Differential Theory on Schemes

Kush Singhal

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Abstract

These are notes I made while learning about the theory of differential calculus on schemes, as well as abelian and non-abelian Hodge theory. Emphasis is on developing the language, rather than gaining intuition. All intuition should come from the analytic perspective of differential geometry and classical Hodge theory.

Contents

Notations, Conventions, and other Remarks

For the classical viewpoint of algebraic geometry in the field of differential geometry and Hodge theory, the book [GH14] is the gold standard.

Throughout, all rings will be commutative rings with identity.

1 De Rham Cohomology

The main references for stuff on the de Rham complex and de Rham cohomology is [Sta23, 0FK4]. For stuff on connections, the best reference is [Kat70]. The expository article [Ill02] is also really good, though its main focus is the theory in characteristic p.

Remark 1.1. A word of warning. Most references, including [Ill02] and [Sta23] usually consider arbitrary Y-schemes and X, and restrict to looking at coherent sheaves on X. However, [Kat70] considers *quasi*-coherent sheaves on *smooth* Y-schemes X. I've tried to be as careful as I can with the assumptions while copying down results, but it's possible that I may have missed some things.

1.1 The de Rham Complex

See [Har13, Section II.8] for details.

Definition. Suppose A is a ring, B an A-algebra, and M a B-module. An *A-derivation of B to M* is an A-linear map $d : B \to M$ such that

$$d(bb') = bdb' + b'db$$

and $da = 0$ for all $a \in A$.

Lemma-Definition. The *module of relative differentials* $\Omega_{B/A}$ is the unique B-module $\Omega_{B/A}$ equipped with an A-derivation $d : B \to \Omega_{B/A}$ that is initial amongst all such B-modules, i.e. for any B-module M with A-derivation $d' : B \to M$, there is a unique B-modules morphism $f : \Omega_{B/A} \to M$ such that $d' = f \circ d$.

One can construct the module of relative differentials in an obvious way. Properties of $\Omega_{B/A}$ can be found in [Har13, Section II.8].

Now, recall that if Z is a closed subscheme of X with closed immersion $i : Z \to X$, the *ideal sheaf* $\mathcal{I}_Z$ of Z is the kernel of the canonical morphism

$$\mathcal{O}_X \to i_*\mathcal{O}_Z.$$

It is a quasi-coherent sheaf of ideals of $\mathcal{O}_X$. Moreover, if $X = \operatorname{Spec} A$ and $Z = \operatorname{Spec} A/\mathfrak{a}$, then $\mathcal{I}_Z = \widetilde{\mathfrak{a}}$ is the sheaf of ideals generated by the ideal $\mathfrak{a}$.

Now suppose $f : X \to Y$ is a morphism of schemes. We have the induced diagonal map $\Delta : X \to X \times_Y X$. It is a fact that Δ is a locally closed immersion, i.e. Δ identifies X with a closed subscheme of an open subscheme W of $X \times_Y X$. We can thus consider the ideal sheaf $\mathcal{I}$ of X in W, i.e.

$$\mathcal{I} = \ker(\mathcal{O}_{X \times_Y X}|_W \to i_*\mathcal{O}_X).$$

Definition. The sheaf of relative differentials of $f : X \to Y$ is the sheaf

$$\Omega_{X/Y} = \Delta^*(\mathcal{I}/\mathcal{I}^2).$$

The natural action of $\mathcal{O}_{\Delta(X)}$ on $\mathcal{I}/\mathcal{I}^2$ induces an action of $\mathcal{O}_X$ on $\Omega_{X/Y}$.

It is known that $\Omega_{X/Y}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules on X. It is coherent if Y is locally Noetherian and f a morphism locally of finite presentation.

Lemma 1.2. *If $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ so that the map $f : X \to Y$ is induced by a map $A \to B$, then $\Omega_{X/Y} = \widetilde{\Omega_{B/A}}$.*

Lemma 1.3. *Suppose $f : X \to Y$ is a morphism. For any point $x \in X$, we have the equality on stalks*

$$\Omega_{X/Y,\,x} = \Omega_{\mathcal{O}_{X,x}/\mathcal{O}_{Y,f(x)}}$$

In particular, by the above lemma, on every open affine $U = \operatorname{Spec} B$ of X with corresponding open affine $V = \operatorname{Spec} A$ of Y, the map $d : B \to \Omega_{B/A}$ induces the map

$$\widetilde{d}_{U,V} : \mathcal{O}_U = \widetilde{B} \to \widetilde{\Omega_{B/A}} = \Omega^1_{V/U}.$$

Glueing these maps together as we range over all open affines, it follows that we have a canonical map

$$d : \mathcal{O}_X \to \Omega_{X/Y}$$

which is a derivation on all fibres. Note that this map d is NOT a morphism of $\mathcal{O}_X$-modules; it is instead a morphism of $f^{-1}\mathcal{O}_Y$-modules, where recall that $f^{-1}\mathcal{O}_Y$ is the sheafification of the presheaf

$$U \mapsto \operatorname*{colim}_{V \text{ open, } f(U) \subset V \subset Y} \mathcal{O}_Y(V).$$

Definition. Suppose $f : X \to Y$, and suppose $\mathcal{M}$ is a $\mathcal{O}_X$-module. A *Y-derivation* is a homomorphism of $f^{-1}\mathcal{O}_Y$-modules $D : \mathcal{O}_X \to \mathcal{M}$ such that on every open subscheme $U \subset X$ the map $D_U : \mathcal{O}_X(U) \to \mathcal{M}(U)$ satisfies Liebniz' rule.

Thus $\Omega_{X/Y}$ is equipped with a canonical Y-derivation $d : \mathcal{O}_X \to \Omega_{X/Y}$.

Suppose now we have a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

of schemes. We have a morphism of $\mathcal{O}_{X'}$-modules

$$g^*\Omega_{X/Y} := \mathcal{O}_{X'} \otimes_{g^{-1}\mathcal{O}_X} g^{-1}\Omega_{X/Y} \to \Omega_{X'/Y'}, \quad 1 \otimes g^{-1}(d_{X/Y}s) \mapsto d_{X'/Y'}(1 \otimes g^{-1}(s)).$$

Here is a collection of results on the sheaf of relative differentials. Proofs for all of these can be found in either [Har13] or [Ill02].

Proposition 1.4. *Suppose $f : X \to Y$ is a morphism of schemes.*

1. *(*Universal Property*) The functor*

$$\mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Ab}, \quad \mathcal{M} \mapsto \mathrm{Der}_Y(\mathcal{O}_X, \mathcal{M}),$$

   taking a $\mathcal{O}_X$-module $\mathcal{M}$ to the (abelian) group of Y-derivations $\mathcal{O}_X \to \mathcal{M}$, is representable by $\Omega_{X/Y}$.

   In other words, for any $\mathcal{O}_X$-module $\mathcal{M}$ with Y-derivation $d' : \mathcal{O}_X \to \mathcal{M}$, there is a unique morphism $f : \Omega_{X/Y} \to \mathcal{M}$ of $\mathcal{O}_X$-modules such that $d' = f \circ d$.

2. *If $g : Y' \to Y$ is another morphism, the sheaf of relative differentials of the induced map $X' := X \times_Y Y' \to Y'$ is*

$$\Omega_{X'/Y'} = p^*\Omega_{X/Y}$$

   where $p : X' \to X$ is the morphism induced by base-changing g.

3. *Suppose $g : Y \to Z$ is another morphism of schemes. There is an exact sequence*

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0,$$

   where the map $f^*\Omega_{Y/Z} \to \Omega_{X/Z}$ is the canonical morphism induced by the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ Z \arrow[r, equal] & Z. \end{tikzcd}$$

4. *If $Y' \to Y$ is another morphism, and $X' := X \times_Y Y'$, then*

$$\Omega_{X'/Y} = p^*\Omega_{X/Y} \oplus q^*\Omega_{Y'/Y}$$

   where $p : X' \to X$ and $q : X' \to Y'$ are the canonical projections. Moreover, the canonical map

$$p^*\Omega_{X/Y} \to \Omega_{X'/Y'}$$

   is an isomorphism of $\mathcal{O}_{X'}$-modules.

5. *Suppose Z is a closed subscheme of X with corresponding ideal sheaf $\mathcal{I}_Z$. Then, there is a natural map $\mathcal{I}_Z/\mathcal{I}_Z^2 \to \Omega_{X/Y} \otimes \mathcal{O}_Z$ fitting into an exact sequence*

$$\mathcal{I}_Z/\mathcal{I}_Z^2 \to \Omega_{X/Y} \otimes \mathcal{O}_Z \to \Omega_{Z/Y} \to 0.$$

6. *For $X = \mathbb{A}^n_Y$ the affine space of dimension n over Y, the $\mathcal{O}_X$-module $\otimes_{X/Y} \cong \mathcal{O}_X^{\oplus n}$ is free (in the obvious way)*

Suppose $f : X \to Y$ a morphism of schemes. Define the $\mathcal{O}_X$-module

$$\Omega^p_{X/Y} := \bigwedge^p \Omega_{X/Y}$$

with $\Omega^0_{X/Y} := \mathcal{O}_X$.

Lemma-Definition. There is a unique family of maps $d : \Omega^p_{X/Y} \to \Omega^{p+1}_{X/Y}$ satisfying

- the degree 0 map $d : \mathcal{O}_X \to \Omega^1_{X/Y}$ is the usual Y-derivation $d : \mathcal{O}_X \to \Omega_{X/Y}$;
- d is a Y-anti-derivation of the exterior algebra $\bigwedge^* \Omega_{X/Y} = \bigoplus_p \Omega^p_{X/Y}$, i.e. d is $f^{-1}(\mathcal{O}_Y)$-linear and
$$d(ab) = da \wedge b + (-1)^i a \wedge db$$
for a a local section of $\mathcal{O}_X$ and b a local section of $\Omega^p_{X/Y}$; and
- the family of maps makes
$$\Omega^\bullet_{X/Y} : \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/Y} \xrightarrow{d} \Omega^2_{X/Y} \to \cdots$$
into a complex, called the *de Rham complex*.

Remark 1.5. Even though each of the objects $\Omega^p_{X/Y}$ are $\mathcal{O}_X$-modules, the complex $\Omega^\bullet_{X/Y}$ is NOT a complex of $\mathcal{O}_X$-modules. This is because the differential maps $d : \Omega^p_{X/Y} \to \Omega^{p+1}_{X/Y}$ are not $\mathcal{O}_X$-linear but are only $f^{-1}\mathcal{O}_Y$-linear. Hence, $\Omega^\bullet_{X/Y}$ is in fact a complex of $f^{-1}(\mathcal{O}_Y)$-modules.

Now, suppose we have a commuting diagram

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y. \end{tikzcd}$$

This square then induces a map
$$\Omega^\bullet_{X/Y} \to g_*\Omega^\bullet_{X'/Y'}$$
of complexes, which is also a morphism of the corresponding differential graded algebra. Of course, if the above square is Cartesian, then this map
$$\Omega^\bullet_{X/Y} \to g_*\Omega^\bullet_{X'/Y'}$$
is an isomorphism.

Finally, let us briefly define the tangent bundle.

Definition. Suppose $f : X \to Y$ is a morphism. The *relative tangent sheaf* $T_{X/Y}$ is the sheaf
$$T_{X/Y} := \mathcal{H}om(\Omega^1_{X/Y}, \mathcal{O}_X).$$

Using the universal property of $\Omega^1_{X/Y}$, we see that for any open subscheme U of X, we have a natural isomorphism
$$T_{X/Y}(U) \cong \mathrm{Der}_Y(\mathcal{O}_U, \mathcal{O}_U).$$

1.2 Smooth Maps

Lemma-Definition. A morphism $f : X \to Y$ of schemes is *flat* if for any open affines $\mathrm{Spec}(A) \subset X$ and $\mathrm{Spec}(B) \subset Y$, the corresponding ring map $B \to A$ is flat. A morphism is *faithfully flat* if it is both surjective and flat.

It is easy to check that flatness is an affine local (in the sense of Vakil) property on the target Y.

Proposition 1.6. *Suppose $f : X \to S$ is a morphism of schemes.*

1. *f is flat if and only if for every $x \in X$ the local ring map $f^\#_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat.*
2. *f is flat if and only if for every $S \to S'$, the pull-back functor $\mathrm{QCoh}(S') \to \mathrm{QCoh}(X \times_S S')$ induced by the map $X \times_S S' \to S'$ is an exact functor.*
3. *Composite of flat morphisms is flat.*
4. *Fibre product of two flat (resp. faithfully flat) morphisms is flat (resp. faithfully flat).*
5. *Flatness and faithful flatness are preserved by base change.*
6. *If f is flat, then for every $x \in X$ and every $s \in S$ such that $f(x) \in \overline{\{s\}}$, there exists $x' \in X$ such that $s = f(x')$ and $x \in \overline{\{x'\}}$.*

7. *If f is flat and locally of finite presentation, then it is universally open, i.e. for every $S' \to S$ the induced map $X \times_S S' \to S'$ is an open map.*

8. *If f is quasi-compact and faithfully flat (i.e. fpqc), then $T \subset S$ is open (resp. closed) iff $f^{-1}(T)$ is open (resp. closed).*

Thus, fpqc maps can be thought of as quotient maps.

Definition. A morphism $f : X \to Y$ is *smooth* (resp. *unramified*, resp. *étale*) if f is locally of finite presentation and if the following condition is satisfied: for any commutative diagram

$$\begin{tikzcd} & & X \arrow[d, "f"] \\ T_0 \arrow[r, "i"'] \arrow[urr, "g_0"] & T_1 \arrow[r] & Y \end{tikzcd}$$

where i is a closed immersion such that the ideal sheaf $\mathcal{I}_{T_0}$ in $\mathcal{O}_{T_1}$ satisfies $\mathcal{I}_{T_0}^2 = 0$, there exists locally in the Zariski topology on T a (resp. at most one, resp. unique) Y-morphism $g : T \to X$ such that $gi = g_0$.

Remark 1.7. We can remove the phrase 'locally in the Zariski topology' from the definition of étale morphism.

We now list basic properties of unramified, smooth, and étale morphisms.

Proposition 1.8. *Suppose $f : X \to Y$ is a morphism of schemes.*

1. *Suppose f is locally of finite presentaiton. Then, f is unramified if and only if any one of the following equivalent conditions hold:*

   (a) *it is locally of finite presentation and for each $x \in X$ and $y = f(x)$, the residue field $k(x)$ is a separable algebraic extension of $k(y)$, and $f_x(\mathfrak{m}_{Y,y})\mathcal{O}_{X,x} = \mathfrak{m}_{X,x}$ where $f_x : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$.*

   (b) *for any affine opens $\mathrm{Spec}(B) \subset X$ and $\mathrm{Spec}(A) \subset Y$ the induced map $f^\# : A \to B$ is formally unramified.*

   (c) *the diagonal map $X \to X \times_Y X$ is an open immersion.*

2. *Composite of unramified morphisms is unramified.*

3. *Base change of an unramified morphism is unramified.*

4. *Open immersions are unramified.*

5. *Unramified-ness is affine local (in the sense of Vakil) on the target Y.*

6. *If $f : X \to Y$ is a morphism of S-schemes where X is unramified over S and Y is locally of finite type over S, then f is unramified.*

7. *Suppose X and Y are S-schemes and $f, g : X \to Y$ morphisms over S. Suppose Y is unramified over S. Let $x \in X$ be such that $f(x) = g(x) =: y$ where the maps $f_x, g_x : k(y) \to k(x)$ induced by f and g are equal. Then, there exists a Zariski open neighbourhood U of x in X such that $f|_U = g|_U$.*

8. *If f is unramified, then $\Omega_{X/Y} = 0$.*

Proposition 1.9. *Suppose $f : X \to S$ is a morphism.*

1. *Suppose f is locally of finite presentation. Then, f is smooth if and only if any one of the following conditions hold:*

   (a) *f is flat and for every S-morphism $\overline{s} : \mathrm{Spec}\, k \hookrightarrow S$ for k algebraically closed, the fibre $X_{\overline{s}} = X \times_S \overline{s}$ is regular.*

   (b) *f is flat and all fibers $f^{-1}(s)$ are regular and remain so after extension of scalars to some perfect extension of $k(s)$.*

2. *Composite of smooth morphisms is smooth.*

3. *Base change of an smooth morphism is smooth.*

4. *Open immersions are smooth.*

5. *Smoothness is affine local (in the sense of Vakil) on the target Y.*

6. *Smooth morphisms is universally open, i.e. f is open and for any base change $Y' \to Y$, the corresponding map $X \times_Y Y' \to Y'$ is also open.*

7. *If f is smooth, then $\Omega_{X/Y}$ is locally free of finite type. As a consequence, for $f : X \to Y$ smooth we have*

$$T_{X/Y}^{\vee} := \mathcal{H}om(T_{X/Y}, \mathcal{O}_X) \cong \Omega_{X/Y}.$$

8. *Suppose $g : Y \to Z$ is another morphism of schemes. If f is smooth, the exact sequence of Proposition 1.4(3) extends to the locally split exact sequence*

$$0 \to f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0.$$

   On the other hand, if $g \circ f$ is smooth and if the above sequence is exact and locally split, then f is smooth.

9. *Suppose Z is a closed subscheme of X with corresponding ideal sheaf $\mathcal{I}_Z$. If $f|_Z$ is smooth, the exact sequence of Proposition 1.4(4) extends to the locally split exact sequence*

$$0 \to \mathcal{I}_Z/\mathcal{I}_Z^2 \to \Omega_{X/Y} \otimes \mathcal{O}_Z \to \Omega_{Z/Y} \to 0.$$

   On the other hand, if f is smooth and the above sequence is exact and locally split, then $f|_Z$ is smooth.

10. *(*Implicit Function Theorem*) Suppose Z is a closed subscheme of X with corresponding ideal sheaf $\mathcal{I}_Z$. Suppose $x \in Z$ is a point such that $f|_Z$ is smooth in some open neighbourhood $U \cap Z$ of x, with $U \subset X$ open. Write $n = \dim(\mathcal{I}_Z/\mathcal{I}_Z^2)_x$ and $n + m = \dim_{k(x)} \Omega_{X/Y,x} \otimes k(x)$. Then, by shrinking U if necessary, there is an étale morphism $U \to \mathbb{A}_Y^{n+m}$ such that*

$$U \cap Z = U \times_{\mathbb{A}_Y^{n+m}} \mathbb{A}_Y^n$$

    where $\mathbb{A}_Y^n \hookrightarrow \mathbb{A}_Y^{n+m}$ is the inclusion into the first n coordinates.

Now for each point $x \in X$, set

$$\dim_x(f) := \dim_{k(x)} \Omega_{X/Y} \otimes k(x)$$

where $k(x)$ is the residue field of x. If f is smooth, $\Omega_{X/Y}$ is locally free and of finite type, and so $\dim_x(f) : X \to \mathbb{Z}_{\geq 0}$ is a locally constant function of x.

Definition. A smooth function $f : X \to Y$ is *pure of relative dimension* r if $\dim_x(f) = r$ for all $x \in X$.

Lemma 1.10. *If f is smooth and pure of relative dimension r, then $\Omega_{X/Y}^p = 0$ for all $p > r$, and $\Omega_{X/Y}^p$ is a locally free $\mathcal{O}_X$-module of rank $\binom{r}{p}$ for all $0 \leq p \leq r$.*

Proposition 1.11. *Suppose $f : X \to Y$ is a morphism.*

1. *f is étale if and only if any one of the following conditions hold:*

   (a) *f is flat and unramified;*

   (b) *f is smooth and unramified;*

   (c) *f is smooth and pure of relative dimension 0;*

   (d) *f is flat, locally of finite presentation, and every fibre $f^{-1}(y)$ is given by the disjoint union $\bigsqcup_{i \in I} \operatorname{Spec} k_{i,y}$ where each $k_{i,y}$ is a finite separable field extension of the residue field $\kappa(y)$;*

   (e) *f is smooth and locally quasi-finite;*

   (f) *f is locally of finite presentation and for any affine opens $\operatorname{Spec}(B) \subset X$ and $\operatorname{Spec}(A) \subset S$ the induced map $f^\# : A \to B$ is formally étale;*

   (g) *for every $x \in X$ there is an open neighbourhood U of X around x and an open affine $V = \operatorname{Spec} A$ around $f(x)$ with $f(U) \subset V$ such that U is V-isomorphic to an open subscheme of $\operatorname{Spec} \left(A[t]/\langle f \rangle\right)_{f'}$ for some monic $f \in A[t]$ (with f' the usual derivative of f).*

2. *Étale morphisms are preserved under composition and base change.*

3. *Being an étale morphism is a local property on both the source and the target.*

4. *Product of a finite family of étale morphisms is étale.*

5. *Suppose $g : Y \to Z$ an unramified map and $f : X \to Y$ a map such that $g \circ f$ is étale. Then, f is étale.*
6. *Any S-morphism between étale S-schemes is étale.*
7. *Étale morphisms are locally quasi-finite.*
8. *Open immersions are étale. Moreover, a morphism is an open immersion if and only if it is étale and universally injective.*
9. *A map $X \to \operatorname{Spec} k$ is étale if and only if X is the disjoint union of $\operatorname{Spec} k'$ for k' a finite separable field extension of k.*
10. *Étale morphisms are open.*

1.3 de Rham Cohomology

Suppose $f : X \to Y$ is a morphism of schemes.

Definition. The *de Rham cohomology of X over S* are the *hyper*-cohomology groups

$$H^p_{dR}(X/Y) := \mathbb{H}^i(X, \Omega^\bullet_{X/Y}) := H^i(R\Gamma(X, \Omega^\bullet_{X/S})).$$

Remark 1.12. The de Rham cohomology groups are NOT the same as the cohomology of the complex $\Omega^\bullet_{X/Y}$. Of course, the cohomology of the complex $\Omega^\bullet_{X/Y}$ would be a sheaf of $f^{-1}\mathcal{O}_Y$-modules.

These de Rham cohomology groups are naturally modules over $\Gamma(Y, \mathcal{O}_Y)$.

Given a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ \Big\downarrow & & \Big\downarrow f \\ Y' & \longrightarrow & Y, \end{array}$$

the canonical maps $\Omega^\bullet_{X/Y} \to g_*\Omega^\bullet_{X'/Y'}$ yields pullback maps

$$g^* : R\Gamma(X, \Omega^\bullet_{X/Y}) \to R\Gamma(X', \Omega_{X'/Y'})$$

and thus maps

$$g^* : H^*_{dR}(X/Y) \to H^*_{dR}(X'/Y').$$

In particular, taking $Y' = Y$, we see that H^q_{dR} defines a contravariant (δ-)functor

$$H^q_{dR} : \mathrm{Sch}_Y \to \mathrm{Mod}\big(\mathcal{O}_Y(Y)\big).$$

Lemma 1.13. *Let $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$, so that the map $f : X \to Y$ is induced by a ring map $A \to B$. Then,*

$$H^*_{dR}(X/Y) \cong H^*(\Omega^\bullet_{B/A})$$

is the usual cohomology of the complex $\Omega^\bullet_{X/Y} = \Omega^\bullet_{B/A}$.

Let us try to compute de Rham cohomology in more generality. Of course, the de Rham cohomology is NOT equal to the usual cohomology $H^*(X, \Omega^q_{X/Y})$ of the bundle $\Omega^q_{X/Y}$ for some specific q, as again we are applying $R\Gamma$ to $\Omega^\bullet_{X/S}$ and then taking cohomology. Thus, in order to compute de Rham cohomology, we need to take an injective resolution $I^{\bullet,\bullet}$ of $\Omega^\bullet_{X/S}$, i.e. $I^{\bullet,\bullet}$ is a double complex of injective objects such that for each p, the complex $I^{p,\bullet}$ is an injective resolution of $\Omega^p_{X/S}$. Writing $I^\bullet_{tot}$ for the total complex of the double complex $I^{\bullet,\bullet}$, our choice of $I^{\bullet,\bullet}$ means that $\Omega^\bullet_{X/S} \to I^\bullet_{tot}$ is a quasi-isomorphism where each term in $I^\bullet_{tot}$ is injective, and so $R\Gamma\Omega^\bullet_{X/S}$ is quasi-isomorphic to $\Gamma(I^\bullet_{tot}) = (\Gamma I)^\bullet_{tot}$. We can compute the cohomology of this total complex by using a spectral sequence on the double complex $\Gamma I^{\bullet,\bullet}$. Since $I^{p,\bullet}$ is an injective resolution of $\Omega^p_{X/Y}$, the cohomology of $\Gamma I^{p,\bullet}$ is precisely $H^q(X, \Omega^p_{X/Y})$. By computing the spectral sequence in two different ways, we get the following result.

Proposition 1.14 (Hodge to de Rham Spectral Sequence)**.** *There is a spectral sequence of $\Gamma(Y, O_Y)$-modules*

$$E_1^{pq} = H^{p+q}(X, \Omega^p_{X/Y}) \Rightarrow H^{p+q}_{dR}(X/Y).$$

The differentials on the first page are the maps

$$d_1^{p,q} : H^q(X, \Omega^p_{X/Y}) \to H^q(X, \Omega^{p+1}_{X/Y})$$

induced by the usual differential $d : \Omega^p_{X/Y} \to \Omega^{p+1}_{X/Y}$.

Corollary 1.14.1. $H^0_{dR}(X/Y) = \ker\left(d : \Gamma(X, \mathcal{O}_X) \to \Gamma(X, \Omega_{X/Y})\right)$

Definition. The $\Gamma(Y, \mathcal{O}_Y)$-modules $H^q(X, \Omega^p_{X/Y})$ are called the *Hodge cohomology groups of X over Y.*

If for example $Y = \operatorname{Spec} k$ and $f : X \to Y$ is proper, then the Hodge cohomology groups are known to be finite dimensional k-vector spaces, and thus the spectral sequence implies that the de Rham cohomology groups are finite dimensional k-vector spaces.

There is another spectral sequence that computes de Rham cohomology (cf. [Sta23, 0FM6]). This second spectral sequence relates the cohomology of the cohomology sheaves of the complex $\Omega^\bullet_{X/Y}$ to de Rham cohomology.

Proposition 1.15 (Conjugate Spectral Sequence for de Rham Cohomology)**.** *There is a spectral sequence*

$$E_2^{p,q} = H^q(X, H^p(\Omega^\bullet_{X/Y})) \Rightarrow H^{p+q}_{dR}(X/Y),$$

where $H^p(\Omega^\bullet_{X/Y}) \in \operatorname{QCoh}(X)$ is the cohomology sheaf of the de Rham complex.

1.4 Connections

We mostly follow [Con23], generalizing to arbitary schemes. This reference also has a lot of computations in local coordinates, linking back to the classical theory.

1.4.1 Definition

Let $p : X \to Y$ be a morphism of schemes. Suppose $\mathcal{E}$ is a quasi-coherent sheaf on X (for example, $\mathcal{E}$ could be a locally free coherent sheaf aka a vector bundle).

Definition. A *connection on $\mathcal{E}$ relative to Y* is a map of abelian sheaves

$$\nabla : \mathcal{E} \to \Omega_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{E}$$

that satisfies the Liebniz rule

$$\nabla(fs) = df \otimes s + f \cdot \nabla s$$

for $f \in \mathcal{O}_X(U)$ and $s \in \mathcal{E}(U)$, for any open subscheme $U \subset X$.

Notice that ∇ is $p^{-1}\mathcal{O}_Y$-linear. Moreover, if ∇ and ∇' are connections, then $\nabla - \nabla' : \mathcal{E} \to \Omega_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{E}$ is easily seen to be $\mathcal{O}_X$-linear. Thus, the space of connections on $\mathcal{E}$ relative to Y is a principal homogeneous space under the group

$$\operatorname{Hom}(\mathcal{E}, \Omega_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{E}).$$

Example 1.16. The canonical differential $d : \mathcal{O}_X \to \Omega_{X/Y} = \Omega_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{O}_X$ is a connection on $\mathcal{O}_X$ relative to Y.

Example 1.17. Suppose Λ is a locally free $p^{-1}\mathcal{O}_Y$-module with finite rank. Set $\mathcal{E} := \mathcal{O}_X \otimes_{p^{-1}\mathcal{O}_Y} \Lambda$. Then, the map

$$\nabla := d \otimes 1 : \mathcal{E} = \mathcal{O}_X \otimes_{p^{-1}\mathcal{O}_Y} \Lambda \to \Omega_{X/Y} \otimes_{p^{-1}\mathcal{O}_Y} \Lambda = \Omega_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{E}$$

satisfies Liebniz rule (because d does), and hence is a connection. Moreover, one sees that under the canonical map $\Lambda \to \mathcal{E}$, the image of Λ is killed by ∇. It turns out that ∇ is the unique connection on $\mathcal{E}$ killing Λ. If p is smooth then it turns out that $\ker \nabla = \Lambda$.

If $Y = \operatorname{Spec} k$, then by definition Λ is a local system (i.e. locally constant sheaf) of k-vector spaces, and the above characterisation of ∇ is a key ingredient of the Riemann-Hilbert correspondence.

If we take $\Lambda = p^{-1}\mathcal{O}_Y$, the above construction recovers the previous example.

Classically, connections are a mechanism to differentiate sections of a vector bundle on a manifold along a vector field. We can recover this construction in this more general case.

Definition. A *vector field of X relative to Y* is a Y-derivation of $\mathcal{O}_X$, i.e. it is an element of $\operatorname{Der}_Y(\mathcal{O}_X, \mathcal{O}_X)$.

By the universal property of the sheaf of relative differentials, a vector field of X relative to Y can also be viewed as a $\mathcal{O}_X$-linear morphism $\Omega_{X/Y} \to \mathcal{O}_X$.

Recall the relative tangent bundle $T_{X/Y}$. We see immediately that for any open subscheme $U \hookrightarrow X$, the space of sections $T_{X/Y}(U)$ is the space of all vector fields of U relative to Y.

Suppose v is a vector field of X relative to Y, which we view as a $\mathcal{O}_X$-linear morphism $v : \Omega_{X/Y} \to \mathcal{O}_X$. If $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{X/Y}$ is a connection on a coherent sheaf $\mathcal{E}$, then we have the composite

$$\nabla_v : \mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{X/Y} \xrightarrow{1\otimes v} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X = \mathcal{E}.$$

A similar statement holds for any open $U \subset X$. Hence, a connection defines (and is defined by) a $\mathcal{O}_X$-linear map

$$T_{X/Y} \to \mathcal{H}om(\mathcal{E}, \mathcal{E}) =: \mathcal{E}nd(\mathcal{E}), \quad v \in T_{X/Y}(U) \mapsto (1 \otimes v) \circ \nabla \in \mathrm{End}(\mathcal{E}|_U).$$

Since ∇ satisfies the Liebniz rule and v is $\mathcal{O}_X$-linear, it follows that ∇_v also satisfies the Liebniz rule

$$\nabla_v(f \cdot s) = v(f) \cdot s + f \cdot \nabla_v(s)$$

for all $f \in \mathcal{O}_X(U)$ and $s \in \mathcal{E}(U)$, for all open $U \subset X$. Here, notice that we are now viewing v as a $p^{-1}\mathcal{O}_Y$-module morphism $v : \mathcal{O}_X \to \mathcal{O}_X$.

1.4.2 Another Perspective on Connections

There is another perspective on connections that is more algebraic-geometric. Consider the diagonal morphism $\Delta : X \to X \times_Y X$, and let $\mathcal{I}_\Delta \subset \mathcal{O}_{X\times_Y X}$ denote the ideal sheaf of this locally closed immersion. Set

$$\mathcal{P}_{X/Y} := \mathcal{O}_{X\times_Y X}/\mathcal{I}_\Delta^2.$$

This is the structure sheaf of the first infinitesimal thickening of the diagonal (see [Ill02, Section 1.1-1.2]). In particular, it is naturally a sheaf on X. Now, the two projections $p_1 : X \times_Y X \to X$ and $p_2 : X \times_Y X \to X$ induce maps

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, shift left, "p_1^*"] \arrow[r, shift right, "p_2^*"'] & \Delta^*\mathcal{O}_{X\times_Y X} \arrow[r, two heads] & \mathcal{P}_{X/Y}, \end{tikzcd}$$

denoted by j_1 and j_2 respectively. These morphisms $j_i : \mathcal{O}_X \to \mathcal{P}_{X/Y}$ induce two different $\mathcal{O}_X$-algebra structures on $\mathcal{P}_{X/Y}$. By definition, $\Omega_{X/Y} = \mathcal{I}_\Delta/\mathcal{I}_\Delta^2$, and so we have an injection $\Omega_{X/Y} \hookrightarrow \mathcal{P}_{X/Y}$. One checks that we have an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \Omega_{X/Y} \arrow[r] & \mathcal{P}_{X/Y} \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[l, bend right, "j_1"'] \arrow[l, bend left, "j_2"] & 0 \end{tikzcd}$$

split by both j_1 and j_2. Here, the map $\mathcal{P}_{X/Y} \to \mathcal{O}_X$ is the map induced by $\Delta^* : \mathcal{O}_{X\times_Y X} \to \mathcal{O}_X$, noting that the kernel of Δ is by definition $\mathcal{I}_\Delta$. Moreover, one checks that $j_2 - j_1 : \mathcal{O}_X \to \Omega_{X/Y}$ coincides with the canonical differential $d : \mathcal{O}_X \to \Omega_{X/Y}$ (that $j_2 - j_1$ lands in the subsheaf $\Omega_{X/Y}$ follows by exactness and the fact that j_1 and j_2 split the exact sequence.)

Proposition 1.18. *The data of a connection on a coherent $\mathcal{O}_X$-module $\mathcal{E}$ is the same as the data of a $\mathcal{P}_{X/Y}$-linear isomorphism*

$$\mathcal{P}_{X/Y} \otimes_{j_2, \mathcal{O}_X} \mathcal{E} \xrightarrow{\sim} \mathcal{E} \otimes_{\mathcal{O}_X, j_1} \mathcal{P}_{X/Y}$$

lifting the identity on $\mathcal{E}$.

Remark 1.19. By coherence of $\mathcal{E}$, any $\mathcal{P}_{X/Y}$-linear map

$$\mathcal{P}_{X/Y} \otimes_{j_2, \mathcal{O}_X} \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X, j_1} \mathcal{P}_{X/Y}$$

lifting the identity on $\mathcal{E}$ is automatically an isomorphism.

Proof. Let

$$\xi : \mathcal{P}_{X/Y} \otimes_{j_2, \mathcal{O}_X} \mathcal{E} \xrightarrow{\sim} \mathcal{E} \otimes_{\mathcal{O}_X, j_1} \mathcal{P}_{X/Y}$$

be any morphism of abelian sheaves. It suffices to specify ξ at the level of the corresponding pre-sheaves, since sheafification is an exact functor. Consider the comutative diagram

$$\begin{tikzcd} & \mathcal{E} \arrow[dl, hook, bend right] \arrow[dr, hook, bend left] & \\ \mathcal{P}_{X/Y} \otimes_{j_2, \mathcal{O}_X} \mathcal{E} \arrow[ur, bend left=50, "\Delta^*\otimes 1"] \arrow[rr, "\xi"'] & & \mathcal{E} \otimes_{\mathcal{O}_X, j_1} \mathcal{P}_{X/Y} \arrow[ul, bend right=50, "1\otimes\Delta^*"'] \end{tikzcd}$$

where $\Delta^* : \mathcal{P}_{X/Y} \to \mathcal{O}_X$. Since $\Delta^* \circ j_1 = \Delta^* \circ j_2 = id_{\mathcal{O}_X}$, it follows that ξ lifts the identity on $\mathcal{E}$ if and only if for any $s \in \mathcal{E}(U)$, the section

$$\xi(1 \otimes s) - s \otimes 1$$

of $\mathcal{E} \otimes_{\mathcal{O}_X, j_1} \mathcal{P}_{X/Y}$ lies in the kernel of

$$1 \otimes \Delta : \mathcal{E} \otimes_{\mathcal{O}_X, j_1} \mathcal{P}_{X/Y} \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X = \mathcal{E},$$

which is precisely $\mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{X/Y}$. Thus, the morphism ξ lifts the identity on $\mathcal{E}$ if and only if we have a well-defined a map

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{X/Y},$$

the correspondence given by $\xi(1 \otimes s) = s \otimes 1 + \nabla s$.

Now, ξ is furthermore $\mathcal{P}_{X/Y}$-linear if and only if $j_2(f) \cdot \xi(1 \otimes s) = \xi(1 \otimes fs)$ for $f \in \mathcal{O}_X(U)$, where $j_2 : \mathcal{O}_X \to \mathcal{P}_{X/Y}$. This equation is equivalent to

$$s \otimes j_2(f) + f \cdot \nabla s = s \otimes j_1(f) + \nabla(fs),$$

which by the formula $d = j_2 - j_1$ is precisely equivalent to ∇ satisfying the Liebniz rule. □

Remark 1.20. This proposition can be viewed as saying that a connection is a first-order descent data on the coherent sheaf $\mathcal{E}$. See [BO15, Chapter 2].

1.5 Curvature, Integrable Connections, and Riemann-Hilbert

Recall that we could extend $d : \mathcal{O}_X \to \Omega_{X/Y}$ to form the differentials of a complex $\Omega^{\bullet}_{X/Y}$. We can do a similar thing for arbitrary connections. As always, fix a Y-scheme X.

Theorem 1.21. *Suppose $(\mathcal{E}, \nabla)$ is a coherent sheaf on X with a connection ∇ relative to Y. Then, for all $p \geq 0$, there is a unique abelian sheaf map*

$$\nabla^p : \Omega^p_{X/Y} \otimes \mathcal{E} \to \Omega^{p+1}_{X/Y} \otimes \mathcal{E}$$

such that $\nabla^0 = \nabla$ satisfying

$$\nabla^{p+q}((\omega_p \wedge \omega_q) \otimes s) = \nabla^p(\omega_p \otimes s) \wedge \omega_q + (-1)^p \omega_p \wedge \nabla^q(\omega_q \otimes s)$$

for local sections s of $\mathcal{E}$, ω_p of $\Omega^p_{X/Y}$, and ω_q of $\Omega^q_{X/Y}$.

The definition of ∇^p is essentially as follows. Suppose $U \subset X$ open, and suppose $s \in \mathcal{E}(U)$ and $\omega \in \Omega^p_{X/S}(U)$ are local sections. Then, we set

$$\nabla^p(\omega \otimes s) := d\omega \otimes s + (-1)^p \omega \wedge \nabla s.$$

Here and throughout, when we write $\omega \wedge \rho$ for a section of $\Omega^i_{X/Y} \otimes_{\mathcal{O}_X} (\Omega^1_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{E}$ we really mean the image of $\omega \otimes \rho \in \Omega^i_{X/Y} \otimes_{\mathcal{O}_X} (\Omega^1_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{E}$ under the canonical map

$$\Omega^i_{X/Y} \otimes_{\mathcal{O}_X} (\Omega^1_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{E} \to \Omega^{i+1}_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{E}, \quad \omega \otimes \tau \otimes e \mapsto (\omega \wedge \tau) \otimes e.$$

Some tedious computation checks that ∇^p is well-defined (for instance, that $\nabla^p((f\omega) \otimes s) = \nabla^p(\omega \otimes (fs))$ for any $f \in \mathcal{O}_X(U)$) and satisfies the required equation. By linearity it can then be extended to a well-defined abelian sheaf map

$$\nabla^p : \Omega^p_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E} \to \Omega^{p+1}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}.$$

Uniqueness is obvious from the given characterising equation. The remaining statements are further tedious calculations.

Some computation yields the following.

Lemma 1.22. *For any ω a local section of $\Omega^i_{X/Y}$ and any s a section of $\mathcal{E}$, one has*

$$(\nabla^{p+1} \circ \nabla^p)(\omega \otimes e) = \omega \wedge (\nabla^1 \circ \nabla^0)(e).$$

Corollary 1.22.1. $\Omega^{\bullet}_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{E}$ *is a complex if and only if $\nabla^1 \circ \nabla^0 = 0$.*

This motivates the following definition.

Definition. The *curvature* K_∇ of a connection ∇ is the map $K_\nabla := \nabla^1 \circ \nabla^0 : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^2_{X/Y}$.

A connection is *integrable* (or *flat*) if its curvature is identically zero. The resulting complex $\Omega^\bullet_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{E}$ with differentials ∇^p is called the *de Rham complex of* $(\mathcal{E}, \nabla)$.

Another corollary of the previous computation is the following suprising (at least at first glance) fact.

Corollary 1.22.2. *The curvature map $K_\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^2_{X/Y}$ is a $\mathcal{O}_X$-linear map of $\mathcal{O}_X$-modules.*

Remark 1.23. Most authors abuse notation by writing ∇ for any of the ∇^p. This abuse of notation is similar in spirit to writing $d : A^p \to A^{p+1}$ for the differentials of a complex $A^\bullet$, for all p.

Example 1.24. The connection d on $\mathcal{O}_X$ is integrable. More generally, for a locally free $\pi^{-1}\mathcal{O}_Y$-module Λ of finite rank (with $\pi : X \to Y$ the structure map), the resulting connection $\nabla_\Lambda = d \otimes 1$ on $\mathcal{O}_X \otimes_{\pi^{-1}\mathcal{O}_Y} \Lambda$ is integrable.

Remark 1.25. One can translate the condition of integrability in terms of the perspective given in Section 1.4.2, and this is done in [BO15, Chapter 2]. Here, they show that a connection is integrable if and only if it can be extended to a certain 'stratification' of the sheaf $\mathcal{E}$.

Lemma 1.26. *If $X \to Y$ is smooth, then ∇ is integrable if and only if $\nabla_{[v,w]} = [\nabla_v, \nabla_w]$ for relative vector fields v, w (i.e. local sections of $T_{X/Y}$).*

Recall that a connection ∇ on $\mathcal{E}$ induces a morphism of sheaves

$$\widetilde{\nabla} : T_{X/Y} \to \mathcal{E}nd(\mathcal{E}).$$

Notice also that both $T_{X/Y}$ and $\mathcal{E}nd(\mathcal{E})$ have $\pi^{-1}\mathcal{O}_Y$-Lie algebra structures on them via the standard commutator pairing. The previous lemma can thus be recast as the following.

Lemma 1.27. *A connection ∇ is integrable if and only if $\widetilde{\nabla}$ is a Lie algebra morphism.*

The importance of integrability is due to the Riemann-Hilbert correspondence.

Theorem 1.28. *Suppose $X \to \operatorname{Spec}\mathbb{C}$ is a smooth map, and suppose $(\mathcal{E}, \nabla)$ is a vector bundle on X with integrable connection. The sheaf $\ker\nabla \subset \mathcal{E}$ is a locally constant sheaf with $\underline{\mathbb{C}}$-rank equal to the $\mathcal{O}_X$-rank of $\mathcal{E}$, and the natural map*

$$\mathcal{O}_X \otimes_{\underline{\mathbb{C}}} \ker\nabla \to \mathcal{E}$$

is an isomorphism identifying ∇ with $\nabla_{\ker\nabla} = d \otimes 1$.

Moreover, the functors $(\mathcal{E}, \nabla) \mapsto \ker\nabla$ and $\Lambda \mapsto (\mathcal{O}_X \otimes_{\underline{\mathbb{C}}} \Lambda, \nabla_\Lambda)$ are inverse equivalences of categories between the category of flat vector bundles and the category of local systems of finite dimensional $\mathbb{C}$-vector spaces.

Bhatt and Lurie also proved a version of Riemann Hilbert in positive characteristic, though they work with étale sheaves rather than Zariski sheaves as above.

1.6 The Category $MC(X/Y)$ of Bundles with Connections

Throughout, we fix $p : X \to Y$.

Definition. Suppose $(\mathcal{E}, \nabla)$ and $(\mathcal{F}, \nabla')$ are quasi-coherent $\mathcal{O}_X$-modules with Y-connections. An $\mathcal{O}_X$-linear mapping $\Phi : \mathcal{E} \to \mathcal{F}$ is *horizontal* if

$$\Phi|_U \circ \nabla_v = \nabla'_v \circ \Phi|_U$$

for all local sections $v \in T_{X/Y}(U)$, for all open $U \subset X$.

Let $MC(X/Y)$ denote the abelian category whose objects are $(\mathcal{E}, \nabla)$ where $\mathcal{E}$ are quasi-coherent $\mathcal{O}_X$-modules with connection ∇, and whose morphisms are horizontal $\mathcal{O}_X$-linear maps.

Let $MC_{int}(X/Y)$ be the full subcategory of $MC(X/Y)$ consisting of sheaves with flat connection.

Of course, in order that $MC(X/Y)$ is an abelian category, we need kernels and cokernels. Suppose $\Phi : (\mathcal{E}, \nabla) \to (\mathcal{F}, \nabla')$ is a horizontal morphism. The kernel of Φ in $\mathrm{QCoh}(X)$ is simply the sheaf $U \mapsto \ker\Phi_U$. Since Φ is horizontal, one checks that the image of

$$\nabla|_{\ker\Phi} : \ker\Phi \to \Omega^1_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{E}$$

is in fact contained in $\Omega^1_{X/Y} \otimes_{\mathcal{O}_X} \ker\Phi$. It is clear that $(\ker\Phi, \nabla|_{\ker\Phi})$ acts as the kernel of Φ in the category $MC(X/Y)$. One can do a similar thing for cokernels, though the sheafification required in the definition of a cokernel of a map of quasicoherent sheaves makes things slightly more annoying.

This abelian category $MC(X/Y)$ has further operations defined on it, such as a direct sum, tensor product, etc.

Definition. Given two quasi-coherent sheaves $\mathcal{E}, \mathcal{E}'$ on X with connections ∇, ∇' respectively, the *direct sum* $\nabla \oplus \nabla'$ is the connection

$$\mathcal{E} \oplus \mathcal{E}' \xrightarrow{\nabla \oplus \nabla'} (\mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{X/Y}) \oplus (\mathcal{E}' \otimes_{\mathcal{O}_X} \Omega_{X/Y}) \cong (\mathcal{E} \oplus \mathcal{E}') \otimes_{\mathcal{O}_X} \Omega_{X/Y}.$$

Lemma-Definition. Given two quasi-coherent sheaves $\mathcal{E}, \mathcal{E}'$ on X with connections ∇, ∇' respectively, the *tensor product* $\nabla \otimes \nabla'$ is the connection

$$\mathcal{E} \otimes \mathcal{E}' \to (\mathcal{E} \otimes \mathcal{E}') \otimes_{\mathcal{O}_X} \Omega_{X/Y}$$

given by

$$s \otimes s' \mapsto \nabla(s) \otimes s' + s \otimes \nabla'(s').$$

Recall that for a quasi-coherent sheaf $\mathcal{E}$, the dual sheaf is

$$\mathcal{E}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X), \quad \text{i.e. given by} \quad \mathcal{E}^\vee(U) := \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{E}|_U, \Omega_{X/Y}|_U).$$

Lemma-Definition. Given a quasi-coherent sheaf $\mathcal{E}$ on X with connection ∇, the *dual connection* $\nabla^\vee$ is the connection

$$\mathcal{E}^\vee \to \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \Omega_{X/Y}$$

given by

$$\nabla^\vee(\ell) := d \circ \ell - (1 \otimes \ell) \circ \nabla \in \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{E}|_U, \Omega_{X/Y}|_U) = \Gamma(U, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \Omega_{X/Y})$$

for any $\ell \in \mathcal{E}^\vee(U)$.

Notice that, for any $U \subset X$ open and any $\ell \in \mathcal{E}^\vee(U)$ and $s \in \mathcal{E}(U)$, we have the section $\ell(s) \in \mathcal{O}_X$. Then, the dual connection $\nabla^\vee$ is constructed so that

$$d\big(\ell(s)\big) = (\nabla^\vee \ell)(s) + (1 \otimes \ell)(\nabla s).$$

One can check that $(\mathcal{E}^\vee)^\vee \cong \mathcal{E}$ and $\nabla^{\vee\vee} = \nabla$.

One also has a internal Hom functor, though certain extra conditions are required.

Lemma-Definition. Suppose given two quasi-coherent sheaves $\mathcal{E}_1, \mathcal{E}_2$ on X with connections ∇_1, ∇_2 respectively. Suppose also that $\mathcal{E}_1$ is locally of finite presentation. We can define a connection ∇ on $\mathcal{E} = \mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2) = \mathcal{E}_2 \otimes \mathcal{E}_1^\vee$ via

$$\nabla = \nabla_2 \otimes \nabla_1^\vee.$$

One checks that for a local section $\varphi : \mathcal{E}_1|_U \to \mathcal{E}_2|_U$ of $\mathcal{E}$, we have

$$\nabla\varphi = \nabla_2 \circ \varphi - (1 \otimes \varphi) \circ \nabla_1 \in \Gamma(U, \mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{X/Y}) = \mathrm{Hom}_U\big(\mathcal{E}_1|_U, (\mathcal{E}_2 \otimes_{\mathcal{O}_X} \Omega_{X/Y})|_U\big).$$

One can check that all of the above constructions are compatible for each other. One can also check that all of the above constructions preserve the category $MC_{int}(X/Y)$.

The next construction gives a sort of functoriality between different $MC(-)$ categories.

Lemma-Definition. Suppose we have a commutative square

$$\begin{array}{ccc} X' & \xrightarrow{h'} & X \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ Y' & \xrightarrow{h} & Y \end{array}$$

and a quasi-coherent sheaf $\mathcal{E}$ on X with a connection ∇ relative to Y. The *pullback connection* ∇' on the quasi-coherent sheaf $h'^*\mathcal{E}$ on X' relative to Y' is the map

$$\nabla' : h'^*\mathcal{E} := \mathcal{O}_{X'} \otimes_{h'^{-1}\mathcal{O}_X} h'^{-1}\mathcal{E} \to \Omega_{X'/Y'} \otimes_{\mathcal{O}_{X'}} h'^*\mathcal{E}$$

given by

$$\nabla'(f' \otimes h'^* s) := df' \otimes h'^{-1} s + f' \cdot (\eta \otimes 1)\big(h'^*(\nabla s)\big)$$

where $\eta : h'^*\Omega_{X/Y} \to \Omega_{X'/Y'}$ is the canonical map.

The above construction yields a functor $(h,h')^* : MC(X/Y) \to MC(X'/Y')$ that sends integrable sheaves to integrable sheaves.

Finally, all of these constructions are also compatible with the above construction of a connection on $\mathcal{O}_X \otimes_{p^{-1}\mathcal{O}_Y} \Lambda$ for Λ a locally free $p^{-1}\mathcal{O}_Y$-module on Y of finite rank. For instance, if we denote ∇_Λ to be the connection on $\mathcal{E}_\Lambda := O_X \otimes_{p^{-1}\mathcal{O}_Y} \Lambda$, then

$$\nabla_\Lambda^\vee = \nabla_{\Lambda^\vee}.$$

Remark 1.29. There is actually a neat way to see the abelian category structure on $MC_{int}(X/Y)$-directly. Suppose $X \to Y$ is fixed, and consider the tangent sheaf $T_{X/Y}$. Let

$$T^\bullet T_{X/Y} := \bigoplus_{n \geq 0} T_{X/Y}^{\otimes n}$$

be the corresponding sheaf of tensor algebras, where $T_{X/Y}^{\otimes 0} := \mathcal{O}_X$. We then define the sheaf $D_{X/Y}$, the sheaf of *PD differential operators*, by taking the sheafification of the quotient of $T^\bullet T_{X/Y}$ under the equivalence relation generated by

$$\partial \cdot f - f \cdot \partial = \partial(f) \quad \text{and} \quad \partial \otimes \partial' - \partial' \otimes \partial = [\partial, \partial']$$

for local sections ∂, ∂' of $T_{X/Y}$ and f a local section of $\mathcal{O}_X$. It turns out that a quasi-coherent sheaf with integrable connection can be viewed as a quasi-coherent $D_{X/Y}$-modules on X, and this gives an equivalence of categories between $MC_{int}(X/Y)$ and $\mathrm{QCoh}_X(\mathcal{D}_{X/Y})$. This perspective also immediately shows that $MC_{int}(X/Y)$ has enough injectives.

1.7 Gauss-Manin Connections on de Rham Cohomology Sheaves

Suppose $\pi : X \to Y$ is a morphism of schemes. Recall that we defined the de Rham cohomology group $H^p_{dR}(X/Y)$ as the image of $\Omega^\bullet_{X/Y}$ under the composition of the derived functors

$$D_{Coh}(X) \xrightarrow{R\Gamma} D(\mathrm{Ab}) \xrightarrow{H^p} \mathrm{Ab}.$$

Here, for a scheme S, we write $D_{Coh}(S)$ to be the derived category of coherent sheaves on S. We can do a similar thing for $R\pi_*$ instead.

Definition. The *p'th relative de Rham cohomology sheaf* $\mathcal{H}^p_{dR}(X/Y)$ is the image of $\Omega^\bullet_{X/Y}$ under the composition

$$D_{Coh}(X) \xrightarrow{R\pi_*} D_{Coh}(Y) \xrightarrow{H^p} \mathrm{Coh}(Y).$$

Notice that the relative de Rham cohomology sheaf is a coherent sheaf on Y.

This has all the usual properties that one would expect from cohomology sheaves.

Proposition 1.30. *The sheafification of the pre-sheaf on* Y

$$V \mapsto H^p_{dR}(\pi^{-1}(V)/Y)$$

is the sheaf $\mathcal{H}^p_{dR}(X/Y)$ *(where recall that* $\pi^{-1}(V) = X \times_Y V$*).*

Proof. Let $\mathcal{I}^{\bullet,\bullet}$ be a Cartan-Eilenberg resolution of $\Omega^\bullet_{X/Y}$, i.e. $\mathcal{I}^{\bullet,\bullet}$ is a double complex of injective sheaves on X such that for every $p \geq 0$, we have an injective resolution $\Omega^p_{X/Y} \to \mathcal{I}^{p,\bullet}$. Then $R\pi_*\Omega^\bullet_{X/Y}$ is quasi-isomorphic to the total complex of the double complex $\pi_*\mathcal{I}^{\bullet,\bullet}$. Thus, $\mathcal{H}^p_{dR}(X/Y)$ can be computed by computing $\pi_*\mathcal{I}^{p,q}$. Notice that $\pi_*\mathcal{I}^{p,q}$ is the sheafification of the pre-sheaf $V \mapsto \Gamma(\pi^{-1}(V), \mathcal{I}^{p,q})$ on Y. Since sheafification is exact and so commutes with cohomology, it follows that $\mathcal{H}^p_{dR}(X/Y)$ is the sheafification of the pre-sheaf

$$V \mapsto H^p\left(\mathrm{Tot}\left(\Gamma(V, \mathcal{I}^{\bullet,\bullet})\right)\right).$$

Here, we also need to use the fact that sheafification commutes with totalisation of a first quadrant double complex, which $\mathcal{I}^{\bullet,\bullet}$ is. This last cohomology group is precisely $H^p_{dR}(\pi^{-1}(V), \Omega^\bullet_{X/Y})$ as required. □

It turns out that we can endow the de Rham cohomology sheaves with a canonical integrable connection, the *Gauss-Manin connection.* Let us restrict to the case $\pi : X \to Y$ a smooth map where X and Y are smooth k-schemes (k some field); we reproduce the construction given in [KO68, Section 2]. For an example of some explicit computations, see Section 3 of loc. cit.

The complex admits $\Omega^\bullet_{X/k}$ admits a canonical filtration

$$\Omega^\bullet_{X/k} = F^0\Omega^\bullet_{X/k} \supset F^1\Omega^\bullet_{X/k} \supset \cdots$$

where $F^i = F^i\Omega^\bullet_{X/k}$ is the complex with terms

$$(F^i)^p = \operatorname{Im}\left(\Omega^{p-i}_{X/k} \otimes_{\mathcal{O}_X} \pi^*\Omega^i_{Y/k} \to \Omega^p_{X/k}\right).$$

Since we have assumed X and Y are smooth over k, the sheaves $\Omega^i_{X/k}$ and $\Omega^i_{Y/k}$ are both locally free of finite type. Also, since π is smooth, we have the exact sequence

$$0 \to \pi^*\Omega_{Y/k} \to \Omega_{X/k} \to \Omega_{X/Y} \to 0.$$

These two facts together imply that the associated graded objects

$$gr^i = gr^i\Omega^\bullet_{X/k} := F^i/F^{i+1}$$

have pth term

$$(gr^i)^p = \pi^*\Omega^i_{Y/k} \otimes_{\mathcal{O}_X} \Omega^{p-i}_{X/Y}.$$

We need to use the following lemma, which is a consequence of abstract nonsense in abelian categories (see [Sta23, 012K] for instance).

Lemma 1.31. *Suppose $K^\bullet$ is a filtered complex with a finite filtration*

$$K^\bullet = F^0K^\bullet \supset F^1K^\bullet \supset F^2K^\bullet \supset \cdots.$$

Set $gr^pK^\bullet = F^pK^\bullet/F^{p+1}K^\bullet$; notice that $gr^pK^\bullet$ is also a complex, and so taking cohomology of $gr^pK^\bullet$ makes sense. There is a spectral sequence

$$E_1^{p,q} = H^{p+q}(gr^pK^\bullet) \Rightarrow H^{p+q}(K^\bullet).$$

We take $K^\bullet = \pi_*\Omega^\bullet_{X/k}$; then we have a spectral sequence as in the lemma with first page

$$E_1^{p,q} = R^{p+q}\pi_*(gr^p),$$

which some computation shows that

$$E_1^{p,q} = \Omega^p_{Y/k} \otimes_{\mathcal{O}_Y} \mathcal{H}^q_{dR}(X/Y).$$

Now, we have a map

$$\nabla = d_1^{0,q} : \mathcal{H}^q_{dR}(X/Y) = E_1^{0,q} \to E_1^{1,q} = \Omega_{Y/k} \otimes_{\mathcal{O}_Y} \mathcal{H}^q_{dR}(X/Y).$$

After some computation with the product structure on the spectral sequence induced by the wedge product $\wedge : F^p \otimes F^q \to F^{p+q}$, one checks that this map ∇ satisfies Liebniz' rule, and so is a connection. However, we see from the explicit form of $E_1^{p,q}$ that the complex $E_1^{\bullet,q}$ is precisely the de Rham complex, which implies that the connection is integrable. This is the Gauss-Manin connection.

There is another (possibly slightly more concrete) description. In fact, the map $d_1^{0,q}$ is the connecting homomorphism of the functor $R^q\pi_*$ in the long exact sequence associated to the short exact sequence

$$0 \to gr^1 \to F^0/F^2 \to gr^0 \to 0.$$

Remark 1.32. In fact, one can upgrade the above construction of the de Rham cohomology sheaf with its Gauss-Manin connection to the following very general setting. Suppose $f : X \to Y$ is a smooth morphism of smooth S-schemes for some base scheme S. We have a functor

$$\mathcal{H}^q_{dR}(X/S, -) : MC_{int}(X/S) \to MC_{int}(Y/S)$$

that sends a flat bundle $\mathcal{E}$ with connection ∇ to $H^q \circ Rf_*(\Omega^\bullet_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E})$, endowed with the Gauss-Manin connection. The functor $\mathcal{H}^q_{dR}(X/S, -)$ is in fact the q'th right-derived functor of $\mathcal{H}^0_{dR}(X/S, -)$. The construction of the Gauss-Manin connection is essentially the same as above.

See [Kat70, (3.0)] for all the detail you may need.

2 Differential Geometry and Hodge Theory in Characteristic 0

To be added. Here are some sample sections I should probably write.

2.1 Connection with Classical Differential Geometry and GAGA

2.2 Canonical Extensions of Connections

3 Differential Geometry and Hodge Theory in Positive Characteristic

Recall that a scheme is of *characteristic* p if it is a scheme over $\operatorname{Spec} \mathbb{F}_p$, or equivalently, if $p\mathcal{O}_X = 0$. Throughout, unless otherwise stated, we will assume all schemes to be over $\mathbb{F}_p$ for a fixed prime p.

Our main reference is [Kat70, (5.0)].

3.1 Frobenius and Cartier Morphisms

For this subsection, see [Ill02, Section 3].

Definition. Suppose X is a $\mathbb{F}_p$-scheme. The *absolute Frobenius* F_X is the endomorphism of the ringed space $(X, \mathcal{O}_X)$ that is the identity on the underlying space X, and acts on $\mathcal{O}_X$ by raising to the pth power.

X is said to be *perfect* if F_X is an isomorphism.

It is easy to see that if $X = \operatorname{Spec} A$, then F_X corresponds to the usual Frobenius morphism $A \to A, a \mapsto a^p$. One can also check easily that if $f : X \to Y$ is a morphism of $\mathbb{F}_p$-schemes, then we have a commuting square

$$\begin{tikzcd} X \arrow[r, "F_X"] \arrow[d, "f"] & X \arrow[d, "f"] \\ Y \arrow[r, "F_Y"] & Y. \end{tikzcd}$$

Definition. Suppose X is a Y-scheme with structure map $f : X \to Y$. The *relative Frobenius twist* is the Y-scheme

$$X^{(p)} = X \times_{Y, F_Y} Y,$$

i.e. it is defined by the commuting square

$$\begin{tikzcd} X^{(p)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f"] \\ Y \arrow[r, "F_Y"] & Y. \end{tikzcd}$$

For any $\mathbb{F}_p$-scheme X, the *absolute Frobenius twist* is the Frobenius twist of X relative to $Y = \operatorname{Spec} \mathbb{F}_p$.

Due to the fibre product, the morphisms $F_X : X \to X$ and $f : X \to Y$ induce a map $F_{X/Y} : X \to X^{(p)}$ sitting inside the following diagram.

$$\begin{tikzcd} X \arrow[dr, "F_{X/Y}"] \arrow[drr, bend left, "F_X"] \arrow[ddr, bend right, "f"'] & & \\ & X^{(p)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f"] \\ & Y \arrow[r, "F_Y"'] & Y \end{tikzcd}$$

Definition. The morphism $F_{X/Y} : X \to X^{(p)}$ is the *relative Frobenius of X over Y*.

Notice that the relative Frobenius is a homeomorphism on the underlying topological space.

Example 3.1. If $Y = \operatorname{Spec} A$ and $X = \mathbb{A}^n_A = \operatorname{Spec} A[t_1, ..., t_n]$, then $X' = \operatorname{Spec} A[t_1, ..., t_n]$ with the morphism $F_{X/Y} : X \to X^{(p)}$ is induced by the ring morphism

$$A[t_1, ..., t_n] \to A[t_1, ..., t_n], \quad t_i \mapsto t_i^p$$

whereas the canonical projection morphism $X^{(p)} \to X$ is induced by the ring morphism

$$A[t_1, ..., t_n] \to A[t_1, ..., t_n], \quad at_i \mapsto a^p t_i.$$

Remark 3.2. While in the above affine case it is true that the relative Frobenius twist of X is isomorphic as a scheme to X, this is not always the case.

Proposition 3.3. *Suppose $f : X \to Y$ is a smooth morphism of pure relative dimension n. Then, the relative Frobenius $F_{X/Y} : X \to X^{(p)}$ is a finite flat morphism, and the $\mathcal{O}_{X^{(p)}}$-algebra $F_{X/Y,*}\mathcal{O}_X$ is locally free of rank p^n.*

Corollary 3.3.1. *If f is étale, then $F_{X/Y}$ is an isomorphism and the square*

$$\begin{tikzcd} X \arrow[r, "F_X"] \arrow[d, "f"] & X \arrow[d, "f"] \\ Y \arrow[r, "F_Y"] & Y. \end{tikzcd}$$

is Cartesian.

Let us now see the interplay of the relative Frobenius morphism with the relative de Rham complex. The basic fact at play is that $d(s^p) = ps^{p-1}ds = 0$ in characteristic p. In particular, the canonical morphism $F_X^*\Omega_{X/Y} \to \Omega_{X/Y}$ coming from the square

$$\begin{tikzcd} X \arrow[r, "F_X"] \arrow[d, "f"] & X \arrow[d, "f"] \\ Y \arrow[r, "F_Y"] & Y \end{tikzcd}$$

is the zero map. Similarly, the canonical morphism $F_{X/Y}^*\Omega_{X^{(p)}/Y} \to \Omega_{X/Y}$ coming from the commuting square

$$\begin{tikzcd} X \arrow[r, "F_{X/Y}"] \arrow[d, "f"] & X^{(p)} \arrow[d] \\ Y \arrow[r, "id"] & Y \end{tikzcd}$$

is the zero map as well.

Another consequence is that the differential of the complex $F_{X/Y,*}\Omega^\bullet_{X/Y}$ is $\mathcal{O}_{X^{(p)}}$-linear. In particular, this forces that the cohomology sheaves $H^qF_{X/Y,*}\Omega^\bullet_{X/Y}$ are all $\mathcal{O}_{X^{(p)}}$-modules. The exterior product then induces a graded anti-commutative $\mathcal{O}_{X^{(p)}}$-algebra structure on

$$\bigoplus_q H^qF_{X/Y,*}\Omega^\bullet_{X/Y}.$$

The following theorem of Cartier is an important fundamental result in the differential theory in positive characteristic. In order to state it, labelling the canonical projection map $X^{(p)} \to X$ (coming from the fibre product) by F', we know that

$$\Omega^1_{X^{(p)}/Y} \cong F'^*\Omega^1_{X/Y} = \mathcal{O}_{X^{(p)}} \otimes_{F'^{-1}\mathcal{O}_X} F'^{-1}\Omega^1_{X/Y}.$$

In particular, if ds is a local section of $\Omega^1_{X/Y}$, we can denote the local section of $\Omega^1_{X^{(p)}/Y}$ by $1 \otimes ds$.

Theorem 3.4. *Suppose Y is a $\mathbb{F}_p$-scheme and $f : X \to Y$ a morphism.*

1. *There exists a unique homomorphism of graded $\mathcal{O}_X$-algebras*

$$\gamma : \bigoplus_q \Omega^q_{X^{(p)}/Y} \to \bigoplus_q H^qF_{X/Y,*}\Omega^\bullet_{X/Y}$$

satisfying the two conditions:

- *for $i = 0$, γ is given by the homomorphism $F_{X/Y}^* : \mathcal{O}_{X^{(p)}} \to F_{X/Y,*}\mathcal{O}_X$; and*
- *for $i = 1$, γ sends $1 \otimes ds \in \Omega^1_{X^{(p)}/Y}$ to the class of $s^{p-1}ds$ in $H^1F_{X/Y,*}\Omega^\bullet_{X/Y}$.*

2. *If f is smooth, then γ is an isomorphism.*

Definition. For X a smooth Y-scheme, the isomorphism γ in the above theorem is called the *Cartier isomorphism*, and is denoted by C^{-1}. The map $C = \gamma^{-1}$ is often called the *Cartier operation.*

Suppose Y is perfect and $f : X \to Y$ is smooth. We have an isomorphism

$$\bigoplus_q \Omega^q_{X/Y} \to \bigoplus_q F'_*\Omega^q_{X^{(p)}/Y}$$

where $F' : X^{(p)} \to X$ is the canonical projection morphism. Composing with the Cartier isomorphism, we get an isomorphism

$$C^{-1}_{abs} : \bigoplus_q \Omega^q_{X/Y} \to \bigoplus_q H^qF_{X,*}\Omega^\bullet_{X/Y}.$$

Definition. The above isomorphism C^{-1}_{abs} is called the *absolute Cartier isomorphism.*

3.2 p-Curvature

As usual, fix $\pi : X \to Y$. Throughout, we will use the identification $T_{X/Y} = \mathrm{Der}_Y(\mathcal{O}_X, \mathcal{O}_X)$.

Suppose D is any derivation of $\mathcal{O}_X$. One can check, by using the fact that $p|\binom{p}{i}$ for $1 \leq i \leq p-1$, that D^p also satisfies Liebniz rule. Thus, the pth iterate of a derivation is also a derivation. Thus, $T_{X/Y}$ is a sheaf of a restricted p-Lie algebra, where we recall the definition of a restricted p-Lie algebra below.

Definition. Suppose V is a Lie algebra over a field k of characteristic p. A *p-operation* is a map $()^p : V \to V$ satisfying the following axioms:

1. $ad_{X^p} = ad_X^p$ the composition of ad_X with itself p times;
2. $(aX)^p = a^p X^p$ for all $a \in k$;
3. $(X+Y)^p = X^p + Y^p + \sum_{i=1}^{p-1} \frac{1}{i} s_i(X,Y)$, where $s_i(X,Y)$ is the coefficient of t^{i-1} in the formal expression $ad_{tX+Y}^{p-1}(X)$.

Here, ad_X is the linear map $ad_X : V \to V, Y \mapsto [X,Y]$.

Notice that for a quasi-coherent sheaf $\mathcal{E}$ on X, the sheaf $\mathcal{E}nd(\mathcal{E})$ is also a sheaf of restricted p-Lie algebras, with the p-operation simply iterating an endomorphism p-times. Thus if ∇ is a flat connection on $\mathcal{E}$, we have a Lie algebra morphism of sheaves

$$\nabla : T_{X/Y} \to \mathcal{E}nd(\mathcal{E})$$

between sheaves of restricted p-Lie algebras, and so one may ask whether ∇-preserves the p-operation. This yields the following definition.

Definition. The *p-curvature* ψ of a connection ∇ is the mapping of sheaves

$$\psi : T_{X/Y} \to \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}), \quad D \mapsto \nabla_D^p - \nabla_{D^p}.$$

Here, it requires a computation to check that $\psi(D)$ is in fact a $\mathcal{O}_X$-linear map, as *a priori* it is only $\pi^{-1}\mathcal{O}_Y$-linear.

The significance of p-curvature comes from the following theorem of Cartier.

Theorem 3.5. *Suppose $\pi : X \to Y$ is a smooth morphism of schemes. Recall the relative Frobenius twist $X^{(p)}$ of X with respect to Y, with relative Frobenius $F_{X/Y} : X \to X^{(p)}$.*

There is an equivalence of categories between the category of quasicoherent sheaves on $X^{(p)}$ and the full subcategory of $MC_{int}(X/Y)$ consisting of sheaves on X with connections of zero p-curvature. The equivalence is given as follows:

- *Given a quasi-coherent sheaf $\mathcal{E}$ on $X^{(p)}$, there is a unique integrable Y-connection ∇ of p-curvature 0 on $F_{X/Y}^*(\mathcal{E})$ such that $\mathcal{E} \cong \ker \nabla$.*
- *For a flat quasi-coherent integrable sheaf $(\mathcal{E}, \nabla)$ with zero p-curvature, the sheaf $\ker \nabla$ is naturally a quasi-coherent sheaf on $X^{(p)}$.*

We have some basic properties of p-curvature.

Proposition 3.6. *Suppose ψ is the p-curvature of a flat connection ∇ on $\mathcal{E} \in \mathrm{QCoh}(X)$.*

1. *(p-linearity) $\psi(gD) = g^p\psi(D)$ for all local sections g of $\mathcal{O}_X$ and D of $T_{X/Y}$.*
2. *For $U \subset X$ open and $D \in T_{X/Y}(U)$, the three elements ∇_D, ∇_{D^p}, and $\psi(D)$ of $\mathcal{E}nd_{\mathcal{O}_U}(\mathcal{E}|_U)$ mutually commute.*
3. *Suppose $X \to Y$ is smooth. If D, D' are any two local sections of $T_{X/Y}$, then $\psi(D)$ and $\psi(D')$ commute.*
4. *Suppose $X \to Y$ is smooth. $\psi(D) : \mathcal{E}|_U \to \mathcal{E}|_U$ is a horizontal map, for all $D \in T_{X/Y}(U)$.*

Corollary 3.6.1. *Suppose ψ is the p-curvature of a flat connection ∇ on $\mathcal{E} \in \mathrm{QCoh}(X)$, and suppose $X \to Y$ is smooth. Let $n \geq 1$ be a given integer. The following are equivalent.*

- *There exists a filtration of $(\mathcal{E}, \nabla)$ of length $\leq n$ whose associated graded objects all have p-curvature 0.*
- *Whenever $D_1, ..., D_n$ are local sections of $T_{X/Y}$, we have $\psi(D_1) \cdots \psi(D_n) = 0$.*

- *There exists a covering of S by affine opens, and on each such open affine U there exist sections $u_1, ..., u_r \in \mathcal{O}_X(U)$ such that $\Omega^1_{X/Y}|_U$ is free on $du_1, ..., du_r$ (these u_i can be viewed as coordinates) such that for every r-tuple $(w_1, ..., w_r)$ of integers with $\sum_i w_i = n$, we have*

$$\nabla^{pw_1}_{\partial_{u_1}} \cdots \nabla^{pw_r}_{\partial_{u_r}} = 0$$

where $\{\partial_{u_i}\}$ is dual to the basis $\{du_i\}$.

Definition. Say that $(\mathcal{E}, \nabla)$ is *nilpotent of exponent* $\leq n$ when one of the above equivalent conditions holds. Say that $(\mathcal{E}, \nabla)$ is nilpotent if it is nilpotent of exponent $\leq n$ for some $n \geq 1$.

Let $Nil(X/Y)$ be the full subcategory of $MC_{int}(X/Y)$ of nilpotent flat sheaves, and let $Nil^n(X/Y)$ be the full subcategory consisting of those flat sheaves that are nilpotent of exponent $\leq n$.

Proposition 3.7. *Suppose $(\mathcal{E}, \nabla) \in MC_{int}(X/Y)$. If $(\mathcal{E}, \nabla)$ is nilpotent, then for any $D \in T_{X/Y}(U)$ (viewed as $\pi^{-1}\mathcal{O}_Y$-linear endomorphism of $\mathcal{O}_X$) which is nilpotent, the corresponding $\pi^{-1}\mathcal{O}_Y|_U$-linear endomorphism ∇_D of $\mathcal{E}|_U$ is nilpotent.*

If $\pi : X \to Y$ is smooth, then the converse holds.

Proposition 3.8. *The category $Nil(X/Y)$ is an exact abelian subcategory of $MC_{int}(X/Y)$, that is stable under internal Hom and the tensor product.*

Each $Nil^n(X/Y)$ is stable under taking sub-objects and taking quotients. If $\mathcal{E} \in Nil^n(X/Y)$ and $\mathcal{F} \in Nil^m(X/Y)$, then $\mathcal{E} \otimes \mathcal{F}$ and $\mathcal{H}om(\mathcal{E}, \mathcal{F})$ (with their corresponding flat connections) are in $Nil^{n+m-1}(X/Y)$

Proposition 3.9. *Suppose $\pi : X \to Y$ is a smooth morphism, and suppose we have a commutative diagram*

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "\pi'"] & X \arrow[d, "\pi"] \\ Y' \arrow[r, "h"] & Y \end{tikzcd}$$

such that π' is also smooth. We assume (as always) that all schemes are over $\mathbb{F}_p$. Then, under the inverse image functor

$$(g, h)^* : MC_{int}(X/Y) \to MC_{int}(X'/Y'),$$

we have for any $n \geq 1$,

$$(g, h)^*\big(Nil^n(X/Y)\big) \subseteq Nil^n(X'/Y').$$

Remark 3.10. There is also a very strong statement ([Kat70, Theorem 5.10]) on the stability of nilpotence under the de Rham cohomology sheaf functor given in Section 1.7. In fact, it not only gives bounded on the nilpotence exponent for the de Rham cohomology sheaves, but also gives information about a certain spectral sequence computing the de Rham cohomology sheaf. We do not reproduce it here, since we did not introduce the de Rham cohomology sheaf functor in that level of generality.

3.3 Going from characteristic p to characteristic 0

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