COURSE ORGANISATION

- Week: Video Lectures, Live Lectorials, Live Tutorials
- Assessments:
  - Oral Presentation – 5%
  - Assignment – 15%
  - Storywall Discussion Questions (Weeks 2, 4, 7, 8, 10) – 6%
  - Final Exam – 74%

M0: Survival Models & the Life Table

- Lifetime: time to the occurrence of a certain event, e.g.
  - Human mortality (time to death)
  - Length of time that a surviving individual will hold an insurance policy
- Survival models are models that describe the probability distribution of such "lifetimes".

A SIMPLE MODEL OF SURVIVAL | CONTINUOUS

- Future lifetime T of a newborn:
  - T is a continuous random variable on the interval $[0, \omega]$, where ω is the limiting age (highest age recorded 122 years and 164 days - Jeanne Calment - see, e.g. this link).
  - The distribution function of T is $F(t) = \Pr[T \leq t]$.
  - The survival function of T is $S(t) = \Pr[T > t] = 1 - F(t)$.
  - The probability density function of T is $f(t) = \frac{dF(t)}{dt}$.
- Future lifetime T_x after age x:
  - This is for a life who has already survived to age x.
  - T_x is a continuous random variable taking values in $[0, \omega - x]$
  - The distribution function of T_x is $F_x(t) = \Pr[T_x \leq t]$.
  - The survival function of T_x is $S_x(t) = \Pr[T_x > t] = 1 - F_x(t)$.

The probability density function of T_x is $f_x(t) = \frac{dF_x(t)}{dt}$.

$$F_x(t) = \Pr[T_x \leq t] = \Pr[T \leq x + t | T > x] = \frac{S(x) - S(x+t)}{S(x)}$$

$$S_x(t) = \Pr[T_x > t] = \Pr[T > x + t | T > x] = \frac{S(x+t)}{S(x)}$$

ACTUARIAL NOTATION

- Probability of death:

$$ {}_tq_x = \Pr[T_x \le t] = F_x(t) = \int_0^t f_x(s)\mathrm{d}s $$

- Probability of survival:

$$ {}_tp_x = \Pr[T_x > t] = 1 - {}_tq_x = S_x(t) $$

- Deferred probability:

$$ {}_{n|m}q_x = Pr[n < T_x < n+m] $$

- Customary notation:

$$ q_x = {}_1q_x,\ p_x = {}_1p_x,\ {}_{|m}q_x = {}_{1|m}q_x $$

- Note:

$$ {}_{s+t}p_x = ({}_tp_x)(\ {}_sp_{x+t}) $$

$$ {}_tp_x = p_x p_{x+1} \cdots p_{x+t-1} $$

FORCE OF MORTALITY

The force of mortality at age x $(0 \le x < \omega)$

$$ \mu_x = \lim_{h \to 0^+} \frac{1}{h} \times \Pr[T \le x+h | T > x] = \frac{\frac{\mathrm{d}F(x)}{\mathrm{d}x}}{S(x)} $$

is an instantaneous measure of mortality at age x. It is known as hazard rate in statistics. Looking forward,

$$ \begin{aligned} \mu_{x+t} &= \lim_{h \to 0^+} \frac{1}{h} \times \Pr[T \le x+t+h | T > x+t] \\ &= \lim_{h \to 0^+} \frac{1}{h} \times \Pr[T_x \le t+h | T_x > t]. \end{aligned} $$

- Cumulative Hazard Function:

$$ H(x) = \int_0^x \mu_t \mathrm{d}t = -\log S(x), $$

so that

$$ S(x) = \exp\left\{-\int_0^x \mu_t \mathrm{d}t\right\} $$

and

$$ {}_tp_x = \exp\left\{-\int_x^{x+t} \mu_s \mathrm{d}s\right\}. $$

- Important result:

$f_x(t) = {}_tp_x \mu_{x+t}$

- Proof:

$$
\begin{aligned}
f_x(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\Pr[T_x \le t] \\
&= \lim_{h\to 0^+} \frac{1}{h} \times (\Pr[T_x \le t+h] - \Pr[T_x \le t]) \\
&= \lim_{h\to 0^+} \frac{1}{h} \times (\Pr[T \le x+t+h | T > x] - \Pr[T \le x+t | T > x]) \\
&= \lim_{h\to 0^+} \frac{\frac{\Pr[T \le x+t+h]-\Pr[T\le x]}{S(x)} - \frac{\Pr[T\le x+t]-\Pr[T\le x]}{S(x)}}{h} \\
&= \lim_{h\to 0^+} \frac{\Pr[T \le x+t+h] - \Pr[T \le x+t]}{S(x)h} \\
&= \frac{S(x+t)}{S(x)} \times \lim_{h\to 0^+} \frac{1}{h}\frac{\Pr[T \le x+t+h] - \Pr[T \le x+t]}{S(x+t)} \\
&= S_x(t)\mu_{x+t} \\
&= {}_tp_x\mu_{x+t}
\end{aligned}
$$

EXPECTATION OF LIFE

- The complete expectation of life at age x is defined by:

$$
\begin{aligned}
\mathring{e}_x &= \mathrm{E}[T_x] \\
&= \int_0^{\omega-x} t f_x(t)\mathrm{d}t \\
&= \int_0^{\omega-x} t(-\frac{\partial}{\partial t}{}_tp_x)\mathrm{d}t \\
&= -[t\,{}_tp_x]_{t=0}^{\omega-x} + \int_0^{\omega-x} {}_tp_x\mathrm{d}t \\
&= \int_0^{\omega-x} {}_tp_x\mathrm{d}t
\end{aligned}
$$

- The curtate future lifetime of a life age x is defined by:

$$K_x = [T_x] \text{ (the integer part of } T_x)$$

  - K_x is a discrete random variable taking integer values $0, 1, \cdots, [\omega - x]$.
  - $\Pr[K_x = k] = \Pr[k \le T_x < k+1] = {}_kp_xq_{x+k}$

- The curtate expectation of life is defined by:

$$
\begin{aligned}
e_x &= \mathrm{E}[K_x] = \sum_{k=0}^{[\omega-x]} k \, {}_k p_x q_{x+k} \\
&= {}_1p_x q_{x+1} \\
&+ {}_2p_x q_{x+2} \qquad + {}_2p_x q_{x+2} \\
&+ {}_3p_x q_{x+3} \qquad + {}_3p_x q_{x+3} \qquad + {}_3p_x q_{x+3} \\
&+ \cdots \\
&+ {}_{[\omega-x]}p_x q_{x+[\omega-x]} + {}_{[\omega-x]}p_x q_{x+[\omega-x]} + \cdots + {}_{[\omega-x]}p_x q_{x+[\omega-x]} \\
&= \sum_{k=1}^{[\omega-x]} \sum_{j=k}^{[\omega-x]} {}_jp_x q_{x+j} = \sum_{k=1}^{[\omega-x]} \sum_{j=k}^{[\omega-x]} {}_{j|}q_x = \sum_{k=1}^{[\omega-x]} {}_kp_x,
\end{aligned}
$$

THE LIFE TABLE

- Radix of the table l_0: a hypothetical cohort of new-born lives
- $l_x = l_0 \times S(x)$: the expected number of survivors to age x out of the original group.
- Notation:
  - ${}_np_x = \frac{l_{x+n}}{l_x}$ conditional probability of surviving to age $x+n$, given alive at age x
  - $d_x = l_x - l_{x+1}$ the expected number who die between ages x and $x+1$
  - ${}_nd_x = l_x - l_{x+n}$ the expected number who die between ages x and $x+n$
  - ${}_nq_x = \frac{{}_nd_x}{l_x}$: conditional probability of dying within n years, given alive at age x
  - $\mu_x = \frac{-\frac{\mathrm{d}}{\mathrm{d}x}S(x)}{S(x)} = \frac{-\frac{\mathrm{d}}{\mathrm{d}x}l_x}{l_x}$
- Deferred quantities:

  Generally

$$
\begin{aligned}
{}_{n|m}q_x &= Pr[n < T_x < n+m] \\
&= \frac{l_{x+n} - l_{x+n+m}}{l_x} \\
&= \frac{l_{x+n}}{l_x} \frac{l_{x+n} - l_{x+n+m}}{l_{x+n}} \\
&= {}_np_x \times {}_mq_{x+n}
\end{aligned}
$$

so that

$$
{}_{n|}q_x = {}_{n|1}q_x = {}_np_x \times q_{x+n}
$$

and

$$
{}_{n|}q_x = Pr[K_x = n] = Pr[n \leq T_x < n+1] = \frac{d_{x+n}}{l_x}
$$

INITIAL AND CENTRAL RATES OF MORTALITY

- Initial rate of mortality:
  - Probability that a life alive at age *x* (the initial time) dies before age *x+1*
  - Denoted $q_x = \frac{d_x}{l_x}$
- Central rate of mortality:
  - Probability that the population dies before age *x+1*
  - Demoted by m_x
  - Useful in population projections

$$
\begin{aligned}
m_x &= \frac{d_x}{\int_x^{x+1} l_z \mathrm{d}z} \\
&= \frac{q_x}{\int_0^1 {}_tp_x \mathrm{d}t}, \\
&= \frac{Pr[T_x \leq 1]}{\int_0^1 {}_tp_x \mathrm{d}t} \\
&= \frac{\int_0^1 {}_tp_x \mu_{x+t} \mathrm{d}t}{\int_0^1 {}_tp_x \mathrm{d}t}
\end{aligned}
$$

ASSUMPTIONS FOR FRACTIONS OF A YEAR

- All these functions ${}_sp_x$ can be determined from a lifetable for integral *x* and *s*, but not for non-integral *s*.
- The determination of functions for non-integral ages requires that values of l_{x+s} be available for all s, $0 \leq s \leq 1$, and integral *x*.
  - To this end, we will assume that l_{x+s} ($0 \leq s \leq 1$, x is any integer) has one of the following mathematical form:
    - Linear form for l_{x+s} (Uniform distribution of deaths)
    - Exponential form for l_{x+s} (Constant force of mortality)
    - Hyperbolic form for l_{x+s} (Balducci assumption/distribution)
  - We also assume that l_{x+s} is differentiable on the open interval $0 < s < 1$, which allows us to evaluate μ_{x+s} and hence the conditional density function $f(s|X > x) = {}_sp_x \mu_{x+s}$.
- Uniform distribution of deaths (UDD): Linear form

  We have

$$
\begin{aligned}
l_{x+s} &= l_x - s \cdot d_x = s \cdot l_{x+1} + (1-s) \cdot l_x \\
& \quad \text{(linear interpolation)} \\
{}_sp_x &= \frac{l_{x+s}}{l_x} = \frac{l_x - s \cdot d_x}{l_x} = 1 - s \cdot q_x \\
f(s|X > x) &= q_x \\
{}_sq_x &= 1 - {}_sp_x = s \cdot q_x \\
\mathring{e}_x &= e_x + \frac{1}{2}
\end{aligned}
$$

$$
{}_tp_{x+s} = \frac{l_{x+s+t}}{l_{x+s}} = \frac{l_x - (s+t) \cdot d_x}{l_x - s \cdot d_x} \text{ for } 0 \leq s, t \leq 1 \text{ and } s + t \leq 1
$$

- Constant force of mortality: Exponential form

  For any integer x and $0 \leq s \leq 1$, assume that

  - l_{x+s} is a continuous exponential function with respect to s on $s \in [0,1]$, i.e. of form $a \cdot b^s$
  - in other words we assume that the log of l_x is linear:

  $$\ln l_{x+s} = \ln a + \ln b \cdot s.$$

  We have then

  $$\begin{aligned} \log l_{x+s} &= (1-s)\log l_x + s \log l_{x+1} \\ \mu_{x+s} &= \mu_x \text{ for } 0 < s < 1 \\ {}_s p_x &= e^{-s\mu_x} \\ {}_{t-s} p_{x+s} &= e^{-(t-s)\mu_x} \end{aligned}$$

- Balducci Assumption: Hyperbolic form

  For any integer x and $s \in [0,1]$, assume that

  - l_{x+s} is a hyperbolic function with respect to s on $s \in [0,1]$, i.e. of form $(a+bs)^{-1}$
  - alternatively, $\frac{1}{l_{x+s}} = (1-s) \cdot \frac{1}{l_x} + s \cdot \frac{1}{l_{x+1}}$ for $0 \leq s \leq 1$ (harmonic interpolation: linear interpolation on the reciprocal of the function)

  Then for any integer x and $s \in [0,1]$,

  - a convenient relationship: ${}_{1-s}q_{x+s} = (1-s)q_x$, $0 \leq s \leq 1$

- Force of mortality under constant force of mortality:

  $$\mu_{x+s} = -\ln p_x, \quad 0 < s < 1$$

- Force of mortality under UDD:

  $$\mu_{x+s} = \frac{q_x}{1 - sq_x}, \quad 0 < s < 1$$

- Force of mortality under Balducci:

  $$\mu_{x+s} = \frac{q_x}{1-(1-s)q_x}, \quad 0 < s < 1$$

SELECT LIFE TABLES

- Assume now mortality also depends on the date of entry in a group
  - $[x]+t$ still means age is $(x+t)$, but
    - the age at date of joining population is x; and
    - the duration from the date of joining the population is t
  - $l_{[x]+t}$: expected number of lives alive at duration t having joined the population at age $[x]$ based on some assumed radix
  - s: the length of the select period
  - After the select period the lives experience ultimate mortality, i.e. $l_{[x]+t} = l_{x+t}$ for $t \geq s$.

Example A 2-year select-and-ultimate mortality table:

$[x]$	$l_{[x]}$	$l_{[x]+1}$	l_{x+2}	$x+2$
30	9907	9905	9901	32
31	9903	9901	9897	33
32	9899	9896	9893	34

M1: Life Insurance Benefits

LIFE INSURANCE CONTRACTS

- Contract between a life insurance company and one or more person's called **policyholders**
- Policyholders agree to pay an amount or series of amounts to the life insurance company called **premiums**
- In return the life insurance company agrees to pay an amount or amounts, called benefits, to the policyholder on the occurrence of a specific event
- Benefits are payable under two main types:
  - **Insurance**: When the benefit is paid as a single lump sum, either on the death or survival of the policyholder to a predetermined maturity date.
  - **Annuity**: A benefit paid in the form of a regular series of payments, usually condition of the survival of the policyholder.
- *Whole life insurance contract* – Benefits are paid on policyholder death
- *Term insurance contract* – A sum assured is paid on or after death provided death occurs during a specified period (called the term of the contract)
- *Pure Endowment insurance contract* – Provides a sum assured at the end of a fixed term, provided the policyholder is then alive
- *Endowment insurance contract* – Combination of a term assurance and a pure endowment assurance.
- *With-profit (aka participating or par) insurance* – Insurance company shares the profits earned on the invested premiums with the policyholders by offering cash dividends, reducing premiums, increasing the sum insured, etc.
- Universal Life Insurance –
  - Combines investment and life insurance
  - The policyholder determines a premium and a level of life insurance cover
  - Some premium is used to fund the life insurance and the remainder is paid into an investment fund
  - Premiums are flexible, as long as they are sufficient to pay for the designated sum insured under the term insurance part of the contract
- Unitised with-profit –
  - Premiums are used to purchase units of an investment fund
  - As the fund earns investment return, the shares increase in value, increasing the benefit entitlement as reversionary bonus
  - The shares will not decrease in value
  - Reversionary bonus and/or terminal bonuses will be paid
- Equity linked insurance –
  - The benefit is linked to the performance of an investment fund

FIXED TERM & WHOLE LIFE INSURANCE | CONTINUOUS CASE

- Life insurance benefits:
  - Denoted by *Z*, the present value, at policy issue, of the benefit payment.
  - $Z = b_{T(x)} v_{T(x)}$ where:
    - T is the future life time of the policyholder
    - $b_{T(x)}$ is called the benefit payment function.
    - $v_{T(x)}$ is the discount function.
- An **n-year term life insurance** provides payment if the insured dies within n years from issue.
- For a unit of benefit payment we have:

$$b_{T(x)} = \begin{cases} 1, & T(x) \leq n \\ 0, & T(x) > n \end{cases} \text{ and } v_{T(x)} = v^{T(x)}.$$

- The present value random variable is therefore:

$$Z = \begin{cases} v^{T(x)}, & T(x) \leq n \\ 0, & T(x) > n \end{cases}$$

- We call E(Z) the actuarial present value (APV) or the expected present value (EPV) of the insurance.
  - We denote the above $E(Z)$ by the actuarial notation: $\bar{A}^{1}_{x:\overline{n|}}$
  - By noting that the probability density function of $T(x)$ is $f_x(t) = {}_tp_x\mu_{x+t}$, we obtain

$$\bar{A}^{1}_{x:\overline{n|}} = E(Z) = \int_0^n v^t {}_tp_x\mu_{x+t}dt.$$

- Rule of moments – For an n-year term life insurance:
  - The j-th moment of Z

$$E(Z^j) = \int_0^n v^{tj} {}_tp_x\mu_{x+t}dt = \int_0^n e^{-(j\delta)t} {}_tp_x\mu_{x+t}dt$$

  - This equals the APV evaluated at the force of interest $j\delta$, i.e., the rule of moment holds:

$$E(Z^j)_{\ @\delta} = E(Z)_{\ @j\delta}.$$

  - Denote ${}^2\bar{A}^{1}_{x:\overline{n|}} = E(Z)_{\ @2\delta}$
  - The variance

$$Var(Z) = {}^2\bar{A}^{1}_{x:\overline{n|}} - (\bar{A}^{1}_{x:\overline{n|}})^2$$

- For a **whole life insurance**, benefits are payable following death at any time in the future.
  - Here, we consider one unit of benefit payment, i.e., $b_{T(x)} = 1$
  - Then the present value random variable is $Z = v^{T(x)}$.
  - APV notation for whole life:

$$\bar{A}_x = E(Z) = \int_0^\infty v^t {}_tp_x \mu_{x+t} dt.$$

  - Variance (using rule of moments):

$$Var(Z) = {}^2\bar{A}_x - (\bar{A}_x)^2.$$

  - Whole life insurance is the limiting case of term life insurance as $n \to \infty$.

ENDOWMENT INSURANCE AND DEFERRED INSURANCE | THE CONTINUOUS CASE

- Consider an **n-year pure endowment insurance**. A benefit is payable at the end of n years if the insured survives at least n years from issue:
  - $b_{T(x)} = \begin{cases} 0, & T(x) \leq n \\ 1, & T(x) > n \end{cases}$ *and* $v_{T(x)} = v^n$
  - The PV r.v. is $Z = \begin{cases} 0, & T(x) \leq n \\ v^n, & T(x) > n \end{cases} = v^n I_{\{T(x)>n\}}$.
  - APV for pure endowment: $A_{x:\overset{1}{\overline{n|}}} = {}_nE_x = v^n {}_np_x$.
  - Variance (using rule of moments):

$$Var(Z) = {}^2A_{x:\overset{1}{\overline{n|}}} - (A_{x:\overset{1}{\overline{n|}}})^2 = v^{2n} {}_np_x \cdot {}_n q_x,$$

  where ${}^2A_{x:\overset{1}{\overline{n|}}} := E(Z)\ {}_{@2\delta}$.

- Consider an **n-year endowment insurance**. A benefit is payable if death is within n years of if the insured survives at least n years from issue, whichever occurs first:
  - $b_{T(x)} = 1$ and $v^{T(x)} = \begin{cases} v^{T(x)}, & T(x) \leq n \\ v^n, & T(x) > n \end{cases}$
  - The PV r.v. is $Z = \begin{cases} v^{T(x)}, & T(x) \leq n \\ v^n, & T(x) > n \end{cases}$.
  - APV of endowment: $\bar{A}_{x:\overline{n|}} = \bar{A}^1_{x:\overline{n|}} + A_{x:\overset{1}{\overline{n|}}}$.
  - Variance (using rule of moments):

$$Var(Z) = {}^2\bar{A}_{x:\overline{n|}} - (\bar{A}_{x:\overline{n|}})^2.$$

- Deferred Insurance – Insurance does not begin to offer death benefit cover until the end of a deferred period.
- For an **m-year deferred whole insurance**, a benefit is payable at the time of death if the insured dies at least m years following policy issue:

- Here, we have $b^{T(x)} = \begin{cases} 0, & T(x) \le m \\ 1, & T(x) > m \end{cases}$ and $v_{T(x)} = v^{T(x)}$
- The PV r.v. is $Z = \begin{cases} 0, & T(x) \le m \\ v^{T(x)}, & T(x) > m \end{cases}$.
- APV for deferred insurance:
  ${}_{m|}\bar{A}_x = E(Z) = \int_m^\infty v^t {}_tp_x \mu_{x+t} dt$.
- ${}_{m|}\bar{A}_x = {}_mp_x v^m \bar{A}_{x+m}$.
- Variance:
  $$Var(Z) = {}^2_{m|}\bar{A}_x - ({}_{m|}\bar{A}_x)^2.$$

INSURANCE BENEFITS PAYABLE AT EOY OF DEATH | ANNUAL CASE

- For insurances payable at the end of year of death:
  - Let $K(x)$ represent the curtate future lifetime:
    $K(x) = \lfloor T(x) \rfloor$.
  - The PV r.v. of benefits, $Z = b_{K(x)+1} v_{K(x)+1}$ where
- Consider an **n-year term insurance** which pays benefits at the end of year of death:
  - $$b_{K(x)+1} = \begin{cases} 1, & K(x) = 0, 1, ..., n-1 \\ 0, & otherwise \end{cases},$$
    $$v_{K(x)+1} = v^{K(x)+1},$$
  - The PV r.v. of benefits:
    $$Z = \begin{cases} v^{K(x)+1}, & K(x) = 0, 1, ..., n-1 \\ 0, & otherwise \end{cases}.$$
  - APV of n-year term insurance:
    $$A^1_{x:\overline{n}|} = E(Z) = \sum_{k=0}^{n-1} v^{k+1} {}_{k|}q_x = \sum_{k=0}^{n-1} v^{k+1} {}_kp_x \cdot q_{x+k}$$
  - Define
    $${}^2A^1_{x:\bar{n}|} = E(Z)_{@2\delta} = \sum_{k=0}^{n-1} e^{-2\delta(k+1)} {}_kp_x \cdot q_{x+k}.$$
  - **Rule of moments also applies**:
    $${}^2A^1_{x:\bar{n}|} = E(Z^2).$$
  - For example,
    $$Var(Z) = {}^2A^1_{x:\overline{n}|} - (A^1_{x:\overline{n}|})^2,$$

- Consider a **whole life insurance** which pays a benefit at the end of year of death:
  - $$b_{K(x)+1} = 1,\ v_{K(x)+1} = v^{K(x)+1},\ \text{and } Z = v^{K(x)+1}.$$
  - APV:
  $$A_x = E(Z) = \sum_{k=0}^{\infty} v^{k+1}{}_{k|}q_x = \sum_{k=0}^{\infty} v^{k+1}{}_k p_x \cdot q_{x+k}$$
  - Define
  $${}^2A_x = E(Z)_{@2\delta} = \sum_{k=0}^{\infty} e^{-2\delta(k+1)}{}_k p_x \cdot q_{x+k}.$$
  - Applying rule of moments,
  $$Var(Z) = {}^2A_x - (A_x)^2,$$
- Recursive relationships –
  - Whole life insurance:
  $$A_x = vq_x + vp_x A_{x+1}$$
  - Term insurance:
  $$A^1_{x:\overline{n}|} = vq_x + vp_x A^1_{x+1:\overline{n-1}|}$$
  - Endowment insurance:
  $$A_{x:\overline{n}|} = vq_x + vp_x A_{x+1:\overline{n-1}|}$$

THE 1/M CASES & INSURANCE WITH VARYING BENEFITS

- The 1/mthly curtate future lifetime random variable –
  - Recall that $K(x)(= \lfloor T(x) \rfloor)$ is the curtate future lifetime of the life (x).
  - The APV of insurance benefit payable at the end of year of death can be expressed in terms of $K(x)$.
  - For any positive integer m, we now define the $1/m$thly curtate future lifetime r.v., $K^{(m)}(x)$, by
  $$K^{(m)}(x) = \frac{\lfloor mT(x) \rfloor}{m}.$$
  - $K^{(m)}(x)$ is the future lifetime of (x) in years rounded to the lower $1/m$th of a year.
  - The most common values of m are 2, 4 and 12, corresponding to half years, quarter years and months.
  - For example, suppose $T(x) = 23.675$. Then $K(x) = 23$, $K^{(2)}(x) = 23.5$, $K^{(4)}(x) = 23.5$, $K^{(12)}(x) = 23.6667$.

- $K^{m}(x)$ is a discrete r.v. that takes values in $\{0, 1/m, 2/m, \cdots\}$.
- $K^{(m)}(x) = k/m$ is equivalent to $T(x) \in [k/m, k/m + 1/m)$. Therefore,

$$P(K^{(m)}(x) = k/m) = P(k \leq T(x) < k/m + 1/m) = {}_{\frac{k}{m}|\frac{1}{m}}q_x.$$

- Consider a whole life insurance where a benefit of 1 is payable at the end of the 1/mth year of death:
  - Benefit is payable at time $K^{(m)} + \frac{1}{m}$
  - The PV r.v. of benefit: $Z = v^{K^{(m)}+1/m}$
  - The EPV of benefit (APV):

$$\begin{aligned} A_x^{(m)} &= v^{\frac{1}{m}}\,{}_{\frac{1}{m}}q_x + v^{\frac{2}{m}}\,{}_{\frac{1}{m}|\frac{1}{m}}q_x + v^{\frac{3}{m}}\,{}_{\frac{2}{m}|\frac{1}{m}}q_x + \cdots \\ &= \sum_{k=0}^{\infty} v^{\frac{k+1}{m}}\,{}_{\frac{k}{m}|\frac{1}{m}}q_x \end{aligned}$$

- Consider an n-year term life insurance where a benefit of 1 is payable at the end of the 1/mth year of death, provided death occurs within n years.
  - In this case, \$1 is payable at $K^{(m)} + 1/m$ if death occurs within n years.
  - The PV of benefit: $Z = \begin{cases} v^{K^{(m)}+1/m} & K^{(m)} \leq n - \frac{1}{m} \\ 0 & K^{(m)} > n - \frac{1}{m} \end{cases}$
  - The EPV of benefit (APV):

$$A^{(m)1}_{x:\overline{n}|} = E[Z] = \sum_{k=0}^{mn-1} v^{\frac{k+1}{m}}\,{}_{\frac{k}{m}|\frac{1}{m}}q_x$$

- Consider an n-year term endowment insurance where a benefit of 1 is payable at the end of the 1/mth year of death, provided death occurs within n years, or at the end of the nth year if the person survives for n years.
  - The PV of benefit: $Z = \begin{cases} v^{K^{(m)}+1/m} & K^{(m)} \leq n - \frac{1}{m} \\ v^n & K^{(m)} \geq n \end{cases}$
  - The EPV of benefit (APV):

$$\begin{aligned} A^{(m)}_{x:\overline{n}|} = E[Z] &= \sum_{k=0}^{mn-1} v^{\frac{k+1}{m}}\,{}_{\frac{k}{m}|\frac{1}{m}}q_x + v^n\,{}_np_x \\ &= A^{(m)1}_{x:\overline{n}|} + A_{x:\overset{\;1}{\overline{n}|}} \end{aligned}$$

- Insurance with varying benefits:
  - Continuously increasing whole life insurance:

$$Z = Tv^T, \quad (\bar{I}\bar{A})_x = \int_0^\infty tv^t \ {}_tp_x \ \mu_{x+t}dt$$

  - Annually increasing Whole Life Insurance

$$Z = \lfloor T+1 \rfloor v^T, \quad APV = (I\bar{A})_x$$

  - Whole Life insurance increasing 1/mthly:

$$Z = \frac{\lfloor Tm+1 \rfloor}{m} v^T, \ APV = (I^{(m)}\bar{A})_x$$

  - Annually increasing and annually payable WL insurance

$$Z = (K+1)v^{K+1}, \ (IA)_x = \sum_{k=0}^{\infty} v^{k+1}(k+1) \ {}_{k|}q_x$$

  - Decreasing n-year term:

$$Z = \begin{cases} (n - \lfloor T \rfloor)v^T, & T \le n \\ 0, & T > n \end{cases} \quad APV = (D\bar{A})^1_{x:\overline{n}|}$$

RELATIONSHIPS BETWEEN CONTINUOUS AND DISCRETE CASES

- Under the UDD assumption (deaths are uniformly distributed during the period between two consecutive integer ages): ${}_sp_k\mu_{k+s} = q_k$ for integer k and $0 < s < 1$.
  - Whole life insurance: $\bar{A}_x = (i/\delta)A_x$.

Proof.

$$\begin{aligned}
&\bar{A}x \\
&= \int_0^\infty v^t {}_tp_x \ \mu_{x+t}dt = \sum_{k=0}^{\infty} \int_k^{k+1} v^t {}_tp_x \ \mu_{x+t}dt \\
&= \sum_{k=0}^{\infty} \int_0^1 v^{k+s} {}_{k+s}p_x \ \mu_{x+k+s}ds = \sum_{k=0}^{\infty} \int_0^1 v^{k+s} {}_kp_x \ {}_sp_{x+k} \ \mu_{x+k+s}ds \\
&= \sum_{k=0}^{\infty} \int_0^1 v^{k+s} {}_kp_x \ q_{x+k}ds = \sum_{k=0}^{\infty} v^{k+1} {}_kp_x \ q_{x+k} \int_0^1 v^{s-1}ds \\
&= A_x \int_0^1 e^{\delta(1-s)}ds = A_x \frac{e^\delta - 1}{\delta} = A_x \frac{i}{\delta}.
\end{aligned}$$

- Under the UDD assumption
  - Term insurance:

$$\bar{A}^1_{x:\overline{n}|} = (i/\delta)A^1_{x:\overline{n}|}$$

  - Increasing term insurance:

$$(I\bar{A})^1_{x:\overline{n}|} = (i/\delta)(IA)^1_{x:\overline{n}|}.$$

  - For whole life insurances payable m-thly,

$$A_x^{(m)} = (i/i^{(m)})A_x.$$

M2: Life Annuities

INTRODUCTION TO LIFE ANNUITIES

- Life annuity – a series of payments to (or from) an individual as long as the individual is alive on the payment date.
- The payments are usually at regular intervals
- The present value of a life annuity is a random variable that depends on the future lifetime.

WHOLE LIFE ANNUITY-DUE | ANNUAL CASE

- Pays a benefit of a unit \$1 at the beginning of each year that the annuitant (x) survives

- The present value random variable of benefits:

$$Y = 1 + v + v^2 + \cdots + v^{K(x)} = \ddot{a}_{\overline{K(x)+1|}}$$

  where $K(x)$ is the curtate future lifetime of (x)

- The actuarial present value of the annuity

$$\begin{aligned} \ddot{a}_x &= E[Y] = E[\ddot{a}_{\overline{K(x)+1|}}] = \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1|}} P(K(x) = k) \\ &= \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1|}} \cdot {}_{k|}q_x = \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1|}} \cdot {}_k p_x \cdot q_{x+k} \end{aligned}$$

- The PV of benefits:

$$Y = 1\, I(T(x) > 0) + v\, I(T(x) > 1) + v^2\, I(T(x) > 2) + \cdots$$

- The current payment technique formula for computing a whole life annuity-due

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k\, {}_k p_x$$

- The ${}_k p_x$ term is the probability of a payment of size 1 being made at time k.

- Relationship to whole life insurance –
  - Recall from interest theory that $\ddot{a}_{\overline{K(x)+1|}} = (1 - v^{K(x)+1})/d$
  - We have the useful relationship:

$$\ddot{a}_x = \frac{1 - A_x}{d}$$

  - Alternatively, we write:

$$A_x = 1 - d\ddot{a}_x$$

  - The variance formula can be calculated as:

$$\begin{aligned} Var(\ddot{a}_{\overline{K(x)+1|}}) &= Var\left(\frac{1 - v^{K(x)+1}}{d}\right) = \frac{Var\left(v^{K(x)+1}\right)}{d^2} \\ &= \frac{{}^2A_x - (A_x)^2}{d^2} \end{aligned}$$

- The recursive relationship –

$$\begin{aligned} \ddot{a}_x &= 1 + vE[\ddot{a}_{\overline{K(x)+1|}} | K(x) \geq 1]P(K(x) \geq 1) \\ &= 1 + vp_x\ddot{a}_{x+1} \end{aligned}$$

TERM ANNUITY-DUE

- Pays a benefit of a unit \$1 at the beginning of each year as long as the annuitant (x) survives for a maximum of n years
- The present value random variable is

$$Y = \begin{cases} \ddot{a}_{\overline{K(x)+1|}}, & K(x) < n \\ \ddot{a}_{\overline{n|}}, & K(x) \geq n \end{cases}$$

- The actuarial present value of the annuity:

$$\ddot{a}_{x:\overline{n|}} = E[Y] = \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1|}}({}_kp_x)q_{x+k} + \ddot{a}_{\overline{n|}}\ {}_np_x$$

- Current payment technique
  - PV of the benefits:

$$\begin{aligned} Y &= I(T(x) > 0) + vI(T(x) > 1) + v^2I(T(x) > 2) + \cdots \\ &\quad + v^{n-1}I(T(x) > n-1) \end{aligned}$$

  - Current payment formula: $\ddot{a}_{x:\overline{n|}} = \sum_{k=0}^{n-1} v^k\ {}_kp_x$
- Recursive formula:

$$\ddot{a}_{x:\overline{n|}} = 1 + vp_x\ddot{a}_{x+1:\overline{n-1|}}$$

- Term annuity-due – relationship to endowment life insurance:
  - Note that $Y = (1 - Z)/d$, where

$$Z = \begin{cases} v^{K(x)+1}, & 0 \leq K(x) < n \\ v^n, & K(x) \geq n \end{cases}$$

    is the PV r.v. for a unit of endowment insurance, payable at EOY or maturity

  - Thus, it follows that

$$\ddot{a}_{x:\overline{n}|} = \frac{1 - A_{x:\overline{n}|}}{d}$$

  - The variance can be represented as

$$Var(Y) = \frac{Var(Z)}{d^2} = \frac{{}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2}{d^2}$$

- Deferred whole life annuity-due:
  - Pays a benefit of a unit \$1 at the beginning of each year while the annuitant (x) survives from $x + n$ onward
  - The PV r.v. is $Y = \begin{cases} 0, & 0 \leq K(x) < n \\ {}_{n|}\ddot{a}_{\overline{K(x)+1-n}|}, & K(x) \geq n \end{cases}$
  - The APV of the annuity is

$${}_{n|}\ddot{a}_x = E(Y) = {}_nE_x \ddot{a}_{x+n} = \ddot{a}_x - \ddot{a}_{x:\overline{n}|} = \sum_{k=n}^{\infty} v^k {}_kp_x$$

- Life annuity-due with guaranteed payments:
  - For a life annuity-due with n-year certain, its PV r.v. is

$$Y = \begin{cases} \ddot{a}_{\overline{n}|}, & 0 \leq K(x) < n \\ \ddot{a}_{\overline{K(x)+1}|}, & K(x) \geq n \end{cases}$$

  - APV of the annuity:

$$\ddot{a}_{\overline{\overline{x:n}|}} = E(Y) = \ddot{a}_{\overline{n}|} \cdot {}_nq_x + \sum_{k=n}^{\infty} \ddot{a}_{\overline{k+1}|} {}_kp_x q_{x+k}$$

  - Current payment technique formula:

$$\ddot{a}_{\overline{\overline{x:n}|}} = \ddot{a}_{\overline{n}|} + \sum_{k=n}^{\infty} v^k {}_kp_x = \ddot{a}_{\overline{n}|} + \ddot{a}_x - \ddot{a}_{x:\overline{n}|},$$

    which equals Guaranteed annuity plus a deferred life annuity!

IMMEDIATE ANNUITIES

- Consider a whole life immediate annuity that pays a benefit of 1 at the end of each year, conditional on the survival of a life to the payment date –
  - The PV r.v. is
$$Y = a_{\overline{K(x)}|}$$
  - The APV of the annuity
$$a_x = E(Y) = \sum_{k=1}^{\infty} a_{\overline{k}|} \cdot {}_k p_x q_{x+k}$$
  - Current payment technique formula:
$$a_x = E(Y) = \sum_{k=1}^{\infty} v^k {}_k p_x$$
  - Relationship to life insurance:
$$Y = \frac{1 - v^{K(x)}}{i} = \frac{1 - (1+i)v^{K(x)+1}}{i}$$
    leads to $1 = ia_x + (1+i)A_x$
- Consider a term immediate annuity that pays a benefit of 1 at the end of each year for a maximum of n years, conditional on the survival of a life to the payment date –
  - The PV r.v. is
$$Y = a_{\overline{\min(K(x),n)}|}$$
  - The APV of the annuity
$$a_{x:\overline{n}|} = E(Y) = \sum_{k=1}^{n-1} a_{\overline{k}|} \cdot {}_k p_x q_{x+k} + a_{\overline{n}|} \cdot {}_n p_x$$
  - Current payment technique formula:
$$a_{x:\overline{n}|} = E(Y) = \sum_{k=1}^{n} v^k {}_k p_x$$
  - Relationship to term life annuity-due:
$$a_{x:\overline{n}|} = \ddot{a}_{x:\overline{n}|} - 1 + v^n \ {}_n p_x$$

ANNUITIES PAYABLE 1/*M*THLY

- Consider an annuity of a total amount of 1 per year, payable in advance *m* times per year throughout the lifetime of (x), with each payment being 1/m –
  - PV r.v. is represented as
$$Y = \ddot{a}^{(m)}_{\overline{K^{(m)}(x)+1/m}|} = \sum_{j=0}^{mK^{(m)}(x)} \frac{1}{m} v^{j/m} = \frac{1 - v^{K^{(m)}(x)+1/m}}{d^{(m)}}$$
    where $K^{(m)}(x) = \frac{\lfloor mT(x) \rfloor}{m}$ is the curtate future lifetime of (x) in years rounded down to the lower $1/m$th of a year.
  - The APV:
$$\ddot{a}^{(m)}_x = E[Y] = \frac{1 - A^{(m)}_x}{d^{(m)}}$$

- Variance is

$$Var(Y) = \frac{Var(v^{K^{(m)}(x)+1/m})}{(d^{(m)})^2} = \frac{{}^2A_x^{(m)} - (A_x^{(m)})^2}{(d^{(m)})^2}$$

- Alternatively,

$$\ddot{a}_x^{(m)} = \frac{1}{m}\sum_{h=0}^{\infty} v^{h/m} \cdot {}_{h/m}p_x$$

ANNUITIES PAYABLE CONTINUOUSLY

- Consider an n-year term life annuity that pays 1 per year continuously while a life aged (x) survives during the next n years –
  - The PV random variable is
    $$Y = \begin{cases} \bar{a}_{\overline{T(x)|}}, & 0 \leq T(x) < n \\ \bar{a}_{\overline{n|}}, & T(x) \geq n \end{cases} = \frac{1-v^{min(T(x),n)}}{\delta}$$
  - The APV of the annuity:
    $$\bar{a}_{x:\overline{n|}} = E(Y) = \int_0^n \bar{a}_{\overline{t|}} \cdot {}_t p_x \mu_{x+t} dt + \bar{a}_{\overline{n|}} \cdot {}_n p_x$$
    $$\bar{a}_{x:\overline{n|}} = \frac{1 - \bar{A}_{x:\overline{n|}}}{\delta}$$
    $$\bar{a}_{x:\overline{n|}} = \int_0^n v^t \, {}_t p_x dt$$
- Recursive formula –
  $$\bar{a}_{x:\overline{n|}} = \bar{a}_{x:\overline{1|}} + vp_x\bar{a}_{x+1:\overline{n-1|}}$$
- Consider a whole life annuity payable continuously at the rate of 1 per year as long as a life aged (x) survives –
  - One can think of it as life annuity payable m-thly per year, with $m \to \infty$
  - The PV r.v is $Y = \bar{a}_{\overline{T(x)|}} = \frac{1-v^{T(x)}}{\delta}$
  - The APV of annuity:
    $$\bar{a}_x = E(Y) = E(\bar{a}_{\overline{T(x)|}}) = \int_0^\infty \bar{a}_{\overline{t|}} \cdot {}_t p_x \mu_{x+t} dt$$
    $$\bar{a}_x = E(\frac{1 - v^{T(x)}}{\delta}) = \frac{1 - \bar{A}_x}{\delta}$$
    $$\bar{a}_x = \int_0^\infty v^t \, {}_t p_x dt$$

- Variance expression:

$$Var(\bar{a}_{\overline{T(x)}|}) = \frac{^2\bar{A}_x - (\bar{A}_x)^2}{\delta^2}$$

- Recursive relation:

$$\bar{a}_x = \bar{a}_{x:\overline{1}|} + vp_x\bar{a}_{x+1}$$

- Relationship to whole life insurance:

$$\bar{A}_x = 1 - \delta\bar{a}_x$$

- Consider a deferred whole life continuous annuity that pays a benefit of 1 each year continuously while the life aged (x) survives from *x + n* onward -
  - The PV r.v. is
    $$Y = \begin{cases} 0 \\ v^n\bar{a}_{\overline{T(x)-n}|} \end{cases} = \begin{cases} 0, & 0 \le T(x) < n \\ \bar{a}_{\overline{T(x)}|} - \bar{a}_{\overline{n}|}, & T(x) \ge n \end{cases}$$
  - The APV of the annuity is

$$_{n|}\bar{a}_x = E[Y] = \ _nE_x\bar{a}_{x+n} = \bar{a}_x - \bar{a}_{x:\overline{n}|}$$
$$_{n|}\bar{a}_x = \int_n^\infty v^t\ _tp_x dt$$

- Consider a whole life annuity with a guarantee of payments for the first n years –
  - Its PV r.v. is

$$Y = \begin{cases} \bar{a}_{\overline{n}|}, & 0 \le T(x) < n \\ \bar{a}_{\overline{T(x)}|}, & T(x) > n \end{cases}$$

  - APV of the annuity:

$$\bar{a}_{\overline{x:\overline{n}|}} = \bar{a}_{\overline{n}|} \cdot {}_n q_x + \int_n^\infty \bar{a}_{\overline{t}|} \cdot {}_t p_x \mu_{x+t} dt$$
$$\bar{a}_{\overline{x:\overline{n}|}} = \bar{a}_{\overline{n}|} + \int_n^\infty v^t\ _tp_x dt = \bar{a}_{\overline{n}|} + \bar{a}_x - \bar{a}_{x:\overline{n}|} = \bar{a}_{\overline{n}|} + {}_{n|}\bar{a}_x$$

INCREASING ANNUITIES

- Arithmetically increasing annuities –
  1. Increasing whole life annuity-due where the amount of $k+1$ is payable at times $k = 0, 1, 2, \cdots$, provided that (x) is alive at time k
     - APV

$$(I\ddot{a})_{x:} = \sum_{k=0}^{\infty}(k+1)v^k\ {}_kp_x$$

  2. Increasing n-year term life annuity-due where the amount of $k+1$ is payable at times $k = 0, 1, 2, \cdots, n-1$ provided that (x) is alive at time k
     - APV

$$(I\ddot{a})_{x:\overline{n|}} = \sum_{k=0}^{n-1}(k+1)v^k\ {}_kp_x$$

  1. Increasing continuously n-year term continuous annuity where the rate of payment at time t is t, provided that (x) is alive at time t
     - APV

$$(\bar{I}\bar{a})_{x:\overline{n|}} = \int_0^n tv^t\ {}_tp_x dt$$

- Geometrically increasing annuities –
  1. An annuitant may be interested in purchasing an annuity that increases geometrically, to offset the effect of inflation on the purchasing power of the income.
  2. Increasing n-year term life annuity-due where the amount of $(1+j)^k$ is payable at times $k = 0, 1, 2, \cdots, n-1$ provided that (x) is alive at time k
     - APV is

$$\sum_{k=0}^{n-1}(1+j)^k v^k\ {}_kp_x = \ddot{a}_{x:\overline{n|}i^*}$$

where $i^* = \frac{i-j}{1+j}$.

APPROXIMATIONS FOR ANNUITIES

- When we have full information about the survival function for a life, we can use summation or numerical integration to compute the EPV of any annuity.
- However, we often only have integer age information, so we need to use approximations for evaluating the EPV of 1/mthly and continuous annuities given only the EPV of annuities of integer ages.

- Under the UDD assumption –
  - Recall
$$A_x^{(m)} = \frac{i}{i^{(m)}} A_x, \quad \bar{A}_x = \frac{i}{\delta} A_x$$
  - $\ddot{a}_x^{(m)} = \alpha(m)\ddot{a}_x - \beta(m)$ where
$$\alpha(m) = \frac{i}{i^{(m)}} \cdot \frac{d}{d^{(m)}}$$
$$\beta(m) = \frac{i - i^{(m)}}{i^{(m)} d^{(m)}}$$
  - $\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m}$ [very widely used]

THE WOOLHOUSE'S FORMULA

- Woolhouse's formula is based on the Euler-Maclaurin formula and expresses $\ddot{a}_x^{(m)}$ in terms of $\ddot{a}_x$
- Euler-Maclaurin formula: for a function g that is differentiable a certain number of times,
$$\int_0^\infty g(t)dt = h\sum_{k=0}^{\infty} g(kh) - \frac{h}{2}g(0) + \frac{h^2}{12}g'(0) - \frac{h^4}{720}g'''(0) + \cdots,$$
  where the terms that involve higher derivatives of g have been omitted.
- The Woolhouse Formula (with three terms) –
$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\delta + \mu_x)$$
- The Woolhouse Formula for term annuities –
$$\begin{aligned}
\ddot{a}_{x:\overline{n}|}^{(m)} &= \ddot{a}_x^{(m)} - v^n {}_np_x \ddot{a}_{x+n}^{(m)} \\
&\approx \left(\ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\delta + \mu_x)\right) \\
&\quad -v^n {}_np_x \left(\ddot{a}_{x+n} - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\delta + \mu_{x+n})\right) \\
&= \ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}(1 - v^n {}_np_x) \\
&\quad - \frac{m^2-1}{12m^2}\left(\delta + \mu_x - v^n {}_np_x(\delta + \mu_{x+n})\right)
\end{aligned}$$

- The Woolhouse formula for continuous annuities,

  letting $m \to \infty$ in the above formulae for $\ddot{a}_{x:\overline{n}|}$ and $\ddot{a}^{(m)}_{x:\overline{n}|}$ we can obtain

$$\bar{a}_x \approx \ddot{a}_x - \frac{1}{2} - \frac{1}{12}(\delta + \mu_x)$$

$$\bar{a}_{x:\overline{n}|} \approx \ddot{a}_{x:\overline{n}|} - \frac{1}{2}(1 - v^n {}_np_x) - \frac{1}{12}\left(\delta + \mu_x - v^n {}_np_x(\delta + \mu_{x+n})\right)$$

- An important difference between the approximation to annuities based on Woolhouse's formula and the UDD approximation is that the Woolhouse approach requires extra information – the force of mortality.
- Woolhouse's formula (with three terms) is very accurate.
  - Even if the full force of mortality is known, it is often a more efficient way to calculate annuity values than the direct formulae.
- Woolhouse's formula may be used for calculating insurance function. E.g. –

$$\begin{aligned}
A_x^{(m)} &= 1 - d^{(m)}\ddot{a}_x^{(m)} \\
&= 1 - d^{(m)}\left(\ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\delta + \mu_x)\right) \\
&= 1 - d^{(m)}\left(\frac{1 - A_x}{d} - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\delta + \mu_x)\right)
\end{aligned}$$

M3: Net Premium Calculations

NET PREMIUMS AND THE EQUIVALENCE PRINCIPLE

- An insurance contract is an agreement between two parties:
  - The insurer agrees to pay for insurance benefits
  - In exchange for insurance premiums to be paid by the insured
- Preliminaries-premiums –
  - The premium may be a single payment or it may be a regular series of payments, possibly annually, quarterly, monthly or weekly.
  - It is common for regular premiums to be a level amount, but they do not have to be.
  - Premiums are payable in advance, wit the first premium payable when the policy is purchased.
  - Regular premiums for a policy on a single life cease to be payable on the death of the policyholder.
- **Net premiums** – Consider only the benefits provided, and not allocated expenses, profit or contingency margins.
- The present value of the net future loss random variable –
  - Denote by $PVFB_0$ the present value, at time of issue, of future benefits to be paid by the insurer.
  - Denote by $PVFP_0$ the present value, at time of issue, of future premiums to be paid by the insured.
  - The insurer's net random future loss is defined by:

    $${}_0L = L = PVFB_0 - PVFP_0$$
- *The equivalence principle*: EPV of benefit outgo = EPV of new premium income
- The net premium is determined according to the equivalence principle by setting: $E[L] = 0$
- The net premium is the amount of premium required to meet the expected cost of the insurance or annuity benefits under a contract, given mortality and interest rate assumptions.
- Example – Consider a life insurance contract with a unit benefit where:
  - level premiums are payable annually with an annual amount of π
  - Z is the PV r.v. associated with the life insurance benefits
  - Y is the PV r.v. associated with the life annuity premium payments assuming that the annual payment is 1

  Then

  $$L = Z - \pi Y.$$

  Hence, the annual premium is

  $$\pi = E(Z)/E(Y)$$

FULLY DISCRETE ANNUAL NET PREMIUMS | WL AND TERM INSURANCE

- Consider the case of a unit WLI with level annual premium payments:
  - The loss random variable is
    $$L = v^{K(x)+1} - \pi \ddot{a}_{\overline{K(x)+1|}}, \qquad \text{for } K(x) = 0, 1, 2, \ldots$$
  - By the principle of equivalence:
    $$E(L) = E(v^{K(x)+1}) - \pi E(\ddot{a}_{\overline{K(x)+1|}}) = 0$$
  - The actuarial notation for the above π is P_x
  - Then
    $$P_x = \pi = \frac{A_x}{\ddot{a}_x}$$
  - The variance of the loss function
    $$Var(L) = \frac{{}^2A_x - (A_x)^2}{(d\ddot{a}_x)^2} = \frac{{}^2A_x - (A_x)^2}{(1 - A_x)^2}$$
- Some formulas:
  - Recall
    $$P_x = \frac{A_x}{\ddot{a}_x}$$
    and
    $$\ddot{a}_x = \frac{1 - A_x}{d}$$
  - Then
    $$\frac{1}{\ddot{a}_x} = P_x + d$$
    $$P_x = \frac{dA_x}{1 - A_x}$$
- WLI with h premium payments:
  - The loss function in this case is:
    $$L = \begin{cases} v^{K(x)+1} - \pi \ddot{a}_{\overline{K(x)+1|}} & \text{for } K(x) = 0, 1, \cdots, h-1 \\ v^{K(x)+1} - \pi \ddot{a}_{\overline{h|}} & \text{for } K(x) = h, h+1, \cdots \end{cases}$$
  - Applying the principle of equivalence, we have:
    $$\pi = {}_hP_x = \frac{A_x}{\ddot{a}_{x:\overline{h|}}}$$
- Term life insurance:
  - *n-yr* term:
    $$L = \begin{cases} v^{K(x)+1} - \pi \ddot{a}_{\overline{K(x)+1|}} & \text{for } K(x) = 0, 1, \cdots, n-1 \\ 0 - \pi \ddot{a}_{\overline{n|}} & \text{for } K(x) = n, n+1, \cdots \end{cases}$$
  - The corresponding premium formula is:
    $$P^1_{x:\overline{n|}} = \frac{A^1_{x:\overline{n|}}}{\ddot{a}_{x:\overline{n|}}}$$

FULLY DISCRETE NET PREMIUMS | OTHER INSURANCE CONTRACTS

- *n-yr* endowment:
  - Loss function:

$$L = \begin{cases} v^{K(x)+1} - \pi \ddot{a}_{\overline{K(x)+1}|} & \text{for } K(x) = 0, 1, \cdots, n-1 \\ v^n - \pi \ddot{a}_{\overline{n}|} & \text{for } K(x) = n, n+1, \cdots \end{cases}$$

  - Premium formula:

$$P_{x:\overline{n}|} = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}$$

- h-pay, n-yr endowment:
  - Loss function:

$$L = \begin{cases} v^{K(x)+1} - \pi \ddot{a}_{\overline{K(x)+1}|} & \text{for } K(x) = 0, 1, \cdots, h-1 \\ v^{K(x)+1} - \pi \ddot{a}_{\overline{h}|} & \text{for } K(x) = h, ..., n-1 \\ v^n - \pi \ddot{a}_{\overline{h}|} & \text{for } K(x) = n, n+1, \cdots. \end{cases}$$

  - Premium formula:

$${}_hP_{x:\overline{n}|} = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{h}|}}$$

- n-yr pure endowment:
  - Loss function:

$$L = \begin{cases} 0 - \pi \ddot{a}_{\overline{K(x)+1}|} & \text{for } K(x) = 0, 1, \cdots, n-1 \\ v^n - \pi \ddot{a}_{\overline{n}|} & \text{for } K(x) = n, n+1, \cdots. \end{cases}$$

  - Premium formula:

$$P_{x:\overset{1}{\overline{n}|}} = \frac{A_{x:\overset{1}{\overline{n}|}}}{\ddot{a}_{x:\overline{n}|}}$$

- n-yr deferred WL annuity with n-yr premium payment:
  - Loss function:

$$L = \begin{cases} 0 - \pi \ddot{a}_{\overline{K(x)+1}|} & \text{for } K(x) = 0, 1, \cdots, n-1 \\ v^n \ddot{a}_{\overline{K(x)+1-n}|} - \pi \ddot{a}_{\overline{n}|} & \text{for } K(x) = n, n+1, \cdots. \end{cases}$$

  - Premium formula:

$$P({}_{n|}\ddot{a}_x) = \frac{A_{x:\overset{1}{\overline{n}|}}\, \ddot{a}_{x+n}}{\ddot{a}_{x:\overline{n}|}}$$

FULLY CONTINUOUS PREMIUMS

- Consider fully continuous level annual premiums for a unit whole life insurance payable immediately upon death of (x).
  - The loss function is expressed as

$$L = v^T - \pi \bar{a}_{\overline{T|}}$$

  - By the principle of equivalence, and denoting the net premium π by $\bar{P}(\bar{A}_x)$, we have

$$\pi = \bar{P}(\bar{A}_x) = \frac{\bar{A}_x}{\bar{a}_x}$$

  - Variance of the insurer's loss function:

$$\begin{aligned} Var(L) &= \left[{}^2\bar{A}_x - (\bar{A}_x)^2 \right] \left[1 + (\bar{P}(\bar{A}_x)/\delta) \right]^2 \\ &= \frac{{}^2\bar{A}_x - (\bar{A}_x)^2}{(\delta \bar{a}_x)^2} = \frac{{}^2\bar{A}_x - (\bar{A}_x)^2}{(1 - \bar{A}_x)^2} \end{aligned}$$

- Endowment insurance premiums (continuous):
  - Loss function:

$$L = \begin{cases} v^T - \pi \bar{a}_{\overline{T|}}, & T \le n \\ v^n - \pi \bar{a}_{\overline{n|}} & T \ge n \end{cases}$$

  - Premium formula:

$$\pi = \bar{P}(\bar{A}_{x:\overline{n|}}) = \frac{\bar{A}_{x:\overline{n|}}}{\bar{a}_{x:\overline{n|}}}$$

  - Variance of the insurer's loss function:

$$\begin{aligned} Var(L) &= \left[{}^2\bar{A}_{x:\overline{n|}} - (\bar{A}_{x:\overline{n|}})^2 \right] \left[1 + (\bar{P}(\bar{A}_{x:\overline{n|}})/\delta) \right]^2 \\ &= \frac{{}^2\bar{A}_{x:\overline{n|}} - (\bar{A}_{x:\overline{n|}})^2}{(\delta \bar{a}_{x:\overline{n|}})^2} = \frac{{}^2\bar{A}_{x:\overline{n|}} - (\bar{A}_{x:\overline{n|}})^2}{(1 - \bar{A}_{x:\overline{n|}})^2} \end{aligned}$$

- Important identities:
  - Whole life insurance:

$$\bar{P}(\bar{A}_x) = \frac{1}{\bar{a}_x} - \delta = \frac{\delta \bar{A}_x}{1 - \bar{A}_x}$$

  - Endowment insurance:

$$\bar{P}(\bar{A}_{x:\overline{n|}}) = \frac{1}{\bar{a}_{x:\overline{n|}}} - \delta = \frac{\delta \bar{A}_{x:\overline{n|}}}{1 - \bar{A}_{x:\overline{n|}}}$$

- n-yr term (continuous):
  - Loss function:

$$L = \begin{cases} v^T - \pi \bar{a}_{\overline{T|}}, & T \le n \\ 0 - \pi \bar{a}_{\overline{n|}} & T \ge n \end{cases}$$

- Premium formula:

$$\bar{P}(\bar{A}^{1}_{x:\overline{n}|}) = \frac{\bar{A}^{1}_{x:\overline{n}|}}{\bar{a}_{x:\overline{n}|}}$$

- h-pay whole life (continuous):
  - Loss function:

$$L = \begin{cases} v^T - \pi\bar{a}_{\overline{T}|}, & T \le h \\ v^T - \pi\bar{a}_{\overline{h}|}, & T > h \end{cases}$$

  - Premium formula:

$${}_h\bar{P}(\bar{A}_x) = \frac{\bar{A}_x}{\bar{a}_{x:\overline{h}|}}$$

- h-pay, n-yr endowment (continuous):
  - Loss function:

$$L = \begin{cases} v^T - \pi\bar{a}_{\overline{T}|}, & T \le h \\ v^T - \pi\bar{a}_{\overline{h}|}, & h < T \le n \\ v^n - \pi\bar{a}_{\overline{h}|}, & T > n \end{cases}$$

  - Premium formula:

$${}_h\bar{P}(\bar{A}_{x:\overline{n}|}) = \frac{\bar{A}_{x:\overline{n}|}}{\bar{a}_{x:\overline{h}|}}$$

- n-yr pure endowment (continuous):
  - Loss function:

$$L = \begin{cases} 0 - \pi\bar{a}_{\overline{T}|}, & T \le n \\ v^n - \pi\bar{a}_{\overline{n}|}, & T > n \end{cases}$$

  - Premium formula:

$$\bar{P}(A_{x:\frac{1}{\overline{n}|}}) = \frac{A_{x:\frac{1}{\overline{n}|}}}{\bar{a}_{x:\overline{n}|}}$$

- n-yr deferred WL annuity:
  - Loss function:

$$L = \begin{cases} 0 - \pi\bar{a}_{\overline{T}|}, & T \le n \\ v^n\bar{a}_{\overline{T-n}|} - \pi\bar{a}_{\overline{n}|}, & T > n \end{cases}$$

  - Premium formula:

$$\bar{P}({}_{n|}\bar{a}_x) = \frac{A_{x:\frac{1}{\overline{n}|}}\,\bar{a}_{x+n}}{\bar{a}_{x:\overline{n}|}}$$

PREMIUMS PAID M TIMES A YEAR

Insurance Plan	Benefit paid	
WLI	**at the EOY of death**	$P_x^{(m)} = A_x / \ddot{a}_x^{(m)}$
	at the moment of death	$P^{(m)}(\bar{A}_x) = \bar{A}_x / \ddot{a}_x^{(m)}$
***n*-year term I**	**at the EOY of death**	$P^{(m)}_{\stackrel{1}{x}:\overline{n\rvert}} = A^{1}_{x:\overline{n\rvert}} / \ddot{a}^{(m)}_{x:\overline{n\rvert}}$
	at the moment of death	$P^{(m)}(\bar{A}^{1}_{x:\overline{n\rvert}}) = \bar{A}^{1}_{x:\overline{n\rvert}} / \ddot{a}^{(m)}_{x:\overline{n\rvert}}$
***n*-year endowment**	**at the EOY of death**	$P^{(m)}_{x:\overline{n\rvert}} = A_{x:\overline{n\rvert}} / \ddot{a}^{(m)}_{x:\overline{n\rvert}}$
	at the moment of death	$P^{(m)}(\bar{A}_{x:\overline{n\rvert}}) = \bar{A}_{x:\overline{n\rvert}} / \ddot{a}^{(m)}_{x:\overline{n\rvert}}$
***h*-pay, WHI**	**at the EOY of death**	${}_hP_x^{(m)} = A_x / \ddot{a}^{(m)}_{x:\overline{h\rvert}}$
	at the moment of death	${}_hP^{(m)}(\bar{A}_x) = \bar{A}_x / \ddot{a}^{(m)}_{x:\overline{h\rvert}}$
***h*-pay, *n*-year endowment**	**at the EOY of death**	${}_hP^{(m)}_{x:\overline{n\rvert}} = A_{x:\overline{n\rvert}} / \ddot{a}^{(m)}_{x:\overline{h\rvert}}$
	at the moment of death	${}_hP^{(m)}(\bar{A}_{x:\overline{n\rvert}}) = \bar{A}_{x:\overline{n\rvert}} / \ddot{a}^{(m)}_{x:\overline{h\rvert}}$

M4: Net Premium Reserves

NET PREMIUM RESERVES

- A reserve is money set aside to cover an insurer's future payments, i.e. benefits to policyholders and expenses.
  - Reserves show up as a liability item in the balance sheet
  - Increases in reserves are an expense item in the income statement
- Reserve calculations may vary because of:
  - Purpose of reserve valuation – statutory (solvency), GAAP (realistic, shareholders/investors), mergers/acquisitions
  - Assumptions and basis (mortality & interest)
- An actuary is responsible for preparing an actuarial opinion and memorandum: that the company's assets are sufficient to back reserves
- Reserves are more often called provisions in Europe
- We hold reserves because for many life insurance contracts:
  - The expected cost of paying the benefits generally increases over the contract term, but
  - The periodic premiums used to fund these benefits are level
- Prospective reserve:
  - At any future time t, the insurer's prospective loss is
  $$ {}_tL = PVFB_t - PVFP_t. $$
  - For most types of policies, it is generally true that for $t \geq 0$, ${}_tL \geq 0$, i.e. $\text{PVFB}_t \geq \text{PVFP}_t$
  (**Discussion** What might happen if the above inequality does not hold for some t?)
  - The prospective *reserves* at time t is given by
  $$ \begin{aligned} {}_tV &= E({}_tL \mid T(x) > t) \\ &= E(PVFB_t \mid T(x) > t) - E(PVFP_t \mid T(x) > t), \end{aligned} $$
    - The prospective reserve is the (smallest) amount for which the insurer is required to hold to be able to cover future obligations.
    - Note here the expectation is conditional on the survival of the life (x) at time t. Because otherwise, there is no need to hold reserves when policy has been paid out.
- The **net premium reserve** is the prospective reserve, where future expenses are ignored, and the premium used in the calculation is the notational net premium (determined by using the equivalence principle at time t = 0)

NET PREMIUM RESERVES | FULLY DISCRETE WLI

- Consider the case of a discrete whole life insurance where the net premium P_x is paid at the beginning of each year and benefit, $1, is paid at the end of the year of death.
- The prospective loss at time k (or age $x + k$) is:

$$ {}_kL = v^{K(x)-k+1} - P_x \ddot{a}_{\overline{K(x)-k+1|}} = v^{K(x)-k+1}(1 + \frac{P_x}{d}) - \frac{P_x}{d}, $$

for $K(x) = k, k+1, \cdots$

- The prospective reserve is thus:

  $_kV_x = A_{x+k} - P_x\ddot{a}_{x+k}$

- Variance:

  $Var(_kL|T(x) > k) = (1 + \frac{P_x}{d})^2({}^2A_{x+k} - (A_{x+k})^2)$

- Other formulas:
  - Ratio of Annuities:

    $$_kV_x = 1 - d\ddot{a}_{x+k} - (\frac{1}{\ddot{a}_x} - d)\ddot{a}_{x+k} = 1 - \frac{\ddot{a}_{x+k}}{\ddot{a}_x}$$

  - $$_kV_x = 1 - \frac{P_x + d}{P_{x+k} + d} = \frac{P_{x+k} - P_x}{P_{x+k} + d}$$

  - $$_kV_x = 1 - \frac{1 - A_{x+k}}{1 - A_x} = \frac{A_{x+k} - A_x}{1 - A_x}$$

  - Other net premium reserves with more obvious interpretations:

    $$_kV_x = 1 - (P_x + d)\ddot{a}_{x+k}$$

    $\Longrightarrow$ Net premium reserves equals sum insured \$1 minus expected PV of future premiums and unused interest.

  - Paid-up insurance formula:

    $$_kV_x = \left(1 - \frac{P_x}{P_{x+k}}\right) A_{x+k}$$

    $\Longrightarrow$ Net premium reserves is used to finance remaining face amount of $1 - P_x/P_{x+k}$.

  - Premium difference formula:

    $$_kV_x = (P_{x+k} - P_x)\ddot{a}_{x+k}$$

    $\Longrightarrow$ Net premium reserves is the shortfall of the premiums, if WLI were to be bought at age $x + k$.

NET PREMIUM RESERVES | OTHER FULLY DISCRETE ENDOWMENTS

- Prospective loss of an endowment insurance:
  - Consider an n-year endowment assurance policy to lives aged x. Assume that premiums are payable annually in advance and death benefits are payable at the end of year of death.
  - The insurer's prospective loss at time $k < n$ (or age $x + k$) is:

    $$_kL = \begin{cases} v^{K(x)-k+1} - P_{x:\overline{n}|}\ddot{a}_{\overline{K(x)-k+1}|} & K(x) = k, , \cdots , n-1 \\ v^{n-k} - P_{x:\overline{n}|}\ddot{a}_{\overline{n-k}|} & K(x) = n, n+1, \cdots \end{cases}$$

    $_nL = 1$ and loss is zero for $k > n$.

- Endowment insurance: the prospective reserve –
  - Recall that the insurer's prospective loss at time $k < n$ is:

$$
{}_kL = \begin{cases} v^{K(x)-k+1} - P_{x:\overline{n}|}\ddot{a}_{\overline{K(x)-k+1}|} & K(x) = k, , \cdots, n-1 \\ v^{n-k} - P_{x:\overline{n}|}\ddot{a}_{\overline{n-k}|} & K(x) = n, n+1, \cdots \end{cases}
$$

  - The prospective (net premium) reserve at time k is:

$$
{}_kV_{x:\overline{n}|} = E[{}_kL|K(x) > k] = A_{x+k:\overline{n-k}|} - P_{x:\overline{n}|} \cdot \ddot{a}_{x+k:\overline{n-k}|}
$$

  - Variance:

$$
Var({}_kL) = \left(1 + \frac{P_{x:\overline{n}|}}{d}\right)^2 \left[{}^2A_{x+k:\overline{n-k}|} - (A_{x+k:\overline{n-k}|})^2\right]
$$

- Other special formulas:
  - Premium difference formula

$$
{}_kV_{x:\overline{n}|} = (P_{x+k:\overline{n-k}|} - P_{x:\overline{n}|})\, \ddot{a}_{x+k:\overline{n-k}|}
$$

  - Paid-up insurance formula

$$
{}_kV_{x:\overline{n}|} = \left(1 - \frac{P_{x:\overline{n}|}}{P_{x+k:\overline{n-k}|}}\right) A_{x+k:\overline{n-k}|}
$$

- Other types of insurances:
  - n-year term:

$$
{}_kV^1_{x:\overline{n}|} = \begin{cases} A^1_{x+k:\overline{n-k}|} - P^1_{x:\overline{n}|}\ddot{a}_{x+k:\overline{n-k}|}, & k < n \\ 0, & k = n \end{cases}
$$

  - h-pay whole life:

$$
{}^h_kV_x = \begin{cases} A_{x+k} - {}_hP_x\ddot{a}_{x+k:\overline{h-k}|}, & k < h \\ A_{x+k}, & k \geq h \end{cases}
$$

  - h-yr pay, n-yr endowment ($h < n$):

$$
{}^h_kV_{x:\overline{n}|} = \begin{cases} A_{x+k:\overline{n-k}|} - {}_hP_{x:\overline{n}|}\ddot{a}_{x+k:\overline{h-k}|}, & k < h \\ A_{x+k:\overline{n-k}|}, & h \leq k < n \\ 1, & k = n \end{cases}
$$

  - n-yr pure endowment:

$$
{}_kV_{x:\overset{1}{\overline{n}|}} = \begin{cases} A_{x+k:\overset{1}{\overline{n-k}|}} - P_{x:\overset{1}{\overline{n}|}}\ddot{a}_{x+k:\overline{n-k}|}, & k < n \\ 1, & k = n \end{cases}
$$

  - n-yr deferred WL annuity:

$$
{}_kV({}_{n|}\ddot{a}_x) = \begin{cases} {}_{n-k|}\ddot{a}_{x+k} - P({}_{n|}\ddot{a}_x)\ddot{a}_{x+k:\overline{n-k}|}, & k < n \\ \ddot{a}_{x+k}, & k \geq n \end{cases}
$$

RETROSPECTIVE RESERVES

- A stream of cashflows may be considered as:
  - **Prospectively** – leading to calculation of present values. This is forward looking.
  - **Retrospectively** – leading to the calculation of accumulations. This is the accumulated value of past premiums and benefits.
- Retrospective accumulations:
  - Suppose that there are N^1 survivors at age $x+n$ out of N "starters" at age x, and that the accumulated funds at age $x+n$ is $F(N)$.
  - The retrospective value at time n is calculated by the accumulation of cashflows over the period $[0,n]$, assuming sharing the resulting fund equally among the survivors of the large group. That is, the retrospective value is defined to be

$$\lim_{N\to\infty} \frac{F(N)}{N^1}.$$

  - It is well known that

$$\lim_{N\to\infty} \frac{F(N)}{N} = E[F(1)] \quad \text{and} \quad \lim_{N\to\infty} \frac{N^1}{N} = {}_n p_x$$

  - So the retrospective accumulation is

$$\lim_{N\to\infty} \frac{F(N)}{N^1} = \frac{E[F(1)]}{{}_n p_x}.$$

- Retrospective accumulation for a term insurance:
  - Consider an n- year term insurance. The distribution for accumulated funds of benefit by a single life at time n can be represented as

$$F(1) = \begin{cases} (1+i)^{n-(k+1)} & \text{if } K_x = k \ (k = 0, 1, \cdots, n-1) \\ 0 & \text{if } K_x \geq n. \end{cases}$$

  - Note that $F(1) = 0$ if the life survives since there is no survival benefit for a term assurance contract.
  - Taking expectation to the above distribution we obtain

$$E[F(1)] = \sum_{k=0}^{n-1} (1+i)^{n-(k+1)} {}_{k|}q_x = (1+i)^n A^1_{x:\overline{n|}}$$

  - Hence the accumulation of the term assurance benefit at time n is

$$\frac{(1+i)^n A^1_{x:\overline{n|}}}{{}_n p_x} \equiv \frac{A^1_{x:\overline{n|}}}{v^n {}_n p_x}.$$

- Retrospective accumulation for an annuity:
  - Consider an annuity-due with a term of n years. The accumulation of the annuity by a single life at time n is $F(1)$:

$$F(1) = \begin{cases} (1+i)^{n-(k+1)}\ddot{s}_{\overline{k+1}|} & \text{if } K_x = k \ (k = 0, 1, \cdots, n-1) \\ \ddot{s}_{\overline{n}|} & \text{if } K_x \geq n. \end{cases}$$

  - Hence

$$\begin{aligned} E[F(1)] &= \sum_{k=0}^{n-1}(1+i)^{n-(k+1)}\ddot{s}_{\overline{k+1}|}\,{}_{k|}q_x + \ddot{s}_{\overline{n}|}\,{}_n p_x \\ &= (1+i)^n \left(\sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|}\,{}_{k|}q_x + \ddot{a}_{\overline{n}|}\,{}_n p_x \right) \\ &= (1+i)^n \ddot{a}_{x:\overline{n}|} \end{aligned}$$

  - The accumulation of an annuity becomes

$$\ddot{s}_{x:\overline{n}|} = \frac{(1+i)^n \ddot{a}_{x:\overline{n}|}}{{}_n p_x} = \frac{\ddot{a}_{x:\overline{n}|}}{v^n \ {}_n p_x}.$$

- Retrospective accumulation example:
  - Accumulated benefits for a life who purchased a life insurance policy at age x, and now aged $x+t$, where the sum assured for t years of the past cover was S, payable at the end of year of death is represented as

$$SA^{1}_{x:\overline{t}|}(1+i)^t / {}_t p_x.$$

  - The accumulated premiums for the life, assuming premiums of P per annum payable annually in advance is:

$$P\ddot{a}_{x:\overline{t}|}(1+i)^t / {}_t p_x = P\ddot{s}_{x:\overline{t}|}$$

- Retrospective reserves - The retrospective reserve is given by:
  - The accumulated value allowing for interest and survivorship of the premiums received to date
    LESS
  - The accumulated value allowing for interest and survivorship of the benefits and expenses paid to date
- Retrospective reserve of a whole life assurance:
  - Consider a fully discrete whole life insurance policy of a life aged x. Under the policy the sum assured is 1 and the annual level premium is determined by the equivalence principle.
  - The retrospective reserve at time k for this insurance is:

$${}_k V_x^R = P_x \ddot{s}_{x:\overline{k}|} - \frac{(1+i)^k}{{}_k p_x} A^{1}_{x:\overline{k}|}.$$

EQUIVALENCE BETWEEN PROSPECTIVE AND RETROSPECTIVE RESERVES

- Prospective and retrospective reserves are equal if:
    - The retrospective and prospective reserves are calculated on the same basis
    - This basis is the same as the basis used to calculate premiums used in the reserve calculation.
- These conditions are rare in practice since the assumptions that are appropriate for the retrospective calculation are not generally appropriate for the prospective calculation.
- The assumptions that were considered appropriate at the time the premium was calculated may not be appropriate for the retrospective or prospective reserve some years later.
- Retrospective reserve is based on actual experience in terms of interest rate, mortality, etc.
    - This will not always be the same as that assumed when premiums were set.
- Equality for reserve for a whole life assurance to a life aged *x*:
    - Assume that sum assured is 1.
    - The prospective reserve is:

$$ {}_tV_x^P = A_{x+t} - P_x \ddot{a}_{x+t}. $$

    - The retrospective reserve is:

$$ {}_tV_x^R = P_x \ddot{s}_{x:\overline{t}|} - \frac{(1+i)^t}{{}_tp_x} A^{1}_{x:\overline{t}|}. $$

    - First term in the retrospective reserve is interpreted as the expected accumulation of premiums received.
    - The second term is the expected accumulated cost of benefits paid.
    - Starting with the prospective reserve we proceed as follows:

$$ \begin{aligned} {}_tV_x^P &= A_{x+t} - P_x \ddot{a}_{x+t} \\ &= A_{x+t} - P_x \ddot{a}_{x+t} - P_x \ddot{s}_{x:\overline{t}|} + P_x \ddot{s}_{x:\overline{t}|} \end{aligned} \qquad (1.1) $$

    - But we know that:

$$ {}_{t|}\ddot{a}_x = v^t\, {}_tp_x \ddot{a}_{x+t} \quad \text{and} \quad \ddot{s}_{x:\overline{t}|} = \frac{(1+i)^t}{{}_tp_x} \ddot{a}_{x:\overline{t}|}. $$

    - Rearranging and substituting into (1.1) we obtain

$$ {}_tV_x^P = P_x \ddot{s}_{x:\overline{t}|} - \frac{(1+i)^t}{{}_tp_x} P_x \left({}_{t|}\ddot{a}_x + \ddot{a}_{x:\overline{t}|} \right) + A_{x+t} \qquad (1.2) $$

- Since ${}_{t|}\ddot{a}_x + \ddot{a}_{x:\overline{t|}} = \ddot{a}_x$ it follows that

$$ {}_tV_x^P = P_x\ddot{s}_{x:\overline{t|}} - \frac{(1+i)^t}{{}_tp_x}P_x\ddot{a}_x + A_{x+t} \qquad (1.3)$$

- From the equation of value, we have $P_x\ddot{a}_x = A_x$. So,

$$\begin{aligned} {}_tV_x^P &= P_x\ddot{s}_{x:\overline{t|}} - \frac{(1+i)^t}{{}_tp_x}A_x + A_{x+t} \qquad (1.4)\\ &= P_x\ddot{s}_{x:\overline{t|}} - \frac{(1+i)^t}{{}_tp_x}\left[A_x - v^t{}_tp_xA_{x+t}\right]. \end{aligned}$$

- Now, since $A_x - v^t{}_tp_xA_{x+t} = A^1_{x:\overline{t|}}$, we obtain

$$ {}_tV_x^P = P_x\ddot{s}_{x:\overline{t|}} - \frac{(1+i)^t}{{}_tp_x}A^1_{x:\overline{t|}},$$

which is same as the retrospective reserve formula.

RECURSIVE CALCULATIONS OF RESERVES

- Consider a general fully discrete insurance issued to (x) for which:
  - Death benefit is payable at the end of the policy year of death
  - Premiums are payable each year at the beginning of the year
  - Death benefit in the *j*-th policy year is b_j
  - Benefit (net) premium payment in the *j*-th policy year is P_{j-1}
- The net premium reserves at time *h* and time *h* + 1 are related by the recursive formula:

  $({}_hV + P_h)(1+i) = b_{h+1}\, q_{x+h} + {}_{h+1}V\, p_{x+h}$
- Other examples of equations of equilibrium:

  $({}_hV_{x:\overline{n|}} + P_{x:\overline{n|}})(1+i) = q_{x+h} + {}_{h+1}V_{x:\overline{n|}}\, p_{x+h}.$

  $({}_hV^1_{x:\overline{n|}} + P^1_{x:\overline{n|}})(1+i) = q_{x+h} + {}_{h+1}V^1_{x:\overline{n|}}\, p_{x+h}.$

  $({}_hV_{x:\overset{1}{\overline{n|}}} + P_{x:\overset{1}{\overline{n|}}})(1+i) = {}_{h+1}V_{x:\overset{1}{\overline{n|}}}\, p_{x+h}.$

DEATH STRAIN AND MORTALITY PROFIT

- Death strain at Risk (DSAR) –
  - The sum ${}_hV + P_h$ is called the **initial benefit reserve** for the policy year $h + 1$.
  - ${}_{h+1}V$ is called the **terminal benefit reserve** for year $h + 1$
  - The **death stain (DS)** in the policy year h to $h + 1$ is defined to be

$$\text{DS}_h = \begin{cases} 0, & \text{for a person alive} \\ b_{h+1} - {}_{h+1}V, & \text{for a person who dies} \end{cases}$$

  - The difference $b_{h+1} - {}_{h+1}V$ is called the net amount at risk (or called **the death strain at risk (DSAR)**).
    - If we think of reserve as money already set aside for the policyholder, DSAR is the amount of extra money that the company would need to pay if the policyholder died during that that policy year.
- The recursive formula can alternatively be written as:

$$({}_hV + P_h)(1 + i) = {}_{h+1}V + (b_{h+1} - {}_{h+1}V)q_{x+h},$$

  where the term $(b_{h+1} - {}_{h+1}V)\, q_{x+h}$ is called the **expected death strain (EDS)** (or expected net amount at risk).
- The observed value at *t + 1* of the random variable is called the actual death strain (ADS)
- The difference between the expected and actual death strain is called mortality profit
  - *Mortality Profit = EDS$_h$ - ADS$_h$*
- For a portfolio of identical policies:
  - *Total EDS = Expected No. of Deaths * DSAR*
  - *Total ADS = Actual No. of Deaths * DSAR*
  - *Total Mortality Profit = Total EDS – Total ADS*
- Mortality profit on a portfolio of policies –

  For a portfolio of policies with all lives are the same age, in the year $t + 1$ (from t to $t + 1$)
  - Total DSAR $= \sum_{\text{all policies at } t} (B_{t+1} - {}_{t+1}V)$
  - Total EDS $= \sum_{\text{all policies at } t} (B_{t+1} - {}_{t+1}V)q_{x+t}$
  - Total ADS $= \sum_{\text{death claims during } (t, t+1)} (B_{t+1} - {}_{t+1}V)$
  - In the above equations, the summation is over all policies that are in force at the *start* of the year
  - Total Mortality Profit = Total EDS- Total ADS
  - If the policies are identical:
    Total EDS= expected no. of deaths during the year $\times$ DSAR
    Total ADS= observed no. of deaths during the year $\times$ DSAR

- Premium decomposition –
  - The premium can be decomposed as
$$P_h = v\, q_{x+h}\, b_{h+1} \ + v\, p_{x+h}\ {}_{h+1}V - \ {}_hV$$
or
$$P_h = v\, q_{x+h}\, (b_{h+1} \ - \ {}_{h+1}V) + v\ {}_{h+1}V - \ {}_hV$$
  - The first term is called *risk premium*:
$$P_h^r = v\, q_{x+h}\, (b_{h+1} \ - \ {}_{h+1}V)$$
    - 1-year term insurance benefit premium for the net amount of risk $b_{h+1} \ - \ {}_{h+1}V$.
  - The second term is called *savings premium*:
$$P_h^s = v\ {}_{h+1}V - \ {}_hV$$
    - Used to increase the net premium reserve.

THIELE'S DIFFERENTIAL EQUATIONS

- Consider the case of a fully continuous whole life insurance policy with unit benefit issued to (*x*).
  - Previously, we learned
$${}_t\bar{V}(\bar{A}_x) = \bar{A}_{x+t} - \bar{P}(\bar{A}_x)\bar{a}_{x+t}$$
  - Denote ${}_t\bar{V}(\bar{A}_x)$ and $\bar{P}(\bar{A}_x)$ by ${}_t\bar{V}_x$ and $\bar{P}_x$, respectively so that now we have
$${}_t\bar{V}_x = \bar{A}_{x+t} - \bar{P}_x\bar{a}_{x+t}$$
  - Thiele's differential equation for the case of a fully continuous reserves (Page 37 of Tables):
$$\bar{P}_x + \delta\, {}_t\bar{V}_x \ = \ \frac{\partial\, {}_t\bar{V}_x}{\partial t} + \mu_{x+t}\, (1 - \ {}_t\bar{V}_x)$$
or
$$\frac{\partial\, {}_t\bar{V}_x}{\partial t} \ = \ \bar{P}_x + \delta\, {}_t\bar{V}_x - \mu_{x+t}\, (1 - \ {}_t\bar{V}_x)$$
- The Thiele's differential equation can be written as:
$$\frac{\partial\, {}_t\bar{V}_x}{\partial t} \ = \ \bar{P}_x + \delta\, {}_t\bar{V}_x + \mu_{x+t}\ \ {}_t\bar{V}_x - \mu_{x+t}$$
  - $\frac{\partial\, {}_t\bar{V}_x}{\partial t}$: the annual rate at which the reserve value for an in-force policy is increasing at duration *t*
  - $\delta\, {}_t\bar{V}_x$: the rate at which the reserve is increasing on account of interest at time *t*
  - $1\mu_{x+t}$ is the annul rate at which money is leaving the fund at time *t* because
    μ_{x+t} is the rate at which people are dying at time t
  - $\mu_{x+t}\ \ {}_t\bar{V}_x$:
    the annual rate at which reserves released by death cause the reserve per survivor to increase at time t

WITH-PROFIT (PARTICIPATING) CONTRACTS

- Conventional whole life or endowment policies can be issues on a without-profit or with-profit basis
- Without-profit contracts have premiums and benefits that are fixed and guaranteed at issue
- In contrast, for with-profit (participating) contracts, premiums and/or benefits can be varied according to the emerging surplus following a valuation
- For example, if there is a surplus of assets over liabilities following a valuation, then a portion of it might be used to reduce premiums and/or increase the benefits or sum insured.
- With-profit business has increased due to:
  - Uncertain interest rates, uncertain mortality rates and uncertain expenses
- With-profit contracts are generally priced more conservatively than without-profit contracts.
- For with-profit contracts, usually we expect the real experience is slightly better than assumed. We would expect the following events to have occurred:
  - Investment returns on assets are greater than assumed
  - Number of death claims is lower than assumed
  - The amount of expenses is lower than assumed
- The profits from with-profit contracts are shared between the policyholder and the insurer
- For all traditional participating insurance, only profits are shared, not losses.

TYPES OF BONUSES

- Some of the most common methods to distribute the profit share are:
  - Cash refunds – distributed at regular intervals, based on the profit emerging in the preceding year
  - Premium reductions – the profit allocated to the policyholder may be used to reduce the premiums due in the period
  - Increased death benefits – determined by applying the emerging profit to purchase additional death benefit cover
  - Additional benefits – these may be purchased, such as extra term insurance
- When surplus is distributed in the form of increased benefits, the additions to the sum assured are called bonuses
- Bonuses are awarded in two stages:
  - **Reversionary bonus** – these are awarded during the term of the contract. Once a reversionary bonus is awarded, it is guaranteed thereafter.
  - **Terminal bonus** – there are awarded when the policy matures. It is used to top up the sum insured when the benefit is finally paid.
- Reversionary and terminal bonuses are determined by the insurer based on the investment performance of the invested premiums.
- Methods of allocating (annual) reversionary bonuses –
  - *Simple* – the rate of bonus each year is a percentage of basic sum insured. Total sum insured then increases linearly over time.

- *Compound* – the rate each year is a percentage of the basic sum insured and the bonuses added in the past. Total sum therefore increases exponentially over time.
- *Super compound* – two different rates apply to the basic sum insured and the bonuses added in the past.

- Bonuses are determined by the emerging profit in the past
- Bonuses added to the death benefit and surrender benefit are the bonuses declared up to the time of death and surrender, respectively.
- Typical breakdown of benefits:
  - In the UK, a typical breakdown of the payment for a 20-year endowment policy is as follows:
    - Initial guaranteed sum insured: 20-25%
    - Total reversionary bonuses: 30-35%
    - Terminal bonus: 40-50%
    - Total: 100%
  - For a 10-year endowment policy, it has been as follows:
    - Initial guaranteed sum insured: 50-60%
    - Total reversionary bonuses: 20-30%
    - Terminal bonus: 15-25%
    - Total: 100%

NET PREMIUM CALCULATIONS | WITH-PROFIT CONTRACTS

- The net premium *P*, for a policy can be determined by:
  - Determining the EPV of future loss RV at the date of issue of policy
  - Setting this EPV equal to 0
  - Solving for *P*
- Consider a whole life policy with a sum assured of 1, premium *P* payable annually in advance, and simple bonus at 100*b*% p.a. Suppose that the first bonus is payable after one year and that deaths in the first year have no bonus payment:
  - The benefit is $(1 + bk)$, payable at the end of year of death if death occurs during age $(x + k, x + k + 1)$
  - The net future loss random variable at time $t = 0$, is:

$$_0L = (1 + bK)v^{K+1} - P\ddot{a}_{\overline{K+1|}}$$

  - The expected value of this random loss is

$$E(_0L) = (1 - b)A_x + b(IA)_x - P\ddot{a}_x$$

  - Therefore, the net annual premium *P* is given by

$$P = \frac{(1 - b)A_x + b(IA)_x}{\ddot{a}_x} = (1 - b)P_x + b\frac{(IA)x}{\ddot{a}_x}$$

- Net premium reserves for with-profit policies:
  - The net premium reserves is defined to be (A) – (B) where:
    - (A) is the expected PV of benefits, allowing for bonuses (declared) to date (but not future bonuses); and
    - (B) is the expected PV of net premiums, where these are calculated at the policy issue on the basic sum insured (ignoring all bonuses).
  - In effect, the net premium reserves for an ordinary whole life insurance contract is:

$$ {}_tV = (S + B_t)A_{x+t} - P\ddot{a}_{x+t} $$

    where S = the basic sum insured, B_t = the total value of bonuses added up to time t, and the net premium P is calculated as

$$ P = \frac{S \cdot A_x}{\ddot{a}_x}. $$

- Rationale for with-profit contracts:
  - Part (A) values all guaranteed sum insured to date, including those declared to date. There is no contractual entitlement for any future bonus, hence they are not included in the reserve calculations
  - Part (B) is conservatively calculated based on net premium, ignoring all bonuses because this is a more prudent and conservative approach – it leads to higher reserves than if bonuses were included
- For with-profit policies:
  - The calculations of prospective net premium reserves will be done according to mortality and interest assumptions specifically chosen for the purpose (referred to as the net premium valuation basis)
  - The net premium valuation basis will normally be different from the underlying basis used to calculate the premiums.

M5: Gross Premiums & Premium Reserves

EXPENSES & GROSS PREMIUMS

- In practice, different expenses are incurred by insurers.
- Investment-related expenses:
    - Analysis
    - Cost of buying
    - Selling
    - Servicing
- Insurance-related expenses:
    - Acquisition (agents' commission, underwriting, preparing new records)
    - Maintenance (premium collection, policyholder correspondence)
    - General (research, actuarial, accounting, taxes)
    - Settlement (claim investigation, legal defence, disbursement)
- There are three main types of expenses associated with policies:
    - **Initial expenses** – incurred by the insurer when a policy is issued. There are two main types of initial expenses: agents' commission and underwriting expenses. There expenses are usually incurred slightly ahead of the date when the first premium is payable. When we calculate a gross premium, it is conventional to assume that the insurer incurs there expenses at exactly the same time that the first premium is payable.
    - **Renewal or maintenance expenses** – Are normally incurred by the insurer each time a premium is payable.
    - **Termination or claim expenses** – occur when a policy expires, typically on the death of a policyholder or on the maturity date of the insurance.
- Gross premium calculations can be done using the principle of equivalence:
    - APV (Future Gross Premiums) = APV (Future Benefits + Expenses)

GROSS PREMIUM RESERVES CALCULATION

- Gross premium reserves at time *t* is the difference between:
    - EPV at time *t* of future benefits and expenses, and
    - EPV at time *t* of future premiums
- We observe generally that:
    - Gross premium reserve < Net premium reserve
- It is more conservative for an insurer to obtain the reserves based on net premium than based on gross premium.

- Consider a 5-year endowment policy issued to a life aged 60 with an initial expense of \$4.50 and the following characteristics:

    1. Sum insured is \$100 payable at the end of the year of death or on maturity;
    2. Renewal expense, including the first, of \$1.50 per policy;
    3. Valuation basis: A1967-70 Ultimate, 4% interest.

    - Denote by G' the gross annual premium
    - According to equivalence principle:

$$G'\ddot{a}_{60:\overline{5}|} = 100A_{60:\overline{5}|} + 1.5\ddot{a}_{60:\overline{5}|} + 4.50$$

$\Rightarrow$

$$\begin{aligned} G' &= 100\frac{A_{60:\overline{5}|}}{\ddot{a}_{60:\overline{5}|}} + 1.5\frac{\ddot{a}_{60:\overline{5}|}}{\ddot{a}_{60:\overline{5}|}} + \frac{4.50}{\ddot{a}_{60:\overline{5}|}} \\ &= P + 1.5 + \frac{4.50}{\ddot{a}_{60:\overline{5}|}} = 18.43 + 1.5 + \frac{4.50}{\ddot{a}_{60:\overline{5}|}} = 20.93 \end{aligned}$$

- Observe the "amortizing" effect of the first year expense over the contract life time.
- The gross premium reserve after t years allowing for initial and renewal expenses becomes:

$$\begin{aligned} & {}_tV' \\ =\; & (100A_{60+t:\overline{5-t}|} + 1.5\ddot{a}_{60+t:\overline{5-t}|}) - G'\ddot{a}_{60+t:\overline{5-t}|} \\ =\; & 100A_{60+t:\overline{5-t}|} + 1.5\ddot{a}_{60+t:\overline{5-t}|} - \quad P + 1.5 + \frac{4.50}{\ddot{a}_{60:\overline{5}|}}\Bigg)\ddot{a}_{60+t:\overline{5-t}|} \\ =\; & 100A_{60+t:\overline{5-t}|} - \quad P + \frac{4.50}{\ddot{a}_{60:\overline{5}|}}\Bigg)\ddot{a}_{60+t:\overline{5-t}|} \\ =\; & 100A_{60+t:\overline{5-t}|} - P\ddot{a}_{60+t:\overline{5-t}|} - (\frac{4.50}{\ddot{a}_{60:\overline{5}|}})\ddot{a}_{60+t:\overline{5-t}|} \\ =\; & \text{Net premium reserve } - 4.50\frac{\ddot{a}_{60+t:\overline{5-t}|}}{\ddot{a}_{60:\overline{5}|}} \end{aligned}$$

- General Situation – Consider an endowment policy of term n years. Assume expense loadings consist of:
    - First year expense: \$ I
    - Renewal expenses: k% of premium (including the first year's) and \$$c$ p.a. (including the first year's).
- Denote by G the gross annual premium:

$$G\ddot{a}_{x:\overline{n}|} = A_{x:\overline{n}|} + I + (k/100)G\ddot{a}_{x:\overline{n}|} + c\ddot{a}_{x:\overline{n}|}$$

$\Rightarrow$

$$G = \frac{1}{(1-k/100)} \quad P_{x:\overline{n}|} + \frac{I}{\ddot{a}_{x:\overline{n}|}} + c\Bigg).$$

ZILMERIZED RESERVES

- Modified reserves – Reserves which account for the first-year expenses.
- Consider an endowment policy of term *n* years. Assume expenses loadings consist of:
  - First year expense: I
  - Renewal expenses: k% of premium (including the first year's) and c p.a. (including the first year's).

  Note

$$G = \frac{1}{(1-k/100)}\left(P_{x:\overline{n}|} + \frac{I}{\ddot{a}_{x:\overline{n}|}} + c\right).$$

  and the modified reserve after t years is

$$\begin{aligned}
{}_tV^{\text{mod}} &= \left\{A_{x+t:\overline{n-t}|} + ((k/100)G + c)\ddot{a}_{x+t:\overline{n-t}|}\right\} - G\ddot{a}_{x+t:\overline{n-t}|} \\
&= A_{x+t:\overline{n-t}|} + c\ddot{a}_{x+t:\overline{n-t}|} - G(1-k/100)\ddot{a}_{x+t:\overline{n-t}|} \\
&= A_{x+t:\overline{n-t}|} + c\ddot{a}_{x+t:\overline{n-t}|} - (P_{x:\overline{n}|} + \frac{I}{\ddot{a}_{x:\overline{n}|}} + c)\ddot{a}_{x+t:\overline{n-t}|} \\
&= A_{x+t:\overline{n-t}|} - (P_{x:\overline{n}|} + \frac{I}{\ddot{a}_{x:\overline{n}|}})\ddot{a}_{x+t:\overline{n-t}|} = {}_tV_{x:\overline{n}|} - I\frac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x:\overline{n}|}}
\end{aligned}$$

  - Here ${}_tV_{x:\overline{n}|}$ is net premium reserve, $(P_{x:\overline{n}|} + \frac{I}{\ddot{a}_{x:\overline{n}|}})$ is the "Zillmerized premium" for the policy, I is often referred to as a "Zillmer expense" and $I\frac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x:\overline{n}|}}$ is referred to as a "Zillmer adjustment".
  - The premium reserve, ${}_tV^{\text{mod}}$, calculated using this modified net premium

$$(P_{x:\overline{n}|} + \frac{I}{\ddot{a}_{x:\overline{n}|}})$$

    is known as a "Zillmerized reserve".
  - Noting the net premium reserve

$${}_kV_{x:\overline{n}|} = A_{x+t:\overline{n-t}|} - P_{x:\overline{n}|}\ddot{a}_{x+t:\overline{n-t}|}.$$

    we conclude

    Zillmerized reserve $\leq$ Net premium reserve

- In many countries, the authorities have made laws specifying methods of calculating reserves for official purposes, including the extent to which reserve values may be modified.

SURRENDER VALUES

- Many policies contain nonforfeiture clauses which provide for cash surrender values. Nonforfeiture benefits are those benefits that will not be lost because of the premature cassation of premium payments, e.g., withdrawal benefits).
- Minimum cash surrender values are usually imposed by regulatory bodies.
- Terminal reserves are used to determine the appropriate policy values; hence cash surrender values are computed similar to reserves.
- The cash surrender value is generally smaller than the terminal reserve: ${}_kCV \leq {}_kV$, the difference is called the surrender charge.

- Surrender values can be computed using retrospective or prospective formulas.
  - For example in the prospective case, we have

$$\begin{aligned} & {}_t\mathrm{CV} \\ = \; & \mathrm{APV(Future\ Benefits)} - \mathrm{APV(Future\ Adjusted\ Premiums)} \\ = \; & A(t) - P^a\, \ddot{a}(t) \\ = \; & {}_tV - (P^a - P)\, \ddot{a}(t) \\ & \textit{or} \end{aligned}$$

$${}_t\mathrm{CV} = {}_t\mathrm{V} - {}_t\mathrm{SC}, \text{ where } {}_t\mathrm{SC} \text{ is the surrender charge}$$

  - In the retrospective case, we have

$$\begin{aligned} {}_t\mathrm{CV} = \; & \text{Accumulated Value (past adjusted premiums)} \\ & - \text{Accumulated Value (past benefits)} \end{aligned}$$

- The calculation basis (mortality/interest assumptions) may be different from the reserve calculation – for conservatism.
- Premiums may also be adjusted to recoup expenses, especially the large first-year initial expense.
- Surrender options:
  - Cash values are available as a lump sum or as an insurance benefit of equal APV.
  - In lieu of receiving cash, some alternative options are generally available such as *paid-up insurance* or *extended term insurance*.
- Paid-up Insurance:
  - In a paid-up insurance surrender option, the idea is to provide for a reduced amount of insurance which becomes paid-up. No further premium is required from surrender date onwards.
  - The general equation for the amount of paid-up insurance, denoted by b_k at time k, is:

$$\begin{aligned} {}_k\mathrm{CV} &= b_k\, A(k) \\ b_k &= \frac{{}_k\mathrm{CV}}{A(k)} \end{aligned}$$

where $A(k)$ is APV(Future Benefits at time k).

- Extended term insurance:
  - In an extended term insurance, the idea is to provide for a term insurance protection, the length of which is determined depending on surrender value. The amount of insurance is therefore maintained.
  - The general equation for the extended term denoted by *s* at time *k*, is:

$${}_k\mathrm{CV} = A^{1}_{x+k:\overline{s}|}.$$

  - If a policy of amount *b* is subject to an outstanding policy loan of amount *L* at the time *k* of premium default, the equation for the extended term denoted by *s* at time *k*, is:

$$b\, {}_k\mathrm{CV} - L = (b - L) A^{1}_{x+k:\overline{s}|}.$$

THE PORTFOLIO PERCENTILE PREMIUM PRINCIPLE

- The portfolio percentile premium principle is an alternative to the equivalence premium principle.
- Consider a large portfolio of *n* identical and independent policies:
  - The policies have the same premium, benefits, term and so on and the policyholders are all subject to the same survival model.
  - The policyholders are independent of each other with respect to mortality.

- Let $L_{0,i}$ represent the future loss random variable for the ith policy in the portfolio.
- $L_{0,i}$, $i = 1, 2, \cdots, n$ are iid
- Let L represent the total future loss in the portfolio.
- Then $L = \sum_{i=1}^{n} L_{0,i}$, $E[L] = nE[L_{0,1}]$, and $Var(L) = nVar(L_{0,1})$.
- The portfolio percentile premium principle **sets the premium so that $Pr(L < 0) = \alpha$, where α is a specified number** .
- Note $Pr(L < 0) = Pr(\frac{L-E[L]}{\sqrt{Var(L)}} < \frac{0-E[L]}{\sqrt{Var(L)}}) \approx \Phi(\frac{-E[L]}{\sqrt{Var(L)}})$, where Φ is the cumulative distribution function of the standard normal distribution.
- **So the premium can be calculated from $\Phi(\frac{-E[L]}{\sqrt{Var(L)}}) = \alpha$, i.e., $\frac{E[L]}{\sqrt{Var(L)}} = -\Phi^{-1}(\alpha)$.**
- Although, the premium, P, does not appear explicitly in the last equation, $E[L]$ and $Var(L)$ are both functions of P.
- The premium depends on the number of policies in the portfolio and the the level of probability we set for the future loss being negative (α).

M6: Profit Testing

INTRODUCTION TO PROFIT TESTING

- Profit testing – the process of analysing emerging costs or cash flows and assessing the profitability.
- The purpose is to identify the profit at the end of each time period.
- Some of the ways in which profit tests are applied in practice:
  - To set premiums
  - To set reserves
  - Measure profitability
  - Stress test profitability
  - Determine distributable surplus
- The process begins by projecting expected cash flows.
- The following information is needed to calculate expected cash flows:
  - Premiums to be paid and when they are paid
  - Expenses and when they are incurred
  - Contingent benefits payable under the contract, e.g. death benefits, annuity payments, etc
  - Other possible benefits such as cash surrender values
  - Taxes, investment fees, provisions for the contract
- The year-end revenue account each year is:

  (+) reserves at the beginning of the year prior to the premium payment
  (+) premium incomes
  (+) investment income
  (–) expenses
  (–) benefit payments (death claims, maturities)
  (–) terminal provisions (reserves) of the year
  = profit gross of tax
- Alternatively:

  (+) premium incomes
  (+) investment income
  (–) expenses
  (–) benefit payments (death claims, maturities)
  (–) increase in provisions/reserves (= reserves at the end of the year – reserve at the start of the year prior to premium income and its interest)
  = profit gross of tax
- Multiple decrement table – A computational tool for dealing with a population subject to more than one independent decrement.
- Examples of decrements:
  - Death and withdrawal
  - Employed, retired, disability, death

- Assume there are m decrements. Some notation:
  - $(al)_x$: total living at age x;
  - $(ad)_x^k$: number of lives removed between x and $x+1$ due to decrement k;
  - $$(al)_{x+1} = (al)_x - \sum_{k=1}^{m}(ad)_x^k$$
- Probabilities:

$$(aq)_x^k = (ad)_x^k/(al)_x$$
$$(aq)_x = \sum_{k=1}^{m}(aq)_x^k$$
$$(ap)_x = 1 - (aq)_x = 1 - \sum_{k=1}^{m}(aq)_x^k$$
$${}_t(ap)_x = (al)_{x+t}/(al)_x = (ap)_x \cdot (ap)_{x+1} \cdots (ap)_{x+t-1}$$

- Construction of the multiple decrement table:

x	$(al)_x$	$(ad)_x^1$	$\cdots$	$(ad)_x^m$	$(aq)_x^1$	$\cdots$	$(aq)_x^m$	$(ap)_x$
.	.	.		.	.		.	.
.	.	.		.	.		.	.
.	.	.		.	.		.	.
⋮	⋮	⋮		⋮	⋮		⋮	⋮
.	.	.		.	.		.	.
.	.	.		.	.		.	.

  - For profit testing purposes, only the values of $(aq)_x^1$, ..., $(aq)_x^m$ and $(ap)_x$ are what we need.

CASH FLOW PROJECTION OF A CONVENTIONAL PRODUCT

- As an example, consider a 5-year regular premium endowment insurance policy issued to a 55-year-old male:
  - Sum insured is \$10,000 payable at the end of the year of death;
  - Expenses: 1st year is 50% of annual premium, renewal expenses are 5% of subsequent premiums;
  - Premiums are payable annually in advance;
  - Surrender benefit is return of premiums paid, with no interest;
  - Reserves based on net premium reserve valuation;
  - Pricing and reserve valuation basis: AM92 ultimate, 4% p.a. interest.

- The double decrement table:

Age x	$(aq)_x^d$	$(aq)_x^w$
55	0.005	0.10
56	0.006	0.05
57	0.007	0.05
58	0.008	0.01
59	0.009	0.00

- Steps in the projection:
  - Compute the gross annual premium payable (if not already given):

$$G = \frac{10,000A_{55:\overline{5|}}}{0.95\ddot{a}_{55:\overline{5|}} - 0.45} = 2,108.81$$

  - Next, compute the emergence of yearly provisions/reserves. One can use the formula

$$_tV = 10,000\left(1 - \frac{\ddot{a}_{55+t:\overline{5-t|}}}{\ddot{a}_{55:\overline{5|}}}\right)$$

    and so we have $_0V = 0$ and

$$\begin{aligned} {}_1V &= 10,000\left(1 - \frac{\ddot{a}_{56:\overline{4|}}}{\ddot{a}_{55:\overline{5|}}}\right) = 1,832.06 \\ {}_2V &= 10,000\left(1 - \frac{\ddot{a}_{57:\overline{3|}}}{\ddot{a}_{55:\overline{5|}}}\right) = 3,740.46 \\ & \text{etc.} \end{aligned}$$

For a policy in force at the start of tth year:

The tth year	1st year	2nd year
(1) Value at start of the year	0	1832.06
(2) Premium (+)	2108.81	2,108.81
(3) Expenses (-)	1054.41	105.44
(paid at start of year)		
(4) Interest (+)	42.18	153.41
(5) Expected death claims (-)		
$=(DB_t + expenses)(aq)_{x+t-1}^d$	50	60
(6) Expected surrender value (-)		
$=(surrender\ benefit_t + expenses)(aq)_{x+t-1}^w$	210.88	210.88
(7) Expected reserve		
needed at year end (-)	1639.69	3530.99
Profit	-803.99	186.97

UNIT-LINKED PRODUCTS

- Key features of unit-linked products:
  - Premiums, after deducting expenses, are invested in investment funds, which form the policyholders' (unit) fund.
  - Regular management charges are deducted from the unit fund and paid into the insurer's fund (non-unit fund)
  - On survival to the end of the contract term the benefit may be just the policyholder's fund and no more, or there may be a guaranteed minimum maturity benefit.

- On death during the term of the policy, the policyholder's estate would receive the policyholder's fund, possibly with an extra amount. There may also be a guaranteed minimum death benefit.

- Also:
  - The policyholder's funds are not mixed with the general assets of the company but are held separately.
  - The policyholder's funds do not contribute to the insurer's profit or less; investment gains and losses are all passed straight to the policyholder.
  - Premiums are allocated to the unit fund and the non-unit fund in specified proportions.
  - Units are bought at the *offer* price and sold at the *bid* price.
  - Bid/offer spread (usually a percentage of premium) – This initial charge accrues to the non-unit fund.
  - Fund management charge is assessed periodically – a percentage deduction from the bid value of the units to cover investment expenses of the insurer accrues to the non-unit fund.
- Further notes:
  - The unit fund is only worth the bid value of the allocated premium.
  - Unit benefits are given to policyholder on surrender, claim or maturity.
  - Non-unit benefits include, death benefit or maturity in excess of the value of the units.
  - The unit fund is what the policyholder sees. The policyholder does not see anything at the non-unit fund level.
  - Sources of income in the non-unit fund: unit-fund charges, unallocated premium, bid/offer spread.
  - Sources of outgo in the non-unit fund: expenses, non-unit benefits.
  - The profit or loss to the company is the balance in the non-unit fund.

CASH FLOW PROJECTION FOR UNIT-LINKED PRODUCTS

- We need to separate the cash flows into the unit and non-unit funds.
- It's the non-unit fund cash flows that are important in pricing, reserving and projections.
- The insurer's income and outgo depend on the balance in the unit fund. Therefore we must first project the cash flows for the unit fund and then use these to project the cash flows in the non-unit fund.
- The projected cash flows for the non-unit fund can then be used to calculate the probability of the contract.
- Mortality charges:
  - F_t, the value of the unit fund
  - On death, the policyholder receives the maximum of the value of the unit fund and a guaranteed minimum payment B_t.
  - Example: Death benefit $= \max(F_t, B_t) = F_t + \max(0, B_t - F_t)$.

- Since the insurer must pay for the additional benefit above the unit fund, the insurer charges all policies for its Expected Death Strain.
- Extra mortality cost (Mortality charge): $MC = (aq)^d_{x+t-1} \cdot (B_t - F_t)$ if $B_t > F_t$ and $MC = 0$ if $B_t \leq F_t$.
- MC is deducted periodically say monthly, annually.

- Illustrative Example:
  - Consider a male select life aged 40 who purchases a *four*-year unit-linked endowment insurance contract with annual premium of \$1,000, payable in advance.
  - In the first year, 50% of the premium is allocated to units and 102.5% in the second and subsequent years. Units are subject to a bid/offer spread of 5%.
  - Annual management charge of 0.5% of the bid value of the units is deducted at the end of each year.
  - Death benefit:
    - paid at the end of the year of death
    - the maximum of \$4,000 and the bid value of the units
    - paid after deduction of mngt charge and mortality charge
  - Surrender benefit:
    - bid value of the units is payable at EOY of surrender
  - Survival benefit:
    - Bid value of the units is payable to the end of the contract term

- Assumptions:

Rate of interest on unit investments:	6% per annum
Rate of interest on non-unit fund cashflows:	4% per annum

Initial expenses: \$150

Renewal expenses: \$50 per annum on the second and subsequent premium dates

Initial commission:	20% of the 1st year premium
Renewal commission:	2.5% of the 2nd and subsequent premiums
Risk discount rate:	8% per annum

- The company assumes the following decrement table

x	$(aq)^d_x$	$(aq)^w_x$	$(ap)_x$	${}_{x-40}(ap)_x$
40	0.000749	0.099961	0.899291	1.000000
41	0.000938	0.049976	0.949086	0.899291
42	0.001076	0.049972	0.948951	0.853504
43	0.001178	0.049970	0.948852	0.809934

- Projecting the **unit** fund:

	Year 1	Year 2	Year 3	Year 4
Value of units at start of the year	0.000	500.983	1,555.400	2,667.495
Premium allocated	500	1,025	1,025	1,025
B/O spread	25.000	51.250	51.250	51.250
Interest	28.500	88.484	151.749	218.475
Management charge	2.518	7.816	13.404	19.299
Value of units at end of the year	500.983	1,555.400	2,667.495	3,840.421

- Projecting the **non-unit** fund:

	Year 1	Year 2	Year 3	Year 4
Premium unallocated (+)	500	-25	-25	-25
B/O spread (+)	25.000	51.250	51.250	51.250
Expenses/comm (-)	350	75	75	75
Interest (+)	7.000	-1.950	-1.950	-1.950
Mngt charge (+)	2.518	7.816	13.404	19.299
Expected extra mortality cost (-)	2.619	2.293	1.434	0.188
Expected extra surrender/maturity cost	0	0	0	0
End of the year cash flow	181.898	-45.177	-38.730	-31.589

PROFIT MEASURES & PROFIT TESTING PRINCIPLES

- **Profit vector** – the vector of the expected profits per policy in force at the start of the year
- **Profit signature** – the vector of the expected profits per policy in force at inception
- Example:
  - Suppose the end of year cash flow is:

	Year 1	Year 2	Year 3	Year 4
End of the year cash flow	181.898	-45.177	-38.730	-31.589

  - The profit vector is: $(181.898, -45.177, -38.730, -31.589)$
  - The profit signature is as follows:

	Year 1	Year 2	Year 3	Year 4
probability in force	1.0000	0.89929	0.85350	0.80993
Profit signature	181.898	-40.627	-33.056	-25.585

- Summary measures of profit:
  - **Net Present Value (NPV)** – The present value of the profit signature, discounted using the risk discount rate.
  - **Profit Margin** – The NPV of the profit signature divided by the EPV of the premium income.
  - *Risk Discount Rate = Risk-free Rate + Margin for Risk*

- Example:

	Year 1	Year 2	Year 3	Year 4
Profit vector	181.898	-45.177	-38.730	-31.589
probability in force	1.0000	0.89929	0.85350	0.80993
Discount factor	0.92592	0.85733	0.79383	0.73502
Profit signature	181.898	-40.627	-33.056	-25.585
EPV profits	168.424	-34.831	-26.241	-18.806
EPV premiums	1,000.000	832.6766	731.74245	642.95174

  - $NPV = 88.54607$
  - $Profit\ Margin = \frac{NPV}{Total\ EPV} = \frac{88.54607}{3207.370861} = 2.76\%$
- Objective of profit testing for conventional business:
  - Set adequate premiums
  - To meet profit criteria under this premium basis
- Objectives of profit testing for unit-linked business:
  - Set adequate charges
  - To meet profit criteria under the premium basis.

ZEROISATION

- Once sold and funded at the outset, a product should be self-supporting.
- That is, the profit signature has a single negative value at inception, which is termed as a "single financing phase at the outset".
- For those products that give profit signature more than one financing phase,
  - Reserves should be established in the non-unit fund at earlier durations to reduce later negative cash flows.
  - These reserves are funded by reducing earlier positive cash flows.
- Good financial management dictates that these reserves should be established as late as possible.
- Example:
  - Consider a 4 year unit-linked endowment insurance contract issued to (40) with the following end of year cash flows in the non-unit account:

	Year 1	Year 2	Year 3	Year 4
End of the year cash flow (CF)	181.898	-45.177	-38.730	-31.589

  - Suppose the non-unit growth rate is 4% and the multiple decrement table is:

x	$(aq)_x^d$	$(aq)_x^w$	$(ap)_x$	$_{t-1}(ap)_x$
40	0.000749	0.099961	0.899291	1.000000
41	0.000938	0.049976	0.949086	0.899291
42	0.001076	0.049972	0.948951	0.853504
43	0.001178	0.049970	0.948852	0.809934

- Reserve at policy duration 4: ${}_4V = 0$.
- Need to set up reserves at policy duration 3 to cover the negative cashflow in year 4:

$$ {}_3V \cdot 1.04 \quad = \quad -[(-31.589) - (ap)_{43} \times {}_4 V] \implies {}_3V = 30.374 $$

  The end of year cashflow at time 4 becomes 0.
- The end of year cashflow at time 3 now (after reserving) becomes

$$ -38.730 - (ap)_{42} \times {}_3 V < 0. $$

  So we need to set up reserves at time 2 to zeroize the negative cashflow in year 3:

$$ {}_2V \cdot 1.04 = -[-38.730 - (ap)_{42} \times {}_3 V] \implies {}_2V = 64.9552 $$

  The end of year cashflow at time 3 becomes 0.
- The end of year cashflow at time 2 after subtracting reserves becomes

$$ -45.177 - (ap)_{41} \times {}_2 V < 0 $$

  So we need to set up reserves at time 1 to zeroize the negative cashflow at time 2:

$$ {}_1V \cdot 1.04 = -[-45.177 - (ap)_{41} \times {}_2 V] \implies {}_1V = 102.7164. $$

  The end of year cashflow at time 2 becomes 0.
- The end of year cashflow at time 1 after subtracting reserves becomes

$$ 181.898 - (ap)_{40} \times {}_1 V = 89.526 $$

- After zeroisation, the end of year cashflows for years 1, 2, 3 and 4 become: 89.526, 0, 0,0.

- Now suppose the expected PV of premiums is 3207.37
- The revised table is as follows:

	Year 1	Year 2	Year 3	Year 4
probability in force	1.0000	0.89929	0.85350	0.80993
revised end of the year cash flow	89.526	0	0	0
discount factor	0.92592	0.85733	0.79383	0.73502
expected p.v. of profits	82.8946	0	0	0
expected p.v. premiums	3,207.370			
profit margin	2.58%			

- General formula for zeroisation:
    - Let CF_k represents the end of year cash flow for year k ignoring reserves given that the policy is in force at the beginning of the year k
    - The reserve at the end of year k using zeroisation is determined by

$$_kV(1 + i_{non-unit}) = |\min((CF)_{k+1} - {}_{k+1}V \cdot (ap)_{x+k}, 0)|$$

    - The end of year cashflow for year k after zeroisation given that the policy is in force at the beginning of the year k:

$$\max((CF)_k - {}_kV \cdot (ap)_{x+k-1}, 0)$$

ZEROISATION FOR CONVENTIONAL PRODUCTS

- Reserves for a conventional life assurance (i.e. non-unit linked) policy can also be determined by zeroising negative cash flows:
    - Project cashflows ignoring reserves.
    - Establish reserves by zeroising negative cashflows.
- For a policy that has a cashflow pattern ignoring reserves, where only the final cashflow is negative, calculating reserves using any of the following will give the same result:
    - EPV of future outgo – EPV of future income
    - Discounted future cashflows
    - Zeroisation of future negative cashflows
- The above is also true when cashflows become increasingly negative over time but is NOT true when the last cashflow is positive.
- Usually we:
    - Calculate cashflows ignoring reserves on a realistic basis
    - Calculate reserves on a prudent basis
    - Use the profit testing cashflows to set the premiums so that we achieve the profitability criterion

UNIVERSAL LIFE INSURANCE

- **Universal Life Insurance** – Generally issued as a whole life contract but with transparent cash values allowing policyholders the flexibility to use it more as an endowment insurance. Policyholder may vary the amount and timing of premiums (within some constraints)
- Key features:
    - **Premiums** are highly flexible but may be subject to some minimum level and payment term. Premiums are deposited in a notional account (assets are not segregated from the insurer's general funds).
    - **Expense charges/Management Expense Rate (MER)** are expressed as a percent of account value (or premiums) and may also include a flat fee. It is deducted from the account value of the notional account.
    - **Cost of insurance (CoI)** is the charge deducted from the policyholder's funds to cover the cost of the death benefit cover.

- The growth/interest rate of the notional account is the credited interest rate determined by the insurer.
    - The insurer shares the profits through the credited interest rate which is declared and applied by the insurer at regular intervals.
    - Credited interest is usually determined at the insurer's discretion but may be based on published rates.
    - A minimum guaranteed annual credited interest rate is specified in the policy contract, regardless of the investment performance of the insurer's assets.
- Account balance/value – The balance of funds in the notional account
- The account value is used to determine the death and survival benefits (surrender benefits), and it represents the insurer's liability
- Death Benefit – Equals the account value of the policy, plus an additional death benefit (ADB)
    - The ADB is required to be a significant portion of the total death benefit, except at very advanced ages, to justify the policy being considered as an insurance contract.
    - The proportions are set through the corridor factor requirement which sets the minimum value for the ratio of the total death benefit to the account value at death.
    - There are two types of death benefits.
    - *Type A*: Total death benefit (i.e. *Account Value + ADB = A Constant*). As the account value increases, the ADB decreases.
    - *Type B*: Level ADB. (i.e. *Total death benefit = Account Value + Level ADB*). Level ADB is selected by the policyholder subject to the corridor factor requirement.
- Surrender Value – If the policyholder chooses to surrender the policy, the surrender value paid will be the policyholder's account balance minus a surrender charge (or zero if greater)
- No-lapse Guarantee – Some policies have this no-lapse guarantee as an additional feature, under which the death benefit cover continues even if the account value declines to 0, provided that the policyholder pays a pre-specified minimum at each premium date.
- The policyholder can see their account (the notional account) value growing and can identify the expense and cost of insurance deductions. They can also see the credited interest rate.

PROFIT TESTING FOR UNIVERSAL LIFE INSURANCE

- Notation (*Type A Death Benefit*):
  - AV_t denotes the policyholder's account value at time t
  - EC_t denotes the expense charge deducted from the account value at the beginning of the tth year
  - CoI_t denotes the cost of insurance deducted from the account value at the beginning of the tth year
  - i_t^c denotes the credited interest rate applied to investment during the tth year
  - P_t denotes the premiums paid at the start of the tth year
  - DB_t denotes the death benefit cover in the tth year
  - CV_t denotes the cash value paid on surrender at the end of the tth year
- The fundamental equation:

  $$AV_t = (AV_{t-1} + P_t - EC_t - CoI_t)(1 + i_t^c)$$
- Also:

  $$ADB_t = DB_t - AV_t$$
  - The CoI charge can be considered as a single premium for a one-year term insurance for a death benefit of of ADB_t.
  - The CoI basis will be specified.
  - Let q_{x+t}^* denote the CoI mortality rate and i_q denote the CoI interest rate. If expenses are ignored,

  $$CoI_t = q_{x+t}^* ADB_t / (1 + i_q)$$