MTH 203 Mathematics III (Multivariate Calculus)

Tutorial Sheet 1

1) Identify the points, if any, where the following functions fail to be continuous:

(i) $f(x,y) = \begin{cases} xy & \text{if } xy \geq 0 \\ -xy & \text{if } xy < 0 \end{cases}$; (ii) $f(x,y) = \begin{cases} xy & \text{if } xy \text{ rational} \\ -xy & \text{if } xy \text{ irrational} \end{cases}$

2) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2y^2 + (x-y)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Show that the function satisfy the following:

(a) The iterated limits $\lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right]$ and $\lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right]$ exist and equal 0

(b) $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist

(c) $f(x,y)$ is not continuous at $(0,0)$

(d) the partial derivatives exist at $(0,0)$

3) Let $f(x,y) = \begin{cases} (x^2+y^2)\sin\frac{1}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

Show that f is differentiable at every point of $\mathbb{R}^2$ but the partial derivatives are not continuous at $(0,0)$

4) Suppose f is a function with $f_x(x,y) = f_y(x,y) = 0$ for all (x,y). Then show that $f(x,y) =$ constant.

5) Let $f(x,y) = \frac{1}{2}(|x|-|y|\,| - |x| - |y|)$. Is f continuous at $(0,0)$? Which directional derivatives of f exist at $(0,0)$? Is f differentiable at $(0,0)$?

6) Let $f(x,y) = \frac{x^2y}{x^2+y^2}$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Show that the directional derivatives of f at $(0,0)$ in all directions exist but f is not differentiable at $(0,0)$.

7) Find the equation of the surface generated by the normals to the surface $x + 2yz + xyz^2 = d$ at all points on the z-axis.