Indian Institute of Technology, Guwahati

Assignment - 2

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Max Mark : 60

Gaussian elimination

A Linear system of n equations in n unknowns $x_1, x_2, ...x_n$ is of the form:

$$\begin{matrix} a_{11}x_1 + ... + a_{1n}x_n = b_1 \\ a_{21}x_2 + ... + a_{2n}x_n = b_2 \\ . \qquad\quad . \qquad . \\ . \qquad\quad . \qquad . \\ . \qquad\quad . \qquad . \\ a_{n1}x_1 + ... + a_{nn}x_n = b_n \end{matrix}$$

Matrix form: The above equations can be written as

Ax = b

where **A** is $n \times n$ coefficient matrix thus,

$$\begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_{n1} & a_{n2} & ... & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{bmatrix}$$

Augmented matrix is obtained by combining **A** and **b** into a single matrix:

$$\begin{bmatrix} \mathbf{A} : \mathbf{b} \end{bmatrix} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & ... & a_{1n} & b_1 \\ a_{21} & a_{22} & ... & a_{2n} & b_2 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{n1} & a_{n2} & ... & a_{nn} & b_n \end{array}\right]$$

Gaussian elimination is a method to solve a system of linear equations $Ax = b$, by converting the augmented matrix into an upper triangular matrix, using elementary row operations, followed by back substitution.
Given system $Ax = b$ is converted to upper triangular form Ux = b' .

$$\left[\mathbf{A} : \mathbf{b}\right] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{array}\right]$$

Upper triangular form :

$$\left[\mathbf{U} : \mathbf{b'}\right] = \left[\begin{array}{cccc|c} a'_{11} & a'_{12} & \dots & a'_{1n} & b'_1 \\ 0 & a'_{22} & \dots & a'_{2n} & b'_2 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & \dots & a'_{nn} & b'_n \end{array}\right]$$

This process is the followed by back substitution to find the unknowns.

$$x_i = \frac{1}{a_{ii}}\{b'_i - \sum_{j=i+1}^{n} a'_{ij}x_j\}$$

Consider the following system:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = d_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = d_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = d_3 \end{aligned}\right\} \quad (1)$$

According to Gauss elimination method, the above system of equations has to be converted into upper triangular form. This is done by going through each column, and making the elements below the diagonal elements equal to zero using the diagonal elements. The diagonal elements are called **pivot elements**.

Consider the first column from the above system. This column consists of three elements, a_{11}, a_{21} and a_{31}. Here a_{11} is the diagonal element (which is the pivot element). a_{21} and a_{31} are elements below the diagonal. These two elements, a_{21} and a_{31} are to be made equal to zero using the pivot element a_{11}, with the use of row operations.

Now, a_{21} is made zero using the pivot a_{11} by the following row operation.

$$R_2 \Rightarrow R_2 - \frac{a_{21}}{a_{11}}R_1$$

The above row operation should be applied to all the elements in row R_2 using the corresponding elements in row R_1.

Similarly, the element a_{31} is made equal to zero, by the following row operation:

$$R_3 \Rightarrow R_3 - \frac{a_{31}}{a_{11}}R_1$$

After these two operations, the system of equations given by (1) will be converted into:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= d_1 \\ b_{22}x_2 + b_{23}x_3 &= e_2 \\ b_{32}x_2 + b_{33}x_3 &= e_3 \end{aligned}\right\} \tag{2}$$

Thus the elements a_{21} and a_{31} are eliminated (equal to zero). Now we move onto the next column. The second column consists of three elements a_{12}, b_{22} and b_{32}. Here the diagonal element is 'b_{22}'. Hence, the new pivot is b_{22}. The element below the pivot is b_{32}. As the elements below the pivot need to be made equal to zero, b_{32} has to be made equal to zero using the pivot element b_{22} in order to convert the above equations into the upper triangular form. This is done using the following row operation:

$$R_3 \Rightarrow R_3 - \frac{b_{32}}{b_{22}} R_2$$

After these operations, we finally obtain the following upper triangular form:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= d_1 \\ b_{22}x_2 + b_{23}x_3 &= e_2 \\ c_{33}x_3 &= f_3 \end{aligned}\right\} \tag{3}$$

Forward elimination phase is over.

Here we can check the consistency of the given system of equations. From the last equation of (3) we have,

$$c_{33}x_3 = f_3$$

On comparing the coefficient c_{33} on left hand side, and f_3 on the right hand side,

(i) If $c_{33} = 0$ and $f_3 \neq 0$, then the system has no solution.
(ii) If $c_{33} = 0$ and $f_3 = 0$, then the system has multiple solutions.
(iii) If $c_{33} \neq 0$, then the system has a unique solution.

When a unique solution exists, we can move onto the next phase of the Gauss elimination method, which is 'Back Substitution'. Solution is found starting from the last equation and subsequently moving back to the first equation.
So from the last equation of (3) we have,

$$c_{33}x_3 = f_3$$
$$x_3 = \frac{f_3}{c_{33}}$$

Thus the value of the x_3 is now found. Let us denote known variables with a star (x_3^*).
Next step is to use this known value of x_3^* to find x_2.
From the second equation in (3), we have

$$b_{22}x_2 + b_{23}x_3^* = e_2$$
$$x_2 = \frac{e_2 - b_{23}x_3^*}{b_{22}}$$

Thus value of x_2 is now found as well .
Similarly, from the first equation of system (3) :

$$a_{11}x_1 + a_{12}x_2^* + a_{13}x_3^* = d_1$$
$$x_1 = \frac{d_1 - a_{13}x_3^* - a_{12}x_2^*}{a_{11}}$$

Example:
Consider the following system of linear equations:

$$x_1 + 4x_2 - x_3 = -5$$
$$x_1 + x_2 - 6x_3 = -12$$
$$3x_1 - x_2 - x_3 = 4$$

Writing the above given system in $Ax = b$ form :

$$\begin{bmatrix} 1 & 4 & -1 \\ 1 & 1 & -6 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ -12 \\ 4 \end{bmatrix}$$

The first column has three elements, a_{11} = 1, a_{21} = 1, and a_{31} = 3. The diagonal element is a_{11} = 1. The elements below it are a_{21} = 1 and a_{31} = 3. Hence these two elements, 1 and 3 are to be made equal to zero by a_{11} = 1 using the following row operation:

$$R_2 \Rightarrow R_2 - R_1$$
$$R_3 \Rightarrow R_3 - 3R_1$$

With these two operations , the above equation will be converted into:

$$\begin{bmatrix} 1 & 4 & -1 \\ 0 & -3 & -5 \\ 0 & -13 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \\ 19 \end{bmatrix}$$

Now we move onto the second column. Here the diagonal (pivot) element is $b_{22} = -3$, and element below it is $b_{32} = -13$. Thus -13 is made equal to zero using -3 through the following row operation:

$$R_3 \Rightarrow R_3 - \frac{13}{3}R_2$$

which results in:

$$\begin{bmatrix} 1 & 4 & -1 \\ 0 & -3 & -5 \\ 0 & 0 & \frac{71}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \\ \frac{148}{3} \end{bmatrix}$$

At this point the above system is in upper triangular form. From the last row of the above matrix, $c_{33} = \frac{71}{3}$ is non-zero. Thus a unique solution exists. Now the equations can be written as:

$$x_1 + 4x_2 - x_3 = -5 \quad \text{(i)}$$
$$-3x_2 - 5x_3 = -7 \quad \text{(ii)}$$
$$\frac{71}{3}x_3 = \frac{148}{3} \quad \text{(iii)}$$

From (iii) we can find x_3

$$\boxed{x_3 = \frac{148}{71}}$$

Now consider equation (ii), substituting the value of x_3, we can find x_2

$$-3x_2 - 5x_3 = -7$$

In the above we can write

$$x_2 = \frac{(-7 + 5x_3^*)}{-3}$$
$$x_2 = \frac{(-7 + 5(\frac{148}{71}))}{-3}$$

$$\boxed{x_2 = -\frac{81}{71}}$$

Now consider the equation (i)

$$x_1 + 4x_2 - x_3 = -5$$

At this point x_2 and x_3 are known and x_1 can be expressed as:

$$x_1 = -5 - 4x_2^* + x_3^*$$
$$x_1 = -5 - 4(-\frac{81}{71}) + \frac{148}{71}$$

$$\boxed{x_1 = \frac{117}{71}}$$

Augmented Matrix

When writing a program to solve a system of linear equations **Ax** = **b** using Gaussian elimination method, we have to use two matrices, square matrix **A** and column matrix **b**. However instead of using separate matrices, we combine **A** and **b** into a single matrix, called augmented matrix **[A:b]**. Since **A** is $n \times n$ matrix , and **b** is $n \times 1$ matrix, the augmented matrix will have n rows and (n+1) columns.

$$
\left[\mathbf{A}:\mathbf{b}\right] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & | & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & | & b_2 \\ . & . & . & . & | & . \\ . & . & . & . & | & . \\ . & . & . & . & | & . \\ a_{n1} & a_{n2} & \dots & a_{nn} & | & b_n \end{bmatrix}
$$

Simple Gaussian Elimination when a zero pivot element appears:

While doing row operations, a problem occurs when a pivot element turns out to be zero ($a_{ii} = 0$). This zero pivot has to be replaced by some other non-zero element. Following steps must be followed to overcome this error.

1. Go to the elements below the diagonal, one by one, in the same column until a non-zero element is found.
2. If a non-zero element is found, then interchange the row with zero pivot with the row containing newly found non-zero element so that the new non-zero element becomes the new pivot.

If there is no non-zero element in the column below the zero diagonal element, it implies that matrix is singular.

Partial Pivoting

When the Gaussian Elimination method is used for large systems (large matrix size of **[A:b]**), the procedure can lead to numerical error propagation, which can affect the final solution.

The procedure of 'Gaussian Elimination method with partial pivoting' is similar to that explained above, however here, before applying the row operations to make the elements below the pivot equal to zero, we need to first search for the element with maximum magnitude below the pivot in the same column. Once it is found, the row corresponding to the present pivot and the row corresponding to this maximum magnitude element are interchanged. Thus this largest element now becomes the pivot. This is called partial pivoting. The rest of the elimination process is same as thet explained above.

Consider the following system of linear equations again:

$$
\begin{aligned}
x_1 + 4x_2 - x_3 &= -5 \\
x_1 + x_2 - 6x_3 &= -12 \\
3x_1 - x_2 - x_3 &= 4
\end{aligned}
$$

Writing the above given system in $Ax = b$ form :

$$
\begin{bmatrix} \boxed{1} & 4 & -1 \\ 1 & 1 & -6 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ -12 \\ 4 \end{bmatrix}
$$

The pivot is $a_{11} = 1$ (boxed element shown above). Now compare this pivot with all the elements below it in the same column.
When largest element is found, do row interchange to make this element the new pivot. Here the third element, a_{31}=3 is the largest element in the column, which belongs to the third row. Thus we need to interchange first and third rows of the matrix, to make this largest element the new pivot. The resulting matrix is:

$$\begin{bmatrix} \boxed{3} & -1 & -1 \\ 1 & 1 & -6 \\ 1 & 4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -12 \\ -5 \end{bmatrix}$$

Now that row interchange is done, continue the elimination process. $R_2 \Rightarrow R_2 - \frac{1}{3}R_1$, and $R_3 \Rightarrow R_3 - \frac{1}{3}R_1$:

$$\begin{bmatrix} 3 & -1 & -1 \\ 0 & \boxed{\frac{4}{3}} & \frac{-17}{3} \\ 0 & \frac{13}{3} & \frac{-2}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ \frac{-40}{3} \\ \frac{-19}{3} \end{bmatrix}$$

Now the pivot here is $a_{22} = \frac{4}{3}$.

Comparing this pivot with the other elements below it in the same column, we see that $a_{32} = \frac{13}{3}$ is the largest element.

Thus exchanging the second and third row to make this largest element the new pivot gives:

$$\begin{bmatrix} 3 & -1 & -1 \\ 0 & \boxed{\frac{13}{3}} & \frac{-2}{3} \\ 0 & \frac{4}{3} & \frac{-17}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ \frac{-19}{3} \\ \frac{-40}{3} \end{bmatrix}$$

Continuing the elimination process ($R_3 \Rightarrow R_3 - \frac{4}{13}R_2$) yields:

$$\begin{bmatrix} 3 & -1 & -1 \\ 0 & \frac{13}{3} & \frac{-2}{3} \\ 0 & 0 & \frac{-71}{13} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ \frac{-19}{3} \\ \frac{-148}{13} \end{bmatrix}$$

Finally back substitution yields the solution.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{117}{71} \\ \frac{-81}{71} \\ \frac{148}{71} \end{bmatrix}$$

Algorithm of simple Gaussian Elimination

NOTE: Arrays in C language start from index 0. It is good practice to leave this 0^{th} index empty and start from index 1, when writing codes for algorithms like Gaussian elimination.

Step-1
Read the Augmented Matrix
For i = 1 to n
    For j = 1 to n +1
        Read $a_{i,j}$
    END for (j)
END for (i)

Step-2
Forward elimination phase:
For j = 1 to n
    For i = 1 to n
        If $i > j$
        constant $=\frac{a_{i,j}}{a_{j,j}}$
        For k = 1 to n + 1
            $a_{i,k} = a_{i,k}$ -constant$\times a_{j,k}$
        END for (k)
    END for (i)
END for (j)

Step-3
Checking consistency of the system of linear equations
If $a_{n,n} \neq 0$
    Print Unique solution
Else
    If $a_{n,n+1} == 0$
        Print Multiple solutions
    Else $a_{n,n+1} \neq 0$
        Print No solution

Step-4
Back substitution phase

Find $x_n = \frac{a_{n,n+1}}{a_{n,n}}$

For i = (n - 1) to 1
    SUM = 0
    For j = (i +1) to n
        SUM = SUM + $a_{i,j} \times x_j$
    END for (j)
    $x_i = (a_{i,n+1} - \text{SUM})/a_{i,i}$
END for (i)

Step-5
Print the result
Print x

Algorithm for zero pivot element problem

Before each row elimination step:
If a_{ii} is 0
    For j=i+1 to n
        If $a_{ji} \neq 0$, then interchange i^{th} row and j^{th} row and break the for loop
        Else (i.e., if a_{ji} = 0)
            If j < n , then j=j+1 and continue the loop
            Else (i.e., if j = n), print Matrix is singular and exit
    End For (j)

Problem 1

Write a program to solve the following systems of linear equations using simple Gauss elimination. Program must print the following:
(i) The final augmented matrix after forward elimination (upper triangular form).
(ii) Whether a system has unique solution, multiple solutions, or no solutions.
(iii) If a unique solution exists, print the solution.

$$(a)\qquad \begin{aligned} x + y - z &= 2\\ -2x + 4y - 6z &= 18\\ 8y + 6z &= -4 \end{aligned}$$

$$(b)\qquad \begin{aligned} x - y + z &= 16\\ -x + 2y - 5z &= 21\\ 3x - 6y + 15z &= 35 \end{aligned}$$

$$(c)\qquad \begin{aligned} x + y + z &= 0\\ 2x - 3y - 3z &= 2\\ 4x - 6y - 6z &= 4 \end{aligned}$$

$$(d)\qquad \begin{aligned} 2x + 3y + z - 11w &= 1\\ x - y + 3z - 3w &= 3\\ 3x + 4y - 7z + 2w &= -7\\ 5x - 2y + 5z - 4w &= 5 \end{aligned}$$

$$(e)\qquad \begin{aligned} 10x + 4y - 2z &= 14\\ -15x + y - 2z - 3w &= 0\\ x + y + w &= 6\\ -5x + 5y - 10z + 8w &= 26 \end{aligned}$$

Instructions: The code has to be written such that the coefficients of augmented matrix of each system are read from each input text file, and the required outputs are printed in corresponding output text file. The names of input text files for each sub-question of problem-1 should be ‘**input_1a.txt**’, ‘**input_1b.txt**’, ‘**input_1c.txt**’ , ‘**input_1d.txt**’ and ‘**input_1e.txt**’. Correspondingly the names of the output text files should be ‘**output_1a.txt**’, ‘**output_1b.txt**’, ‘**output_1c.txt**’, ‘**output_1d.txt**’ and ‘**output_1e.txt**’. There should be a single C program file for problem-1. The name of the C program file should be ‘**problem_1.c**’.

marks: 25

Problem 2

Write a program to solve the following system of linear equations using Gauss elimination with zero diagonal element situation. The program should be able to handle the situation where a zero pivot element is encountered. The program must print the following:
(i) The final augmented matrix after forward elimination (upper triangular form).
(ii) The solution.

$$4x + 2y + 6z = 19$$
$$2x + y - 7z = -1$$
$$10x - 3y + 7z = 15$$

Instructions: The code has to be written such that the coefficients of augmented matrix of the above system are read from an input text file, and the required outputs are printed in an output text file. The name of input text file for problem-2 should be '**input_2.txt**', and the name of the output text file should be '**output_2.txt**'. The name of the C code file should be '**problem_2.c**'.

marks: 15

Problem 3

Write a program to solve the following system of linear equations using Gauss elimination incorporating partial pivoting. The program must print the following:
(i) The final augmented matrix after forward elimination (upper triangular form).
(ii) The solution.

$$x + 1.5y = 4.5$$
$$0.01x + 0.02y + 0.03z = 0.12$$
$$y + 2z + 3t = 21$$
$$10z + 20t = 100$$

Hint for partial pivoting:

- Incorporate two nested loops
- First loop is to compare the pivot element with the elements below it in that column, to find the largest element.
- If a largest element is found, the second loop interchanges the pivot row with the one containing this largest element.

Instructions: The code has to be written such that the coefficients of augmented matrix of the above system are read from an input text file, and the required outputs are printed in an output text file. The name of input text file for problem-3 should be '**input_3.txt**', and the name of the output text file should be '**output_3.txt**'. The name of the C code file should be '**problem_3.c**'.

marks: 20