Automorphic Vector Bundles on Shimura Varieties

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2023

Contents

1 Shimura Varieties

A good reference with examples is [Mil05], though the structure of exposition follows [Har13].

1.1 Shimura Data

We fix some notation. Let $\mathbb{S} = R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}}$ denote the Deligne torus. Then we have

$$\mathbb{S}(R) = \big(R[j]/\left\langle j^2+1\right\rangle\big)^\times \cong (R \oplus Rj)^\times,$$

and $\mathbb{S}_\mathbb{C} \cong \mathbb{G}_{m,\mathbb{C}}^2$ under the isomorphism

$$(R \oplus Rj)^\times = \mathbb{S}_\mathbb{C}(R) \xrightarrow{\sim} \mathbb{G}_{m,\mathbb{C}}^2 = (R^\times)^2, \quad a + bj \mapsto (a + bi, a - bi)$$

where $i \in \mathbb{C} \subset R$ is a square root of unity. Let $X^*(-) := \mathrm{Hom}(-, \mathbb{G}_m)$ and $X_*(-) := \mathrm{Hom}(\mathbb{G}_m, -)$ be the character and co-character groups. Then $X^*(\mathbb{S})_\mathbb{C} \cong \mathbb{Z}^2$ with basis $X^*(\mathbb{S})_\mathbb{C} \cong \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ where $e_1(a+bj) = a+bi$ and $e_2(a + bj) = a - bi$. Complex conjugation acts on $X^*(\mathbb{S})_\mathbb{C}$ by swapping e_1 and e_2.

We have the weight homomorphism $w : \mathbb{G}_{m,\mathbb{R}} \hookrightarrow \mathbb{S}$ given by $w(a) = a + 0j$. This induces the map

$$w^* : \mathbb{Z}^2 \cong X^*(\mathbb{S}_\mathbb{C}) \to X^*(\mathbb{G}_{m,\mathbb{R}}) \cong \mathbb{Z}, \quad (p, q) \mapsto p + q.$$

Let G be a connected reductive group over $\mathbb{Q}$. Let its Lie algebra be $\mathfrak{g}$. We have the adjoint rep $\mathrm{Ad} : G \to GL(\mathfrak{g})$ with image $G^{ad} \cong G/Z_G$ where Z_G is the center of G. We write $G(\mathbb{R})^+$ for the connected component of $G(\mathbb{R})$ in the real topology, and we set $G(\mathbb{Q})^+ = G(\mathbb{R})^+ \cap G(\mathbb{Q})$. We let $G(\mathbb{R})_+$ denote the preimage of $G^{ad}(\mathbb{R})^+$ under the map $G(\mathbb{R}) \to G^{ad}(\mathbb{R})$, and we set $G(\mathbb{Q})_+ = G(\mathbb{R})_+ \cap G(\mathbb{Q})$. Let A_G denote the connected component of $Z_G(\mathbb{R})$ in the real topology.

Definition. A *Shimura datum* is a pair (G, X) where G is a reductive group over $\mathbb{Q}$ and X is a $G(\mathbb{R})$-conjugacy class of homomorphisms $h : \mathbb{S} \to G_{\mathbb{R}}$ satisfying the following axioms.

SD1 For any $h \in X$, $\mathrm{Ad}\big(h(j)\big)$ is a *Cartan involution* of G^{ad}, i.e. the group

$$\{g \in G^{ad}(\mathbb{C}) : h(j) \cdot \bar{g} \cdot h(j)^{-1} = g\}$$

is a compact Lie group, where $\bar{-}$ denotes complex conjugation.

SD2 For $h \in X$, only the characters $(-1, 1), (0, 0), (1, -1) \in X^*(\mathbb{S}_{\mathbb{C}})$ occur in the adjoint representation $ad \circ g : \mathbb{S} \to GL(\mathfrak{g}_{\mathbb{C}})$, i.e. we can write

$$\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{p}_h^- \oplus \mathfrak{p}_h^+ \oplus \mathfrak{m}_h$$

where $z \in \mathfrak{S}(\mathbb{R}) = \mathbb{C}^\times$ acts on $\mathfrak{p}_h^-$ as multiplication by $\bar{z}/z$, on $\mathfrak{p}_h^+$ as multiplication by $z/\bar{z}$, and acts trivially on $\mathfrak{m}_h$.

SD3 (*optional*) G^{ad} has no non-trivial $\mathbb{Q}$-rational factor G_0 such that the projection of h on G_0 is trivial.

Remark 1.1. Brian Conrad's notes, especially Section 5, are an excellent place to gain intuition for why the axioms are the way they are.

Notice that $h \circ w : \mathbb{G}_{m,\mathbb{R}} \to G_{\mathbb{R}}$ acts trivially on $\mathfrak{g}$ by SD2, and so $h \circ w$ has image in $Z_{G,\mathbb{R}}$. Since we are taking $G(\mathbb{R})$-conjugacy classes of morphisms $h \in X$, we see that $h \circ w = h' \circ w$ for any other $h' \in X$. We thus have a well-defined *weight morphism* $w_X : \mathbb{G}_{m,\mathbb{R}} \to Z_{G_{\mathbb{R}}}$ attached to our Shimura datum (G, X).

Remark 1.2. Milne defines the weight morphism differently. In his notation, the weight morphism is the *inverse* of our weight morphism w_X.

Now suppose K is an open compact subgroup $K \subset G(\mathbb{A}_{\mathbb{Q}}^\infty)$, we have

$$Sh_K(G, X) := G(\mathbb{Q})\backslash X \times \big(G(\mathbb{A}_{\mathbb{Q}}^\infty)/K\big).$$

In fact, equipping $G(\mathbb{A}_{\mathbb{Q}}^\infty)$ with the adelic topology, we have a homeomorphism

$$Sh_K(G, X) \cong \bigsqcup_i \Gamma_i \backslash X^+$$

where $\Gamma_i := gKg^{-1} \cap G(\mathbb{Q})_+$, X^+ is a connected component of X, and where the disjoint union runs over a set of coset representatives of the double quotient space

$$G(\mathbb{Q})_+\backslash G(\mathbb{A}_{\mathbb{Q}}^\infty)/K$$

This double coset space is known to be finite.

Remark 1.3. One checks that $G(\mathbb{R})_+$ is the subgroup of $G(\mathbb{R})$ that stabilizes X^+, so that in fact $X^+ \cong G(\mathbb{R})_+/G_h(\mathbb{R})$ where $h \in X$ is fixed, $M_h(\mathbb{R})$ is the stabilizer of h in $G(\mathbb{R})$, and X^+ is the connected component of X containing h.

1.2 Hecke Correspondences

We consider the family $Sh_K(G, X)$ as K varies through the cofiltered poset of compact opens K in $G(\mathbb{A}_{\mathbb{Q}}^\infty)$. It is known that there is a cofinal sub-poset consisting of those K such that $Sh_K(G, X)$ are smooth manifolds (we can take for instance the *neat* subgroups; see [GH22]). For $K' \subset K$ compact open neat subgroups of $G(\mathbb{A}_{\mathbb{Q}}^\infty)$, it is known that the natural map $Sh_{K'}(G, X) \to Sh_K(G, X)$ is smooth.

Now suppose K is a compact open subset of $G(\mathbb{A}_{\mathbb{Q}}^\infty)$, and $g \in G(\mathbb{A}_{\mathbb{Q}}^\infty)$. Then gKg^{-1} is another compact open subset of $G(\mathbb{A}_{\mathbb{Q}}^\infty)$, and we get an isomorphism

$$Sh_{gKg^{-1}}(G, X) \xrightarrow{\sim} Sh_K(G, X), \quad [x, a \cdot gKg^{-1}] \mapsto [x, ag \cdot K].$$

Thus, if $K' \subset gKg^{-1}$ is another compact open subgroup, then we can compose the above two maps to get the family of maps

$$T_g : Sh_{K'}(G, X) \to Sh_K(G, X)$$

which on points is given by

$$G(\mathbb{Q}) \cdot (x, hK') \mapsto G(\mathbb{Q}) \cdot (x, hgK).$$

These maps are finite étale if K and K' are neat (see [GH22, Section 15.2]). This family of maps T_g gives a right action of $G(\mathbb{A}_{\mathbb{Q}}^\infty)$ on the inverse system $(Sh_K(G, X))_K$, called the *Hecke action*.

Definition. The *Shimura variety* attached to the Shimura datum (G, X) is the inverse system of varieties $(Sh_K(G, X))_K$ equipped with the given Hecke action. We set

$$Sh(G, X) := \varprojlim_K Sh_K(G, X),$$

this is a scheme over $\mathbb{C}$.

Proposition 1.4. *For any Shimura datum (G, X), we have*

$$Sh(G, X) = \big(G(\mathbb{Q})/Z_G(\mathbb{Q})\big)\Big\backslash X \times \big(G(\mathbb{A}_{\mathbb{Q}}^{\infty})/\overline{Z_G(\mathbb{Q})}\big)$$

where $\overline{Z_G(\mathbb{Q})}$ is the closure of $Z_G(\mathbb{Q})$ in the adelic topology of $Z_g(\mathbb{A}_{\mathbb{Q}}^{\infty})$.

If $Z_G(\mathbb{Q})$ is discrete in $Z(\mathbb{A}_{\mathbb{Q}}^{\infty})$, then we simply have

$$Sh(G, X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_{\mathbb{Q}}^{\infty}).$$

Until now, we have just considered these spaces as topological manifolds. However more is true, as we see in the next subsection.

1.3 The Geometry of $Sh_K(G, X)$

Now fix $h \in X$. Suppose M_h is the stabilizer in $G_{\mathbb{R}}$ of the image of $h : \mathbb{S} \to G_{\mathbb{R}}$; then $M_h(\mathbb{R}) = A_G K_h$ where K_h is a maximal connected compact subgroup of $G(\mathbb{R})$ (follows from SD2), and we have $X \cong G(\mathbb{R})/A_G K_h$. Then one immediately sees that $\mathrm{Lie}(M_h) = \mathfrak{m}_h$. Also, $\mathrm{ad}h(i)$ is an involution and so acts by either ± 1 on each element of $\mathfrak{g}$; we see immediately that $\mathrm{ad}h(i)$ acts on $\mathfrak{g}_{\mathbb{C}}$ as -1 on $\mathfrak{p}_h^+ \oplus \mathfrak{p}_h^+$ and as 1 on $\mathfrak{m}_h$. Thus, the -1-eigenspace $\mathfrak{g}^{ad=-1}$ of $\mathrm{ad}h(i)$ in $\mathfrak{g}$ satisfies $\mathfrak{g}_{\mathbb{C}}^{ad=-1} = \mathfrak{p}_h^+ \oplus \mathfrak{p}_h^-$, and we have the Cartan decomposition

$$\mathfrak{g} = \mathfrak{g}_h \oplus \mathfrak{g}^{ad=-1}$$

corresponding to the Cartan involution $\mathrm{Ad}h(i)$. One checks from a Hodge decomposition argument that $\mathfrak{p}_h^-$ is a commutative Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$.

Now, we have an obvious bijection $G(\mathbb{R})/A_G K_h \to X$; this endows X with the structure of a smooth manifold. Moreover, we see that $\mathfrak{p}_h$ is the tangent space of X at h. We can endow X with a complex structure by setting the action of i on the tangent space $\mathfrak{p}_h$ to be given by $\mathrm{ad}(h(\zeta_4))$, where ζ_4 is a square root of i (that this actually yields a complex structure is non-trivial). Hence, X is a complex manifold. Since $\Gamma_i := gKg^{-1} \cap G(\mathbb{Q})_+$ acts discretely on X for K small enough, we see that $\Gamma_i \backslash X^+$ are complex manifolds for all i, and hence $Sh_K(G, X)$ is a complex manifold for all K small enough.

We can in fact do something better.

Proposition 1.5. *For K sufficiently small, $Sh_K(G, X)$ is a quasi-projective complex algebraic variety.*

Remark 1.6. This is shown by taking the Baily-Borel compactification of $Sh_K(G, X)$, which can be described easily if G^{ad} has no factors of dimension 3. In this case, if Ω^1 is the sheaf of differentials on $Sh_K(G, X)$ and we set $\omega := \bigwedge^d \Omega^1$ where $d = \dim X$, then we have the graded ring

$$A := \bigoplus_{n \geq 0} \Gamma\big(Sh_K(G, X), \omega^{\otimes n}\big),$$

and we have a canonical inclusion $Sh_K(G, X) \hookrightarrow \mathrm{Proj}A$. The closure of the image of this map is the Baily-Borel compactification of $Sh_K(G, X)$.

If however G^{ad} has factors of dimension 3, then we must replace $\Gamma\big(Sh_K(G, X), \omega^{\otimes n}\big)$ with the group of sections having at worst logarithmic singularities along the boundary of some smooth compactification of $Sh_K(G, X)$.

Proposition 1.7 (Borel)**.** *Let G be a reductive group over $\mathbb{Q}$. Then, the space Sh_K is compact for some $K \leq G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ (equivalently, for all such compact open K) if and only if G^{der} is $\mathbb{Q}$-anisotropic.*

Now, let P_h be the subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{p}_h := \mathfrak{p}_h^- \oplus \mathfrak{m}_h$ (this is the 0th step in the Hodge filtration of $\mathfrak{g}_{\mathbb{C}}$ induced by μ_h), and let $U_h \subset P_h$ be the subgroup with Lie algebra $\mathfrak{p}_h^-$. We have $G(\mathbb{R}) \cap P_h(\mathbb{C}) = M_h(\mathbb{R})$

Proposition 1.8 ([Mil90, Chap 3, Prop 1.1]). *The subgroup P_h is a parabolic subgroup of $G_{\mathbb{C}}$ with Levi component $M_{h,\mathbb{C}}$ and with unipotent radical U_h. We thus have a smooth open embedding*

$$\beta = \beta_h : X \hookrightarrow G(\mathbb{C})/P_h(\mathbb{C}),$$

called the Borel embedding, *of X into the projective complex algebraic variety $\check{X} := (G/P_h)(\mathbb{C})$. This embedding is equivariant with respect to the $G(\mathbb{R})$-action on X and the $G(\mathbb{C})$-action on $\check{X}$.*

Definition. We call $\check{X}$ the *compact dual of X*.

Note that $(G/P_h)(\mathbb{C})$ is indeed a generalized flag variety. We can view this embedding on points as a certain embedding of X into a flag variety directly as follows.

For $h \in X$, consider the cocharacter μ_h defined by

$$\mu_h : \mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{C}}, \quad \mu_h(z) = h(z, 1)$$

where we identify $\mathbb{S}_{\mathbb{C}} = \mathbb{G}_{m,\mathbb{C}}^2$ as above. We describe how this cocharacter defines a filtration functor, following [Mil90].

In general, if μ is a cocharacter of an algebraic group G over a field k of characteristic 0, then for any k-representation $\rho : G \to GL(V)$ we can attach a filtration

$$\cdots \supset F^p V \supset F^{p+1} V \supset \cdots, \quad F^p V := \oplus_{q \geq p} V^q$$

on V where

$$V = \bigoplus_{q \in \mathbb{Z}} V^q$$

where $\rho \circ \mu : \mathbb{G}_{m,k} \to GL(V)$ acts on V^q by the character $z \mapsto z^q$. One checks that this defines a symmetric monoidal functor

$$\mathrm{Filt}(\mu) : \mathrm{Rep}_k(G) \to \mathrm{FiltVect}_k$$

such that the diagram of symmetric monoidal functors

$$\begin{tikzcd} \mathrm{Rep}_k(G) \arrow[rr, "\mathrm{Filt}(\mu)"] \arrow[dr] & & \mathrm{FiltVect}_k \arrow[dl] \\ & \mathrm{Vect}_k & \end{tikzcd}$$

where the vertical functors are the forgetful functors.

Remark 1.9. Though we will not need this fact, every such functor $\mathrm{Rep}_k(G) \to \mathrm{FiltVect}_k$ arises (non-uniquely) from a cocharacter μ in such a way.

Now, suppose we fix a faithful representation $\rho : G \to GL(V)$. Then to each point $h \in X$ we have an associated $G(\mathbb{C})$filtration $\mathrm{Filt}(\mu_h)(V)$ on V, and so this gives a map from X to a Grassmann variety. One checks that P_h is the stabilizer of the filtration $\mathrm{Filt}(\mu_h)(V)$, and so in fact the $G(\mathbb{C})$ conjugacy class of filtrations of V containing $\mathrm{Filt}(\mu_h(V))$ for all $h \in X$ is precisely $G(\mathbb{C})/P_h(\mathbb{C}) \cong \check{X}$. Hence, the embedding $X \hookrightarrow \check{X}$ can be viewed as the map $h \mapsto \mathrm{Filt}(\mu_h)(V)$.

1.4 Canonical Models

A lot can be said about canonical models of Shimura varieties. We summarize things here.

Consider the map $X \to X_*(G)_{\mathbb{C}}$, given by sending h to μ_h. For $h, h' \in X$, we see that μ_h and $\mu_{h'}$ are $G(\mathbb{C})$-conjugate in $X_*(G)_{\mathbb{C}}$. Thus the Shimura datum in particular picks out a $G(\mathbb{C})$-conjugacy class M_X of cocharacters in $X_*(G)_{\mathbb{C}}$. However, G is defined over $\mathbb{Q}$. It thus follows that there exists a minimal number field $E = E(G, X)$ such that M_X is the base-change to $\mathbb{C}$ of a $G(E)$-conjugacy class of cocharacters in $X_*(G)_E$.

Definition. Such a minimal number field E is the *reflex* field of the Shimura datum (G, X).

It is a hard fact of Deligne's that every Shimura variety admits a *canonical model* over E, i.e. we can find a unique inverse system $M(G, X) = (M_K(G, X))_K$ of varieties over K with a $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ action such that there is a $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$-equivariant isomorphism of $\mathbb{C}$-varieties $Sh_K(G, X) \cong M_K(G, X) \times_E \mathbb{C}$ and $Sh(G, X) \cong M(G, X) \times_E \mathbb{C}$. There are other requiremens for a model over E to be a canonical model, but we ignore them for now.

It is also known that the projective variety $\check{X}$ is in fact also defined over the reflex field E.

2 Local Systems and Vector Bundles on Shimura Varieties

2.1 Review of Local Systems and the Riemann-Hilbert Correspondence

Suppose for now S is a complex manifold.

Definition. A *k-local system of vector spaces* on S is a locally constant sheaf of k-vector spaces on S.

Given a k-local system $\mathcal{V}$ of vector spaces we get a so-called *monodromy representation*, as follows. Notice first that if two points lie in the same (path-)connected component, the fibres over the two points will be (non-canonically) isomorphic via the point. Moreover, this isomorphism between the fibres is uniquely determined by (and uniquely determines) the path we chose between the two points. Thus, if S is connected, specifying a local system $\mathcal{V}$ is the same as (upon picking a base point $x \in S$) specifying a vector space $V \cong \mathcal{V}_x$ and isomorphisms $\sigma_\gamma : V \cong V$ along every loop γ such that if two paths are homotopic, then the isomorphism must be the identity. Therefore, a k-local system $\mathcal{V}$ on a connected space S is the same as a k-representation $\sigma : \pi_1(S, x) \to GL(V)$ where $V := \mathcal{V}_x$.

Remark 2.1. Similar definitions can be made for a scheme S. In this case, we also obtain a monodromy representation from a local system, though this time the representation is of the étale fundamental group.

We now consider connections and flat bundles on schemes. We follow [Fon], though Brian Conrad's notes also seem to be pretty good. Suppose X is a scheme over S with structure map $\pi : X \to S$, and let $\mathbb{V}$ be a quasi-coherent $\mathcal{O}_X$-module (for instance, $\mathbb{V}$ could be a locally free sheaf, i.e. a vector bundle on X).

Definition. A connection on $\mathbb{V}$ is a morphism of $\pi^{-1}\mathcal{O}_S$-modules

$$\nabla : \mathbb{V} \to \mathbb{V} \otimes_{\mathcal{O}_X} \Omega^1_{X/S},$$

satisfying the Leibniz rule

$$\nabla(f \otimes v) = v \otimes df + f \cdot \nabla(v)$$

for $f \in \mathcal{O}_X(U)$ and $v \in \Gamma(U, \mathbb{V})$, where U is any open S-subscheme of X.

Let $T_{X/S} := \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{O}_X)$ be the relative tangent sheaf (sections of this sheaf are vector fields). A connection induces (and is uniquely determined by) a morphism

$$T_{X/S} \to \underline{\mathrm{End}}_{\mathcal{O}_S}(\mathbb{V}), \qquad \text{written } v \mapsto \nabla_v$$

where $\nabla_v(e) = \langle v, \nabla e \rangle$ is the *covariant derivative of e along v*. Notice that

$$\nabla_v(fe) - v(f)e + f\nabla_v e$$

where $v(f) = \langle v, df \rangle$ is the canonical pairing of vector fields with 1-forms.

Example 2.2. $\mathcal{O}_X$ can be equipped with the connection $d : \mathcal{O}_X \to \Omega^1_{X/S} = \mathcal{O}_X \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$.

Definition. A connection is *flat* if for any local sections v, w of $T_{X/S}$ we have

$$\nabla_{[v,w]} = \nabla_v \circ \nabla_w - \nabla_w \circ \nabla_v.$$

A bundle is *flat* if it can be equipped with a flat connection.

Of course, all of these definitions carry over to complex manifolds as well (i.e. we take $S = \mathbb{C}$). In fact, these definitions are compatible with Serre's GAGA.

Theorem 2.3 (Riemann-Hilbert Correspondence)**.** *Suppose X is a complex manifold. Then, the category of flat vector bundles on M is equivalent to the category of $\mathbb{C}$-local systems of finite rank.*

Let us write down this categorical equivalence. Suppose $(\mathbb{V}, \nabla)$ is a flat vector bundle. We have a $\mathbb{C}$-local system $\mathcal{V} = \mathbb{V}^\nabla$ given by

$$\mathcal{V}(U) = \{e \in \mathbb{V}(U) : \nabla e = 0\}.$$

On the other hand, suppose $\mathcal{V}$ is a $\mathbb{C}$-local system of finite rank. Set $\mathbb{V} := \mathcal{V} \otimes_{\mathbb{C}} \mathcal{O}_X$. Then, the map

$$\mathrm{id} \otimes d : \mathcal{V} \otimes_{\mathbb{C}} \mathcal{O}_X \to \mathcal{V} \otimes_{\mathbb{C}} \Omega^1_X = (\mathcal{V} \otimes_{\mathbb{C}} \mathcal{O}_X) \otimes_{\mathcal{O}_X} \Omega^1_X$$

is in fact a flat connection

$$\nabla : \mathcal{V} \to \mathcal{V} \otimes_{\mathcal{O}_X} \Omega^1_X.$$

2.2 Automorphic Vector Bundles

Our exposition will (mostly) be pulling from [Su18], [Har85], and [Mil90]. The following is a key definition.

Definition. Let S be an algebraic variety over a field k equipped with an action of an algebraic group G. A *G-vector bundle* on S is a vector bundle $\mathbb{V}$ on S together with an action of G on the total space of $\mathbb{V}$ (as an algebraic variety), such that

- $p(g \cdot v) = g \cdot p(v)$ for all $g \in G$ and $v \in \mathbb{V}$, where $p : \mathbb{V} \to S$ is the projection map from the total space of the vector bundle onto S; and
- the maps $g : \mathbb{V}_s \to \mathbb{V}_{gs}$ are linear for all $s \in S$.

We have the obvious analog for vector bundles on manifolds with a Lie group action.

The idea behind automorphic vector bundles is simply this: we want to take a $G_{\mathbb{C}}$ vector bundle $\mathcal{V}$ on $\check{X}$. We want to descend this vector bundle to get a vector bundle $\mathcal{V}_K$ on $Sh_K(G,X)$ for each level K. This bundle turns out to be algebraic. Manifestly, the global sections of $\mathcal{V}_K$ are functions on X that are invariant with respect to discrete subgroups of the form $K \cap G(\mathbb{Q})$, which is precisely what we want for automorphic forms. However, there are technical complications that can arise when constructing $\mathcal{V}_K$ from $\mathcal{V}$, since it is unclear whether after quotienting by K the vector space structure on the stalks still survives.

A stupid obstruction to constructing $\mathcal{V}_K$ is if the groups $gKg^{-1} \cap G(\mathbb{Q})_+$ fail to act discretely with no fixed points, for then even $Sh_K(G,X)$ fails to be a manifold. Thus, for instance, we would need to assume that our level K is neat. Further technical obstructions are in fact closely related to this one; we need the groups $gKg^{-1} \cap G(\mathbb{Q})_+$ to act discretely without fixed points on the vector bundle $\mathcal{V}$ as well! Different authors impose slightly (at least superficially) different conditions:

1. Let Z_G^s denote the largest subtorus of Z_G that splits over $\mathbb{R}$ such that no subtorus of Z_G^s splits over $\mathbb{Q}$. Then, we require that $Z_g^s(\mathbb{C})$ acts trivially on $\mathcal{V}$ (this is how it is phrased in [Mil90]).
2. The maximal $\mathbb{Q}$-split torus in Z_G is also the maximal $\mathbb{R}$-split torus in Z_G (this is how it is phrased in [Su18]).

From now on, we will suppose that one of the two conditions hold, and that also G is connected. Fix a point $o \in X$, and let P_o be the corresponding parabolic subgroup of $G_{\mathbb{C}}$ so that $\check{X} \cong G(\mathbb{C})/P_o(\mathbb{C})$. Then the Levi factor of P_o is $M_{o,\mathbb{C}}$, where M_o is the stabilizer in $G_{\mathbb{R}}$ of the image of $h : \mathbb{S} \to G_{\mathbb{R}}$. Recall that $M_o(\mathbb{R}) = A_G K_o$ where K_o is a maximal compact subgroup of $G(\mathbb{R})$.

Suppose that $\mathcal{V}$ is a $G_{\mathbb{C}}$-vector bundle on $\check{X}$. The Borel embedding $\beta_o : X \hookrightarrow \check{X}$ is an open embedding, and so the sheaf $\beta_o^* \mathcal{V}$ is still a vector bundle on X. We then obtain a $G(\mathbb{A}_{\mathbb{Q}}^\infty)$-homogeneous holomorphic vector bundle $\beta^* \mathcal{V} \times \big(G(\mathbb{A}_{\mathbb{Q}}^\infty)/K\big)$ over $X \times \big(G(\mathbb{A}_{\mathbb{Q}}^\infty)/K\big)$. Under the above assumptions, for a neat level K the group $G(\mathbb{Q})$ acts freely on $X \times \big(G(\mathbb{A}_{\mathbb{Q}}^\infty)/K\big)$, and so we obtain a vector bundle

$$\mathcal{V}_K := G(\mathbb{Q}) \backslash \beta_o^* \mathcal{V} \times G(\mathbb{A}_{\mathbb{Q}}^\infty)/K$$

on $Sh_K(G,X)$.

Definition. An *automorphic vector bundle* is a bundle $\mathcal{V}_K$ on $Sh_K(G,X)$ obtained by the above construction from a $G_{\mathbb{C}}$-vector bundle $\mathcal{V}_K$ on $\check{X}$.

For each $g \in G(\mathbb{A}_{\mathbb{Q}}^\infty)$ and pair of neat levels K and K' such that $K' \subset gKg^{-1}$, we get a morphism

$$\mathcal{V}_K \to \mathcal{V}_{K'}, \quad [x,a] \mapsto [x, ag]$$

just as we had for $Sh_K(G,X)$. It is clear that the following diagram commutes.

$$\begin{tikzcd} \mathcal{V}_K \arrow[r] \arrow[d] & \mathcal{V}_{K'} \arrow[d] \\ Sh_{K'}(G,X) \arrow[r, "T_g"] & Sh_K(G,X) \end{tikzcd} \tag{1}$$

Proposition 2.4. *The vector bundles $\mathcal{V}_K$ and the maps $\mathcal{V}_K \to \mathcal{V}_{K'}$ are algebraic.*

Proposition 2.5. *If G has no factors of dimension 3, then every holomorphic section of $\mathcal{V}_K$ is algebraic, and the space of such sections is finite-dimensional over $\mathbb{C}$.*

Remark 2.6. The requirement that G has no factors of dimension 3 is due to the same issues that show up in Baily-Borel compactification. In this case, X^+ has a factor isomorphic to the unit disk (or what is essentially the same thing, the upper half-plane). Thus, one has to take care of logarithmic singularities occurring at the cusps, and so extra conditions are required.

2.3 Computing Automorphic Vector Bundles

We can construct $G(\mathbb{C})$-vector bundles as follows. If $\rho : P_o \to GL(V)$ is an algebraic representation with V a finite dimensional $\mathbb{C}$-vector space, then we set

$$\tilde{V} := G(\mathbb{C}) \times^{P_o(\mathbb{C})} V = \big(G(\mathbb{C}) \times V\big)/P_o(\mathbb{C})$$

where $p \in P_o(\mathbb{C})$ acts on $(g, v) \in G(\mathbb{C}) \times V$ by $p \cdot (g, v) = (gp^{-1}, \rho(p)v)$. There is an obvious projection map $\tilde{V} \to \check{X}$, and it is easy to see that $\tilde{V}$ is a vector bundle over $\check{X}$. The $G(\mathbb{C})$-action given by multiplying on the left on the first coordinate makes $\tilde{V}$ a $G(\mathbb{C})$-homogeneous bundle. We thus have a vector bundle $\tilde{V}_K$ on $Sh_K(G, X)$.

Remark 2.7. There is actually an equivalence of (symmetric monoidal) categories between the category of finite-dimensional representations of P_o, and the category of $G(\mathbb{C})$-vector bundles on $\check{X} = G(\mathbb{C})/P_o(\mathbb{C})$ [EH17]. If $\mathcal{V}$ is a $G(\mathbb{C})$-vector bundle on $\check{X}$, then the fibre $\mathcal{V}_o$ at the base point $o \in X \subset \check{X}$ has a natural action of $P_o(\mathbb{C})$.

Now, if V is a $M_{o\mathbb{C}}$-representation, then under the Levi projection $P_o \twoheadrightarrow M_o$ we view V as a P_o-representation, and so we have a $G(\mathbb{C})$-vector bundle on $\check{X}$. It turns out that this vector bundle is semi-simple, and in fact, the above equivalence restricts to an equivalence between the category of finite-dimensional representations of $M_{o\mathbb{C}}$ and *semi-simple* $G(\mathbb{C})$-vector bundles on $\check{X}$ [EH17].

Remark 2.8. I think $\beta_o^* \tilde{V} \cong G(\mathbb{R}) \times^{K_o(\mathbb{R})} V$ as $G(\mathbb{R})$-homogeneous vector bundles on X, where the action of $K_o(\mathbb{R})$ on V is obtained by restriction.

Example 2.9. If we take $V := \bigwedge^p(\mathfrak{g}_{\mathbb{C}}/\mathrm{Lie}(P_o))$, then $\tilde{V} = \Omega^p_{\check{X}}$ is the bundle of smooth p-forms on $\check{X}$. It then follows that

$$\tilde{V}_K = \Omega^p_{Sh_K(G,X)}$$

is the bundle of smooth p-forms on $Sh_K(G, X)$.

Let $\mathcal{O}$ denote the sheaf of holomorphic functions on the complex manifold $Sh_K(G, X)$. As usual with algebraic vector bundles, we can view $\tilde{V}_K$ as a locally free sheaf of $\mathcal{O}$ modules as follows. Consider the quotient map

$$\pi_K : G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q}})/A_G K = G(\mathbb{Q})\backslash\big(G(\mathbb{R})/A_G \times G(\mathbb{A}^\infty_{\mathbb{Q}})/K\big) \twoheadrightarrow Sh_K(G, X).$$

Let $G(\mathbb{A}_{\mathbb{Q}}) = A_G G^1(\mathbb{A}_{\mathbb{Q}})$ where G^1 is the intersection of the kernels of all $\mathbb{Q}$-characters of G. Write $\mathfrak{g}^1 := Lie(G^1)$, and write $\mathfrak{p}^1_o = \mathfrak{p}_o \cap \mathfrak{g}^1$; then writing $\mathfrak{a}_G = Lie(A_G)$ we have $\mathfrak{p}_h = \mathfrak{p}_o \oplus \mathfrak{a}_G$. There is a natural $(\mathfrak{p}^1_o, K_o)$-module structure on the space of smooth functions

$$C^\infty(\pi_K^{-1}(U)) \otimes V,$$

for $U \subset Sh_K(G, X)$ open, where V is a $P_o(\mathbb{C})$-module and so is a $(\mathfrak{p}^1_o, K_o)$-module, whereas $\mathfrak{p}^1_o$ acts on $C^\infty(\pi_K^{-1}(U))$ by right differentiation and K_o acts on $C^\infty(\pi_K^{-1}(U))$ by the right regular representation.

Proposition 2.10. *$\tilde{V}_K$, viewed as a sheaf of $\mathcal{O}$-modules, is the sheaf*

$$U \mapsto \big(C^\infty(\pi_K^{-1}(U)) \otimes V\big)^{(\mathfrak{p}^1_o, K_o)},$$

where for concreteness

$$\big(C^\infty(\pi_K^{-1}(U)) \otimes V\big)^{(\mathfrak{p}^1_o, K_o)} = \big\{f : \pi_K^{-1}(U) \to V : f \text{ smooth, } v(f) = 0\ \forall v \in \mathfrak{p}^1_o, \text{ and } f(gk) = f(g)\ \forall k \in K_o\big\}$$

with V endowed with the usual analytic topology induced by the $\mathbb{C}$-vector space structure on V.

Now suppose K, K' are compact open subgroups of $G(\mathbb{A}^\infty_{\mathbb{Q}})$. The commuting square (1) yields a natural isomorphism $T_g^* \tilde{V}_K \cong \tilde{V}_{K'}$. By the pull-back–push-forward adjunction, we thus have a natural map

$$\tilde{V}_K \to (T_g)_* \tilde{V}_{K'}.$$

It is straightforward to check that as sheaf maps, this is simply the map

$$\begin{gathered}\big(C^\infty(\pi_K^{-1}(U)) \otimes V\big)^{(\mathfrak{p}^1_o, K_o)} = \tilde{V}_K(U) \to (T_g)_* \tilde{V}_{K'}(U) = \Big(C^\infty\big(\pi_{K'}^{-1}(T_g^{-1}(U))\big) \otimes V\Big)^{(\mathfrak{p}^1_o, K_o)} \\ f \mapsto g f(\cdot\, g)\end{gathered}$$

for every open $U \subset Sh_K(G, X)$. Here, we can define an action of $G(\mathbb{A}^\infty_{\mathbb{Q}})$ action on V as follows: for any $g \in G(\mathbb{A}^\infty_{\mathbb{Q}})$, write $a_g \in A_G$ to be the unique element such that $g \in a_g G^1(\mathbb{A}^\infty_{\mathbb{Q}})$. Then g acts on V by multiplication by a_g^{-1}.

2.4 Automorphic Local Systems

Suppose now V is a representation of $G_{\mathbb{C}}$. Then we have a local system

$$\underline{V}_K := G(\mathbb{Q})\backslash(V \times X \times G(\mathbb{A}_{\mathbb{Q}}^{\infty})/K)$$

on $Sh_K(G,X)$, for every compact open neat subgroup $K \leq G(\mathbb{A}_{\mathbb{Q}}^{\infty})$. Here, $G(\mathbb{Q})$ acts on V via the inclusion $G(\mathbb{Q}) \hookrightarrow G(\mathbb{C})$.

Example 2.11. For V the trivial representation of $G_{\mathbb{C}}$, we have $\underline{V}_K = \underline{\mathbb{C}}$, the sheafification of the constant sheaf $\mathbb{C}$.

Of course, any representation $\rho : G_{\mathbb{C}} \to GL(V)$ of $G_{\mathbb{C}}$ restricts to a representation of P_o, and so defines the automorphic vector bundle $\mathcal{V}_K$. It turns out that under the Riemann-Hilbert correspondence, the local system $\underline{V}_K$ corresponds to the vector bundle $\mathcal{V}_K$ equipped with a certain flat connection. We can construct this flat connection from ρ directly (we follow the construction in Section 5 of [GP02]). In fact, we define a $G(\mathbb{R})$-invariant connection

$$\nabla : \tilde{V}_K \to \tilde{V}_K \otimes_{\mathcal{O}} \Omega^1_{Sh_K(G,X)}$$

by defining a morphism $T_{Sh_K} \to \underline{\mathrm{End}}(\tilde{V}_K)$. First, notice that $\rho : G_{\mathbb{C}} \to GL(V)$ induces a map $d\rho : \mathfrak{g}_{\mathbb{C}} \to \mathrm{End}(V)$. Now, consider any local vector field v on Sh_K and local (holomorphic) section f of $\tilde{V}_K$. We view f as a smooth map $f : \pi_K^{-1}(U) \to V$ such that $d\rho|_{\mathfrak{p}_o^1} \cdot f \equiv 0$ and $f(gk) = \rho(k^{-1})f(g)$, where

$$\pi_K : G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q}})/A_G K \to Sh_K.$$

Then, we set $\nabla_v f$ to be the section

$$(\nabla_v f)(g) := \tilde{v}(f)(g) + \rho(g) d\rho(g^{-1} \cdot \tilde{v}) \cdot f(g),$$

where $\tilde{v}$ is any lift of v under the map π_K, and we consider $\rho(g)d\rho(g^{-1} \cdot \tilde{v}_g) \in \mathrm{End}(V)$ acting on $f(g) \in V$ (where g^{-1} pushes $v_g \in T_g G(\mathbb{C})$ to $T_0 G(\mathbb{C}) = \mathfrak{g}$).

2.5 A Note on Automorphy Factors

Classically, automorphic forms were defined using so-called automorphy factors. That is, an automorphic form for Γ of type J, for J an automorphy factor, was a function $f : X^+ \to V$ such that $f(\gamma x) = J(\gamma, x) f(x)$ and satisfying other nice conditions (holomorphicity, growth conditions at ∞, etc). For instance, the automorphy factor for a discrete subgroup Γ of $SL_2(\mathbb{Z})$ acting on the upper half plane with values in $\mathbb{C}$ given by

$$J\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), z\right) = (cz+d)^k$$

is classically used to define weight k-modular forms.

Suppose $\rho : P_o \to GL(V)$ is a representation with corresponding automorphic vector bundle $\tilde{V}_K$. Fix a base point $x_0 \in X$ (say $x_0 = [1] \in G(\mathbb{R})/K_o(\mathbb{R})$).

Definition. A (holomorphic) *automorphy factor for* $\tilde{V}_K$ is a smooth map $J : G(\mathbb{R}) \times X \to GL(V)$ such that

- $J(g, -) : X \to GL(V)$ is holomorphic for all $g \in G(\mathbb{R})$,
- $J(gg', x) = J(g, g'x) J(g', x)$ for all $g, g' \in G(\mathbb{R})$ and $x \in X$, and
- $J(k, x_0) = \rho(k)$ for all $k \in P_o(\mathbb{C}) \cap G(\mathbb{R}) = K_o(\mathbb{R})$.

An automorphy factor J determines a holomorphic trivialization

$$\Phi_J : \beta_o^* \tilde{V} \cong G(\mathbb{R}) \times^{K_o(\mathbb{R})} V \to X \times V, \quad [h, v] \mapsto (hx_0, J(h, x_0)v)$$

where the action of $G(\mathbb{R})$ is $g \cdot (x, v) = (gx, J(g, x)v)$.

2.6 Hecke Action on Cohomology of $\tilde{V}_K$

We now construct an action of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ on the cohomology of bundles on Shimura varieties. Throughout, we fix a representation (ρ, V) of K_o. We follow the construction in [Nic20]. A slightly less general construction is carried out in [GH22], where some concrete discussion also takes place.

Let K and K' be arbitrary compact open neat subgroups of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$, and let $g \in G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ be arbitrary. We have the finite étale map

$$\pi_{K \cap g^{-1}K'g, K} : Sh_{K \cap g^{-1}K'g} \twoheadrightarrow Sh_K,$$

which induces a map on cohomology

$$\pi^*_{K\cap g^{-1}K'g,K} : H^\bullet(Sh_K, \tilde{V}_K) \to H^\bullet\left(Sh_{K\cap g^{-1}K'g}, \tilde{V}_{K\cap g^{-1}K'g}\right).$$

Now, we have the right multiplication isomorphism

$$[g^{-1}] : Sh_{gKg^{-1}\cap K'} \xrightarrow{\cong} Sh_{K\cap g^{-1}K'g},$$

which yields an isomorphism

$$[g^{-1}]^* : H^\bullet(Sh_{K\cap g^{-1}K'g}, \tilde{V}_{K\cap g^{-1}K'g}) \to H^\bullet\left(Sh_{gKg^{-1}\cap K'}, \tilde{V}_{gKg^{-1}\cap K'}\right).$$

We finally construct a *trace* map

$$Tr_{gKg^{-1}\cap K',K'} : H^\bullet\left(Sh_{K'}, \tilde{\pi}_{gKg^{-1}\cap K',K'*}V_{gKg^{-1}\cap K'}\right) \to H^\bullet(Sh_{K'}, \tilde{V}_{K'}).$$

Write $K'' := gKg^{-1} \cap K'$ for notational simplicity. Since $\pi_{K'',K'}$ is finite étale, the functors $(\pi_{K'',K'})_!$ and $(\pi_{K'',K'})_*$ coincide. Under the shriek-pushforward–pullback adjunction, the identity map on $\pi^*_{K'',K'}\tilde{V}_{K'}$ induces a map

$$\pi_{K'',K'*}\pi^*_{K'',K'}\tilde{V}_{K'} \to \tilde{V}_{K'},$$

which on fibres is simply the map

$$\oplus_{x'\in\pi^{-1}_{K'',K'}(x)}V \cong \left(\pi_{K'',K'}{}_*\pi^*_{K'',K'}\tilde{V}_{K'}\right)_x \to V, \quad (v_{x'})_{x'\in\pi^{-1}_{K'',K'}(x)} \mapsto \sum_{x'\in\pi^{-1}_{K'',K'}(x)} v_{x'};$$

here, we have identified $(\tilde{V}_{K'})_x \cong V$. Since $\pi^*_{K'',K'}\tilde{V}_{K'} \cong \tilde{V}_{K''}$, we thus have a map

$$\pi_{K'',K'*}\tilde{V}_{K''} \to \tilde{V}_{K'},$$

and hence a trace map on cohomology

$$Tr_{K'',K'} : H^\bullet\left(Sh_{K'}, \pi_{K'',K'*}\tilde{V}_{K''}\right) \to H^\bullet(Sh_{K'}, \tilde{V}_{K'}).$$

Of course, the push-forward also induces the map

$$\pi_{K'',K'*} : H^\bullet\left(Sh_{K''}, \tilde{V}_{K''}\right) \to H^\bullet\left(Sh_{K'}, \pi_{K'',K'*}\tilde{V}_{K''}\right).$$

Remark 2.12. This is a general construction, the *Méthode de la trace*.

Composing these four maps together, we get the *Hecke operator* on cohomology

$$T_{KgK'} = \pi_{K'',K'*} \circ Tr_{K'',K'} \circ [g^{-1}]^* \circ \pi^*_{g^{-1}K''g,K} : H^\bullet(Sh_K, \tilde{V}_K) \to H^\bullet(Sh_{K'}, \tilde{V}_{K'}).$$

In particular, taking $K' = K$, we get Hecke operators

$$T_g : H^\bullet(Sh_K, \tilde{V}_K) \to H^\bullet(Sh_K, \tilde{V}_K).$$

It thus follows that we have an action of $G(\mathbb{A}^\infty_{\mathbb{Q}})$ on the system

$$H^\bullet(Sh(G,X), \tilde{V}) := \varinjlim_{K\leq G(\mathbb{A}^\infty_{\mathbb{Q}})} H^\bullet(Sh_K, \tilde{V}_K).$$

For a fixed k, since $H^k(Sh_K, \tilde{V}_K)$ is finite dimensional, and since $H^k(Sh(G,X), \tilde{V})^K = H^k(Sh_K, \tilde{V}_K)$ (resulting from the Hochschild-Sere spectral sequence), it follows that the $G(\mathbb{A}^\infty_{\mathbb{Q}})$ on the profinite space $H^k(Sh(G,X), \tilde{V})$ is an admissible representation.

2.7 Using $(\mathfrak{g}/\mathfrak{a}_G, K_o)$-Cohomology

2.7.1 Definition

In order to prevent confusion with previous notation, consider a Lie group H with Lie algebra $\mathfrak{h}$ and maximal compact C. Let $\mathfrak{c}$ be the Lie subalgebra of $\mathfrak{h}$ corresponding to C. Suppose W is a $(\mathfrak{h}, C)$-module. For a given $q \geq 0$, set $C^q(\mathfrak{h}, C; W)$ to be the space of linear functions $f : \bigwedge^q(\mathfrak{h}/\mathfrak{c}) \to W$ satisfying

$$\sum_{i=1}^{q} f(x_1, ..., x_{i-1}, [x, x_i], x_{i+1}, ..., x_q) = x \cdot f(x_1, ..., x_q)$$

for all $x \in \mathfrak{c}$ and $x_1, ..., x_q \in \mathfrak{h}$, and satisfying

$$\sum_{i=1}^{q} f(x_1, ..., x_{i-1}, Ad_h(x_i), x_{i+1}, ..., x_q) = h \cdot f(x_1, ..., x_q)$$

for all $h \in C$ and all $x_1, ..., x_q \in \mathfrak{h}$. Strictly speaking, the first condition is a special case of the second condition via the exponential map, but we write both conditions separately for emphasis. We have natural differential maps

$$d : C^q(\mathfrak{h}, C; W) \to C^{q+1}(\mathfrak{h}, C; W)$$

given by

$$(df)(x_0, ..., x_q) := \sum_{i=1}^{q} (-1)^i x_i \cdot f(x_0, ..., \hat{x}_i, ..., x_q) + \sum_{1 \leq i < j \leq q} (-1)^{i+j} f([x_i, x_j], x_0, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_q).$$

The cohomology of this complex $C^\bullet(\mathfrak{h}, C; W)$ is denoted by $H^*(\mathfrak{h}, C; W)$.

2.7.2 Cohomology of Automorphic Vector Bundles as $(\mathfrak{g}, K_o)$-cohomology

Fix a level K, and let V be a complex representation of P_o, so that we have the bundle $\tilde{V}_K$ on Sh_K. Consider the Dolbeault complex $\mathcal{A}^{0,\bullet}$ whose differential maps are $\bar{\partial}$. It is a general fact that $\tilde{V}_K \otimes_{\mathcal{O}_{Sh_K}} \mathcal{A}^{0,\bullet}$ is a fine resolution of $\tilde{V}_K$, and so can be used to compute the sheaf cohomology $H^*(Sh_K, \tilde{V}_K)$. We thus try to compute the cohomology of the complex $\Gamma(Sh_K, \tilde{V}_K \otimes_{\mathcal{O}_{Sh_K}} \mathcal{A}^{0,\bullet})$. Let Ω^q denote the sheaf of smooth q forms, so that

$$\Omega^q = \bigoplus_{i+j=q, i,j \geq 0} \mathcal{A}^{i,j}.$$

First, we know that $\tilde{V}_K \otimes_{\mathcal{O}_{Sh_K}} \Omega^0$ (where recall $\Omega^0 = \mathcal{C}^\infty$) is the sheaf

$$U \mapsto \left(C^\infty(\pi_K^{-1}(U)) \otimes V\right)^{K_o},$$

where π_K is the projection

$$\pi_K : G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q})/A_G K = G(\mathbb{Q})\backslash\big(G(\mathbb{R})/A_G \times G(\mathbb{A}_\mathbb{Q}^\infty)/K\big) \twoheadrightarrow Sh_K(G, X).$$

Suppose $U \subset Sh_K$ is open. Let us define a map

$$\Phi_U : \tilde{V}_K \otimes_{\mathcal{O}_{Sh_K}} \Omega^q(U) \to \text{Hom}\left(\bigwedge^q(\mathfrak{g}_\mathbb{C}/\mathfrak{a}_G), C^\infty(\pi_K^{-1}(U)) \otimes V\right)$$

as follows. Suppose $\omega \in \tilde{V}_K \otimes_{\mathcal{O}_{Sh_K}} \Omega^q(U)$. Suppose $v_1, ..., v_q \in \mathfrak{g}_\mathbb{C}/\mathfrak{a}_G$; then via left-translation ℓ_p we can view $v_1, ..., v_q$ as elements of $T_p\pi_K^{-1}(U)$ for all $p \in \pi_K^{-1}(U)$. We then get the element

$$\omega_{\pi_K p}\left(\pi_{K*}\ell_{p,*}v_1, ..., \pi_{K*}\ell_{p,*}v_q\right) \in V.$$

Thus, we have a smooth map

$$\pi_K^{-1}(U) \to V, \quad p \mapsto \omega_{\pi_K p}\left(\pi_{K*}\ell_{p,*}v_1, ..., \pi_{K*}\ell_{p,*}v_q\right),$$

and so $v_1, ..., v_q$ have defined an element of $C^\infty(\pi_K^{-1}(U)) \otimes V$. This map is obviously linear, and so to ω we have assigned an element $\Phi_U\omega \in \text{Hom}\left(\bigwedge^q(\mathfrak{g}_\mathbb{C}/\mathfrak{a}_G), C^\infty(\pi_K^{-1}(U)) \otimes V\right)$. We have thus defined the map Φ_U. If some

$v_j \in \mathfrak{k}_o = \mathfrak{m}_o/\mathfrak{a}_G$, then $\pi_{K*}v_j = 0$ since π_K is the quotient by K_o map. Thus, $\Phi_U\omega(v_1, ..., v_q) = 0$ if any one of the $v_i \in \mathfrak{m}_o/\mathfrak{a}_G$. Hence, Φ_U actually defines a map

$$\Phi_U : \tilde{V}_K \otimes_{\mathcal{O}_{Sh_K}} \Omega^q(U) \to \operatorname{Hom}\left(\bigwedge^q(\mathfrak{g}_{\mathbb{C}}/\mathfrak{m}_o), C^\infty(\pi_K^{-1}(U)) \otimes V\right).$$

Suppose $k \in K_o$ and $p \in \pi_K^{-1}(U)$. Since π_K is the quotient by K_o map, it follows that for any $v \in \mathfrak{g}/\mathfrak{a}_G$ we have

$$\pi_{K*}\ell_{p,*}v = \pi_{K*}\ell_{pk^{-1},*}Ad(k) \cdot v,$$

and so in fact the map Φ is K_o-equivariant. Thus, we have a map

$$\Phi_U : \tilde{V}_K \otimes_{\mathcal{O}_{Sh_K}} \Omega^q(U) \to C^q(\mathfrak{g}_{\mathbb{C}}/\mathfrak{a}_G, K_o; C^\infty(\pi_K^{-1}(U)) \otimes V).$$

Since π_K is a principal K_o-bundle, it follows that Φ_U actually induces an isomorphism. Thus, if $C^\infty(Sh_K, -)$ denotes the functor of global smooth sections, we get the following result, where for simplicity, set

$$[G] := G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q}})/A_G.$$

Proposition 2.13. *With notation as above, we have*

$$R^qC^\infty(Sh_K, \tilde{V}_K) \cong H^q\left(\mathfrak{g}_{\mathbb{C}}/\mathfrak{a}_G, K_o; C^\infty\left([G]/K\right)^{K_o-fin} \otimes V\right).$$

Let us now consider the functor $\Gamma(Sh_K, -)$ of *holomorphic* global sections. Since, by definition, the complex structure on Sh_K is induced by the decomposition $\mathfrak{g}_{\mathbb{C}}/\mathfrak{m}_o \cong \mathfrak{p}_o^+ \oplus \mathfrak{p}_o^-$, and since $\mathfrak{p}_o^- = \mathfrak{p}_o^1/\mathfrak{k}_o$, it follows that Φ_U induces an isomorphism

$$\tilde{V} \otimes_{O_{Sh_K}} \mathcal{A}^{0,q}(U) \cong C^q\left(\mathfrak{p}_o^1, K_o; C^\infty(\pi^{-1}(U)) \otimes V\right).$$

We thus have the following result.

Proposition 2.14. *With notation as above, we have*

$$H^q(Sh_K, \tilde{V}_K) \cong H^q\left(\mathfrak{p}_o^1, K_o; C^\infty\left([G]/K\right)^{K_o-fin} \otimes V\right).$$

Taking limits, we then have the following.

Proposition 2.15. *There is a $G(\mathbb{A}_{\mathbb{Q}}^\infty)$-equivariant isomorphism*

$$H^q(Sh, \tilde{V}) \cong H^q\left(\mathfrak{p}_o^1, K_o; C^\infty\left([G]\right)^{K_o-fin} \otimes V\right),$$

where the right hand side has the $G(\mathbb{A}_{\mathbb{Q}}^\infty)$ action induced by its diagonal action on $C^\infty([G])^{K_o-fin} \otimes V$.

As a reminder, $G(\mathbb{A}_{\mathbb{Q}}^\infty)$ acts on V via projection to A_G.

3 Toroidal Compactifications and Canonical Extensions of Bundles

4 Chern Classes and Geometry of Automorphic Vector Bundles

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