

Problem Set 4

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Throughout, p is going to denote a prime number. Unless otherwise mentioned, all lattices will be assumed to be regular.

1. (*Corollary 7.9.3*) Suppose L is unimodular. Show that $Q(L) \supseteq \mathbb{Z}_p^*$ whenever $\text{rank} L \geq 2$, and that $Q(L) = \mathbb{Z}_p$ for $\text{rank} L \geq 3$.
2. (*Proposition 7.11 and Exercise 7.12*) Go through the proof of Proposition 7.11 as given in the lecture notes. Then, by proving that every element of $O(L)$ is the product of at most 2 reflections in $O(L)$ whenever $\text{rank} L = 2$, modify the proof of Proposition 7.11 to prove the following stronger statement:

Theorem. *If L is a regular lattice of rank $n \geq 2$, then every element of $O(L)$ is the product of at most $2n - 2$ reflections in $O(L)$.*

3. (*Exercise 7.15*) Show the following equivalences, where $u \in \{1, 3, 5, 7\}$ can be arbitrary.

$$\langle 1 \rangle \perp \langle 1 \rangle \cong \langle 5 \rangle \perp \langle 5 \rangle; \quad \langle 1 \rangle \perp \langle 2 \rangle \cong \langle 3 \rangle \perp \langle 6 \rangle; \quad \langle 1 \rangle \perp \langle 4 \rangle \cong \langle 5 \rangle \perp \langle 20 \rangle; \quad \langle u \rangle \perp \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle \cong \langle u \rangle \perp \langle 1 \rangle \perp \langle -1 \rangle;$$

$$\langle u \rangle \perp \langle \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \rangle \cong \langle 3u \rangle \perp \langle -u \rangle \perp \langle -u \rangle; \quad \langle \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \rangle \perp \langle \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \rangle \cong \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle \perp \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle.$$

(as an added challenge, try not to use Theorem 7.16).

4. For $p \in \{2, 3, 5\}$ (and any other prime of your choice), compute the Jordan decomposition of the following quadratic forms. Which of these lie in the same \mathbb{Z}_p -class?

$$f_1(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + 2x_1x_2 - 45x_3^2,$$

$$f_2(x_1, x_2, x_3) = x_1x_2 + x_2x_3 + x_3x_1$$

$$f_3(x_1, x_2, x_3) = 2x_1x_2 + 2x_2x_3 + 2x_3x_1 - x_1^2 - x_2^2 - x_3^2,$$

$$f_4(x_1, x_2, x_3) = 5x_1^2 + 13x_2^2 + 11x_3^2 + 2x_2x_3 + 2x_3x_1 + 16x_1x_2$$

Can you compute the set $f_i(\mathbb{Z}_p^3)$?

5. Over which p are the following quadratic forms \mathbb{Z}_p -equivalent?

$$x_1^2 + 2x_2^2 + 6x_3^2 + 6x_2x_3 \quad \text{and} \quad 2x_1^2 + 3x_2^2 + 5x_3^2.$$

6. (*Exercise 8.11*) Show that the lattices $\langle 1 \rangle \perp \langle 11 \rangle$ and $\langle \begin{pmatrix} 3 & -1 \\ -1 & 4 \end{pmatrix} \rangle$ are isomorphic over \mathbb{Z}_p for all primes p , and yet they are not in the same class over \mathbb{Z} .
7. (*Exercise 8.12*) Show that $L = \langle 1 \rangle \perp \langle 11 \rangle$ represents 3 p -adically for all primes p , but does not represent 3 over \mathbb{Q} .
8. Let V be the rational quadratic space with quadratic form $x^2 + y^2 + z^2$. Construct a full \mathbb{Z} -lattice L in V such that $L_2 \cong \langle \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \rangle \perp \langle 12 \rangle$, $L_3 \cong \langle 2 \rangle \perp \langle 3 \rangle \perp \langle 6 \rangle$, and $L_p \cong \langle 1 \rangle \perp \langle 1 \rangle \perp \langle 1 \rangle$ for all $p \geq 5$.
9. Compute class representatives of the genus of the quadratic form $x^2 + 14y^2$.