# Minimum Bounding Circle optimization proof for Diffusion Limited Aggregation

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At each iteration of DLA, we spawn a point $C_1$ at a randomly selected position $(x, y)$ on the border of our m x m canvas. This point follows Brownian motion in the canvas, till it meets the termination criterion.

As m increases, the area in which $C_1$ can move also increases, making simulations slow. In order to quicken the process, we would like to decrease the search space. One way to do is, is to spawn as close as possible to the current group of 1s in the figure.

We can do this by finding a group of points $L_2$ such that the probability that a point that spawned on the square border reaches any of the points in $L_2$ is the same. Formally, this would be a locus such that any two points $l_{21}$ and $l_{22}$ in it satisfy the condition

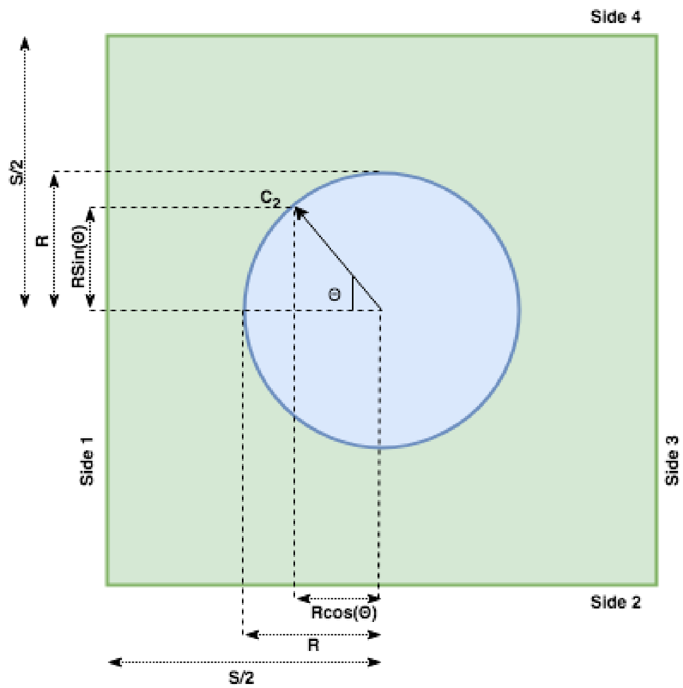
$$\sum_{x \in L_1} W_{reach}(x, l_{21}) = \sum_{x \in L_1} W_{reach}(x, l_{22})$$

$L_1$ is the set of points which make up the square boundary. $W_{reach}(a, b)$ is the ratio of the number of ways of reaching $b$ from $a$ to the total number of ways of reaching $b$ from all points in $L_1$. We take a ratio, since there are infinitely many ways of reaching $b$ from any point in $L_1$.

$$W_{reach}(a, b) = \frac{n(a, b)}{\sum_{c \in L_1} n(c, b)}$$

If the area covered by locus $L_2$ is less than the area of the square boundary, then we can hope to see a faster DLA.

Intuitively it seems $L_2$ can be a circle. We shall be proving it below.




$W_{total}$ of a point $C_2$ on the circle is calculated by summing $W_{reach}(c, C_2)$ for all points $c$ on the square.

$$W_{total}(C_2) = \sum_{c \in side1} W_{reach}(c, C_2) \quad + \sum_{c \in side2} W_{reach}(c, C_2)$$

$$+ \sum_{c \in side3} W_{reach}(c, C_2) \quad + \sum_{c \in side4} W_{reach}(c, C_2)$$

Now we shall show that $W_{total}$ is equal for all points on the circle. There are two assumptions we make to simplify our analysis

- Instead of considering discrete coordinates, we shall consider all points on the 2D plane.
- We define $W_{reach}(a, b) = k(|a_x - b_x| + |a_y - b_y|)$, where $k$ is a constant.

Using our two assumptions,

$$\sum_{C_1 \in side1} W_{reach}(c, C_2) = \int_{x=0}^{\frac{S}{2}-R\sin\theta} k(\frac{S}{2} - R\cos\theta + x)dx + \int_{x=0}^{\frac{S}{2}+R\sin\theta} k(\frac{S}{2} - R\cos\theta + x)dx$$

$$= k\frac{S}{2}\int_{x=0}^{\frac{S}{2}-R\sin\theta} dx - kR\cos\theta \int_{x=0}^{\frac{S}{2}+R\sin\theta} dx + \int_{x=0}^{\frac{S}{2}-R\sin\theta} xdx + \int_{x=0}^{\frac{S}{2}+R\sin\theta} xdx$$

$$= k\frac{S}{2}(\frac{S}{2} - R\sin\theta) - kR\cos\theta(\frac{S}{2} + R\sin\theta) + \int_{x=0}^{\frac{S}{2}-R\sin\theta} xdx + \int_{x=0}^{\frac{S}{2}+R\sin\theta} xdx$$

$$= k\frac{S^2}{2} - kSR\cos\theta + \int_{x=0}^{\frac{S}{2}-R\sin\theta} x + \int_{x=0}^{\frac{S}{2}+R\sin\theta} x$$

$$= k\frac{S^2}{2} - kSR\cos\theta + (\frac{S}{2} - R\sin\theta)^2 + (\frac{S}{2} + R\sin\theta)^2$$

$$= k\frac{S^2}{2} - kSR\cos\theta + \frac{S^2}{4} + R^2\sin^2\theta - SR\sin\theta + \frac{S^2}{4} + R^2\sin^2\theta + SR\sin\theta$$

$$= k\frac{S^2}{2} - kSR\cos\theta + \frac{S^2}{2} + 2R^2\sin^2\theta$$

Similarly,

$$\sum_{C_1 \in side2} W_{reach}(c, C_2) = \int_{x=0}^{\frac{S}{2}-R\cos\theta} k(\frac{S}{2} + R\sin\theta + x)dx + \int_{x=0}^{\frac{S}{2}+R\cos\theta} k(\frac{S}{2} + R\sin\theta + x)dx$$

$$= k\frac{S^2}{2} + kSR\sin\theta + \frac{S^2}{2} + 2R^2\cos^2\theta$$

$$\sum_{C_1 \in side3} W_{reach}(c, C_2) = \int_{x=0}^{\frac{S}{2}-R\sin\theta} k(\frac{S}{2} + R\cos\theta + x)dx + \int_{x=0}^{\frac{S}{2}+R\sin\theta} k(\frac{S}{2} + R\cos\theta + x)dx$$

$$= k\frac{S^2}{2} + kSR\cos\theta + \frac{S^2}{2} + 2R^2\sin^2\theta$$

$$\sum_{C_1 \in side4} W_{reach}(c, C_2) = \int_{x=0}^{\frac{S}{2}-R\cos\theta} k(\frac{S}{2} - R\sin\theta + x)dx + \int_{x=0}^{\frac{S}{2}+R\cos\theta} k(\frac{S}{2} - R\sin\theta + x)dx$$

$$= k\frac{S^2}{2} - kSR\sin\theta + \frac{S^2}{2} + 2R^2\cos^2\theta$$

Therefore,

$$\begin{aligned}
W_{total}(C_2) = &\sum_{c \in side1} W_{reach}(c, C_2) \quad + \sum_{c \in side2} W_{reach}(c, C_2) \\
&+ \sum_{c \in side3} W_{reach}(c, C_2) \quad + \sum_{c \in side4} W_{reach}(c, C_2) \\
= &\, 4\frac{S^2}{2} - kSR\cos\theta + kSR\sin\theta + kSR\cos\theta - kSR\sin\theta \\
&+2R^2\cos^2\theta + 2R^2\sin^2\theta + 2R^2\cos^2\theta + 2R^2\sin^2\theta \\
= &\, 2S^2 + 4R^2
\end{aligned}$$

Clearly $W_{total}(C_2)$ is a constant, for every $C_2$ on the circle with radius $R$. Hence the probability of reaching any point on a circle with radius $R$ is the same. **As a result, we can chose to spawn at any point on the circle with a radius $R$ instead of the square boundary.**

The only constraint is that all the points outside the circle should be free. And obviously, we would like $R$ to be as small as possible. Hence while spawning a new point, we chose the smallest circle that encloses the current image (minimum bounding circle).