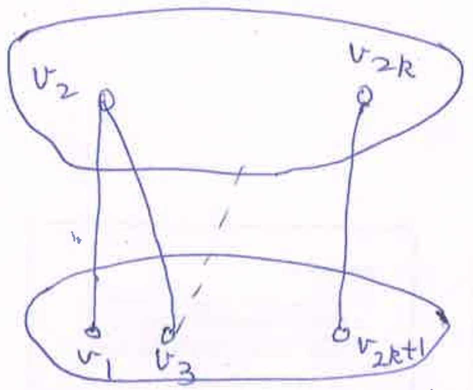
Proposition 5: A graph G with at least two vertices is bipartite if and only if it has no odd cycle.

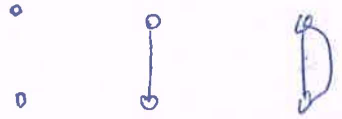
Proof: [$\Rightarrow$] Suppose G is bipartite with K_1, K_2 as its partite sets. Suppose BWOC that G has an odd cycle $v_1, v_2, \ldots, v_{2k+1}, v_1$.
WOLOG, $v_1 \in K_1$.



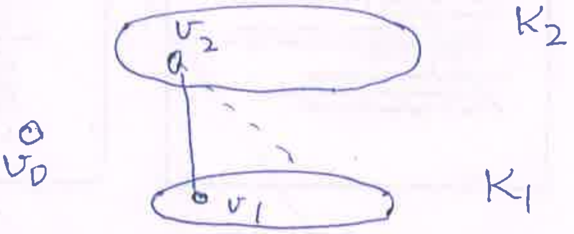
The vertices are distinct and alternately in K_1 and K_2 (see diagram). But then we get an edge $v_{2k+1}v_1$ in which the end-points are in the same partite set K_1. $\Rightarrow\Leftarrow$

[$\Leftarrow$] Suppose G has no odd cycle; we need to show it is bipartite. We proceed by induction on the order of G, i.e. $|V(G)| = n$.

Base Case: $n = 2$.
Then G has two independent sets (provided there is no loop) and either no edges, one edge, or ≥ 2 edges. In all such cases it is bipartite with no odd cycle.



Induction Step: Suppose the result holds for all graphs with n vertices ($n \geq 2$) and G has $n+1$ vertices.
Let $P = v_0 v_1 \ldots v_k$ be a maximal path in G, and let $H = G - v_0$.
Then, H has n vertices, and no odd cycles. Thus, by IH, H is bipartite with partite sets K_1, K_2, say.
WOLOG, $v_1 \in K_1$.



Claim: $\{v_0\} \cup K_2$ is an independent set in G.
To prove the claim, suppose BWOC that $v_0 v$ is an edge in G with $v \in K_2$.
Case 1: $v \notin P$.
Then $v v_0 \ldots v_k$ is a path in G containing P $\Rightarrow\Leftarrow$.
Case 2: $v \in P$.
Then $v = 2m$ for some m.
But then: $v_0 v_1 v_2 \cdots \underset{v}{\underset{\shortparallel}{v_{2m}}} v_0$
is an odd cycle in G.
This proves the claim and hence the result.