LAB 11 : Hidden Markov Model

In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd
```

What is HMM?

Hidden Markov Model is a Markov Chain which is mainly used in problems with temporal sequence of data. Markov Model explains that the next step depends only on the previous step in a temporal sequence. In Hidden Markov Model the state of the system is hidden (invisible), however each state emits a symbol at every time step.

Please refer to the following article to understand Hidden Markov Model

Here we will be dealing with 3 major problems :

1. Evaluation Problem
2. Learning Problem
3. Decoding Problem

1. Evaluation Problem : Implementation of Forward and Backward Algorithm

Model:
$\theta : s, v, a_{ij}, b_{jk}$
a_{ij} = Transition Probabilities
b_{jk} = Emission Probabilities
Given the model (θ) and Sequence of visible/observable symbol (V^T), we need to determine the probability that a particular sequence of visible states/symbols (V^T) that was generated from the model (θ).

Forward and Backward Algortihm

As we have seen that for evaluation problem we have to do the following steps:

- First we need to find all possible sequences of the state S^M where M is the number of Hidden States.
- Then from all those sequences of S^M, find the probability of which sequence generated the visible sequence of symbols V^T.
  We will a recursive dynamic programming approach.
  #### Forward: In Forward Algorithm , we will use the computed probability on current time step to derive the probability of the next time step.
  $\alpha_j(t) = p(v(1) \ldots v(t), s(t) = j)$
  In general-
  $\alpha_j(t) = b_{jk} \sum_{i=1}^{M} \alpha_i(t-1) a_{ij}$

Backward: Backward Algorithm is the time-reversed version of the Forward Algorithm. In Backward Algorithm we need to find the probability that the machine will be in hidden state s_i at time step t and will generate the remaining part of the sequence of the visible symbol V^T.
$\beta_i(t) = \sum_{j=0}^{M} a_{ij} b_{jk}(t+1)\beta_j(t+1)$

In [2]:

```
data =
pd.read_csv('/Users/kushagrakhatwani/Downloads/data_python.csv')
## Read the data, change the path accordingly
print(data.head())
V = data['Visible'].values
H = data['Hidden'].values
print("Vshape",V.shape)
# Transition Probabilities
a = np.array(((0.54, 0.46), (0.49, 0.51)))

# Emission Probabilities
b = np.array(((0.16, 0.26, 0.58), (0.25, 0.28, 0.47)))

# Equal Probabilities for the initial distribution
initial_distribution = [0.5,0.5]## Write your code here



def forward(V, a, b, initial_distribution):
    alpha = np.zeros((V.shape[0], a.shape[0]))

    ## Write your code here
    alpha[0, :] = initial_distribution * b[:, V[0]]

    for t in range(1, V.shape[0]):
        for j in range(a.shape[0]):
            alpha[t, j] = alpha[t - 1].dot(a[:, j]) * b[j, V[t]]
    return alpha



alpha = forward(V, a, b, initial_distribution)



def backward(V, a, b):
    beta = np.zeros((V.shape[0], a.shape[0]))

    ## Write your code here
    beta[V.shape[0] - 1] = np.ones((a.shape[0]))
```

```
    for t in range(V.shape[0] - 2, -1, -1):
        for j in range(a.shape[0]):
            beta[t, j] = (beta[t + 1] * b[:, V[t + 1]]).dot(a[j,
:])
    return beta
beta = backward(V, a, b)
print("Alpha:",alpha)

print("Beta:",beta)
```

```
  Hidden  Visible
0      B        0
1      B        1
2      B        2
3      B        2
4      B        2
Vshape (500,)
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2. Learning Problem : Implementation of Baum Welch Algorithm

Objective of the problem-

- The objective of the Learning Problem is to estimate for a_{ij} and b_{jk} using the training data.
- The standard algorithm for Hidden Markov Model training is the Forward-Backward or Baum-Welch Algorithm.
  ### Steps to follow-
- Start with initial probability estimates (A,B). Initially set equal probabilities or define them randomly.
- Compute expectation of how often each transition/emission has been used.
- Re-estimate the probabilities (A,B).
  Repeat until convergence.
  #### Derivation of a_{ij_pred}: If we know the probability of a given transition from i to j at time step t, then we can sum over all the T times to estimate for the numerator in our equation for A_{pred}.
  We can define this as the probability of being in state i at time t and in state j at time t+1,

given the observation sequence and the model as-

$p(s(t)=i,s(t+1)=j|V^T,\theta)$

$p(s(t)=i,s(t+1)=j|V^T,\theta)=\frac{p(s(t)=i,s(t+1)=j,V^T|\theta)}{p(V^T|\theta)}$

The numerator of the equation can be expressed using Forward and Backward Probabilities-

$p(s(t)=i,s(t+1)=j,V^T|\theta)=\alpha_i(t)a_{ij}b_{jk(t+1}\beta_j(t+1)$

The denominator $p(V^T|\theta)$ is the probability of the observation sequence V^T by any path given the model θ. It can be expressed as the marginal probability:

$p(V^T|\theta)=\sum_{i=1}^{M}\sum_{j=1}^{M}\alpha(t)a_{ij}b_{jk}(t+1)\beta_j(t+1)$

We will define ξ as the latent variable representing $p(s(t)=i,s(t+1)=j|V^T,\theta)$. We can now define $\xi_{ij}(t)$ as:

$\xi(t)=\frac{\alpha_i(t)a_{ij}b_{jk}(t+1)\beta j(t+1)}{\sum_{i=1}^{M}\sum_{j=1}^{M}\alpha_i(t)a_{ij}b_{jk}(t+1)\beta_j(t+1)}$

Then estimate a_{ij} as:

$a_{ij\_pred}=\frac{\sum_{t=1}^{T-1}\xi_{ij}(t)}{\sum_{t=1}^{T-1}\sum_{j=1}^{M}\xi_{ij}(t)}$ #### Derivation of b_{jk_pred}: b_{jk} is the probability of a given symbol vk from the observations V given a hidden state j.

$\gamma_j(t)=\frac{\alpha_j(t)\beta_j(t)}{\sum_{j=1}^{M}\alpha_j(t)\beta_j(t)}$

We can compute b_{jk_pred} using $\gamma_j(t)$,

$b_{jk\_pred}=\frac{\sum_{t=1}^{T-1}\xi_j(t)1(v(t)=k)}{\sum_{t=1}^{T}\gamma_j(t)}$

where 1(v(t)=k) is the indicator function.

In [3]:

```
def baum_welch(V, a, b, initial_distribution, n_iter=100):
  ## Write your code here
    M = a.shape[0]
    T = len(V)

    for n in range(n_iter):
        alpha = forward(V, a, b, initial_distribution)
        beta = backward(V, a, b)

        xi = np.zeros((M, M, T - 1))
        for t in range(T - 1):
            denominator = np.dot(np.dot(alpha[t, :].T, a) * b[:, 
V[t + 1]].T, beta[t + 1, :])
            for i in range(M):
                numerator = alpha[t, i] * a[i, :] * b[:, V[t + 
1]].T * beta[t + 1, :].T
                xi[i, :, t] = numerator / denominator

        gamma = np.sum(xi, axis=1)
        a = np.sum(xi, 2) / np.sum(gamma, axis=1).reshape((-1, 1))
```

```
        # Add additional T'th element in gamma
        gamma = np.hstack((gamma, np.sum(xi[:, :, T - 2],
axis=0).reshape((-1, 1))))

        K = b.shape[1]
        denominator = np.sum(gamma, axis=1)
        for l in range(K):
            b[:, l] = np.sum(gamma[:, V == l], axis=1)

        b = np.divide(b, denominator.reshape((-1, 1)))
    return (a,b)

data =
pd.read_csv('/Users/kushagrakhatwani/Downloads/data_python.csv')

V = data['Visible'].values

# Transition Probabilities
a = np.ones((2, 2))
a = a / np.sum(a, axis=1)
# Emission Probabilities
b = np.array(((1, 3, 5), (2, 4, 6)))
b = b / np.sum(b, axis=1).reshape((-1, 1))

# Equal Probabilities for the initial distribution
initial_distribution = np.array((0.5, 0.5))

a,b = baum_welch(V, a, b, initial_distribution, n_iter=100)
print("a:",a)
print("b:",b)
```

```
a: [[0.53816345 0.46183655]
 [0.48664443 0.51335557]]
b: [[0.16277513 0.26258073 0.57464414]
 [0.2514996  0.27780971 0.47069069]]
```

3. Decoding Problem : Implementation of Viterbi Algorithm

Given a sequence of visible symbol V^T and the model (θ→{A,B}) find the most probable sequence of hidden states S^T.
The following equation represents the highest probability along a single path for first t observations which ends at state i.
$\omega_i(t) = \max s_1, \ldots, s_{T-1} p(s1, s2 \ldots . s_T = i, v1, v2 \ldots v_T|\theta)$
We can use the same approach as the Forward Algorithm.

Now to find the sequence of hidden states we need to identify the state that maximizes $\omega_i(t)$ at each time step t.

In [4]:

```
def viterbi(V, a, b, initial_distribution):

    ## Write your code here
    T = V.shape[0]
    M = a.shape[0]

    omega = np.zeros((T, M))
    omega[0, :] = np.log(initial_distribution * b[:, V[0]])

    prev = np.zeros((T - 1, M))

    for t in range(1, T):
        for j in range(M):
            # Same as Forward Probability
            probability = omega[t - 1] + np.log(a[:, j]) +
np.log(b[j, V[t]])

            # This is our most probable state given previous state
at time t (1)
            prev[t - 1, j] = np.argmax(probability)

            # This is the probability of the most probable state
(2)
            omega[t, j] = np.max(probability)

    # Path Array
    S = np.zeros(T)

    # Find the most probable last hidden state
    last_state = np.argmax(omega[T - 1, :])

    S[0] = last_state

    backtrack_index = 1
    for i in range(T - 2, -1, -1):
        S[backtrack_index] = prev[i, int(last_state)]
        last_state = prev[i, int(last_state)]
        backtrack_index += 1

    # Flip the path array since we were backtracking
    S = np.flip(S, axis=0)
```

```
    # Convert numeric values to actual hidden states
    result = []
    for s in S:
        if s == 0:
            result.append("A")
        else:
            result.append("B")
    return result
```

In [5]:

```
data =
pd.read_csv('/Users/kushagrakhatwani/Downloads/data_python.csv')

V = data['Visible'].values

# Transition Probabilities
a = np.ones((2, 2))
a = a / np.sum(a, axis=1)

# Emission Probabilities
b = np.array(((1, 3, 5), (2, 4, 6)))
b = b / np.sum(b, axis=1).reshape((-1, 1))

# Equal Probabilities for the initial distribution
initial_distribution = np.array((0.5, 0.5))

a, b = baum_welch(V, a, b, initial_distribution, n_iter=100)

result = viterbi(V, a, b, initial_distribution)
print(result)
acc = (np.sum(H!=result))/len(H)
```

```
['B', 'B', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A',
'A', 'A', 'A', 'A', 'A', 'A', 'B', 'B', 'B', 'B', 'A', 'A', 'A', 'B', 'A',
'B', 'B', 'A', 'A', 'A', 'B', 'A', 'B', 'A', 'A', 'B', 'A', 'A', 'A', 'B',
'B', 'B', 'B', 'B', 'A', 'A', 'A', 'A', 'B', 'A', 'A', 'A', 'A', 'A', 'B',
'B', 'B', 'B', 'A', 'A', 'A', 'B', 'B', 'B', 'B', 'A', 'B', 'B', 'B', 'B',
'A', 'B', 'B', 'A', 'B', 'B', 'B', 'A', 'B', 'B', 'B', 'A', 'B', 'B', 'B',
'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A',
'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A',
'B', 'B', 'B', 'B', 'B', 'B', 'A', 'B', 'B', 'B', 'B', 'B', 'B', 'B', 'A',
'B', 'A', 'A', 'B', 'B', 'B', 'B', 'B', 'A', 'A', 'B', 'A', 'B', 'B', 'B',
'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A',
'A', 'A', 'B', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'B', 'A', 'A', 'A',
'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'B', 'A', 'A', 'A', 'A',
'A', 'A', 'A', 'A', 'A', 'A', 'A', 'B', 'A', 'B', 'B', 'A', 'B', 'B', 'B',
'B', 'B', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'B', 'B', 'B', 'B',
```

```
'A', 'B', 'B', 'A', 'A', 'A', 'A', 'A', 'A', 'B', 'B', 'B', 'A', 'A', 'A',
'B', 'B', 'B', 'B', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A',
'B', 'A', 'B', 'B', 'B', 'A', 'B', 'B', 'A', 'A', 'A', 'B', 'B', 'B', 'B',
'B', 'A', 'A', 'A', 'B', 'B', 'A', 'B', 'A', 'A', 'B', 'B', 'B', 'B', 'B',
'B', 'B', 'B', 'A', 'A', 'B', 'B', 'A', 'A', 'A', 'B', 'B', 'B', 'A', 'A',
'B', 'B', 'A', 'A', 'A', 'A', 'A', 'A', 'B', 'B', 'A', 'A', 'B', 'A', 'B',
'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'B', 'B', 'B',
'A', 'A', 'A', 'A', 'A', 'B', 'B', 'A', 'B', 'B', 'B', 'B', 'B', 'A', 'A',
'A', 'A', 'A', 'A', 'A', 'B', 'A', 'B', 'B', 'A', 'A', 'A', 'A', 'A', 'A',
'B', 'B', 'B', 'B', 'B', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A',
'A', 'A', 'A', 'A', 'A', 'B', 'B', 'A', 'B', 'B', 'A', 'A', 'A', 'B', 'A',
'B', 'B', 'A', 'A', 'B', 'A', 'A', 'B', 'A', 'A', 'A', 'A', 'B', 'A', 'A',
'A', 'A', 'A', 'B', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'B', 'B',
'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'B', 'A',
'A', 'A', 'A', 'A', 'A', 'A', 'A', 'A', 'B', 'A', 'A', 'A', 'A', 'B', 'B',
'B', 'B', 'A', 'B', 'B', 'A', 'A', 'A', 'B', 'B', 'B', 'B', 'A', 'A', 'A',
'A', 'A', 'B', 'A', 'A', 'A', 'A', 'B', 'B', 'A', 'B', 'B', 'B', 'B', 'B',
'B', 'B', 'A', 'B', 'B', 'B', 'A', 'A', 'A', 'A', 'B', 'B', 'B', 'A', 'A',
'A', 'A', 'A', 'A', 'A']
```

1. Use the built-in **hmmlearn** package to fit the data and generate the result using the decoder

In [6]:

```
#!pip install hmmlearn
```

In [7]:

```
## Write your code here
from hmmlearn import hmm
model = hmm.GaussianHMM(n_components=2, covariance_type="full",
n_iter=100)
model.fit(np.reshape(V,(len(V),1)))
hidden = model.predict(np.reshape(V,(len(V),1)))
print("Hidden: \n",hidden)
print("A:\n",model.transmat_)
print("Using Decoder:\n",model.decode(np.reshape(V,(len(V),1))))
print("Probability of getting above sequence:",np.exp(-
model.decode(np.reshape(V,(len(V),1)))[0]))
```

```
Hidden:
 [1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 1 1 1 1 0 1 0 1 0 1 1 0 1 0 1 0
 1 0 0 1 0 0 0 1 1 1 1 1 0 0 0 0 1 0 0 0 1 0 1 1 1 1 0 1 0 1 1 1 1 0 1 1 1
 1 0 1 1 0 1 1 1 0 1 1 1 0 1 1 1 0 1 0 1 1 0 1 0 0 0 1 0 1 0 1 1 1 0 0 1 0
 0 1 0 0 0 0 0 0 0 1 1 1 1 1 1 0 1 1 1 1 1 1 1 0 1 0 0 1 1 1 1 1 0 0 1 0 1
 1 1 0 0 0 0 0 1 0 0 0 0 0 1 0 1 0 1 0 1 0 0 0 0 0 0 0 0 1 0 1 0 1 1 1 0 0
 0 0 0 0 0 1 0 0 0 0 0 1 0 1 0 1 0 1 0 1 1 0 1 1 1 1 1 0 0 0 0 0 0 0 0 0 1
 1 1 1 0 1 1 0 0 1 0 0 0 1 1 1 0 0 0 1 1 1 1 0 0 1 0 0 0 1 0 0 0 0 1 0 1 1
 1 0 1 1 0 0 0 1 1 1 1 1 0 1 0 1 1 0 1 0 0 1 1 1 1 1 1 1 1 0 0 1 1 0 0 0 1
 1 1 0 0 1 1 0 0 0 0 0 0 1 1 0 0 1 0 1 0 0 0 1 0 0 0 1 1 1 1 0 1 1 1 0 0 1
 1 0 1 1 0 1 1 1 1 1 0 1 0 1 0 0 0 1 0 1 1 0 0 0 0 0 0 1 1 1 1 1 0 1 0 0 0
 0 0 1 1 0 0 0 0 0 0 1 1 0 1 1 0 1 0 1 0 1 1 0 0 1 0 0 1 0 0 0 0 1 0 0 0 1
 0 1 0 1 0 0 0 1 0 0 0 1 1 0 1 0 0 0 0 0 0 0 1 0 0 0 1 0 0 1 0 0 0 0 1 0 1
 0 1 0 0 1 1 1 1 0 1 1 0 0 0 1 1 1 1 0 1 0 0 0 1 0 1 0 0 1 1 0 1 1 1 1 1 1
 1 0 1 1 1 0 1 0 0 1 1 1 0 1 1 0 0 1 0]
A:
```

```
 [[0.55548478 0.44451522]
 [0.49149477 0.50850523]]
Using Decoder:
 (577.1457377057429, array([1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1,
0, 0, 0, 0, 1,
       1, 1, 1, 0, 1, 0, 1, 0, 1, 1, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 0, 0,
       1, 1, 1, 1, 1, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 1, 1, 1, 0, 1, 0,
       1, 1, 1, 1, 0, 1, 1, 1, 1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 0, 1,
       1, 1, 0, 1, 0, 1, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 1, 1, 0, 0, 1,
       0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1,
       1, 1, 0, 1, 0, 0, 1, 1, 1, 1, 1, 0, 0, 1, 0, 1, 1, 1, 0, 0, 0, 0,
       0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0,
       1, 0, 1, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0,
       1, 0, 1, 0, 1, 0, 1, 1, 0, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0,
       0, 1, 1, 1, 1, 0, 1, 1, 0, 0, 1, 0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 1,
       1, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 1, 1, 1, 0, 1, 1, 0,
       0, 0, 1, 1, 1, 1, 1, 0, 1, 0, 1, 1, 0, 1, 0, 0, 1, 1, 1, 1, 1, 1,
       1, 1, 0, 0, 1, 1, 0, 0, 0, 1, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0,
       1, 1, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 1, 1, 1, 0, 1, 1, 1,
       0, 0, 1, 1, 0, 1, 1, 0, 1, 1, 1, 1, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0,
       1, 1, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 0, 1, 0, 0, 0, 0, 0, 1, 1,
       0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 0, 1, 0, 1, 0, 1, 1, 0, 0, 1, 0,
       0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0,
       1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0,
       0, 1, 0, 1, 0, 1, 0, 0, 1, 1, 1, 1, 0, 1, 1, 0, 0, 0, 1, 1, 1, 1,
       0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 0, 1,
       1, 1, 0, 1, 0, 0, 1, 1, 1, 0, 1, 1, 0, 0, 1, 0], dtype=int32))
Probability of getting above sequence: 2.2324968789538278e-251
```

In [ ]: