भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

First Lecture on Numerical Methods for Solving ODEs

(MA-5363)

Motivation:

ordinary differential equations frequently occur in mathematical models that arises in many branches of Science, engineering and economics. However there are many ordinary differential equations (variable coefficients and nonlinear), which cannot be solved exactly using analytical techniques. So, for solving these ODES, instead of exact solutions, we can look for approximate solutions. To get these approximate solutions, we can consider numerical methods.

In this Section, we are more focused to Solve the initial value problem (IVP) by using numerical methods.

IVP:

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0, \quad x \in [x_0, b] \quad \text{--- (1)}$$

where $f(x,y)$ is real valued function of two variable.

To get the solution from the IVP ①, first we need to know that whether the solution exists or not and if the solution exists, whether the solution is unique or not.

To verify that we need to recall the

Picard's Existence and uniqueness Theorem

I already discussed in the ODE course.

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0, \quad x \in [x_0, b] \qquad \text{--- (1)}$$

Assumption: $f(x,y)$ satisfies the conditions of Picard's Theorem:

(i) Suppose $f(x,y)$ is continuous function on a rectangular domain $D: |x-x_0| \le a, \; |y-y_0| \le c$

and $|f(x,y)| \le M$.

(ii) $f(x,y)$ is Lipschitz continuous w.r.t y i.e, $f(x,y)$ satisfies the Lipschitz condition

$$|f(x,y_2) - f(x,y_1)| \le L\,|y_2 - y_1|,$$

where L is Lipschitz Constant and $(x, y_1), (x, y_2)$ $\in D$. Then the solution to the IVP (1) exists and it is unique. Therefore this solution is defined atleast for all x in the interval $I: |x - x_0| < h$, where $h = \min\{a, b/M\}$

So by picard's existence and uniqueness theorem we can check the existence and uniqueness of the solution to the IVP.

Now we are proceeding to get the approximate solution by using the semi analytical technique.

Semi analytical method :

Picard's method of successive approximation :

$$\frac{dy}{dx} = f(x, y) \quad \text{--- (1)}$$

Integrating (1) from x_0 to x, we have or $y(x) - y(x_0) = \int_{x_0}^{x} f(x,y)\,dx$

or $\boxed{y(x) = y(x_0) + \int_{x_0}^{x} f(x,y)\,dx}$ — 2

To solve 2, we can consider an iterative method to get the approximate soln. Now we will find out a sequence of approximation to the soln of IVP.

Let $y^{(0)}(x) = y(x_0) = y_0$ be an initial approximation to the solution of the IVP.

we have

$$y(x) = y(x_0) + \int_{x_0}^{x} f(x, y^{(0)}(x))\, dx$$

Now we consider the approximate solution $y(x)$ as first picard's approximation $y^{(1)}(x)$.

So $$\boxed{y^{(1)}(x) = y(x_0) + \int_{x_0}^{x} f(x, y^{(0)}(x))\, dx. \quad \text{--- (3)}}$$

Now use $y^{(1)}(x)$ to get second approximation

$y^{(2)}(x)$ as

$$\boxed{y^{(2)}(x) = y(x_0) + \int_{x_0}^{x} f(x, y^{(1)}(x))\, dx \quad \text{--- (4)}}$$

Here $y^{(2)}(x)$ is better approximation to $y(x)$ than $y^{(1)}(x)$.

Similarly we can find next approximations.

Now the n^{th} approximation is obtained as

$$\boxed{y^{(n)}(x) = y(x_0) + \int_{x_0}^{x} f\left(x, y^{(n-1)}(x)\right) dx} \quad \text{--- (5)}$$

we obtain the sequence of Picard's Successive iteration as $\{y^{(n)}\}_{n=1}^{\infty}$. If $\{y^{(n)}\}_{n=1}^{\infty}$ converges as $n \to \infty$ and it converges to the solution $y(x)$ of IVP.

(*) Picard's Theorem ensures that the solution to the IVP exists and unique and converges to the exact solution $y(x)$.

Example: $\frac{dy}{dx} = 1+y^2$, $y(0)=0$. Find $y(0.1)$.

Solⁿ: Picard's nth approximation

$$y^{(n)}(x) = y(x_0) + \int_{x_0}^{x} f\left(x, y^{(n-1)}(x)\right) dx$$

Here $f(x,y) = 1+y^2$, $x_0 = 0$, $y^{(0)}(x) = y(0) = 0$

n=1: $$y^{(1)}(x) = y(0) + \int_0^x \left(1 + y(0)^2\right) dx$$

$$= x.$$

$n=2$:

$$y^{(2)}(x) = y(0) + \int_0^x \left(1 + y^{(1)\,2}\right) dx$$

$$= \int_0^x (1+x^2)\, dx = x + x^3/3$$

$n=3$:

$$y^{(3)}(x) = y(0) + \int_0^x \left(1 + y^{(2)\,2}\right) dx$$

$$= \int_0^x \left[1 + \left(x + x^3/3\right)^2\right] dx$$

$$= x + x^3/3 + \frac{2}{15}x^5 + x^7/63$$

If we compute the higher approximations, we can see that it converges to the exact solution $\tan x$.

$$\boxed{y^{(n)}(x) \sim \tan x \quad \text{as } n \to \infty}$$

Now we can determine the value of $y(0.1)$ by using this approximations.

$$y(0.1) \approx y^{(3)}(0.1) = \underline{0.100335}.$$

Example: $y' = 1 + xy$, $y(0) = 1$. Solve it by using Picard's method upto third approximation. Find $y(0.2)$. (Home work!)

Question: When do we stop in this iterative process?

Ans: We need to check the accuracy of the approximated solutions.

If ε is prescribed, we need to check $\boxed{|y^{(k+1)}(x) - y^{(k)}(x)| < \varepsilon}$ at some given point $x \in [x_0, b]$ then we can say that the Picard's method converges upto assigned accuracy ε.

See the first example:

we have $y^{(1)}(x) = x$,

$y^{(2)}(x) = x + x^3/3$, $y^{(3)}(x) = x + x^3/3 + \dfrac{2x^5}{15} + x^7/63$.

Here we need to find the value $y(0.1)$. So we will take the point $x = 0.1$.

$$\left| y^{(2)}(0.1) - y^{(1)}(0.1) \right| = 0.0003$$

and $\left| y^{(3)}(0.1) - y^{(2)}(0.1) \right|$

$$= 1.33 \times 10^{-6}.$$

If ε is given as 10^{-5}. We can see that

$$\left| y^{(3)}(0.1) - y^{(2)}(0.1) \right| < 10^{-5}$$

So, we will stop at third approximation

$y^{(3)}(x)$ to the solution $y(x)$.

Now we will know the other numerical approaches to find the numerical solution to the IVP.

Numerical methods for Solving IVP

Single Step method

Multistep method

Single Step method

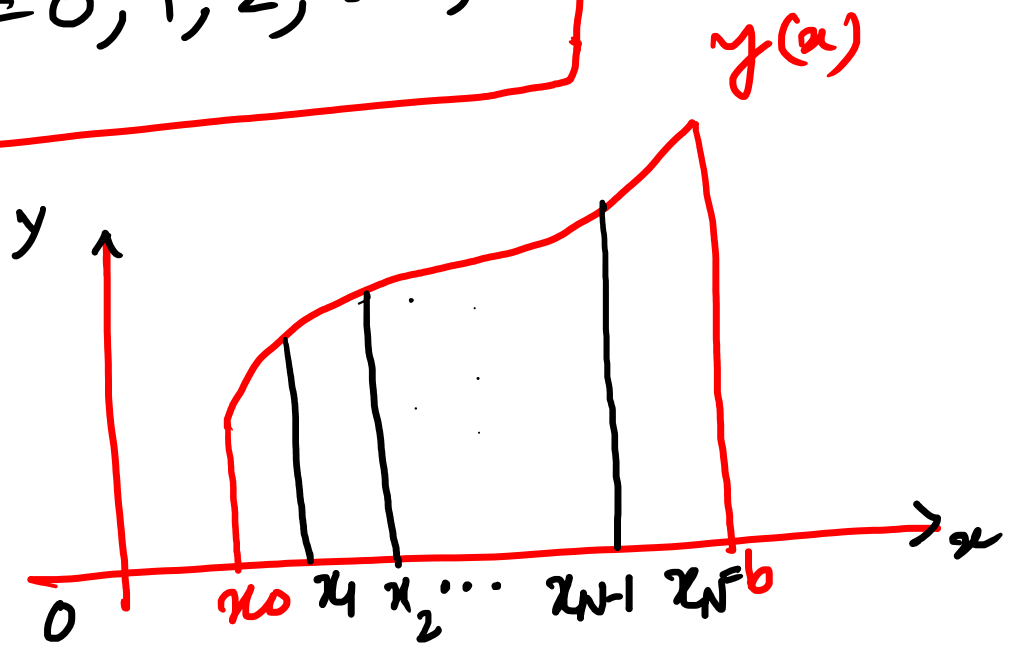
<u>Ivp</u>: $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$, $x \in [x_0, b]$ → $f(x,y)$ satisfies the Condition of Picard's Theorem.

Starting from the initial point x_0, we need the solution at different points in some interval $[x_0, b]$. We partition the interval $[x_0, b]$ into a finite N number of subintervals using the points $x_0, x_1, \ldots, x_N$ such that $x_0 < x_1 < x_2 \ldots < x_N = b$. These points are called the grid points or the mesh points.

We assume that the grid points are equispaced with spacing h, which is called the step size or the step length. Hence the grid points are defined by

$$\boxed{x_n = x_0 + nh, \quad n = 0, 1, 2, \ldots, N}$$

$$\underline{x_{n+1} = x_n + h.}$$



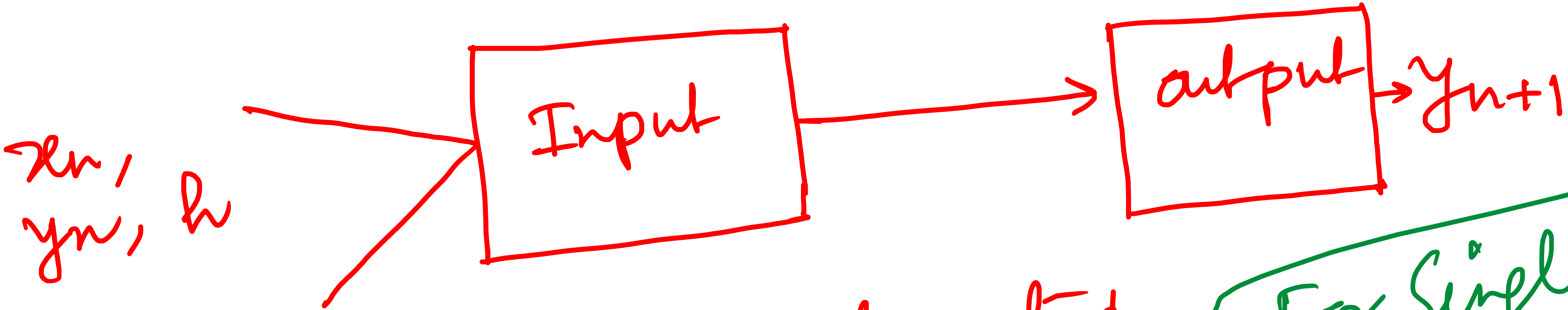
A general single step method can be written as

$$\boxed{y_{n+1} = y_n + h\,\phi(x_n, y_n; h)}$$

where y_{n+1} is the approximated value of the exact value $y(x_{n+1})$ and similarly $y(x_n) \approx y_n$.

$\phi(x_n, y_n; h)$ is the increment function.

Def^n: To get the solution at the grid point x_{n+1}, if the numerical method demands only one past value at the grid point x_n, then this method is called as single step method.



Single step method

Example:

(i) $y_{n+1} = y_n + x_n h$

(ii) $y_{n+1} = 2y_n - 3y_n'$

For Single step methods, y_{n+1} is expressed in terms of previous value y_n.

Single step method

Explicit method | Implicit method

- **Explicit single step method:**

$$\boxed{y_{n+1} = y_n + h\,\phi(x_n, y_n; h)}$$

here y_{n+1} is expressed in terms of previous value y_n.

- **Implicit single step method:**

$$\boxed{y_{n+1} = y_n + h\,\phi(x_n, x_{n+1}, y_{n+1}; h)}$$

Here y_{n+1} is expressed in terms of previous value y_n and present value y_{n+1}.

Here we will discuss the single step method. First I want to Taylor Series method.

Taylor Series method :

Taylor Series method is the fundamental numerical method for finding the approximated solution of the IVP $\frac{dy}{dx} = f(x,y),\ y(x_0) = y_0,$
$x \in [x_0, b]$.

Assumption :

(i) Existence and uniqueness : The IVP has a unique solⁿ $y(x)$ on $[x_0, b]$

$f(x,y)$ satisfies the conditions of Picard's Theorem

(ii) The solution $y(x)$ has continuous partial derivative of order $(p+1)$ (say) on $[x_0, b]$

$p \geq 1$.

Then the solution $y(x)$ of (1) can be expanded in a Taylor Series about any point say $x = x_0$ as follows —

$$y(x) = y(x_0) + (x-x_0)\,y'(x_0) + \frac{(x-x_0)^2}{2!}\,y''(x_0) + \cdots + \frac{(x-x_0)^p}{p!}\,y^{(p)}(x_0) + \frac{(x-x_0)^{p+1}}{(p+1)!}\,y^{(p+1)}(\xi),$$

$$x_0 < \xi < x.$$

Remainder term: Remainder term

$$= \frac{1}{p!}\int_{x_0}^{x} (x-t)^p\, y^{(p+1)}(t)\, dt.$$

Remainder term $= \frac{1}{p!}\int_{x_0}^{x}(x-t)^p\, y^{(p+1)}(t)\,dt$

$$= \frac{1}{p!}\, y^{(p+1)}(\xi)\int_{x_0}^{x}(x-t)^p\,dt = \frac{(x-x_0)^{p+1}}{(p+1)!}\, y^{(p+1)}(\xi)$$

[By Integral Mean value Theorem,

Let $g(x)$ be non negative and integrable on $[a,b]$ and let $f(x)$ be continuous on $[a,b]$. Then

$$\boxed{\int_a^b f(x)\,g(x)\,dx = f(\xi)\int_a^b g(x)\,dx}$$ for some $\xi \in [a,b]$]

Expanding $y(x)$ in Taylor Series about any point x_n

$$y(x) = y(x_n) + (x - x_n)\, y'(x_n) + \frac{(x-x_n)^2}{2!}\, y''(x_n) + \cdots$$

$$+ \frac{(x-x_n)^p}{p!}\, y^{(p)}(x_n) + \frac{(x-x_n)^{p+1}}{(p+1)!}\, y^{(p+1)}(\xi_n)$$

$$x_n < \xi_n < x.$$

Now at $x = x_{n+1} = x_n + h$

$$y(x_{n+1}) = y(x_n) + (x_{n+1} - x_n)\, y'(x_n) + \frac{(x_{n+1}-x_n)^2}{2!}\, y''(x_n)$$

$$+ \cdots + \frac{(x_{n+1}-x_n)^p}{p!}\, y^{(p)}(x_n) + \frac{(x_{n+1}-x_n)^p}{(p+1)!}\, y^{(p+1)}(\xi_n),$$

$n = 0, 1, \cdots N-1.$

$$\underline{x_{n+1} - x_n = h}, \quad n = 0, 1, 2, \ldots, N-1$$

$$y(x_{n+1}) = y(x_n) + h\,y'(x_n) + \frac{h^2}{2!}\,y''(x_n) + \cdots$$

$$+ \frac{h^p}{p!}\,y^{(p)}(x_n) + \frac{h^{p+1}}{(p+1)!}\,y^{(p+1)}(\xi_n)$$

$$= y(x_n) + h\,y'(x_n) + \frac{h^2}{2!}\,y''(x_n) + \cdots + \frac{h^p}{p!}\,y^{(p)}(x_n)$$

$$+ \epsilon_n, \qquad n = 0, 1, 2, \ldots N-1$$

where ε_n $= \dfrac{h^{p+1}}{(p+1)!} y^{(p+1)}(\xi_n)$ [Truncation error]

$$\boxed{y(x_{n+1}) = y(x_n) + h\,\phi(x_n, y(x_n); h) + \varepsilon_n}$$

where $\phi(x_n, y(x_n); h)$

$$= y'(x_n) + \frac{h}{2!} y''(x_n) + \cdots + \frac{h^{p-1}}{p!} y^{(p)}(x_n)$$

By ignoring the truncation error, we can get the approximated solution at x_{n+1}

$$y(x_{n+1}) \approx y_{n+1} = y_n + h y_n' + \frac{h^2}{2!} y_n'' + \cdots + \frac{h^p}{p!} y_n^{(p)}$$

$$\Rightarrow \boxed{y_{n+1} = y_n + h\, \phi(x_n, y_n; h)}, \text{ where}$$

$$\phi(x_n, y_n, h) = y_n' + \frac{h}{2!} y_n'' + \cdots + \frac{h^{p-1}}{p!} y_n^{(p)}$$

The Taylor Series Method is given by

$$y_{n+1} = y_n + h\,\phi(x_n, y_n; h), \quad n = 0, 1, \ldots, N-1$$

where $\phi(x_n, y_n, h)$ is the increment function and defined by —

$$\phi(x_n, y_n; h) = y_n' + \frac{h}{2!} y_n'' + \cdots + \frac{h^{p-1}}{p!} y_n^{(p)}$$

This is an explicit Single Step method.

Question : How do we compute the higher order derivatives of $y(x)$ at $x = x_n$?

The higher order derivatives have to be computed from the given differential eqⁿ $\boxed{y' = f(x,y).}$

To compute these, we need to assume that $f(x,y)$ is differentiable as many times as we require.

$$y'(x_n) \approx y_n' = f(x_n, y_n) \quad \left[\text{since } y' = f(x, y)\right]$$

$$y'' = \frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

$$= f_x + f\, f_y \,.$$

$$y_n'' = \left[f_x + f\, f_y\right]\Big|_{(x_n, y_n)}$$

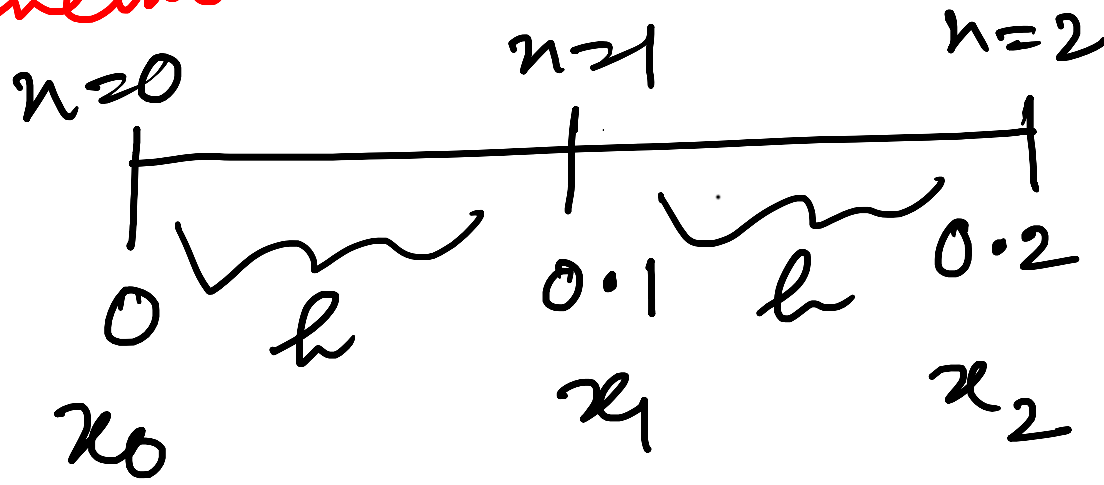
$$y''' = \frac{dy''}{dx} = \left(\frac{\partial}{\partial x} + f\frac{\partial}{\partial y}\right)\left(f_x + f f_y\right)$$

$$y''' = f_{xx} + 2 f f_{xy} + f^2 f_{yy} + f_y (f_x + f f_y)$$

$$y'''_n = \left[f_{xx} + 2 f f_{xy} + f^2 f_{yy} + f_y (f_x + f f_y) \right]_{(x_n, y_n)}$$

In this way, we can compute the higher order derivatives of $y(x)$ at x_n.

Example: Solve $y' = 3x + y^2$, $y(0) = -1$, to compute $y(0\cdot 2)$ using Taylor Series with $h = 0\cdot 1$. Consider only first three terms of the Taylor Series method.



Sol^n: $f(x,y) = 3x + y^2$.

$x_0 = 0$,

$y_0 = -1$, $h = 0\cdot 1$, $x_1 = 0\cdot 1$, $x_2 = 0\cdot 2$

Our aim is to find $y(x_2) = y(0\cdot 2) = \underline{y_2}$.

Taylor Series method with three terms

$$\boxed{y_{n+1} = y_n + h y_n' + \frac{h^2}{2!} y_n''}$$

$n=0$: $y_1 = y_0 + h y_0' + \frac{h^2}{2!} y_0''$

$y_0 = -1$, $h = 0.1$, $y_0' = ?$ $y_0'' = ?$

$y' = f(x,y) = 3x + y^2 \Rightarrow y_0' = 3x_0 + y_0^2$

$= 1$

$$y'' = \frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y} = 3 + 2y\left(3x + y^2\right)$$

$$y_0'' = 3 + 2y_0\left(3x_0 + y_0^2\right) = 1.$$

Now $y_1 = -1 + (0.1) \cdot 1 + \frac{(0.1)^2}{2!} \cdot 1$

$$= -0.895.$$

$\underline{n=1:}$ $\quad y_2 = y_1 + hy_1' + \dfrac{h^2}{2!} y_1''$

Here $y_1 = -0.895$, $y_1' = 1.1010$

$y_1'' = 1.4574.$

Now find y_2 ! (Try!)

Ex: $y' = -2xy^2$, $y(1) = 1$, obtain the approximate value of $y(1.3)$ for the given IVP using first three terms of the Taylor Series method with stepsize $h = 0.1$. Compare with the exact solution $y = \frac{1}{x^2}$.

Try!

Ans: $y(1.3) = 0.5967$. (approximated Solⁿ)

Exact Solⁿ: $y(1.3) = 0.5917$.

Error $= \underline{0.0050}$.

$$\boxed{\text{global error} = \text{Exact} - \text{Approx}}$$

Truncation Error :

Local Truncation Error

Global Truncation Error

Local Truncation error :

$$E_n = h\,T_n = \frac{h^{p+1}}{p!}\, y^{(p+1)}(x_n + \theta h)$$

$$0 < \theta < 1$$

where T_n is the local truncation error and it is defined by

$$T_n = \frac{h^p}{(p+1)!} y^{(p+1)}(x_n + \theta h)$$

Also we can write it from the Taylor Series expression as —

$$y(x_{n+1}) = y(x_n) + h\,\phi(x_n, y(x_n); h) + E_n$$

$$= y(x_n) + h\,\phi(x_n, y(x_n); h) + h\,T_n$$

or, $$\boxed{T_n = \frac{y(x_{n+1}) - y(x_n)}{h} - \phi(x_n, y(x_n); h)}$$

This is the concept of local Truncation error.

Global Truncation Error:

$$T = \max_{0 \leq n \leq N-1} |T_n|$$

or we can say in this way that accumulation of all truncation error will give the global truncation error.

⊘ Remark:

Q. How many number of terms should we consider in the Taylor Series method?

Ans: If we take more number of term in the Taylor Series method, we will get more accurate approximate solution.

Note: The truncation error is playing a vital role for getting more accurate, consistent and convergent numerical solution. We will see later in this section.

- If $\varepsilon > 0$ is a preassigned number to bound the truncation error, then $\left| h T_n \right| < \varepsilon$

as $\left| \dfrac{h^{p+1}}{(p+1)!} y^{(p+1)}(x_n + \theta h) \right| < \varepsilon$

as $\left| \dfrac{h^{p+1}}{(p+1)!} f^{(p)}(x_n + \theta h, y(x_n + \theta h)) \right| < \varepsilon \quad —(2)$

we assume that it is known.

(i) For a given h and ε, (2) will determine p, i.e. the number of terms in the Taylor Series method.

(ii) If p and ε are specified, then it will given an upper bound on h.

Bound on Truncation error:

$$|h T_n| \le \frac{h^{p+1}}{(p+1)!} M_{p+1}$$

where $M_{p+1} = \max\limits_{x_0 \le x \le b} |y^{(p+1)}(x)|$

Global Error : (Total Error)

In order to assess the accuracy of the numerical method, we define the global error

$$\boxed{\begin{aligned} e_n &= \text{Exact value} - \text{Approximate value} \\ &= y(x_n) - y_n. \end{aligned}}$$

Remember: This global error e_n is associated with all type of errors in the numerical method, i.e it is combined with truncation and round off errors.

Example: Given the IVP —

$$\frac{du}{dt} = t^2 + u^2, \quad u(0) = 0.$$

Determine the first three nonzero terms in the Taylor series for $u(t)$ and hence obtain the value for $u(1)$. Also determine t when the error in $u(t)$ obtained from the first two non-zero terms is to be less than 10^{-6} after rounding. (Homework!)

Ans: $u(t) = t^3/3 + \dfrac{t^7}{63} + \dfrac{2}{2079} t^{11}.$

$u(1) = 0.350168.$

$$\left| \frac{2}{2079} t^{11} \right| < 0.5 \times 10^{-7}$$

$$\Rightarrow t \simeq 0.41$$

(Try!)

Ex: Find the three term Taylor Series Solution for the third order IVP —

$$w''' + ww'' = 0, \quad w(0) = 0$$
$$w'(0) = 0, \quad w''(0) = 1.$$

Find the bound on the error for $t \in [0, 0.2]$

Ans: $w(t) = \dfrac{t^2}{2!} - \dfrac{t^5}{5!} + \dfrac{11}{8!} t^8 + T.E.$

where $|T.E.| \leq \max |w^{(9)}(t)| \dfrac{t^9}{9!}$

bound is $\underline{1.06 \times 10^{-11}}$

i.e $|T.E.| \leq 1.06 \times 10^{-11}$

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Second Lecture on Numerical Methods for Solving ODEs

(MA-5363)

Contents for this class

- ✓ General Single-Step Method
- ✓ Local Truncation Error
- ✓ Global Error
- ✓ Boundness of Global Error
- ✓ Consistency of Numerical Method
- ✓ Order of the Method
- ✓ Convergence of Numerical Solution

General Single Step method:

Generally a one-step method can be written as —

$$\boxed{y_{n+1} = y_n + h\,\phi(x_n, y_n ; h), \quad n=0,1,2,\dots,N-1}$$

with $y(x_0) = y_0$, $x_n = x_0 + nh$, $n=0,1,\dots,N$ and

where $y(x_n + h) = y(x_{n+1}) \approx y_{n+1}$, $y(x_n) \approx y_n$,

$\phi(x_n, y_n ; h)$ is an increment function.

Local Truncation Error:

We know from earlier studies on Taylor Series method (See my first class note), the local truncation error is defined by

$$T_n = \frac{h^p}{(p+1)!} y^{p+1}(\xi_n), \quad \text{where } x_n < \xi_n < x_{n+1}$$

Also one can write it as

$$\boxed{T_n = \frac{y(x_{n+1}) - y(x_n)}{h} - \phi\left(x_n, y(x_n); h\right)}$$

Global Error:

I already discussed this error in last classnote (please read).

The global error is defined by

$$e_n = \text{Exact value} - \text{Approximate value}$$
$$= y(x_n) - y_n$$

$$\therefore \boxed{e_n = y(x_n) - y_n}$$

Bound on Truncation Error:

$$|T_n| = \left| \frac{h^p y^{(p+1)}(\xi_n)}{(p+1)!} \right| \leq \frac{h^p}{(p+1)!} M_{p+1}$$

where $M_{p+1} = \max\limits_{\xi \in [x_0, b]} \left| y^{(p+1)}(\xi) \right|$

* The next theorem will give a bound on the magnitude of the global error in terms of truncation error.

Theorem: Consider a general single step method

$$\boxed{y_{n+1} = y_n + h\,\phi(x_n, y_n; h)}, \quad n = 0, 1, 2, \ldots, N-1, \quad y(x_0) = y_0,$$

where ϕ is continuous function of its arguments and ϕ is assumed to satisfy a Lipschitz condition with respect to its second argument i.e., $\boxed{|\phi(x, u; h) - \phi(x, v; h)| \le L_\phi |u - v|}$ for $0 \le h \le h_0$ and for all (x, u) and (x, v) in the rectangle

$$D = \{(x, y) : x_0 \le x \le b,\ |y - y_0| \le c\}$$

with Lipschitz constant L_ϕ.

Then we have

$$\boxed{|e_n| \le \frac{T}{L_\phi}\left(e^{L_\phi (x_n - x_0)} - 1\right), \quad n = 0, 1, 2, \ldots, N \qquad \text{where } T = \max_{0 \le n \le N-1} |T_n|}$$

Proof: we have

$$T_n = \frac{y(x_{n+1}) - y(x_n)}{h} - \phi(x_n, y(x_n); h)$$

or $$\boxed{y(x_{n+1}) = y(x_n) + h\,\phi(x_n, y(x_n); h) + h\,T_n} \quad \text{— (1)}$$

From the single step method, we have

$$\boxed{y_{n+1} = y_n + h\,\phi(x_n, y_n; h)} \quad \text{— (2)}$$

Substracting ② from ①, we have

$$y(x_{n+1}) - y_{n+1} = y(x_n) - y_n + h\left[\phi(x_n, y(x_n); h) - \phi(x_n, y_n; h)\right] + hT_n$$

$$\text{or, } e_{n+1} = e_n + h\left[\phi(x_n, y(x_n); h) - \phi(x_n, y_n; h)\right] + hT_n$$

$$\text{or, } |e_{n+1}| = \left|e_n + h\left[\phi(x_n, y(x_n); h) - \phi(x_n, y_n; h)\right] + hT_n\right|$$

$$\leq |e_n| + h\left|\phi(x_n, y(x_n); h) - \phi(x_n, y_n; h)\right| + |hT_n|$$

$$\leq |e_n| + hL_\phi|y(x_n) - y_n| + |hT_n|, \quad n = 0, 1, 2, \ldots, N-1$$

$$|e_{n+1}| \le |e_n| + hL_\phi |e_n| + hT, \quad \text{where } |T| = \max_{0\le n\le N-1} |T_n|$$

$$= |e_n|(1+hL_\phi) + hT,$$

$$\therefore \boxed{|e_{n+1}| \le |e_n|(1+hL_\phi) + hT, \quad n=0,1,2,\dots,N-1}$$

For $n=0$:

$$|e_1| \le |e_0|(1+hL_\phi) + hT.$$

Since $e_0 = y(x_0) - y_0 = 0$,

we have

$$|e_1| \le hT.$$

For $n=1$:

$$|e_2| \le |e_1|(1+hL_\varphi) + hT$$

$$\le hT(1+hL_\varphi) + hT$$

$$= 2hT + h^2 L_\varphi T$$

$$= \frac{T}{L_\varphi}\left[2hL_\varphi + h^2 L_\varphi^2 + 1 - 1\right]$$

$$\boxed{|e_2| \le \frac{T}{L_\varphi}\left[(1+hL_\varphi)^2 - 1\right]}$$

Similarly we have

$$\boxed{|e_n| \leq \frac{T}{L_\varphi}\left[(1+hL_\varphi)^n - 1\right]}, \quad n = 0, 1, 2, \dots N.$$

Since $(1+hL_\varphi) \leq e^{hL_\varphi}$, we have

$$|e_n| \leq \frac{T}{L_\varphi}\left[e^{nhL_\varphi} - 1\right],$$

$$x_n - x_0 = nh.$$

So we have

$$\boxed{|e_n| \le \frac{T}{L\varphi}\left[e^{L\varphi(x_n - x_0)} - 1\right]}$$

(Proved)

Example: Consider the IVP $y' = \tan^{-1} y$, $y(0) = y_0$, where y_0 is a given real number. Apply the Taylor Series method with first two terms and find the upper bound on the global error.

Sol^n: IVP: $y' = \tan^{-1} y$, $y(x_0) = y_0$.

Taylor Series method with first two terms:

$$y_{n+1} = y_n + h y_n', \quad \text{with error term} = \frac{h^2}{2!} y''(\xi_n)$$

$$x_n < \xi_n < x_{n+1}$$

Local Truncation error: $T_n = \frac{h^2}{2!} y''(\xi_n)$.

$$\Rightarrow \boxed{y_{n+1} = y_n + h\, f(x_n, y_n)}, \quad \text{Since } y' = f(x, y).$$

If we compare the general single step method, then we have $\phi(x_n, y_n; h) = f(x_n, y_n)$.

This is Euler method, we will know later about this method.

we know

$$\boxed{|e_n| \le \frac{T}{L_\phi}\left(e^{L_\phi (x_n - x_0)} - 1\right)}, \quad n = 0, 1, 2, \dots, N.$$

Global error $[y(x_n) - y_n]$

now $T_n = \frac{h}{2!} y''(\xi_n), \quad n = 0, 1, \dots, N-1$

$$|T_n| = \left| \frac{h}{2!} y''(\xi_n) \right| \leq \frac{h}{2} M_2$$

where $M_2 = \max\limits_{\xi \in [x_0, b]} |y''(\xi)|$

$\therefore |T_n| \leq T, \quad n = 0, 1, \ldots N-1$

where $\boxed{T = \frac{h}{2} M_2.}$

Remember since $\phi(x_n, y_n; h) = f(x_n, y_n)$,

$\boxed{L\varphi = L}$, where L is the Lipschitz constant for Lipschitz continuity of function $f(x,y)$ w.r.t. y.

Now the global error bound can be obtained as

$$\boxed{|e_n| \leq \frac{M_2}{2}\left[\frac{e^{L(x_n - x_0)} - 1}{L}\right] h}, \quad n = 0, 1, 2, \ldots, N$$

Now find M_2 and L from the given IVP.:

Here $f(x,y) = \tan^{-1} y$,

$$|f(x,u) - f(x,v)| = \left|\frac{\partial f}{\partial y}(x,\eta)(u-v)\right| \quad \text{[By Mean value Theorem]}$$

$$\left|f(x,u)-f(x,v)\right| = \left|\frac{\partial f}{\partial y}(x,\eta)\right|\left|u-v\right|,\ \text{where } u<\eta<v.$$

$$\left|\frac{\partial f}{\partial y}(x,y)\right| = \left|\frac{1}{1+y^2}\right| \leq 1 .$$

So $\boxed{\left|f(x,u)-f(x,v)\right| \leq \left|u-v\right|}$

$\Rightarrow \underline{L=1}$

To find M_2, we need to obtain a bound on $|y''|$.

$$y'' = \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\tan^{-1}y}{(1+y^2)}, \Rightarrow |y''(x)| \leq M_2$$

where $M_2 = \max_{\xi \in [x_0, b]} |y''(\xi)| = \pi/2$.

Now we have $|e_n| \leq \frac{M_2}{2}\left[\frac{e^{L(x_n - x_0)} - 1}{L}\right] h$

$\Rightarrow |e_n| \leq \pi/4 \, (e^{x_n} - 1) h, \quad n = 0, 1, \dots, N$

$$\boxed{|e_n| \le \frac{\pi}{4}\left(e^{x_n}-1\right)h, \quad n=0,1,\dots,N}$$

Now we want a specific tolerance to bound this error, i.e. if ε is a tolerance we have

$$\boxed{|e_n| \le \varepsilon}$$

To get the error which does not exceed the given tolerance ε, we need to choose such h satisfying –

$$\boxed{h \leq \frac{4}{\pi(e^{b}-1)}\,\varepsilon}$$

For such h, we have

$$|y(x_n) - y_n| = |e_n| \leq \varepsilon .$$

$n = 0, 1, \ldots, N.$

Thus, in this way, we can find the numerical solution to arbitrarily high accuracy by choosing a sufficiently small step size h.

(*) **Note:** A numerical result shows that this error estimate is rather pessimistic.

For example, Take $y_0 = 1$, $b = 1$, and $\varepsilon = 0.01$.

Now $|y(x_n) - y_n| \leq 0.01$, when $h \leq 0.0074$.

(Calculate from $h \leq \dfrac{4}{\pi(e^b - 1)} \times \varepsilon$)

Hence it will give $\boxed{N \geq 135.}$

(*) However, even with $N = 27$, we can achieve the error bounded by given $\varepsilon = 0.01$.

(*) So this error estimate has predicted the use of a stepsize which is five times smaller than is actually required.

Consistency :

Definition : The general single step is consistent with the given IVP if for any $\varepsilon > 0$, $\exists$ a positive $h(\varepsilon)$ such that

$$\boxed{|T_n| < \varepsilon \quad \text{for } 0 < h < h(\varepsilon) \text{ and any pair of points } (x_n, y(x_n)), (x_{n+1}, y(x_{n+1})) \text{ on any solution curve in } D.}$$

Observations from this definition: Truncation error $\to 0$ as $h \to 0$.

It also imply that

Global error $\to 0$ as $h \to 0$.

(*) For the general single step method, in the limit of $h \to 0$ and $n \to \infty$ with

$$\lim_{n\to\infty} x_n = x \in [x_0, b],$$ we have —

$$\boxed{\lim_{n\to\infty} T_n = y'(x) - \phi(x, y(x); 0)}$$

$$\left[\begin{aligned}
&\text{Since } T_n = \frac{y(x_{n+1}) - y(x_n)}{h} - \phi(x_n, y(x_n); h) \\
&\qquad = \frac{y(x_n+h) - y(x_n)}{h} - \phi(x_n, y(x_n); h) \\
&\text{by taking } h\to 0 \text{ and } n\to\infty \text{ and } \lim_{n\to\infty} x_n = \underline{x}.
\end{aligned}\right]$$

Now the single step method is consistent if and only if

$$\boxed{\phi(x, y; 0) = f(x, y)}$$

(Since for consistency of the numerical method, $T_n \to 0$ as $n \to \infty$ and $h \to 0$ and $y' = f(x, y)$)

This condition is also taken as the definition of <u>consistency</u>.

Order of a method :

Definition : The single step method is said to have order of accuracy p if p is the largest positive integer and for any sufficiently smooth solution curve $(x, y(x))$ in D of the IVP, $\exists$ constants K and h_0 such that

$$\boxed{|T_n| \le K h^p, \text{ for } 0 < h \le h_0 \text{ and for any pair of points } (x_n, y(x_n)) \text{ and } (x_{n+1}, y(x_{n+1})) \text{ on the solution curve.}}$$

Also one can say this def^n

$$\boxed{T_n \in O(h^p) \text{ as } h \to 0}$$

Example : Consider the Taylor Series method with three terms

$$y_{n+1} = y_n + h y_n' + \frac{h^2}{2!} y_n'' \text{, with error } \frac{h^3}{3!} y'''(\xi_n),$$

$x_n < \xi_n < x_{n+1}$

Here $T_n = \frac{h^2}{2!} y'''(\xi_n)$

$$|T_n| \le K h^2, \text{ where } K = \frac{1}{2} \max_{\xi \in [x_0, b]} |y'''(\xi)|$$

So this method is second order.

(*) Now, we want to state the theorem for convergence of a numerical solution obtained by single step method.

Convergence :

Theorem : Assume that the IVP satisfies the conditions of Picard's Theorem and also that its approximation generated from the single step method when $h \le h_0$ lies in the region D. Assume further that the function ϕ is continuous on $D \times [0, h_0]$, and satisfies the consistency condition $\phi(x, y; 0) = f(x, y)$ and the Lipschitz condition –

$$|\phi(x, u; h) - \phi(x, v; h)| \le L_\phi |u - v| \text{ on } D \times [0, h_0]$$

Then if successive approximation sequences $\{y_n\}$, generated by using the mesh points $x_n = x_0 + nh$, $n = 1, \ldots N$, are obtained from the single step method with successively smaller values of h (each h less than h_0), we have convergence of the numerical solution to the solution of the IVP, i.e., $\lim_{n\to\infty} y_n = y(x)$ as $x_n \to x \in [x_0, b]$ when $h \to 0$ and $n \to \infty$.

Proof: Suppose that $h = \dfrac{b - x_0}{N}$, where N is a positive integer.

We shall assume that N is sufficiently large so that $h \le h_0$.

Since $y(x_0) = y_0$ and therefore $e_0 = 0$.

Aim: Our aim is to show that

$$|y(x) - y_n| < \varepsilon \;, \quad n = 1, 2, \ldots, N.$$

$$|y(x) - y_n| = |y(x) - y(x_n) + (y(x_n) - y_n)|$$

$$\leq |y(x) - y(x_n)| + |y(x_n) - y_n|$$

Now in the limit of $h \to 0$ and $n \to \infty$ with $x_n \to x$ $\in [x_0, b]$, we have $\lim_{n\to\infty} y(x_n) = y(x)$, since $y(x)$ is continuous function on $[x_0, b]$.

So, the first term $|y(x) - y(x_n)|$ is very small when $h \to 0$ and $n \to \infty$.

Now see the second term

$|y(x_n) - y_n|$ and we have to make it very small when $h \to 0$ and $n \to \infty$ in order to get the convergence of the numerical solution y_n.

$$|y(x_n) - y_n| = |e_n| \leq \frac{T}{L_\phi}\left[e^{L_\phi(x_n - x_0)} - 1\right] \quad [\text{By theorem}]$$

where $T = \max\limits_{0 \leq n \leq N-1} |T_n|$

So we have

$$\boxed{\left| y(x_n) - y_n \right| \leq \frac{1}{L\varphi}\left[e^{L\varphi(b-x_0)} - 1 \right] \max_{0 \leq n \leq N-1} |T_n|}$$

Note that $\frac{1}{L\varphi}\left[e^{L\varphi(b-x_0)} - 1\right]$ is constant and it is independent of h and n.

So our next step to show $|T_n|$ is very small quantity when $h \to 0$ and $n \to \infty$

$$|T_n| = \left| \frac{y(x_{n+1}) - y(x_n)}{h} - \phi(x_n, y(x_n); h) \right|$$

$$= \left| \frac{y(x_n+h) - y(x_n)}{h} - \phi(x_n, y(x_n); 0) + \phi(x_n, y(x_n); 0) + \phi(x_n, y(x_n); h) \right|$$

By using the consistency condition $\phi(x_n, y_n; 0) = f(x_n, y_n)$

$$= \left| \left(\frac{y(x_n+h) - y(x_n)}{h} - f(x_n, y(x_n)) \right) + \left(\phi(x_n, y(x_n); 0) - \phi(x_n, y(x_n); h) \right) \right|$$

$$\frac{y(x_n+h) - y(x_n)}{h} - f(x_n, y_n)$$

$$= y'(\xi_n) - y'(x_n)$$ [By using the Mean Value Theorem]

where $\xi_n \in [x_n, x_{n+1}]$

By Picard's Theorem, y' is continuous on $[x_0, b]$ therefore it is uniformly continuous on this interval.

Hence for each $\varepsilon > 0$ $\exists$ $h_1(\varepsilon)$ such that

$$\boxed{\left| y'(\xi_n) - y'(x_n) \right| \le \varepsilon/2 \quad \text{for } h < h_1(\varepsilon), \quad n = 0, 1, 2, \dots, N-1}$$

Since ϕ is a continuous function on $D \times [0, h_0]$ and therefore it is uniformly continuous on $D \times [0, h_0]$.

Hence for each $\varepsilon > 0$ $\exists\, h_2(\varepsilon)$ such that

$$\left| \phi(x_n, y(x_n); 0) - \phi(x_n, y(x_n); h) \right|$$

$$\leq \varepsilon/2, \quad \text{for } h < h_2(\varepsilon), \; n = 0, 1, \ldots, N-1$$

Now Define $h(\varepsilon) = \min\{h_1(\varepsilon), h_2(\varepsilon)\}$.

we have $$|T_n| \leq |y'(\xi_n) - y'(x_n)| + |\phi(x_n, y(x_n); 0) - \phi(x_n, y(x_n); h)|$$

$$\leq \varepsilon/2 + \varepsilon/2 = \varepsilon, \text{ for } h < h(\varepsilon), \; n = 0, 1, \ldots, N-1.$$

$\therefore \ |T_n| \leq \varepsilon$, for $h < h(\varepsilon)$, $n = 0, 1, \ldots, N-1$

Therefore $|y(x_n) - y_n| \leq \varepsilon \cdot \dfrac{e^{L\varphi(b-x_0)} - 1}{L\varphi}$, for $h < h(\varepsilon)$,

Now $|y(x) - y_n| \leq |y(x) - y(x_n)| + \varepsilon \cdot \dfrac{e^{L\varphi(b-x_0)} - 1}{L\varphi}$

This term is also small when $h \to 0$ and $n \to \infty$.

This term can be made arbitrary small by letting $\varepsilon \to 0$.

Therefore in the limit of $h \to 0$ and $n \to \infty$ with $x_n \to x \in [x_0, b]$, we have

$$\boxed{\lim_{n \to \infty} y_n = y(x)}$$

(Proved)

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Third Lecture on Numerical Methods for Solving ODEs

(MA-5363)

Contents for this class

- ✓ Euler method
- ✓ Backward Euler method
- ✓ Modified Euler Method
- ✓ Trapezium Method
- ✓ Euler-Cauchy Method

Note: Taylor Series method can give you the best approximated solution. However, from application point of view, Taylor Series method has a major disadvantage. The method requires evaluation of derivatives of higher order for the function $f(x,y)$ of two variables x and y.

These higher order derivatives have to be obtained manually for each problem.

Therefore, we need to develope such methods which don't require the evaluation of higher order derivatives.

Now see such methods next.

Euler method:

If you consider the Taylor series method with two terms only i.e., first order Taylor series method, then we can get the Euler method.

The Euler method can be written as —

$$\boxed{y_{n+1} = y_n + h y_n' = y_n + h f(x_n, y_n)}, \quad n = 0, 1, 2, \dots N-1$$

It is known as single step explicit method.

Assumption: The step size h in this method is taken to be very small so that $O(h^2)$ can be neglected in the Taylor series method.

$$\boxed{y(x_{n+1}) = y(x_n) + h\,y'(x_n) + O(h^2)}$$ — Taylor Series method with two terms.

$$\Rightarrow y(x_{n+1}) = y(x_n) + h\,y'(x_n) + \frac{h^2}{2!}\,y''(\xi_n), \text{ where } x_n < \xi_n < x_{n+1}$$

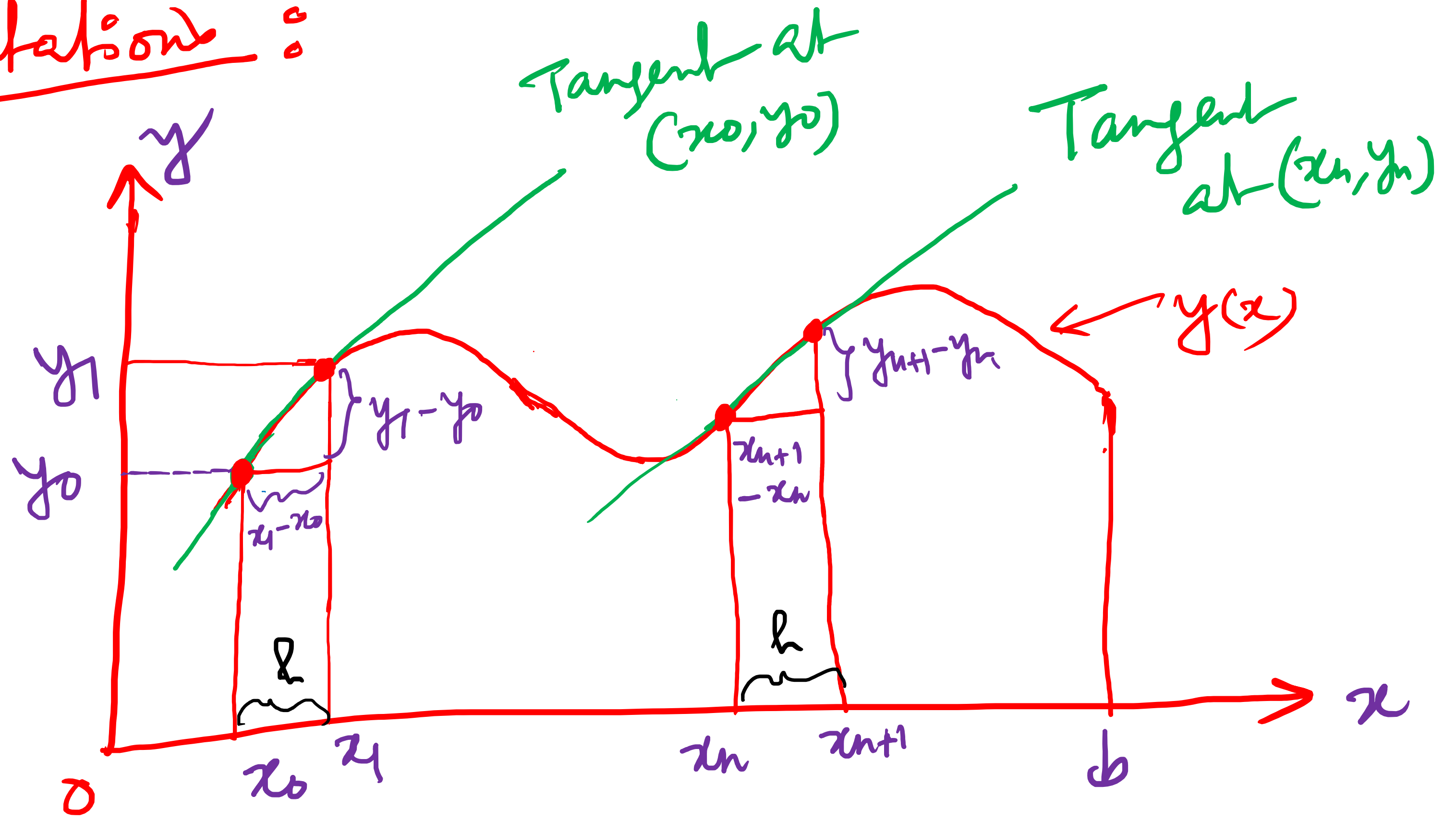
Now neglecting the $O(h^2)$ term i.e. $\frac{h^2}{2!}\,y''(\xi_n)$, we have

$$\boxed{y_{n+1} = y_n + h\,y_n' = y_n + h\,f(x_n, y_n)}, \quad n = 0, 1, \dots, N-1$$

This is Euler method.

Note: Here the step size h is taken as very small.

Geometrical Representation:



Using tangent at (x_0, y_0) as an approximation to the solution curve $y(x)$ in $[x_0, x_1]$, we have

$$y_0' = \frac{y_1 - y_0}{x_1 - x_0}.$$

Since $x_1 - x_0 = h$ and $y_0' = f(x_0, y_0)$

$$\therefore \boxed{y_1 = y_0 + h y_0' = y_0 + h f(x_0, y_0)},$$

Similarly using the tangent at (x_n, y_n) as an approximation to the

Solution curve $y(x)$ in $[x_n, x_{n+1}]$, we have

$$\boxed{y_{n+1} = y_n + h f(x_n, y_n)}\quad , n = 0, 1, 2, \ldots, N-1.$$

(*) Here we are approximating the solution curve $y(x)$ by a slope at a point (x_n, y_n).

Truncation Error: The local truncation error

is $$T_n = \frac{y(x_{n+1}) - y(x_n)}{h} - \phi(x_n, y(x_n); h)$$

$$= \frac{y(x_n+h) - y(x_n)}{h} - f(x_n, y_n)$$

[Since $\phi(x_n, y(x_n); h) = f(x_n, y_n)$]

$$= \frac{1}{h}\left[y(x_n) + h\,y'(x_n) + \frac{h^2}{2!}y''(\xi_n) - y(x_n) \right] - f(x_n, y_n), \quad x_n < \xi_n < x_{n+1}$$

[By using the Taylor Series expansion]

$$= \frac{1}{h}\left[h\,f(x_n, y_n) + \frac{h^2}{2!}y''(\xi_n) \right] - f(x_n, y_n)$$

$$= \frac{h}{2!}y''(\xi_n), \quad x_n < \xi_n < x_{n+1}$$

$$T_n = \frac{h}{2} y''(\xi_n), \quad x_n < \xi_n < x_{n+1}, \quad n = 0, 1, \ldots, N-1$$

$$\boxed{|T_n| \le Kh}, \quad \text{where } K = \frac{1}{2} \max_{\xi \in [x_0, b]} |y''(\xi)|$$

This shows that this method is of order 1.

First order method

Global Error: $e_n = y(x_n) - y_n, \quad n = 0, 1, \ldots, N$

we know that $|e_n| \le \frac{T}{L_\phi}\left(e^{L_\phi (x_n - x_0)} - 1\right)$, where

$$T = \max_{0 \le n \le N-1} |T_n|.$$

and L_φ is the Lipschitz constant for the function $\phi(x_n, y_n; h)$.

For this method, $\varphi(x_n, y_n; h) = f(x_n, y_n)$.

we have $|\phi(x_n, u_n; h) - \phi(x_n, v_n; h)|$

$= |f(x_n, u_n) - f(x_n, v_n)|$

$\leq L|u_n - v_n|$, where L is the Lipschitz constant for the function $f(x,y)$ and it is known to us.

So $\boxed{L_\varphi = L}$

$$|T_n| = \left| \frac{h}{2!} y''(\xi_n) \right| \le \frac{h}{2} M_2$$

where $M_2 = \max\limits_{\xi \in [x_0, b]} |y''(\xi)|$

$\therefore |T_n| \le T,$ where $T = \max\limits_{0 \le n \le N-1} |T_n|$

$$= \frac{h}{2} M_2 .$$

Now $|e_n| \le \frac{M_2}{2} \left[\frac{e^{L(b-x_0)} - 1}{L} \right] h = K_1 h$

where $K_1 = \frac{M_2}{2} \left[\frac{e^{L(b-x_0)} - 1}{L} \right]$
$=$ Constant

$\therefore \boxed{|e_n| \le K_1 h}$ $\Rightarrow$ The global error is also of $O(h)$.

which also shows that the method is first order.

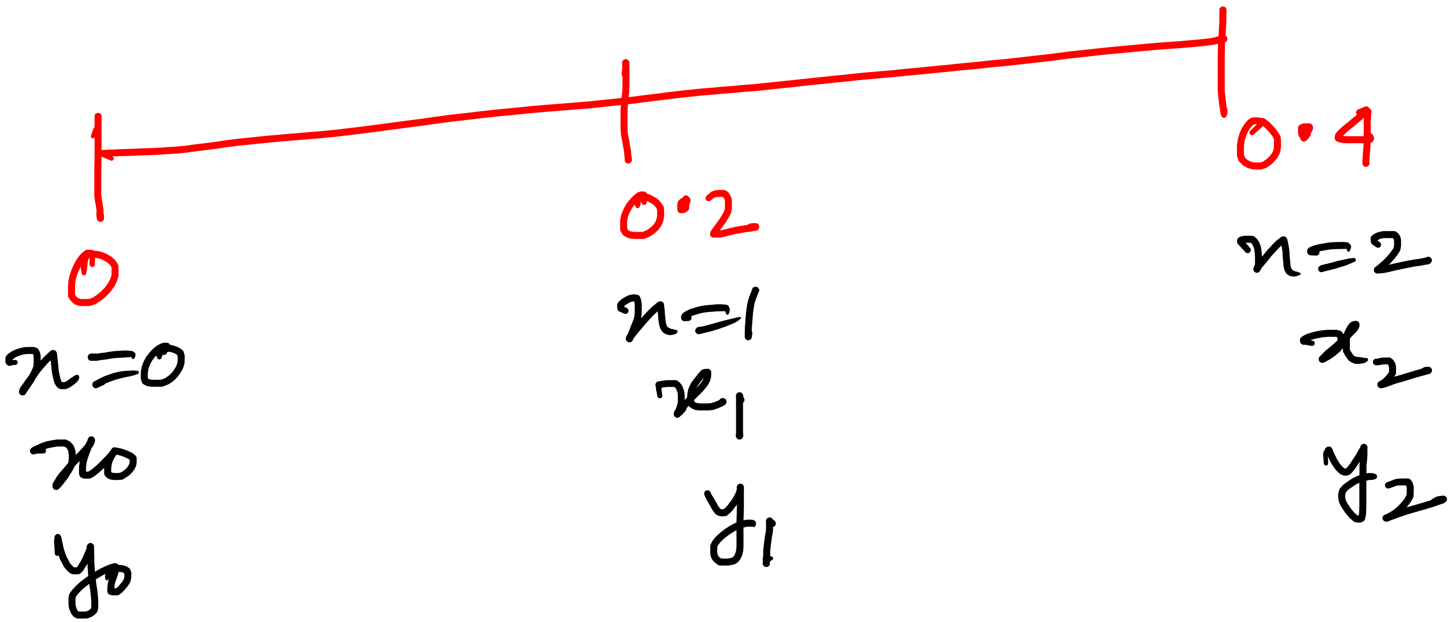
Example: Solve IVP $yy' = x$, $y(0) = 1$ by using the Euler method in $0 \le x \le 0.4$ with $h = 0.2$ and $h = 0.1$. Compare the results with exact solution at $x = 0.4$.

Solⁿ: $y' = x/y$, here $f(x,y) = x/y$.

Euler method: $y_{n+1} = y_n + h\, f(x_n, y_n)$

$= y_n + h \cdot x_n / y_n$.

$h = 0.2$:



$n=0$:

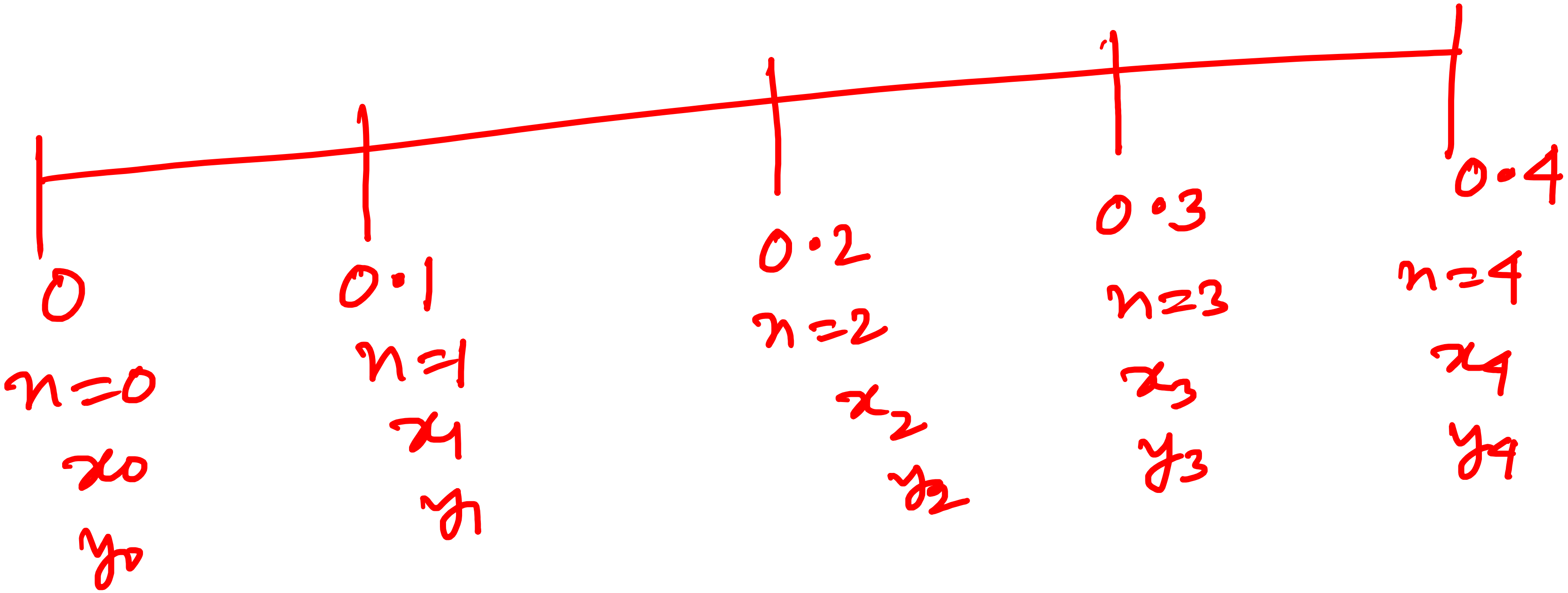
$$y_1 = y_0 + h\,{x_0}/{y_0}, \qquad y_0 = 1,\ x_0 = 0.$$

$$= y_0 + 0.2 \times {x_0}/{y_0} = 1$$

$n=1$:

$$y_2 = y_1 + 0.2 \times \frac{x_1}{y_1} = 1.64$$

$h=0.1$:



$n=0$: $y_1 = y_0 + 0.1 \times \frac{x_0}{y_0} = 1$

$n=1$: $y_2 = 1.01$,

$n=2$: $y_3 = 1.02980$,

$n=3$: $y_4 = 1.05893$.

Exact Sol^n: $y(x) = \sqrt{x^2+1}$, $y(0.4) = 1.07703$.

Error:

$h=0.2:$ $\Rightarrow |1.07703 - 1.04|$

$= 0.03703$

$= 0.04$

$h=0.1:$ $\Rightarrow |1.07703 - 1.05893|$

$= 0.01810 = 0.02.$

It is clear that if h is going to be small, then the error is also small.

Example: Use the Euler method to solve numerically the initial value problem

$$u' = -2tu^2, \quad u(0) = 1,$$

with $h = 0.2$, 0.1 and 0.05 on the interval $[0, 1]$. Determine the bound for the truncation error.

Try! Homework!

Ans: $h = 0.2 \Rightarrow u(1) = 0.50706$.

$h = 0.1 \Rightarrow u(1) = 0.50364$

$h = 0.05 \Rightarrow u(1) = 0.50179$

The differential eqⁿ is given by

$$\frac{dy}{dx} = f(x,y) \quad \text{— (1)}$$

Integrating (1) in the interval $[x_n, x_{n+1}]$, we have —

$$\int_{x_n}^{x_{n+1}} \frac{dy}{dx} \cdot dx = \int_{x_n}^{x_{n+1}} f(x,y)\,dx$$

or $\boxed{y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f(x,y)\,dx}$ — (2)

Applying the Integral mean value theorem to the right hand side, we have

$$y(x_{n+1}) = y(x_n) + f(\xi_n, y(\xi_n)) \int_{x_n}^{x_{n+1}} dx, \quad x_n < \xi_n < x_{n+1}$$

$$= y(x_n) + (x_{n+1} - x_n) f(\xi_n, y(\xi_n))$$

$$= y(x_n) + h f(\xi_n, y(\xi_n)) \quad \text{--- (3)}$$

$$= y(x_n) + h f(x_n + \theta h, y(x_n + \theta h)), \quad 0 < \theta < 1$$

In eqn (2), $f(x,y)$ is the slope of the solution curve and it changes continuously in $[x_n, x_{n+1}]$.

Now we will approximate the continuously varying slope in $[x_n, x_{n+1}]$ by a fixed slope or by a linear combination of slopes at some intermediate points in $[x_n, x_{n+1}]$ and in this way, we will obtain different type of Numerical methods.

Case 1: $\theta = 0$: From eqn ③, by setting $\theta = 0$, we have

$$\boxed{y_{n+1} = y_n + h f(x_n, y_n)}$$

$n = 0, 1, \cdots, N-1$

, This is Euler method.

This is first order method.

Note: Here we are approximating the continuously varying slope $f(x,y)$ in $[x_n, x_{n+1}]$ by a fixed slope at $x = x_n$. i.e. $f(x,y) \approx f(x_n, y_n)$ in $[x_n, x_{n+1}]$.

Case II: $\theta = 1$: by setting $\theta = 1$, we have from ③

$$y_{n+1} = y_n + h\, f(x_n + h, y(x_n + h))$$
$$= y_n + h\, f(x_{n+1}, y_{n+1}), \quad n = 0, 1, \ldots, N-1$$
$$= y_n + h\, f$$

This method is Backward Euler method. It is a implicit single step method.

This is also a first order method.

Here we are approximating the continuously varying slope in $[x_n, x_{n+1}]$ by a fixed slope at $x = x_{n+1}$ in $[x_n, x_{n+1}]$.

i.e. $f(x, y) \approx f(x_{n+1}, y_{n+1})$ in $[x_n, x_{n+1}]$.

<u>Backward Euler Method:</u> $\boxed{y_{n+1} = y_n + h\, f(x_{n+1}, y_{n+1})}$

One can change this implicit scheme to a explicit scheme by using the Euler method

$$y_{n+1} = y_n + h\, f(x_n, y_n)$$

$$\Rightarrow \quad y_{n+1} = y_n + h\, f\left(x_{n+1},\, y_n + h f(x_n, y_n)\right).$$

Solution Procedure:

The backward Euler method is given by

$$\boxed{y_{n+1} = y_n + h\, f(x_{n+1}, y_{n+1})} \quad \text{--- (4)}$$

Since $f(x,y)$ is nonlinear, (4) produces a nonlinear algebraic eq^n for the solution of y_{n+1}.

For the solution of this nonlinear eqn, we can use the Newton Raphson method.

Let $F(y_{n+1}) = y_{n+1} - y_n - h f(x_{n+1}, y_{n+1})$,

here y_n, x_{n+1} and h are known to us.

Let $y_{n+1}^{(0)}$ be the initial approximation to the solution y_{n+1} and it can be obtained by using Euler method

$$\boxed{y_{n+1}^{(0)} = y_n + h f(x_n, y_n)}$$

one can consider $y_{n+1}^{(0)} = y_n$

Now applying the Newton Raphson method we have —

$$\boxed{y_{n+1}^{(k+1)} = y_{n+1}^{(k)} - \frac{f\left(y_{n+1}^{(k)}\right)}{f'\left(y_{n+1}^{(k)}\right)}, \quad k = 0, 1, \ldots}$$

Stop Criterion: we will not stop this iteration until $\left| y_{n+1}^{(k+1)} - y_{n+1}^{(k)} \right| \le \varepsilon$, where ε is given error tolerance.

Note: The initial approximation can be also taken by $y_{n+1}^{(0)} = y_n$.

Through this process, we can get y_{n+1} and it will give you the approximation to the solution $y(x)$ at the point x_{n+1}.

Truncation Error:

$$T_n = \frac{y(x_{n+1}) - y(x_n)}{h} - f(x_{n+1}, y(x_{n+1}))$$

$$= \frac{1}{h}\left[y(x_n) + h y'(x_n) + \frac{h^2}{2!} y''(x_n) + \dots - y(x_n) \right]$$

$$- f(x_n + h, y_n + h f(x_n, y_n))$$

$$= \left[y'(x_n) + \frac{h}{2!} y''(x_n) + \cdots \right] - \left[f(x_n, y_n) \right.$$

$$\left. + h \left(\frac{\partial f}{\partial x} + f \frac{\partial f}{\partial y} \right) \Big|_{(x_n, y_n)} + \cdots \right]$$

$$= f(x_n, y_n) + \frac{h}{2} \left(\frac{\partial f}{\partial x} + f \frac{\partial f}{\partial y} \right) \Big|_{(x_n, y_n)} + O(h^2)$$

$$- f(x_n, y_n) - h \left(\frac{\partial f}{\partial x} + f \frac{\partial f}{\partial y} \right) \Big|_{(x_n, y_n)} + O(h^2)$$

$$= -\frac{h}{2}\left(\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right)\Big|_{(x_n, y_n)} + o(h^2)$$

$$= -\frac{h}{2}\, y''(x_n) + o(h^2)$$

$$= -\frac{h}{2}\, y''(\xi_n), \quad \underline{x_n < \xi_n < x_{n+1}}$$

$$\therefore \quad T_n = -\frac{h}{2}\, y''(\xi_n), \quad |T_n| \le Kh,$$

where $K = \frac{1}{2}\max_{\xi \in [a,b]} |y''(\xi)|$.

So this method is of **first order.**

Example: Solve the initial value problem $y' = -2xy^2$, $y(0)=1$, with $h=0.2$, on the interval $[0, 0.4]$, using backward Euler method.

Solⁿ: The backward Euler method gives —

$$y_{n+1} = y_n + h\, f(x_{n+1}, y_{n+1})$$

$$= y_n - h \cdot 2\, x_{n+1} \cdot y_{n+1}^2$$

$$\text{or} \quad \boxed{y_{n+1} = y_n - 2h\, x_{n+1}\, y_{n+1}^2}, \quad n = 0, 1, \ldots$$

This is an implicit eqⁿ in y_{n+1} and can be solved by using iterative method. We generally use Newton Raphson method.

Denote, $f(y_{n+1}) = y_{n+1} - y_n + 0.4\, x_{n+1} \cdot y_{n+1}^2$

we have $f'(y_{n+1}) = 1 + 0.8\, x_{n+1} y_{n+1}$

$$\boxed{y_{n+1}^{(k+1)} = y_{n+1}^{(k)} - \frac{f(y_{n+1}^{(k)})}{f'(y_{n+1}^{(k)})}, \quad k = 0, 1, \ldots}$$

Taking $y_{n+1}^{(0)} = y_n$, we have —

$\underline{n=0}$: $x_1 = 0.2$, $y_1^{(0)} = y_0 = 1$

$f(y_1^{(0)}) = 0.08$, $f'(y_1^{(0)}) = 1.16$, $y_1^{(1)} = 0.93103448$

$f(y_1^{(1)}) = 0.00038050$, $f'(y_1^{(1)}) = 1.14896552$

$y_1^{(2)} = \underline{0.93070332}$.

$$F(y_1^{(2)}) = 0.14 \times 10^{-7}, \quad F'(y_1^{(2)}) = 1.14891253,$$

$$y_1^{(3)} = 0.93070331.$$

$$\text{Therefore } y(0.2) \approx y_1 = \underline{0.93070331}$$

$$\underline{\underline{n=1:}} \quad x_2 = 0.4, \quad y_2^{(0)} = y_1 = 0.93070331$$

$$F(y_2^{(0)}) = 0.13859338, \quad F'(y_2^{(0)}) = 1.29782506$$

$$y_2^{(1)} = 0.82391436.$$

$$F\left(y_2^{(1)}\right) = 0.00182463, \quad F'\left(y_2^{(1)}\right) = 1.26365260$$

$$y_2^{(2)} = 0.82247043,$$

$$F\left(y_2^{(2)}\right) = 0.00000034, \quad F'\left(y_2^{(2)}\right) = 1.26319054,$$

$$y_2^{(3)} = 0.82247016.$$

$$F\left(y_2^{(3)}\right) = -0.4 \times 10^{-8}, \quad F'\left(y_2^{(3)}\right) = 1.26319045$$

$$u_2^{(4)} = 0.82247016.$$

Therefore $y(0.4) \approx y_2 = 0.82247016.$

The exact values are

$$y(0.2) = 0.96153846,$$

and $y(0.4) = 0.86206897.$

Case III : $\theta = \frac{1}{2}$.

we know,
$y(x_n + h/2)$
$= y_n + \frac{h}{2} f(x_n, y_n)$

By setting $\theta = \frac{1}{2}$, we have from eqn ③

$$y_{n+1} = y_n + h f\left(x_n + h/2,\ y(x_n + h/2)\right)$$

This is by using Euler method

$$\Rightarrow \boxed{y_{n+1} = y_n + h f\left(x_n + h/2,\ y_n + h/2 f(x_n, y_n)\right)}$$

$n = 0, 1, 2, \ldots, N-1$

This method is known to be as modified Euler method or Midpoint method.

Here we are approximating the continuously varying slope by a fixed slope at $x = x_n + h/2$ in $[x_n, x_{n+1}]$, i.e.

$$f(x,y) \approx f\left(x_n + h/2,\ y(x_n + h/2)\right) \text{ in } [x_n, x_{n+1}]$$

Truncation error :

$$T_n = \frac{y(x_{n+1}) - y(x_n)}{h} - \phi\left(x_n, y(x_n); h\right)$$

Here $\phi\left(x_n, y(x_n); h\right) = f\left(x_n + h/2,\ y(x_n + h/2)\right)$

$$T_n = \frac{1}{h}\left[y(x_n) + h y'(x_n) + \frac{h^2}{2!} y''(x_n) + \cdots - y(x_n)\right]$$

$$- f\left(x_n + h/2,\ y_n + \frac{h}{2} f(x_n, y_n)\right)$$

$$= y'(x_n) + \frac{h}{2} y''(x_n) + \frac{h^2}{6} y'''(x_n) + O(h^3) - f(x_n, y_n) - \frac{h}{2} \frac{\partial f}{\partial x}\Big|_{(x_n, y_n)}$$

$$- \frac{h}{2} f(x_n, y_n) \cdot \frac{\partial f}{\partial y}\Big|_{(x_n, y_n)}$$

$$- \frac{1}{2!}\left[\frac{h^2}{4} \frac{\partial^2 f}{\partial x^2}\Big|_{(x_n, y_n)} + 2 \cdot \frac{h}{2} \cdot \frac{h}{2} f(x_n, y_n) \frac{\partial^2 f}{\partial x \partial y}\Big|_{(x_n, y_n)} + \frac{h^2}{4} f^2 \cdot \frac{\partial^2 f}{\partial y^2}\Big|_{(x_n, y_n)}\right] - O(h^3)$$

$$T_n = f(x_n, y_n) + \frac{h}{2}\left(f_x + f f_y\right)\Big|_{(x_n, y_n)}$$

$$+ \frac{h^2}{6}\left(f_{xx} + 2f f_{xy} + f^2 f_{yy} + f_y\left(f_x + f f_y\right)\right)\Big|_{(x_n, y_n)}$$

$$+ O(h^3) - f(x_n, y_n) - \frac{h}{2}\left(f_x + f f_y\right)\Big|_{(x_n, y_n)}$$

$$- \frac{h^2}{8}\left(f_{xx} + 2f f_{xy} + f^2 f_{yy}\right)\Big|_{(x_n, y_n)} + O(h^3)$$

$$= \left(\frac{h^2}{6} - \frac{h^2}{8}\right)\left(f_{xx} + 2f f_{xy} + f^2 f_{yy}\right)\Big|_{(x_n, y_n)}$$

$$+ \frac{h^2}{6} f_y \left(f_x + f f_y\right)\Big|_{(x_n, y_n)} + O(h^3)$$

$$= \frac{h^2}{24}\left(f_{xx} + 2f f_{xy} + f^2 f_{yy}\right)\Big|_{(x_n, y_n)}$$

$$+ \frac{h^2}{6} f_y \left(f_x + f f_y\right)\Big|_{(x_n, y_n)} + O(h^3)$$

$$= \frac{h^2}{24}\left(f_{xx} + 2f f_{xy} + f^2 f_{yy}\right)\Big|_{(\xi_n,\, y(\xi_n))}$$

$$+ \frac{h^2}{6} f_y\left(f_x + f f_y\right)\Big|_{(\xi_n,\, y(\xi_n))}, \quad x_n < \xi_n < x_{n+1}$$

which implies that the truncation error is $O(h^2)$.

$$\Rightarrow \boxed{|T_n| \le K h^2}$$

where $K = \max\limits_{\xi \in [x_0, b]} \left| \frac{1}{24}\left[f_{xx} + 2f f_{xy} + f^2 f_{yy}\right]_{(\xi,\, y(\xi))} + \left[\frac{f_y}{6}\left(f_x + f f_y\right)\right]_{(\xi,\, y(\xi))} \right|$

$$\Rightarrow \boxed{|T_n| \le T}$$

$$\text{So } T = \max_{\xi \in [x_0, b]} \left| \frac{h^2}{24}\left[f_{xx} + 2 f f_{xy} + f^2 f_{yy} + \frac{h^2 f_y}{6}\left(f_x + f f_y \right) \right]_{(\xi, y(\xi))} \right|$$

To find Global error:

$$\boxed{e_n = y(x_n) - y_n}$$

we know

$$|e_n| \le \frac{T}{L_\phi}\left(e^{L_\phi(x_n - x_0)} - 1\right)$$

where $T = \max_{0 \le n \le N-1} |T_n|$ is known and L_ϕ is Lipschitz constant for $\phi(x_n, y_n; h)$ which has to be determined.

we have

$$y_{n+1} = y_n + h\,\varphi(x_n, y_n; h)$$

where $\varphi(x_n, y_n; h)$

$$= f\left[x_n + h/2,\ y_n + \tfrac{h}{2} f(x_n, y_n)\right)$$

$$\left|\varphi(x_n, u_n; h) - \varphi(x_n, v_n; h)\right|$$

$$= \left| f\left(x_n + h/2,\ u_n + \tfrac{h}{2} f(x_n, u_n)\right) - f\left(x_n + h/2,\ v_n + \tfrac{h}{2} f(x_n, v_n)\right)\right|$$

$$\leq L_f \left| u_n + \frac{h}{2} f(x_n, u_n) - v_n - \frac{h}{2} f(x_n, v_n) \right|$$

where L_f is known Lipschitz constant for the function $f(x, y)$.

$$\leq L_f |u_n - v_n| + \frac{h}{2} L_f \left| f(x_n, u_n) - f(x_n, v_n) \right|$$

$$\leq L_f |u_n - v_n| + \frac{h}{2} L_f \cdot L_f |u_n - v_n|$$

$$= \left(L_f + \frac{h}{2} L_f^2\right) |u_n - v_n|$$

$$\boxed{\begin{aligned} &|\phi(x_n, u_n; h) - \varphi(x_n, v_n; h)| \\ &\quad \leq L_\varphi |u_n - v_n| \end{aligned}}$$

where $L_\varphi = \left(L_f + h/_2 L_f^2\right)$.

Now $|e_n| \le \frac{T}{L\varphi}\left(e^{L\varphi(b-x_0)} - 1\right)$

$\le K_1 h^2$, where K_1 is constant.

Since $T = \max\limits_{0 \le n \le N-1} |T_n|$ and

$T_n \in O(h^2)$.

$\Rightarrow$ $e_n \in O(h^2)$. | So this method is of order 2.

Case 4: We approximate the continuously varying slope in $[x_n, x_{n+1}]$ by the mean of slopes at the points x_n and x_{n+1}.

Therefore we have

$$\boxed{y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]}$$

This method is implicit method and it is known as Trapezium method.

Here we need to use the Newton-Raphson method as seen for Euler Backward method.

Similarly we can apply here to compute the approximations for y_{n+1}.

I am skiping this procedure.

To find Global error:

$$e_n = y(x_n) - y_n, \quad n = 0, 1, \ldots, N$$

we know that $$|e_n| \le \frac{T}{L_\phi}\left(e^{L_\phi(x_n - x_0)} - 1\right)$$

where $$T = \max_{0 \le n \le N-1} |T_n|$$

and L_ϕ is the Lipschitz constant for the function $\phi(x_n, y_n; h)$

For this method,

$$y_{n+1} = y_n + h\,\phi(x_n, y_n; h)$$

where $\phi(x_n, y_n; h)$

$$= \frac{1}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$$

$$\phi(x_n, y_n; h) = \frac{1}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_n + h\phi)\right]$$

we have $|\phi(x_n, u_n; h) - \phi(x_n, v_n; h)|$

$$= \frac{1}{2}\Big| f(x_n, u) + f\big(x_n + h,\ u + h\varphi(x_n, u; h)\big) - f(x_n, v) - f\big(x_n + h,\ v + h\varphi(x_n, y_n; h)\big)\Big|$$

$$\leq \frac{1}{2}\big| f(x_n, u) - f(x_n, v)\big| + \frac{1}{2}\big| f(x_n + h, u + h\varphi) - f(x_n + h, v + h\varphi)\big|$$

$$\leq \frac{1}{2} L_f |u - v| + \frac{1}{2} L_f \big| u + h\phi(x_n, u_n; h) - v - h\varphi(x_n, v_n; h)\big|$$

where L_f is the Lipschitz constant for the function $f(x, y)$.

and L_f is known to us.

$$\le \frac{1}{2} L_f |u-v| + \frac{1}{2} L_f |u-v| + \frac{1}{2} L_f h \left| \varphi(x_n, u_n; h) - \varphi(x_n, v_n; h) \right|$$

a, $\left(1 - \frac{h}{2} L_f\right) \left| \phi(x_n, u_n; h) - \phi(x_n, v_n; h) \right|$

$$\le L_f |u-v|$$

$$\Rightarrow \left| \varphi(x_n, u_n; h) - \varphi(x_n, v_n; h) \right| \le \frac{L_f}{\left(1 - h/2 L_f\right)} |u-v|$$

Provided $\left(1-\frac{h}{2}L_f\right) > 0$.

Here $L_\varphi = \dfrac{L_f}{1-\frac{h}{2}L_f}$.

Now truncation error T_n has to be calculated.

$|T_n| \leq K h^2$

$\Rightarrow T_n \in O(h^2)$

we can get $|T_n| \leq \dfrac{h^2}{12} M_3$, where

$M_3 = \max_{\xi \in [x_0, b]} |y'''(\xi)|$

$\Rightarrow \boxed{T = \frac{h^2}{12} M_3}$.

So the global error

$$|e_n| \le \frac{h^2}{12} M_3 \cdot \frac{1}{L_\varphi} \left[e^{L_\varphi \cdot (b-x_0)} - 1 \right]$$

where $L_\varphi = \dfrac{L_f}{1 - \frac{h}{2} L_f}$.

$\therefore \boxed{|e_n| \le K_1 h^2}$, where K_1 is constant

$$= \frac{M_3}{12} \frac{1}{L_\varphi} \left[e^{L_\varphi (b-x_0)} - 1 \right]$$

$\Rightarrow e_n \in O(h^2) \longrightarrow$ The method is of order 2.

Now we want to get the explicit method from this implicit scheme.

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$$

$$= y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_n + hf(x_n, y_n))\right]$$

Here by using Euler method

$y_{n+1} = y_n + hf(x_n, y_n)$, we can

get the Explicit method and it is known as Euler-Cauchy method or Heun's method.

This method is also of order 2.

Justify this statement by finding the truncation error and also try to get the global error.

(Try!) (Homework!)

Homework: Try to find the global error for Backward Euler method.

Note:

Euler method: First order method. Here one fixed slope at $x = x_n$ is considered. Explicit

Backward Euler method: First order method. Here one fixed slope at $x = x_{n+1}$ is considered to approximate $f(x,y)$ in $[x_n, x_{n+1}]$

Implicit

Modified Euler method :
mid point method

Second order method but only one Slope at mid point in $[x_n, x_{n+1}]$ i.e at $x = x_n + h/2$ is considered.

Explicit

Trapezium method: Second order method and average of two slopes at $x = x_n$ and $x = x_{n+1}$ are considered. Implicit.

Euler-Cauchy method: Second order method and two slopes are considered. Explicit

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

4^{th} Lecture on Numerical Methods for Solving ODEs

(MA-5363)

Contents for this class

- ✓ Explicit Runge Kutta (RK) methods
  - ✓ Second-order RK method
  - ✓ Third-order RK method
  - ✓ Fourth-order RK method
- ✓ Errors in RK methods

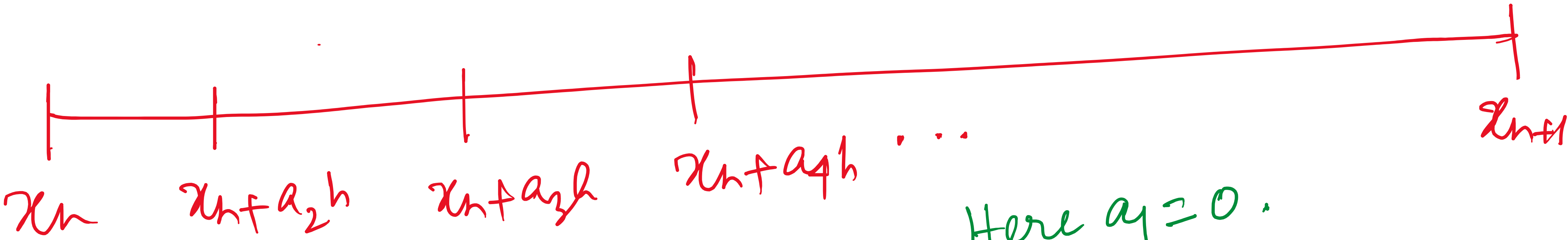
Generalization of the Single step methods

$$y_{n+1} = y_n + h\left(\text{weighted average of the slopes at intermediate points in } [x_n, x_{n+1}]\right)$$

$n = 0, 1, 2, \ldots, N-1.$

(iii) Runge Kutta methods: (Explicit)

The Runge-Kutta methods use the weighted average of slopes on the given interval $[x_n, x_{n+1}]$.



Here $a_1 = 0$.

Consider the continuously varying slope $f(x,y)$ in $[x_n, x_{n+1}]$ is approximated by weighted average of l number of slopes at intermediate points $x_n + a_i h$, $i = 1, 2, \ldots l$, in $[x_n, x_{n+1}]$.

Here $\underline{a_1 = 0}$.

The Runge-Kutta method is most generalized single step methods. From this method, one can derive different single step methods.

Now the Runge-Kutta method can be written as —

$$y_{n+1} = y_n + h \sum_{i=1}^{l} w_i \, y'(x_n + a_i h) \quad \left[\text{since } y' = f(x,y)\right]$$

$$= y_n + h \sum_{i=1}^{l} w_i \, f(x_n + a_i h, \, y(x_n + a_i h)), \quad n = 0, 1, \ldots, N-1.$$

Here w_i, a_i, $i = 1, 2, \ldots, l$ are constants and need to be obtained.

This method is having l slopes.

now find the slopes —

$\underline{i=1:}$ $f(x_n + a_1 h,\ y(x_n + a_1 h))$, $\underline{a_1 = 0}$.

$= f(x_n, y_n) = K_1$ (say).

one last slope at $x = x_n + a_1 h$

$\underline{i=2:}$ $f(x_n + a_2 h,\ y(x_n + a_2 h))$

$= f(x_n + a_2 h,\ y_n + a_2 h y_n')$

$= f(x_n + a_2 h,\ y_n + a_2 h f(x_n, y_n))$

first two terms of Taylor Series,

$$f(x_n + a_2 h,\ y(x_n + a_2 h))$$

$$= f(x_n + a_2 h,\ y_n + a_{21} h K_1) = K_2.$$

$i = 3$:

$$f(x_n + a_3 h,\ y(x_n + a_3 h))$$

$$= f\left(x_n + a_3 h,\ y_n + a_3 h \cdot \frac{b_{31}\, y(x_n + a_1 h) + b_{32}\, y(x_n + a_2 h)}{b_{31} + b_{32}}\right)$$

weighted average of two last slopes

$$= f\left(x_n + a_3 h,\ y_n + a_3 h \cdot \frac{b_{31} K_1 + b_{32} K_2}{b_{31} + b_{32}}\right)$$

$$f(x_n + a_3 h,\ y(x_n + a_3 h))$$
$$= f(x_n + a_3 h,\ y_n + h(a_{31} k_1 + a_{32} k_2))$$
$$= k_3 .$$

Similarly for $i = l$, we have

$$f(x_n + a_l h,\ y(x_n + a_l h))$$
$$= f(x_n + a_l h,\ y_n + h(a_{l1} k_1 + a_{l2} k_2 + \cdots + a_{l,l-1} k_{l-1}))$$

l-slopes: $K_1 = f(x_n, y_n)$

and

$$K_i = f\left(x_n + a_i h,\; y_n + h \underbrace{\sum_{\substack{j=1 \\ j<i}}^{l-1} a_{ij} K_j}\right),$$

weighted average of $(l-1)$ slopes at previous $(l-1)$ no. of intermediate points in $[x_n, x_{n+1}]$

$$i = 2, 3, \ldots, l$$

Here all a_i and all a_{ij} are unknown to us.

Now the Runge-Kutta method can be written as

$$y_{n+1} = y_n + h(\omega_1 K_1 + \omega_2 K_2 + \dots + \omega_l K_l)$$

where $K_1 = f(x_n, y_n)$

and

$$K_i = f\left(x_n + a_i h,\ y_n + h \sum_{\substack{j=1 \\ j<i}}^{l-1} a_{ij} K_j\right) \quad i = 2, \dots, l.$$

⊘ Second order R-K method: (Explicit)

To get the second order R-K method, we will consider two slopes.

Taking two slopes, we have

$$y_{n+1} = y_n + h(w_1 K_1 + w_2 K_2)$$

where $K_1 = f(x_n, y_n)$

and $K_2 = f(x_n + a_2 h,\ y_n + h a_{21} K_1)$

$= f(x_n + \alpha h,\ y_n + \beta h K_1)$

For sake of simplicity we consider $a_2 = \alpha$ and $a_{21} = \beta$

Here $\omega_1, \omega_2, \alpha$ and β are unknown parameters and these have to be determined.

$$y_{n+1} = y_n + h\left[\omega_1 f(x_n, y_n) + \omega_2 f(x_n + \alpha h, y_n + \beta h k_1)\right]$$

This is a single step method where

$$\phi(x_n, y_n; h) = \omega_1 f(x_n, y_n) + \omega_2 f(x_n + \alpha h, y_n + \beta h k_1)$$

The condition for consistent method is

$$\phi(x_n, y_n; 0) = f(x_n, y_n)$$ [we know from earlier classes]

So we have

$$\boxed{\phi(x_n, y_n; 0) = (w_1 + w_2) f(x_n, y_n)}$$

$\Rightarrow$ The method will be consistent if and only if $\boxed{w_1 + w_2 = 1}$ — (1)

To determine the unknown parameters α and β, we will find the truncation error in this method.

Here our main target is to get atleast second order accurate method.

Truncation error:

$$T_n = \frac{y(x_{n+1}) - y(x_n)}{h} - \phi(x_n, y(x_n); h)$$

we know so far —

$$y' = f(x,y), \quad y'' = f_x + f f_y,$$

$$y''' = (f_{xx} + 2 f f_{xy} + f^2 f_{yy}) + f_y (f_x + f f_y)$$

now $\dfrac{y(x_{n+1}) - y(x_n)}{h}$

$$= \frac{1}{h}\left[y(x_n) + h\, y'(x_n) + \frac{h^2}{2!} y''(x_n) + \frac{h^3}{3!} y'''(x_n) + O(h^4) - y(x_n) \right]$$

$$= y'(x_n) + \frac{h}{2} y''(x_n) + \frac{h^2}{6} y'''(x_n) + O(h^3)$$

$$= f(x_n, y_n) + \frac{h}{2}\left\{f_x + f f_y\right\}\Big|_{(x_n, y_n)}$$

$$+ \frac{h^2}{6}\left\{f_{xx} + 2 f f_{xy} + f^2 f_{yy}\right.$$

$$\left. + f_y\left(f_x + f f_y\right)\right\}\Big|_{(x_n, y_n)}$$

$$+ O(h^3)$$

$$\phi(x_n, y(x_n); h)$$

$$= w_1 f(x_n, y_n) + w_2 f(x_n + \alpha h, y_n + \beta h k_1)$$

$$= w_1 f(x_n, y_n) + w_2 \Big[f(x_n, y_n) + \alpha h f_x\big|_{(x_n, y_n)}$$

$$+ \beta h k_1 f_y\big|_{(x_n, y_n)} + \frac{1}{2!} \Big\{ \alpha^2 h^2 f_{xx} + 2\alpha h \cdot \beta h k_1 \cdot f_{xy}$$

$$+ \beta^2 h^2 k_1^2 f_{yy} \Big\} \Big]\Big|_{(x_n, y_n)} + O(h^3)$$

$$= (w_1 + w_2) f(x_n, y_n) + w_2 h (\alpha f_x + \beta f f_y)\big|_{(x_n, y_n)}$$

$$+ w_2 \cdot \frac{1}{2} h^2 (\alpha^2 f_{xx} + 2\alpha\beta f f_{xy} + \beta^2 f^2 f_{yy})\big|_{(x_n, y_n)} + O(h^3)$$

$$T_n = \frac{y(x_{n+1}) - y(x_n)}{h} - \phi(x_n, y(x_n); h)$$

$$= f(x_n, y_n) + \frac{h}{2}(f_x + f f_y)\Big|_{(x_n,y_n)} + \frac{h^2}{6}\Big(f_{xx} + 2 f f_{xy}$$

$$+ f^2 f_{yy} + f_y(f_x + f f_y)\Big)\Big|_{(x_n,y_n)} - (\omega_1 + \omega_2) f(x_n, y_n)$$

$$- \omega_2 h\,(\alpha f_x + \beta f f_y)\Big|_{(x_n,y_n)} - \omega_2 \frac{h^2}{2}\Big(\alpha^2 f_{xx} + 2\alpha\beta f f_{xy}$$

$$+ \beta^2 f^2 f_{yy}\Big)\Big|_{(x_n,y_n)} + O(h^3)$$

Since $w_1 + w_2 = 1$, we have

$$T_n = \frac{h}{2}\left(f_x + f f_y\right)\Big|_{(x_n, y_n)} + \frac{h^2}{6}\Big(f_{xx} + 2 f f_{xy}$$

$$+ f^2 f_{yy} + f_y\left(f_x + f f_y\right)\Big)\Big|_{(x_n, y_n)}$$

$$- w_2 h\left(\alpha f_x + \beta f f_y\right)\Big|_{(x_n, y_n)} - \frac{w_2 h^2}{2}\Big(\alpha^2 f_{xx}$$

$$+ 2\alpha\beta f f_{xy} + \beta^2 f^2 f_{yy}\Big)\Big|_{(x_n, y_n)} + O(h^3)$$

$$T_n = \left(\frac{h}{2} - \omega_2 \alpha h\right) f_x(x_n, y_n) + \left(\frac{h}{2} - \omega_2 \beta h\right) f(x_n, y_n)$$

$$f_y(x_n, y_n) + \left(\frac{h^2}{6} - \frac{\omega_2 \alpha^2 h^2}{2}\right) f_{xx}(x_n, y_n)$$

$$+ \left(\frac{h^2}{3} - \omega_2 h^2 \alpha \beta\right) f(x_n, y_n) f_{xy}(x_n, y_n) \quad \leftarrow \; *$$

$$+ \left(\frac{h^2}{6} - \frac{\omega_2 h^2 \beta^2}{2}\right) f^2(x_n, y_n) \; f_{yy}(x_n, y_n)$$

$$+ \frac{h^2}{6}\left(f_y\left(f_x + f f_y\right)\right)\Big|_{(x_n, y_n)} + O(h^3)$$

To get the truncation error belongs to $O(h^2)$, we need to vanish the $O(h)$ from ⊛. By considering the coefficients of h equal to zero, we have —

$$\boxed{\begin{array}{l} \frac{1}{2} - w_2\alpha = 0 \quad \text{---} \; (2) \\ \text{and } \frac{1}{2} - w_2\beta = 0 \quad \text{---} \; (3) \end{array}}$$

$$\Rightarrow w_2 = \frac{1}{2\alpha} \text{ and } w_2 = \frac{1}{2\beta}.$$

Take $\beta = \alpha$, we have from ①, ② and ③

$$w_2 = \frac{1}{2\alpha} \text{ and } w_1 = 1 - \frac{1}{2\alpha}, \quad \alpha \neq 0$$

is an arbitrary.

Now we have the R-K method with two slopes

$$\boxed{\begin{gathered} y_{n+1} = y_n + h\left[\left(1-\frac{1}{2\alpha}\right)k_1 + \frac{1}{2\alpha}k_2\right] \text{ where } \quad (4) \\ k_1 = f(x_n, y_n),\ k_2 = f(x_n + \alpha h,\ y_n + \alpha h k_1) \end{gathered}}$$

Note: we can not vanish terms $O(h^2)$ from the truncation error ⊛, since the term $\frac{1}{6} f_y (f_x + f f_y)$ in ⊛ can not be zero at any situation.

Here $w_2 = \frac{1}{2\alpha}$, $\alpha \neq 0$

$\Rightarrow$ This implies that $\boxed{T_n \notin O(h^3)}$

i.e. always we can get $T_n \in O(h^2)$

So by taking two slopes, we can get upto second order accuracy R-K methods.

The R-K method (4) can derive infinite no. of 2nd order R-K methods by suitable choice of arbitrary constants α.

Now we can have some cases as follows —

Case-I: If $\alpha = \frac{1}{2}$, then $w_1 = 0$ and $w_2 = 1$ $\Rightarrow$

$$\boxed{\begin{aligned} y_{n+1} &= y_n + h k_2 \\ \text{where } k_2 &= f\left(x_n + \tfrac{h}{2},\, y_n + \tfrac{h}{2} k_1\right) \end{aligned}}$$

This is Modified Euler method which is

Second order method.

Case-II: If $\alpha = 1$, then $w_1 = \frac{1}{2}$ and $w_2 = \frac{1}{2}$. So we have

$$\boxed{\begin{aligned} y_{n+1} &= y_n + \frac{h}{2}\left(K_1 + K_2\right) \\ \text{where } K_1 &= f(x_n, y_n) \\ \text{and } K_2 &= f(x_n + h,\ y_n + hK_1) \end{aligned}}$$

This method is known as Euler-Cauchy or Heun's method which is second order method.

These two methods explicit methods

Here we are talking about Explicit second order R-K methods only.

Case III : Second order method with minimum truncation error (optimal truncation error) :

To get the minimum truncation error T_n, we need to vanish the $O(h^2)$ terms in $(*)$ as many as possible.

But we can't vanish all the terms $O(h^2)$ that we know.

Here we can make the coefficients of h^2 equal to zero.

$$\Rightarrow \quad \left.\begin{aligned} \frac{1}{6} - \frac{w_2\alpha^2}{2} &= 0 \\ \frac{1}{3} - w_2\alpha\beta &= 0 \\ 1/6 - \frac{w_2\beta^2}{2} &= 0 \end{aligned}\right\} \Rightarrow \boxed{\alpha = 2/3.}$$

Since $\beta = \alpha$, $w_2 = \frac{1}{2\alpha}$

If we consider $\alpha = 2/3$, we get the optimal truncation error

$$\boxed{T_n = \frac{h^2}{6} f_y \left(f_x + f f_y \right) \Big|_{(x_n, y_n)} + O(h^3)}$$

Therefore this optimal second order R-K method can be written as

$$y_{n+1} = y_n + h\left[\frac{k_1}{4} + \frac{3}{4}k_2\right]$$

where $k_1 = f(x_n, y_n)$

and $k_2 = f\left(x_n + \frac{2h}{3}, y_n + \frac{2h}{3}k_1\right)$

This optimal second order R-K method will give more accurate solution than other second order R-K method.

Third order explicit R-K method:

We will consider three slopes to get third order R-K method.

$$y_{n+1} = y_n + \sum_{i=1}^{3} w_i K_i$$

where $K_1 = f(x_n, y_n)$

$K_2 = f(x_n + a_2 h,\ y_n + h a_{21} K_1)$

$K_3 = f(x_n + a_3 h,\ y_n + h(a_{31} K_1 + a_{32} K_2))$

Expanding k_2, k_3 in Taylor Series about (x_n, y_n) and calculating the Truncation error, we have the following six equations in eight parameters $w_1, w_2, w_3, a_2, a_3, a_{21}, a_{31}$ and a_{32}. The methods contain two arbitrary parameters. The eq^{ns} are

$$a_{21} = a_2, \quad a_{31} + a_{32} = a_3, \quad w_1 + w_2 + w_3 = 1$$

$$a_2 w_2 + a_3 w_3 = \tfrac{1}{2}, \quad a_2^2 w_2 + a_3^2 w_3 = \tfrac{1}{3}$$

$$a_2 a_{32} w_3 = \tfrac{1}{6}.$$

Now we can obtain the unknown parameters $a_2, a_3, a_{21}, a_{31}, a_{32}, w_1, w_2$ and w_3. By suitable choice of these parameters, we can have different type of numerical methods. These are following :—

⊛ Nystrom method :

$$y_{n+1} = y_n + \frac{h}{8}\left[2K_1 + 2K_2 + 2K_3\right]$$

$K_1 = f(x_n, y_n)$, $K_2 = f\left(x_n + \frac{2h}{3}, y_n + \frac{2h}{3}K_1\right)$

$K_3 = f\left(x_n + \frac{2h}{3}, y_n + \frac{2h}{3}K_2\right)$.

⊛ Classical method:

$$y_{n+1} = y_n + \frac{h}{6}\left(k_1 + 4k_2 + k_3\right)$$

$$k_1 = f(x_n, y_n), \quad k_2 = f\left(x_n + h/2,\ y_n + hk_1/2\right)$$

$$k_3 = f\left(x_n + h,\ y_n - hk_1 + 2hk_2\right)$$

⊛ Heun method:

$$y_{n+1} = y_n + h/4\left(k_1 + 3k_3\right)$$

$$k_1 = f(x_n, y_n)$$

$$k_2 = f\left(x_n + h/3,\ y_n + hk_1/3\right)$$

$$K_3 = f\left(x_n + \frac{2h}{3},\ y_n + \frac{2h}{3} K_2\right)$$

⊛ Nearly optimal:

$$y_{n+1} = y_n + \frac{h}{9}\left(2K_1 + 3K_2 + 4K_3\right)$$

$$K_1 = f(x_n, y_n)$$

$$K_2 = f\left(x_n + \frac{h}{2},\ y_n + \frac{hK_1}{2}\right)$$

$$K_3 = f\left(x_n + \frac{3h}{4},\ y_n + \frac{3h}{4} K_2\right)$$

(*) **Note:** All these methods are of third order. i.e $T_n \in O(h^3)$

So if we take three slopes in Explicit R-K methods, the order of these methods is 3.

(*) Last derived "nearly optimal method" will provide less truncation error. So we can get good third order approximated solution.

③ Fourth order explicit R-K methods:

If we take four slopes, we will get fourth order explicit R-K methods.

⊛ I am skipping the procedure all to get these fourth order methods. However you can follow the same procedure as adopted in second order RK method.

with four slopes, we have the method as -

$$\begin{cases} y_{n+1} = y_n + h\sum_{i=1}^{4} w_i K_i \\ \text{where } K_1 = f(x_n, y_n) \\ K_i = f\left(x_n + a_i h,\ y_n + h\sum_{\substack{j=1 \\ j<i}}^{3} a_{ij} K_j\right), \end{cases}$$

$i = 2, 3, 4.$

Here we have 13 unknowns such as
$w_1, w_2, w_3, w_4, a_2, a_3, a_4, a_{21}, a_{31}, a_{32}, a_{41}, a_{42}, a_{43}.$

Now we can obtain 11 eq^{n}s ⟶

$$a_2 = a_{21}, \quad a_3 = a_{31} + a_{32},$$

$$a_4 = a_{41} + a_{42} + a_{43}, \quad w_1 + w_2 + w_3 + w_4 = 1$$

$$a_2 w_2 + a_3 w_3 + a_4 w_4 = \frac{1}{2}$$

$$a_2^2 w_2 + a_3^2 w_3 + a_4^2 w_4 = \frac{1}{3}$$

$$w_2 a_2 a_{32} + w_4 (a_2 a_{42} + a_3 a_{43}) = \frac{1}{6}$$

$$w_2 a_2^3 + w_3 a_3^3 + w_4 a_4^3 = \frac{1}{4}$$

$$w_3 a_2^2 a_{32} + w_4 (a_2^2 a_{42} + a_3^2 a_{43}) = \frac{1}{12}$$

$$w_3 a_2 a_3 a_{32} + w_4 (a_2 a_{42} + a_3 a_{43}) a_4 = \frac{1}{8}$$

$$w_4 a_2 a_{32} a_{43} = \frac{1}{24}.$$

Now choose suitable choice of these unknown parameters, we will get different fourth order methods.

Classical fourth order R-K method:

The simplest solution of these 11 equations is given by

$a_2 = a_3 = 1/2$, $a_4 = 1$, $w_2 = w_3 = 1/3$,

$w_1 = w_4 = 1/6$, $a_{21} = 1/2$, $a_{31} = 0$, $a_{32} = 1/2$

$a_{41} = 0$, $a_{42} = 0$, $a_{43} = 1$.

Thus the fourth order method becomes

$$\boxed{\begin{aligned} y_{n+1} &= y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\ k_1 &= f(x_n, y_n) \\ k_2 &= f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right) \\ k_3 &= f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right) \\ k_4 &= f(x_n + h, y_n + hk_3). \end{aligned}}$$

This method is the best to consider for solving IVPs. This method is of order 4.

(iii) Kutta method: This fourth order method is

$$y_{n+1} = y_n + \frac{h}{8}(k_1 + 3k_2 + 3k_3 + k_4)$$

$$k_1 = f(x_n, y_n), \quad k_2 = f\left(x_n + \frac{h}{3}, y_n + \frac{hk_1}{3}\right)$$

$$k_3 = f\left(x_n + \frac{2h}{3}, y_n - \frac{hk_1}{3} + hk_2\right)$$

$$k_4 = f(x_n + h, y_n + hk_1 - hk_2 + hk_3)$$

Example: Solve the initial value problem $y' = -2xy^2$, $y(0) = 1$, with $h = 0.2$ on the interval $[0, 0.4]$. Estimate $y(0.4)$ by using

(i) modified Euler method

(ii) Euler-Cauchy or Heun's method (second order RK method)

(iii) Fourth order classical R-K method.

Sol^n:

(i) Modified Euler method:

$$\boxed{y_{n+1} = y_n + h f\left(x_n + h/2,\ y_n + \frac{h}{2} f(x_n, y_n)\right)}$$

This is second order R-K method,

So one can write it as

$$\boxed{\begin{gathered} y_{n+1} = y_n + h K_2, \text{ where } K_1 = f(x_n, y_n), \\ \text{and } K_2 = f\left(x_n + h/2,\ y_n + \frac{h}{2} K_1\right) \end{gathered}}$$

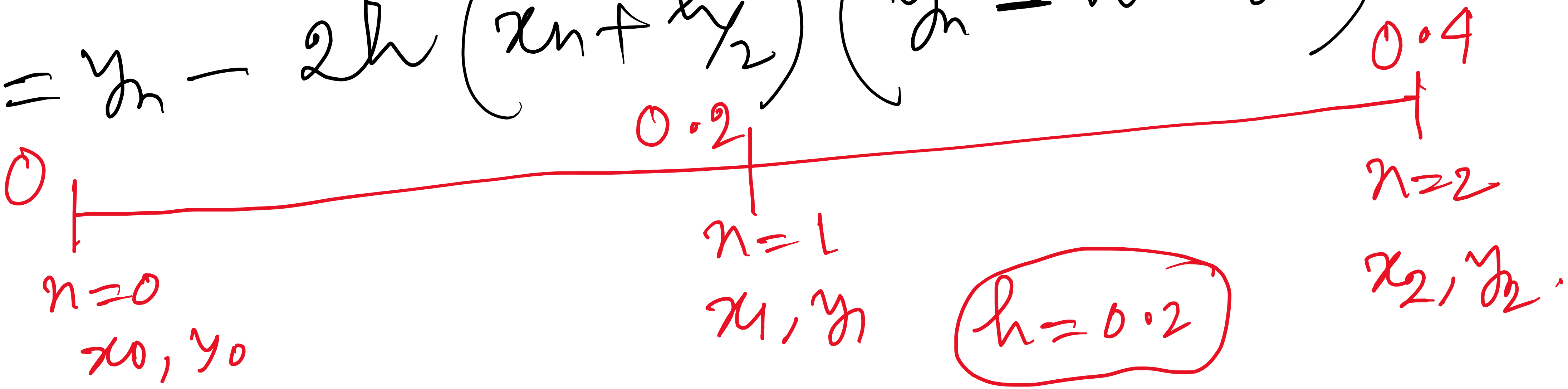
So we have

$$y_{n+1} = y_n + h f\left(x_n + \frac{h}{2}, \; y_n + \frac{h}{2} f(x_n, y_n)\right)$$

$$= y_n - 2h\left(x_n + \frac{h}{2}\right)\left(y_n + \frac{h}{2} \cdot -2x_n y_n^2\right)^2$$

Since $f(x,y) = -2xy^2$.

$$= y_n - 2h\left(x_n + \frac{h}{2}\right)\left(y_n - h x_n y_n^2\right)^2$$



Here $x_0 = 0$, $y_0 = 1$, $h = 0.2$

$n = 0$:

$$y_1 = y_0 - 2h\left(x_0 + h/2\right)\left(y_0 - h x_0 y_0^2\right)^2$$

$$= 1 - 0.4\left(0 + 0.1\right)\left(1 - 0.2 \times 0 \times 1^2\right)^2$$

$$= 0.96.$$

$n = 1$:

$$y_2 = y_1 - 2h\left(x_1 + h/2\right)\left(y_1 - h x_1 y_1^2\right)^2$$

$$= 0.96 - 0.4\left(0.2 + 0.1\right)\left(0.96 - 0.2 \times 0.2 \times (0.96)^2\right)^2$$

$$= 0.857738$$

$$= 0.85774.$$

(ii) Heun's method (Euler - Cauchy method):

$$y_{n+1} = y_n + \frac{h}{2}\left(f(x_n, y_n) + f\left(x_{n+1}, y_n + h f(x_n, y_n)\right)\right)$$

$n = 0, 1, 2, \ldots, N-1$

One can write it as

$$\boxed{\begin{gathered} y_{n+1} = y_n + h\left(\tfrac{1}{2}k_1 + \tfrac{1}{2}k_2\right) \\ k_1 = f(x_n, y_n),\quad k_2 = f(x_n + h,\ y_n + h k_1) \end{gathered}}$$

Now we have

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f\left(x_n + h, y_n + h f(x_n, y_n)\right) \right]$$

$$= y_n + \frac{h}{2}\left[-2x_n y_n^2 - 2(x_n + h)\left(y_n - h \cdot 2x_n y_n^2\right)^2 \right]$$

$n=0$: $y_1 = 0.96$

$n=1$: $y_2 = 0.86030$

(try to find out)!

Exact solution is $y(x) = \dfrac{1}{1+x^2}$,

$\Rightarrow \quad y(0.2) = 0.96154$

and $y(0.4) = 0.86207$.

Error in modified Euler method:

$|y(0.2) - y_1| = 0.00154$

and $|y(0.4) - y_2| = 0.00433$

Error in Euler-Cauchy method:

$$|y(0.2) - y_1| = 0.00154$$

and $$|y(0.4) - y_2| = 0.00177.$$

(iii) Fourth order R-K method:

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = f(x_n, y_n), \quad k_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{hk_1}{2}\right)$$

$$k_3 = f\left(x_n + \frac{h}{2},\ y_n + h\frac{k_2}{2}\right)$$

$$k_4 = f\left(x_n + h,\ y_n + hk_3\right)$$

Now try to find out k_1, k_2, k_3 and k_4 for $n = 0$ and $n = 1$. Therefore find y_1 and y_2. (Home work)

The solutions are by R-K fourth order classical method

$$y_1 = 0.9615328,$$

$$y_2 = 0.8620525.$$

The absolute errors are

$$\varepsilon(0.2) = |y(0.2) - y_1| = 0.0000006$$

and $$\varepsilon(0.4) = |y(0.4) - y_2| = 0.0000016.$$

Now you see the errors in these methods —

errors in Modified Euler method > errors in Euler-Cauchy method > errors in R-K 4th order classical method

So R-K 4th order classical method is the best for solving IVP.

Example: Solve the IVP

$y' = x^3 + y$, $y(0) = 2$, with $h = 0.2$, on the interval $[0, 0.4]$. Estimate $y(0.4)$ by using any second order RK method and fourth order RK method.

(Home work!)

5^{th} & 6^{th} Lectures on Numerical Methods for Solving ODEs

MA-5363

Contents for this class

- Stability analysis for Single step methods
- Implicit R-K Methods

Stability analysis of the Single step methods:

Importance of this Study:

There are mainly two types of errors generated during the numerical computation. One is truncation error and another is round off error. The truncation error is in the hand of user, and it can be controlled by choosing higher order methods. However, the round off error is not in the hand of user, and it is generated by machine. It can grow and finally destroy the true solution. In such case, this method is numerically unstable. This happens when one choose the large step length than the allowed limiting value. So the idea is that to get a good approximated solution, we need a numerically stable method. That's why we will study the stability analysis of any kind of numerical methods.

Definition of Stability:

A numerical method is stable if the cumulative effect of all errors including round-off errors is bounded, independent of the number of the mesh points.

The stability concept will be discussed mathematically later in this talk.

Note: One idea to overcome from the numerical unstability is that if one can bound the round-off error. This idea will be carried forward later in this study.

Now the concept of stability can be understood by analysing the numerical solutions of the test equation $y' = \lambda y$ which is the linearised form of the nonlinear equation $y' = f(x, y)$.

② Analysis of the numerical solution of the Test equation $y' = \lambda y$:

Before going to start this analysis, the question can be asked that how do we linearise the nonlinear equation $y' = f(x,y)$?

$\Rightarrow$ The behaviour of the solution of the IVP in the neighbourhood of any point $(\bar{x}, \bar{y})$ can be predicted by considering the linearised form of the differential eqn.

now $f(x,y) = f(\bar{x},\bar{y}) + (x-\bar{x}) \left.\frac{\partial f}{\partial x}\right|_{(\bar{x},\bar{y})} + (y-\bar{y}) \left.\frac{\partial f}{\partial y}\right|_{(\bar{x},\bar{y})}$

$+ \cdots$

$$f(x,y) = y\,\underbrace{\frac{\partial f}{\partial y}\bigg|_{(\bar{x},\bar{y})}}_{} + \underbrace{f(\bar{x},\bar{y}) - \bar{y}\frac{\partial f}{\partial y}\bigg|_{(\bar{x},\bar{y})} + (x-\bar{x})\frac{\partial f}{\partial x}\bigg|_{(\bar{x},\bar{y})} + \cdots}_{\text{constants.}}$$

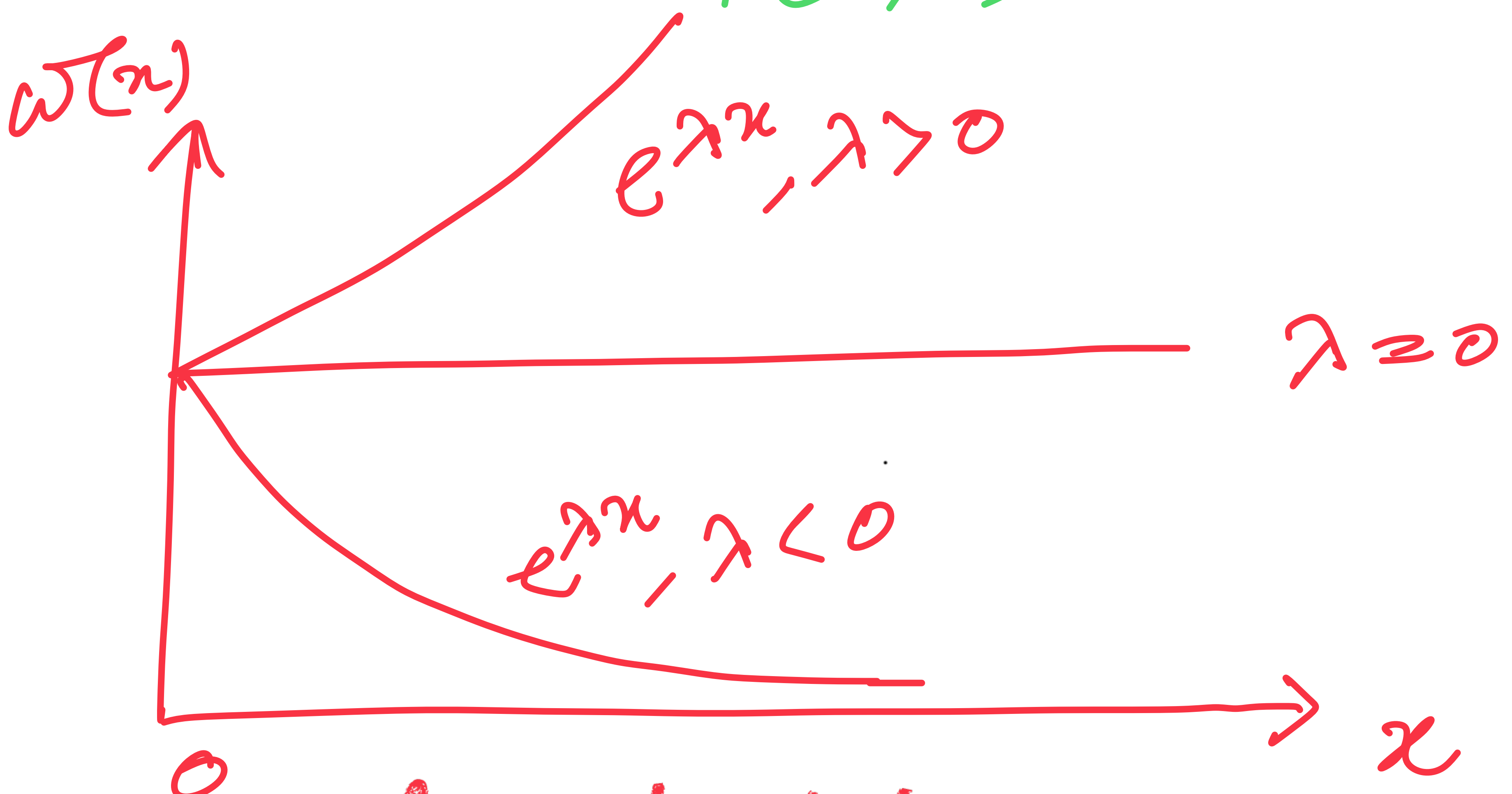
y → dependent variable.

Now $y' = f(x,y) = \lambda y + c$, where $c = f(\bar{x},\bar{y}) - \bar{y}\frac{\partial f}{\partial y}\Big|_{(\bar{x},\bar{y})} + (x-\bar{x})\frac{\partial f}{\partial x}\Big|_{(\bar{x},\bar{y})}$

and $\lambda = \frac{\partial f}{\partial y}\Big|_{(\bar{x},\bar{y})}$

$\Rightarrow \boxed{w' = \lambda w}$ where $w = y + c/\lambda$,

$\Rightarrow w(x) = Ke^{\lambda x}$, K constant.



Analyze the behaviour of the solution.

Example: $y' = x^2 - y^2$, $y(0) = 1$.

Analyze the behaviour of the solution around $(1,-1)$ and $(0,2)$.

Sol^n: $f(x,y) = x^2 - y^2$, $\frac{\partial f}{\partial y} = -2y$,

$\frac{\partial f}{\partial y}(1,-1) = 2 \Rightarrow \lambda > 0 \rightarrow$ the solution grows

and $\frac{\partial f}{\partial y}(0,2) = -4 \Rightarrow \lambda < 0 \rightarrow$ the solution decay.

In this way, one can analyze the behaviour of the solution in the neighbourhood of the points after linearising the given differential eq^n.

Now we want to solve the linearised form $y' = \lambda y$ with $y(x_0) = y_0$, $x \in [x_0, b]$.

The exact solution is obtained as

$$\boxed{y(x) = y(x_0)\, e^{\lambda(x - x_0)}}$$

Now in order to compute $y(x)$ at $x_n = x_0 + nh$, $n = 1, 2, \dots, N$

we have

$$y(x_1) = y_0\, e^{\lambda(x_1 - x_0)} = y_0\, e^{\lambda h}$$

$$y(x_2) = y_0\, e^{\lambda(x_2 - x_0)} = y_0\, e^{2\lambda h} = y_0\, e^{\lambda h} \cdot e^{\lambda h} = y(x_1)\, e^{\lambda h}$$

Similarly, we have

$$\boxed{y(x_{n+1}) = y(x_n)\, e^{\lambda h}}, \quad n = 0, 1, 2, \dots, \underline{N-1}$$

Exact Solution

Difficult to compute, need to be approximated

$$e^{\lambda h} = 1 + \lambda h + \frac{\lambda^2 h^2}{2!} + \frac{\lambda^3 h^3}{3!} + \cdots$$

$\approx E(\lambda h)$ [This is the approximation of $e^{\lambda h}$]

Numerical Solⁿ: we can get the numerical solution as

$$\boxed{y_{n+1} = y_n E(\lambda h)}$$

→ numerical approximation to the exact $y(x_{n+1}) = y(x_n)\, e^{\lambda h}$

Now we want to find the exact form of $E(\lambda h)$ at each numerical scheme.

① Euler method: $y_{n+1} = y_n + h f(x_n, y_n)$

First order explicit single step method

we have $f(x, y) = \lambda y$ since $y' = f(x, y)$ is linearised to $y' = \lambda y$.

$$y_{n+1} = y_n + h \cdot \lambda y_n$$
$$= (1+\lambda h) y_n .$$
$$\Rightarrow \boxed{y_{n+1} = E(\lambda h)\, y_n}, \text{ where } E(\lambda h) = 1+\lambda h$$

Note: If you pick any explicit first order single step numerical method, always you will get the numerical solution in the linearised form as

$$\boxed{y_{n+1} = y_n E(\lambda h), \text{ where } E(\lambda h) = 1+\lambda h}$$

It is only applicable to explicit single step method.

Modified Euler method : (2nd order explicit single step method)

$$y_{n+1} = y_n + h f\left(x_n + h/2,\ y_n + \frac{h}{2} f(x_n, y_n)\right)$$

$$= y_n + h \cdot \lambda \left(y_n + \frac{h}{2} \cdot \lambda y_n\right)$$

$$= y_n + \lambda h y_n + \frac{\lambda^2 h^2}{2} y_n$$

$$= \left(1 + \lambda h + \frac{\lambda^2 h^2}{2}\right) y_n$$

$$\Rightarrow \boxed{y_{n+1} = y_n E(\lambda h)} \quad \text{where } E(\lambda h) = 1 + \lambda h + \frac{\lambda^2 h^2}{2}$$

Note: For any 2^{nd} order explicit single step method, we will get $E(\lambda h)$ as $1 + \lambda h + \frac{\lambda^2 h^2}{2}$.

⊘ 3rd order explicit single step methods: (R-K 3rd order method)

$$y_{n+1} = E(\lambda h)\, y_n, \quad \text{where}$$

$$E(\lambda h) = 1 + \lambda h + \frac{\lambda^2 h^2}{2!} + \frac{\lambda^3 h^3}{3!}.$$

⊘ 4th order explicit single step methods: (R-K 4th order methods)

$$y_{n+1} = E(\lambda h)\, y_n, \quad \text{where } E(\lambda h) = 1 + \lambda h + \frac{\lambda^2 h^2}{2!} + \frac{\lambda^3 h^3}{3!} + \frac{\lambda^4 h^4}{4!}$$

⊘ Note: For Implicit single step methods, the same will not happen.

⊘ Backward Euler method (First order implicit single step method):

$$y_{n+1} = y_n + h f(x_{n+1}, y_{n+1})$$

$$y_{n+1} = y_n + h f(x_{n+1}, y_{n+1})$$
$$= y_n + h\lambda y_{n+1}$$
$$\Rightarrow (1-\lambda h) y_{n+1} = y_n \Rightarrow y_{n+1} = \frac{1}{1-\lambda h} y_n$$
$$\Rightarrow \boxed{y_{n+1} = E(\lambda h) y_n, \text{ where } E(\lambda h) = \frac{1}{1-\lambda h}}$$

$$E(\lambda h) = \frac{1}{1-\lambda h} = 1 + \lambda h + \lambda^2 h^2 + \lambda^3 h^3 + \cdots$$

Trapezium method:

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$$
$$= y_n + \frac{h}{2}\left[\lambda y_n + \lambda y_{n+1}\right]$$
$$= 1 + \frac{\lambda h}{2} y_n + \frac{\lambda h}{2} y_{n+1}$$

or, $\left(1-\frac{\lambda h}{2}\right) y_{n+1} = \left(1+\frac{\lambda h}{2}\right) y_n$

or, $y_{n+1} = \left(\dfrac{1+\frac{\lambda h}{2}}{1-\frac{\lambda h}{2}}\right) y_n$,

or $\boxed{y_{n+1} = E(\lambda h)\, y_n, \text{ where } E(\lambda h) = \dfrac{1+\lambda h/2}{1-\lambda h/2}}$

(2) **Stability:** A numerical method is said to be stable if for any $\varepsilon > 0$ $\exists$ $\delta = \delta(\varepsilon)$ such that for two different numerical solution (y_n) and (z_n) generated by same methods but different starting points y_0 and z_0, respectively, we have $|y_n - z_n| < \varepsilon$, whenever $|y_0 - z_0| < \delta(\varepsilon)$, $\forall\, 0 < h \leq h_0$ and $x_n \leq b$.

OR A numerical method is stable if $\exists$ a constant K such that for any two solution sequences (y_n) and (z_n) generated by the same methods but different starting points y_0 and z_0, respectively, we have

$$\boxed{|y_n - z_n| \le K\,|y_0 - z_0|, \quad \text{for } x_n \le b \text{ and } 0 < h \le h_0.}$$

Stability of the Solution:

For your better understanding I am going to explain the stability of the solution to the IVP.

The stability of the solution $y(x)$ is examined when the IVP is changed by a small amount.

Consider a small perturbation in a starting point.

Consider the perturbed problem as —

Perturbed form:

$$\boxed{\begin{array}{l} y' = f(x,y) \\ \text{with } y(x_0) = y_0 + \varepsilon \text{, where } \varepsilon \text{ is the small perturbation.} \\ x \in [x_0, b]. \end{array}}$$

Clearly the solution must depend on ε, so we can rewrite the IVP as —

$$\boxed{\begin{array}{l} y'(x;\varepsilon) = f(x, y(x;\varepsilon)) \\ \text{with } y(x_0;\varepsilon) = y_0 + \varepsilon \end{array}} \quad — \quad (1)$$

original form :

$$\begin{cases} y' = f(x,y) \\ \text{with } y(x_0) = y_0 \,, \; x \in [x_0, b] \end{cases} \qquad \text{— ②}$$

Now our job is check the difference between the original solution and perturbed solution. If the difference is so small, that mean the solution is stable. with the small change in the starting point, if the perturbed solution changes quite large, then the solution is not stable.

Let $\boxed{Z(x;\varepsilon) = y(x;\epsilon) - y(x)}$, → this is the difference between original and perturbed solution.

Now $z'(x;\varepsilon) = y'(x;\varepsilon) - y'(x)$

$$= f(x, y(x;\varepsilon)) - f(x,y)$$

$$= \frac{\partial f}{\partial y} z(x;\varepsilon)$$

$$\Rightarrow \boxed{z'(x;\varepsilon) = \frac{\partial f}{\partial y} z(x;\varepsilon)}$$

Lets find the solution $z(x;\varepsilon)$

with $z(x_0;\varepsilon) = y(x_0;\varepsilon) - y(x_0) = \varepsilon$.

$$\boxed{z(x;\varepsilon) = \varepsilon\, e^{\int_{x_0}^{x} \frac{\partial f}{\partial y} dt}}$$

Sol^n

If $z(x;\varepsilon)$ is very small, then we can get the stable solution.

If the partial derivative satisfies $\frac{\partial f}{\partial y} \leq 0$, then we have that $z(x,\varepsilon)$ remains bounded by ε as x increases.

In this case, we say that the IVP is well-conditioned.

- For a problem to be well-conditioned, we want the integral $\int_{x_0}^{x} \frac{\partial f}{\partial y}\, dt$ to be bounded by zero or a small positive number, as x increases. Then the perturbation $Z(x;\varepsilon)$ will be bounded by some constant times ε, with the constant not too large, i.e., $\boxed{|Z(x;\varepsilon)| = \left|\varepsilon\, e^{\int_{x_0}^{x} \frac{\partial f}{\partial y} dt}\right| \le K\varepsilon,}$ where K is constant.

Example: Ill-conditioned problem:

$y' = 100y - 101e^{-x}$, $y(0) = 1$ has the solution $y(x) = e^{-x}$.

The perturbed problem $y' = 100y - 101e^{-x}$, $y(0) = 1+\varepsilon$ has the solution $y(x;\varepsilon) = e^{-x} + \varepsilon e^{100x}$ which rapidly departs from the true solution. So the given IVP is ill-conditioned problem.

Now we will study the stability analysis for numerical solution.

(i) Original IVP: $y' = f(x, y)$, $y(x_0) = y_0$, $x \in [x_0, b]$. —— (1)

(ii) Perturbed IVP: $u' = f(x, u)$, $u(x_0) = u_0 = y_0 + \varepsilon$, $x \in [x_0, b]$ — (2)

where ε is very small.

The numerical solution to the original IVP is

$$\boxed{y_{n+1} = y_n + h\,\phi(x_n, y_n; h)} \quad \text{—— (3)}$$

The numerical solution to the perturbed IVP is

$$\boxed{u_{n+1} = u_n + h\,\phi(x_n, u_n; h)} \quad \text{—— (4)}$$

Now for the stability of the numerical solution, we will check the difference between true solution and the perturbed solution.

Let $\varepsilon_{n+1} = u_{n+1} - y_{n+1}$ and $\varepsilon_n = u_n - y_n$

$$u_{n+1} - y_{n+1} = u_n - y_n + h\left[\phi(x_n, u_n) - \phi(x_n, y_n)\right]$$

$$\text{or } e_{n+1} = e_n + h\left[\phi(x_n, u_n) - \phi(x_n, y_n)\right]$$

$$|e_{n+1}| = \left|e_n + h\left[\phi(x_n, u_n) - \phi(x_n, y_n)\right]\right|$$

$$\leq |e_n| + h\left|\phi(x_n, u_n) - \phi(x_n, y_n)\right|$$

$$\leq |e_n| + hL_\phi|u_n - y_n| \quad \left[\text{Since } \phi \text{ is Lipschitz continuous and } L_\phi \text{ is Lipschitz constant}\right]$$

$$= (1 + hL_\phi)|e_n|$$

$$\Rightarrow |e_{n+1}| \leq (1 + hL_\phi)|e_n|$$

$\underline{n=0}$: $|e_1| \leq (1 + hL_\phi)|e_0|$

$$e_0 = x_0 - y_0 = \varepsilon.$$

$$|e_1| \leq (1+hL_\varphi)\varepsilon, \quad |e_2| \leq (1+hL_\varphi)|e_1|$$
$$\leq (1+hL_\varphi)^2 \varepsilon$$

$$\boxed{|e_n| \leq (1+hL_\varphi)^n \varepsilon}$$

$$|e_n| \leq (1+hL_\varphi)^n \varepsilon \leq e^{nhL_\varphi} \cdot \varepsilon$$

Since $(1+hL_\varphi) \leq e^{hL_\varphi}$

$$\Rightarrow |e_n| \leq e^{L_\varphi(x_n - x_0)} \varepsilon \leq e^{L_\varphi(b-x_0)} \cdot \varepsilon$$

$$\therefore |e_n| \leq e^{L_\varphi(b-x_0)} \varepsilon. \Rightarrow \boxed{|e_n| \leq K\varepsilon}$$ whenever $|x_0 - y_0| = \varepsilon$, a small quantity.

where $K = e^{L_\varphi(b-x_0)}$

$|e_n| \le K \varepsilon$

$\Rightarrow \boxed{|u_n - y_n| \le K|u_0 - y_0|}$, where K is constant for $x_n \le b$ and $0 < h \le h_0$.

$\Rightarrow$ The numerical method is stable.

We will find the condition for stability of the numerical methods.

To get this we need to check the error.

Error: Exact sol^n $\Rightarrow$ $\boxed{y(x_{n+1}) = y(x_n)\, e^{\lambda h}}$

Numerical sol^n $\Rightarrow$ $\boxed{y_{n+1} = y_n E(\lambda h)}$

error :

$$e_{n+1} = y(x_{n+1}) - y_{n+1}$$

$$\left[e_n = y(x_n) - y_n \right.$$
$$\Rightarrow y_n = y(x_n) - e_n$$

$$= y(x_n)\, e^{\lambda h} - y_n E(\lambda h)$$

$$= y(x_n)\, e^{\lambda h} - \left(y(x_n) - e_n\right) E(\lambda h)$$

$$= y(x_n)\, e^{\lambda h} - y(x_n) E(\lambda h) + e_n E(\lambda h)$$

$$= \underbrace{y(x_n)\left[e^{\lambda h} - E(\lambda h)\right]}_{\text{Local Truncation error}} + \underbrace{e_n E(\lambda h)}_{\text{propagation error}}$$

Local Truncation error
it can be made small

propagation error and $E(\lambda h)$ is called as propagating factor.

$$e_{n+2} = y(x_{n+2}) - y_{n+2}$$

$$= y(x_{n+1})\, e^{\lambda h} - y_{n+1} E(\lambda h)$$

$$= y(x_n)\, e^{2\lambda h} - y_n E^2(\lambda h) = y(x_n)\left(e^{2\lambda h} - E^2(\lambda h)\right) + e_n E^2(\lambda h)$$

$$e_{n+k} = y(x_n)\left(e^{kh\lambda} - E^k(\lambda h)\right) + e_n E^k(\lambda h)$$

It is noted that the truncation error is in our control while the propagation error is not in our control.

At $(n+1)$ step, the second term in the error is of the form $E(\lambda h)\, e_n$
at $(n+2)$ step, the second term in the error is of the form $E^2(\lambda h)\, e_n$ and at $(n+k)$ step, the second term is of the form $E^k(\lambda h)\, e_n$. So, the $E(\lambda h)$ term is carried over at each step.

As $n \to \infty$, the obtained results will be meaningful only if the propagation error decays or is atleast bounded. Hence the propagating factor

$E(\lambda h)$ should satisfy the condition —

$$\boxed{|E(\lambda h)| < 1}$$

- **Absolute Stability:** A single step numerical method is said to be absolutely stable if $|E(\lambda h)| < 1$, when $\lambda < 0$.

- **Relatively Stable:** A single step numerical method is said to be relatively stable if $|E(\lambda h)| \leq e^{\lambda h}$, $\lambda > 0$.

Note: when $\lambda < 0$, the exact solution decrease as x increase and the necessary condition is absolute stability, since the numerical solution must also decrease with x.

when $\lambda > 0$, the exact solution increases with x and we do not need the condition $|E(\lambda h)| < 1$, so that the relative stability is the necessary condition to be satisfied.

⊘ **Periodically Stable:** A single step numerical method is periodically stable (P-stability) if $|E(\lambda h)| = 1$, when λ is pure imaginary.

⊘ **Asymptotically Stable:** A single step numerical method is asymptotically stable if $y_n \to 0$ as $n \to \infty$. This implies that the stability interval is $h\lambda \in (-\infty, 0)$, that is the entire left half $h\lambda$ plane.

Ex: Backward Euler method and Trapezium method

Find absolute stability region for different single step methods:

(1) Euler method (First order Explicit R-K method):

(λ is real and $\lambda < 0$)

We know $E(\lambda h) = 1 + \lambda h$

For absolute stability, $|E(\lambda h)| < 1 \Rightarrow -2 < \lambda h < 0$

$\Rightarrow \lambda h \in (-2, 0)$.

So $(-2, 0)$ is the absolute stability region.

(2) Second order explicit R-K method: (λ is real and $\lambda < 0$)

$|E(\lambda h)| = \left|1 + \lambda h + \frac{\lambda^2 h^2}{2}\right| < 1 \Rightarrow -1 < 1 + \lambda h + \frac{\lambda^2 h^2}{2} < 1$

$\Rightarrow -1 < \frac{1}{2}(\lambda^2 h^2 + 2\lambda h + 1) + \frac{1}{2} < 1$

$$\Rightarrow -1 < \frac{1}{2}(1+\lambda h)^2 + \frac{1}{2} < 1$$

$$\Rightarrow -2 < (1+\lambda h)^2 + 1 < 2$$

The left inequality is always satisfied. The right inequality gives — $(\lambda h+1)^2 < 1 \Rightarrow |\lambda h+1| < 1$

$$\Rightarrow \lambda h \in (-2, 0)$$

Hence the stability interval for second order explicit RK method is also $(-2, 0)$.

(λ is real and $\lambda < 0$)

③ Third order explicit R-K method:

$$|E(\lambda h)| = \left| 1 + \lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{3!} \right| < 1$$

$$\Rightarrow \lambda h \in (-2.5, 0)$$

stability interval

④ Fourth order explicit R-K method:

$$\left|E(\lambda h)\right| = \left|1 + \lambda h + \frac{\lambda^2 h^2}{2!} + \frac{\lambda^3 h^3}{3!} + \frac{\lambda^4 h^4}{4!}\right| < 1$$

(λ is real and $\lambda < 0$)

$$\Rightarrow \lambda h \in (-2.78, 0)$$

(*) All the explicit methods have restrictions on the step length h. So these methods are conditionally stable.

⑤ First order implicit R-K method:

Backward Euler method:

$$\left|E(\lambda h)\right| = \left|\frac{1}{1-\lambda h}\right| < 1 \Rightarrow |1-\lambda h| > 1, \quad \lambda \text{ is real and } \lambda < 0$$

$$\Rightarrow \lambda h \in (-\infty, 0),$$ which is known unconditionally stable method.

This method is also A-stable.

(6) Trapezium method:

$$E(\lambda h) = \frac{1+\lambda h/2}{1-\lambda h/2}, \quad \lambda < 0 \text{ and } \lambda \text{ is real.}$$

$$|E(\lambda h)| < 1 \Rightarrow \boxed{\lambda h \in (-\infty, 0)},$$

unconditionally stable.

This method is also A-stable.

- Many implicit methods have no restriction on the step length h. So such methods are called unconditionally stable.

Question: What will be the stability region for the numerical method with complex number λ?

Euler Method: Let λ be complex with $Re(\lambda)<0$

Let $\bar{h} = \lambda h = x+iy$.

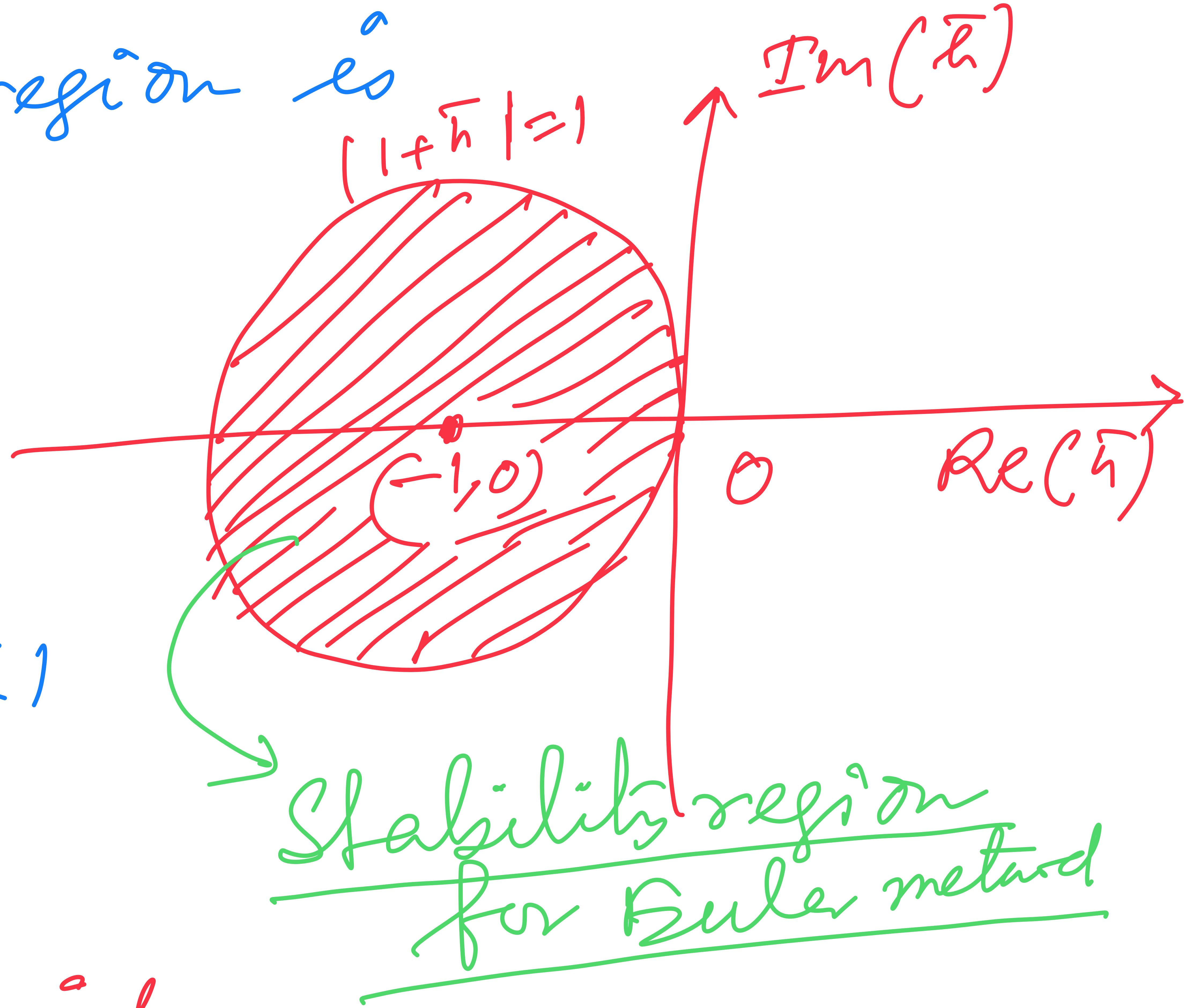
The absolute stability region is

$$|E(\lambda h)| < 1$$

$$\Rightarrow |1+\lambda h| < 1$$

$$\Rightarrow |1+x+iy| < 1$$

$$\Rightarrow (1+x)^2+y^2 < 1$$



The stability region is inside the circle with centre at $(-1,0)$ and radius 1.

Backward Euler method:

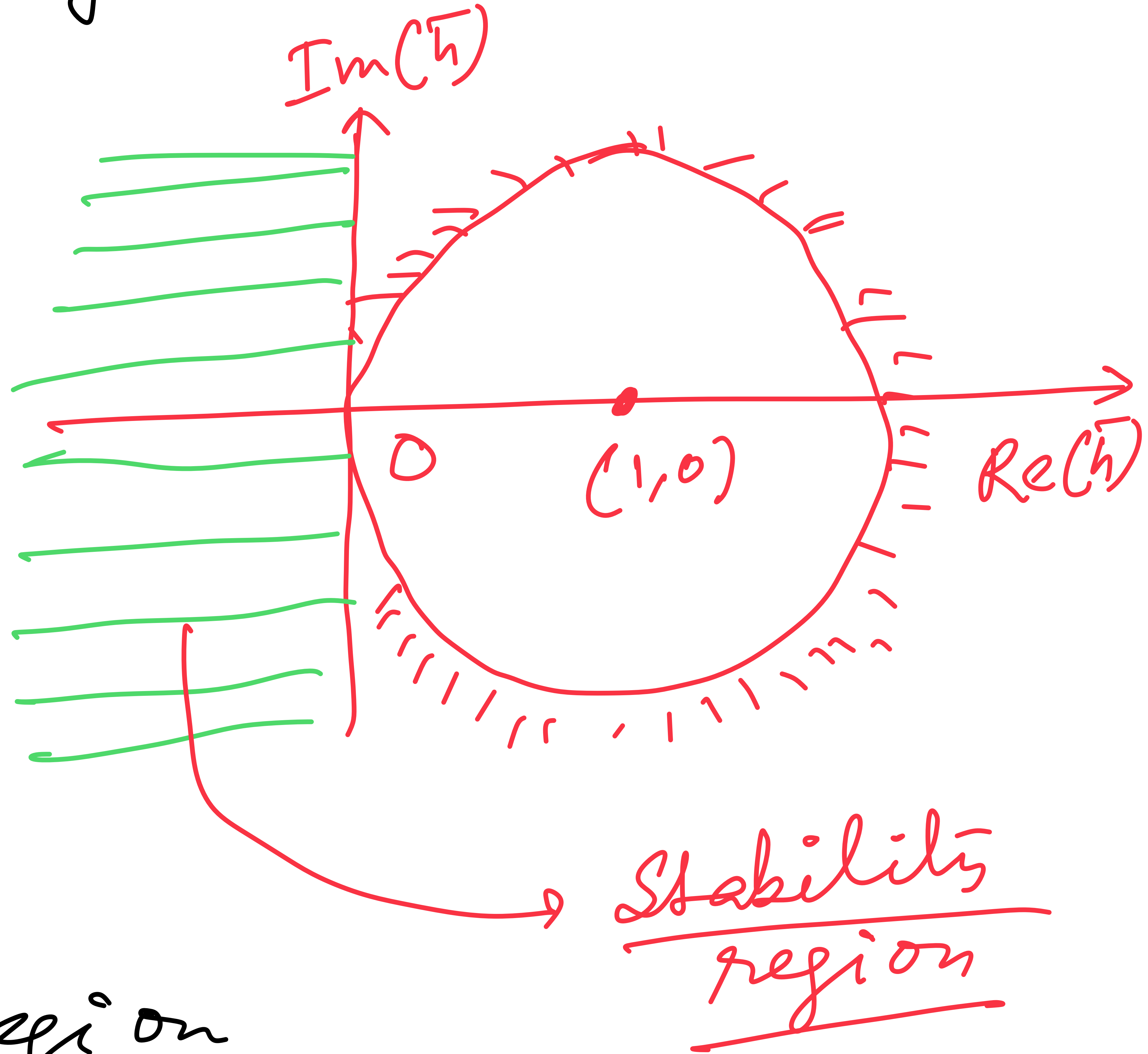
When λ is complex, $\bar{h} = \lambda h = x + iy$, with $Re(\lambda) < 0$.

$$|E(\lambda h)| = \left|\frac{1}{1-\lambda h}\right| < 1$$

$$= \left|\frac{1}{1-x-iy}\right| < 1$$

$$\text{or } |1-x-iy| > 1$$

$$\Rightarrow \boxed{(1-x)^2 + y^2 > 1}$$



which shows that the stability region occurs outside the circle with centre at (1,0) and radius 1. Since $Re(\lambda h) = x < 0$, the stability region is the entire left half of λh-plane.

[This method is A-stable]

Implicit Runge-Kutta Method:

The implicit Runge-Kutta method using ℓ slopes is defined as $\quad y_{n+1} = y_n + h\sum_{i=1}^{\ell} w_i K_i,$

where $\quad K_i = f\left(x_n + a_i h,\; y_n + h\sum_{j=1}^{\ell} a_{ij} K_j\right),$

$i = 1, 2, \dots$

$\ell = 1$: with one slope:

$$\boxed{\begin{aligned} y_{n+1} &= y_n + h w_1 K_1 \\ K_1 &= f(x_n + a_1 h,\; y_n + h a_{11} K_1) \end{aligned}}$$

$\ell = 2$: with two slopes: $\quad y_{n+1} = y_n + h(w_1 K_1 + w_2 K_2)$

where $k_1 = f\left(x_n + a_1 h,\ y_n + h\left(a_{11}k_1 + a_{12}k_2\right)\right)$

$k_2 = f\left(x_n + a_2 h,\ y_n + h\left(a_{21}k_1 + a_{22}k_2\right)\right)$

(2) The implicit R-K methods considering only one slope ($l=1$)

$$y_{n+1} = y_n + h\, w_1 k_1$$

where $k_1 = f\left(x_n + \alpha h,\ y_n + \alpha h k_1\right) \Rightarrow$

Here we need to determine the unknowns w_1 and α to get implicit methods

Single step method: $y_{n+1} = y_n + h\,\phi(x_n, y_n; h)$

Here $\phi(x_n, y_n; h) = w_1 k_1 = w_1 f\left(x_n + \alpha h,\ y_n + \alpha h k_1\right)$.

Consistency Condition: $\phi(x_n, y_n; 0) = f(x, y)$

$$\Rightarrow \boxed{w_1 = 1}$$

Truncation error: $T_n = \dfrac{y(x_{n+1}) - y(x_n)}{h} - \phi(x_n, y(x_n); h)$

$$\frac{y(x_{n+1}) - y(x_n)}{h} = \frac{1}{h}\left[y(x_n) + h y'(x_n) + \frac{h^2}{2!} y''(x_n) + \frac{h^3}{3!} y'''(x_n) + O(h^4) - y(x_n)\right]$$

$$= y'(x_n) + \frac{h}{2} y''(x_n) + \frac{h^2}{6} y'''(x_n) + O(h^3)$$

$$\phi(x_n, y(x_n); h) = f(x_n + \alpha h, y_n + \alpha h k_1)$$

$$= f(x_n, y_n) + \left[\alpha h f_x + \alpha h k_1 f_y\right]_{(x_n, y_n)}$$
$$+ \left[\alpha^2 h^2 f_{xx} + 2\alpha^2 h^2 k_1 f_{xy} + \alpha^2 h^2 k_1^2 f_{yy}\right]_{(x_n, y_n)}$$
$$+ O(h^3)$$

$$y''(x_n) = f_x + f f_y \Big|_{(x_n, y_n)}, \quad y'''(x_n) = \left(f_{xx} + 2 f f_{xy} + f^2 f_{yy}\right) + f_y\left(f_x + f f_y\right)\Big|_{(x_n, y_n)}$$

Now $T_n = f(x_n, y_n) + \frac{h}{2}\left(f_x + f f_y\right)\Big|_{(x_n,y_n)} + \frac{h^2}{6}\Big(f_{xx} + 2 f f_{xy} + f^2 f_{yy}$

$+ f_y (f_x + f f_y)\Big)\Big|_{(x_n,y_n)}$

$+ O(h^3) - f(x_n, y_n) - \left(\alpha h f_x + \alpha h k_1 f_y\right)\Big|_{(x_n,y_n)}$

$- \left(\alpha^2 h^2 f_{xx} + 2\alpha^2 h^2 k_1 f_{xy} + \alpha^2 h^2 k_1^2 f_{yy}\right)\Big|_{(x_n,y_n)}$

$+ O(h^3)$

$$= \left[\frac{h}{2}(f_x + f f_y) - \left(\alpha h f_x + \alpha h f_y \cdot f(x_n + \alpha h, y_n + h\alpha k_1)\right)\right]_{(x_n,y_n)}$$

$$+ h^2\Big[\frac{1}{6}\left(f_{xx} + 2 f f_{xy} + f^2 f_{yy}\right) + f_y (f_x + f f_y) - \alpha^2 f_{xx} - 2\alpha^2 f_{xy} \cdot f(x_n + \alpha h, y_n + \alpha h k_1) - \alpha^2 f_{yy} \cdot \left(f(x_n + \alpha h, y_n + \alpha h k_1)\right)^2\Big]_{(x_n,y_n)}$$

$+ O(h^3)$

Since

$k_1 = f(x_n + \alpha h, y_n + \alpha h k_1)$

Now

$$T_n = \left[\frac{h}{2}(f_x + f f_y) - \left\{ \alpha h f_x + \alpha h f_y \cdot (f + \alpha h f_x + \alpha h k_1 f_y + \cdots) \right\} \right]_{(x_n, y_n)} + O(h^2)$$

$$= h \left[\left(\frac{1}{2} - \alpha \right) f_x + \left(\frac{1}{2} - \alpha \right) f f_y \right] + O(h^2)$$

To vanish the $O(h)$ term in the error, we need to consider $\alpha = \frac{1}{2}$.

Then truncation error is $O(h^2)$, which imply that if you consider $\alpha = \frac{1}{2}$, the method will be 2nd order.

Now the 2nd order Implicit R-K method is given by

$$\boxed{\begin{aligned} y_{n+1} &= y_n + h K_1 \\ \text{where } K_1 &= f\left(x_n + \tfrac{h}{2},\ y_n + \tfrac{h}{2} K_1\right) \end{aligned}}$$

This is the method with one <u>slope</u>.

<u>Note</u>: To get these unknowns w_1 and α, one can compare the method

$$y_{n+1} = y_n + h f(x_n + \alpha h, y_n + \alpha h K_1)$$

with the Taylor Series method

$$y_{n+1} = y_n + h y'(x_n) + \frac{h^2}{2!} y''(x_n) + \frac{h^3}{3!} y'''(x_n) + \cdots$$

For these case, you compare the coefficients of h, h^2, h^3 of both methods.

The implicit R-K method with two slopes ($l=2$):

$$y_{n+1} = y_n + h\left(w_1 k_1 + w_2 k_2\right)$$

$$k_1 = f\left(x_n + a_1 h,\ y_n + a_{11} h k_1 + a_{12} h k_2\right)$$

$$k_2 = f\left(x_n + a_2 h,\ y_n + a_{21} h k_1 + a_{22} h k_2\right)$$

By applying the same procedure, we can get the unknowns as

$$w_1 = \frac{1}{2},\ w_2 = \frac{1}{2},\ a_1 = \frac{3-\sqrt{3}}{6},\ a_2 = \frac{3+\sqrt{3}}{6},$$

$$a_{11} = \frac{1}{4},\ a_{12} = \frac{3-2\sqrt{3}}{12},\ a_{21} = \frac{3+2\sqrt{3}}{12}$$

$$a_{22} = \frac{1}{4}$$

The truncation error is $O(h^4)$. So the method is of order 4.

The method is given by

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$

$$k_1 = f\left(x_n + \frac{3-\sqrt{3}}{6}h,\ y_n + \frac{h}{4}k_1 + \frac{3-2\sqrt{3}}{12}h k_2\right)$$

$$k_2 = f\left(x_n + \frac{3+\sqrt{3}}{6}h,\ y_n + \frac{3+2\sqrt{3}}{12}h k_1 + \frac{h}{4}k_2\right)$$

Clearly it is seen that with one slope, we have reached to second order method and with two slopes, we have reached to fourth order method, whereas in explicit scheme, to reach fourth order RK method, we need four slopes.

So this is one of the clear advantage of the implicit scheme.

(*) Also we have seen in stability analysis that the implicit methods have the large stability region than explicit methods. This is also another advantage of the implicit scheme.

(i) Explicit R-K methods:

Slopes	$l=1$	$l=2$	$l=3$	$l=4$	$l=5$
order	1	2	3	4	4

(ii) Implicit R-K methods:

Slopes	$l=1$	$l=2$
order	2	4

Example: Solve the IVP $y' = -2xy^2$, $y(0) = 1$ with $h = 0.2$ on the interval $[0, 0.4]$. Use the second order implicit Runge-Kutta method.

Sol^n: The second order implicit Runge Kutta method is given by

$$y_{n+1} = y_n + hK_1$$

$$K_1 = f\left(x_n + \frac{h}{2}, y_n + \frac{hK_1}{2}\right)$$

$$\boxed{K_1 = -2\left(x_n + \frac{h}{2}\right) \cdot \left(y_n + \frac{hK_1}{2}\right)^2}$$

This is an implicit eq^n in K_1 and can be solved by using iterative method. we generally use the Newton-Raphson method

we can write

$$F(K_1) = K_1 + 2\left(x_n + \frac{h}{2}\right)\left(y_n + \frac{hK_1}{2}\right)^2$$

$$F'(K_1) = 1 + 4\left(x_n + \frac{h}{2}\right)\left(y_n + \frac{hK_1}{2}\right) \cdot \frac{h}{2}$$

$$= 1 + 2h\left(x_n + \frac{h}{2}\right)\left(y_n + \frac{hK_1}{2}\right).$$

The Newton Raphson method gives —

$$K_1^{(s+1)} = K_1^{(s)} - \frac{F(K_1^{(s)})}{F'(K_1^{(s)})}, \quad s = 0, 1, 2 \ldots$$

Assume $K_1^{(0)} = f(x_0, y_0) = -2x_0 y_0^2 = \underline{0.}$

Now find $K_1^{(1)}, K_1^{(2)}$. Stop the iteration if $|K_1^{(s+1)} - K_1^{(s)}| < \varepsilon$. where ε is very small.

We will see $|K_1^{(2)} - K_1^{(1)}| < 10^{-3}$.

Therefore $K_1 = K_1^{(2)}$ (Find)! (Homework!)

$$\Rightarrow \boxed{y_1 = y_0 + hK_1 = 0.96152433.}$$

Now find y_2!

You need to find again K_1.

Take $K_1^{(0)} = f(x_1, y_1) = -2x_1y_1^2$

Again find $K_1^{(1)}, K_1^{(2)}, K_1^{(3)}$. This will be K_1. Therefore $y(0.4) = y_2 = y_1 + hK_1$
$= 0.86179013.$

Lecture 7 and 8 on Numerical Methods

Class contents

- Multi-step methods
- Explicit and implicit form
- Adams-Bashforth methods
- Nystrom methods
- Adams-Moulton methods
- Milne methods
- Milne-Simpson methods

Multistep methods:

- For single step methods, only one past value y_n at $x = x_{n+1}$ is required to compute y_{n+1} at next grid point $x = x_{n+1}$, whereas for multistep methods, it takes several past values.

- **Definition:** A numerical method for solving IVP $y' = f(x, y)$, $y(x_0) = y_0$ is called a multistep method or k step method if it uses the values of $y(x)$ and $y'(x)$ at $(k+1)$ successive mesh points $x_{n+1}, x_n, x_{n-1}, \ldots, x_{n-k+1}$, i.e, the values $y_{n+1}, y_n, y_{n-1}, \ldots, y_{n-k+1}$ to determine $y(x)$ at x_{n+1}.

$$\text{Input} \longrightarrow \boxed{y_{n+1},\ y_n,\ y_{n-1},\ y_{n-2}, \ldots, y_{n-k+1}} \xrightarrow{\text{output}} y_{n+1}$$

Multistep methods

Explicit | Implicit

General form:

$$y_{n+1} = a_1 y_n + a_2 y_{n-1} + \cdots + a_k y_{n-k+1} + h\,\phi(x_{n+1}, x_n, \ldots, x_{n-k+1}, y_{n+1}, y_n, \ldots, y_{n-k+1})$$

$$= a_1 y_n + a_2 y_{n-1} + \cdots + a_k y_{n-k+1} + h\left(b_0 y'_{n+1} + b_1 y'_n + \cdots + b_k y'_{n-k+1}\right)$$

$$\boxed{y_{n+1} = \sum_{i=1}^{k} a_i y_{n-i+1} + h \sum_{i=0}^{k} b_i y'_{n-i+1}}$$

Remarks: (i) If $b_0 = 0$, then we have

$$\boxed{y_{n+1} = \sum_{i=1}^{k} a_i y_{n-i+1} + h\sum_{i=1}^{k} b_i y'_{n-i+1}}$$

$\Rightarrow$ Explicit multistep method

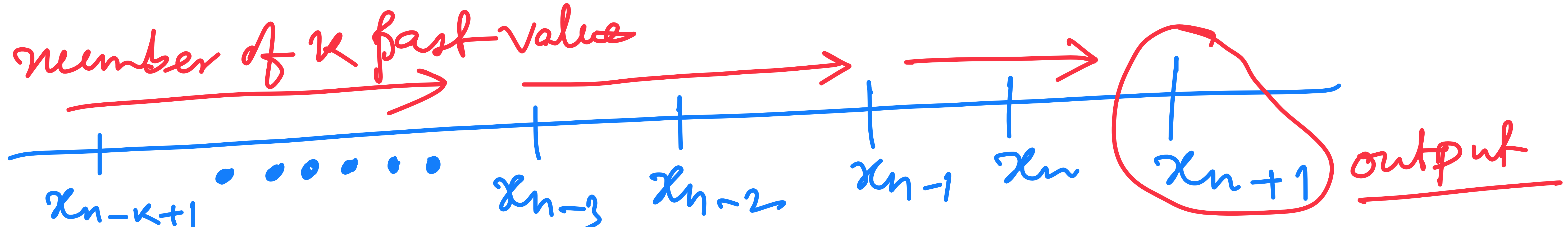
(ii) If $b_0 \neq 0$, then we have an implicit multistep method.

Explicit multistep methods:

Consider IVP — $y' = f(x,y),\ y(x_0) = y_0$ — (1)

now integrating between x_n to x_{n+1}, we get

$$y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f(x,y)\,dx \qquad \text{— (2)}$$



To evaluate $y(x_{n+1})$, we will approximate $f(x,y)$ by a polynomial that interpolates $f(x,y)$ at k points (x_n, y_n'), (x_{n-1}, y_{n-1}'), ..., (x_{n-k+1}, y_{n-k+1}'). We will use Newton backward difference interpolating polynomial of degree $(k-1)$.

$$P_{k-1}(x) = f_n + \frac{(x-x_n)}{h}\nabla f_n + \frac{(x-x_n)(x-x_{n-1})}{2!\,h^2}\nabla^2 f_n + \cdots$$

$$+ \frac{(x-x_n)\cdots(x-x_{n-k+2})}{(k-1)!\,h^{k-1}}\nabla^{k-1} f_n$$

$$+ \frac{(x-x_n)\cdots(x-x_{n-k+1})}{k!} f^{(k)}(\xi) \quad \text{--- } (3)$$

where $f_n = f(x_n, y_n)$

Now changing the variable in ③ by

$$\boxed{\frac{x - x_n}{h} = u} \Rightarrow x - x_n = uh \Rightarrow x = x_n + uh$$

$$\text{or } x = x_{n-1} + h + uh$$

$$= x_{n-1} + (u+1)h$$

$$\text{or } \boxed{\frac{x - x_{n-1}}{h} = u+1}$$

$$\frac{x - x_{n-2}}{h} = u+2, \qquad \frac{x - x_{n-k+2}}{h} = u+k-2$$

$$\frac{x - x_{n-k+1}}{h} = u+k-1$$

$$P_{k-1}(x) = f_n + u\nabla f_n + \frac{u(u+1)}{2!}\nabla^2 f_n + \cdots + \frac{u(u+1)\cdots(u+k-2)}{(k-1)!}\nabla^{k-1} f_n$$

$$+ \frac{(x-x_n)(x-x_{n-1})\cdots(x-x_{n-k+1})}{k!} f^{(k)}(\xi)$$

$$y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f(x,y)\,dx \quad , \quad \frac{x - x_n}{h} = u$$

$$\Rightarrow x = x_n + uh$$

$$\Rightarrow dx = h\,du$$

$$x = x_n \Rightarrow u = 0$$

$$x = x_{n+1} \Rightarrow u = 1$$

$$\simeq \; y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} P_{k-1}(x)\,dx$$

$$= y_n + h\int_0^1 \left[f_n + u\,\nabla f_n + \frac{u(u+1)}{2}\nabla^2 f_n + \cdots \frac{u(u+1)\cdots(u+k-2)}{(k-1)!}\nabla^{k-1} f_n \right] du$$

Where $\boxed{\text{error term} = \int_0^1 \frac{h^k\, u(u+1)\cdots(u+k-1)}{k!} f^{(k)}(\xi)\,du}$

$$\Rightarrow y_{n+1} = y_n + h\left[f_n + \frac{1}{2}\nabla f_n + \frac{5}{12}\nabla^2 f_n + \frac{3}{8}\nabla^3 f_n + \frac{251}{720}\nabla^4 f_n + \cdots + \nabla^{k-1} f_n \int_0^1 \frac{u(u+1)\cdots(u+k-2)}{(k-1)!}\,du \right]$$

These methods are known as Bashforth methods.

Error term:

$$T_K = \int_0^1 h^k \frac{u(u+1)\cdots(u+k-1)}{k!} f^{(k)}(\xi)\,du$$

$$= h^k \int_0^1 g(u)\, f^{(k)}(\xi)\,du$$

[Since $g(u)$ does not change its sign, apply integral mean value theorem]

$$= h^k f^{(k)}(\xi_1) \int_0^1 g(u)\,du, \qquad 0 < \xi_1 < 1$$

$$\boxed{T_K = h^K f^{(K)}(\xi_1) \cdot A}$$

$\Rightarrow$ This shows that this method is order K if we use K data points. Anyway we will check this now.

Explicit Adam Bashforth methods:

$$y_{n+1} = y_n + h\left[f_n + \frac{1}{2}\nabla f_n + \frac{5}{12}\nabla^2 f_n + \frac{3}{8}\nabla^3 f_n + \frac{251}{720}\nabla^4 f_n + \cdots\right]$$

$K=1$: $y_{n+1} = y_n + h f_n$, Euler method, order is 1

② $K=2$:

$$y_{n+1} = y_n + h\left[f_n + \frac{1}{2}\nabla f_n\right] \qquad \nabla f_n = f_n - f_{n-1}$$

$$= y_n + h\left[f_n + \frac{1}{2}(f_n - f_{n-1})\right]$$

$$= y_n + h\left[\frac{3f_n}{2} - \frac{1}{2}f_{n-1}\right]$$

Error:

Truncation error

$$T_n = \frac{y(x_{n+1}) - \sum_{i=1}^{K} a_i\, y(x_{n-i+1}) - h\sum_{i=1}^{K} b_i\, y'(x_{n-i+1})}{h}$$

$$= \frac{y(x_{n+1}) - \sum_{i=1}^{2} a_i\, y(x_{n-i+1}) - h\sum_{i=1}^{2} b_i\, y'(x_{n-i+1})}{h} \quad (\text{here } K=2)$$

$$= \frac{y(x_{n+1}) - y(x_n) - h\left[\frac{3}{2}y'(x_n) - \frac{1}{2}y'(x_{n-1})\right]}{h}$$

here $a_1 = 1,\ a_2 = 0$
$b_1 = 3/2,$
$b_2 = -1/2$

$$= \frac{y(x_n+h) - y(x_n) - h\left[\frac{3}{2}y'(x_n) - \frac{1}{2}y'(x_n-h)\right]}{h}$$

$$= \frac{1}{h}\Big[y(x_n) + h y'(x_n) + \frac{h^2}{2!}y''(x_n) + \frac{h^3}{3!}y'''(x_n) + O(h^4) - y(x_n) - h\Big[\frac{3}{2}y'(x_n) - \frac{1}{2}y'(x_n) + \frac{h}{2}y''(x_n) - \frac{h^2}{4}y'''(x_n) + O(h^3)\Big]\Big]$$

$$= \frac{1}{h}\Big[y(x_n) + h y'(x_n) + \frac{h^2}{2!}y''(x_n) + \frac{h^3}{3!}y'''(x_n) + O(h^4) - y(x_n) - \frac{3}{2}h y'(x_n) + \frac{h}{2}y'(x_n) - \frac{h^2}{2}y''(x_n) + \frac{h^3}{4}y'''(x_n) + O(h^4)\Big]$$

$$= \frac{1}{h}\Big[h y'(x_n) - h y'(x_n) + \frac{h^2}{2!}y''(x_n) - \frac{h^2}{2!}y''(x_n) + \frac{h^3}{6}y'''(x_n) + \frac{h^3}{4}y'''(x_n) + O(h^4)\Big]$$

$$= \frac{5h^2}{12}y'''(x_n) + O(h^3) = \frac{5h^2}{12}y'''(\xi_n), \quad x_{n-1} < \xi_n < x_{n+1}$$

order is 2.

K=3:

$$y_{n+1} = y_n + \frac{h}{12}\left[23 f_n - 16 f_{n-1} + 5 f_{n-2}\right]$$

error: $\boxed{T_n = \frac{3}{8} h^3 y^{(4)}(\xi_n), \quad x_{n-2} < \xi_n < x_{n+1}}$

you try!

$\Rightarrow$ order is 3

K=4:

$$y_{n+1} = y_n + h\left[f_n + \frac{1}{2}\nabla f_n + \frac{5}{12}\nabla^2 f_n + \frac{3}{8}\nabla^3 f_n\right]$$

$$= y_n + h/24\left[55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3}\right]$$

error: $\boxed{T_n = \frac{251}{720} h^4 y^{(5)}(\xi_n), \quad x_{n-3} < \xi_n < x_{n+1}}$

you try!

order is 4

This imply that k step explicit method gives a method of order k.

② **Example**: Find the approximate value of $y(0.3)$ using the Adams-Bashforth method of third order for the IVP

$$y' = x^2 + y^2, \quad y(0) = 1, \quad \text{with } h = 0.1.$$

Calculate the starting values using the corresponding Taylor Series method with the same step length.

Solⁿ: $f(x,y) = x^2 + y^2$, $x_0 = 0$, $y_0 = 1$.

The Adams Bashforth method of third order is given by

$$y_{n+1} = y_n + \frac{h}{12}\left[23f_n - 16f_{n-1} + 5f_{n-2}\right]$$

0	0.1	0.2	0.3
x_0	x_1	x_2	x_3
y_0	y_1	y_2	y_3

To get y_3 i.e, $y(0.3)$, we need y_0, y_1 and y_2.

The third order Taylor Series method is given by –

$$y_{n+1} = y_n + h y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n'''$$

$$y' = x^2 + y^2, \quad y'' = 2x + 2yy', \quad y''' = 2 + 2\left[yy'' + y'^2\right]$$

$x_0 = 0$, $y_0 = 1$, $y_0' = 1$, $y_0'' = 2$, $y_0''' = 8$. Apply Taylor Series method –

$\underline{n=0:}$ $y(0.1) \approx y_1 = y_0 + h y_0' + \frac{h^2}{2} y_0'' + \frac{h^3}{6} y_0'''$

$= 1.111333$

$\underline{n=1:}$ $y_2 = y(0.2) = 1.252625$

$\left[y_1' = 1.245061,\ y_1'' = 2.967355,\ y_1''' = 11.695793 \right]$

Now apply Adams-Bashforth method of fourth order —

$\underline{n=2:}$ $y(0.3) = y_3 = y_2 + \frac{h}{12}\left[23 f_2 - 16 f_1 + 5 f_0\right]$

$= y_2 + \frac{h}{12}\left[23 y_2' - 16 y_1' + 5 y_0'\right]$

$= 1.436688$

Example: Find the approximate value of $y(0.4)$ using the Adams – Bashforth method of fourth order for the initial value problem — $y' = x + y^2$, $y(0) = 1$ with $h = 0.1$.

Calculate the starting values using the Euler's method with the same step length.

Ans: $y(0.4) = 1.664847$

Nystrom method :

$$\text{IVP} - \quad y' = f(x,y), \quad y(x_0) = y_0$$

$$\Rightarrow \int_{x_{n-1}}^{x_{n+1}} y'\,dx = \int_{x_{n-1}}^{x_{n+1}} f(x,y)\,dx$$

$$\Rightarrow y(x_{n+1}) = y(x_{n-1}) + \int_{x_{n-1}}^{x_{n+1}} f(x,y)\,dx$$

To evaluate $y(x_{n+1})$, we will approximate $f(x,y)$ by a polynomial that interpolates $f(x,y)$ at K points (x_n, y_n'), (x_{n-1}, y_{n-1}'), $\ldots$, (x_{n-K+1}, y_{n-K+1}')

we will use Newton backward difference interpolating polynomial of degree $(K-1)$.

$$P_{k-1}(x) = f_n + \frac{(x-x_n)}{h}\nabla f_n + \frac{(x-x_n)(x-x_{n-1})}{2!\,h^2}\nabla^2 f_n + \cdots$$

$$+ \frac{(x-x_n)\cdots(x-x_{n-k+2})}{(k-1)!\,h^{k-1}}\nabla^{k-1} f_n$$

$$+ \frac{(x-x_n)\cdots(x-x_{n-k+1})}{k!} f^{(k)}(\xi)$$

where $f_n = f(x_n, y_n)$

Now apply the new variable u defined by —

$$\boxed{\frac{x-x_n}{h} = u}$$

$$y(x_{n+1}) = y(x_n) + \int_{x_{n-1}}^{x_{n+1}} f(x,y)\,dx \quad , \qquad \frac{x - x_n}{h} = u$$

$$\Rightarrow x = x_n + uh$$

$$\Rightarrow dx = h\,du$$

$$\approx \quad y_{n+1} = y_{n-1} + \int_{x_{n-1}}^{x_{n+1}} P_{k-1}(x)\,dx$$

$$x = x_{n-1} \Rightarrow u = -1$$

$$x = x_{n+1} \Rightarrow u = 1$$

$$= y_{n-1} + h\int_{-1}^{1} \left[f_n + u\nabla f_n + \frac{u(u+1)}{2}\nabla^2 f_n + \cdots \frac{u(u+1)\cdots(u+k-2)}{(k-1)!}\nabla^{k-1} f_n \right] du$$

where $\boxed{\text{error term} = \int_{-1}^{1} \frac{h^k\, u(u+1)\cdots(u+k-1)}{k!} f^{(k)}(\xi)\,du}$

$$y_{n+1} = y_{n-1} + h\left[2f_n + 0 \times \nabla f_n + \frac{1}{3}\nabla^2 f_n + \frac{1}{3}\nabla^3 f_n + \frac{29}{90}\nabla^4 f_n + \cdots \right]$$

This is Nystrom method.

K=1 :

$$\boxed{y_{n+1} = y_{n-1} + 2h f_n}$$, This is of 2nd order.

Truncation error : $T_n = \dfrac{y(x_{n+1}) - \sum_{i=1}^{K} a_i y(x_{n-i+1}) - h \sum_{i=1}^{K} b_i y'(x_{n-i+1})}{h}$

$$T_n = \frac{y(x_{n+1}) - y(x_{n-1}) - 2h y'(x_n)}{h}$$

now simplify to get the order of the method (Try!)

$$\boxed{T_n = \frac{h^2}{3} y'''(\xi) , \quad x_{n-1} < \xi < x_{n+1}}$$

$\Rightarrow \; |T_n| \leq M_2 h^2$, where $M_2 = \frac{1}{3} \max\limits_{\xi \in [x_0, b]} |y'''(\xi)|$

$\Rightarrow$ This shows that the method is of 2nd order.

K=2: we will have same 2nd order method.

If we take K=3, we will get 3rd order method.

Note: 1 step nystrom method gives 2nd order method but 2 step nystrom method gives again a 2nd order method. So there is no further improvement in the accuracy of the method. This is a clear drawback of the method.

So now we are looking for a explicit multi-step method to get a better numerical method.

Implicit method :

The general implicit multistep method is —

$$\boxed{y_{n+1} = \sum_{i=1}^{K} a_i\, y_{n-i+1} + h \sum_{i=0}^{K} b_i\, y'_{n-i+1}}$$

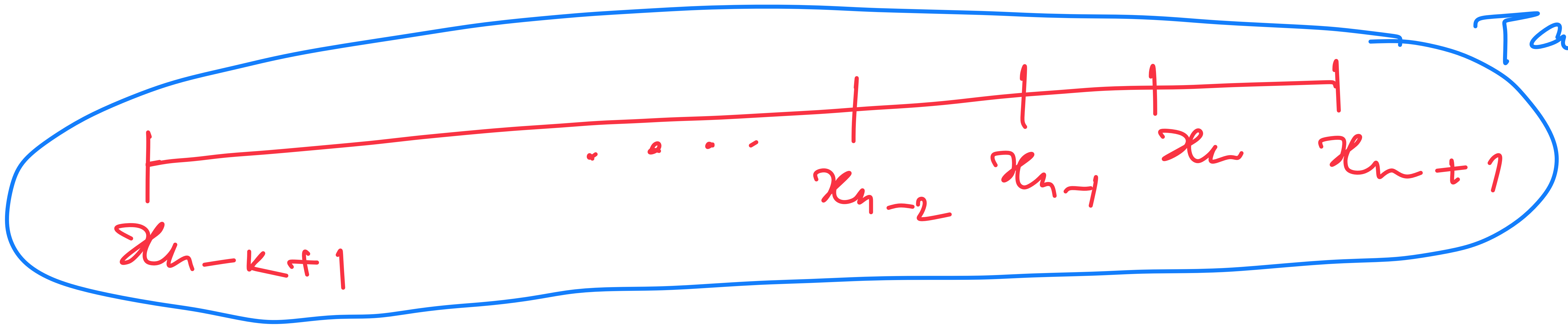
It takes $(K+1)$ data i.e, $(x_{n+1}, y'_{n+1}), (x_n, y'_n), (x_{n-1}, y'_{n-1}), \ldots, (x_{n-K+1}, y'_{n-K+1})$ to compute y_{n+1}.

IVP: $y' = f(x,y),\ y(x_0) = y_0,\ x \in [x_0, b]$.

Integrating between x_n and x_{n+1}, we have –

$$\boxed{y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f(x,y)\,dx}$$

To evaluate $y(x_{n+1})$, we will approximate $f(x,y)$ by a polynomial that interpolates $f(x,y)$ at $(k+1)$ points $(x_{n+1}, y'_{n+1}), (x_n, y'_n), \dots, (x_{n-k+1}, y'_{n-k+1})$. We will use Newton backward difference interpolating polynomial of degree k.



Take all data to compute y_{n+1}

$$P_k(x) = f_{n+1} + \frac{x - x_{n+1}}{h} \nabla f_{n+1} + \frac{(x-x_{n+1})(x-x_n)}{2!\, h^2} \nabla^2 f_{n+1}$$

$$+ \cdots + \frac{(x-x_{n+1})(x-x_n)\cdots(x-x_{n-k+2})}{k!\, h^k} \nabla^k f_{n+1}$$

$$+ \frac{(x-x_{n+1})\cdots(x-x_{n-k+1})}{(k+1)!} f^{k+1}(\xi)$$

error of interpolation and this will lead to truncation error.

Now take a new variable u as $u = \frac{x - x_n}{h}$.

$$\Rightarrow \frac{x - x_{n+1}}{h} = u - 1,$$

$$y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} P_K(x)\,dx$$

$\underline{x \to u}$

$$= y_n + h\int_0^1 P_K(u)\,du$$

You try to get this (Home work!)

$$\boxed{y_{n+1} = y_n + h\left[f_{n+1} - \frac{1}{2}\nabla f_{n+1} - \frac{1}{12}\nabla^2 f_{n+1} - \frac{1}{24}\nabla^3 f_{n+1} - \frac{19}{720}\nabla^4 f_{n+1} - \cdots\right]}$$

This is Adams Moulton method.

$\underline{K=1}$:
$$y_{n+1} = y_n + h\left[f_{n+1} - \frac{1}{2}\nabla f_{n+1}\right]$$
$$= y_n + \frac{h}{2}\left[f_n + f_{n+1}\right] \to \text{Trapezium method}$$

Truncation error $= -\frac{h^3}{12} y'''(\xi)$, $x_n < \xi < x_{n+1}$

(Try to find it)

$\Rightarrow$ This method is of 2nd order.

K=2:

$$y_{n+1} = y_n + h\left[f_{n+1} - \frac{1}{2}\nabla f_{n+1} - \frac{1}{12}\nabla^2 f_{n+1}\right]$$

$$= y_n + \frac{h}{12}\left[5f_{n+1} + 8f_n - f_{n-1}\right]$$

T. E. : $T_n = -\frac{h^3}{24} y^{(4)}(\xi)$, $x_{n-1} < \xi < x_{n+1}$

$\Rightarrow$ This method is of order 3.

Try!

K=3: $y_{n+1} = y_n + h\left[f_{n+1} - \frac{1}{2}\nabla f_{n+1} - \frac{1}{12}\nabla^2 f_{n+1} - \frac{1}{24}\nabla^3 f_{n+1}\right]$

$$\boxed{y_{n+1} = y_n + \frac{h}{24}\left[9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}\right]}$$

Truncation error: $TE = -\frac{19}{720}h^4 y^{(5)}(\xi), \quad x_{n-2} < \xi < x_{n+1}$

The method is of fourth order.

Note: K step implicit multistep method gives (K+1) order method, whereas K step explicit multistep method gives K order method.

This implicit Adams moulton methods clearly provide better accurate approximated solution than by the Adams-Bashforth methods.

Still we will search another best method to solve IVP:

Milne-Simpson method:

$$y' = f(x,y), \quad y(x_0) = y_0, \quad x \in [x_0, b]$$

$\Rightarrow$ Now we will proceed same as Nystrom method if we get any better approximated method.

Integrating between x_{n-1} and x_{n+1}, we get

$$\int_{x_{n-1}}^{x_{n+1}} y'\,dx = \int_{x_{n-1}}^{x_{n+1}} f(x,y)\,dx$$

or $$\boxed{y(x_{n+1}) = y(x_{n-1}) + \int_{x_{n-1}}^{x_{n+1}} f(x,y)\,dx.}$$

Now approximate $f(x,y)$ by Newton backward difference interpolating polynomial of degree k with $(k+1)$ data

(x_{n+1}, y'_{n+1}), (x_n, y'_n), $\ldots$ (x_{n-k+1}, y'_{n-k+1}).

Apply a same procedure as considered in Adams-Moulton methods.

now we will get

$$y_{n+1} = y_{n-1} + h\int_{-1}^{1}\left[f_{n+1} + (u-1)\nabla f_{n+1} + \frac{(u-1)u}{2}\nabla^2 f_{n+1} + \cdots\right] du$$

$$= y_{n-1} + h\left[2f_{n+1} - 2\nabla f_{n+1} + \frac{1}{3}\nabla^2 f_{n+1} + 0\cdot\nabla^3 f_{n+1} - \frac{\nabla^4}{90} f_{n+1} - \cdots\right]$$

This methods are called as <u>Milne's methods</u>.

$K=0$: $y_{n+1} = y_{n-1} + 2h f_{n+1}$, order 1

$K=1$: $y_{n+1} = y_{n-1} + h\left[2 f_{n+1} - 2 \nabla f_{n+1}\right]$

$= y_{n-1} + 2h f_n$, This is order 2

This is an explicit method.

$K=2$: $y_{n+1} = y_{n-1} + h\left[2 f_{n+1} - 2 \nabla f_{n+1} + \frac{\nabla^2}{3} f_{n+1}\right]$

$= y_{n-1} + \frac{h}{3}\left[f_{n+1} + 4 f_n + f_{n-1}\right]$

This method is Milne-Simpson's method.

$$TE = -\frac{h^4}{90} y^{(5)}(\xi), \quad x_{n-1} < \xi < x_{n+1}$$

$\Rightarrow$ This method is of fourth order.

Remarks: 2 step Milne method gives the 4th order Milne-Simpson's method. So this is a clear improvement in numerical scheme.

The best method in multistep methods is implicit Milne-Simpson's method.

Example: For the IVP $y' = x + y$, $y(0) = 1$, estimate $y(0.5)$ using

(i) Third order Adams-Moulton method

(ii) Milne Simpson fourth order method

with $h = 0.1$ and compare the results with exact solution.

Try!

(Homework)

Ans: (i) $y(0.5) = 1.7974827$,

(ii) $\boxed{y(0.5) = 1.7974430}$

clearly this solution is the best.

Exact: 1.7974425

Lecture 9, 10, 11 and 12
On Numerical Methods

Class contents

- Predictor and corrector methods
- Local truncation error for linear multistep methods
- Order of the methods
- Consistency of the methods
- Convergency
- Stability analysis

Predictor - Corrector methods
(P-C methods)

we have solved the IVP $y' = f(x,y)$, $y(x_0) = y_0$, by applying explicit and implicit single and multi step methods.

If we perform the analysis for stability (numerical stability) then we find that all the explicit methods converge only if h is sufficiently small. Thefore to get the numerical solution to the IVP, we need a lots of step if the integration of the IVP is to be performed over a large interval.

On the other hand most of the Implicit methods have strong stability properties, that is one can choose sufficiently large values of h for integration. However, we need to solve a nonlinear eqn by iteration at each step, which is also expensive.

Hence we combine the explicit and implicit methods to define a new class of methods called Predictor — Corrector methods.

The explicit methods are called as Predictor and implicit methods are called as Corrector.

- Predictor (P): Predict an approximation $y_{n+1}^{(P)}$ to compute y_{n+1} using explicit methods.

- Corrector methods (C): Correct using an implicit method to obtain $y_{n+1}^{(c)}$.

- Remarks: The order of predictor should be less than or equal to the order of corrector.

If both are of the same methods, then we may need only 1 or 2 iterations of correctors.

However if the predictor is of lower order, then we need to use more iterations of the corrector.

Example - 1 :

Let us consider —

Predictor (P) : $y_{n+1}^{(P)} = y_n + hf(x_n, y_n)$ [Euler method]

Corrector (C) : $y_{n+1}^{(C)} = y_n + hf(x_{n+1}, y_{n+1}^{(P)})$ [Backward Euler method]

So we have —

initial approximation $\Rightarrow$ $y_{n+1}^{(0)} = y_n + hf(x_n, y_n)$ $\longrightarrow$ Predictor

1st approx. $\Rightarrow$ $y_{n+1}^{(1)} = y_n + hf(x_{n+1}, y_{n+1}^{(0)})$

2nd approx. $\Rightarrow$ $y_{n+1}^{(2)} = y_n + hf(x_{n+1}, y_{n+1}^{(1)})$

$\vdots$

$(K+1)^{th}$ approx $\Rightarrow$ $y_{n+1}^{(K+1)} = y_n + hf(x_{n+1}, y_{n+1}^{(K)})$ $\longrightarrow$ This will be the solution

Correctors

Iteration Stop Criteria :

$$\left| y_{n+1}^{(K+1)} - y_{n+1}^{(K)} \right| < \varepsilon$$

where ε is very small quantity.

Example - 2:

$$P: \quad y_{n+1}^{(P)} = y_n + h f(x_n, y_n) \quad [\text{Euler method}]$$

First order

$$C: \quad y_{n+1}^{(C)} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^{(P)})\right] \quad [\text{Trapezium method}]$$

Second order

Therefore use the procedure adopted in last example.

Example - 3:

P: Adams-Bashforth method of fourth order

$$y_{n+1}^{(P)} = y_n + \frac{h}{24}\left[55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3}\right]$$

C: Adams Moulton method of fourth order

$$y_{n+1}^{(C)} = y_n + \frac{h}{24}\left[f_{n-2} - 5 f_{n-1} + 19 f_n + 9 f(x_{n+1}, y_{n+1}^{(P)})\right]$$

Example-4: Milne's methods:

P: Adams-Bashforth method of fourth order

$$y_{n+1}^{(p)} = y_{n-3} + \frac{4h}{3}\left[2f_n - f_{n-1} + 2f_{n-2}\right]$$

C: Milne-Simpson's method of fourth order

$$y_{n+1}^{(c)} = y_{n-1} + \frac{h}{3}\left[f\left(x_{n+1}, y_{n+1}^{(p)}\right) + 4f_n + f_{n-1}\right]$$

Ex: Using the Adams-Bashforth Predictor-Corrector eqns evaluate $y(1.4)$, if y satisfies

$$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}, \text{ and } y(1) = 1, \; y(1.1) = 0.996$$

$y(1.2) = 0.986$, $y(1.3) = 0.972$.

Ans: P: $y_{n+1}^{(P)} = y_n + \frac{h}{24}\left[55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3}\right]$ → Adams Bashforth method of fourth order

C: $y_{n+1}^{(C)} = y_n + \frac{h}{24}\left[9 f(x_{n+1}, y_{n+1}^{(P)}) + 19 f_n - 5 f_{n-1} + f_{n-2}\right]$ → Adams Moulton method of fourth order.

$n=0$	$n=1$	$n=2$	$n=3$	$n=4$
1	1.1	1.2	1.3	1.4
x_0	x_1	x_2	x_3	x_4
y_0	y_1	y_2	y_3	y_4

$h = 0.1$

$f(x,y) = \frac{1}{x^2} - \frac{y}{x}$, $y_0 = 1$, $y_1 = 0.996$, $y_2 = 0.986$, $y_3 = 0.972$.

Predictor: $n=3$:

$$y_4^{(0)} = y_4^{(P)} = y_3 + \frac{h}{24}\left[55 f_3 - 59 f_2 + 37 f_1 - 9 f_0\right]$$

$f_0 = 0$, $f_1 = -0.079008$, $f_2 = -0.127222$,

$f_3 = -0.155976$.

$y_4^{(0)} = 0.955357$

Corrector: $y_4^{(1)} = y_4^{(c)} = y_3 + \frac{h}{24}\left[9 f(x_4, y_4^{(0)}) + 19 f_3 - 5 f_2 + f_1\right]$

First iteration

$= 0.955516$

Second iteration: $y_4^{(2)} = y_3 + \frac{h}{24}\left[9 f(x_4, y_4^{(1)}) + 19 f_3 - 5 f_2 + f_1\right]$

$= 0.955512$

$y_4^{(2)} = 0.955512 \longrightarrow$ Correct up to 5 decimal places.

$\left| y_4^{(2)} - y_4^{(1)} \right| = 0.0000004 < 5 \times 10^{-6}$

Example: Given $y' = x^3 + y$, $y(0) = 2$, the values $y(0.2) = 2.073$, $y(0.4) = 2.452$ and $y(0.6) = 3.023$ are obtained by Runge Kutta method of fourth order. Find $y(0.8)$ by Milne's Predictor – Corrector method taking $h = 0.2$.

$\boxed{y(0.8) = 3.7689}$, Correct up to three decimal places.

Try! [You apply Example – 4 Predictor – Corrector method]

Local Truncation Error:

$$T_n = \frac{y(x_{n+1}) - \sum_{i=1}^{k} a_i\, y(x_{n-i+1}) - h \sum_{i=0}^{k} b_i\, y'(x_{n-i+1})}{h}$$

$$x_{n-i+1} = x_n + (1-i)h$$

Expanding in Taylor Series :-

$$y(x_{n-i+1}) = y(x_n + (1-i)h) = y(x_n) + (1-i)h\,y'(x_n) + \frac{(1-i)^2}{2!} h^2 y''(x_n)$$

$$+ \cdots + \frac{(1-i)^p h^p}{p!} y^{(p)}(x_n) + O(h^{p+1}) = \sum_{q=0}^{p} \frac{(1-i)^q}{q!} h^q y^{(q)}(x_n) + O(h^{p+1})$$

$$y'(x_{n-i+1}) = y'(x_n + (1-i)h) = y'(x_n) + (1-i)h\,y''(x_n) + \frac{(1-i)^2}{2!} h^2 y'''(x_n)$$

$$+ \cdots + \frac{(1-i)^p h^p}{p!} y^{(p+1)}(x_n) + O(h^{p+1})$$

$$y'(x_{n-i+1}) = \sum_{q=1}^{p+1} \frac{(1-i)^{q-1}}{(q-1)!} h^{q-1} \, y^{(q)}(x_n) + o(h^{p+1})$$

$$h\,y'(x_{n-i+1}) = \sum_{q=1}^{p} \frac{(1-i)^{q-1}}{(q-1)!} h^{q} y^{(q)}(x_n) + o(h^{p+1})$$

$$y(x_{n+1}) = y(x_n + h) = y(x_n) + h\,y'(x_n) + \frac{h^2}{2!} y''(x_n) + \cdots + \frac{h^p}{p!} y^{(p)}(x_n) + o(h^{p+1})$$

$$= \sum_{q=0}^{p} \frac{h^q}{q!} y^{(q)}(x_n)$$

Now

$$T_n = \frac{y(x_{n+1}) - \sum_{i=1}^{K} a_i\, y(x_{n-i+1}) - h \sum_{i=0}^{K} b_i\, y'(x_{n-i+1})}{h}$$

$$= \frac{1}{h}\left[\sum_{q=0}^{p} \frac{h^q}{q!} y^{(q)}(x_n) - \sum_{i=1}^{K} a_i \sum_{q=0}^{p} \frac{(1-i)^q}{q!} h^q y^{(q)}(x_n) - \sum_{i=0}^{K} b_i \sum_{q=1}^{p} \frac{(1-i)^{q-1}}{(q-1)!} h^q y^{(q)}(x_n) + o(h^{p+1}) \right]$$

$$T_n = \frac{1}{h}\left[\sum_{q=0}^{p}\frac{h^q}{q!}y^{(q)}(x_n) - \sum_{i=1}^{K}a_i\sum_{q=0}^{p}\frac{(1-i)^q}{q!}h^q y^{(q)}(x_n) - \sum_{i=0}^{K}b_i\sum_{q=1}^{p}\frac{(1-i)^{q-1}}{(q-1)!}h^q y^{(q)}(x_n) + O(h^{p+1})\right]$$

$$= \frac{1}{h}\Bigg[\underbrace{\left\{y(x_n) - \sum_{i=1}^{K}a_i \cdot y(x_n)\right\}}_{\text{For } q=0} + \left\{\sum_{q=1}^{p}\frac{h^q}{q!}y^{(q)}(x_n) - \sum_{i=1}^{K}a_i\sum_{q=1}^{p}\frac{(1-i)^q}{q!}h^q y^{(q)}(x_n)\right.$$

$$\left. - \sum_{i=0}^{K}b_i\sum_{q=1}^{p}\frac{(1-i)^{q-1}}{(q-1)!}h^q y^{(q)}(x_n)\right\} + O(h^{p+1})\Bigg]$$

$$= \frac{1}{h}\left[\left\{\left(1-\sum_{i=1}^{K}a_i\right)y(x_n)\right\} + \sum_{q=1}^{p}\left\{\frac{1}{q!} - \sum_{i=1}^{K}a_i\frac{(1-i)^q}{q!} - \sum_{i=0}^{K}b_i\frac{(1-i)^{q-1}}{(q-1)!}\right\}h^q y^{(q)}(x_n) + O(h^{p+1})\right]$$

$$T_n = \frac{1}{h}\left[C_0\, y(x_n) + \sum_{q=1}^{p} C_q\, h^q y^{(q)}(x_n) + O(h^{p+1})\right]$$

where $C_0 = 1 - \sum_{i=1}^{K} a_i$

$$C_q = \frac{1}{q!}\left[1 - \sum_{i=1}^{K} a_i (1-i)^q\right] - \frac{1}{(q-1)!}\sum_{i=0}^{K} b_i (1-i)^{q-1},$$

$q = 1, 2, \ldots, p.$

Order :

Def^n: The linear multistep method is said to be order p if $C_0 = C_1 = C_2 = \cdots = C_p = 0$ and $C_{p+1} \neq 0$.

$\Rightarrow$ If the method is of order p, then we have –

$$T_n = \frac{1}{h}\left[C_{p+1} h^{p+1} y^{(p+1)}(x_n) + O(h^{p+2})\right]$$

$$= C_{p+1} h^{p} \cdot y^{(p+1)}(x_n) + O(h^{p+1}).$$

Example: Derive a 2^{nd} order method of the form

$$y_{n+1} = a\, y_{n-2} + h\left(b\, y_n' + c\, y_{n-1}'\right)$$

and find the corresponding error.

Try!

Home work!

Hints: 2^{nd} order method means $C_0 = C_1 = C_2 = 0$ - from these three conditions, you will get three eqns, then find three unknowns a, b, and c.

Characteristic Polynomial :

Linear multistep method :

$$y_{n+1} = \sum_{i=1}^{K} a_i y_{n-i+1} + h \sum_{i=0}^{K} b_i y'_{n-i+1}$$

$$\Rightarrow \quad y_{n+1} - \sum_{i=1}^{K} a_i y_{n-i+1} - h \sum_{i=0}^{K} b_i y'_{n-i+1} = 0$$

$E y_i = y_{i+1}$, $\boxed{E y_{n-1} = y_n, \quad E y_n = y_{n+1} \quad E^2 y_{n-1} = y_{n+1}}$

Forward operator or shift operator

$$\left.\begin{aligned} E y_{n-K+1} &= y_{n-K+2} \\ E^2 y_{n-K+1} &= y_{n-K+3} \\ &\vdots \\ E^{K-1} y_{n-K+1} &= y_n \\ E^K y_{n-K+1} &= y_{n+1} \end{aligned}\right\}$$

we have

$$\rho(E)\, y_{n-k+1} - h\,\sigma(E)\, y'_{n-k+1} = 0$$

where $\rho(E) = E^k - a_1 E^{k-1} - a_2 E^{k-2} - \cdots - a_k E^0$

and $\sigma(E) = b_0 E^k + b_1 E^{k-1} + \cdots + b_k$

- 1st characteristic polynomial:

$$\rho(\xi) = \xi^k - a_1 \xi^{k-1} - a_2 \xi^{k-2} - \cdots - a_k$$

- 2nd characteristic polynomial:

$$\sigma(\xi) = b_0 \xi^k + b_1 \xi^{k-1} + \cdots + b_k$$

Example: $y_{n+1} = y_n + \frac{h}{12}\left(23y_n' - 16y_{n-1}' + 5y_{n-2}'\right)$

Solⁿ: $y_{n+1} - y_n - \frac{h}{12}\left(23y_n' - 16y_{n-1}' + 5y_{n-2}'\right) = 0$

$Ey_{n-2}' = y_{n-1}'$, $E^2y_{n-2}' = y_n'$

$E^2y_{n-2} = y_n$, $E^3y_{n-2} = y_{n+1}$

$$E^3y_{n-2} - E^2y_{n-2} - \frac{h}{12}\left[23E^2y_{n-2}' - 16Ey_{n-2}' + 5E^0y_{n-2}'\right] = 0$$

$$\Rightarrow \boxed{\rho(E)\,y_{n-2} - h\,\sigma(E)\,y_{n-2}' = 0}$$

where $\rho(E) = E^3 - E^2$ and $\sigma(E) = \frac{23}{12}E^2 - \frac{16}{12}E + \frac{5}{12}$

1st characteristic polynomial: $\rho(\xi) = \xi^3 - \xi^2$

2nd characteristic polynomial: $\sigma(\xi) = \frac{23}{12}\xi^2 - \frac{4}{3}\xi + \frac{5}{12}$

Consistency:

Defn: The linear multistep method is said to be consistent if it has order $p \geq 1$.

Since $p \geq 1$, we have $C_0 = C_1 = 0$ [for $p=1$]

$$C_0 = 1 - \sum_{i=1}^{k} a_i = 0 \Rightarrow \boxed{\rho(1) = 0}$$

Since $\rho(1) = 1 - \sum_{i=1}^{k} a_i$

$$C_q = \frac{1}{q!}\left[1 - \sum_{i=1}^{k} a_i (1-i)^q\right] - \frac{1}{(q-1)!}\sum_{i=0}^{k} b_i (1-i)^{q-1}, \quad q = 1, 2, \ldots p$$

This is for k step method.

$$\boxed{C_1 = \left[1 + a_2 + 2a_3 + \cdots + (k-1)a_k\right] - \left[b_0 + b_1 + \cdots + b_k\right] = 0}$$

$\Rightarrow \quad C_1 = 0$

$$\Rightarrow 1 + \left(a_1 + 2a_2 + \cdots + k a_k\right) - \left(a_1 + a_2 + \cdots + a_k\right) - \sigma(1) = 0$$

or $\quad 1 + \rho'(1) - 1 - \sigma(1) = 0 \quad \Rightarrow \boxed{\rho'(1) = \sigma(1)}$

Therefore for a consistent of the method, we have the relations in terms of characteristic polynomials

$$- \quad \boxed{\rho(1) = 0, \quad \rho'(1) = \sigma(1)}$$

Example : **Adams- Bashforth method of 2nd order :**

$$y_{n+1} = y_n + \frac{h}{2}\left(3f_n - f_{n-1}\right), \quad \boxed{T_n = \frac{5}{12}h^2 y'''(\xi)}$$

Truncation error

$$\Rightarrow \left(E^2 y_{n-1} - E y_{n-1}\right) - h\left[\frac{3}{2} E y'_{n-1} - \frac{1}{2} y'_{n-1}\right] = 0$$

$$\left.\begin{aligned} \Rightarrow \rho(\xi) &= \xi^2 - \xi \\ \text{and } \sigma(\xi) &= \frac{3}{2}\xi - \frac{1}{2} \end{aligned}\right\} \Rightarrow$$

These are satisfying the following relation

$\rho(1) = 0$

and $\rho'(1) = \sigma(1)$

Also truncation error shows that the method is order 2

by the def^n

So, this method is consistent

② **Example:** Adams – Moulton method of 2^{nd} order:

$$y_{n+1} = y_n + \frac{h}{12}\left[5y'_{n+1} + 8y'_n - y'_{n-1}\right]$$

Homework!

Try to show that it is consistent method.

Convergence and Stability of the multi-step methods:

(i) **Convergence:** The linear multistep method is said to be convergent if for all IVPs subject to the hypothesis of Picard's existence theorem, we have

$$\lim_{h \to 0} y_n = y(x_n), \quad 0 \le n \le N \quad \forall\, x \in [x_0, b] \text{ and}$$

provided the rounding errors arising from all the initial conditions tend to zero.

A necessary and sufficient condition for the convergence of the method are the Consistency and Zero-Stability (Later it is written in Dahlquist's Equivalence Theorem)

⇒ Now we will know about Zero stability.

Zero stability:

To apply a linear k-step method to the IVP, we need k-starting values, $y_0, y_1, y_2, \ldots y_{k-1}$. Note that y_0 is given by initial condition and other values $y_1, \ldots, y_{k-1}$ have to be computed by using a suitable single step methods (i.e. R-K methods). So, these computed numerical values $y_1, \ldots y_{k-1}$

contain numerical errors and it is important to know how these will affect further approximation y_n, $n \geq k$, which are calculated by mean of linear multistep methods. Thus we need to consider the stability of the numerical method w.r.t. small perturbations in the starting conditions.

Definition: A linear k-step method is said to be zero-stable if exists a constant M such that, for any two sequences $\{y_n\}$ and $\{z_n\}$ which have been generated by the same formulae but different starting values $y_0, y_1, \ldots, y_{k-1}$ and $z_0, z_1, \ldots, z_{k-1}$, respectively, we have

$$\boxed{|y_n - z_n| \leq M \max\{|y_0 - z_0|, |y_1 - z_1|, \ldots, |y_{k-1} - z_{k-1}|\}}$$

for $x_n \leq b$ and h tends to 0.

Note:

Zero-stability of the method can be determined by merely considering its behaviour when applied to the differential eqⁿ $y' = 0$, corresponding to $f(x,y) \equiv 0$.

i.e one can decide whether a method is zero-stable or not by considering only $y' = 0$ i.e, $f(x,y) = 0$.

For this reason, the concept of stability is known as zero stability.

However to check the zero stability of the linear multistep methods, it would be very tedious exercise by using the definition of zero stability. Instead of that we can use the <u>Root Condition</u> to check the zero stability.

Root condition is an algebraic method which is equivalent to the analysis of $\rho = 0$

$\Rightarrow$ $\boxed{\text{Root Condition} \longrightarrow \text{Zero-Stability.}}$

Root Condition: The linear multistep method is said to satisfy the root condition if all roots of the eqn $\rho(\zeta) = 0$, where $\rho(\zeta)$ is the first characteristic polynomial, lie inside the closed unit disc in the complex plane and are simple if they lie on the unit circle.

The Root Condition is an alternative way to define zero stability.

Note: The linear multistep method is zero stable for any IVP $y' = f(x,y)$, $y(x_0) = y_0$, $x \in [x_0, b]$, where f satisfies the hypotheses of Picard's theorem, if and only if it satisfies the Root Condition.

- If Root condition is violated then the method is not zero stable.

- If we have all roots ξ_j of $\rho(\xi) = 0$, $j = 0, 1 \ldots p$, then

  when the degree of the polynomial is $p+1$.

  Root Conditions $\Rightarrow$ $\underline{|\xi_j| \leq 1}$

① Example : Euler method :

$$y_{n+1} = y_n + h y_n'$$
$$\Rightarrow E y_n - y_n + h y_n' = 0$$
$$\Rightarrow (E-1) y_n + h y_n' = 0$$

1st characteristic polynomial
$\rho(\xi) = \xi - 1$

$\rho(\xi) = \xi - 1 = 0$
$\Rightarrow$ root is $\xi = 1$

So this root lie on the unit circle

$\Rightarrow$ This method is zero stable

Similarly it is same for Backward Euler method.

② Adams Bashforth method of fourth order :

$\rho(\xi) = \xi^4 - \xi^3 = \xi^3(\xi - 1)$, $\xi = 0$ multiple root and $\xi = 1$ simple root

So this method is zero-stable.

Example: Three step method:

$$11y_{n+3} + 27y_{n+2} - 27y_{n+1} - 11y_n = 3h\left(f_{n+3} + 9f_{n+2} + 9f_{n+1} + f_n\right)$$

is not zero stable.

roots: $\xi_1 = 1$, $|\xi_1| = 1$

$\xi_2 = -0.32$, $|\xi_2| < 1$

$\xi_3 = -3.14$, $|\xi_3| > 1$

we know that $|\xi_i| \le 1$ for satisfying root condition.

So, root condition is not satisfied

$\Rightarrow$ not zero stable.

Example: Three step method:

$$y_{n+3} + y_{n+2} - y_{n+1} - y_n = 2h(f_{n+2} + f_{n+1})$$

Is this method zero stable? Try! (Homework!)

Dahlquist's Equivalence Theorem: For a linear k-step method that is consistent with IVP, where $f(x,y)$ is assumed to satisfy the Lipschitz condition, and with consistent starting values, zero-stability is necessary and sufficient for convergence. Moreover if the solution y has continuous derivative of order $(p+1)$ and truncation error $O(h^p)$ of the method then the global error $e_n = y(x_n) - y_n$ is also $O(h^p)$.

Theorem on Convergence: (Convergence Theorem)

The linear multistep method is convergent if and only if the method is consistent and satisfies the root condition.

Note that: $\boxed{\text{Root Condition} \Rightarrow \text{Zero Stability}}$

Convergence $\Leftrightarrow$ Consistency + Root Condition or Zero stability

② Dahlquist's Barrier Theorem:

The order of accuracy of a zero-stable k-step method cannot exceed $k+1$ if k is odd, or $k+2$ if k is even.

Examples: ① Zero-stable 4-step Adams Bashforth method has order 4.

② 3step Adams Moulton method has order 4.

These two methods satisfy the Barrier Theorem.

Stability :

Stability $\longrightarrow$ with respect to $h \to 0$ and $n \to \infty$.

Here we are checking behaviour by applying $y' = \lambda y$

This is the linearization of $f(x,y)$.

Linear multistep method :

$$y_{n+1} = \sum_{i=1}^{K} a_i y_{n-i+1} + h \sum_{i=0}^{K} b_i y'_{n-i+1}$$

$$\Rightarrow y_{n+1} - \sum_{i=1}^{K} a_i y_{n-i+1} - h \sum_{i=0}^{K} b_i y'_{n-i+1} = 0$$

or $\quad \rho(E)\, y_{n-i+1} - h\, \sigma(E)\, y'_{n-i+1} = 0$

where $\rho(E) = E^K - a_1 E^{K-1} - a_2 E^{K-2} - \cdots - a_K$

and $\sigma(E) = b_0 E^K + b_1 E^{K-1} + \cdots + b_K$

Now apply the test eqn $\boxed{y' = \lambda y}$, we have

$$\rho(E)\, y_{n-i+1} - h\,\sigma(E)\,\lambda y_{n-i+1} = 0$$

$$\text{or}\ \left\{\rho(E) - \lambda h\,\sigma(E)\right\} y_{n-i+1} = 0$$

$$\text{or}\ \left\{\rho(E) - \bar{h}\,\sigma(E)\right\} y_{n-i+1} = 0, \quad \text{where } \bar{h} = \lambda h.$$

Here we have the stability polynomial

$$\boxed{\pi(\xi;\bar{h}) = \rho(\xi) - \bar{h}\,\sigma(\xi), \quad \text{where } \bar{h} = \lambda h}$$

Absolute Stability:

A linear multistep method is said to be absolutely stable for a given values of λh if each root $\xi_i = \xi_i(\lambda h)$ of the associated stability polynomial $\pi(\xi; \lambda h)$ satisfies $|\xi_i(\lambda h)| < 1$.

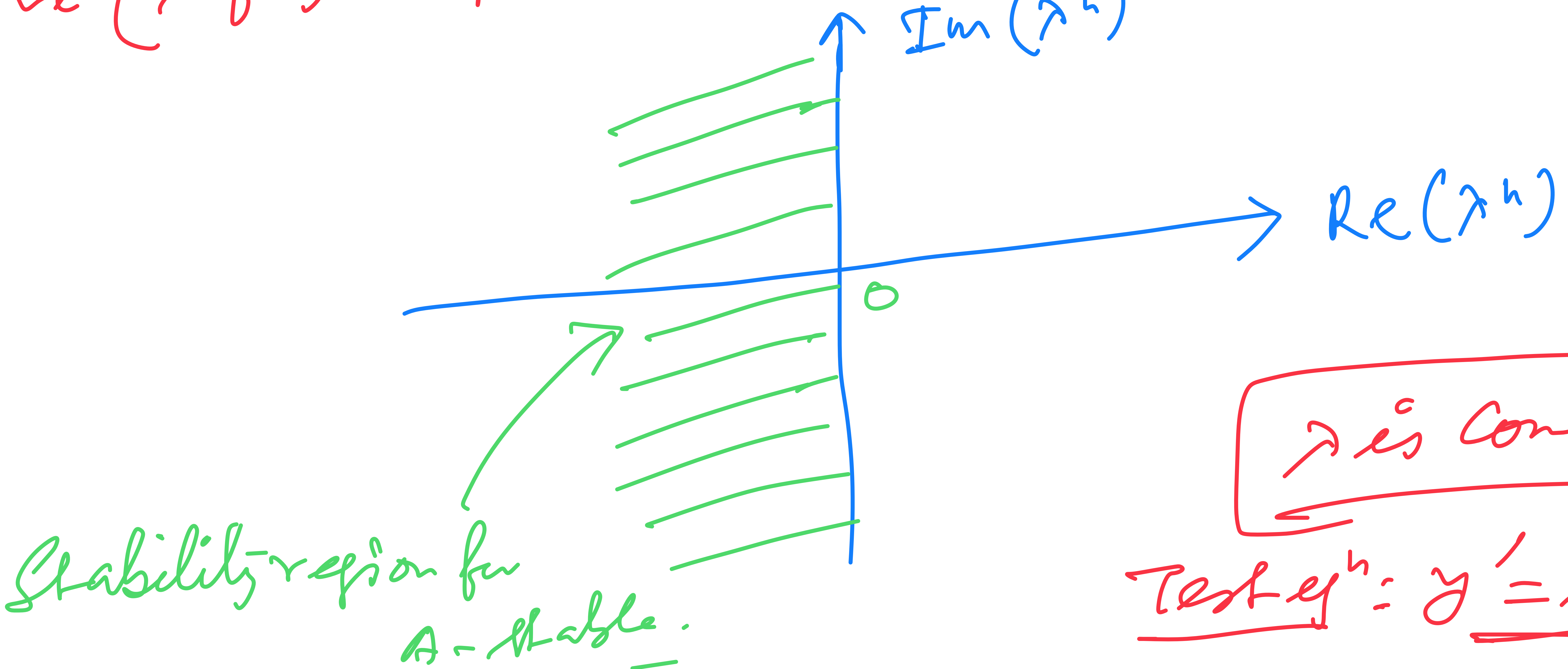
Test eqn: $\boxed{y' = \lambda y, \ Re(\lambda) < 0}$

Absolute Stability region:

The region of absolute stability of a linear multistep method is set of all points λh in the complex plane for which the method is absolutely stable.

Note: The region of absolute stability of the method should admit all values of λ, $\underline{Re(\lambda) < 0}$.

A-stable: A linear multistep method is said to be A-stable if its region of absolute stability contains the negative (left) complex half plane.



$\boxed{\lambda \text{ is complex}}$

Test eq^n: $y' = \lambda y$.

Second Barrier Theorem: This theorem says that

(i) No explicit linear multistep method is A-stable

(ii) No A-stable linear multistep method can have order greater than 2.

(iii) The second order A-stable linear multistep method with the smallest error constant is the Trapezium rule method.

So one can say that,

The order p of an A-stable linear multistep method cannot exceed 2 and the method must be implicit

Interval of absolute stability

General method: This is straight forward method to find the absolute stability interval.

Ex: Euler method: $y_{n+1} = y_n + h y_n'$

$= y_n + \lambda h y_n$, $\quad y' = \lambda y$, $\lambda < 0$

$= (1 + \lambda h) y_n$.

The stability polynomial is

$\pi(\xi; \bar{h}) = \rho(\xi) - \bar{h}\,\sigma(\xi) = (\xi - 1) - \bar{h}$

where $\rho(\xi) = \xi - 1$ and $\sigma(\xi) = 1$.

The characteristic eqⁿ is

$$\pi(\xi;\bar{h}) = 0$$

$$\Rightarrow \xi - 1 - \bar{h} = 0 \quad \Rightarrow \xi = (1+\bar{h}).$$

The root of characteristic eqⁿ must satisfy the condition $|\xi| < 1$ for having absolute stability region.

$$\Rightarrow |(1+\bar{h})| < 1 \quad \Rightarrow \quad \bar{h} \in (-2,0) \text{ is the absolute stability region,}$$

Ex: Second order Adams – Bashforth method:

$$y_{n+1} = y_n + \frac{h}{2}\left[3y_n' - y_{n-1}'\right], \quad n \geq 1$$

The characteristic eq^n is

$$\xi^2 - \left(1 + \frac{3}{2}\bar{h}\right)\xi + \frac{1}{2}\bar{h} = 0$$

$$\Rightarrow \xi_1 = \frac{1}{2}\left\{1 + \frac{3}{2}\bar{h} + \sqrt{1 + \bar{h} + \frac{9}{4}\bar{h}^2}\right\}$$

$$\text{and } \xi_2 = \frac{1}{2}\left\{1 + \frac{3}{2}\bar{h} - \sqrt{1 + \bar{h} + \frac{9}{4}\bar{h}^2}\right\}$$

For absolute stability, $|\xi_1| < 1$ and $|\xi_2| < 1$.

However it is very tedious to find the stability region for this case.

So we need some other procedure to get the absolute stability region easily.

Next page we will discuss on such methods.

$$\boxed{\text{Anyway the absolute stability region is } \bar{h} \in (-1, 0)}$$

Interval of absolute stability

- Now we need to know some other methods for finding the absolute stability.

- ## Schur Criterion:

Consider the polynomial

$$\phi(\xi) = c_k \xi^k + c_{k-1} \xi^{k-1} + \cdots + c_1 \xi + c_0 ,$$

$c_k \neq 0$, $c_0 \neq 0$ with complex coefficients.

Then polynomial ϕ is said to be a Schur polynomial if each of its roots ξ_i satisfy $|\xi_i| < 1$, $i = 1, 2, \ldots, K$.

Let us consider the polynomial

$$\hat{\varphi}(\xi) = \overline{c_0}\,\xi^{k} + \overline{c_1}\,\xi^{k-1} + \cdots + \overline{c_{k-1}}\,\xi + \overline{c_k}$$

where $\overline{c_i}$ are complex conjugates of c_i.

Let us define $\varphi_1(\xi) = \frac{1}{\xi}\left[\hat{\varphi}(0)\varphi(\xi) - \varphi(0)\hat{\varphi}(\xi)\right]$

that has degree $\leq k-1$.

Theorem: The polynomial φ is a Schur polynomial if and only if $|\hat{\varphi}(0)| > |\varphi(0)|$ and φ_1 is a Schur polynomial.

Example: $y_{n+2} - y_n = \frac{h}{3}(f_{n+1} + 2f_n)$. Find the absolute stability interval.

Solⁿ:

$$y_{n+2} - y_n = \frac{h}{3}(f_{n+1} + 2f_n)$$

$$= \frac{h}{3}(y'_{n+1} + 2y'_n)$$

$y' = \lambda y$.

or, $y_{n+2} - y_n - \frac{h}{3}(\lambda y_{n+1} + 2\lambda y_n)$,

$$\Rightarrow (E^2 - 1)y_n - \lambda h\left(\frac{1}{3}E y_n + \frac{2}{3}y_n\right) = 0$$

or, $\{\rho(E) - \bar{h}\,\sigma(E)\}y_n = 0$, $\bar{h} = \lambda h$

The stability polynomial $\Pi(\xi; \bar{h}) = \rho(\xi) - \bar{h}\,\sigma(\xi)$

where $\rho(\xi) = \xi^2 - 1$, $\sigma(\xi) = \frac{1}{3}(\xi + 2)$.

$$\Pi(\xi, \bar{h}) = \xi^2 - 1 - \bar{h}/_3(\xi + 2)$$

$$= \xi^2 - \bar{h}/_3\,\xi - (1 + 2\bar{h}/_3)$$

Here $c_0 = -(1 + 2/_3\bar{h})$, $c_1 = -\bar{h}/_3$, $c_2 = 1$.

$$\hat{\Pi}(\xi, \bar{h}) = -(1 + 2/_3\bar{h})\xi^2 - \frac{1}{3}\bar{h}\,\xi + 1$$

$$|\hat{\Pi}(0)| > |\Pi(0)| \Rightarrow \bar{h} \in (-3, 0)$$

Now we have to show that Π_1 is Schur polynomial.

$$\pi_1(\xi) = \frac{1}{\xi}\left[\hat{\pi}(0)\,\pi(\xi) - \pi(0)\,\hat{\pi}(\xi)\right]$$

$$= -\frac{1}{3}\left(2+2\bar{h}/3\right)(2\xi-1)$$

The root of the eqn $\pi_1(\xi)=0$ is $\underline{\xi=\frac{1}{2}}$.

$$\boxed{|\xi|<1}$$

Hence $\pi_1(\xi)$ is a Schur polynomial.

This imply that the method is absolutely Stable and the Stability region is $\underline{(-3,0)}$

Example: $y_{n+2} - y_n = \frac{h}{2}(f_{n+1} + 3f_n)$

Find the absolute stability region.

$\boxed{\text{Home work!}}$

Routh-Hurwitz criterion:

This is the best method for finding absolute stability.

Consider the mapping $z = \frac{\zeta+1}{\zeta-1}$ of the open unit disk $|\zeta|<1$ of the complex plane to the left half plane $Re(z)<0$ of the complex z-plane. Then the inverse mapping is $\zeta = \frac{1+z}{1-z}$.

under the transformation

$$\xi = \frac{1+z}{1-z}, \text{ the stability polynomial}$$

$\pi(\xi;\bar{h}) = \rho(\xi) - \bar{h}\,\sigma(\xi)$ becomes

$$\pi(z;\bar{h}) = \rho\left(\frac{1+z}{1-z}\right) - \bar{h}\,\sigma\left(\frac{1+z}{1-z}\right)$$

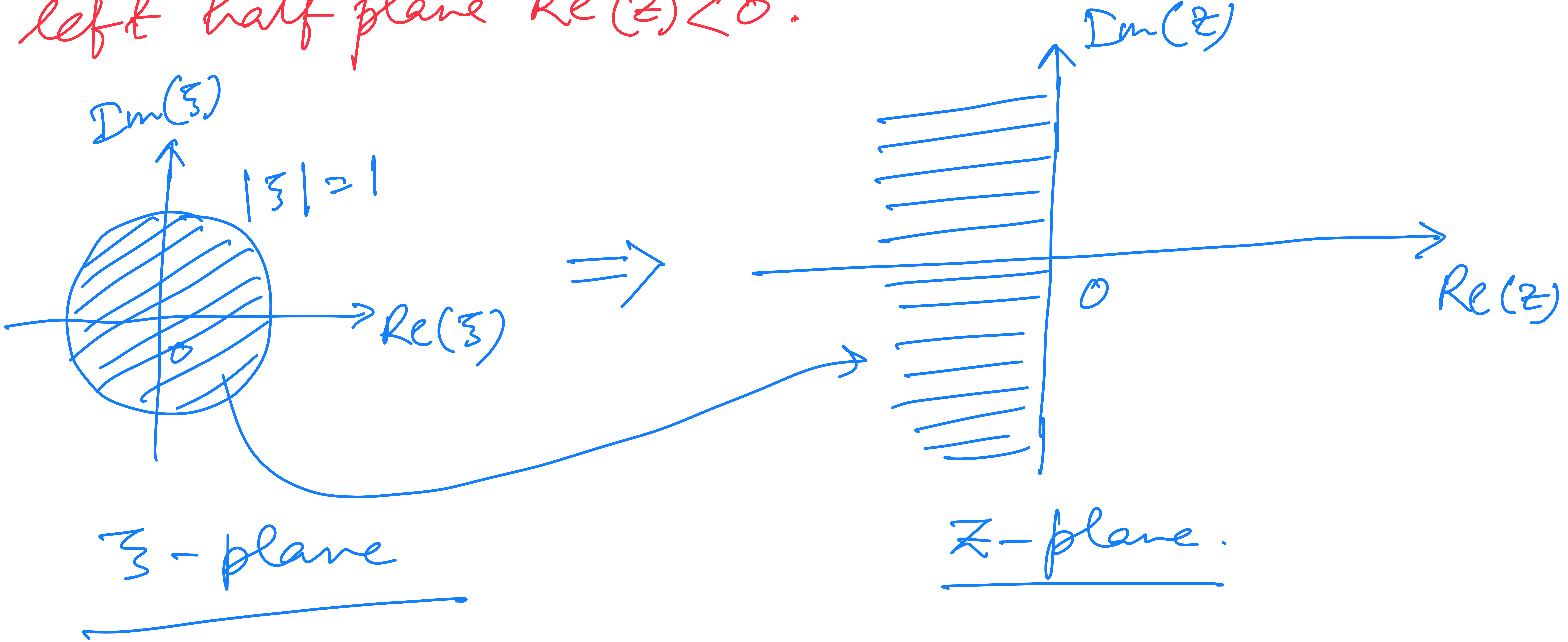
now the characteristic eqn is

$$\pi(z;\bar{h}) = 0$$

$$\Rightarrow \rho\left(\frac{1+z}{1-z}\right) - \bar{h}\,\sigma\left(\frac{1+z}{1-z}\right) = 0$$

$$\Rightarrow a_0 z^k + a_1 z^{k-1} + \cdots + a_k = 0 \quad — \; *$$

The roots of the stability polynomial $\pi(\xi;\bar{h})$ lie inside the open unit disk $|\xi|<1$ if and only if the roots of characteristic eqn ⊛ lie in the open left half plane $Re(z)<0$.



Theorem: (Routh - Hurwitz criterion):

The roots of eqⁿ ⊛ lie in the open left half plane if and only if the leading principal minors of the $K \times K$ matrix

$$\begin{pmatrix} a_1 & a_3 & a_5 & \cdots & a_{2K-1} \\ a_0 & a_2 & a_4 & \cdots & a_{2K-2} \\ 0 & a_1 & a_3 & \cdots & a_{2K-3} \\ 0 & a_0 & a_2 & \cdots & a_{2K-4} \\ \cdots & \cdots & \cdots & & \cdots \\ 0 & 0 & 0 & \cdots & a_K \end{pmatrix}_{K \times K}$$

are positive and $a_0 > 0$.

$\boxed{\text{we assume that } a_j = 0 \text{ for } j > K.}$

Conditions

K=1: $a_0 > 0, \quad a_1 > 0$

K=2: $\begin{pmatrix} a_1 & a_3 \\ a_0 & a_2 \end{pmatrix}_{2\times 2} \rightarrow \underbrace{a_0 > 0, \quad a_1 > 0, \quad a_2 > 0}_{\text{leading principle minors}}$

K=3: $\begin{pmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{pmatrix}_{3\times 3} \rightarrow a_0 > 0, \quad a_1 > 0, \quad a_2 > 0, \quad a_3 > 0, \quad a_1 a_2 - a_3 a_0 > 0$

Conditions

K=4: $\begin{pmatrix} a_1 & a_3 & a_5 & a_7 \\ a_0 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 & a_4 \end{pmatrix}_{4\times 4} \rightarrow$

conditions.

$a_1 > 0, \quad a_2 > 0, \quad a_3 > 0, \quad a_4 > 0$

$a_1 a_2 a_3 - a_0 a_3^2 - a_4 a_1^2 > 0$

These are the necessary and sufficient condition for ensuring that all roots lie in the left half plane.

Example: $y_{n+2} - y_n = \frac{h}{2}\left(f_{n+1} + 3f_n\right)$

Find the absolute stability region.

Sol^n: The stability polynomial is

$$\Pi(\xi; \bar{h}) = \xi^2 - 1 - \frac{\bar{h}}{2}(\xi + 3)$$

$$= \xi^2 - \xi\frac{\bar{h}}{2} - \left(1 + \frac{3}{2}\bar{h}\right).$$

The characteristic eqn is

$$\pi(z; \bar{h}) = 0$$

$$\Rightarrow \left(\frac{1+z}{1-z}\right)^2 - \left(\frac{1+z}{1-z}\right)\bar{h}/2 - \left(1 + \tfrac{3}{2}\bar{h}\right) = 0$$

$$\Rightarrow a_0 z^2 + a_1 z + a_2 = 0$$

where $a_0 = -\bar{h}$, $a_1 = 4 + 3\bar{h}$, $a_2 = -2\bar{h}$

Now the Routh Hurwitz Criterion is

$a_0 > 0$, $a_1 > 0$ and $a_2 > 0$

[Note here $k = 2$ quadratic polynomial.]

$\Rightarrow \bar{h} \in (-4/3, 0)$ is the region of the absolute stability.

Example : Third order Adams – Moulton method

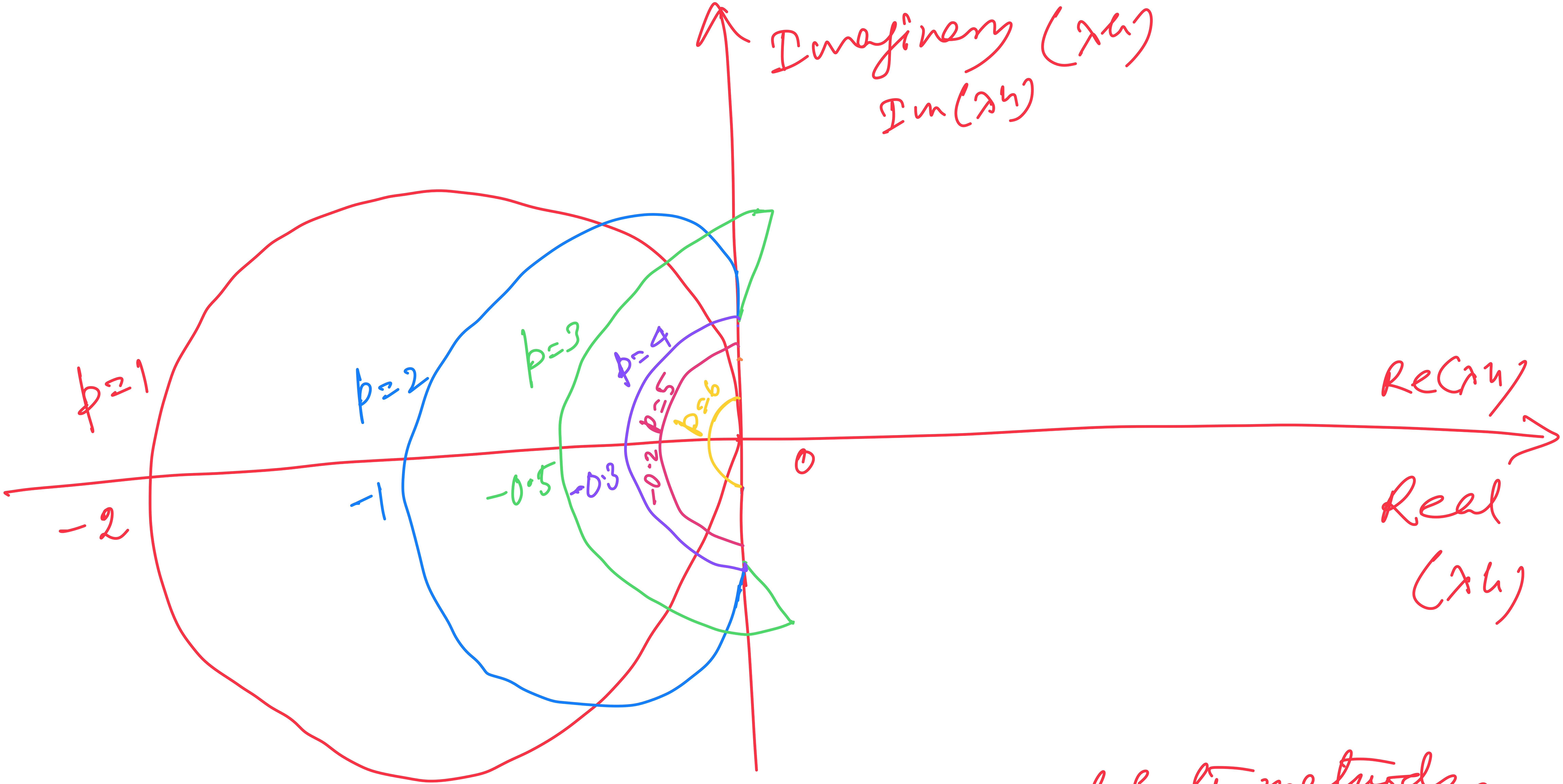
$$y_{n+1} = y_n + h/_{12}\left(5y'_{n+1} + 8y'_n - y'_{n-1}\right).$$

Find the absolute stability region.

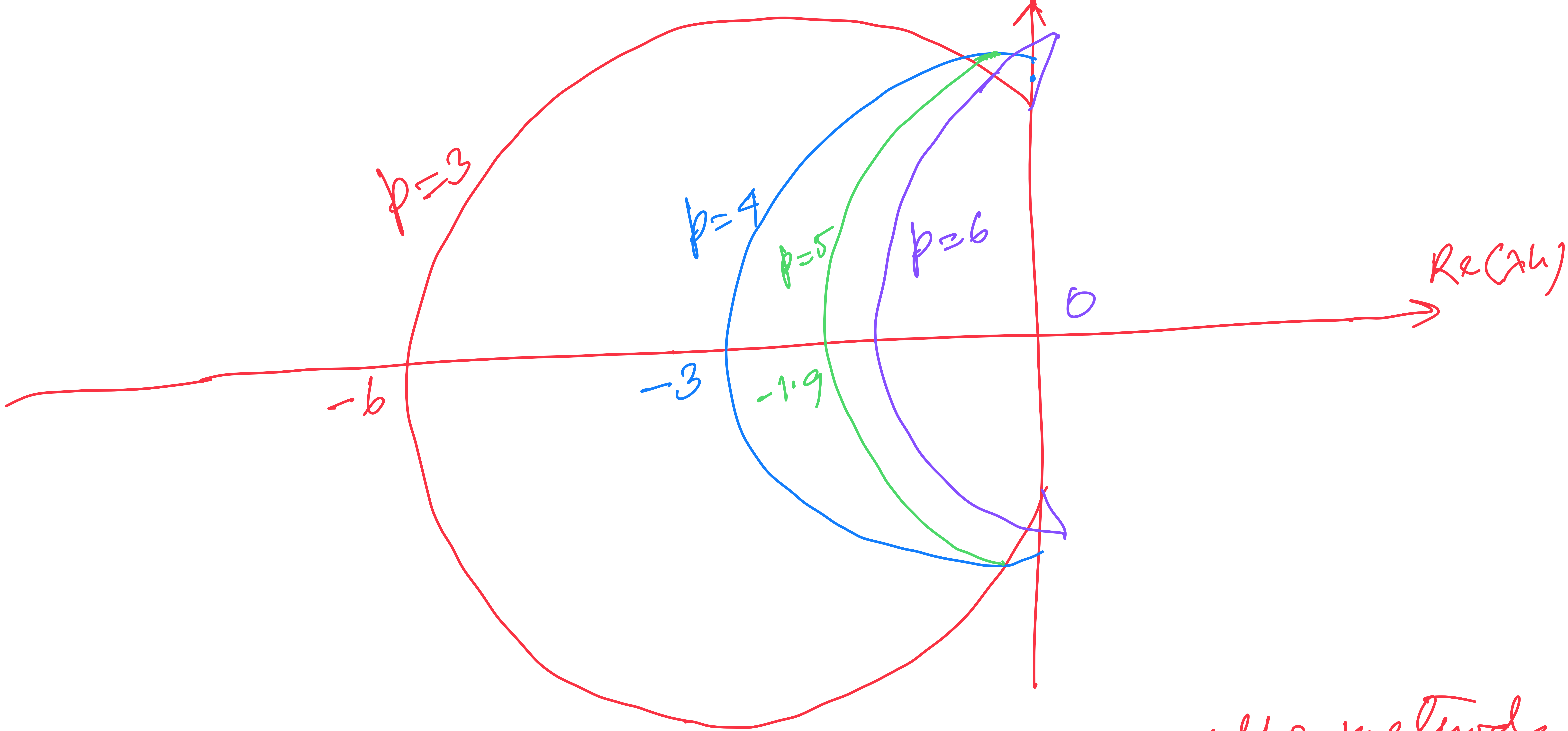
Homework 1 (Ans: $(-6,0)$). Try!

Intervals of absolute stability :

order (p)	Adams-Bashforth	Adams Moulton
1	$(-2, 0)$	$(-\infty, 0)$, A-stable
2	$(-1, 0)$	$(-\infty, 0)$, A-stable
3	$(-0.5, 0)$	$(-6, 0)$
4	$(-0.3, 0)$	$(-3, 0)$
5	$(-0.2, 0)$	$(-1.9, 0)$



Stability region for Adams-Bashforth methods. The order of p is stable inside the region indicated left of origin.



Stability regions for Adams-Moulton methods. The method of order p is stable inside the region indicated.

Weak stability :

Defn : The linear multistep method is said to be weakly stable if there is more than one simple root of the polynomial eqn $\rho(\xi) = 0$ on the unit circle (and strongly stable if not).

Example Second order Nyström method or mid point method : (Check the stability)

Second order Nyström method or mid-point method is given by —

$$y_{n+1} = y_{n-1} + 2h f_n$$

$y_{n+1} = y_{n-1} + 2h\, y_n'$, use the test eq^n $y' = \lambda y$

$\Rightarrow y_{n+1} = y_{n-1} + 2\lambda h\, y_n = y_{n-1} + 2\bar{h}\, y_n$

$\Rightarrow (E^2 - 1) y_{n-1} - 2\bar{h} E y_{n-1} = 0$,

$\Rightarrow \rho(\xi) = \xi^2 - 1$ and $\sigma(\xi) = 2\xi$.

Now by the def^n we have $\rho(\xi) = 0 \Rightarrow \xi^2 - 1 = 0$

$\Rightarrow \xi = \pm 1$.

$\xi_1 = 1$, $\xi_2 = -1 \Rightarrow |\xi_1| = |\xi_2| = 1$.

So we have two double root on the unit circle

$\Rightarrow$ This method is weakly stable.

Now we want to study more rigorously on weak stability.

We have $(E^2-1)y_{n-1} - 2\bar{h}\,E y_{n-1} = 0$,

$\Rightarrow$ Stability polynomial $\pi(\xi;\bar{h}) = (\xi^2-1) - 2\bar{h}\xi$

$$= \xi^2 - 2\bar{h}\xi - 1.$$

The characteristic eqn is $\xi^2 - 2\bar{h}\xi - 1 = 0$.

Note that $y' = \lambda y$ gives the exact soln

$$y(x) = c\,e^{\lambda x}$$

at $y(x_0) = y_0$

$\Rightarrow y(x) = y_0\, e^{\lambda(x-x_0)}$

$\Rightarrow y(x_{n+1}) = y(x_n)\, e^{\lambda h} = y(x_n)\, e^{\bar{h}}$

This is exact soln

$$\Rightarrow \xi = \bar{h} \pm (\bar{h}^2+1)^{1/2}$$

$$= \bar{h} \pm \left(1 + \frac{\bar{h}^2}{2}\right) + O(\bar{h}^3)$$

Now $\xi_1 = \bar{h} + 1 + \bar{h}^2/2 + O(\bar{h}^3)$

$$= 1 + \bar{h} + \bar{h}^2/2 + O(\bar{h}^3)$$

$\simeq e^{\bar{h}}$, This soln behaves as exact soln.

and another root $\longrightarrow$ This root is known as extraneous root.

$$\xi_2 = \bar{h} - 1 - \frac{\bar{h}^2}{2} + O(\bar{h}^3)$$

$$= -\left(1 - \bar{h} + \frac{\bar{h}^2}{2}\right) + O(\bar{h}^3)$$

$$\simeq -e^{-\bar{h}}$$

The general solⁿ is $y_n = c_1 \xi_1^n + c_2 \xi_2^n$, where

$\xi_1 \simeq e^{\bar{h}}$ and $\xi_2 \simeq -e^{-\bar{h}}$.

Case-I:

λ is real and positive:

$$\xi_1 > |\xi_2| > 0$$

Then ξ_2^n increases less rapidly than ξ_1^n

$\Rightarrow c_1 \xi_1^n$ dominates.

So $y_n \simeq c_1 (e^{\bar{h}})^n + c_2 (-1)^n (e^{-\bar{h}})^n$ → It is also known as extraneous part.

exact sol^n

Parasitic term, however in this case it does not create any problem.

— This imply that overall sol^n grows as per the exact sol^n.

For this case the method is **stable** for $\bar{h} > 0$.

This type of stability is known as Relative Stability. we will discuss next.

② Case II: λ is real and $\lambda < 0$:

For this case, ξ_1 decreases as does the exact sol^n but ξ_2 oscillates with increasing amplitude.

Here parasitic term creates problem. Stability does not occur.

The mid point is unstable for $\bar{h} < 0$.

However for $\bar{h} = 0$, the characteristic eq^n becomes

$$\xi^2 - 1 = 0,$$ whose roots are $\xi = \pm 1$.

So by the def^n, the mid point method is weakly stable, it is not strongly stable.

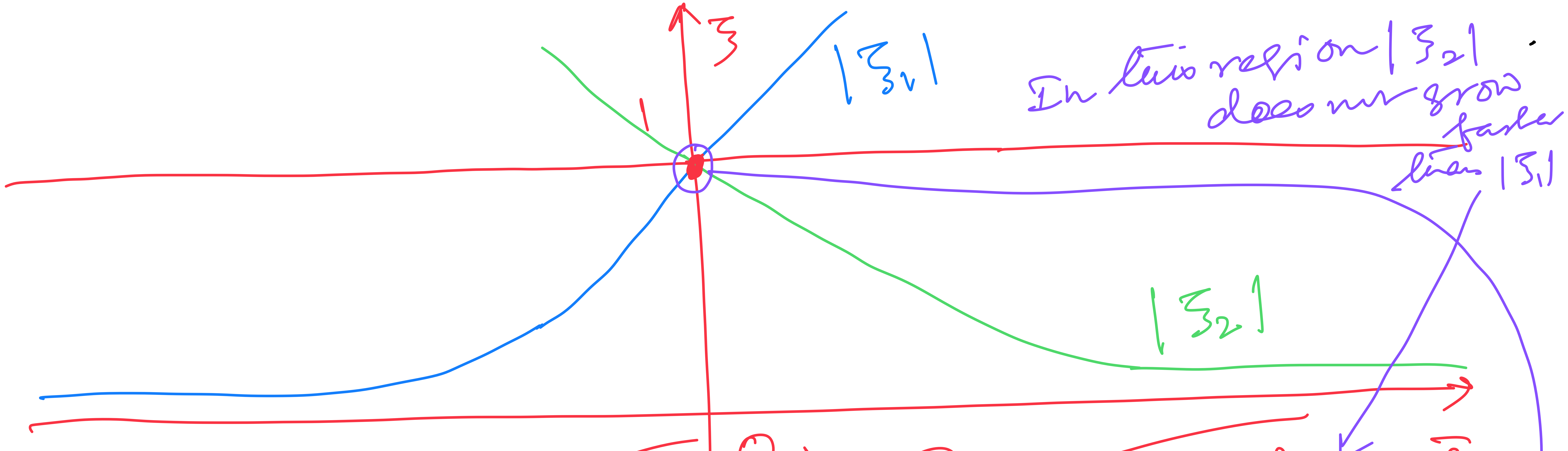
Case III: λ is pure imaginary:

Let $\lambda = i x$, x is real

$$\bar{h} = \lambda h = i \lambda x.$$

The characteristic eq^n become $(\xi^2 - 1)^2 = -4h^2 x^2 \xi$

$$\Rightarrow \left(\xi - \frac{1}{\xi}\right)^2 = -4h^2 x^2.$$



unstable

$\lambda<0$, $\bar{h}<0$, $|\xi_2|>1$

relatively stable

$\lambda>0$, $\bar{h}>0$, $|\xi_2| \leq |\xi_1|$

-1

at $\bar{h}=0$, weakly stable.

$|\xi_1|=|\xi_2| = 1$

more than one simple root.

This figure shows the characteristic roots of the midpoint method.

Setting $\xi = e^{i\theta}$, we get $h^2\lambda^2 = \sin^2\theta$

Hence the interval of stability is

$$0 < |h\lambda| \leq 1.$$

Example: Discuss the stability analysis of Milne Simpson's method.

(i) Is it weakly stable?

(ii) Discuss the stability for all cases of λ.

Homework! Try it!

Relatively Stable :

Defn : The linear multistep method is said to be relatively stable if the roots ξ_i $(i=1,2,\dots k)$ of the stability polynomial eqn i.e, characteristic eqn $\pi(\xi;\bar{h})=0$ satisfy the following condition —

$$\boxed{|\xi_i| \le |\xi_1|, \quad i=2,3,4,\dots,k}$$

note that this root is related to exact solution.

Principal root

The region of the relative stability is defined to be set of points in the $\bar{h}$-plane for which the method is relatively stable.

Example: For $\lambda > 0$, we have seen the relative stability for second order Nystrom or mid point method.

What is the difference between absolute and relative stabilities?

To discuss this, let us apply the second order Adams-Bashforth method

$$y_{n+1} = y_n + \frac{h}{2}\left(3y_n' - y_{n-1}'\right)$$

to the test eqn $y' = \lambda y$.

The characteristic eq^n is —

$$\xi^2 - \left(1 + \frac{3\bar{h}}{2}\right)\xi + \bar{h}/2 = 0 \quad , \text{ where } \bar{h} = \lambda h.$$

Two roots are

$$\xi_1 = \frac{1}{4}\left[2 + 3\bar{h} + \sqrt{4 + 4\bar{h} + 9\bar{h}^2}\right]$$

and $$\xi_2 = \frac{1}{4}\left[2 + 3\bar{h} - \sqrt{4 + 4\bar{h} + 9\bar{h}^2}\right]$$

$$\xi_1 = \frac{1}{4}\left[2 + 3\bar{h} + \sqrt{4 + 4\bar{h} + 9\bar{h}^2}\right]$$

$$= 1 + \bar{h} + \frac{\bar{h}^2}{2} + O(\bar{h}^3) \simeq e^{\bar{h}} + O(\bar{h}^3)$$

is the principal root approximating $e^{\bar{h}}$, i.e. the exact solution.

$$\xi_2 = \frac{1}{4}\left[2+3\bar{h} - \sqrt{4+4\bar{h}+9\bar{h}^2}\right]$$

$$= \bar{h}/2 - \frac{\bar{h}^2}{2} + \frac{\bar{h}^3}{4} + o(\bar{h}^4)$$

$$= \bar{h}/2\left(1 - \bar{h} + \frac{\bar{h}^2}{2}\right) + o(\bar{h}^4)$$

$$= \bar{h}/2\, e^{-\bar{h}} + o(\bar{h}^4).$$

This is the extraneous root.

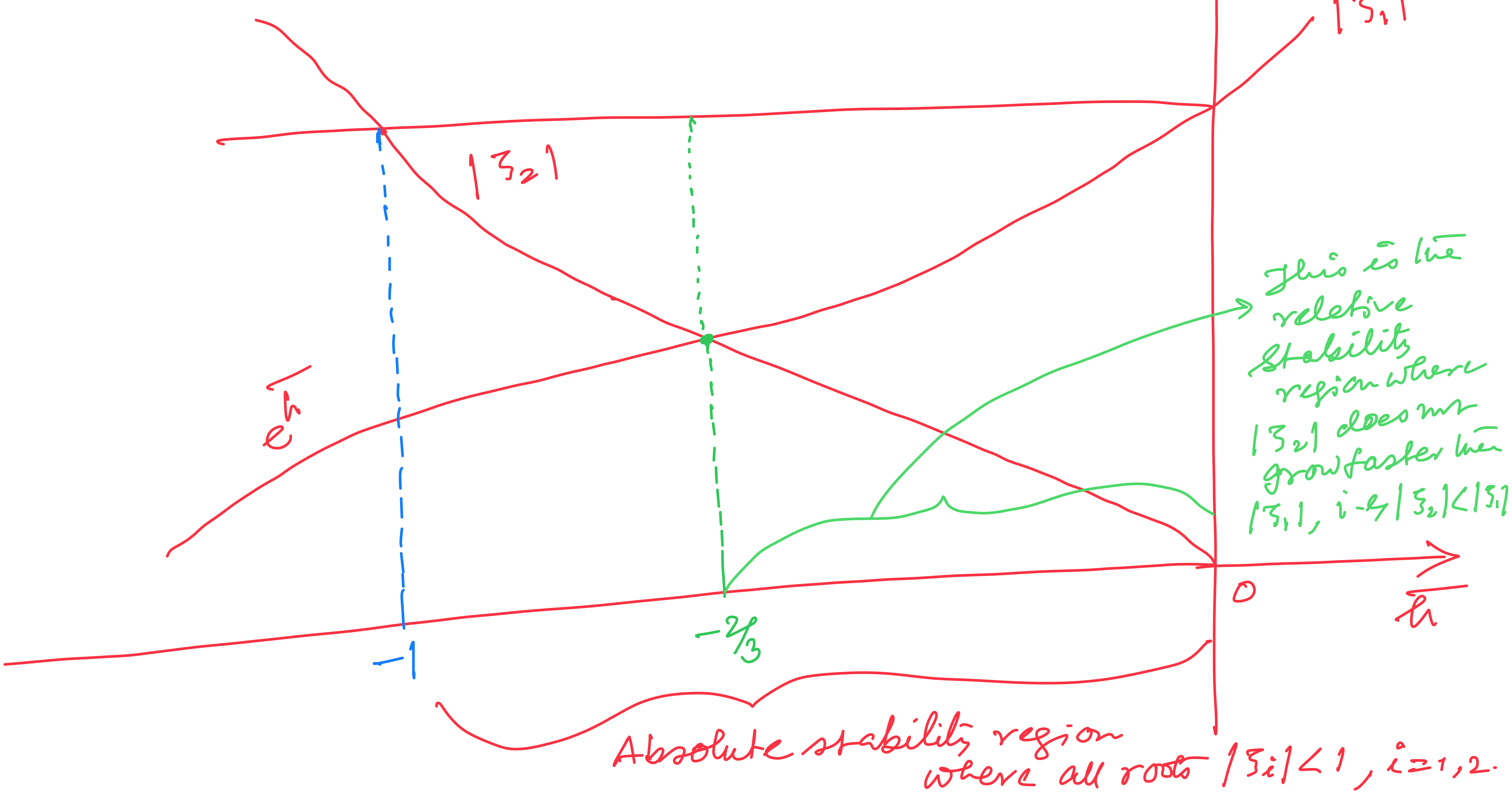
Now the solution is given by

$$y_n = c_1 \xi_1^n + c_2 \xi_2^n.$$

extraneous part.

Now the roots ξ_1 and ξ_2 are folled for better understanding.

This figure shows the roots of the characteristic eqn of Adams-Bashforth method graphically.



we find by plotting the graph of ξ_1 and ξ_2 (using mathematica Software)

that $|\xi_2| > 1$ for $\bar{h} < -1$

$\Rightarrow$ The method is unstable for $\bar{h} < -1$.

It is clear that the method is absolutely stable for $-1 < h < 0$.

For $-2/3 < \bar{h} < 0$, we have $|\xi_2| \leq |\xi_1|$, so in this region the method is relatively stable.

Note that in the common interval, $-2/3 < \bar{h} < 0$, the method is both absolutely and relatively stable.

Lecture 13 and 14 on Numerical Methods

Class contents

- Higher order IVPs
- BVPs
- Shooting Method

Higher order initial value problem

The general form of mth order IVP is

$$\phi\left(x, y, y', y'', \ldots, y^{(m)}\right) = 0$$

$$\Rightarrow \quad a_0(x)\, y^{(m)}(x) + a_1(x)\, y^{(m-1)}(x) + \cdots + a_{m-1}(x)\, y'(x)$$

$$+ a_m(x)\, y(x) = r(x).$$

where $a_0(x), a_1(x), \ldots, a_m(x)$ and $r(x)$ are constants or continuous functions of x

with initial conditions

$$y(x_0) = b_0, \quad y'(x_0) = b_1, \ldots, y^{(m-1)}(x_0) = b_{m-1}$$

⑥ Reduction of Second order eqⁿ to a first order problem:

The given second order IVP is

$$a_0(x)y''(x) + a_1(x)y'(x) + a_2(x)y(x) = r(x)$$

$$y(x_0) = b_0, \quad y'(x_0) = b_1$$

To solve this problem, we need to reduce it to two first order IVPs.

Let $y' = z$.

Therefore $a_0(x)z' + a_1(x)z + a_2(x)y(x) = r(x)$

$$\Rightarrow z' = \frac{1}{a_0(x)}\left[r(x) - a_1(x)z - a_2(x)y(x)\right]$$

So we have two IVPs :

<u>First IVP :</u> $y' = z \,, \quad y(x_0) = b_0 \quad \text{--- (1)}$

<u>Second IVP :</u> $z' = \frac{1}{a_0(x)} \left[r(x) - a_1(x) z - a_2(x) y(x) \right] \,,$

$$z(x_0) = b_1 \quad \text{--- (2)}$$

Now we need to solve these IVPs by means of <u>Single step or multi step methods.</u>

(*) In general we will pick fourth order classical Runge-Kutta method.

If you pick Euler method

Let, $y' = z = f(x, y, z)$, $y(x_0) = b_0$

and $z' = \frac{1}{a_0}\left[r - a_1 z - a_2 y\right] = g(x, y, z)$, $z(x_0) = b_1$

Now apply Euler method

$$\left.\begin{aligned} y_{n+1} &= y_n + h f(x_n, y_n, z_n) \\ \text{and } z_{n+1} &= z_n + h g(x_n, y_n, z_n) \end{aligned}\right]$$

now find the solution $y(x)$ and $z(x)$

⊘ If you pick 4th order R-K method:

$$y_{n+1} = y_n + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right),$$

and

$$z_{n+1} = z_n + \frac{h}{6}\left(l_1 + 2l_2 + 2l_3 + l_4\right)$$

where $k_1 = f(x_n, y_n, z_n)$; $l_1 = g(x_n, y_n, z_n)$

$$k_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{hk_1}{2}, z_n + \frac{hl_1}{2}\right)$$

$$l_2 = g\left(x_n + \frac{h}{2}, y_n + \frac{hk_1}{2}, z_n + \frac{hl_1}{2}\right)$$

$$k_3 = f\left(x_n + \frac{h}{2}, y_n + \frac{hk_2}{2}, z_n + \frac{hl_2}{2}\right)$$

$$l_3 = g\left(x_n + \frac{h}{2}, y_n + \frac{hk_2}{2}, z_n + \frac{hl_2}{2}\right)$$

$$k_4 = f\left(x_n + h, y_n + hk_3, z_n + hl_3\right)$$

$$l_4 = g\left(x_n + h, y_n + hk_3, z_n + hl_3\right)$$

Example: $y' = x + yz$, $y(0) = 1$

and $z' = y + xz$, $z(0) = -1$, $h = 0.2$

find $y(0.4)$. By Euler method

Solⁿ: Let $y' = f(x,y,z)$ and $z' = g(x,y,z)$

where $f(x,y,z) = x + yz$, and

$g(x,y,z) = y + xz$,

now by Euler method, we have

$$y_{n+1} = y_n + h f(x_n, y_n, z_n)$$

and $z_{n+1} = z_n + h\, g(x_n, y_n, z_n)$

$x_0 = 0$, $y_0 = 1$,

$z_0 = -1$, $h = 0.2$.

$\underline{n=0}$:

$$y_1 = y(0.2) = y_0 + h f(x_0, y_0, z_0)$$
$$= 1 + 0.2 \times (0 + 1(-1)) = 0.8$$

$$z_1 = z(0.2) = z_0 + h g(x_0, y_0, z_0) = -0.8$$

$\underline{n=1}$:

$$y_2 = y(0.4) = y_1 + h f(x_1, y_1, z_1) = 0.612$$

$$z_2 = -0.672$$

⊘ **Example**: $y'' + 2y' - xy = 0$, $y(1) = 1$, $y'(1) = -1$, find $y(1.6)$ by Heun's method considering $h = 0.6$

$\underline{\text{Sol}^n}$: Let $y' = z$, $y(1) = 1$
and $z' = (xy - 2z)$, $z(1) = -1$.

Heun's method gives —

$$y_{n+1} = y_n + \frac{h}{4}(k_1 + 3k_3)$$

and $z_{n+1} = z_n + \frac{h}{4}(l_1 + 3l_3)$

where $k_1 = f(x_n, y_n, z_n)$

$$l_1 = g(x_n, y_n, z_n)$$

$$k_2 = f\left(x_n + \frac{h}{3}, y_n + \frac{h}{3}k_1, z_n + \frac{h}{3}l_1\right)$$

$$l_2 = g\left(x_n + \frac{h}{3}, y_n + \frac{h}{3}k_1, z_n + \frac{h}{3}l_1\right)$$

$$k_3 = f\left(x_n + \frac{2h}{3}, y_n + \frac{2hk_2}{3}, z_n + \frac{2h}{3}l_2\right)$$

$$l_3 = g\left(x_n + \frac{2h}{3}, y_n + \frac{2hk_2}{3}, z_n + \frac{2hl_2}{3}\right)$$

Here $f(x,y,z) = z$, $g(x,y,z) = (xy - 2z)$

<u>$n=0$</u>: $k_1 = z_0 = -1$

$l_1 = 1.1 - 2(-1) = 3.$

$k_2 = z_0 + h/3\, l_1 = -0.4$

$l_2 = 1.76$, $k_3 = -0.296$, $l_3 = ?$

$y(1.6) = y_1 = y_0 + h/4\,(k_1 + 3k_3) = ?$ (Find!).

Ex: Compute approximations to $y(0.4)$ and $y'(0.4)$ for the IVP $y'' + 4y = \cos t$, $y(0) = 1$, $y'(0) = 0$. Using Runge-Kutta method of fourth order with step length $h = 0.2$, $y(t) = (2\cos 2t + \cos t)/3$. Find the errors.

$y(0.4) = 0.771546$, Exact 0.771491.

Try! Home work!

Boundary Value Problem:

Two point BVP — $y'' = f(x, y, y'), \quad x \in (a, b).$

with boundary conditions:

(i) First kind: $y(a) = \gamma_1, \quad y(b) = \gamma_2$

(ii) Second kind: $y'(a) = \gamma_1, \quad y'(b) = \gamma_2$

(iii) Third kind: $c_0\, y(a) + c_1\, y'(a) = \gamma_1$

$$d_0\, y(a) + d_1\, y'(a) = \gamma_2$$

The boundary value problem can be solved by –

(i) Shooting method

(ii) Finite difference method

(iii) Finite element method

Shooting method:

Two point BVP:

$$\left.\begin{aligned} & y'' + p(x)y' + q(x)y = r(x), \\ & y(a) = \gamma_1, \quad y(b) = \gamma_2 \end{aligned}\right\} \quad \text{--- } (*)$$

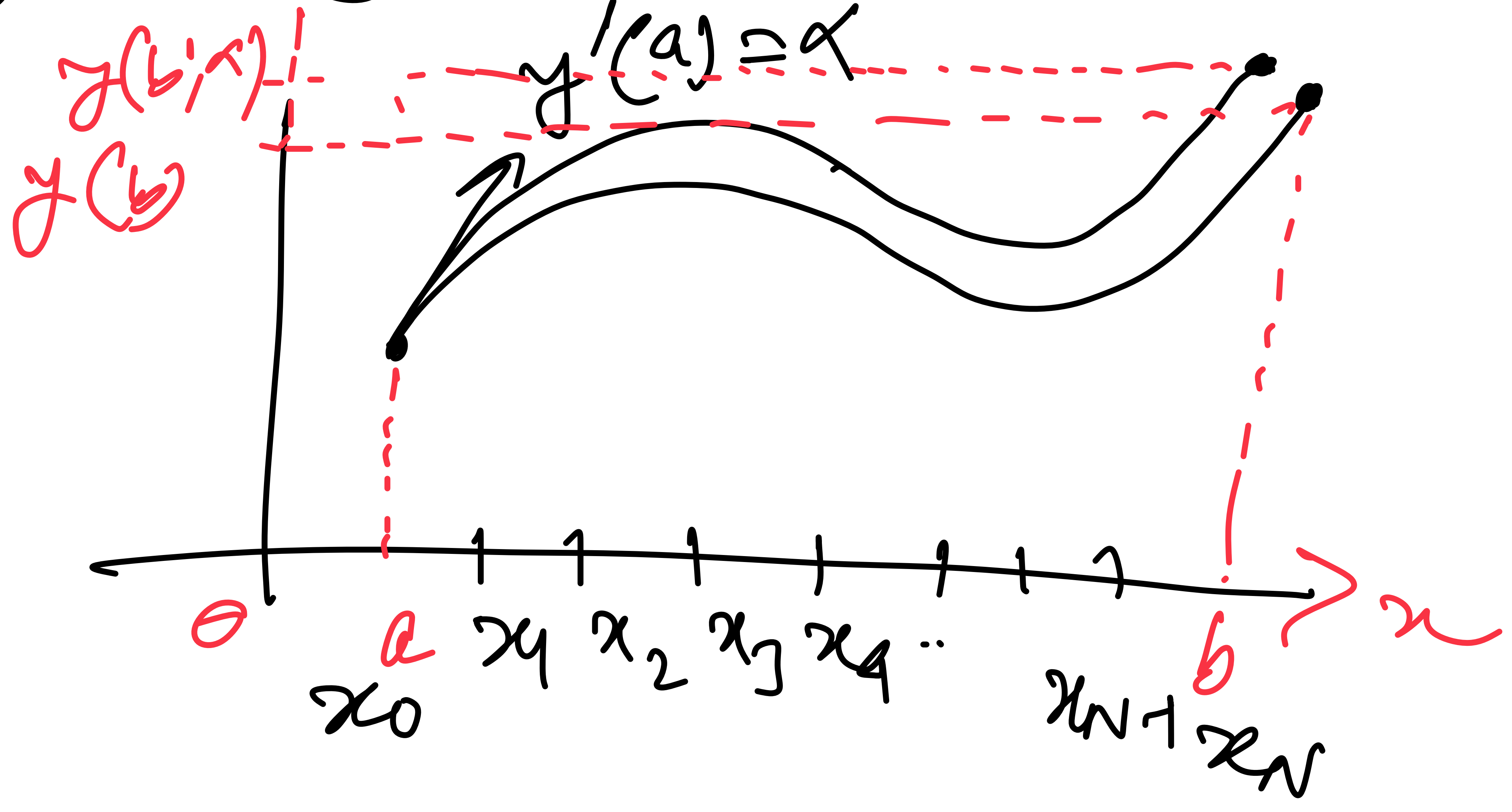
first task: Convert this $(*)$ to IVP.
and supply initial condition $y'(a)$ or $y(b)$

Let $y'(a) = \alpha$.

Therefore we have

IVP can be solved by using a suitable R-K or Taylor series method

$$\left.\begin{aligned} & y'' + p(x)y' + q(x)y = r(x) \\ & y(a) = \gamma_1 \quad \text{and} \quad y'(a) = \alpha \end{aligned}\right\} \leftarrow \text{IVP}$$



Note: the slope α is unknown is to us,

now our target is to find this α.

$x_0 = a$	x_1	x_2	----	$x_N = b$
$y(a)$	$y(x_1)$	$y(x_2)$	---	$y(b)$
$y'(a) = \alpha \rightarrow$	$y(x_1;\alpha)$	$y(x_2;\alpha)$	--	$y(b;\alpha)$

To find this α, we have to make

$$y(b;\alpha) \simeq y(b).$$

i.e, $$|y(b;\alpha) - y(b)| < \varepsilon$$

$$\Rightarrow |y(b;\alpha) - \gamma_2| < \varepsilon.$$

So $$y(b;\alpha) - \gamma_2 \simeq 0$$

now find the solution of

$$\varphi(\alpha) = y(b;\alpha) - \gamma_2 = 0 . \quad \text{---} \ (**)$$

i.e, find the root α of eqn $(**)$.

Apply root-finding method.

Two best methods are available in our hand

(1) Secant method (the rate of convergence is 1.62)

(2) Newton - Raphson method (the rate of convergence is 2)

$$x_{n+1} = x_n - \frac{\varphi(x_n)}{\varphi'(x_n)},$$

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})}{\varphi(x_n) - \varphi(x_{n-1})}\varphi(x_n)$$

Let's consider Secant method.

For Secant method, we need two approximations x_0 and x_1

Two initial slopes define the IVPs.

IVP1: $y'' + py' + qy = r$
$y(a) = \gamma_1, \; y'(a) = \alpha_0$

IVP2: $y'' + py' + qy = r$
$y(a) = \gamma_1, \; y'(a) = \alpha_1$

$$\left.\begin{array}{l} \text{IVP1} \\ \text{IVP2} \end{array}\right\} \text{— } ***$$

Solve two IVPs we get
$y(b;\alpha_0)$ and $y(b;\alpha_1)$

$$\left.\begin{array}{l} \varphi(\alpha_0) = y(b;\alpha_0) - \gamma_2 \\ \varphi(\alpha_1) = y(b;\alpha_1) - \gamma_2 \end{array}\right\} \Rightarrow \alpha_2 \text{ using Secant method.}$$

$$\alpha_2 = \alpha_1 - \frac{(\alpha_1 - \alpha_0)}{\varphi(\alpha_1) - \varphi(\alpha_0)} \cdot \varphi(\alpha_1)$$

Solve IVP3 :

$$y'' + py' + qy = r, \quad y(a) = \gamma_1, \; y'(a) = \alpha_2$$

now find $y(b; \alpha_2)$

check if $|y(b; \alpha_2) - \gamma_2| < \varepsilon$

If yes then the BVP is solved.

If no, then refine α_2 again by Secant method.

Example: $y'' = 6y^2 - x$, $y(0) = 1$, $y(1) = 5$, $h = 1/3$

Solve BVP. use secant method by choosing two initial slopes $\alpha_0 = 1.2$ and $\alpha_1 = 1.5$.

Solve:

IVP 1: $y'' = 6y^2 - x$, $y(0) = 1$, $y'(0) = 1.2$

IVP 2: $y'' = 6y^2 - x$, $y(0) = 1$, $y'(0) = 1.5$

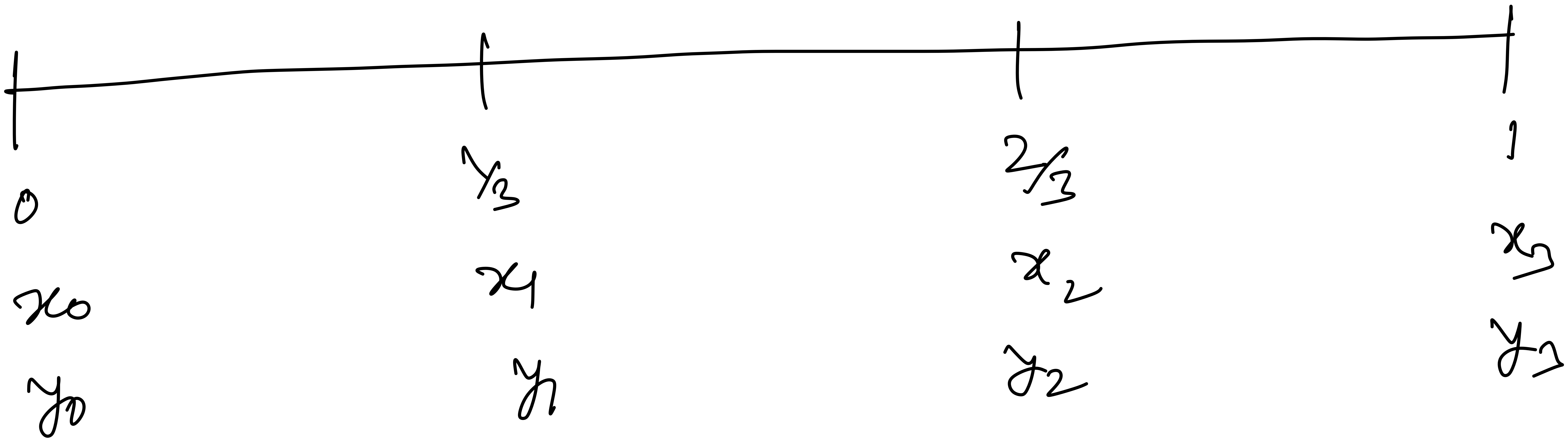
using Euler's method for the IVPs,

$$y'' = 6y^2 - x, \quad y(0) = 1, \quad y'(0) = 1.2$$

Let $y' = z \Rightarrow z' = 6y^2 - x$

IVP 1A: $y' = z,\ y(0) = 1$

IVP 1B: $z' = 6y^2 - x,\ y'(0) = z(0) = 1.2.$



$$y_{n+1} = y_n + h z_n, \quad z_{n+1} = z_n + h(6y_n^2 - x_n)$$

$$y_1 = y(1/3) = y_0 + h z_0 = 1.4, \quad z_1 = 3.2, \quad y_2 = 2.466$$

$$z_2 = 7.01, \quad y_3 = 4.7966 = y(1; 1.2) = y(b; \alpha_0)$$

for <u>IVP2</u>: $y(0) = 1,\ z(0) = 1.5,$

$y_1 = 1.5,\ z_1 = 3.5,\ y_2 = 2.666,\ z_2 = 7.89,$

$y_3 = 5.29 = y(1; 1.5)$

<u>target</u>: $y(1) = 5$

$$\varphi(\alpha_0) = y(1; 1.2) - 5 = 4.7966 - 5 = -0.2034$$

$$\varphi(\alpha_1) = y(1; 1.5) - 5 = 5.29 - 5 = 0.29$$

$$\alpha_{n+1} = \alpha_n - \frac{(\alpha_n - \alpha_{n-1})}{\varphi(\alpha_n) - \varphi(\alpha_{n-1})}\varphi(\alpha_n), \quad n = 1, 2, \ldots$$

$\underline{n=1}$

$$\alpha_2 = \alpha_1 - \frac{\alpha_1 - \alpha_0}{\varphi(\alpha_1) - \varphi(\alpha_0)} \varphi(\alpha_1)$$

$$= 1.5 - \frac{1.5 - 1.2}{0.494} \times 0.29$$

$$= 1.32$$

Solve the IVP $\begin{cases} y' = z, & y(0) = 1 \\ z' = 6y^2 - x, & z(0) = \alpha_2 = 1.32. \end{cases}$

$y_1 = 1.44, \quad z_1 = 3.32$

$y_2 = 2.51, \quad z_2 = 7.3672$

$y_3 = 4.965$

$= y(1; 1.32)$

$$\varphi(\alpha_2) = y(1; 1.32) - \gamma_2 = -0.035$$

now check $|\varphi(\alpha_2)| < \varepsilon$, if $\varepsilon = 0.05$ (given)

So we can stop here, otherwise we will proceed further to find α_3 by secant method.

and continue to obtain $\varphi(\alpha_3)$.

So the desired sol's are

$y(1/3) = 1.44$ and $y(2/3) = 2.51$

Solve by N-R method:

$y'' = f(x, y, y')$, $y(a) = \gamma_1$, $y(b) = \gamma_2$ $\longrightarrow$ $y'(a) = \alpha$ (Say)

$$\varphi(\alpha) = y(b; \alpha) - \gamma_2 = 0$$

Solve for α using N-R method

$$\alpha_{n+1} = \alpha_n - \frac{\varphi(\alpha_n)}{\varphi'(\alpha_n)}, \quad n = 1, 2, \ldots$$

Now $\varphi'(\alpha_n)$ is unknown to us.

Task: find $\varphi'(\alpha_n)$

$$\varphi(\alpha) = y(b;\alpha) - \gamma_2$$

$$\varphi'(\alpha) = \frac{\partial y(b;\alpha)}{\partial \alpha} = \eta_\alpha(b), \text{ (say)}$$

$$y'' = f(x, y, y')$$

$$y''(b) = f(b, y(b;\alpha), y'(b;\alpha))$$

now differentiating w.r.t α

$$\frac{\partial y''}{\partial \alpha} = \frac{\partial f}{\partial \alpha} \quad \text{at } x = b$$

$$\frac{d^2}{dx^2}\left(\frac{\partial y}{\partial \alpha}\right) = \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\cdot\frac{\partial y'}{\partial \alpha} \quad \text{at } x=b$$

$$\Rightarrow \eta_\alpha'' = \frac{\partial f}{\partial y}\cdot\eta_\alpha + \frac{\partial f}{\partial y'}\cdot\eta_\alpha' , \quad \left[\text{Since } \frac{\partial y}{\partial \alpha} = \eta_\alpha \text{ at } x=b.\right]$$

$$y(a) = \gamma_1 \Rightarrow \eta_\alpha(a) = 0$$

$$y'(a) = \alpha \Rightarrow \eta_\alpha'(a) = 1$$

$$\boxed{\begin{gathered}\eta_\alpha'' = \frac{\partial f}{\partial y}\cdot\eta_\alpha + \frac{\partial f}{\partial y'}\cdot\eta_\alpha' \\ \eta_\alpha(a) = 0,\ \eta_\alpha'(a) = 1\end{gathered}} \rightarrow \underline{\text{IVP}}$$

$(*)$

On Solving (*), we obtain η_α which is $\varphi'(\alpha)$

After finding it, now we can apply N-R method to get best α.

$$\boxed{\alpha_{n+1} = \alpha_n - \frac{\varphi(\alpha_n)}{\varphi'(\alpha_n)}}, \quad n = 1, 2, \dots$$

<u>Example:</u> $y'' = y$, $y(0) = 1$, $y(1) = 0$. $h = 1/3$.

<u>Soln:</u> Let $f(x, y, y') = y$. $\frac{\partial f}{\partial y} = 1$, $\frac{\partial f}{\partial y'} = 0$

$$\eta_\alpha'' = \frac{\partial f}{\partial y} \cdot \eta_\alpha + \frac{\partial f}{\partial y'} \cdot \eta_\alpha' = \eta_\alpha$$

$y(0) = 1 \Rightarrow \eta_\alpha(0) = 0$, $y'(0) = \alpha \Rightarrow \eta_\alpha'(0) = 1$.

Let $\eta_\alpha' = u$, $\eta_\alpha(0) = 0$

$n=0$	$n=1$	$n=2$	$n=3$
x_0	x_1	x_2	x_3
y_0	y_1	y_2	y_3.

$u' = \eta_\alpha$, $u(0) = 1$

$$\left.\begin{aligned} \eta_{\alpha_{n+1}} &= \eta_{\alpha_n} + h\, u_n \\ u_{n+1} &= u_n + h\, \eta_{\alpha_n} \end{aligned}\right\} \text{Euler method}$$

$\eta_{\alpha_1} = 1/3$, $\eta_{\alpha_2} = 2/3$, $u_1 = 1$, $u_2 = 10/9$

$\eta_{\alpha_3} = 28/27$, $u_3 = 4/3$,

Consider $\eta_{\alpha_3} = \eta_\alpha$ at $x = b$

$= \varphi'(\alpha) = \underline{\underline{28/27}}$, Now apply N-R method

$$\alpha_{n+1} = \alpha_n - \frac{\varphi(\alpha_n)}{\varphi'(\alpha_n)}$$

$$= \alpha_n - \frac{y(b;\alpha_n) - \gamma_2}{\eta_\alpha(b)}$$

Consider initial approx. $\rightarrow$ $\alpha_0 = 1$, $y(1;\alpha_0) = 0.32$ $\leftarrow$ you try to find it by solving the IVP $y'' = y$, $y(0) = 1$ and $y'(0) = \alpha_0 = 1$.

$$\varphi(\alpha_0) = y(1;\alpha_0) - \gamma_2 = 0.32$$

$|\varphi(\alpha_0)| = |y(1;\alpha_0) - \gamma_2| = 0.32 \not< \varepsilon = 0.05$. find α_1 to check $\varphi(\alpha_1)$.

now, $$\alpha_1 = 1 - \frac{0.32 - 0}{28/27} = 0.92$$

Now after getting α_1, we have to solve the

IVP $\quad y'' = y, \quad y(0) = 1, \quad y'(0) = \alpha_1$

find $\boxed{y(1/3) = y_1, \quad y(2/3) = y_2}$ and $y(1) = y_3$.

check $\varphi(\alpha_1) = y(1; \alpha_1) - \gamma_2$

If you get $|\varphi(\alpha_1)| = |y(1; \alpha_1) - \gamma_2| < \varepsilon = 0.05$

then y_1 and y_2 will be the desired solution, otherwise repeat this process.

Simple pendulum :

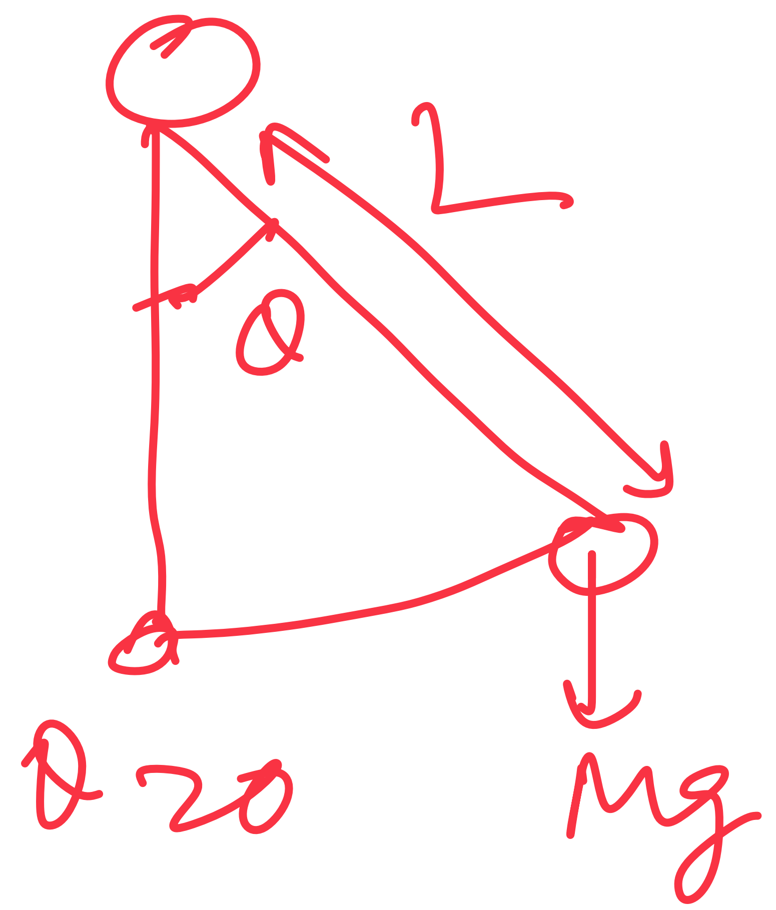
Practicle example

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta = K\sin\theta$$

$$\Rightarrow \ddot{\theta} = K\sin\theta$$

boundary condition

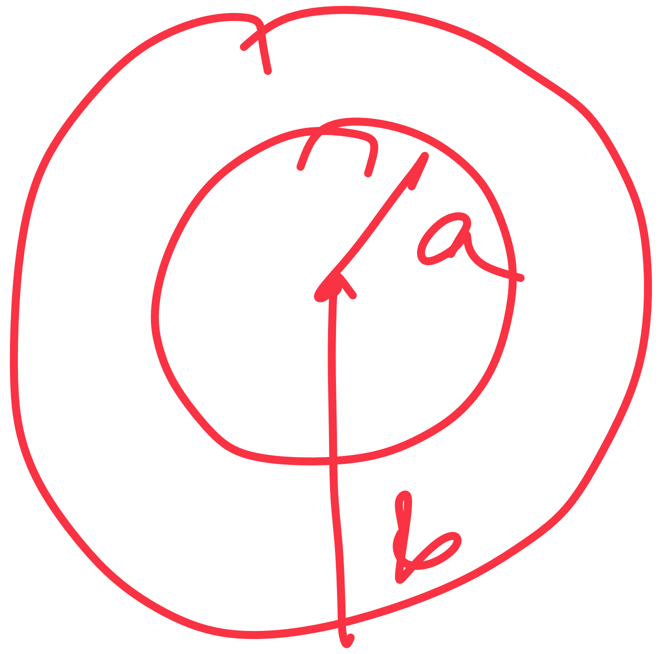
$$\theta(0) = \pi/4, \quad \theta(1) = \pi/2, \quad h = 1/3.$$



Try!

Practical Problem :

Consider a pressure vessel that is being tested in the laboratory to check its ability to withstand pressure. For a thick pressure vessel of inner radius a and outer radius 'b', the differential eqⁿ for the radial displacement of a point along the thickness is given by



$$\frac{d^2 P}{dr^2} + \frac{1}{r}\frac{dP}{dr} - \frac{P}{r^2} = 0$$

with $P|_{r=a} = 0.00387$, $a = 5$

$P|_{r=b} = 0.00307$, $b = 8$.

Try!