0.0.1 Importing necessary libraries

[1]:
```
import numpy as np
from sklearn.datasets import make_spd_matrix
import matplotlib.pyplot as plt
```

0.1 Problem 1

0.1.1 Problem 1 (a)

$f(x) = x^T Ax + b$
$\Rightarrow \nabla f(x) = 2Ax$
where $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, $b = [1]$
By setting the gradient to zero $\Rightarrow 2Ax = 0$
$\Rightarrow x = 0$

[2]:
```
A_a = np.matrix([[2, -1, -1], [-1, 2, 0], [-1, 0, 1]])
b_a = np.matrix([[1]])

def f_a(x):
    return x.T @ A_a @ x + b_a

def gradient_f_a(x):
    return 2* A_a @ x

def gradient_descent(x0, learning_rate, num_iterations):
    x = x0
    for i in range(num_iterations):
        x = x - learning_rate * gradient_f_a(x)
    return x

N = 1000
t = 0.02

x0 = np.random.rand(3,1)
result_gradient_descent = gradient_descent(x0, t, N)
print("Minimizer using gradient descent:", result_gradient_descent)
print("Value of f_a at minimizer:", f_a(result_gradient_descent)[0,0])
result_gradient_zero = (1/2) * A_a.I @ np.matrix([[0],[0],[0]])
print("Minimizer using gradient zero:", result_gradient_zero)
print("Value of f_a at minimizer using gradient zero:",␣
↪f_a(result_gradient_zero)[0,0])
```

```
Minimizer using gradient descent: [[1.54960309e-04]
 [8.59964837e-05]
 [1.93232345e-04]]
Value of f_a at minimizer: 1.000000013616153
```

```
Minimizer using gradient zero: [[0.]
 [0.]
 [0.]]
Value of f_a at minimizer using gradient zero: 1.0
```

0.1.2 Problem 1 (b)

$f(x) = ||Ax - b||^2$
$\Rightarrow \nabla f(x) = 2A^T(Ax - b)$
where $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$
By setting the gradient to zero $\Rightarrow 2A^T(Ax - b) = 0$
$\Rightarrow A^T(Ax - b) = 0$
$\Rightarrow A^TAx = A^Tb$
$\Rightarrow x = (A^TA)^{-1}A^Tb$

[3]:
```
A_b = np.matrix([[1,2],[2,4],[3,1]])
b_b = np.matrix([[1],[3],[1]])

def f_b(x):
    return x.T @ A_b.T @ A_b @ x - b_b.T @ A_b @ x - x.T @ A_b.T @ b_b + b_b.T␣
 ↪@ b_b

def gradient_f_b(x):
    return 2* A_b.T @ (A_b @ x - b_b)

def gradient_descent(x0, learning_rate, num_iterations):
    x = x0
    for i in range(num_iterations):
        x = x - learning_rate * gradient_f_b(x)
    return x

N = 1000
t = 0.02
x0 = np.random.rand(2,1)
result_gradient_descent = gradient_descent(x0, t, N)
print("Minimizer using gradient descent:", result_gradient_descent)
print("Value of f_b at minimizer:", f_b(result_gradient_descent)[0,0])
result_gradient_zero = (A_b.T @ A_b).I @ A_b.T @ b_b
print("Minimizer using gradient zero:", result_gradient_zero)
print("Value of f_b at minimizer using gradient zero:",␣
 ↪f_b(result_gradient_zero)[0,0])
```

```
Minimizer using gradient descent: [[0.12]
 [0.64]]
Value of f_b at minimizer: 0.20000000000000107
Minimizer using gradient zero: [[0.12]
```

 [0.64]]
Value of f_b at minimizer using gradient zero: 0.20000000000000107

0.1.3 Problem 1 (c)

$f(x) = ||Ax - b||^2$
$\Rightarrow \nabla f(x) = 2A^T(Ax - b)$
where $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 3 & 1 & 9 \\ 4 & 1 & 0 \\ 2 & 1 & 4 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \\ 9 \end{bmatrix}$
By setting the gradient to zero $\Rightarrow 2A^T(Ax - b) = 0$
$\Rightarrow A^T(Ax - b) = 0$
$\Rightarrow A^TAx = A^Tb$
$\Rightarrow x = (A^TA)^{-1}A^Tb$

[4]:
```
A_c = np.matrix([[1,2,1],[2,4,2],[3,1,9],[4,1,0],[2,1,4]])
b_c = np.matrix([[1],[3],[1],[0],[9]])

def f_c(x):
    return x.T @ A_c.T @ A_c @ x - b_c.T @ A_c @ x - x.T @ A_c.T @ b_c + b_c.T␣
 ↪@ b_c

def gradient_f_c(x):
    return 2* A_c.T @ (A_c @ x - b_c)

def gradient_descent(x0, learning_rate, num_iterations):
    x = x0
    for i in range(num_iterations):
        x = x - learning_rate * gradient_f_c(x)
    return x

N = 1000
t = 0.002 # learning rate is made smaller than previous problems due to out of␣
 ↪bound errors
x0 = np.random.rand(3,1)
result_gradient_descent = gradient_descent(x0, t, N)
print("Minimizer using gradient descent:", result_gradient_descent)
print("Value of f_c at minimizer:", f_c(result_gradient_descent)[0,0])
result_gradient_zero = (A_c.T @ A_c).I @ A_c.T @ b_c
print("Minimizer using gradient zero:", result_gradient_zero)
print("Value of f_c at minimizer using gradient zero:",␣
 ↪f_c(result_gradient_zero)[0,0])
```

```
Minimizer using gradient descent: [[0.05744939]
 [0.65686105]
 [0.33915902]]
```

```
Value of f_c at minimizer: 56.990482782488314
Minimizer using gradient zero: [[0.05744939]
 [0.65686105]
 [0.33915902]]
Value of f_c at minimizer using gradient zero: 56.99048278248832
```

0.2 Problem 2

Generating A randomly using `sklearn.datasets.make_spd_matrix`,and generate b and c using `np.random.rand` function of numpy.

[5]:
```
A = np.matrix(make_spd_matrix(10))
b = np.matrix(np.random.randn(10,1))
c = np.matrix(np.random.randn(1,1))
```

$f(x) = x^T Ax - 2b^T x + c$
$\Rightarrow \nabla f(x) = 2Ax - 2b$
$\Rightarrow \nabla^2 f(x) = 2A$
$\nabla^2 f(x) \geq 0 \Rightarrow f$ is convex
$2 * A \geq 0 \Rightarrow A \geq 0 \Rightarrow A$ is positive semi definite $\Rightarrow f$ is convex
So we need to check if A is positive semi definite or not to know if f is convex or not.

[6]:
```
def f(x):
    return x.T @ A @ x -2 * b.T @ x + c

def gradient_f(x):
    return 2 * A @ x - 2 * b

def grad_grad_f(x):
    return 2 * A

def is_pos_sem_def(x):
    return np.all(np.linalg.eigvals(x) >= 0)
```

0.2.1 Problem 2 (a)

Analytical solution

[7]:
```
if is_pos_sem_def(A):
    print("A is semi positive definite, hence f is convex")
```

```
A is semi positive definite, hence f is convex
```

By setting the gradient of f to zero, we can find the analytical solution $\Rightarrow \nabla f(x) = 0$
$\Rightarrow 2Ax - 2b = 0$
$\Rightarrow x = A^{-1}b$

[8]:
```
result_gradient_zero = np.matrix(A.I @ b)
```

[9]:
```
analytical_solution = f(result_gradient_zero)[0,0]
```

[10]:
```
print("Minimizer using gradient zero:", result_gradient_zero)
print("Analytical solution by setting gradient to zero:", analytical_solution)
```

```
Minimizer using gradient zero: [[-9.15256852]
 [ 2.78083746]
 [-1.18661204]
 [ 1.32064519]
 [-7.09335175]
 [13.35628316]
 [-3.38601289]
 [ 2.42614784]
 [-0.79290161]
 [ 9.7378983 ]]
Analytical solution by setting gradient to zero: -28.24615806045249
```

0.2.2 Problem 2 (b)

Gradient descent Initial point $x_0 = [1/10, 1/10, 1/10, 1/10, 1/10, 1/10, 1/10, 1/10, 1/10, 1/10]^T$ where $x_0 \in \mathbb{R}^{10}$

[11]:
```
x0 = np.matrix([[1/10],[1/10],[1/10],[1/10],[1/10],[1/10],[1/10],[1/10],[1/
 ↪10],[1/10]])
```

Step size $l_{GD} = \frac{1}{2||A||+||b||_2}$ where $||A||$ is the spectral norm of A and $||b||_2$ is the 2-norm of b

[12]:
```
a_norm = np.linalg.norm(A, ord=2)
b_norm = np.linalg.norm(b)
step_size = 1 / ((2*a_norm) + b_norm)
```

[13]:
```
def gradient_descent(x0, step_size, num_iterations):
    """
    Gradient descent algorithm
    :param x0: np.matrix - Initial point
    :param step_size: float - step size
    :param num_iterations: int - number of iterations
    :return: value of x after num_iterations, list of objective values
    """
    x = x0
    f_list = []
    for i in range(num_iterations):
        x = x - step_size * gradient_f(x)
        f_list.append(f(x).tolist()[0][0])
    return x, f_list
```

[14]:
```
result_gradient_descent, f_list = gradient_descent(x0, step_size, 1000)
```

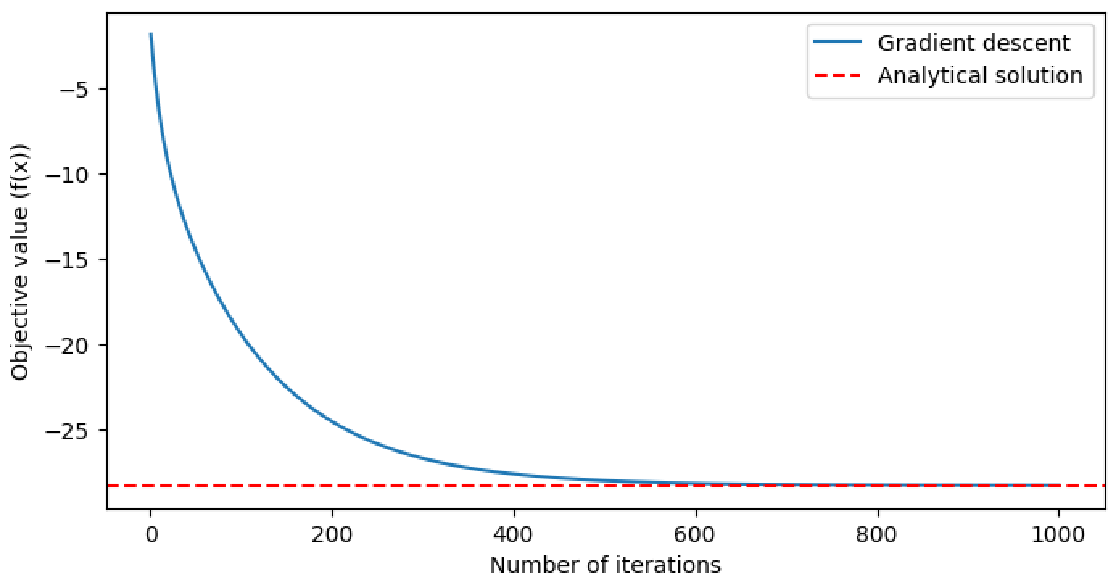
[15]:
```
optimal_solution = f(result_gradient_descent)[0,0]
```

[16]:
```
print("Minimizer using gradient descent:", result_gradient_descent)
print("Optimal Solution by using gradient descent:", optimal_solution)
```

```
Minimizer using gradient descent: [[-9.03581218]
 [ 2.72650626]
 [-1.14045911]
 [ 1.29836243]
 [-6.98249168]
 [13.19409598]
 [-3.3210739 ]
 [ 2.37920814]
 [-0.77443266]
 [ 9.6221097 ]]
Optimal Solution by using gradient descent: -28.242278949955974
```

In the following plot, the blue line shows the objective value after each iteration and the red dashed line shows the analytical solution.

[17]:
```
plt.figure(figsize=(8,4))
plt.plot(range(1,1001), f_list)
plt.xlabel("Number of iterations")
plt.ylabel("Objective value (f(x))")
plt.axhline(y=analytical_solution, color='r', linestyle='--')
plt.legend(["Gradient descent", "Analytical solution"])
plt.show()
```



You can see that the objective value decreases after each iteration and converges to the analytical solution.