SOLUTION SET

(Not necessarily in same order)

MTH303 – CLASS TEST 13/04/2015

TIME: 1 HOUR **MAXIMUM MARKS: 50**

NB: You may use any known result (i.e. propositions and lemmas) without proof; however, it should be identified clearly. This does not apply if you have been asked to prove a know result. Marks will depend on the correctness and completeness of your proofs.

1. a) Show that distinct blocks of a simple graph are edge disjoint. *(3 marks)*
   b) What can you say about a common vertex of two distinct blocks ? Justify your answer in detail. *(7 marks)*

2. a) Let $G = K_{m,n}$ and let S be the set containing a vertices from the first partite set and b vertices from the second partite set. Find the number of edges in the cut set $[S, V(G) - S]$ in terms of a formula involving a, b, m, n. *(5 marks)*

   b) Show that every set of seven edges in $K_{3,3}$ is an edge disconnecting set, but no set of seven edges is a cut set. *(5 marks)*

3. Show that if a simple graph G has at least 11 vertices, then G and its complement cannot both be planar graphs. *(10 marks)*

4. Let G be a k-chromatic graph such that $\chi(G - v) = k - 1$ for every vertex v of G. Show that $k - 1 \le \delta(G)$. Give an example to show that the converse doe not hold, i.e. there exists a k-chromatic graph G with $k - 1 \le \delta(G)$ but $\chi(G - v) = k - 1$ does not hold for every vertex v of G. *(10 marks)*

5. Let G be a simple graph with vertex set $\{v_1, v_2, \dots, v_n\}$ and let $G_i = G - v_i$. Show that G is connected if and only if at least two of the G_i are connected. *(10 marks)*

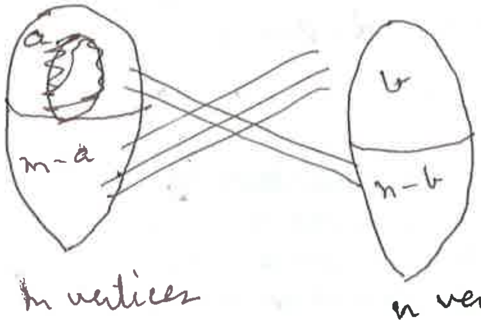
Q 1(a) From Proposition 34, 2
two distinct blocks of a graph have at most one common vertex. Hence, they must be edge-disjoint, i.e. the blocks constitute a partition of the edge-set of the graph

(b) If v is a common vertex of two blocks, then v must be a cut-vertex.
Let $v \in B_1$ and B_2 which are distinct blocks. We may assume the graph G under consideration is connected. Then v is incident with an edge $vv_1 \in B_1$ and an edge $vv_2 \in B_2$, say e_1 and e_2.

Suppose that v is not a cut vertex. Then, there is a $v_1 v_2$ path P not containing v. Then P together with v and the edges e_1, e_2 make up a cycle containing e_1 and e_2. However, this is not possible since e_1 and e_2 are in distinct blocks of G.

Alternatively, if $v \in B_1 \cap B_2$, ~~The~~ and v is not a cut-vertex, then ~~B~~ $B_1 \cup B_2 - v$ is contained ~~connect~~ in a component of $G - v$, contradicting maximality of a block.

Q2(a) The diagram below is helpful :-



The vertices of S constitute a vertices from the m-partite set and the b vertices from the n-partite set, and we are only interested in the edges from these $(a+b)$ vertices to the remaining vertices. Since there are no edges inside a partite set, we get:

$$a(n-b) + b(m-a) = an + bm - 2ab \text{ ~~vertices~~ edges}$$

i.e. $|[S,\bar{S}]| = na + mb - 2ab$ (1)

(b) In $K_{3,3}$, any edge set containing 7 edges, contains ≥ 3 edges from a single vertex, hence it is an edge-disconnecting set. Let S be any ~~edge~~ cut-set, we use (1) to show that $|[S,\bar{S}]| \neq 7$. Let S contain a vertices from the first partite set and b vertices from the second partite set, and note that $m = n = 3$, while $0 \leq a, b \leq 3$.

If either ~~a, b~~ a or $b = 0$ or either a, or $b = 3$, then RHS of (2) is divisible by 3, and hence cannot be 7.

For convenience, put $|[S,\bar{S}]| = x$.

So we have to only consider 3 possibilities (using symmetry)

Case 1: $a = 2, b = 2$.

Then $x = 3(2) + 3(2) - 2(2)(2)$
$= 6 + 6 - 8 = 4 \neq 7$

Case 2: $a = 2, b = 1$

Then $x = 3(2) + 3(1) - 2(2)(1)$
$= 6 + 3 - 4 = 5 \neq 7$

Case 3: $a = 1, b = 1$.

Then $x = 3(1) + 3(1) - 2(1)(1)$
$= 3 + 3 - 2 = 4 \neq 7$

Result follows

3. Show that if a graph G has at least 11 vertices, then G and its complement cannot both be planar graphs.

Ans:- Suppose G has n vertices and m edges, then its complement G' has $m' = \binom{n}{2} - m$ edges.

If both G and G' are planar, then $m \leq 3(n) - 6$ (1)

and $m' \leq 3(n) - 6$ (2)

for $n \geq$ [illegible]

so that:

$m + m' = \dfrac{n(n-1)}{2} \leq 6n - 12$

i.e. $n^2 - n \leq 12n - 24$ (3)

i.e. $n^2 - 13n + 24 \leq 0$ (4)

Putting $f(x) = x^2 - 13x + 24$, → pg 3

~~we see that $f'(n) = 2n - 13 \geq 0$ for $n \geq \frac{13}{2}$~~

~~and~~ ~~From (4), the roots of $f(n)$ are $\dfrac{13 \pm \sqrt{169-96}}{2}$~~

~~$= \dfrac{13}{2} \pm \dfrac{\sqrt{73}}{2}$~~

~~$f(10) = 100 - 130 + 24 \leq 0$~~

~~Now, $f(11) = 121 - 143 + 24 \geq 0$~~

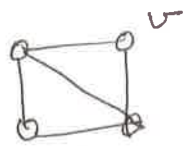
~~Thus, if $n \geq 11$, (4) doesn't hold~~

Q4: Given: G is k-chromatic with $\chi(G-v) = k-1$ for all $v \in V(G)$. Show that $k-1 \le \delta(G)$. Show the converse doesn't hold.

Answer:- Suppose BWOC that $\delta(G) < k-1$. Let $v \in V(G)$ with $d(v) = \delta = \delta(G)$.
The graph $G-v$ is $(k-1)$-colorable by assumption, giving a vertex partition $\{V_1, V_2, \ldots, V_{k-1}\}$ where each V_j is independent. of $G-v$. Since $d(v) < k-1$, there is some set V_i s.t. v is not adjacent to any vertex in V_i. If we assign color i to v, we get a proper $(k-1)$-coloring of G, contradicting that

$$\chi(G) = k.$$

For the counter-example, let $G = K_4 - e$, where e is an edge.



G

Then $\chi(G) = 3 = k$
Then $\delta(G) = 2 \ge k-1$.

However, $\chi(G-v) = 3 \ne k-1$

Q3. cont'd: ~~we need to~~ ~~We need~~ ~~have $n^2 \le 13n - 24$~~

i.e. Now, $f'(x) = 2x - 13 = 0$ when $x = \frac{13}{2}$ ~~...~~ and $f''(x) = 2 > 0$. $\therefore$ $f(x)$ has a min. at $x = \frac{13}{2}$

Now, $f(11) = 121 - 13(11) + 24 = 121 - 143 + 24 > 0$. $\therefore f(n) > 0$ for $n \ge 11$

Q5. Given $V(G) = \{v_1, \ldots, v_n\}$ and $G_i = G - v_i$. Show that G is ~~connected~~ connected if and only if at least two of the G_i are connected.

Ans: $[\Rightarrow]$ Suppose G is connected. Let T be a spanning tree of G.
Then T has at least two vertices, ~~say~~ v_1 and v_2, of degree 1 (leaf-vertices).
Since $T - v_i$, $i = 1, 2$ is connected, so are G_1, G_2 ($\because$ they ~~[illegible]~~ have spanning trees).

$[\Leftarrow]$ Conversely, suppose G_i and G_j are connected subgraphs. Any vertex ~~[illegible]~~ other than v_i, v_j belongs to both G_i and G_k. Hence, there is a path b/w v_k and v_j in the connected graph G_i and a path b/w v_k and v_i in the connected graph G_j, i.e. there is a path from v_i to v_j in G.

For any pair of vertices, v_k, v_m not v_i, v_j, there is clearly a path in G b/w v_k and v_m.