1.1) a) $\hat{x} = \underset{x \in \mathbb{R}^n}{\text{Arg min}} \; \|Ax - b\|^2$

b) $\text{dist}(Ax, b)^2 = (Ax-b)^2 = \sum_{i=1}^{m} (Ax_i - b_i)^2$

$= (Ax-b)^T (Ax-b)$

$= x^T A^T A x - 2b^T A x + b^T b$

$\frac{\partial(\;)}{\partial x} = 0$

$\Rightarrow \quad 2A^T A x - 2A^T b = 0$

$\Rightarrow \quad x = (A^T A)^{-1} A^T b$

c) Since rank$(A) = m$ & $m < n$ there are infinitely many solns, or no solns, because there are m eqn's and n unknowns and $m < n$.

1.2) a) $x = (x_1, x_2, \ldots x_n)^T_{n\times 1}$

$w = (w_1, w_2, \ldots w_n)_{n\times 1}$

$\phi(x) = (1, x_1, x_2, \ldots, x_n)^T_{(n+1)\times 1}$

$w' = (b, w_1, w_2, \ldots w_n)_{(n+1)\times 1}$

$w^T x + b = (w')^T \phi(x)$

Here $\phi(x)$ is feature map of x

1.2) b) $(x-a)^2 + (y-b)^2 = r^2$

Let $z = \begin{bmatrix} x \\ y \end{bmatrix}$ $c = \begin{bmatrix} a \\ b \end{bmatrix}$

$\therefore \|z-c\|_2^2 = r^2$

$(z^T - c^T)(z-c) = r^2 \Rightarrow z^T z - 2z^T c + c^T c - r^2 = 0$

$\Rightarrow \begin{bmatrix} 1 & -2c & c^T c - r^2 \end{bmatrix} \begin{bmatrix} z^T z \\ z^T \\ 1 \end{bmatrix} = 0$

$\therefore \phi(z) = \begin{bmatrix} z^T z \\ z^T \\ 1 \end{bmatrix}$

2.1) a) $\min \frac{1}{m} \sum_{i=1}^{m} (w^T \phi(x_i) - y_i)^2$

$$\frac{1}{m} \sum_{i=1}^{m} (\phi(x_i)^T w - y_i^T)(w^T \phi(x_i) - y_i)$$

$$\frac{1}{m} \sum_{i=1}^{m} \left[\phi(x_i)^T w w^T \phi(x_i) + y_i^T y_i - y_i^T w^T \phi(x_i) - \phi(x_i)^T w y_i\right]$$

Differentiating wrt w & eqating to zero

$$\frac{2}{m} \sum_{i=1}^{m} \left(\phi(x_i)^T \phi(x_i) w^* - \phi(x_i)^T y_i\right) = 0$$

$$\sum_{i=1}^{m} \phi(x_i)^T \phi(x_i)\, w^* = \sum_{i=1}^{m} \phi(x_i)^T y_i$$

$$w^* = \left(\phi(X)^T \phi(X)\right)^{-1} \phi(X)^T y$$

b) For $\phi(x) = x$

$$w^* = (X^T X)^{-1} X^T y$$

w^* is same as in 1.1 (b)

c) Consider $\phi(x) = (1, x)$

$$\therefore \phi(X) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \phi(X)^T \quad y = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$$

$$\therefore \phi(X)^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$w^* = \left(\phi(x)^T \phi(x)\right)^{-1} \phi(x)^T y$$

$$= \phi(x)^{-1} y$$

$$= \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

2.2) Let $D = \{(1,1), (2,4), (3,9), (4,16)\}$

$$\therefore \phi = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} \qquad y = \begin{bmatrix} 1 \\ 4 \\ 9 \\ 16 \end{bmatrix}$$

$$\phi^T = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\phi^T \phi = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 30 & 10 \\ 10 & 4 \end{bmatrix} \Rightarrow (\phi^T \phi)^{-1} = \begin{bmatrix} 1/5 & -1/2 \\ -1/2 & 3/2 \end{bmatrix}$$

$$\phi^T y = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 9 \\ 16 \end{bmatrix} = \begin{bmatrix} 100 \\ 30 \end{bmatrix}$$

$$\therefore w^* = (\phi^T \phi)^{-1} \phi^T y$$

$$w^* = \begin{bmatrix} 1/5 & -1/2 \\ -1/2 & 3/2 \end{bmatrix} \begin{bmatrix} 100 \\ 30 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$$

2.3) a)

$$\min \frac{1}{m} \sum_{i=1}^{m} (w^T \phi(x_i) - y_i)^2 + \frac{\lambda}{2} \|w\|^2$$

$$\frac{1}{m} \sum_{i=1}^{m} (\phi(x_i)^T w - y_i^T)(w^T \phi(x_i) - y_i) + \frac{\lambda}{2} w^T w$$

$$\frac{1}{m} \sum_{i=1}^{m} \left[\phi(x_i)^T w w^T \phi(x_i) + y_i^T y_i - y_i^T w^T \phi(x_i) - \phi(x_i)^T w y_i\right] + \frac{\lambda}{2} w^T w$$

Differentiating wrt w & eqating to zero

$$\frac{2}{m} \sum_{i=1}^{m} \left(\phi(x_i)^T \phi(x_i) w^* - \phi(x_i)^T y_i\right) + \frac{2\lambda}{2} w^* = 0$$

$$\sum_{i=1}^{m} \phi(x_i)^T \phi(x_i) w^* + \frac{\lambda m}{2} w^* = \sum_{i=1}^{m} \phi(x_i)^T y_i$$

$$w^* = \left(\phi(X)^T \phi(X) + \frac{\lambda m}{2} I\right)^{-1} \phi(X)^T y$$

b) $$\lim_{\lambda \to 0} w^*_R = \lim_{\lambda \to 0} \left(\phi(X)^T \phi(X) + \frac{\lambda m}{2} I\right)^{-1} \phi(X)^T y$$

$$= \left(\phi(X)^T \phi(X)\right)^{-1} \phi(X)^T y$$

$$= w^*_L$$

2.4) a)

$$\min_{w \in \mathbb{R}^n} C \sum_{i=1}^{m} \ell(w^T \phi(x_i), y_i) + \frac{1}{2} \|w\|^2$$

Differentiating objective function wrt w and equating to zero.

$$\frac{\partial}{\partial w} \left[C \sum_{i=1}^{m} \ell(w^T \phi(x_i), y_i) + \frac{1}{2} \|w\|^2 \right] = 0$$

$$C \sum_{i=1}^{m} \frac{\partial \ell(w^T \phi(x_i), y_i)}{\partial w} \cdot \phi(x_i) + w = 0$$

Let $\alpha_i = -C \dfrac{\partial \ell(w^T \phi(x_i), y_i)}{\partial w}$

$$\therefore \; w = \sum_{i=1}^{m} -C \frac{\partial \ell(w^T \phi(x_i), y_i)}{\partial w} \cdot \phi(x_i)$$

$$w = \sum_{i=1}^{m} \alpha_i \phi(x_i)$$

2.4) b)

$$\min_{w \in \mathbb{R}^n} C \sum_{i=1}^{m} \ell(w^T \phi(x_i), y_i) + \frac{1}{2} \|w\|^2$$

$$\min_{w \in \mathbb{R}^n} C \sum_{i=1}^{m} \ell\left(\sum_{j=1}^{m} \alpha_j \phi(x_j)^T \phi(x_i), y_i \right) + \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i \alpha_j \phi(x_j)^T \phi(x_i)$$

3.1) a) $A_{2\times 2} = \begin{bmatrix} 2 & 0 & -3 & 0 \\ 0 & -1/\sqrt{2} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1/\sqrt{2} \\ -3 & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$

$$= \begin{bmatrix} 13 & 0 \\ 0 & 1 \end{bmatrix}$$

Clearly 13 & 1 are eigen values.

$$\begin{bmatrix} 13 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 13 \begin{bmatrix} x \\ y \end{bmatrix}$$

$\therefore x = 1, y = 0$

$\therefore$ Principal component $= \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

b) $\therefore [2 . 0] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2$

3-2) a) $\text{Min} \sum_{i=1}^{m} \| x_i - UWx_i \|_2^2$

$U \in \mathbb{R}^{n \times 1}$

$W \in \mathbb{R}^{1 \times n}$

Encoding $\rightarrow Wx$

Decoding $\rightarrow UWx$

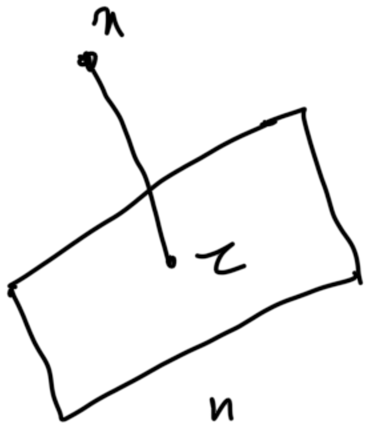
b) c)

Let $f(y) = \|x - Vy\|^2$

$= (x - Vy)^T (x - Vy)$

$= x^T x + y^T V^T V y - 2 x^T V y$

$\frac{\partial f}{\partial y} = 0 \Rightarrow y = V^T x$



$\exists y \in \mathbb{R}^n$ st

$\underset{n \times 1}{Z} = Vy$

$\sum_{j=1}^{N} \| x_j - VV^T x_j \| \leq \sum_{j=1}^{N} \| x_j - UWx_j \|$

$\text{Min} \sum_{j=1}^{N} \| x_j - UWx_j \|^2$

$\Downarrow$

$\text{Min} \sum_{j=1}^{N} \| x_j - UU^T x_j \|^2$

$U^T U = I_{1 \times 1}$

d) e) f) $\| x_j - UU^T x_j \| = (x_j - UU^T x_j)^T (x_j - UU^T x_j)$

$$= \|x\|^2 + x^T U \underbrace{U^T U}_{I} U^T x - 2x^T UU^T x$$

$$= \|x\|^2 - x^T UU^T x$$

$$= \|x\|^2 - \text{Trace}(U^T x x^T U)$$

$$\therefore \sum_{j=1}^{m} \| x_j - UU^T x_j \|^2 = \sum_{j=1}^{m} \|x_j\|^2 - \sum_{j=1}^{m} \text{Trace}(U^T x x^T U)$$

$$= \sum_{j=1}^{m} \|x_j\|^2 - \text{Trace}\left(U^T \sum_{j=1}^{m} x_j x_j^T U\right)$$

$$= \sum_{j=1}^{m} \|x_j\|^2 - \text{Trace}(U^T A U)$$

$$\underset{n\times n}{A} = XX^T \qquad X = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_m \\ | & | & & | \end{bmatrix}_{n\times m}$$

$$\therefore \max_{U^T U = I} U^T A U$$

$$L = U^T A U - \lambda (U^T U - I_k)$$

$$\frac{\partial L}{\partial U} = 0 \Rightarrow 2AU - 2\lambda U = 0$$

$$\Rightarrow AU = \lambda U$$

$\therefore$ U is eigen vector of A

Let $\lambda_1 \geq \lambda_2 \geq \lambda_3 \ldots\ldots \geq 0$ be eigen values of A and let $u_1, u_2, \ldots u_n$ be corresponding eigen vectors.

Since we are reducing from n to 1. We take highest eigen vector ie u_1.

$\therefore$ $U = [u_1]$ $\rightarrow$ optimal soln

Encoded data $\rightarrow U^T X$

Decoded data $\rightarrow U U^T X$

Optimal encoder $\rightarrow u_1 = U$

4.1) $\quad \text{Min } \frac{1}{2}\|w\|^2$

$\text{s.t. } \quad y_i w^T \phi(x_i) \geq 1 \qquad \forall i = 1, \ldots m$

$\Rightarrow \text{Min } \frac{1}{2} w^T w$

$\text{s.t. } \quad y_i w^T \phi(x_i) \geq 1 \qquad \forall i = 1 \ldots m$

$$\|w\|^2 = w^T w = \left(\sum_{j=1}^{m} \alpha_j \phi(x_j)^T \right) \left(\sum_{j=1}^{m} \alpha_j \phi(x_j) \right)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i \alpha_j \phi(x_j)^T \phi(x_i)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i \alpha_j \langle \phi(x_j), \phi(x_i) \rangle$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i \alpha_j G_{ij} = \alpha^T G \alpha$$

$\Rightarrow \text{Min } \frac{1}{2} \alpha^T G \alpha$

$\text{s.t. } \quad y_j \sum_{i=1}^{m} \alpha_i \phi(x_i)^T \phi(x_j) \geq 1 \qquad \forall j = 1 \ldots m$

4.2) a) $\phi(x_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\phi(x_2) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$$G = \begin{bmatrix} [1 \ 1]\begin{bmatrix} 1 \\ 1 \end{bmatrix} & [1 \ 1]\begin{bmatrix} 2 \\ 2 \end{bmatrix} \\ [2 \ 2]\begin{bmatrix} 1 \\ 1 \end{bmatrix} & [2 \cdot 2]\begin{bmatrix} 2 \\ 2 \end{bmatrix} \end{bmatrix}$$

$$G = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$$

$1(\alpha_1 \phi(x_1)^T \phi(x_1) + \alpha_2 \phi(x_2)^T \phi(x_1)) \geq 1 \Rightarrow 2\alpha_1 + 4\alpha_2 \geq 1 \rightarrow 1$

$-1(\alpha_1 \phi(x_1)^T \phi(x_2) + \alpha_2 \phi(x_2)^T \phi(x_2)) \geq 1 \Rightarrow 4\alpha_1 + 8\alpha_2 \leq -1 \rightarrow 2$

From 1 & 2 you can see there's no such α_1 & α_2 satisfying 1 & 2.

Hence no solution i.e no $\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$

4.2) b) $\phi(x_1) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\phi(x_2) = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$

$$G = \begin{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 5 & 9 \end{bmatrix}$$

$$1 \cdot \left(\alpha_1 \phi(x_1)^T \phi(x_1) + \alpha_2 \phi(x_2)^T \phi(x_1)\right) \geq 1 \Rightarrow 3\alpha_1 + 5\alpha_2 \geq 1$$

$$-1 \left(\alpha_1 \phi(x_1)^T \phi(x_2) + \alpha_2 \phi(x_2)^T \phi(x_2)\right) \geq 1 \Rightarrow 5\alpha_1 + 9\alpha_2 \leq -1$$

$\therefore$ Min $\frac{1}{2} \alpha^T G \alpha$ $\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$

s.t. $C_1^T \alpha \geq 1$ $C_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$C_2^T \alpha \leq -1$

$C_2 = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$

KKT condns :-

① $G\alpha - \lambda_1 C_1 + \lambda_2 C_2 = 0$

② $\lambda_1 \left(1 - C_1^T \alpha\right) = 0$

$\lambda_2 \left(1 + C_2^T \alpha\right) = 0$

Take $\lambda_1 = 7$, $\lambda_2 = 4 \Rightarrow C_1^T \alpha = 1$, $C_2^T \alpha = 1$

we get $\alpha_1 = 7$, $\alpha_2 = -4$

$$\therefore \alpha = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$$

4.3) a)b) $\phi(x_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\phi(x_2) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\phi(x_3) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\phi(x_4) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$$\text{Min } \frac{1}{2} \|w\|^2$$

$$\text{s.t } y_i w^T \phi(x_i) \geq 1 \quad \forall\ i = 1, 2, 3, 4$$

$$\Rightarrow \text{Min } \frac{1}{2} \|w\|^2$$

$$\text{s.t. } w_1 \geq 1$$

$$-w_2 \geq 1$$

KKT condns

① $w - \lambda_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ +1 \\ 0 \end{bmatrix} = 0$

② $\lambda_1 [1 - w_1] = 0$

$\lambda_2 [1 + w_2] = 0$

Keeping $\lambda_1 = \lambda_2 = 1$ we get $w_1 = 1$ $w_2 = -1$ $w_3 = 0$

$\therefore w = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

c) $g(x) = w^T \phi(x)$

$$= \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}^T \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1 x_2 \end{bmatrix}$$

$$= x_1^2 - x_2^2$$

4.4) Let point 'a' lie on $w^T \phi(x) = 1$

point 'b' lie on $w^T \phi(x) = -1$

$\therefore\ w^T a = 1 \quad w^T b = -1$

$w^T (a-b) = 2$

We know that we can reach point a to b by moving in direction of w.

$\therefore\ a + tw = b$

$\therefore\ w^T(-tw) = 2 \Rightarrow -t\,\|w\|^2 = 2 \Rightarrow t = \frac{-2}{\|w\|^2}$

$\therefore\ \|a-b\| = \|-tw\| = |t|\,\|w\| = \frac{2}{\|w\|^2}\|w\| = \frac{2}{\|w\|}$

4.5) $\text{Min } \frac{1}{2}\|w\|^2$ Considering $\phi(x) = x$

s.t $y_i w^T \phi(x_i) \geq 1 \quad \forall i = 1,2,3$

$$\begin{cases} 1(w^T)3 \geq 1 \\ 1(w^T)1 \geq 1 \Rightarrow w^T \geq 1 \\ -1(w^T)(-1) \geq 1 \end{cases}$$

$\downarrow$
Intersection

$\Rightarrow \text{Min } \frac{1}{2}\|w\|^2$

s.t. $w^T \geq 1$

Clearly $w^* = 1$

$\therefore f(x) = x$

$\therefore f(2) = 2 > 0 \quad \therefore y(2) = +1$

4.6) a) Here 0 lies in between class 1 & -1 which is element of class 1. A contradiction. Hence data is not linearly seperable.

b) $\phi(x_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\phi(x_2) = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$ $\phi(x_3) = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$

Yes Data is linearly seperable.

$$\min \frac{1}{2} \|w\|^2$$

s.t $y_i w^T \phi(x_i) \geq 1$

$$\Rightarrow \min \frac{1}{2} \|w\|^2$$

s.t. $w_1 \geq 1$ $\Rightarrow w^T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \geq 1$

$-w_1 + \sqrt{2} w_2 - w_3 \geq 1 \Rightarrow w^T \begin{bmatrix} -1 \\ \sqrt{2} \\ -1 \end{bmatrix} \geq 1$

$-w_1 - \sqrt{2} w_2 - w_3 \geq 1 \Rightarrow w^T \begin{bmatrix} -1 \\ -\sqrt{2} \\ -1 \end{bmatrix} \geq 1$

KKT condns:-

① $w^* - \lambda_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \lambda_2 \begin{bmatrix} -1 \\ \sqrt{2} \\ -1 \end{bmatrix} - \lambda_3 \begin{bmatrix} -1 \\ -\sqrt{2} \\ -1 \end{bmatrix} = 0$

② $\lambda_1 \left[1 - w^* \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right] = 0$

$\lambda_3 \left[1 - w^* \begin{bmatrix} -1 \\ -\sqrt{2} \\ -1 \end{bmatrix} \right] = 0$

$\lambda_2 \left[1 - w^* \begin{bmatrix} -1 \\ \sqrt{2} \\ -1 \end{bmatrix} \right] = 0$

For $\lambda_1 = 3 \quad \lambda_2 = \lambda_3 = 1$

$$w^* = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \Rightarrow f(x) = w^{*T}\phi(x)$$

if $f(x) > 0$ then $y = +1$

$f(x) < 0$ then $y = -1$

5-i) a) $K(x,y) = xy = \langle \phi(x), \phi(y) \rangle$

$$\therefore \phi(x) = x$$

b) $K(x,y) = 1 + xy + (xy)^2 + (xy)^3$

$$= \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \begin{bmatrix} 1 \\ y \\ y^2 \\ y^3 \end{bmatrix} = \langle \phi(x), \phi(y) \rangle$$

$$= \phi(x)^T \phi(y)$$

$$\therefore \phi(x) = \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{bmatrix}$$

c) $K(x,y) = x^T y = \langle \phi(x), \phi(y) \rangle = \phi(x)^T \phi(y)$

$\therefore \phi(x) = x$

e) $x^2y^2 + 2xy + 1 = x^2y^2 + \sqrt{2}x \cdot \sqrt{2}y + 1 = K(x,y)$

$$= \begin{bmatrix} x^2 & \sqrt{2}x & 1 \end{bmatrix} \begin{bmatrix} y^2 \\ \sqrt{2}y \\ 1 \end{bmatrix} = \phi(x)^T \phi(y)$$

$$\therefore \phi(x) = \begin{bmatrix} x^2 \\ \sqrt{2}x \\ 1 \end{bmatrix}$$

f) $K(x,y) = x^T A y = x^T \sqrt{A} \cdot \sqrt{A}\, y = x^T B B y = (Bx)^T (By)$

$\Rightarrow K(x,y) = (\sqrt{A}x)^T (\sqrt{A}y) = \phi(x)^T \phi(y)$

$\therefore \phi(x) = \sqrt{A}\, x$

g) $K(x,y) = c K_1(x,y) = \sqrt{c}\, \phi_1(x)^T \sqrt{c}\, \phi_1(y)$

$$= \left(\sqrt{c}\, \phi_1(x)\right)^T \left(\sqrt{c}\, \phi_1(y)\right)$$

$$= \phi(x)^T \phi(y)$$

$\therefore \phi(x) = \sqrt{c}\, \phi_1(x)$

5.2) $G_{ij} = \langle \phi(x_i), \phi(x_j) \rangle = K(x_i, x_j)$

$$x^T G x = \sum_{i=1}^{m} \sum_{j=1}^{m} x_i G_{ij} x_j = \sum_{i=1}^{m} \sum_{j=1}^{m} x_i \langle \phi(x_i), \phi(x_j) \rangle x_j$$

$$= \sum_{i=1}^{m} x_i \langle \phi(x_i), \sum_{j=1}^{m} \phi(x_j) x_j \rangle$$

$$= \langle \sum_{i=1}^{m} \phi(x_i) x_i , \sum_{j=1}^{m} \phi(x_j) x_j \rangle$$

$$= \langle X, X \rangle = \|X\|^2 \geq 0$$

$x^T G x = 0 \Rightarrow \sum \phi(x_i) x_i = 0 \Rightarrow x_i = 0 \ \forall i$

$\therefore x^T G x > 0$ for all x non zero vector

$\therefore$ Hence Positive Definite.

5.3) $$C \sum_{i=1}^{m} l\left(w^T \phi(x_i), y_i\right) + \frac{1}{2} w^T w$$

$$\Rightarrow C \sum_{i=1}^{m} l\left(\sum_{j=1}^{m} \alpha_i \phi(x_j)^T \phi(x_i), y_i\right) + \frac{1}{2}\left(\sum_{i=1}^{m} \alpha_i \phi(x_i)^T\right)\left(\sum_{j=1}^{m} \alpha_j \phi(x_j)\right)$$

$$K(x_i, x_j) = \langle \phi(x_i), \phi(x_j) \rangle = \phi(x_i)^T \phi(x_j)$$

$$\Rightarrow C \sum_{i=1}^{m} l\left(\sum_{j=1}^{m} \alpha_i K(x_j, x_i), y_i\right) + \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i \alpha_j K(x_i, x_j)$$

5.4) a) Kernilized Ridge Regression:-

$$\min \frac{1}{m}\sum_{i=1}^{m}\left(\sum_{j=1}^{m}\alpha_j K(x_j,x_i)-y_i\right)^2+\frac{\lambda}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\alpha_i\alpha_j K(x_i,x_j)$$

b) Kernilized logistic Regression:-

$$\min \frac{1}{m}\sum_{i=1}^{m}\log\left(1+e^{-y_i\sum_{j=1}^{m}\alpha_j K(x_j,x_i)}\right)+\frac{\lambda}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\alpha_i\alpha_j K(x_i,x_j)$$

c) Kernilized SVM:-

$$\min \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\alpha_i\alpha_j y_i y_j K(x_i,x_j)$$