UNIT 4 SIMPLE STRESSES AND STRAINS

Structure

4.1 INTRODUCTION

You have already learnt how to analyse a given truss and find the forces induced in the various individual members. Later on you will be introduced to various other types of structures and will learn how to find the forces induced in the members due to the forces applied on the structure. Such analysis forms the essence of Structural Analysis. Our concern in Strength of Materials is what happens to the individual members, when they have to carry these forces. As you advance in your learning of strength of materials, you will realise that the two subjects are closely interrelated and together they form the foundation for the design of a large variety of structures. In this unit, you will be introduced to the analysis of simple stresses and strains induced in members.

Objectives

After studying this unit, you should be able to

- grasp the concepts of stress, strain and deformation,
- elaborate the effect of forces on deformable solids and the relationship between stress and strain for typical materials,
- explain the elastic properties of solids and their deformations due to direct forces,
- explain the methods of analysing composite bars,
- explain the deformations and stresses caused by temperature variation,

- calculate stresses on oblique sections, and
- define the elastic constants and their interrelationships.

When you have clearly learnt these concepts, you will be able to use your knowledge to understand the behaviour of solids under simple loadings and will be able to design such members. You will also be able to verify whether such members are safe under the loads they are subjected to. In addition, you will also be able to find the load carrying capacity of members whose geometric and elastic properties are known.

4.2 BASIC CONCEPTS

4.2.1 Rigid and Deformable Solids

The subject matter of Strength of Materials is also learnt under the title Applied Mechanics, and Mechanics of Solids. Applied Mechanics is the general title in which Fluid Mechanics and Solid Mechanics are branches. What we study in Strength of Materials is the effect of forces acting on solids of different geometrical and elastic properties, and hence more people now choose to call it solid mechanics.

Let us now learn two terms used to describe two types of solids. A **rigid solid** is one which does not undergo any change in its geometry, size or shape. On the other hand, a deformable solid is one in which change in size, shape or both will occur when it is subjected to a force. The geometrical changes produced are called **deformations** and hence the name deformable solid. Many of the common solid articles we handle in our day to day life exhibit perceptible deformations when subjected to loads. In many other solids, though we may not perceive any deformations with our naked eye, with measuring instruments of sufficient precision we can see that they also get deformed under applied forces. A more careful scrutiny will reveal that all solids are deformable and the idea of rigid solids is only a conceptual idealization.

Look at the common swing shown in Figure 4.1 which you would have enjoyed playing with during your earlier days. When a child sits on the swing we may not perceive any appreciable deformations. But if the iron bars be replaced by bamboo sticks and the iron chain be replaced by rubber strings, we can observe considerable deformations as shown in Figure 4.2. We may then conclusively say that whenever forces are applied on solids deformations are introduced in them. All solids are deformable solids; hence, the subject which once was called **Mechanics of Deformable Solids** is now simply called Mechanics of Solids.

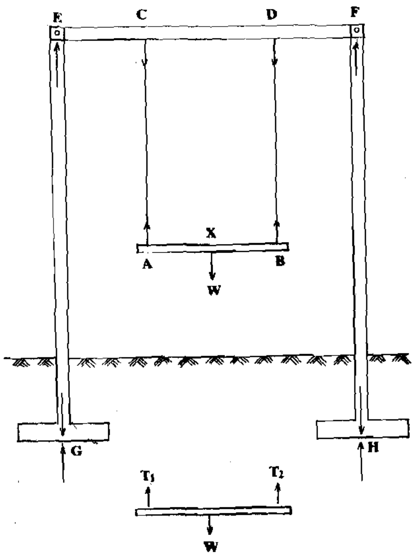


Figure 4.1

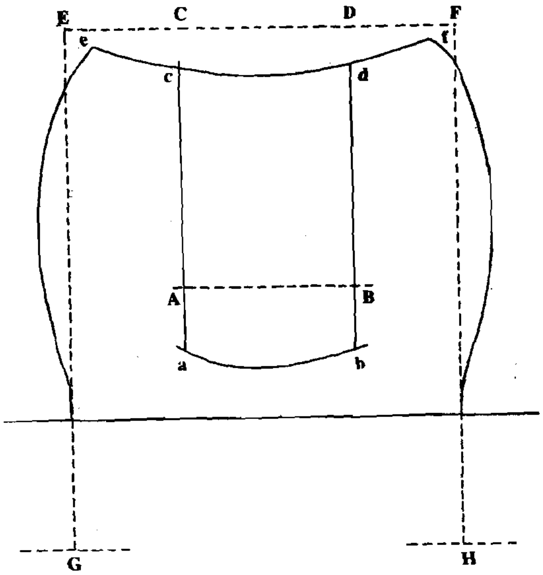


Figure 4.2

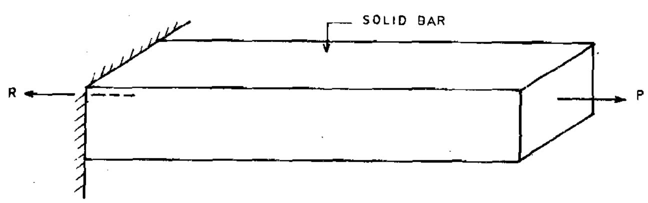
You may wonder as to why we should have the term rigid solid at all, if in reality, there are no such solids. It has been already mentioned that the term rigid solid is only a conceptual idealization. There are a few important uses for such idealization. Consider the analysis of forces in the members of some of the plane trusses you have studied in Units 1 and 2. You were given the geometry of a structure (which in its totality may be considered as a solid of complex geometry), on which some forces have been presented to act. While analysing these structures (solids) for finding out the member forces, the forces in the members have been resolved into their horizontal and vertical components, assuming that the member orientations have not changed due to the application of external forces. In reality, the orientations of most of the members change, as the structure deforms under the loads. However, these deformations and hence, the resulting changes in member orientations are so small that the error in computed member forces are negligibly small. You may think that even these errors may be eliminated by taking into account the deformations too. But an attempt to do so will reveal the complexity of the calculations involved. The corrections thus effected are called **secondary effects** and may be neglected in most of the cases. In the analysis of other types of structures also when the overall equilibrium is considered the entire structure or any part under consideration may be treated as a rigid solid.

In many cases, known as determinate problems, the analysis of forces within a solid can be calculated treating it as rigid.

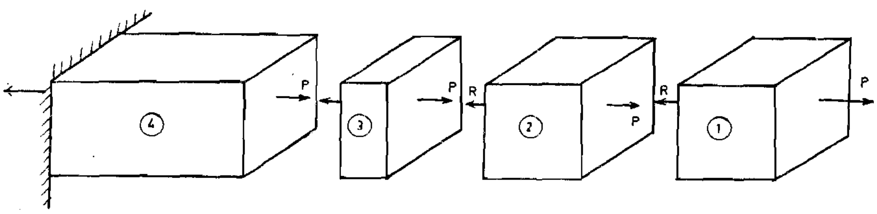
4.2.2 Stresses and Strains

When we apply forces on solids, deformations are produced if the solid is prevented from motion (with acceleration = force / mass of solid) either fully or partially. If the solid is not restrained it may undergo displacements without change in shape or size and these displacements are termed as rigid body displacements. If the solid is restrained by some other force, known as **reaction**, which keeps the solid in equilibrium, the force will be transmitted through the medium of the solid to the restraining support.

Consider the solid shown in Figure 4.3 (a) which is pulled by the force P at the right hand side and restrained by a support at the left hand side end. If the solid is to remain in static equilibrium we know that the support should exert an equal pulling force in the direction opposite to that of force P. This restraining force, termed as support reaction is shown as R in Figure 4.3 (a). Now consider the solid being divided into a number of small bits or elements of different lengths as shown in Figure 4.3 (b). The force P applied at the L.H.S. end of element 1 is transmitted to the support through the medium of elements 2, 3 and 4. The same thing is true even if you divide the solid into hundreds or even millions of elements.



(a)



(b)

Figure 4.3

Now, to keep the element 1 in static equilibrium element 2 should exert a force R $(=P)$ on element 1. At the same time element 1 exerts a force P on element 2 so that equilibrium exists at the point or surface of contact. The element 2 is again pulled by the equilibrant R exerted by element 3. Thus equilibrium exists at each and every point or surface of contact and each of the elements are kept in equilibrium.

The force P (or R) shown in the figures is only the resultant of forces applied over some area (small or even full). On any area the force may not be distributed uniformly. However, we may consider an infinitesimly small area ΔA over which a small part of the total force ΔP may be considered to be uniformly acting. Here, the ratio $\frac{dp}{dA}$ $\left(\text{or rather } \underset{\Delta A \to 0}{\text{Lt}} \frac{\Delta P}{\Delta A} \right)$ may be called as pressure (suctional or compressive) at the point. Considering a small element having the area dA on one of its faces as shown in Figure 4.4, we realise that this pressure is accompanied by the equilibrating pressure on the opposite face of the element.

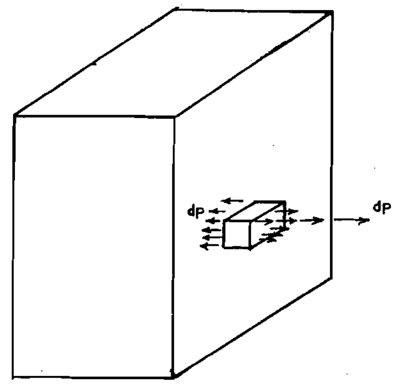


Figure 4.4

Such a set of internal pressures acting on solids is defined as stress. Depending on which set of planes such pressures are applied, there can be a number of such sets and they may be called stress components. (More about this we shall learn later.)

To take the simplest case, if the total force P is uniformly distributed over the cross section A as shown in Figure 4.3 (a), then the magnitude of stress in the solid is P/A. This entity, stress is denoted by the symbol 'σ' and usually expressed in N/mm^2 units, also denoted as Mega Pascal (MPa) as both are numerically equal.

Let us now turn our attention to deformations. The set of forces P and R [Figure 4.3 (a)] are pulling the solid in opposite directions. Consequently, the solid gets elongated and let this elongation be denoted by the symbol 'δ'. If the forces be applied on the individual elements, each element will elongate [Figure 4.5 (a)] and it has been experimentally established that,

1. the magnitude of individual elongations are proportional to the length of each of the elements, and
2. the total elongation of all the elements add up exactly to δ.

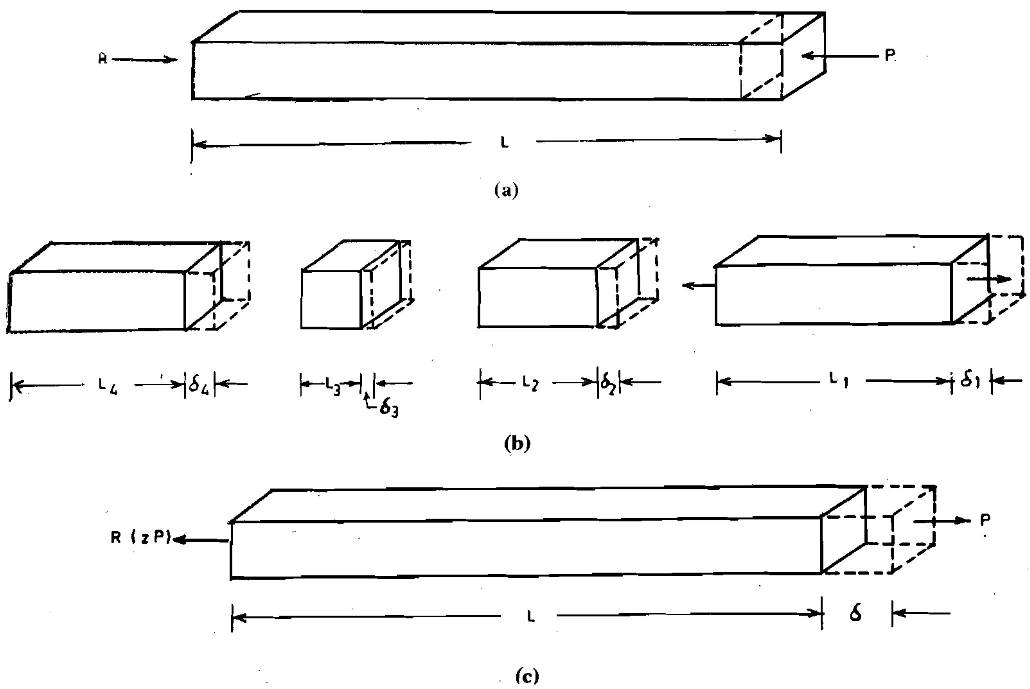


Figure 4.5 (a), (b) and (c)

We may conclude that the ratio of elongation to length of the elements $\dfrac{\delta_1}{L_1}, \dfrac{\delta_2}{L_2} \ldots$ is invariable and equal to $\dfrac{\delta}{L}$. This ratio is defined as **strain** (more specifically as longitudinal strain) and denoted by ε. If the forces applied are in the opposite direction as shown in Figure 4.5 (b), the solid will get its length shortened and the deformation and strain in such a case are considered negative.

4.2.3 Tensile, Compressive and Shear Stresses

We have observed that forces produce elongations or contractions on solids depending on how they act. The forces that produce elongation on solids are called **tensile forces** and the stresses induced by them are called **tensile stresses**. Tensile forces and tensile stresses are both considered positive. Similarly, the forces which cause shortening of length are called **compressive forces** and the stresses induced by them are called **compressive stresses**. Both Compressive Forces and Compressive Stresses are considered negative.

We may also observe that both tensile stresses and compressive stresses act normal to the surface on which they act. For this reason they are both classified as normal stresses. However, there are many instances of load applications when the stresses are induced in directions other than that of the normal to the surface. A few such cases are shown in Figure 4.6. A small element is shown enlarged in scale. Beside the element the coordinate system is also given. The plane *BEFC* is normal to the *x* axis and is hence, defined as *x* plane. Similarly the planes *DCFG* and *ABCD* are defined as *y* and *z* planes respectively.

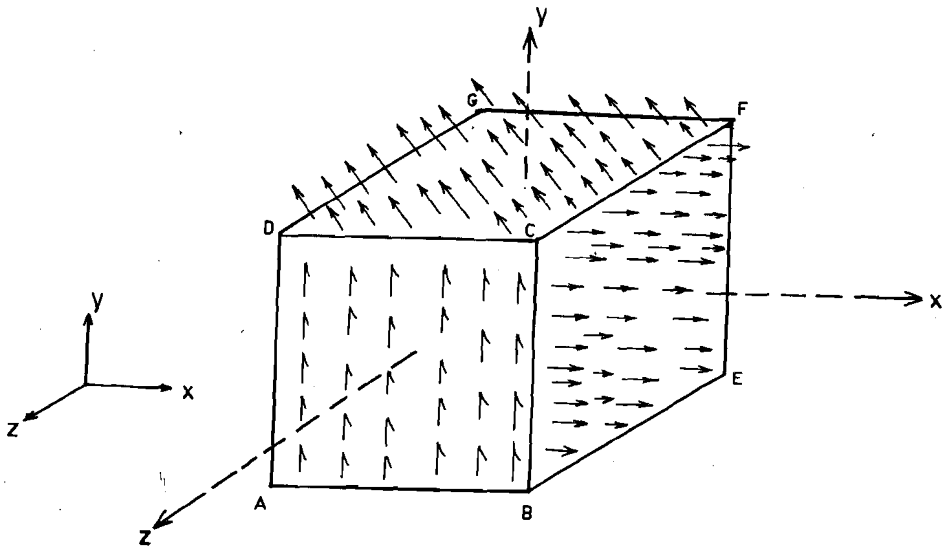


Figure 4.6

You may observe that the stress on *x* plane is normal (tensile), stress on *y* plane is inclined and that on *z* plane is parallel to the plane itself. The stresses that are acting parallel to the plane on which they are applied are called **shear stresses**. On further consideration we may resolve the stress on the *y* plane into components parallel and normal to the surface. Hence, on any plane, there may be normal stresses, shear stresses or both. Further, while the direction of normal to a plane is uniquely defined, there are infinitely large number of directions in which shear stresses may be applied on a plane.

Whatever may be the direction of shear stress, it can be further resolved into components in two mutually perpendicular directions, and having done this, you would have completely defined the stresses acting on a plane. Defining such states of stress on all the planes will furnish the complete state of stress on the small element (or at a point).

Let us now consider a few examples of solids in which shear stresses are induced. Figure 4.7 shows how two steel plates are connected together by a riveted joint. When the connected plates are pulled, the pull from plate *A* is transmitted to plate *B* through the rivet. The rivet may be considered to consist of two parts. Part *C* of the rivet carries the force *P* from plate *A* by diametral compression or bearing, while part *D* of the rivet transmits the load *P* to plate *B* by bearing. The transfer of force from part *C* to its counterpart *D* takes place at the common interface namely its cross sectional area at midlength. On this area the force *P* is applied parallel and hence, the rivet is in shear.

Again consider two wooden blocks connected together by some glue and also connected to a rigid base similarly, as shown in Figure 4.8. If a horizontal force *P* is applied on the *x* plane of the block *A* at the top, this force is transmitted to the base through a horizontal surface which is the common interface between the base and the block *D*. As the force is parallel to the surface on which it is acting, it is a shear force and the stress induced by it also should be shear stress. On further scrutiny you will find that the transmission of force from one block to another in the assembly is only as shear force and not only on interfaces between two adjacent blocks, but on any horizontal.

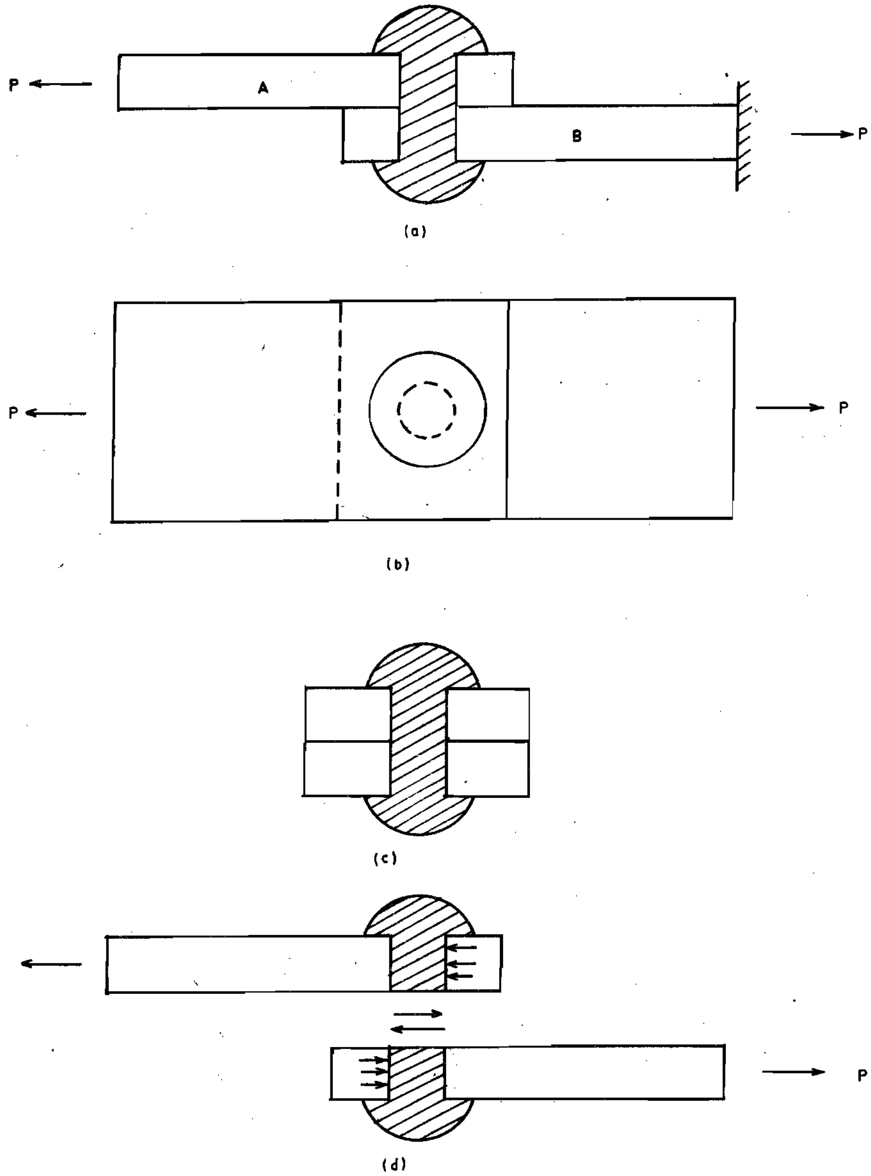


Figure 4.7

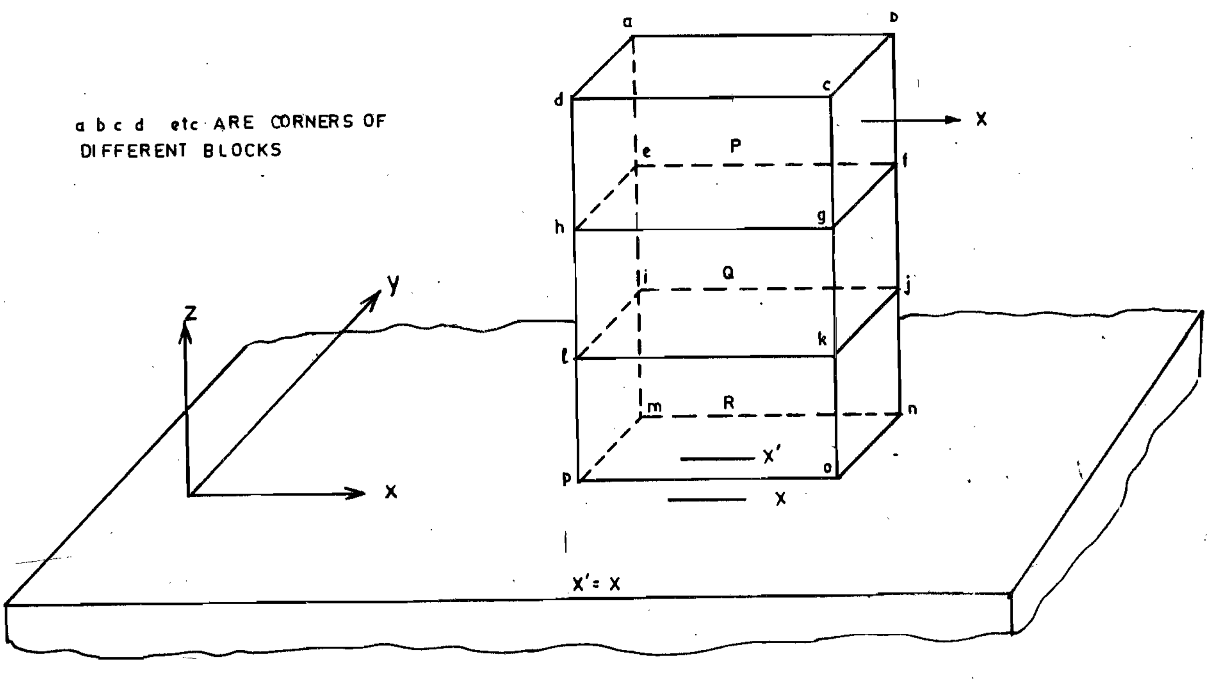


Figure 4.8

4.2.4 Complementary Shear Stresses

Consider the equilibrium of the solid element of dimensions l, b and h and acted upon by shear stress components τ_{xy} and τ_{yx} as shown in Figure 4.9. By inspection, we may recognize that the components τ_{xy} on a pair of vertical planes and the pair of τ_{yx} on two horizontal planes satisfy independently the force equilibrium equations :

$$\Sigma F_x = 0 \text{ and } \Sigma F_y = 0$$

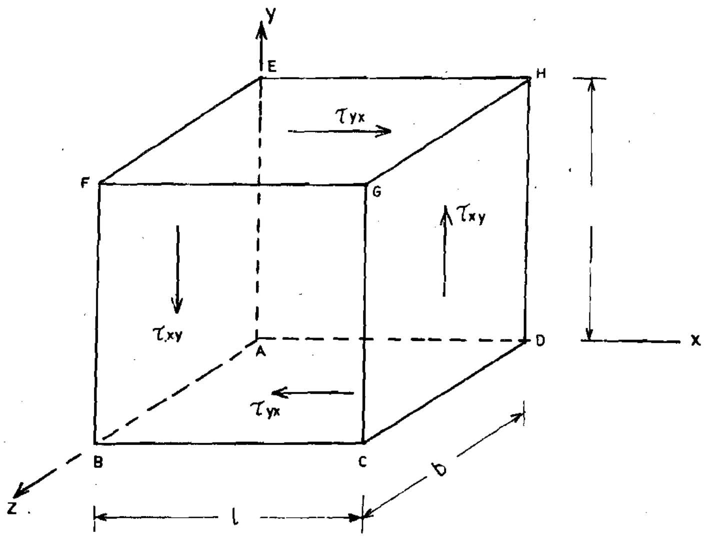


Figure 4.9

Let us now consider the moment equilibrium equations. τ_{xy} components on planes *CDHC* and *ABFE* have resultant forces $Q_{xy} = \tau_{xy} . b . h$ which form a couple equal to $(\tau_{xy} . b . h) \times l$. Similarly, the moment of the couple formed by the shear stress component $\tau_{xy} = (\tau_{yx} . l . b) h$. For equilibrium,

$$(\tau_{xy} . b . h)l + (\tau_{yx} . l . b)h = 0$$

or $$\tau_{xy} = - \tau_{yx} .$$

That is τ_{xy} is always accompanied by $- \tau_{yx}$ and this pair of shear stresses is called **complementary shear stresses.**

When a piece of carrot is cut into two pieces, the force exerted by the knife acts on surface parallel to the force and hence the carrot is sheared into two. Likewise, you may think over the various cases of load applications in our day to day life and identify the cases in which shear stresses are involved.

4.3 MECHANICAL BEHAVIOUR OF MATERIALS

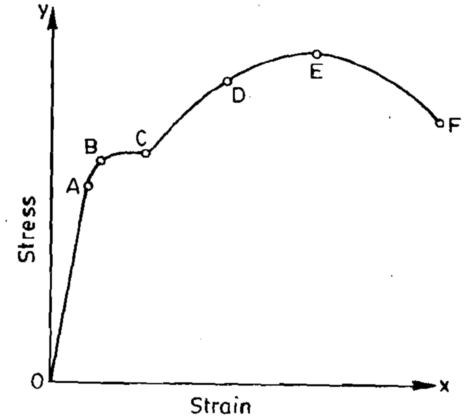
4.3.1 Stress-Strain Curves

As we have earlier observed, the behaviour of solids, when subjected to loads, depends upon its geometrical and mechanical properties. The mechanical properties of solids have to be investigated through laboratory tests. Let us consider one such test which you might have conducted in the Strength of Materials laboratory.

Tension Test on a Mild Steel Rod

A mild steel rod of known diameter and length of about 400 mm to 500 mm is firmly gripped in a universal testing machine and a deformeter is fixed on the rod over a measured gauge of 150 mm to 200 mm. The rod is subjected to axial tensile load. The load at any instant is indicated by the load dial of the UTM while the elongation of the bar over the gauge length is measured through the deformeter.

The load is gradually applied on the bar in suitable increments and the corresponding elongations of the bar are measured at each stage. The test is carried out until the specimen ultimately fails by rupture. The load values at each stage are divided by the cross sectional area of the bar to get the stress values and the corresponding elongations of the bar divided by the gauge length give the strain values. A graph is drawn relating stress and corresponding strain. A typical stress-strain curve for mild steel is shown in Figure 4.10.



Cardinal Points

A : Proportionality Limit

B : Elastic Limit

C : Yield Point

C - D : Ductile Extension Zone

E : Ultimate Strength

F : Rupture Strength

Figure 4.10 : Stress-Strain Curve for Mild Steel

4.3.2 Hooke's Law, Elastic Limit

From the stress-strain curve of a material a number of essential mechanical properties of the material can be studied. When you look at the stress-strain curve (Figure 4.10), you can identify segments of different nature. The initial segment of the curve is steep and straight and the strain in this range is proportional to the stress. Hence, the terminal stage of this range is called proportional limit. At this stage, we find some small fluctuations followed by a horizontal segment, indicating that the strain in the solid is continuously increasing without any increase in stress. Here, the material yields at the end of proportional limit and exhibits plasticity (i.e. being strained without increase in stress) on further pulling. This range of strain is called plastic range, and is about 12 to 13 times the proportional range of strain. The stage where proportional range ends and plastic range begins is called yield point.

When the end of plastic range is reached further loading results in increase of both stress and strain. But here the stress-strain curve is not so steep as in the initial range and is also non-linear. The material having exhibited plasticity once again hardens to take additional stress and hence this range is called **strain hardening** range. Straining further leads to ultimate failure by rupture.

To understand the stress strain relationship of the solid more clearly, further tests are carried out by loading, unloading and reloading of the solid at various stages and the results of such an investigation are shown in Figure 4.11.

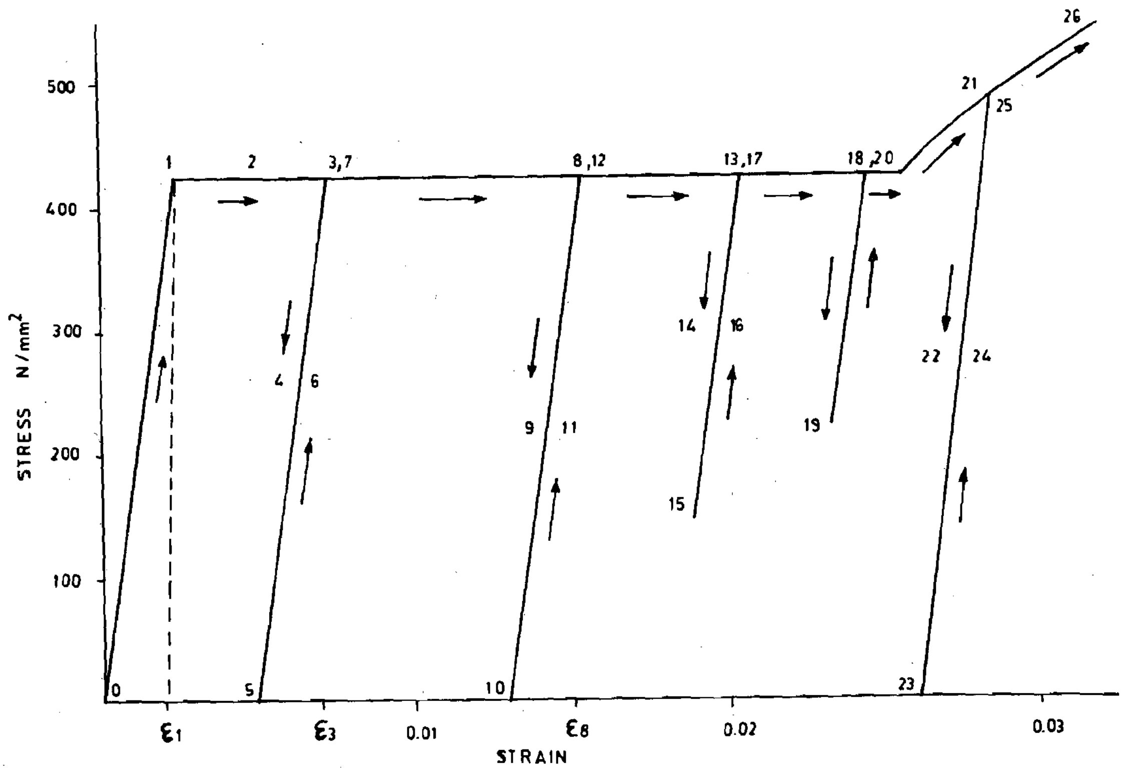


Figure 4.11

From the results of such a study the following observations may be made.

(i) When the loading is done within the proportional limit of the solid, the strain (deformation) caused is fully recovered when the stress is removed. (If the load is partially removed, a proportional amount of strain gets relieved.)

(ii) When the solid is loaded beyond yield limit and then unloaded, only a part of the strain is relieved and a considerable part of strain still persists thus effecting permanent changes in the geometry of the solid. The strain remaining unrelieved is known as **permanent set.**

(iii) When we reload a solid which is already subjected to permanent set, the stress strain curve for reloading is linear upto the maximum stage to which it was subjected to earlier.

From these observations what does a designer learn. If a structure or any component is loaded beyond proportional limit, the geometry gets permanently changed and if repeatedly loaded beyond the proportional limit, it will go completely out of shape after a few repetitions. On the other hand, you can load it within their proportional limit any number of times and the original geometry recover on removal of the load. Hence, in any design, one has to be careful to avoid loading the component beyond the proportional limit. Within this limit the solid is said to be elastic. The property called elasticity consists of two aspects, normally complete removal of strains on unloading and linearity of stress-strain relationship. Hence, the part of stress-strain curve upto yield point is also called as elastic limit. To ensure that the solid is always within this elastic limit, designers permit stresses in solid much lower than the elastic limit. The ratio of the maximum stress a material can withstand to the maximum stress which a designer chooses to allow is called the **factor of safety.** In short,

$$\text{Factor of Safety} = \frac{\text{maximum stress the material can withstand}}{\text{maximum stress permissible}}$$

If the numerator is yield point stress a factor of safely of 2 is usually allowed and if it is ultimate stress, factor of safety may be 3 or more.

All materials do not behave as mild steel. The stress-strain curves of a few materials are given in Figure 4.12. Irrespective of the variety of shapes seen in these curves, one can identify in each of them a segment where the curve may be considered linear. This limit is taken as the elastic limit for the material.

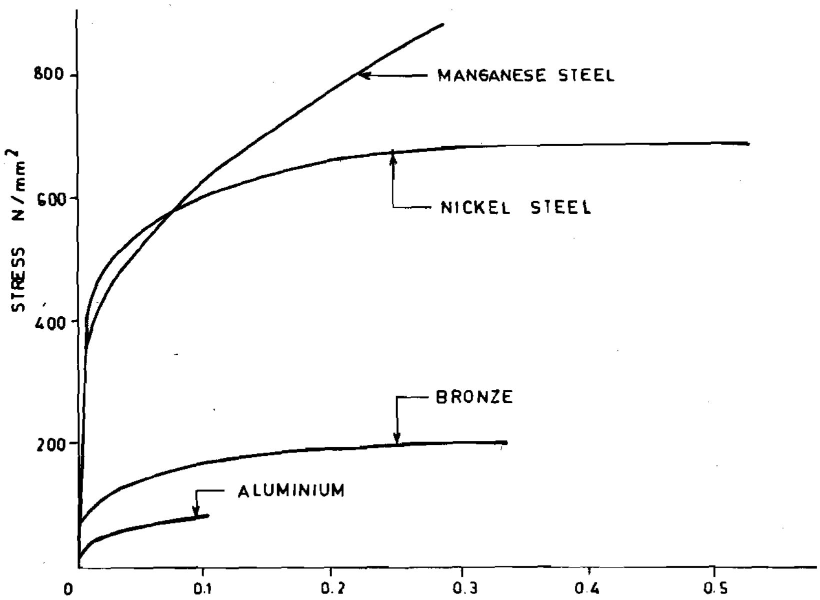


Figure 4.12 : Stress-Strain Relationship for a Few Materials

The overall behaviour of the material and its complete stress strain relationship is discussed here only in order to caution the designer to keep the stress level well within the elastic limit. Hence, for practical purposes the stress strain relationship is taken as linear and during the rest of this course you will not be dealing with nonlinearity of material behaviour. Hence, for a designer stress is always proportional to the strain and the proportionality constant is defined as **Elastic Modulus.**

Thus, Stress, σ = Elastic Modulus, E $\times$ Strain, ε

or
$$\sigma = E \, . \, \varepsilon \tag{4.1}$$

From Eq. (4.1), we get,

$$E = \frac{\sigma}{\varepsilon} \tag{4.2}$$

Eq. (4.2) indicates that the Elastic Modulus of a material may be calculated by measuring the slope of the stress-strain curve (within the elastic limit). The fact that Eq. (4.1) holds good within elastic limit which has already been taught to you as **Hooke's law,** which we may restate as follows :

> *Within elastic limit the strain produced in a solid is always proportional to the stress applied.*

4.3.3 Elastic Constants

As there are different types of stresses possible, such as linear (normal) stress, shear stress, volumetric (bulk) stress, correspondingly different modulii of elasticity namely Young's modulus, shear modulus and bulk modulus are defined as elastic constants of a material. The term E used in Eqs. (4.1) and (4.2) in the context of normal stresses and corresponding strains is known as Young's Modulus.

In addition to the three modulii of elasticity an important elastic constant, used in defining the mechanical properties of a solid, is known as Poisson's ratio. While testing the material for stress strain relationship, you observe that strains are produced not only in the direction of the applied stress, but also in direction perpendicular (lateral) to it. On further investigations, you will find that the lateral strain is always proportional to the longitudinal strain and the proportionality constant is negative. This proportionality constant is defined as Poisson's Ratio and denoted by ν.

$$\therefore \text{ Poisson's Ratio, } \nu = \frac{-\text{ lateral strain}}{\text{longitudinal strain}}$$

While applying tensile force on a rod we found that the deformations produced involved change of volume of the solid as well as its shape. Such deformations may be found in many other cases of loading. Deformations involving change of volume and shape are more common. However, we may identify two special cases of deformations, dilatation and distortion.

Dilatation is defined as change in volume (bulk) of the solid without change of shape. If normal stress components of equal magnitude and same nature (all tensile or all compressive) are applied on all the three mutually perpendicular planes of a solid element as shown in Figure 4.13, the solid will undergo only volume change without any change in shape. Such a state of stress (as may be obtaining in the case of solid subjected to hydrostatic pressure) is called **volumetric stress** and the change of volume produced per unit volume of the solid is called the volumetric stress. The ratio between volumetric stress and volumetric strain is defined as Bulk Modulus.

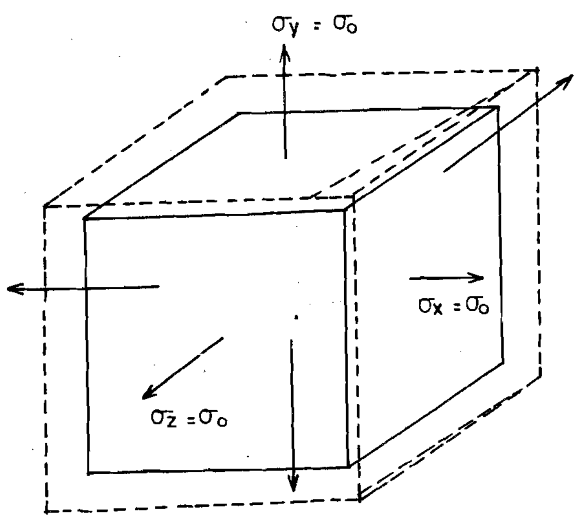


Figure 4.13 : Bulk Stress and Dilotation

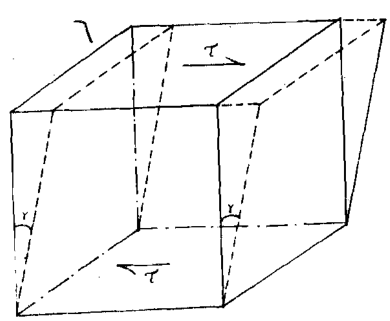


Figure 4.14 : Shear Stress and Shear Strain

Thus, Bulk Modulus, $K = \dfrac{\text{Volumetric stress}}{\text{Volumetric strain}}$ (4.3)

in which Volumetric strain $= \dfrac{\text{Change of volume}}{\text{Original volume}}$

The state of stress, with equal values of stress components in all the directions is also known as **spherical state of stress**.

The state of deformation involving change of shape without any change of volume is defined as **Distortion.** Though distortion may be introduced in many ways let us consider a simple case. When shear stresses are applied on a solid as shown in Figure 4.14, we find that angular deformations are introduced. The angular strain or change in angle produced is called shear strain denoted by γ (greek gamma) and its magnitude is expressed in radians. The ratio of the applied shear stress and the shear strain produced by it is defined as Rigidity Modulus or Shear Modulus of the material, i.e.

$$\text{Rigidity Modulus, } G = \frac{\text{Shear Stress } (\tau)}{\text{Shear Strain } (\gamma)} \quad (4.4)$$

We have so far defined four elastic constants, namely, Young's Modulus, (E), Poisson's Ratio (ν) , Bulk Modulus (K), and Shear Modulus (G). Later on we shall learn that these four are not independent constants, but are related to each other. We can show that there are only two independent elastic constants and the other two are dependent on them.

4.4 DEFORMATION OF BARS

4.4.1 Bars of Uniform Section

With the knowledge gained already, you will now be able to learn calculation of deformations in solids due to applications of simple stresses. Let us start with the example of a prismatic bar of length 'L', uniform cross section of area A carrying an axial load 'P' as shown in Figure 4.15.

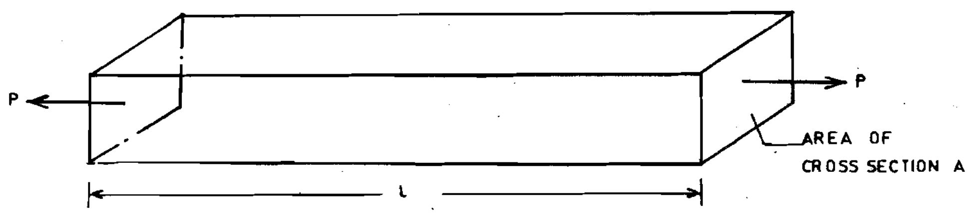


Figure 4.15

Stress in the solid is in the longitudinal direction and its magnitude may be calculated as,

$$\sigma = \frac{\text{Load}}{\text{Area}} = \frac{P}{A}$$

If the Young's Modulus of the material, E is known, the strain induced may be calculated as

$$\varepsilon = \frac{\sigma}{E} = \frac{P}{AE}$$

As strain is only deformation per unit length, the total elongation of the bar is calculated as

$$\delta = \varepsilon . L = \frac{P}{AE} L$$

Thus, Elongation $\delta = \dfrac{PL}{AE}$ (4.6)

You may please note that the strain ε calculated here is in the longitudinal direction and is accompanied by strains in the lateral directions also whose magnitude is given by $- \nu\varepsilon$. Let us now consider a few numerical examples.

Example 4.1

If the bar shown in Figure 4.15 is 2 m long with rectangular cross section of 300 mm deep and 400 mm wide, calculate the change in volume of the solid due to a longitudinal compressive force of 720 kN, if the elastic constants E and ν for the material are known as 120 kN/mm^2 and 0.2 respectively.

Solution

Area of cross section of the member $= 300 \times 400 = 120000\ \text{mm}^2$

Longitudinal strain $\varepsilon = \dfrac{P}{A\,E} = \dfrac{-720 \times 1000}{120000 \times 120 \times 10^3} = -0.00005$

(Note that all values have to be converted to consistent units; here, it is N for forces and mm for length.)

$\therefore$ Total change in length $\delta = 1000 \times (-0.00005) = -0.05$ mm.

Lateral strain $\varepsilon_l = -\nu\varepsilon = -0.2 \times (-0.00005) = 0.00001$

Change in depth $= 0.00001 \times 300 = 0.003$ mm

Change in width $= 0.00001 \times 400 = 0.004$ mm

$\therefore$ Change in volume of the solid,

$$= (1000 - 0.05)\,(300 + 0.003)\,(400 + 0.004) - (1000 \times 400 \times 300)$$

$$= 999.95 \times 300.003 \times 400.004 - (1000 \times 400 \times 300)$$

$$= -3600.109\ \text{mm}^3$$

Let us consider an alternate approximate method also.

Change in volume, $dV = V + dV - V$

$$= (l + \Delta l)\,(b + \Delta b)\,(d + \Delta d) - l\,.\,b\,.\,d$$

where Δl, Δb and Δd are changes in length, breadth and depth of the solid.

i.e. $\quad dV = l(1 + \varepsilon_1) \times b(1 + \varepsilon_2) \times d(1 + \varepsilon_3) - l\,.\,b\,.\,d$

where ε_1, ε_2 and ε_3 are the strains in the three mutually perpendicular directions.

$$\therefore \quad dV = lbd \times (1 + \varepsilon_1)\,(1 + \varepsilon_2)\,(1 + \varepsilon_3) - lbd$$

$$= lbd \times (1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1 + \varepsilon_1\varepsilon_2\varepsilon_3) - lbd$$

$$= lbd \times (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1 + \varepsilon_1\varepsilon_2\varepsilon_3)$$

Neglecting the second order products,

$$dV = V \times (\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \qquad (4.7)$$

Now let us calculate the change in volume of the given solid using Eq. (4.7).

Change in volume, $dV = V \times (\varepsilon_1 + \varepsilon_2 + \varepsilon_3))$

$$= 1000 \times 300 \times 400\,(-0.00005 + 0.00001 + 0.00001)$$

$$= 3600\ \text{mm}^3$$

Though there is a small error, the approximation is quite satisfactory. (As an exercise you may calculate the percentage error in the value.) If you are very particular about accuracy, use the following formulation :

$$dV = V \times (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1 + \varepsilon_1\varepsilon_2\varepsilon_3)$$

SAQ 1

(a) Taking the dimensions of the solid in the Example 4.1 as same, calculate the change in volume of the solid if additional tensile forces of 1200 kN, and 2000 kN are applied in the lateral directions namely horizontal and vertical respectively.

(b) A concrete cylinder of height 300 mm and diameter 150 mm is tested for compression in a Universal Testing Machine. Within the elastic limit, the cylinder was found to be shortened by 0.12 mm and its diameter was found to be increased by 0.01 mm under an axial load of 90 kN. Calculate the Youngs Modulus E and Poisson's Ratio ν for the specimen.

4.4.2 Bars of Varying Cross Section

Let us now consider a few cases of a little more complexity. To give you a feeling of solid reality all solids have so far been represented by pictorial views. Hereafter you should be able to understand the problems when represented in projected views. Figure 4.16 represents the projected view of a bar whose cross sectional area differs in different segments. Taking the elastic modulus of the material as 60 kN/mm^2, let us evaluate the total elongation of the bar.

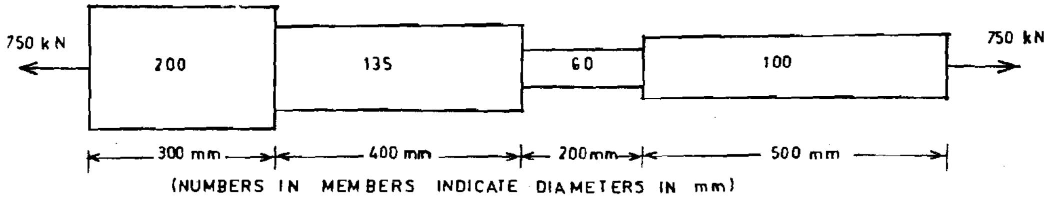


Figure 4.16

Although the bar has no uniform cross-section, it consists of segments of uniform cross section. Hence, the total elongation of the bar can be calculated as a sum of the elongations of individual segments.

Thus, $$\delta = \Sigma \delta_i = \frac{P_1 L_1}{A_1 E_1} + \frac{P_2 L_2}{A_2 E_2} + \ldots + \frac{P_i L_i}{A_i E_i} + \ldots$$

In the numerical example under consideration

$$\delta = \frac{750 \times 300}{\frac{\pi}{4} \times 200^2 \times 60} + \frac{750 \times 400}{\frac{\pi}{4} \times 135^2 \times 60} + \frac{750 \times 200}{\frac{\pi}{4} \times 60^2 \times 60} + \frac{750 \times 500}{\frac{\pi}{4} \times 100^2 \times 60}$$

$$= \frac{750}{\frac{\pi}{4} \times 60} \left[\frac{300}{200^2} + \frac{400}{135^2} + \frac{200}{60^2} + \frac{500}{100^2} \right] = 2.14865 \text{ mm}$$

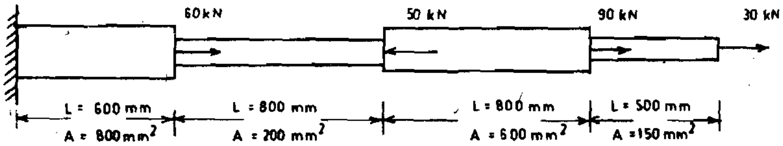
SAQ 2

Taking the Elastic modulus E = 80 kN/mm^2 and Poisson's ratio of the bar as 0.24, calculate the change in volume of the bar shown in Figure 4.16.

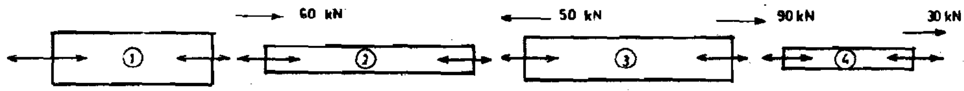
In the bar shown in Figure 4.16, the axial pull is applied at the ends and hence the axial force in all the members is the same. On the bar shown in Figure 4.17 (a), external forces are applied at intermediate sections also. In such cases, the axial force in each member should be evaluated first, before any deformation calculations. This can be accomplished by consideration of equilibrium of each of the sections where external loads are applied. Though no external load has been prescribed at the LHS end, the support reaction has to be calculated and taken as the external load. Equilibrium analysis can be easily carried out by treating each segment as a free body as shown in Figure 4.17 (b).

For example, consider the equilibrium of the segment 4 in Figure 4.17 (b). At the RHS end of the member a point load of 60 kN is applied. Hence, for the member to be in equilibrium a force of –60 kN should be applied at the RHS end of the member. Hence, the member is subjected to a tensile force of 60 kN which is represented by the internal arrows in accordance with the sign conventions you have already learnt. Member 3 is pulled with a tensile force of 60 kN exerted by member 4 and in addition the external force of 80 kN also pulls the member in the same direction, resulting in the member carrying a total tensile force of 140 kN. Proceeding thus, the arial forces in all the members can be calculated. To simplify the graphical representation we may show the member forces along with external forces as shown in Figure 4.17 (c).

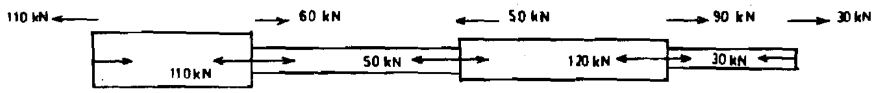
Now let us calculate the total elongation of the bar, taking the elastic modulus E as 200 kN/mm^2.



(a) (Not to the Scale)



(b)



(c)

Figure 4.17 (a : Not to the Scale), (b) and (c)

$$\delta = \Sigma\delta_i = \sum \frac{P_i L_i}{A_i E_i}$$

$$= \frac{160 \times 600}{800 \times 200} + \frac{100 \times 800}{200 \times 200} + \frac{140 \times 800}{600 \times 200} + \frac{60 \times 500}{150 \times 200}$$

$$= 0.6 + 2.0 + 0.9333 + 1.0$$

$$= 4.53333 \text{ mm.}$$

SAQ 3

Calculate the total elongation (or shortening) of the non-uniform bars (with loads), shown in Figures 4.18 and 4.19.

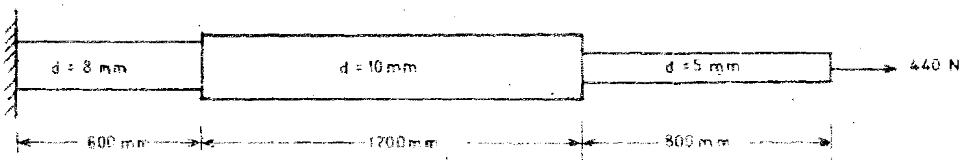


Figure 4.18 : (E = 180 GPa)

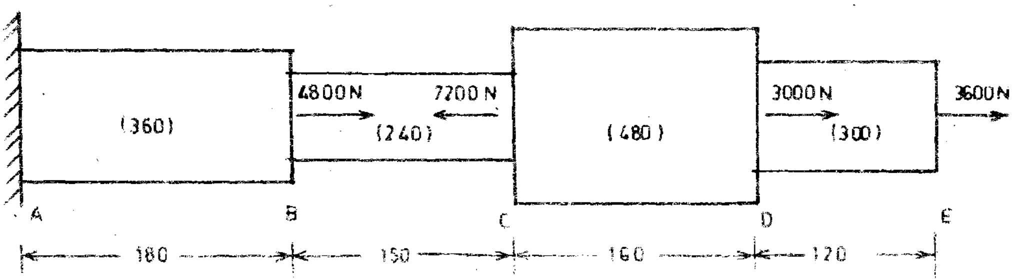


Figure 4.19 : (E = 200 GPa)

Now, let us consider the case of a bar whose cross section is varying continuously as shown in Figure 4.20. The bar of (truncated) conical shape is subjected to an axial force of P. As in the case of stepped bars, here also, we may apply the equation $\delta = \Sigma\delta_i = \sum \frac{P_i L_i}{A_i E_i}$, if we recognize that L_i is sufficiently small and A_i may be taken as uniform over the length of the small segment. That is, the expression $\sum \frac{P_i L_i}{A_i E_i}$, now becomes $\int_0^L \frac{P_i \, dx}{A_i E_i}$.

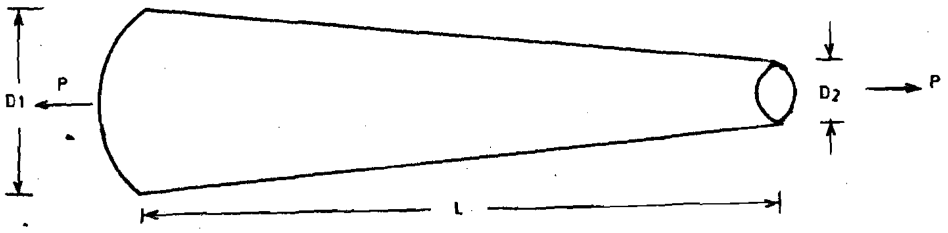


Figure 4.20

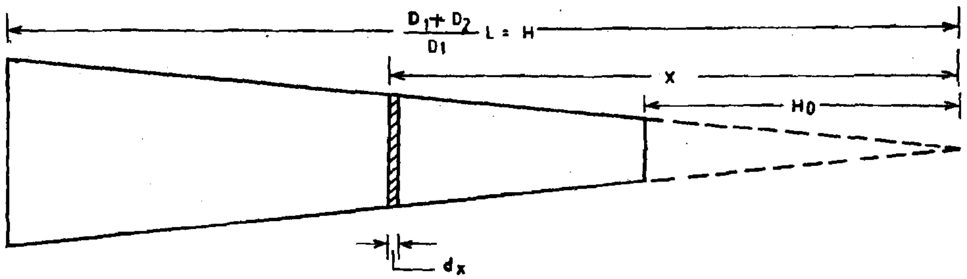


Figure 4.21

However, to simplify the expression for A_i, the origin may be conveniently be shifted to the apex of the produced cone as shown in Figure 4.21. Thus,

$$\delta = \int_{H_0}^{H} \frac{P\,dx}{\frac{\pi}{4}\left(\frac{D_2 x}{H}\right)^2 E}$$

The integral in above equation may be evaluated for solids having stepped as well as continuous variation in cross section. Stepped bars require no special treatment except that the equilibrium and deformations of each segment may be analysed separately and added wherever necessary. Even a prismatic segment of the bar may have to be considered as two or more members, if external loads are applied at interior points of the segment.

SAQ 4

Taking the Youngs Modulus E as 120 kN/mm^2, find the total elongation of the bar shown in Figure 4.22.

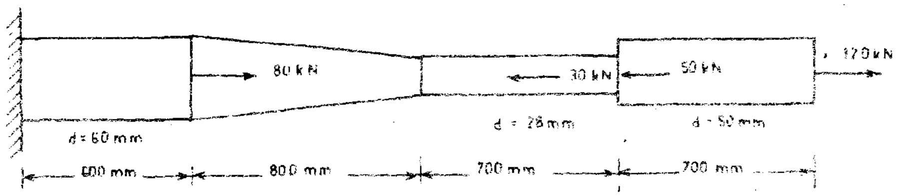


Figure 4.22

4.4.3 Bars of Uniform Strength

If we calculate the stresses induced in the various segments of the bar shown in Figure 4.17 (a), we get the following results :

$$\sigma_1 = \frac{110 \times 1000}{800} = 137.5 \text{ N/mm}^2$$

$$\sigma_2 = \frac{50 \times 1000}{200} = 250 \text{ N/mm}^2$$

$$\sigma_3 = \frac{120 \times 1000}{600} = 200 \text{ N/mm}^2$$

$$\sigma_4 = \frac{30 \times 1000}{150} = 200 \text{ N/mm}^2$$

The results show that the bar should be made of a material with permissible stress not less than 250 N/mm^2. Such a material (should be specially manufactured, if required) is wasted in segments where the stress induced is much less. For instance, in segment 1, the stress induced is only 137.5 N/mm^2. If we reduce the cross sectional area of segment 1, 3 and 4 to 440 mm^2, 480 mm^2 and 120 mm^2 respectively, the stress in all segments will be uniformly 250 N/mm^2, the bar is safe and we would have effected considerable economy of material. Such a bar is called a bar of uniform strength. A designer will strive to achieve a bar of uniform strength, whenever possible. However, due to other constraints and variations in loadings it may not always be possible to provide a bar of uniform strength. Thus, the concept of bar of uniform strength is a design ideal aiming to effect maximum saving in the material.

4.5 COMPOSITE BARS

So far we have considered the cases of solids made of a single material. However, in a number of engineering applications we use components made of two or more materials. Bars made of two or more materials are called **composite bars**. This section will introduce to you the basic principles applied in analysing such bars.

4.5.1 Modular Ratio

Consider the composite bar shown in Figure 4.23 in which an Aluminum bar is encased just snug in a Copper tube which in itself is encased just snug in a steel tube. Let us analyse how the three different components of the composite bar share the load applied axially on the composite bar.

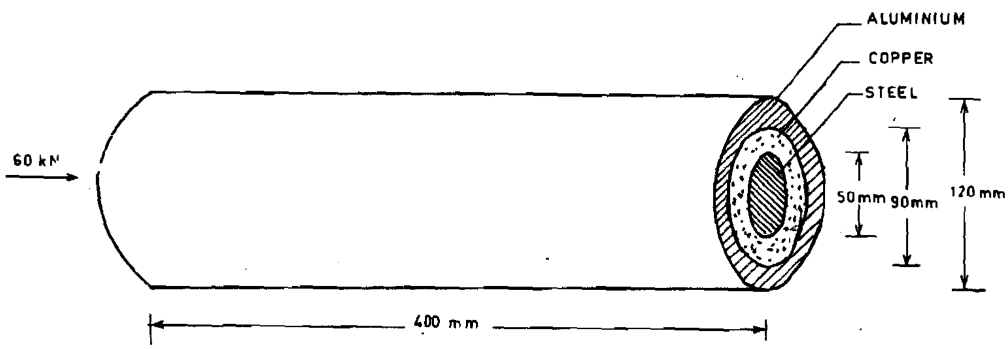


Figure 4.23

The basic requirement in such a case is that all the three components of the bar should undergo the same amount of deformation while carrying the load. Otherwise, the bar which shortens the least alone will be in contact with the load which is physically impossible, since a component which looses its contact will not be carrying any load and hence there can be no shortening of that component which contradicts the assumption that it shortens more. Hence, it is logically necessary that all the three bars should undergo the same amount of deformation. This condition is called the compatibility condition or condition of consistent deformation. Satisfaction of this condition is the basis on which the problem of composite bars is analysed.

Let the total axial force applied on the composite member P be shared by the different members as P_1, P_2 and P_3. If the deformations produced are δ_1, δ_2 and δ_3, then we get,

$$\delta_1 = \frac{P_1 L_1}{A_1 E_1}$$

$$\delta_2 = \frac{P_2 L_2}{A_2 E_2}$$

$$\delta_3 = \frac{P_3 L_3}{A_3 E_3}$$

The compatibility condition requires that

$$\frac{P_1 L_1}{A_1 E_1} = \frac{P_2 L_2}{A_2 E_2} = \frac{P_3 L_3}{A_3 E_3} \quad (4.8)$$

Let the elastic modulii of the materials be redefined as E_1, m_2E_1 and m_3E_1 where,

$$m_2 = \frac{E_2}{E_1} \text{ and } m_3 = \frac{E_3}{E_1}$$

Eq. (4.8) may be rewritten as follows,

$$\frac{L_1}{E_1} \times \frac{P_1}{A_1} = \frac{L_2}{m_2 E_1} \times \frac{P_2}{A_2} = \frac{L_3}{m_3 E_1} \times \frac{P_3}{A_3} \quad (4.9)$$

Recognising that $L_1 = L_2 = L_3$, we may rewrite Eq. (4.9) as

$$\frac{P_1}{A_1} = \frac{P_2}{m_2 A_2} = \frac{P_3}{m_3 A_3} \quad (4.10)$$

from which we get,

$$P_2 = \frac{P_1}{A_1} m_2 A_2 \text{ and}$$

$$P_3 = \frac{P_1}{A_1} m_3 A_3 \quad (4.11)$$

The equilibrium condition required to be satisfied is,

$$P_1 + P_2 + P_3 = P$$

or

$$P_1 + \frac{P_1}{A_1} m_2 A_2 + \frac{P_1}{A_1} m_3 A_3 = P$$

or

$$P_1 \left(1 + \frac{m_2 A_2}{A_1} + \frac{m_3 A_3}{A_1} \right) = P$$

$$P_1 = \frac{P}{\left(1 + \frac{m_2 A_2}{A_1} + \frac{m_3 A_3}{A_1} \right)} \quad (4.12)$$

After calculating P_1 using Eq. (4.12), the loads shared by other members P_2 and P_3 may be calculated using the Eq. (4.11).

In the process of the above solution we have introduced two terms, namely, m_2 and m_3 whose values are the ratios $\frac{E_2}{E_1}$ and $\frac{E_3}{E_1}$ respectively. These ratios are the ratios of Elastic modulii of two different materials and hence, are called **Modular Ratios**.

Let us now solve the problem of load sharing in the composite member shown in Figure 4.23.

Here, we have,

$$P = 60 \text{ kN}$$

$$L = 400 \text{ mm}$$

$$A_1 = \frac{\pi}{4} \times 50^2 = 1963.5 \text{ mm}^2$$

$$A_2 = \frac{\pi}{4} \times (90^2 - 50^2) = 4398.23 \text{ mm}^2$$

$$A_3 = \frac{\pi}{4} (120^2 - 90^2) = 4948 \text{ mm}^2$$

Elastic modulii for the materials, namely Aluminum, Copper and Steel may be taken as 80 kN/mm^2, 120 kN/mm^2 and 200 kN/mm^2 respectively.

$$m_2 = \frac{E_2}{E_1} = \frac{120}{80} = 1.5$$

$$m_3 = \frac{E_3}{E_1} = \frac{200}{80} = 2.5$$

Substituting these values in Eq. (4.12),

$$P_1 = \frac{60}{\left(1 + 1.5 \times \frac{4398.23}{1963.5} + 2.5 \times \frac{4948}{1963.5}\right)}$$

$$= 5.6285 \text{ kN.}$$

Now, using Eq. (4.11),

$$P_2 = \frac{5.62852}{1963.5} \times 1.5 \times 4398.23 = 18.912 \text{ kN}$$

$$P_3 = \frac{5.62852}{1963.5} \times 2.5 \times 4948 = 35.4595 \text{ kN}$$

4.5.2 Equivalent Area of a Composite Section

On reconsideration, we may rewrite Eq. (4.12) as follows :

$$P_1 = \frac{PA_1}{[A_1 + m_2 A_2 + m_3 A_3 + \ldots]}$$

or

$$\frac{P_1}{A_1} = \frac{P}{[A_1 + m_2 A_2 + m_3 A_3 + \ldots]} \qquad (4.13)$$

In Eq. (4.13), we recognise that $\frac{P_1}{A_1}$ is the stress induced in material 1 and hence,

$$\sigma_1 = \frac{P}{[A_1 + m_2 A_2 + m_3 A_3 + \ldots]} \qquad (4.14)$$

The denominator in Eq. (4.14) has Area unit and is called the equivalent area of the composite bar, as if entirely made of material 1 alone. In other words if the composite bar in Figure 4.23 is entirely made of Aluminum, then the equivalent area is given as

$$A_{eq} = A_1 + m_2 A_2 + m_3 A_3$$

$$= 1963.5 + (1.5 \times 4398.23) + (2.5 \times 4948)$$

$$= 20930.86 \text{ mm}^2$$

That is, the composite bar shown in Figure 4.23 is equivalent to an Aluminum bar of cross sectional area 20930.86 mm^2, in resisting axial forces. (An aluminum bar of 163.25 mm diameter will give this area.)

SAQ 5

(a) Show that the composite bar in Figure 4.23 is equivalent to a steel bar of cross sectional area 8372.344 mm^2.

(b) If the composite bar shown in Figure 4.23 is to be replaced by a copper bar, calculate the equivalent area of cross section of the copper bar.

One of the practical examples of composite bars is the R.C.C. column, a typical cross section of which is shown in Figure 4.24. Here, a 400 mm × 400 mm square cross section of a R.C.C column reinforced with 8 numbers of 32 mm diameter mild steel bars. Taking the Elastic modulii of steel and concrete as E_s and E_c and the modular ratio, $m = \left(\frac{E_s}{E_c}\right)$ as 18, let us find the loads shared by steel and concrete when an axial load of 600 kN is applied on the column.

Area of steel bars, $A_s = 8 \times \frac{\pi}{4} \times 32^2 = 6434 \text{ mm}^2$

Area of concrete, $A_c = (400 \times 400 - 6434) = 153566 \text{ mm}^2$

Modular Ratio, $m = 18$

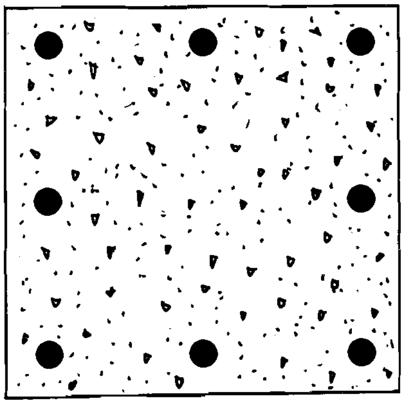


Figure 4.24

Equivalent Area of concrete $= (153566 + 18 \times 6434) = 269378 \text{ mm}^2$

Total axial load = 600 kN.

$$\text{Load shared by concrete} = \frac{600}{269378} \times 153566 = 342.046 \text{ kN}$$

$$\text{Load shared by steel} = \frac{600}{269378} \times 18 \times 6434 = 257.954 \text{ kN}$$

4.5.3 Stresses in Composite Bars and Load Carrying Capacity of Composite Bars

You have learnt how a total load applied on a composite bar is shared by different components of a composite. Though this is certainly an enlightenment on the behaviour of composite bars, we need to learn more in order to apply the concept in practically more useful way. That is, we need to learn

(i) what will be the total load that a composite bar can carry, and

(ii) to carry a given load how should a composite bar be proportioned.

For this purpose, we should know the strength of each material and also we should be able to calculate the stresses induced in the various components of a composite bar.

We have already learnt the compatibility condition that the axial deformations undergone by all the components of a composite bar should be equal.

As the lengths of components are also equal, the strain in each component should also be equal. Taking the strain of the composite bar as ε, stress in any component may be expressed as

$$\sigma_i = E_i \times \varepsilon \qquad (4.15)$$

or

$$\sigma_i = \frac{E_i}{E_1} \times E_1 \times \varepsilon$$

or

$$= m_i \times E_1 \times \varepsilon.$$

Since, $E_1 \times \varepsilon = \sigma_1$, then we can write,

$$\sigma_i = m_i \times \sigma_1 \qquad (4.16)$$

Eq. (4.16) expresses the relationship that the stress induced in any component of a composite bar should be proportional to its elastic modulus or modular ratio. For example, in the R.C.C. column shown in Figure 4.24, the stress in steel σ_s will be 18 times the stress in concrete. You could have seen this by yourself, if you had divided the load shared by each member by its area of cross section. Use of Eq. (4.16) saves some computational effort.

We can get stress in concrete $= \dfrac{342.046 \times 1000}{153566} = 2.227355 \text{ N/mm}^2$.

Thus, stress in steel $= 18 \times 2.227355$

$= 40.0924 \text{ N/mm}^2$

SAQ 6

Evaluate the stress induced in each component of the composite bar shown in Figure 4.23.

In order to calculate the load carrying capacity, we need to know the strength (allowable stress) of all the components. For example, let the allowable stress in concrete be 4 N/mm^2 and the allowable stress in steel be 120 N/mm^2. There is another restriction, you can notice, that even though the strength of steel is 120 N/mm^2, we cannot stress it to the full, because if we stress steel to its full strength, i.e. 120 N/mm^2, then concrete will be stressed to 6.667 N/mm^2, i.e. 120/18, which is not permissible. Hence, the maximum stress that may be induced in steel is only 72 N/mm^2, i.e. 4×18.

Now, we can calculate the maximum load the column can support or the load carrying capacity of the column as follows :

$$P = \sigma_s \times A_s + \sigma_c \times A_c$$

$$= 72 \times 6434 + 4 \times 153566$$

$$= 1077512 \text{ N or } 1077.512 \text{ kN.}$$

Another type of problem that a designer faces is to decide the amount of steel required if the column is to support a known magnitude of load. For instance, if the column is required to carry an axial load of 960 kN, let us calculate the steel requirement.

Maximum stress in steel $= 72$ N/mm^2

Maximum stress in concrete $= 4$ N/mm^2

Let the area of steel be A_s.

Area of concrete $= 160000 - A_s$.

$$\therefore \quad 72 \times A_s + 4\,(160000 - A_s) = 960 \times 1000$$

$$(72 - 4)\, A_s + 640000 = 960000$$

$$\therefore \quad A_s = \frac{(960000 - 640000)}{(72 - 4)} = 4706 \text{ mm}^2$$

This may be provided suitably (say, 8 nos of 28 mm dia bars, which will be a little more than sufficient).

SAQ 7

(a) In the case of composite bar shown in Figure 4.23, if the permissible stresses in Aluminum, Copper and Steel respectively are 50 N/mm^2, 60 N/mm^2 and 120 N/mm^2, calculate the load carrying capacity of the composite bar.

(b) If the column shown in Figure 4.24 is required to carry an axial load of 880 kN, find the requirement of steel, with mechanical properties as given in worked example.

(c) Provide a graphical chart to relate the strength of column shown in Figure 4.24 to the percentage ratio of $\frac{A_s}{A}$ where $A + A_s + A_c$. Take $\sigma_c = 4$ MPa, $\sigma_s = 140$ MPa and $m = 18$.

(d) If the column shown in Figure 4.24 is cast with higher quality of concrete with a permissible stress of 5 N/mm^2 and $E_c = 0.06 E_s$, what is its load carrying capacity and what is the quantity of steel required if the column has to carry an axial load of 1200 kN ?

4.6 TEMPERATURE STRESSES

You have studied in physics about the expansion of solids due to rise in temperature. You may recall that the linear expansion of a solid is proportional to the rise in temperature and the proportionality constant is a material property called Coefficient of Linear Thermal Expansion denoted by α, its units are expressed as m/m/°C. If α is the coefficient of linear thermal expansion of a solid, its linear dimensions will increase by α m per every meter of its original length for every 1°C rise in temperature.

We may symbolically represent this as,

$$\delta = L \times \alpha \times \Delta T \qquad (4.17)$$

where, ΔT is the rise in temperature in °C units.

When the temperature of a solid is raised, expansion or thermal deformation is certainly produced, but it need not always produces stresses.

Thermal stresses are produced in a solid only if the expansion is restrained either fully or partially. This section will introduce to you the methods of analysis of thermal stresses in solids.

4.6.1 Indeterminate Bars

With reference to thermal stresses, bars whose thermal expansion is not restrained in any manner are called **determinate bars** and the bars whose thermal expansion is restrained in any manner are called **indeterminate bars**. As in the case of many of the other indeterminate problems thermal stress analysis of indeterminate bars may be carried out easily by the method of compatibility or consistent deformations.

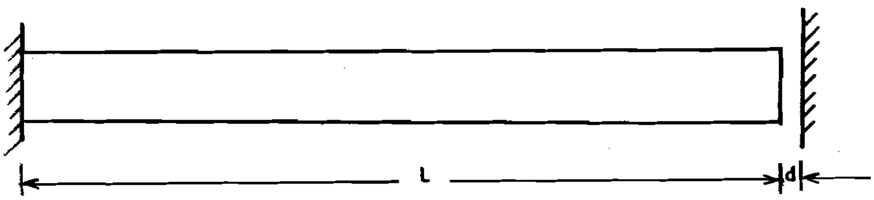


Figure 4.25

Consider the prismatic bar of length 'L' shown in Figure 4.25. The bar is connected to a rigid support at LHS and free at the other end. But at a small distance 'd' from the free end of the bar there is a rigid body which would prevent any expansion of the bar beyond the small gap of width 'd'. If the thermal expansion of the bar is less than or even equal to 'd' no stresses will be induced in the bar and the problem remains determinate. If the bar is heated further, then temperature stresses will be produced. As the bar is heated it will put against the rigid support which will restrain the expansion by exerting a compressive force on the bar, which must produce elastic shortening equal to the excess thermal expansion.

Let us illustrate the principle by a numerical example. Let the length and cross sectional area of the bar be 750 mm and 1200 mm^2 respectively. Let us also assume, $d = 0.12$ mm, $E = 200$ kN/mm^2, $\alpha = 12 \times 10^{-6}$ m/m/°C and $\Delta T = 32$°C. Using the above data, we can readily calculate the free thermal expansion of the bar as

$$\delta_t = L \times \alpha \times \Delta T$$

$$= 750 \times 12 \times 10^{-6} \times 32 = 0.288 \text{ mm}$$

But this free expansion cannot take place due to the presence of the rigid restraint after a gap of 0.12 mm.

Assuming the elastic force in the bar as P, the elastic deformation δ_e may be calculated as

$$\delta_e = \frac{PL}{AE}$$

Anyhow the total deformation δ of the bar should be only equal to d.

Hence, the compatibility condition may be expressed as

$$\delta_t + \delta_e = d \qquad (4.19)$$

$$(L \times \alpha \times \Delta T) + \frac{PL}{AE} = d$$

from which

$$P = -(L \times \alpha \times \Delta T - d)\frac{AE}{L} \tag{4.20}$$

or

$$P = -\left(\alpha \times \Delta T - \frac{d}{L}\right) AE \tag{4.21}$$

Substituting the numerical values, in Eq. (4.20),

$$P = -(0.288 - 0.12)\frac{1200 \times 200}{750}$$

$$= -53.76 \text{ kN.}$$

Now, stress induced $\sigma_t = \dfrac{-53.76 \times 1000}{1200} = 44.8 \text{ N/mm}^2$.

In many cases, the bar may already be kept snug between supports and 'd' is zero so that we may write as follows :

$$P = \alpha \times \Delta T \times AE$$

or thermal stress

$$\sigma_t = E \times \alpha \times \Delta T \tag{4.22}$$

Eq. (4.22) implies that the thermal stress induced is a function of rise in temperature and elastic & thermal properties and not dependent on the geometric properties of the bar.

SAQ 8

(a) If the non-uniform bar shown in Figure 4.16 is placed between two rigid supports with a small gap of 0.2 mm and the bar is heated by 30°C, calculate the maximum thermal stress induced in the bar. Take $\alpha = 12 \times 10^{-6}$ m/m/°C and $E = 200$ GPa.

(b) If the non-uniform bar shown in Figure 4.17 is placed just snug between two rigid restraints, calculate the thermal stresses induced in the bar due to temperature rise of 24°C. α and E may be taken as 12×10^{-6} m/m/°C and 200 GPa.

(c) If the permissible stress in the material of the bar shown in Figure 4.17 is 105 N/mm^2, find out the maximum temperature rise to which the bar may be safely subjected to.

4.6.2 Compound Bars

In the case of thermal stress analysis in compound bars, the following points should be clearly understood and carefully applied :

(i) When the temperature of the compound bar is raised, each of its components tend to expand differently, depending on their α values.

(ii) How much of the thermal expansion in each component is decided based on compatibility conditions.

(iii) When once the deformation to be suppressed in the case of each component is known, the stress induced may be calculated for each bar according to its E value.

Exercise 4.2

Consider the compound bar shown in Figure 4.26 consisting of a steel bolt of diameter 18 mm, surrounded by a copper tube of outer and inner diameters 30 mm and 18 mm respectively. The assembly is just snug at 15°C. The material properties are given as.

Young's modulus of steel, $E_s = 200$ kN/mm^2

Young's modulus of copper, $E_c = 120$ kN/mm^2

Coefficient of linear thermal expansion of steel, $\alpha_s = 12 \times 10^{-6}$ m/m/°C

Coefficient of linear thermal expansion of copper $\alpha_c = 18 \times 10^{-6}$ m/m/°C

Calculate the thermal stresses in steel and copper when the temperature of the assembly is raised to 45°C.

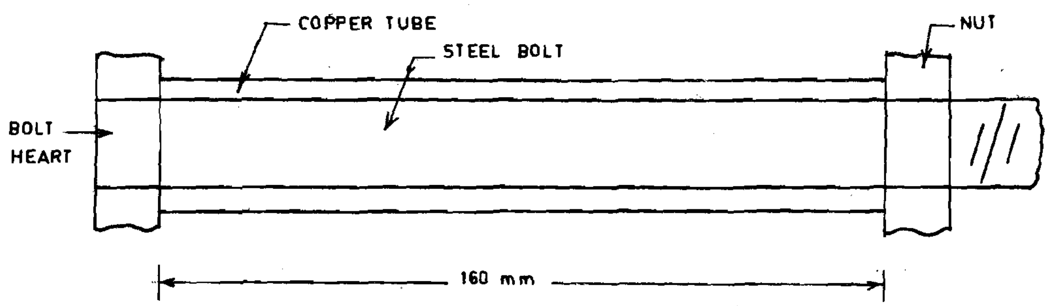


Figure 4.26

Solution

In this case, when the temperature of the assembly is raised, copper will expand more than steel and hence, will exert a thrust on the steel nut and bolt head thus producing tension in the steel. Since there is no other external force this tension has to be kept in equilibrium by the compressive force induced in copper. Hence by formulating the compatibility condition of total strains (thermal + elastic) the problem can be solved.

Let the forces induced in steel and copper be designated as P_s and P_c respectively in kN units.

$$\text{Thermal strain in steel, } \varepsilon_{st} = \alpha_s \times \Delta T$$
$$= 12 \times 10^{-6} \times (40 - 15)$$
$$= 3 \times 10^{-4} \text{ m/m.}$$

$$\text{Thermal strain in copper } = \alpha_c \times \Delta T$$
$$= 18 \times 10^{-6} \times (40 - 15)$$
$$= 4.5 \times 10^{-4} \text{ m/m.}$$

$$\text{Elastic strain in steel, } \delta_{se} = \frac{P_s}{A_s E_s}$$
$$= \frac{P_s}{\frac{\pi}{4} \times 18^2 \times 200}$$
$$= 1.965 \times 10^{-5} P_s.$$

$$\text{Elastic strain in copper } \delta_{ce} = \frac{P_c}{A_c E_c}$$
$$= \frac{P_c}{\frac{\pi}{4} \times (30^2 - 18^2)\, 120}$$
$$= 1.8421 \times 10^{-5} P_c$$

Compatibility Condition

Total strain in steel = Total strain in copper

i.e. $\quad 3 \times 10^{-4} + 1.965 \times 10^{-5} P_s = 4.5 \times 10^{-4} + 1.8421 \times 10^{-5} P_c.$

Recognising that $P_c = -P_s$, and dividing both sides by 10^{-5}, we get

$$30 + 1.965Ps = 45 - 1.8421\, P_s$$

or $$P_s = \frac{45-30}{(1.965+1.8421)} = 3.94 \text{ kN}.$$

$$\therefore \text{ Stress in steel } = \frac{3.94 \times 1000}{\frac{\pi}{4} \times 18^2} = 15.5 \text{ N/mm}^2$$

$$\therefore \text{ Stress in copper } = \frac{-3.94 \times 1000}{\frac{\pi}{4}(30^2 - 18^2)} = -8.71 \text{ N/mm}^2$$

SAQ 9

(a) The compound bar shown in Figure 4.23 is placed between two rigid supports with a small gap of 0.1 mm and its temperature is raised by 24°C. Calculate the thermal stresses induced in Aluminum, Copper and Steel if

$$\alpha_a = 23 \times 10^{-6} \text{ m/m/°C},$$
$$\alpha_c = 18 \times 10^{-6} \text{ m/m/°C, and}$$
$$\alpha_s = 12 \times 10^{-6} \text{ m/m/°C}.$$

(b) If the bolt in the assembly showing in Figure 4.26 is loosened by 0.02 mm, calculate the rise in temperature required to induce a stress of 12 N/mm^2 in steel bolt and also calculate the corresponding stress in the copper tube.

(c) For an assembly similar to the one shown in Figure 4.26, if thermal stresses in steel and copper are to be equal what should be the outer diameter of the copper tube.

Note

Rise in temperature will cause thermal stresses only if α of tube material is greater than α of bolt material. It α of tube material is smaller, the assembly will become loose due to rise in temperature, while thermal stresses will be induced due to fall in temperature.

4.6.3 Yielding of Supports

In our earlier analysis of thermal stresses we have assumed the bars to be placed between rigid supports. As we have already realised that there are no rigid solids, there are also no rigid supports. The supports may be much stiffer than the bars, but they too undergo deformation, to a smaller extent, and in many cases these deformations may be too small to be considered.

Consider the bar shown in Figure 4.25. Without changing any other data, let us assume the **stiffness of the support,** k_s to be 80 kN/mm (i.e. the support will yield by 1 mm for a thrust of 80 kN applied on it).

If P is the force developed then Eq. (4.19) may be rewritten as follows :

$$\delta_t + \delta_e = d + \frac{P}{k_s} \tag{4.22}$$

(Equilibrium requires that P is same in the support and bar.)

Substituting numerical data in Eq. (4.22),

$$0.288 - \frac{P \times 750}{1200 \times 200} = 0.12 + \frac{P}{80}$$

$$P\,(0.0125 + 0.003125) = 0.288 - 0.12$$

$$P = \frac{0.168}{0.015625} = 10.752 \text{ kN}$$

Note that in the case of steel bolt and copper tube assembly (Figure 4.26), the steel bolt is serving as an yielding support to the copper tube.

The support has yielded considerably so that most of the stress in the bar is relieved.

SAQ 10

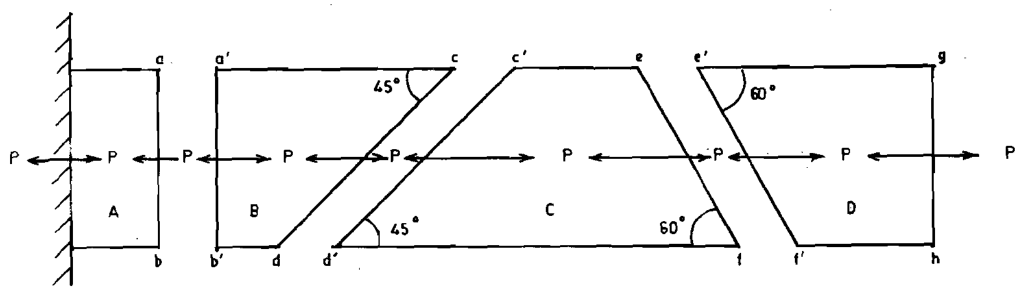
(a) Solve the above example with values of k_s equal to 200 kN/mm, 500 kN/mm, 1000 kN/mm and 5000 kN/mm. Draw a graph, P vs k_s and find the values of k_s for which percentage relief of thrust in the bar are 40 and 25 respectively.

(b) Repeat the above exercise with $d = 0$ and all other data remaining the same.

(c) If the compound bar described in Figure 4.23 is placed just snug between two supports of stiffness 1200 kN/mm and heated up by 20°C, calculate the thermal stresses induced.

4.7 STRESSES ON OBLIQUE SECTIONS

We have already seen that the equilibrium of a rigid body should be satisfied overall and in addition if we divide it into a number of small rigid bodies each of these small elemental bodies should also be in equilibrium individually. The satisfaction of equilibrium does not depend on the way elements are divided. Even if the solid is divided by inclined or even curved surfaces, equilibrium must be satisfied for each of the elements thus divided.



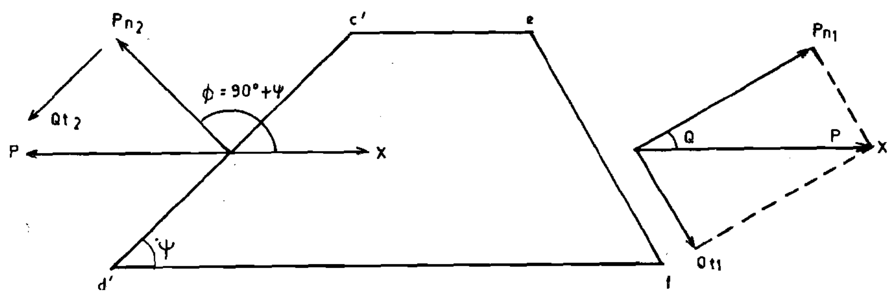


Figure 4.27

Consider the solid shown in Figure 4.27 where the solid is divided into small elements by inclined planes. The inclination (or orientation) of a plane is defined by its **aspect angle,** defined as the angle made by its normal to the longitudinal axis of the original bar. Let the aspect angle of the plane be θ. Since the width of the plane b is unaltered and length of the plane h is increased to $\frac{h}{\cos\theta}$, the area of the inclined plane is $\frac{A}{\cos\theta}$, where A is area of cross section of the original solid.

If σ_x is the stress acting on the plane normal to x axis (longitudinal axis), then the axial force $P = \sigma_x \times A$. This force acting on the inclined plane may be resolved into normal and tangential components.

$$\text{Normal component on the plane} = P\cos\theta$$

$$= \sigma_x \times A\cos\theta$$

$$\text{Tangential component} = -P\sin\theta$$

$$= -\sigma_x \times A\sin\theta$$

$$\text{Normal stress on the plane} = \frac{\sigma_x \times A \cos\theta}{\frac{A}{\cos\theta}} = \sigma_x \times \cos^2\theta \qquad (4.23)$$

$$\text{Shear stress on the plane} = \frac{-\sigma_x \times A \sin\theta}{\frac{A}{\cos\theta}} = \sigma_x \times \cos\theta \sin\theta \qquad (4.24)$$

If on the cross sectional area of the original solid shear stress τ_{xy} is applied, its components on the inclined plane may be evaluated as,

$$\text{Normal stress component} = \tau_{xy} \cos\theta \sin\theta \qquad (4.25)$$

$$\text{Shear stress component} = \tau_{xy} \cos^2\theta \qquad (4.26)$$

By a similar analysis (your exercise), you may verify the following :

If a normal stress of σ_y is applied on the solid, then the stress components on a plane whose normal is inclined at θ to the x axis are given by

$$\text{Normal stress} = \sigma_y \sin^2\theta \qquad (4.27)$$

$$\text{Shear stress} = \sigma_y \sin\theta \cos\theta . \qquad (4.28)$$

If a shear stress of τ_{yx} is applied on the solid the stress components on the inclined plane are given by (with $\tau_{yx} = -\tau_{xy}$).

$$\text{Normal stress} = \tau_{xy} \sin\theta \cos\theta \qquad (4.29)$$

$$\text{Shear stress} = \tau_{xy} \sin^2\theta \qquad (4.30)$$

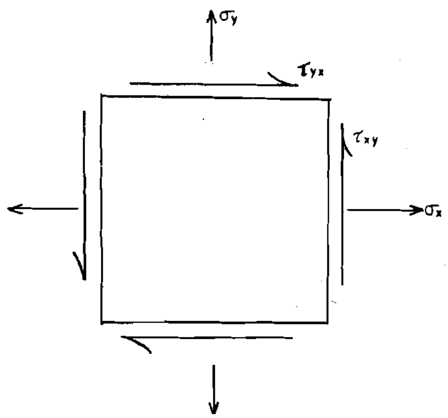


Figure 4.28 : General State of Stress in Two Dimensions

Since a general state of stress in two dimensions is defined by the stress components σ_x , σ_y and τ_{xy} as shown in Figure 4.28, general expressions for normal and shear stress components may be obtained by algebraic sum of the respective components from Eqs. (4.23) to (4.30), as given below :

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\,\tau_{xy} \cos\theta \sin\theta$$

$$\tau_{nt} = -\sigma_x \cos\theta \sin\theta + \sigma_y \cos\theta \sin\theta + \tau_{xy} (\cos^2\theta - \sin^2\theta).$$

On further simplification, we obtain,

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \qquad (4.31)$$

$$\tau_{nt} = \left(\frac{\sigma_y - \sigma_x}{2}\right) \sin 2\theta + \tau_{xy} \cos 2\theta \qquad (4.32)$$

Application of Eqs. (4.31) and (4.32) will be elaborately dealt with in Unit 5. However, the significance of the equations should be stressed at this stage. When we are carrying out stress analysis on solids, we may be evaluating stress components on a set of mutually perpendicular planes, such as σ_x, σ_y and τ_{xy}. The magnitudes of these components may not always be sufficient to decide whether the solid is safe or not, and we require evaluation of stresses components on other specific planes, or determination of planes on which the stress components have extreme value. Such an analysis can be easily carried out with the help of Eqs. (4.31) and (4.32).

4.8 RELATIONSHIP BETWEEN ELASTIC CONSTANTS

In Section 4.3.3, we have defined the four Elastic Constants E, ν, K and G and also stated, that they are not independent. Now we shall establish the relationship between them.

4.8.1 Relationship Between E and K

Figure 4.13 shows the spherical state of stress in which the normal stress component is the same, σ_0 in any direction and shear stress components are zero on any plane. Such a state of stress is also called as volumetric stress. If the Young's Modulus and Poisson's Ratio of the solid are known as E and ν, the strain components are given by

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} - \nu \frac{\sigma_z}{E} \qquad (4.33)$$

Since, $\sigma_x = \sigma_y = \sigma_z = \sigma_0$

$$\varepsilon_x = \frac{\sigma_0}{E}(1 - 2\nu) \qquad (4.34)$$

Similarly, $\varepsilon_y = \frac{\sigma_0}{E}(1 - 2\nu)$ and $\varepsilon_z = \frac{\sigma_0}{E}(1 - 2\nu)$

From Eq. (4.7), we can express the change in volume of the solid as

$$dV = V(\varepsilon_x + \varepsilon_y + \varepsilon_z)$$

Volumetric strain $\varepsilon_v = \frac{dV}{V} = \varepsilon_x + \varepsilon_y + \varepsilon_z$

Bulk modulus, $K = \frac{\text{Volumetric Stress}}{\text{Volumetric Strain}}$

$$= \frac{\sigma_0}{\varepsilon_x + \varepsilon_y + \varepsilon_z} = \frac{\sigma_0}{3\,\varepsilon_x}$$

$$= \frac{\sigma_0}{\frac{\sigma_0}{E} \times 3\,(1 - 2\nu)}$$

$$K = \frac{E}{3\,(1 - 2\nu)} \qquad (4.35)$$

4.8.2 Relationship Between E and G

Figure 4.28 shows what is known as the state of pure shear. The deformation of the body due to the shear stress is shown in Figure 4.29 (projected view).

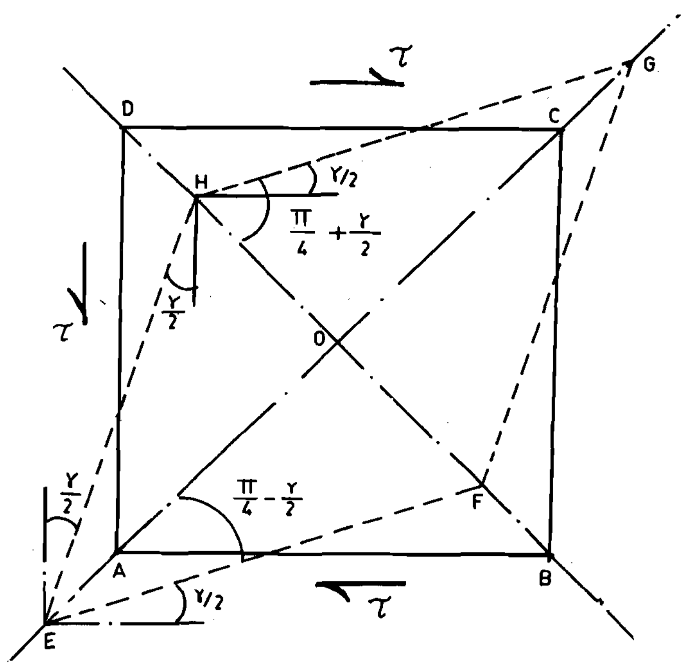


Figure 4.29

The state of stress on the element is given by the stress component S, $\tau_{xy} = -\tau_{yx} = \tau$. Due to shearing strain the original plane $ABCD$ (a square) gets distorted into a rhombns EFGH. The magnitude of the shear strain γ is given by change in the right angles $\angle BAB$, $\angle ABC$, $\angle BCD$ and $\angle CDA$. This change is produced by rotations of pairs of lines by $\gamma/2$ such as $\angle DEH$, $\angle FEB$ etc.

Using Eqs. (4.31) and (4.32), let us find the stress components on the diametral planes DB and AC whose aspect angles are $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ respectively. On plane DB, we get,

$$\text{Normal stress component } \sigma = \tau \sin 2\theta$$

$$= \tau \sin \frac{2\pi}{4} = -\tau$$

$$\text{Shear stress component} = \tau \cos 2\theta$$

$$= \tau \cos \frac{2\pi}{4} = 0$$

Similarly, on plane AC,

$$\text{Normal stress component} = \tau \sin 2\theta$$

$$= \tau \sin 2 \times \frac{3}{4}\pi = -\tau$$

$$\text{Shear stress companent} = \tau \cos 2\theta$$

$$= \tau \cos 2 \times \frac{3}{4}\pi = 0$$

Hence, the distortion of the square $ABCD$ by shear stress may also be treated as due to the elongation of the diagonal AC due to normal stress of $+\tau$ on planes parallel to BD and shortening of the diagonal due to normal stress of $-\tau$ on planes parallel to AC.

$$\varepsilon_{OA} = \frac{\tau}{E} - \nu\left(-\frac{\tau}{E}\right) = \tau\frac{(1-\nu)}{E} \tag{4.36}$$

$$\varepsilon_{OB} = -\frac{\tau}{E} - \nu\left(\frac{\tau}{E}\right) = -\tau\frac{(1-\nu)}{E} \tag{4.37}$$

$$OE = OB\,(1 - \varepsilon_{OB})$$

$$OF = OB\,(1 + \varepsilon_{OB})$$

$$\phi = OEF = \frac{\pi}{4} - \frac{\gamma}{2}$$

$$\tan\phi = \tan\left(\frac{\pi}{4} - \frac{\gamma}{2}\right) = \frac{1-\gamma/2}{1+\gamma/2} \tag{4.38}$$

From Figure 4.29,

$$\tan\phi = \frac{OF}{OE} = \frac{OB\,(1-\varepsilon_{OB})}{OA\,(1+\varepsilon_{OB})}$$

$$\tan\phi = \frac{1 + \frac{\tau(1-\nu)}{E}}{1 - \frac{\tau(1+\nu)}{E}} \tag{4.39}$$

On comparing Eqs. (4.38) and (4.39),

$$\frac{\gamma}{2} = \frac{\tau}{E}(1-\nu)$$

$$\text{or} \quad \text{shear strain} \quad \gamma = \frac{2\tau(1-\nu)}{E}$$

$$\text{Rigidity Modulus} \quad G = \frac{\text{Shear Stress}}{\text{Shear Strain}} = \frac{\tau}{\frac{2\tau}{E}(1-\nu)}$$

$$G = \frac{E}{2(1-\nu)} \tag{4.40}$$

4.8.3 Significance of the Relationships

We have already mentioned that there are only two independent elastic constants. But this is true only in the case of isotropic solids. The term isotropy means same property in all directions. We have defined Elastic Modulus as ratio between stress and strain and implied (without stating) that the ratio holds good for all directions, i.e. $\frac{\sigma_x}{\varepsilon_x} = \frac{\sigma_y}{\varepsilon_y} = \frac{\sigma_z}{\varepsilon_z}$. Materials for which this is true are called isotropic materials and throughout this course you will be learning only about isotropic solids.

When any two of the elastic constants are known, the other two may be calculated using Eqs. (4.35) and (4.40). Let us have an example.

Example 4.3

In separate experiments, Young's Modulus and Rigidity Modulus of a material have been determined as 120 GPa and 50 GPa respectively. Calculate the Poisson's Ratio and Bulk Modulus of the material.

Solution

Here, Young's Modulus $E = 120$ GPa.

Let Poisson's ratio be ν.

$$\text{Rigidity Modulus, } G = \frac{E}{2(1+\nu)} = 50 \text{ GPa}$$

$$(1+\nu) = \frac{E}{2G} = \frac{120}{2 \times 50} = 1.2$$

$$\therefore \text{ Poisson's Ratio, } \nu = 1.2 - 1 = 0.2$$

$$\text{Bulk Modulus, } \quad K = \frac{E}{3(1-2\nu)}$$

$$= \frac{120}{3\,(1 - 2 \times 0.2)}$$

$$= 66.667 \text{ GPa}$$

SAQ 11

(a) Through separate experiments the elastic constants [illegible] G and K were determined as 75 GPa, 0.247, 30 GPa and [illegible] respectively. Taking the values of E and G as correct, find the [illegible] errors in the values of Poisson's Ratio and Bulk Modulus.

(b) The values of Young's Modulus and Rigidity Modulus of a material are known to be 20.8 GPa and 8 GPa respectively. If a spherical ball [illegible] 150 mm made of the material is immersed in water to a depth of [illegible], find the change in volume of the ball.

(c) By eliminating Poisson's ratio ν from Eqs. (4.35) and (4.40), show that

$$G = \frac{3KE}{9K - E}$$

4.9 SUMMARY

In this unit you have learnt the concepts of stress, strain, elastic modulus and the basic concepts in application to analysis of stresses and deformation in simple solids. Any designer of engineering systems has to have a clear perception of how forces are resisted by solids and the effects produced in solids so as to enable him to produce designs with safety and stability. The study of these aspects will require learning further new concepts and methods, understanding of which will require the knowledge you acquired in this unit. This unit is therefore like the first step in a full flight of stairs climbing which will provide you with a powerful tool of engineering.

4.10 ANSWERS TO SAQs

SAQ 1

(a) Change in volume = – 1800 mm^3.

(b) Young's Modulus E = 12.7324 kN/mm^2, and

Poisson's Ratio, ν = 0.16667.

SAQ 2

dV_1 = 1462.5 mm^3, dV_2 = 1950 mm^3, dV_3 = 975mm^3, dV_4 = 2437.5 mm^3.

Total change in volume dV = 6825 mm^3.

SAQ 3

(a) For bar given in Figure 4.18, δ = 24.92 mm.

(b) For bar given in Figure 4.19, δ = 0.0268 mm.

SAQ 5

(b) Equivalent area of Copper bar = 13953.91 mm^2.

SAQ 6

Stress in Steel = 7.16645 N/mm^2
Stress in Copper = 4.3 N/mm^2
Stress in Aluminum = 2.8666 N/mm^2

SAQ 7

(a) Load carrying capacity of bar = 837.23 kN.

(b) Area of reinforcement steel required = 3529.4 mm^2.

(c) The chart will be straight line with the equation

$$P = 640\left(1 + 17\frac{A_s}{A}\right) \text{ kN}$$

(d) Load carrying capacity of given column = 1305 kN.
Area of steel required, = 5106.4 mm^2 if P = 1200 kN.

SAQ 8

(a) σ_{max} = 125.1 N/mm^2 in the 60 mm dia segment.

(b) σ_1 = 20.644 N/mm^2, σ_2 = 82.577 N/mm^2.
σ_3 = 27.526 N/mm^2, and σ_4 = 110.103 N/mm^2.

(c) ΔT_{max} = 22.88766 °C.

SAQ 9

(a) Stress in Aluminum = 24.16 N/mm^2
Stress in Copper = 21.84 N/mm^2
Stress in Steel = 7.6 N/mm^2

(b) Required rise in temperature, ΔT = 43.333 °C, and
Corresponding stress in copper tube = 9 N/mm^2.

(c) Outer diameter of copper tube = 25.456 mm.

SAQ 11

(a) Error in Poisson's Ratio = 1.2%
Error in Bulk Modulus = 1.6 %

(b) Change in volume of Spherical ball = 122341 mm^3 or 6.823 %.