Karthik Kurugodu MA20BTECH11008

1) Apply Euler cauchy method with step size 'h' to IVP

$$y' = -y \quad y(0)=1$$

Now determine an explicit expression y_n.

For what values of 'h', the sequence $\{y_n\}$ is bounded.

<u>Sol</u>:-

$$y_{n+1} = y_n + \frac{h}{2}\left(f(x_n, y_n) + f(x_{n+1}, y_{n+1}) \right)$$

$$y_{n+1} = y_n + \frac{h}{2}\left(-y_n + f(y_n + h(-y_n))\right)$$

$$y_{n+1} = y_n + \frac{h}{2}\left(-y_n + f(x_n, y_n - hy_n)\right)$$

$$y_{n+1} = y_n + \frac{h}{2}\left(-y_n + hy_n - y_n\right)$$

$$y_{n+1} = y_n - \frac{hy_n}{2} + \frac{h^2 y_n}{2} - \frac{hy_n}{2}$$

$$\boxed{y_{n+1} = y_n - hy_n + \frac{h^2 y_n}{2}}$$

To be bounded $\{y_n\}$

$$\left|\frac{y_{n+1}}{y_n}\right| \le 1$$

$$\left|\frac{y_n - hy_n + \frac{h^2 y_n}{2}}{y_n}\right| < 1$$

$$\left|1 - h + \frac{h^2}{2}\right| \le 1$$

$$1 - h + \frac{h^2}{2} \le 1 \qquad h - 1 - \frac{h^2}{2} \le 1$$

$$2 - 2h + h^2 - 2 \le 0 \qquad h^2 - 2h + 4 \ge 0$$

$$h(h-2) \le 0 \qquad (h-1)^2 + 3 \ge 0$$

↳ This is always true ~~for~~ any h

$$h \in [0, 2]$$

2) Find the implicit RK method of the form

$$y_{n+1} = y_n + h(w_1 k_1 + w_2 k_2),$$

where $k_1 = f(y_n)$, $k_2 = f(y_n + ah(k_1 + k_2))$ for soln of IVP $y' = f(y)$, $y(x_0) = y_0$. Therefore obtain interval of absolute stability when method is applied on $y' = \lambda y$, $\lambda < 0$.

$\therefore \phi(x_n, y_n, 0) = f(y_n) \rightarrow$ For consistency

$$w_1 k_1 + w_2 k_2 = f(y_n)$$

$$w_1 + w_2 = 1$$

Truncation error: $T_n = \dfrac{y(x_{n+1}) - y(x_n)}{h} - \phi(x_n, y(x_n), h)$

$$\frac{y(x_{n+1}) - y(x_n)}{h} = \frac{1}{h}\left[\cancel{y(x_n)} + hy'(x_n) + \frac{h^2}{2!}y''(x_n) + \frac{h^3}{3!}y'''(x_n) + O(h^4) - y(x_n)\right]$$

$$= y'(x_n) + \frac{h}{2!}y''(x_n) + \frac{h^2}{3!}y'''(x_n) + O(h^3)$$

$$k_1 = \lambda y_n$$

$$k_2 = \lambda\left(y_n + ah(k_1 + k_2)\right)$$

$$k_2 = \lambda\left(y_n + ah(\lambda y_n + k_2)\right)$$

$$k_2 = \lambda y_n + ah\lambda^2 y_n + \lambda k_2 ah$$

$$k_2 = \lambda y_n + ah\lambda^2 y_n + \lambda ah\left(\lambda y_n + ah\lambda^2 y_n + \lambda k_2 ah\right)$$

$$k_2 = \lambda y_n + 2\lambda^2 ahy_n + \lambda^3 a^2 h^2 y_n + \lambda^2 a^2 h^2 k_2$$

$$k_2 = \lambda y_n + 2\lambda^2 ahy_n + 2\lambda^3 a^2 h^2 y_n + h^3 2\lambda^4 a^3 y_n + \ldots$$

$$k_2 = \lambda y_n + 2\lambda^2 ahy_n + 2\lambda^3 a^2 h^2 y_n + O(h^3)$$

$$\phi(x_n, y_n, h) = w_1\lambda y_n + w_2\lambda y_n + w_2 2\lambda^2 ahy_n + w_2 2\lambda^3 a^2 h^2 y_n + O(h^3)$$

$$\frac{y(x_{n+1}) - y(x_n)}{h} = \lambda y_n + \frac{\lambda^2 y_n h}{2} + \frac{\lambda^3 y_n h^2}{6} + O(h^3)$$

Comparing terms of $\phi(x_n, y_n, h)$ & $\frac{y(x_{n+1}) - y(x_n)}{h}$

$$\lambda = \lambda(w_1 + w_2) \Rightarrow w_1 + w_2 = 1$$

$$\frac{\lambda^2}{2} = 2\lambda^2 a w_2 \Rightarrow w_2 a = \frac{1}{4}$$

$$\frac{\lambda^3}{6} = 2\lambda^3 w_2 a^2 \Rightarrow w_2 a^2 = \frac{1}{12}$$

$$\Rightarrow a = \frac{1}{3} \Rightarrow w_2 = \frac{3}{4} \Rightarrow w_1 = \frac{1}{4}$$

$$\therefore y_{n+1} = y_n + h\left(\frac{k_1}{4} + \frac{3k_2}{4}\right)$$

$$y_{n+1} = y_n + \frac{h}{4}\left(\lambda y_n + 3\left(\frac{3\lambda y_n + \lambda^2 h y_n}{3 - \lambda h}\right)\right)$$

$$y_{n+1} = \left(1 + \frac{h}{4}\left(\lambda + \left(\frac{9\lambda + 3\lambda^2 h}{3 - \lambda h}\right)\right)\right) y_n$$

$$\therefore E(\lambda h) = 1 + \frac{h}{4}\left(\frac{12\lambda + 2\lambda^2 h}{3 - \lambda h}\right)$$

absolute
For stability $|E(\lambda h)| < 1$

$$\left|1+\frac{h}{4}\left(\frac{12\lambda+2\lambda^2 h}{3-\lambda h}\right)\right| < 1$$

$$1+\frac{h}{4}\left(\frac{12\lambda+2\lambda^2 h}{3-\lambda h}\right) < 1 \qquad -1 < \frac{h}{4}\left(\frac{12\lambda+2\lambda^2 h}{3-\lambda h}\right) < 1$$

$$\cancel{\frac{12+h}{4}\left(\frac{8\lambda+\lambda^2 h}{3-\lambda h}\right) < 0}$$

$$\Rightarrow \lambda^2 h^2 + 2\lambda h + 12 > 0$$

$$\frac{\lambda h(2\lambda h+12)}{3-\lambda h} < 0 \qquad \Rightarrow (\lambda h+1)^2 + 11 > 0$$

This is true for every h

$$\Rightarrow \lambda h \in (-6, 0)$$

$$\therefore \lambda h \in (-6, 0)$$