Differentiability : When $f:\mathbb{R}\to\mathbb{R}, x\in\mathbb{R}$ we define

$$f^{'}(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \qquad (*)$$

provided the limit exists. In case $f:\mathbb{R}^3\to\mathbb{R}$ and $X=(x_1,x_2,x_3)\in\mathbb{R}^3$ the above definition of the differentiability of functions of one variable (*) cannot be generalized as we cannot divide by an element of $\mathbb{R}^3$. So, in order to define the concept of differentiability, what we do is that we rearrange the above definition (*) to a form which can be generalized.

Let $f:\mathbb{R}\to\mathbb{R}$. Then f is differentiable at x if and only if there exists $\alpha\in\mathbb{R}$ such that

$$\frac{\mid f(x+h)-f(x)-\alpha\cdot h\mid}{\mid h\mid}\to 0 \quad as \quad h\to 0.$$

When f is differentiable at x, α has to be $f^{'}(x)$. We generalize this definition to the functions of several variables.

Definition : Let $f:\mathbb{R}^3\to\mathbb{R}$ and $X=(x_1,x_2,x_3)$. We say that f is *differentiable* at X if there exists $\alpha=(\alpha_1,\alpha_2,\alpha_3)\in\mathbb{R}^3$ such that the error function

$$\varepsilon(H)=\frac{f(X+H)-f(X)-\alpha\cdot H}{\parallel H\parallel}$$

tends to 0 as $H\to 0$.

In the above definition $\alpha\cdot H$ is the scalar product. Note that the derivative $f'(X)=(\alpha_1,\alpha_2,\alpha_3)$.

Theorem 26.2: Suppose f is differentiable at X. Then the partial derivatives $\frac{\partial f}{\partial x}\mid_X, \frac{\partial f}{\partial y}\mid_X$ and $\frac{\partial f}{\partial z}\mid_X$ exist and the derivative

$$f^{'}(X)=(\alpha_1,\alpha_2,\alpha_3)=(\frac{\partial f}{\partial x}\mid_X, \frac{\partial f}{\partial y}\mid_X, \frac{\partial f}{\partial z}\mid_X).$$

Proof : Suppose f is differentiable at X and $f'(X)=(\alpha_1,\alpha_2,\alpha_3)$. Then by taking $H=(t,0,0)$, we have

$$\varepsilon(H)=\frac{f(X+H)-f(X)-\alpha_1 t}{|t|}\to 0 \quad as \quad t\to 0, \quad i.e., \quad \frac{f(X+H)-f(X)-\alpha_1 t}{t}\to 0$$

This implies that $\alpha_1=\frac{\partial f}{\partial x}\mid_X$. Similarly we can show that $\alpha_2=\frac{\partial f}{\partial y}\mid_X$ and $\alpha_3=\frac{\partial f}{\partial x}\mid_X$. □

Example 3 : Let

$$\begin{aligned} f(x,y) &= xy\frac{x^2-y^2}{x^2+y^2} \quad at \quad (x,y)\neq(0,0)\\ &= 0 \quad at \quad (0,0)\end{aligned}$$

To verify that f is differentiable at $(0,0)$, let us choose $\alpha=(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y})\mid_{(0,0)}$ and verify that $\epsilon(H)\to 0$ as $H=(h,k)\to 0$. In this case $\alpha=(0,0)$ and

$$\mid\varepsilon(H)\mid=\mid\frac{f(0+H)-f(0)-(0,0)\cdot H}{\parallel H\parallel}\mid\leq\mid\frac{hk}{\sqrt{h^2+k^2}}\mid\leq\sqrt{h^2+k^2}\to 0 \quad as \quad H\to 0.$$

Hence f is differentiable at $(0,0)$. □