3rd January 2023 Tuesday.

limit of a fn. $f(x)$ at a point a exist only if it's left hand limit & RH limit exist & are equal.

$$\lim_{x\to a} f(x) = \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$$

A function $f(x)$ is continuous if

$$\lim_{x\to a} f(x) = f(a)$$

i.e, $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$.

Function

f is said to be a fn. $f : A \to B$ if $\exists!$ b in B s.t. $f(a) = b$, where $\forall\, a \in A$.

Real valued function

A fn. is said to be real valued fn. if codomain is $\mathbb{R}$, ie the output must be a real no.

$f : D \subseteq \mathbb{R} \to \boxed{\mathbb{R}}$, defined by $\boxed{f(x)} = x^2$ $\quad f(x) = x^2$ $\quad f(x)$

$f(a) = b$; $b \in \mathbb{R}$ $\quad \boxed{n = 1, 2, 3}$

$f : \mathbb{N} \to \mathbb{R}$ $\quad$ Domain $f(1) = 1$

$f(x) = x^2$ $\quad f(2) = 4$

$f(3) = 9$

D

$\boxed{\mathbb{R}}$

Neighbourhood of a Point

$[1, 3]$

6/1/2023 Friday.

Open Set

$[a, b]$

(a, b)

A set A is an open set if $A = A^{int}$

$\Phi = \Phi^{int}$

$[1, 3]$

$(1, 3)$

$(1, 3)$

Empty set & $\mathbb{R}$ are open set.

$(1, 3]$

$[1, 3)$

Boundary Point

A point x is said to be a boundary point of A if the δ-Neighbourhood of x consist of some points in A & some points that are not in A.

In $A = [a, b]$, a & b are the boundary point.

x may or may not belong to A.

eg:- $A = (a, b)$

$A^{bry} = \{a, b\} \rightarrow$ set of all boundary points of A.

Exterior Point

Let A be any set. Then $c \in \bar{A}$ is an exterior point. if $\exists$ $N_\delta(c)$ s.t. $c \in N_\delta(c) \subset \bar{A}$.

7/1/2023 Saturday.

Functions of Several Variables

A real-valued fns. of several independent real variables are defined analogously to fns. in the single variable case.

Points in the domains are ordered pairs [triples, quadruples, n tuples] of real nos. & values in the range are real nos.

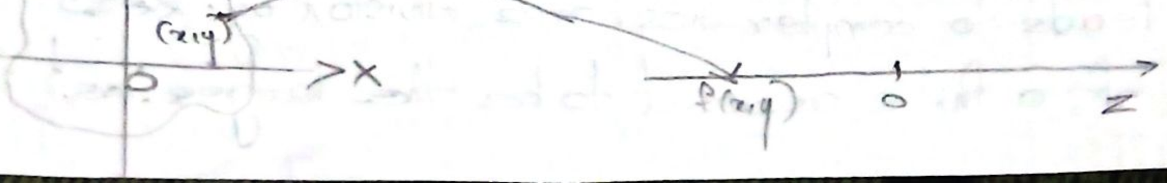
Real Valued Function

Suppose D is the set of n tuples of real nos. $(x_1, x_2, \cdots x_n)$. A real valued fn. on f on D is a rule that assigns a unique real no. $\omega = f(x_1, x_2 \cdots x_n)$ to each element in D.

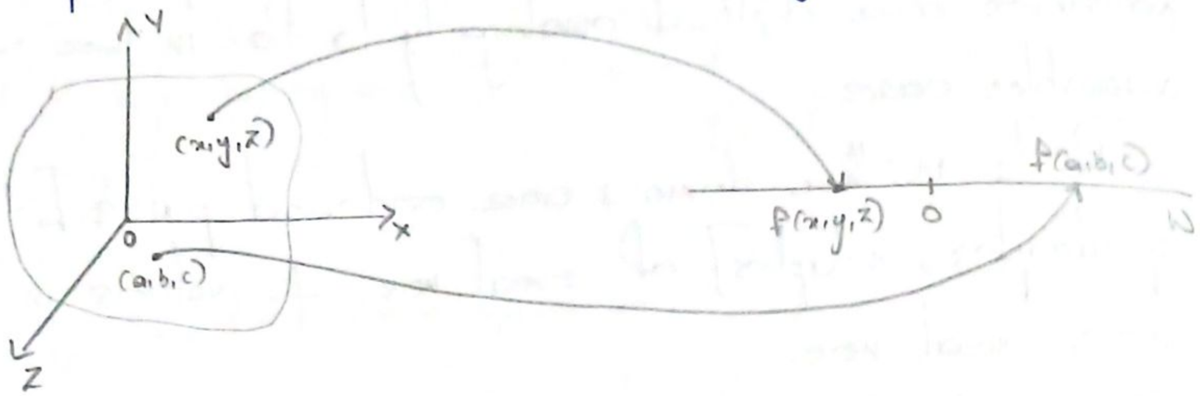
The set D is the fns. domain.

The set of ω values taken on by f is the fns. range. The symbol ω is the dependent variable of f & f is said to be a fn. of the n independent variables $x_1, \ldots x_n$. We also call the x_i's the fns. input variables & call ω's the function's output variables.

If f is a fn. of two independent variables, we usually call the independent variables x & y and dept. variable z & we picture domain of f as a region in the XY plane.



If f is a fn. of 3 independent variables, we call the indept. variables x, y, z & depent. variable w & we picture the domain as a region in the space.



1. Find the value of

$$f(x,y,z) = \sqrt{x^2+y^2+z^2}$$

at the point $(3,0,4)$

$x = 3$

$y = 0$

$z = 4$

$$f(3,0,4) = \sqrt{3^2+0^2+4^2}$$
$$= \sqrt{9+16}$$
$$= \sqrt{25}$$
$$= 5$$

Domains & Ranges

In defining a functions of more than one variable we follow the usual practice of excluding the inputs that leads to complex nos. or division by zero. The domain of a fn. is assumed to be the larges set for which the defining rule generates real nos., unless the domain is otherwise specified explicitly.

eg :- * $f(x,y) = \sqrt{y-x^2}$

Domain :- $y \geq x^2$

If $f(x,y) = \sqrt{y-x^2}$, y can't be less than x^2 and

* If $f(x,y) = \frac{1}{xy}$, $xy \neq 0$ is the domain.

Range

The range consist of the set of output values for the dependent variables.

eg :-

Find the domain & range for the following functions.

	Function	Domain	Range
1.	$z = \sqrt{y-x^2}$	$y \geq x^2$	$[0, \infty)$
2.	$z = \frac{1}{xy}$	$xy \neq 0$	$(-\infty, 0) \cup (0, \infty)$
3.	$z = \sin xy$	Entire plane	$[-1, 1]$
4.	$w = \sqrt{x^2+y^2+z^2}$	Entire ~~plane~~ space	$[0, \infty)$
5.	$w = \frac{1}{x^2+y^2+z^2}$	$(x,y,z) \neq (0,0,0)$	$(0, \infty)$

6. $w = xy \ln z$	$z > 0$ Half space	$(-\infty, \infty)$

text page 79 Thomas calculus

10/1/2022
Tuesday

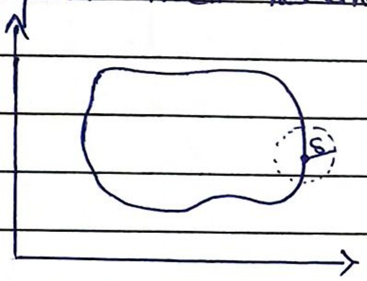
Maths Note

Interior Point

A point (x_0, y_0) in a region R in the XY plane is an interior point of R if it is the centre of a disc of +ve radius that lies entirely in R

Boundary Point

A point (x_0, y_0) is a boundary point of a region R if every disc centred at (x_0, y_0) contains points that lies outside of R as well as points that lies in R. [The boundary point itself neednot belong to R.



Interior of a region

The interior points of a region as a set make up the interior of a region.

Boundary

The region's boundary points make up its boundary

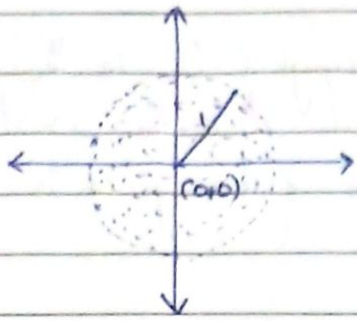
Open Region

A region is open if it consists entirely of interior points.

Closed Region

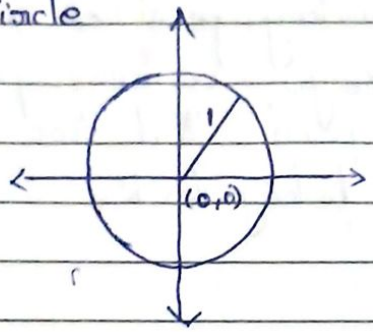
A region is closed if it contains all its boundary points.

Open unit disk : Every point is an interior point.



$\{(x,y) / x^2+y^2<1\}$

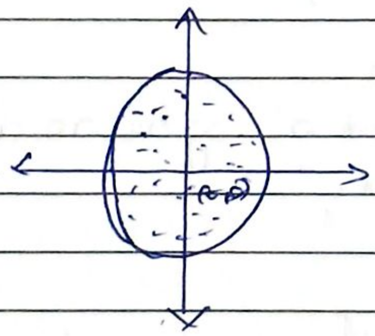
Circle



Neigther open nor closed

$\{(x,y) / x^2+y^2=1\}$

Closed unit disk



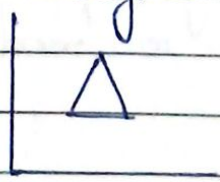
$\{(x,y) / x^2+y^2\leq 1\}$

Bounded Function

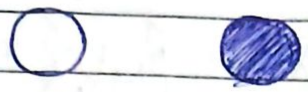
A region in a plane is bounded if it lies inside a disk of a finite region radius.

A region is unbounded if it is not bounded.

eg :- Examples of bounded set in the plane includes line segment. Triangles Interior of Δ^{les}



Circles   Disk

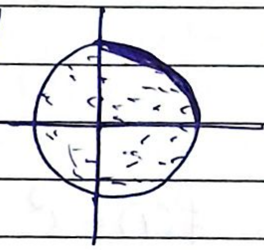


Unbounded sets in the plane includes any line ||el x/y axis lines, coordinate axes, the graphs of a fn. defined on infinite intervals, quadrants, half plane, the plane itself.

Φ & XY plane are both open & closed sets.

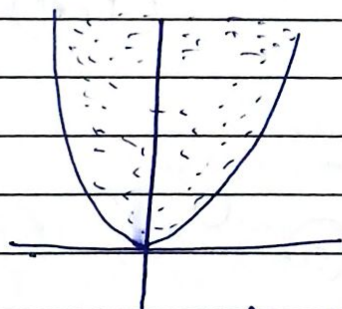
Empty region & XY plane are both open as well as closed regions.

Neither open nor closed



boundary point of only 1st quadrant are included

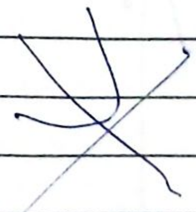
Unbounded Closed Region



$y \geq x^2$

Graph of a fn. f $y = f(x)$ is $(x, f(x))$.

$y = x^2$
$f(x) = x^2$
(x, x^2)



12/1/2022 Thursday.

Graph

Graph of a fn. $y=f(x)$ is the set of all points $(x, f(x))$.

Level Curve

The set of points in the plane where a fn. $f(x,y)$ has a constant value $f(x,y)=c$ is called a level curve of f.

Graph

The set of all points $(x,y,f(x,y))$ in space for (x,y) in the domain of f is called graph of f.

Level surface

The set of points (x,y,z) in the space where a fn $f(x,y,z)$ has a constant value $f(x,y,z)=c$ is called a level surface of f.

Practice :- Finding domain, range & level curve page no:- 799

13/1/2022 Friday.

Limit & Continuity In Higher Dimensions

Write Definitions

eg:-

1. Solve the following

(i) $\lim\limits_{(x,y)\to(0,1)} \dfrac{x-xy+3}{x^2y+5xy-y^3}$

(ii) $\lim\limits_{(x,y)\to(3,-4)} \sqrt{x^2+y^2}$

(i) $\lim\limits_{(x,y)\to(0,1)} \dfrac{0-0+3}{0+0-1^3} = \dfrac{3}{-1} = \underline{\underline{-3}}$

(ii) $\lim\limits_{(x,y)\to(3,-4)} \sqrt{x^2+y^2} = \sqrt{9+16} = \sqrt{25} = \underline{\underline{5}}$

(iii) $\lim\limits_{(x,y)\to(0,0)} \dfrac{4xy^2}{x^2+y^2} = \lim\limits_{x\to 0} \dfrac{4x(mx)^2}{x^2+(mx)^2}$

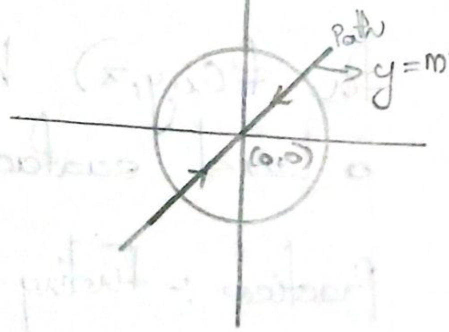
$= \lim\limits_{x\to 0} \dfrac{4x^3m^2}{x^2+m^2x^2}$

$= \lim\limits_{x\to 0} \dfrac{\cancel{x^2}(4xm^2)}{\cancel{x^2}(1+m^2)}$

$= \lim\limits_{x\to 0} \dfrac{4m^2x}{1+m^2}$

$\underline{\underline{= 0}}$

choosing path



(iv) $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2-y^2+5}{x^2+y^2+2} = \underline{\underline{\dfrac{5}{2}}}$

2. $\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{y}} = ?$

3. $\lim_{(x,y)\to(3,4)} \sqrt{x^2+y^2-1} = \sqrt{9+16-1} = \underline{\sqrt{24}}$

4. $\lim_{(x,y)\to(2,-3)} \left(\frac{1}{x}+\frac{1}{y}\right)^2 = \left(\frac{1}{2}-\frac{1}{3}\right)^2 = \left(\frac{1}{6}\right)^2 = \frac{1}{36}$

5. $\lim_{(x,y)\to(0,\frac{\pi}{4})} \sec x \tan y = 1.1 = \underline{1}$

6. $\lim_{(x,y)\to(0,0)} \frac{x^2+y^3}{x+y+1} = \underline{0}$

14/1/2023 Saturday

$\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{y}} =$

Two Path Test for Non Existence of a Limit

If a fn. $f(x,y)$ has different limits along two different paths in the domain of f as (x,y) approaches to (x_0,y_0), then $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ doesnot exist.

eg:- $f(x,y) = \frac{2x^2y}{x^4+y^2}$ as has no limit as $(x,y)\to(0,0)$

Let the path be $y = mx^2$

$\lim_{(x,y)\to(x_0,y_0)} 2x^2y/x^4+y^2 = \lim_{x\to 0} \frac{2x^2y}{x^4+m^2x^4} = \lim_{x\to 0} \frac{2x^2y}{\cancel{x^2}(x^2+m^2x^2)}$

$$= \lim_{x \to 0} \frac{2mx^2}{x^2 + m^2x^2} = \lim_{x \to 0} \frac{2m}{1+m^2}$$

$$= \frac{2m}{1+m^2}$$

For different values of 'm' we will get different limiting values. But the limit of a fn. should be unique.

$\therefore$ For the given fn., limit doesnot exist.

1. Evaluate the followings

a) $\lim_{(x,y) \to (0, \ln 2)} e^{x-y} = e^{0 - \ln 2} = e^{\ln 2^{-1}} = 2^{-1} = \frac{1}{2}$

b) $\lim_{(x,y) \to (\frac{\pi}{2}, 0)} \frac{\cos y + 1}{y - \sin x}$

$$= \frac{\cos \cancel{\frac{\pi}{2}} (0) + 1}{0 - \sin \frac{\pi}{2}}$$

$$= \frac{1+1}{-1}$$

$$= -2$$

c) $\lim_{\substack{(x,y) \to (1,1) \\ x \neq y}} \frac{x^2 - 2xy + y^2}{x - y} = \lim_{(x,y) \to (1,1)} \frac{(x-y)^{\cancel{2}}}{\cancel{(x-y)}}$

$$= 1 - 1$$

$$= 0$$

d) $\lim\limits_{\substack{(x,y)\to(1,1) \\ x\neq y}} \dfrac{x^2-y^2}{x-y} = \lim\limits_{(x,y)\to(1,1)} \dfrac{\cancel{(x-y)}(x+y)}{\cancel{x-y}}$

$= \underline{\underline{2}}$

e) $\lim\limits_{p\to(1,3,4)} \left(\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}\right) = \dfrac{1}{1}+\dfrac{1}{3}+\dfrac{1}{4}$

$= \dfrac{4}{3}+\dfrac{1}{4}$

$= \underline{\underline{\dfrac{19}{12}}}$

f) $\lim\limits_{p\to(\pi,\pi,0)} \left(\sin^2 x+\cos^2 y+\sec^2 z\right)$

$= \sin^2 \pi + \cos^2 \pi + \sec^2(0)$

$= 1 + \dfrac{1}{\cos^2(0)}$

$= 1+1$

$= \underline{\underline{2}}$

2. $f(x,y) = -\dfrac{x}{\sqrt{x^2+y^2}}$ find limit as $(x,y)\to(0,0)$

3. $\lim\limits_{(x,y)\to(0,0)} x^4-y^2/x^4+y^2$

4. $g(x,y)$
$(x,y)\to(0,0)$

2. $\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}}$

Let $y = mx$

$\therefore \lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}} = \lim_{x\to 0} \frac{x}{\sqrt{x^2+m^2x^2}}$

$= \lim_{x\to 0} \frac{\cancel{x}}{\cancel{x}\sqrt{1+m^2}}$

$= \lim \frac{1}{\sqrt{1+m^2}}$

$\therefore$ limit doesnot exist.

3. $f(x,y) = \frac{x^4-y^2}{x^4+y^2}$

Let $y = mx^2$

$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{x^4-m^2x^4}{x^4+m^2x^4}$

$= \lim_{x\to 0} \frac{\cancel{x^4}(1-m^2)}{\cancel{x^4}(1+m^2)}$

$= \frac{1-m^2}{1+m^2}$

limit doesnot exist

4. $g(x,y) = \dfrac{x-y}{x+y}$

Let $y = mx$

$$\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y} = \lim_{x\to 0} \frac{x-mx}{x+mx} = \frac{1-m}{1+m}$$

5. $h(x,y) \dfrac{x^2y}{x^4+y^2}$ $(x,y)\to(0,0)$

let $y = mx^2$

$$\lim_{(x,y)\to(0,0)} h(x,y) = \lim_{x\to 0} \frac{x^2 \cdot m x^2}{x^4+m^2x^4}$$

$$= \lim_{x\to 0} \frac{\cancel{x^4}\, m}{\cancel{x^4}(1+m^2)}$$

$$= \frac{m}{1+m^2}$$

6. Let $f(x,y) = \begin{cases} 1 & , y \geq x^4 \\ 1 & , y \leq 0 \\ 0 & ; \text{otherwise} \end{cases}$

Find each of the following limits (or) explain that the limit doesnot exist.

(a) $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 1$ since $1 \geq 0^4 \Rightarrow 1 \geq 0$

(b) $\lim_{(x,y)\to(2,3)} f(x,y) = 0$ $\because$ $3 \not\geq 2^4$ $3 \not< 0$

$3 \leq 0$ $y \geq x^4$ $3 \not\geq 2^4$

(c) $\lim_{(x,y)\to(0,0)} f(x,y) = 1$ $\because$ y $0 \leq 0$ & $0 \geq 0$

$y \geq x^4$ $0 \geq 0$ $y \leq 0$ $0 \leq 0$

Continuity of a Function at a Point

A fn. $f(x,y)$ is said to be continuous at a point (x_0,y_0) if

(i) $f(x_0,y_0)$ is exist.

(ii) $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exist

(iii) $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

Continuous Function

A fn. f is said to be continuous in a region R if it is continuous at every point in R.

1. $f(x,y) = \begin{cases} \dfrac{2x^2y}{x^4+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$

Given function is not continuous because limit doesnot exists.

Algebra of Continuous functions

Let $f(x,y)$ & $g(x,y)$ be two fns. continuous at (x_0,y_0). Then

1) $f(x,y) \pm g(x,y)$ is continuous

2) $f(x,y) \cdot g(x,y)$ is continuous

3) $k f(x,y)$ is continuous ; where k is a constant.

4) $\frac{f(x,y)}{g(x,y)}$; $g(x,y) \neq 0$ is continuous.

5) $g(x,y) = g(f(x,y))$; is continuous.

24/1/2023 Tuesday.

Partial Derivatives

$z = f(x,y)$

The partial derivative of $f(x,y)$ w.r.t x at the point (x_0,y_0) is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h}$$

provided that the limit exist. h is any real no.

Different notations

$$\frac{\partial f}{\partial x}, \quad f_x(x_0,y_0), \quad \left.\frac{dz}{dx}\right|_{(x_0,y_0)}$$

The partial derivative of $f(x,y)$ w.r.t y at the point (x_0,y_0) is

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0,y_0)}{h}$$

provided that the limit exist.

1. Find the values of $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ at $(4,-5)$ if $f(x,y) = x^2+3xy+y^{-1}$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2+3xy+y-1)$$

$$\frac{\partial f}{\partial x} = \underline{2x+3y}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(4,-5)} = 2\times 4 + 3\times -5$$

$$= 8-15$$

$$= \underline{\underline{-7}}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2+3xy+y-1)$$

$$= \underline{\underline{3x+1}}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(4,-5)} = 3\times 4+1$$

$$= \underline{\underline{13}}$$

2. Find $\frac{\partial f}{\partial y}$ as a fn. of $f(x,y) = y\sin xy$ & also $\frac{\partial f}{\partial x}$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y\sin(xy))$$

$$= y\cdot\cos(xy)\cdot x + \sin(xy)$$

$$= \underline{\underline{xy\cos(xy) + \sin(xy)}}$$

$$\frac{\partial f}{\partial x} = y\cdot\cos(xy)\cdot y$$

$$= \underline{\underline{y^2\cos(xy)}}$$

3. Find $\frac{\partial z}{\partial x}$ if the eqn. $yz - \ln z = x + y$.

4. Find $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ of the following

(i) $f(x,y) = \sqrt{x^2+y^2}$

(ii) $f(x,y) = e^{(x+y+1)}$

(iii) $f(x,y) = e^{xy} \ln y$

(iv) $f(x,y) = \cos^2(3x-y^2)$

4.(i) $f(x,y) = \sqrt{x^2+y^2}$

$$\frac{\partial f}{\partial x} = \frac{1}{\cancel{2}\sqrt{x^2+y^2}} \cdot \cancel{2}x$$

$$= \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

3. $\frac{\partial}{\partial x}(yz - \ln z) = \frac{\partial}{\partial x}(x+y)$

$$y \cdot \frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} = 1 + 0$$

$$\frac{\partial z}{\partial x}\left(y - \frac{1}{z}\right) = 1$$

$$\frac{\partial z}{\partial x} = \frac{z}{yz-1}$$

$\cos 2x = \cos^2 x - \sin^2 x$

4 (ii) $f(x,y) = e^{(x+y+1)}$

$$\frac{\partial f}{\partial x} = e^{(x+y+1)}$$

$$\frac{\partial f}{\partial y} = e^{(x+y+1)}$$

(iii) $f(x,y) = e^{xy} \cdot \ln y$

$$\frac{\partial f}{\partial x} = \ln y \cdot e^{xy} \cdot y$$

$$= \underline{\underline{y \ln y \cdot e^{xy}}}$$

$$\frac{\partial f}{\partial y} = e^{xy} \cdot \frac{1}{y} + \ln y \cdot e^{xy} \cdot x$$

$$= \frac{1}{y} e^{xy} + x \ln y \cdot e^{xy}$$

(iv) $f(x,y) = \cos^2(3x - y^2)$

$$\frac{\partial f}{\partial x} = 2\cos(3x-y^2) \cdot \cancel{A} - \sin(3x-y^2) \cdot 3$$

$$= -6\cos(3x-y^2)\sin(3x-y^2) = -3\sin 2(3x-y^2)$$

$$= -3\sin(6x-2y^2)$$

$$\frac{\partial f}{\partial y} = 2\cos(3x-y^2) \cdot -\sin(3x-y^2) \cdot -2y$$

$$= 2y \cdot 2\cos(3x-y^2)\sin(3x-y^2)$$

$$= \underline{\underline{2y \sin(2(3x-y^2))}}$$

5. If x, y, z are independent variables &

$f(x,y,z) = x \sin(y+3z)$

$$\frac{\partial f}{\partial x} = \sin(y+3z)$$

$$\frac{\partial f}{\partial y} = x \cdot \cancel{\sin} \cos(y+3z).$$

27/1/2023 Friday.

<u>Note :-</u>

All differentiable fns. are continuous. But not vice versa.

eg :- $f(x) = |x| = \begin{cases} x & : x \geq 0 \\ -x & : x \leq 0 \end{cases}$

$f(x)$ is continuous at $x=0$, but not differentiable.

$f_x(x,y)$

1. Find $f(x)$ & $f(y)$ for the following fns.

$$f(x,y) = \frac{2y}{y+\cos x}$$

2. If $f(x,y) = x\cos y + ye^x$, find $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial y^2}$ & $\frac{\partial^2 f}{\partial x \partial y}$

3. Find $\frac{\partial^2 \omega}{\partial x \partial y}$ if $\omega = xy + \frac{e^y}{y^2+1}$

<u>Ans :-</u>

1. $f_x(x,y) = \frac{\partial f(x,y)}{\partial x}$

$$= 2y \cdot \frac{-1}{(y+\cos x)^2} \cdot -\sin x = \frac{2y\sin x}{(y+\cos x)^2}$$

$$\frac{\partial f}{\partial y} = \frac{(y+\cos x)2 - 2y}{(y+\cos x)^2}$$

$$= \frac{2y + 2\cos x - 2y}{(y+\cos x)^2}$$

$$= \frac{2\cos x}{(y+\cos x)^2}$$

2. $f(x,y) = x\cos y + ye^x$

$$\frac{\partial f}{\partial x} = \cos y + ye^x$$

$$\frac{\partial^2 f}{\partial x^2} = ye^x$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}(\cos y + ye^x)$$

$$= -\sin y + e^x$$

$$\frac{\partial f}{\partial y} = -x\sin y + e^x$$

$$\frac{\partial^2 f}{\partial y^2} = -x\cos y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(-x\sin y + e^x)$$

$$= -\sin y + e^x$$

3. $\omega = xy + \frac{e^y}{y^2+1}$

$$\frac{\partial \omega}{\partial y} = x + \frac{e^y \cdot 2y - (y^2+1)e^y}{(y^2+1)^2}$$

$$= x + \frac{2ye^y - y^2e^y - e^y}{(y^2+1)^2}$$

$$\frac{\partial^2 \omega}{\partial x \partial y} = \underline{\underline{1}}$$

4. Find f_{yxyz} if $f(x,y,z) = 1 - 2xy^2z + x^2y$

$$\frac{\partial f}{\partial z} = -2xy^2$$

$$\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial}{\partial y}(-2xy^2)$$

$$= \underline{\underline{-4xy}}$$

$$\frac{\partial^3 f}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}(-4xy)$$

$$= -4y$$

$$\frac{\partial^4 f}{\partial y \partial x \partial y \partial z} = \frac{\partial}{\partial y}(-4y)$$

$$= \underline{\underline{-4}}$$

5. Verify that $\omega_{xy} = \omega_{yx}$

(i) $\omega = \ln(2x+3y)$

$$\frac{\partial \omega}{\partial y} = \frac{1}{2x+3y} \cdot 2$$

$$\frac{\partial^2 \omega}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{3}{2x+3y}\right)$$

$$= \frac{-3}{(2x+3y)^2} \cdot 2$$

$$= \frac{-6}{(2x+3y)^2}$$

$$\frac{\partial^2 \omega}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} \ln(2x+3y)\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{2}{2x+3y}\right)$$

$$= \frac{-2}{(2x+3y)^2} \cdot 3$$

$$= \frac{-6}{(2x+3y)^2}$$

$\therefore \omega_{xy} = \omega_{yx}$

(ii) $\omega = e^{x} + x\ln y + y\ln x$

$$\omega_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(e^{x} + x\ln y + y\ln x\right)\right)$$

$$= \frac{\partial}{\partial x}\left(x \cdot \frac{1}{y} + \ln x\right)$$

$$= \underline{\underline{\frac{1}{y} + \frac{1}{x}}}$$

$$\omega_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(e^{x} + x\ln y + y\ln x\right)\right)$$

$$= \frac{\partial}{\partial y}\left(e^{x} + \ln y + \frac{y}{x}\right)$$

$$= \underline{\underline{\frac{1}{y} + \frac{1}{x}}}$$

(iii) $\omega = xy^{2} + x^{2}y^{3} + x^{3}y^{4}$

$$\omega_{xy} = \frac{\partial}{\partial x}\left(2xy + 3x^{2}y^{2} + 4x^{3}y^{3}\right)$$

$$= 2y + \underline{6xy^{2}} + 12x^{2}y^{3}$$

$$\omega_{yx} = \frac{\partial}{\partial y}\left(y^{2} + 2xy^{3} + 3x^{2}y^{4}\right)$$

$$= 2y + \underline{\underline{6xy^{2}}} + 12x^{2}y^{3}$$

$\therefore \underline{\underline{\omega_{xy} = \omega_{yx}}}$

(iv) $\omega = x\sin y + y\sin x + xy$

$$\omega_{xy} = \frac{\partial}{\partial x}\left(x\cos y + \sin x + x\right)$$

$$= \cancel{x}\cos y + \cos x + 1$$

$$\omega_{yx} = \frac{\partial}{\partial y}\,\frac{\partial}{\partial x}\left(x\sin y + y\sin x + xy\right)$$

$$= \cancel{x\cos y}$$

$$= \frac{\partial}{\partial y}\left(\sin y + y\cos x + y\right)$$

$$= \cos y + \cos x + 1$$

$$\omega_{xy} = \omega_{yx}$$

6. If the resistors of $R_1, R_2 + R_3$ ohms are connected in parallel to make an R-ohm resistor, the value of R can be found from the equation $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$ Find the values of $\frac{\partial R}{\partial R_2}$ when $R_1 = 30$, $R_2 = 45$ + $R_3 = 90\,\Omega$.

$$\frac{1}{R} = \frac{R_2R_3 + R_1R_3 + R_1R_2}{R_1R_2R_3}$$

$$R = \frac{R_1 R_2 R_3}{R_1 R_2 + R_2 R_3 + R_3 R_1}$$

$$\frac{\partial R}{\partial R_2} = \frac{(R_1 R_2 + R_2 R_3 + R_3 R_1) R_1 R_3 - R_1 R_2 R_3 (R_1 + R_3)}{(R_1 R_2 + R_2 R_3 + R_3 R_1)^2}$$

$$= \cancel{\frac{(1350 + 1350 + 900)\, 900 - 40500 \times 60}{}}$$

$$= \frac{(1350 + 4050 + 2700)\, 2700 - 121{,}500\,(120)}{(8{,}100)^2}$$

$$= 0.111$$

$$= \frac{1}{9}$$

$$\frac{\partial}{\partial R_2}\left(\frac{1}{R}\right) = \frac{\partial}{\partial R_2}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)$$

$$= \frac{-1}{R_2^2}. \ ?$$

28/1/2023 Saturday

The 5^{th} order partial derivative $\cancel{\frac{\partial^5 f}{\partial x^2 y^3}}$ $\frac{\partial^5 f}{\partial x^2 \partial y^3}$ is zero for each of the following functions. To show this as quickly as possible, which variable should you differentiate w.r.t to 1^{st} x or y? Try to answer.

1. $f(x,y) = y^2 x^4 e^x + 2$
2. $f(x,y) = y^2 + y\sin x - x^4$

3. $f(x,y) = x^2 + 5xy + \sin x + 7e^x$

4. $f(x,y) = xe^{y^2/2}$

1. $f(x,y) = y^2x^4e^x + y$

$$\frac{\partial f}{\partial y} = 2yx^4e^x + 1$$

$$\frac{\partial^2 f}{\partial y^2} = 2x^4e^x$$

$$\frac{\partial^3 f}{\partial y^3} = 0$$

$$\frac{\partial^5 f}{\partial x^2 \partial y^3} = 0$$

2. $f(x,y) = y^2 + y\sin x - x^4$

$$\frac{\partial f}{\partial y} = 2y + \sin x - x^4$$

$$\frac{\partial^2 f}{\partial y^2} = 2 + 0$$

$$\frac{\partial^3 f}{\partial y^3} = 0$$

3. $f(x,y) = x^2 + 5xy + \sin x + 7e^x$

$$\frac{\partial f}{\partial y} = 5x$$

$$\frac{\partial^2 f}{\partial y^2} = 0$$

4. $\frac{\partial f}{\partial x} = e^{y^2/2}$

$\frac{\partial^2 f}{\partial y \, \partial x} = e^{y^2/2} \cdot y$

$\frac{\partial^3 f}{\partial y^2 \partial x} =$

Chain Rule

Let $w = f(x)$ & $x = g(t)$. Then $w = f(g(t))$.
Then,

$$\frac{dw}{dt} = \frac{dw}{dx} \cdot \frac{dx}{dt}$$

where w - dependent variable
x - Intermediate variable
t - Independent variable.

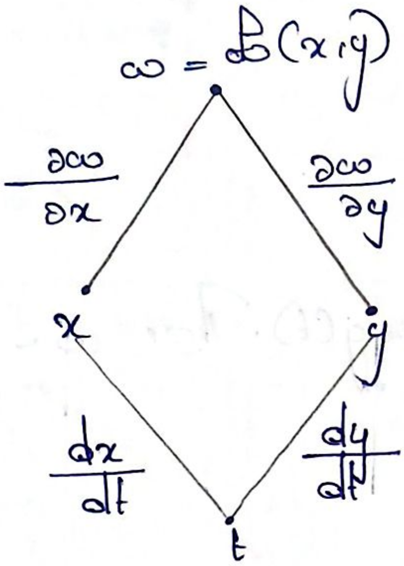
Chain Rule for functions of one Independent Variable & Two Intermediate Variable

If $w = f(x,y)$ is differentiable & if $x = x(t)$, $y = y(t)$ are differentiable functions of t. Then the composite fn. $w = f(x(t), y(t))$ is differentiable fn. of f ; ~~then the~~ f derivative of ω w.r.t t is

$$\frac{d\omega}{dt} = f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t)$$

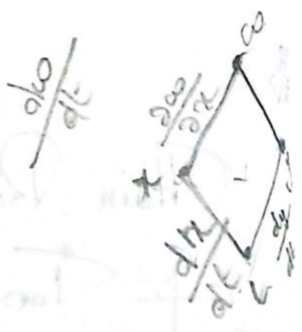
$$= \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

<u>Branch Diagram</u>



Intermediate variables

1. Use this chain rule to find derivative of $\omega = xy$ w.r.t t along the path $x = \cos t$, $y = \sin t$.
What is the derivative's value at $t = \frac{\pi}{2}$



$\omega = xy$

$\frac{\partial \omega}{\partial x} = y \qquad \frac{dx}{dt} = -\sin t$

$\frac{\partial \omega}{\partial y} = x \qquad \frac{dy}{dt} = \cos t$

$$\frac{d\omega}{dt} = \frac{\partial \omega}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial \omega}{\partial y} \cdot \frac{dy}{dt}$$

$$= y \cdot -\sin t + x \cdot \cos t$$

$$= -\sin^2 t + \cos^2 t$$

$$= \underline{\underline{\cos^2 t - \sin^2 t}}$$

$$\left.\frac{dw}{dt}\right|_{t=\frac{\pi}{2}} = \cos^2\frac{\pi}{2} - \sin^2\frac{\pi}{2}$$

$$= \underline{\underline{-1}}$$

2. Find $\frac{dw}{dt}$ if $w = xy + z$ where $x = \cos t$, $y = \sin t$, $z = t$.

$$\frac{\partial w}{\partial x} = y \qquad \frac{dx}{dt} = -\sin t$$

$$\frac{\partial w}{\partial y} = x \qquad \frac{dy}{dt} = \cos t$$

$$\frac{\partial w}{\partial z} = 1 \qquad \frac{dz}{dt} = 1$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$

$$= y \cdot -\sin t + x \cdot \cos t + 1$$

$\sin^2 t + \cos^2 t = 1$

$$= -\sin^2 t + \cos^2 t + 1$$

$$= \cos^2 t + 1 - \sin^2 t$$

$$= \cos^2 t + \cos^2 t$$

$$= \underline{\underline{2\cos^2 t}}$$

Evaluate $\frac{dw}{dt}$ at the given value of t & express $\frac{dw}{dt}$ as a fn. of t, both by using the chain rule & by expressing w in terms of t & differentiating direc w.r.t t.

1. $w = x^2+y^2$; $x = \cos t$, $y = \sin t$, $t = \pi$

2. $w = x^2+y^2$, $x = \cos t + \sin t$, $y = \cos t - \sin t$, $t = 0$

3. $w = \frac{x}{z} + \frac{y}{z}$, $x = \cos^2 t$, $y = \sin^2 t$, $z = \frac{1}{t}$, $t = 3$

4. $w = \ln(x^2+y^2+z^2)$, $x = \cos t$, $y = \sin t$, $z = 4\sqrt{t}$, $t = 3$

5. $w = 2ye^x - \ln z$, $x = \ln(t^2+1)$, $y = \tan^{-1} t$, $z = e^t$, $t = 1$

6. $w = z - \sin xy$, $x = t$, $y = \ln t$, $z = e^{t-1}$, $t = 1$

7. $z = \tan^{-1}(x/y)$, $x = u\cos v$, $y = u\sin v$, $(u, v) = (1.3, \pi/6)$

1. $w = x^2+y^2$

(i) $\frac{\partial w}{\partial x} = 2x$ $\quad \frac{dx}{dt} = -\sin t$

$\frac{\partial w}{\partial y} = 2y$ $\quad \frac{dy}{dt} = \cos t$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} = 2x \cdot -\sin t + 2y\cos t$$

$$= -2\cos t \sin t + 2\cos t \sin t$$

$$= \underline{\underline{0}}$$

(ii) $x = \cos t$

$y = \sin t$

$$= \frac{-a^2 \sin^2 v}{a^2} + \frac{-a^2 \cos^2 v}{a^2}$$

$$= -\left(\sin^2 v + \cos^2 v\right)$$

$$= \underline{\underline{-1}}$$

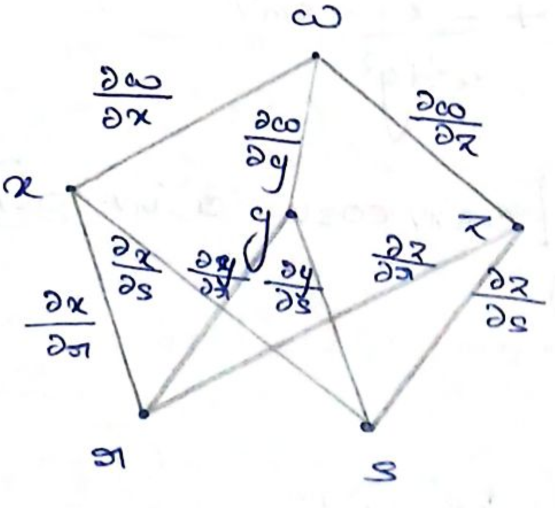
31/1/2023 Tuesday.

Chain Rule for 2 Independent Variable & 3 Intermediate Variables

Suppose $w = f(x, y, z)$ & $x = g(r, s)$, $y = h(r, s)$, $z = k(r, s)$. If all 4 fns. are differentiable, then w has partial derivative w.r.t r & s given by the formulas

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial s}$$



1. Express $\frac{\partial \omega}{\partial r}$ & $\frac{\partial \omega}{\partial s}$ in terms of r & s if

$\omega = x + 2y + z^2$; $x = \frac{r}{s}$, $y = r^2 + \ln s$, $z = 2r$.

$\frac{\partial \omega}{\partial x} = 1$ $\qquad \frac{\partial x}{\partial r} = \frac{1}{s}$ $\qquad \frac{\partial x}{\partial s} = \frac{-r}{s^2}$

$\frac{\partial \omega}{\partial y} = 2$ $\qquad \frac{\partial y}{\partial r} = 2r$ $\qquad \frac{\partial y}{\partial s} = \frac{1}{s}$

$\frac{\partial \omega}{\partial z} = 2z$ $\qquad \frac{\partial z}{\partial r} = 2$ $\qquad \frac{\partial z}{\partial s} = 0$

$$\frac{\partial \omega}{\partial r} = \frac{\partial \omega}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial \omega}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial \omega}{\partial z} \cdot \frac{\partial z}{\partial r}$$

$$= 1 \cdot \frac{1}{s} + 2 \cdot 2r + 2z \cdot 2$$

$$= \frac{1}{s} + 4r + 8r$$

$$= \underline{\underline{\frac{1}{s} + 12r}}$$

$$\frac{\partial \omega}{\partial s} = \frac{\partial \omega}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial \omega}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial \omega}{\partial z} \cdot \frac{\partial z}{\partial s}$$

$$= 1 \cdot \frac{-r}{s^2} + 2 \cdot \frac{1}{s} + 2z \times 0$$

$$= \frac{-r}{s^2} + \frac{2}{s}$$

<u>Note:-</u>

If ω is a fn. of 2 variable x & y. ie,

$\omega = f(x,y)$ & $x = g(r,s)$, $y = h(r,s)$, then

$$\frac{\partial \omega}{\partial r} = \frac{\partial \omega}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial \omega}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial \omega}{\partial s} = \frac{\partial \omega}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial \omega}{\partial y} \cdot \frac{\partial y}{\partial s}$$

1. Express $\frac{\partial \omega}{\partial r}$ & $\frac{\partial \omega}{\partial s}$ in terms of r & s if

$\omega = x^2+y^2$, $x = r-s$, $y = r+s$.

$$\frac{\partial \omega}{\partial r} = \frac{\partial \omega}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial \omega}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$= 2x \cdot 1 + 2y \cdot 1$$

$$= 2(r-s) + 2(r+s)$$

$$= 2r - \cancel{2s} + 2r + \cancel{2s}$$

$$= \underline{\underline{4r}}$$

$$\frac{\partial \omega}{\partial s} = \frac{\partial \omega}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial \omega}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$= 2x \cdot -1 + 2y \cdot 1$$

$$= -2(r-s) + 2(r+s)$$

$$= 2[-\cancel{r}+s+\cancel{r}+s]$$

$$= \underline{\underline{4s}}$$

<u>Note :-</u>

If $\omega = f(x)$ & $x = g(r,s)$, then

$$\frac{\partial \omega}{\partial r} = \frac{d\omega}{dx} \frac{\partial x}{\partial r}$$

$$\frac{\partial \omega}{\partial s} = \frac{d\omega}{dx} \cdot \frac{\partial x}{\partial s}$$

2. Express $\frac{\partial \omega}{\partial r}$ & $\frac{\partial \omega}{\partial s}$ if

$\omega = x^2 + 2x, \quad x = r - s$

$$\frac{\partial \omega}{\partial r} = \frac{d\omega}{dx} \cdot \frac{\partial x}{\partial r}$$

$$= (2x+2)\, 1$$

$$= 2(r-s) + 2$$

$$= \underline{\underline{2(r-s+1)}}$$

$$\frac{\partial \omega}{\partial s} = \frac{d\omega}{dx} \cdot \frac{\partial x}{\partial s}$$

$$= (2x+2) - 1$$

$$= -2(r-s) - 2$$

$$= \underline{\underline{-2(r-s+1)}}$$

Implicit Differentiation

H.W. Define implicit & explicit fns. Give 2 examples for each.

Suppose the fn. $F(x,y)$ is differentiable & that eqn. $F(x,y)=0$ defines y as a differentiable fn. of x. Then at any point where partial derivative of F w.r.t

y i.e, $F_y \neq 0$.

$$\frac{dy}{dx} = \frac{-F_x}{F_y}$$

1. Find $\frac{dy}{dx}$ if $y^2 - x^2 - \sin xy = 0$

$F(x,y) = y^2 - x^2 - \sin xy = 0$

$F_x = -2x - \cos xy \cdot y$

$= -2x - y\cos xy$

$F_y = 2y - \cos xy \cdot x$

$= 2y - x\cos xy$

$$\frac{dy}{dx} = \frac{-F_x}{F_y}$$

$$= \frac{2x + y\cos xy}{2y - x\cos xy}$$

Note:

Suppose that $F(x,y,z) = 0$ defines the variable z implicitly as a fn. $z = f(x,y)$. Then for all values of (x,y) in the domain of f, we have

$F(x, y, f(x,y)) = 0$.

Assuming that F & P are differentiable fns. we can use the chain rule to differentiate the eqn. $f(x,y,z) = 0$.

$$\therefore \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \& $$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

1. Find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ at $(0,0,0)$ if $x^3+z^2+ye^{xz}+z\cos y=0$

$$F_x = 3x^2 + ye^{xz}\left(z \cancel{+ x\frac{\partial z}{\partial x}}\right)$$

$$= 3x^2 + yze^{xz}$$

$$\cancel{F_x = 3x^2 +}$$

$$F_y = e^{xz} - z\sin y$$

$$F_z = 2z + yxe^{xz} + \cos y$$

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-(3x^2+yze^{xz})}{2z+yxe^{xz}+\cos y}$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-e^{xz}+z\sin y}{2z+yxe^{xz}+\cos y}$$

$$\left.\frac{\partial z}{\partial x}\right|_{(0,0,0)} = 0$$

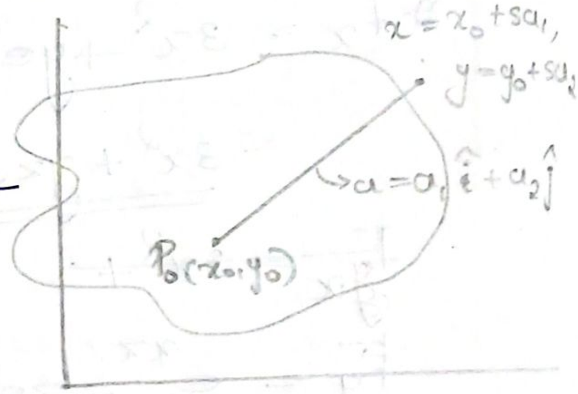
$$\left.\frac{\partial z}{\partial y}\right|_{(0,0,0)} = \underline{-1}$$

2/2/2023 Thursday

Directional Derivative & Gradient Vectors

Let $f(x,y)$ be a fn. where $x = h(s)$ & $y = g(s)$ are fns. of s. Then the derivative of f with respect to s at P_0

$$\left(\left.\frac{df}{ds}\right|_{a}\right)_{P_0} = \lim_{s \to 0} \frac{f(x_0 + s a_1, y_0 + s a_2) - f(x_0, y_0)}{s}$$



$$\left(\frac{df}{ds}\right)_{P_0} = \left(\frac{\partial f}{\partial x}\right)_{P_0}\frac{dx}{ds} + \left(\frac{\partial f}{\partial y}\right)_{P_0}\frac{dy}{ds}$$

$$= \left(\frac{\partial f}{\partial x}\right)_{P_0} a_1 + \left(\frac{\partial f}{\partial y}\right)_{P_0} a_2$$

$$= \left[\left(\frac{\partial f}{\partial x}\right)_{P_0}\hat{i} + \left(\frac{\partial f}{\partial y}\right)_{P_0}\hat{j}\right] \cdot \left[a_1\hat{i} + a_2\hat{j}\right]$$

$$= \text{Gradient} \cdot \text{unit vector}$$

$$= \underline{\nabla f \cdot a}$$

Definition :- Directional Derivative.

The derivative of f at point $P_0(x_0, y_0)$ in the direction of unit vector $a = a_1\hat{i} + a_2\hat{j}$ is the no. given by

$$\left(\frac{df}{ds}\right)_{a, P_0} = \lim_{s \to 0} \frac{f(x_0 + sa_1, y_0 + sa_2) - f(x_0, y_0)}{s}$$

provided the limit exist.

Other notations : $(D_a f)_{P_0}$

The derivative of f at P_0 in the direction of a

s - Arc length from the point P_0

1. Using the definition find the derivative of fn. $f(x, y)$ $f(x, y) = x^2 + xy$ at $P_0(1, 2)$ in the direction of the unit vector $a = \left(\frac{1}{\sqrt{2}}\right)\hat{i} + \left(\frac{1}{\sqrt{2}}\right)\hat{j}$

$\frac{df}{d}$ $\quad a_1 = a_2 = \frac{1}{\sqrt{2}}$

$$\bar{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$

$$f\left(1 + s \cdot \frac{1}{\sqrt{2}}, 2 + s\frac{1}{\sqrt{2}}\right) = \left(1 + \frac{s}{\sqrt{2}}\right)^2 + \left(1 + \frac{s}{\sqrt{2}}\right)\left(2 + \frac{s}{\sqrt{2}}\right)$$

$$f(x_0, y_0) = f(1, 2) = \underline{\underline{3}}$$

$$\left(\frac{df}{ds}\right)_{a, P_0} = \lim_{s \to 0} f\left(1 + \frac{s}{\sqrt{2}}\right)\left(1 + \frac{s}{\sqrt{2}} + 2 + \frac{s}{\sqrt{2}}\right)$$

$$= \lim_{s \to 0} \frac{\left(1 + \frac{s}{\sqrt{2}}\right)\left(3 + \sqrt{2}s\right)}{s} - S \quad \frac{1}{\sqrt{2}s} \quad \frac{2\sqrt{2} + 2s + 3s + \sqrt{2}s^2}{2}$$

$$= \frac{5}{\sqrt{2}}$$

Gradient Vector (Gradient)

orthogonal to all level curves

The gradient vector of $f(x,y)$ at a point $P_0(x_0,y_0)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$

obtained by evaluating the partial derivatives of f at P_0.

Direction Derivative

The directional derivative is a dot product. If $f(x,y)$ is differentiable in open region containing $P_0(x_0,y_0)$ then directional derivative of f w.r.t parameter s is

$$\left(\frac{df}{ds}\right)_{P_0} = (\nabla f)_{P_0} \cdot u$$

In breif,

$$\underline{\underline{(D_u f) = \nabla f \cdot u}}$$

$$\nabla f \cdot u = |\nabla f||u|\cos\theta$$

$$= |\nabla f|_{P_0}\cos\theta \quad [\because u \text{ is unit vector}].$$

1. Find the derivative of $f(x,y) = xe^y + \cos(xy)$ at the point $(2,0)$ in the direction of $V = 3\hat{i} - 4\hat{j}$.

$$u = \frac{v}{|v|} = \frac{3\hat{i} - 4\hat{j}}{\sqrt{9+16}} = \underline{\underline{\frac{3}{5}\hat{i} + \frac{-4}{5}\hat{j}}}$$

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$

$$= e^y - \sin(xy)\cdot y\,\hat{i} + xe^y + -\sin xy \cdot x$$

$$= [e^y - y\sin(xy)]\hat{i} + [xe^y - x\sin(xy)]\hat{j}$$

$$\nabla f_{P_0} = (e^0 - 0)\hat{i} + (2\cdot e^0 - 2\cdot\sin(0))\hat{j}$$

$$= \boxed{\underline{\underline{1\hat{i} + 2\hat{j}}}}$$

$$\left(\frac{df}{ds}\right)_{P_0} = \nabla f_{P_0} \cdot u$$

$$= (\hat{i} + 2\hat{j}) \cdot \left(\frac{3}{5}\hat{i} + \frac{-4}{5}\hat{j}\right)$$

$$= \frac{8}{5} + \frac{-8}{5}$$

$$= \underline{\underline{\frac{-5}{5}}} = \underline{\underline{-1}}$$

2. Find the direction in which $f(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$

(a) Increases most rapidly at the point $(1,1)$.

The fn. ↑ most rapidly in the direction of ∇f at $(1,1)$.

$$\nabla f = x\hat{i} + y\hat{j}$$

$\nabla f|_{(1,1)} = 1\hat{i} + 1\hat{j} = \hat{i}+\hat{j}$

Direction is given by unit vector.

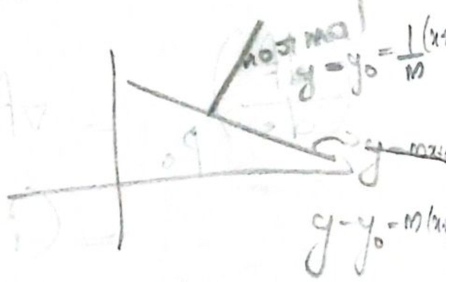
$\therefore \quad u = \frac{\nabla f}{|\nabla f|} = \frac{\hat{i}+\hat{j}}{\sqrt{1+1}}$

$u = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$

(b) The fn. ↓ most rapidly

$-u = -\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$

(c) The direction of zero change at (1,1) are the direction orthogonal to ∇f.



~~$n = |\nabla f| \cdot |u| \cdot \cos$~~

$n = -\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$

$-n = \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$

3. Find the derivative of the fn. at P_0 in the direction of u.

$g(x,y) = \frac{x-y}{xy+2}$, $P_0(1,-1)$, $v = 12i + 5j$

Direction of v $=$, $\hat{v} = \dfrac{v}{|v|}$

$$= \frac{12\hat{i} + 5\hat{j}}{\sqrt{144+25}}$$

$$a = \frac{12}{13}\hat{i} + \frac{5}{13}\hat{j}$$

$$\nabla g = \frac{\partial g}{\partial x}\hat{i} + \frac{\partial g}{\partial y}\hat{j}$$

$$= \left[\frac{(xy+2) - (x-y)y}{(xy+2)^2}\right]\hat{i} + \left[\frac{(xy+2)\cdot -1 - (x-y)\,x}{(xy+2)^2}\right]\hat{j}$$

$$= \frac{(\cancel{xy} + 2 - \cancel{xy} + y^2)}{(xy+2)^2}\hat{i} + \frac{(\cancel{-xy} - 2 - x^2 + \cancel{xy})}{(xy+2)^2}\hat{j}$$

$$= \frac{(y^2+2)}{(xy+2)^2}\hat{i} + \frac{-(x^2+2)}{(xy+2)^2}\hat{j}$$

$$\nabla g\Big|_{(1,-1)} = \frac{3}{1}\hat{i} - \frac{3}{1}\hat{j}$$

$$= 3\hat{i} - 3\hat{j}$$

$$D_a g\Big|_{P_0} = \nabla g \cdot a$$

$$= (3\hat{i} - 3\hat{j})\cdot\left(\frac{12}{13}\hat{i} + \frac{5}{13}\hat{j}\right)$$

$$= \frac{36}{13} - \frac{15}{13} = \frac{21}{13}$$