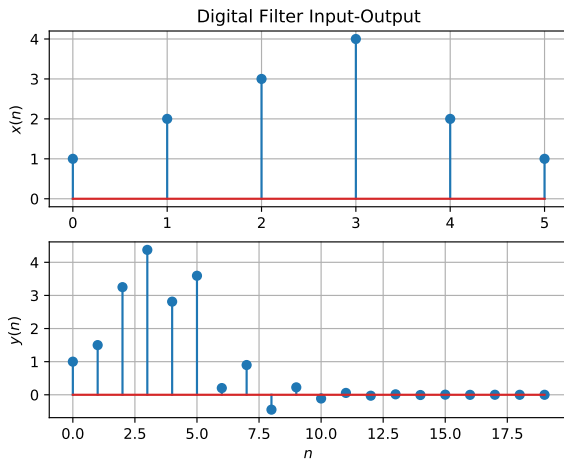


Fig. 3.2

3.3 Repeat the above exercise using a C code.

**Solution:** The following code yields Fig. 3.3.

wget [https://github.com/kurugodukarthik11/EE3900/blob/main/Assignments/Assignment\\_1/codes/xny.cpp](https://github.com/kurugodukarthik11/EE3900/blob/main/Assignments/Assignment_1/codes/xny.cpp)

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \quad (4.2)$$

and find

$$\mathcal{Z}\{x(n-k)\} \quad (4.3)$$

**Solution:** From (4.1),

$$\mathcal{Z}\{x(n-1)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \quad (4.4)$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-(n-1)} = z^{-1} \sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (4.5)$$

Similarly,

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-k)z^{-n} \quad (4.6)$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-k} \quad (4.7)$$

$$= z^{-k} \sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (4.8)$$

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \quad (4.9)$$

4.2 Obtain  $X(z)$  for  $x(n)$  defined in problem 3.1.

**Solution:** We know from (4.1)

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (4.10)$$

### 4 Z-TRANSFORM

4.1 The Z-transform of  $x(n)$  is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (4.1)$$

Substitute values of  $x(n)$  from (3.1) in (4.1)

$$X(z) = \mathcal{Z}\{x(n)\} \quad (4.11)$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (4.12)$$

$$X(z) = \sum_{n=0}^5 x(n)z^{-n} \quad (4.13)$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \quad (4.14)$$

from (3.2) assuming that the Z-transform is a linear operation.

**Solution:** Applying (4.9) in (3.2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \quad (4.15)$$

$$\Rightarrow \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \quad (4.16)$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \quad (4.17)$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (4.18)$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \quad (4.19)$$

**Solution:**

$$\mathcal{Z}\{\delta(n)\} = \sum_{n=-\infty}^{\infty} \delta(n)z^{-n} \quad (4.20)$$

$$= \delta(0)z^0 \quad (4.21)$$

$$= 1 \quad (4.22)$$

Thus

$$\delta(n) \stackrel{\mathcal{Z}}{=} 1 \quad (4.23)$$

and from (4.18),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \quad (4.24)$$

$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \quad (4.25)$$

using the formula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n) \stackrel{\mathcal{Z}}{=} \frac{1}{1 - az^{-1}} \quad |z| > |a| \quad (4.26)$$

**Solution:** Infinite GP:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1 - r}, \quad |r| < 1 \quad (4.27)$$

$$U(z) = \sum_{n=0}^{\infty} a^n z^{-n} \quad (4.28)$$

$$= \frac{1}{1 - az^{-1}}, \quad |z| > |a| \quad (4.29)$$

4.6 Let

$$H(e^{j\omega}) = H(z = e^{j\omega}). \quad (4.30)$$

Plot  $|H(e^{j\omega})|$ . Is it periodic? If so, find the period.  $H(e^{j\omega})$  is known as the *Discrete Time Fourier Transform* (DTFT) of  $x(n)$ .

**Solution:**

$$H(e^{j\omega}) = \frac{1 + e^{-2j\omega}}{1 + \frac{1}{2}e^{-j\omega}} \quad (4.31)$$

$$|H(e^{j\omega})| = \frac{|1 + \cos 2\omega - j \sin 2\omega|}{|1 + \frac{1}{2} \cos \omega - \frac{1}{2} j \sin \omega|} \quad (4.32)$$

$$= \frac{\sqrt{(1 + \cos 2\omega)^2 + (\sin 2\omega)^2}}{\sqrt{(1 + \frac{1}{2} \cos \omega)^2 + (\frac{1}{2} \sin \omega)^2}} \quad (4.33)$$

$$= \sqrt{\frac{2 + 2 \cos 2\omega}{\frac{5}{4} + \cos \omega}} \quad (4.34)$$

$$= \sqrt{\frac{2(2 \cos^2 \omega)4}{5 + 4 \cos \omega}} \quad (4.35)$$

$$= \frac{4 |\cos \omega|}{\sqrt{5 + 4 \cos \omega}} \quad (4.36)$$

Therefore the period of above equation is  $2\pi$  since lcm of period of numerator( $\pi$ ) and period of denominator( $2\pi$ ) is  $2\pi$ .

The following code plots Fig4.6.

```
wget https://github.com/kurugodukarthik11/
EE3900/blob/main/Assignments/
Assignment_1/codes/dtft.py
```

4.7 Express  $h(n)$  in terms of  $H(e^{j\omega})$ . **Solution:** We

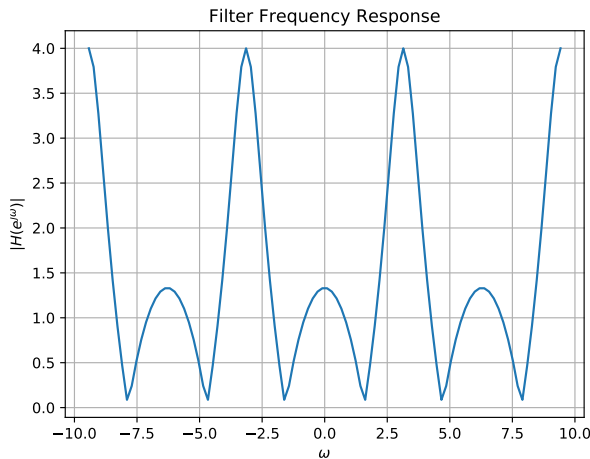


Fig. 4.6:  $|H(e^{j\omega})|$

have,

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k} \quad (4.37)$$

However,

$$\int_{-\pi}^{\pi} e^{j\omega(n-k)} d\omega = \begin{cases} 2\pi & n = k \\ 0 & \text{otherwise} \end{cases} \quad (4.38)$$

and so,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega \quad (4.39)$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{-\pi}^{\pi} h(k) e^{j\omega(n-k)} d\omega \quad (4.40)$$

$$= \frac{1}{2\pi} 2\pi h(n) = h(n) \quad (4.41)$$

which is known as the Inverse Discrete Fourier Transform. Thus,

$$h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega \quad (4.42)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 + e^{-2j\omega}}{1 + \frac{1}{2}e^{-j\omega}} e^{j\omega n} d\omega \quad (4.43)$$

## 5 IMPULSE RESPONSE

5.1 Using long division, find

$$h(n), \quad n < 5 \quad (5.1)$$

for  $H(z)$  in (4.16).

**Solution:** For long division, substitute

$$x := z^{-1}$$

$$\begin{array}{r} 2x - 4 \\ \frac{1}{2}x + 1 \overline{) x^2 + 1} \\ \underline{-x^2 - 2x} \phantom{+ 1} \\ -2x + 1 \\ \underline{2x + 4} \\ 5 \end{array}$$

Therefore,

$$H(z) = -4 + 2z^{-1} + \frac{5}{1 + \frac{1}{2}z^{-1}} \quad (5.2)$$

$$= -4 + 2z^{-1} + 5 \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n z^{-n} \quad (5.3)$$

$$= 1 - \frac{1}{2}z^{-1} + 5 \sum_{n=2}^{\infty} \left(-\frac{1}{2}\right)^n z^{-n} \quad (5.4)$$

$$= \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n z^{-n} + 4 \sum_{n=2}^{\infty} \left(-\frac{1}{2}\right)^n z^{-n} \quad (5.5)$$

$$= \sum_{n=-\infty}^{\infty} u(n) \left(-\frac{1}{2}\right)^n z^{-n} + \sum_{n=-\infty}^{\infty} u(n-2) \left(-\frac{1}{2}\right)^{n-2} z^{-n} \quad (5.6)$$

Now, from (4.1), we get

$$h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad (5.7)$$

5.2 Find an expression for  $h(n)$  using  $H(z)$ , given that

$$h(n) \stackrel{Z}{\Longleftrightarrow} H(z) \quad (5.8)$$

and there is a one to one relationship between  $h(n)$  and  $H(z)$ .  $h(n)$  is known as the *impulse response* of the system defined by (3.2).

**Solution:** From (4.16),

$$H(z) = \frac{1}{1 + \frac{1}{2}z^{-1}} + \frac{z^{-2}}{1 + \frac{1}{2}z^{-1}} \quad (5.9)$$

$$\Rightarrow h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad (5.10)$$

ROC:  $(-\infty, -1/2) \cup (-1/2, \infty)$  using (4.26) and (4.9).

5.3 Sketch  $h(n)$ . Is it bounded? Justify theoretically.

**Solution:**

We know

$$a) \left| \left(-\frac{1}{2}\right)^n \right| \leq 1 \quad (5.11)$$

$$b) |u(n)| \leq 1 \quad (5.12)$$

$$c) \left| \left( \frac{-1}{2} \right)^{n-2} \right| \leq 1 \quad (5.13)$$

$$d) |u(n-2)| \leq 1 \quad (5.14)$$

Therefore

$$\left| \left( \frac{-1}{2} \right)^n u(n) \right| \leq 1 \quad (5.15)$$

$$\left| \left( \frac{-1}{2} \right)^{n-2} u(n-2) \right| \leq 1 \quad (5.16)$$

Therefore

$$\left| \left( \frac{-1}{2} \right)^n u(n) + \left( \frac{-1}{2} \right)^{n-2} u(n-2) \right| \leq 2 \quad (5.17)$$

Hence Bounded

The following code plots Fig. 3.

```
wget https://github.com/kurugodukarthik11/
EE3900/blob/main/Assignments/
Assignment_1/codes/hn.py
```

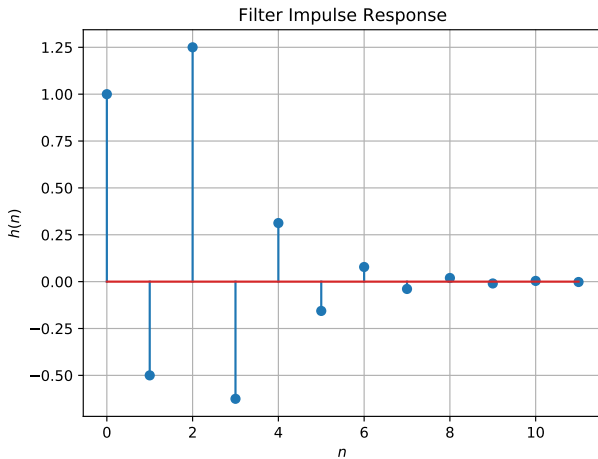


Fig. 5.3:  $h(n)$  as the inverse of  $H(z)$

5.4 Convergent? Justify using the ratio test.

**Solution:** Using the ratio test for convergence

$$\lim_{n \rightarrow \infty} \left| \frac{h(n+1)}{h(n)} \right| \quad (5.18)$$

$$= \lim_{n \rightarrow \infty} \left| \frac{\left( \frac{-1}{2} \right)^{n+1} u(n+1) + \left( \frac{-1}{2} \right)^{n-1} u(n-1)}{\left( \frac{-1}{2} \right)^n u(n) + \left( \frac{-1}{2} \right)^{n-2} u(n-2)} \right| \quad (5.19)$$

$$\leq \lim_{n \rightarrow \infty} \left| \frac{\left( \frac{-1}{2} \right)^{n-1} \left( \frac{1}{4} + 1 \right)}{\left( \frac{-1}{2} \right)^{n-2} \left( \frac{1}{4} + 1 \right)} \right| \quad (5.20)$$

$$= \lim_{n \rightarrow \infty} \left| -\frac{1}{2} \right| \quad (5.21)$$

$$= \frac{1}{2} < 1 \quad (5.22)$$

Hence by ratio test  $h(n)$  is convergent

5.5 The system with  $h(n)$  is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \quad (5.23)$$

Is the system defined by (3.2) stable for the impulse response in (5.8)?

**Solution:**

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=-\infty}^{\infty} \left( \frac{-1}{2} \right)^n u(n) + \sum_{n=-\infty}^{\infty} \left( \frac{-1}{2} \right)^{n-2} u(n-2) \quad (5.24)$$

$$= \sum_{n=0}^{\infty} \left( \frac{-1}{2} \right)^n + \sum_{n=2}^{\infty} \left( \frac{-1}{2} \right)^{n-2} \quad (5.25)$$

$$= \sum_{n=0}^{\infty} \left( \frac{-1}{2} \right)^n + \sum_{n=2}^{\infty} \left( \frac{-1}{2} \right)^{n-2} \quad (5.26)$$

From Infinite GP

$$= \frac{2}{3} + \frac{2}{3} \quad (5.27)$$

$$= \frac{4}{3} < \infty \quad (5.28)$$

Thus System is stable from (5.23)

5.6 Verify the above result using a python code.

**Solution:**

```
https://github.com/kurugodukarthik11/EE3900
/blob/main/Assignments/Assignment_1/
codes/5_6.py
```

5.7 Compute and sketch  $h(n)$  using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \quad (5.29)$$

This is the definition of  $h(n)$ .

**Solution:** The following code plots Fig. 3. Note that this is the same as Fig. 3.

```
wget https://github.com/kurugodukarthik11/
EE3900/blob/main/Assignments/
```

Assignment\_1/codes/hndef.py

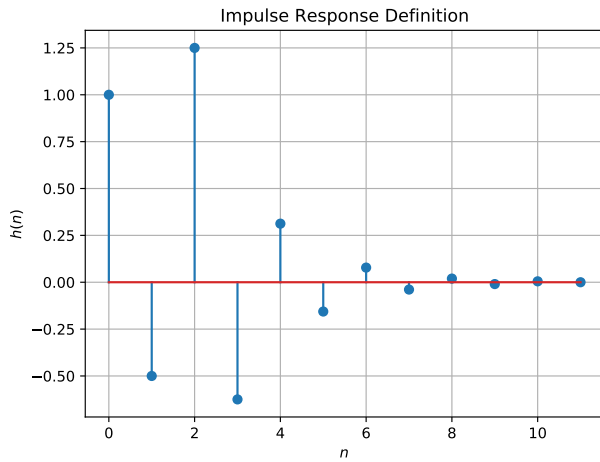


Fig. 5.7:  $h(n)$  from the definition

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad (5.30)$$

Comment. The operation in (5.30) is known as *convolution*.

**Solution:** The following code plots Fig. 3. Note that this is the same as  $y(n)$  in Fig. 3.3.

```
wget https://raw.githubusercontent.com/
gadepall/EE1310/master/filter/codes/
ynconv.py
```

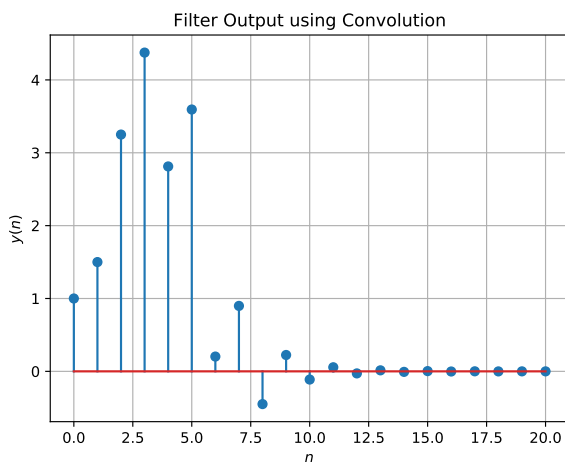


Fig. 5.8:  $y(n)$  from the definition of convolution

5.9 Express the above convolution using a Teoplitz matrix.

**Solution:**

The Toeplitz matrices for convolution are,

$$\mathbf{y} = \mathbf{x} \otimes \mathbf{h} \quad (5.31)$$

$$\mathbf{y} = \begin{pmatrix} h_1 & 0 & \cdot & \cdot & \cdot & 0 \\ h_2 & h_1 & \cdot & \cdot & \cdot & 0 \\ h_3 & h_2 & h_1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & h_3 & h_2 & h_1 \\ 2 & \cdot & \cdot & \cdot & h_2 & h_1 \\ 3 & \cdot & \cdot & \cdot & 0 & h_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad (5.32)$$

5.10 Show that

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \quad (5.33)$$

**Solution: Solution:** From eqn (5.30)

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad (5.34)$$

Replace 'k' with 'n-k'

$$y(n) = x(n) * h(n) \quad (5.35)$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(n-(n-k)) \quad (5.36)$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \quad (5.37)$$

Hence Proved

## 6 DFT AND FFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \quad (6.1)$$

and  $H(k)$  using  $h(n)$ .

6.2 Compute

$$Y(k) = X(k)H(k) \quad (6.2)$$

6.3 Compute

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) \cdot e^{j2\pi kn/N}, \quad n = 0, 1, \dots, N-1 \quad (6.3)$$

**Solution:** The following code plots Fig.

3. Note that this is the same as  $y(n)$  in Fig. 3.3.

```
wget https://raw.githubusercontent.com/
gadepall/EE1310/master/filter/codes/yndft.
py
```

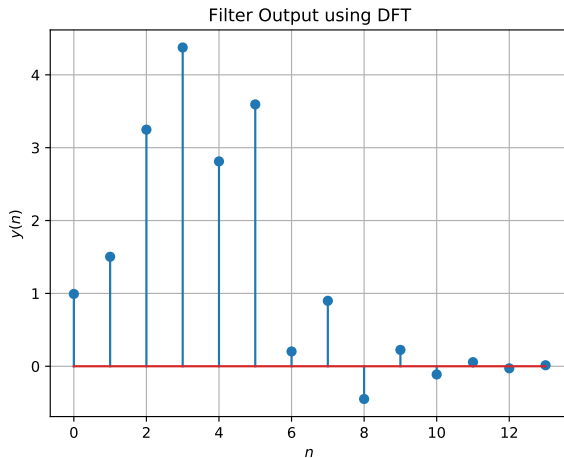


Fig. 6.3:  $y(n)$  from the DFT

- 6.4 Repeat the previous exercise by computing  $X(k)$ ,  $H(k)$  and  $y(n)$  through FFT and IFFT.
- 6.5 Wherever possible, express all the above equations as matrix equations.
- 6.6 Verify the above equations by generating the DFT matrix in python.

## 7 EXERCISES

Answer the following questions by looking at the python code in Problem 2.3.

7.1 The command

```
output_signal = signal.lfilter(b, a,
    input_signal)
```

in Problem 2.3 is executed through the following difference equation

$$\sum_{m=0}^M a(m) y(n-m) = \sum_{k=0}^N b(k) x(n-k) \quad (7.1)$$

where the input signal is  $x(n)$  and the output signal is  $y(n)$  with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.

7.2 Repeat all the exercises in the previous sections for the above  $a$  and  $b$ .

7.3 What is the sampling frequency of the input signal?

**Solution:** Sampling frequency( $f_s$ )=44.1kHz.

7.4 What is type, order and cutoff-frequency of the <https://www.overleaf.com/project/62ef9e6ba73a18aa9841> butterworth filter

**Solution:** The given butterworth filter is low pass with order=2 and cutoff-frequency=4kHz.

7.5 Modifying the code with different input parameters and to get the best possible output.