Proposition 54: $\chi'(G) \leq 2\Delta(G) - 1$

Proof:- The idea is basically a greedy coloring procedure for edges. Order the edges, and assign to each edge the lowest indexed color not already appearing on edges incident to it.

Now, each edge is incident with at most $(\Delta - 1)$ edges on one end-pt. and at most $(\Delta - 1)$ edges on the other end-point, hence it is incident with $2(\Delta - 1)$ edges in all, and hence can be assigned the $(2\Delta - 1)$-st color.

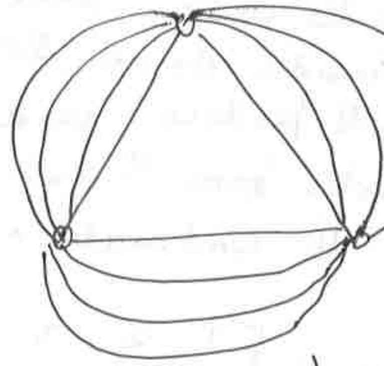
Proposition 55: If G is bipartite, then $\chi'(G) = \Delta(G)$.

Proof: By ~~Proposition~~ a mid-sem exam problem, every k-regular bipartite graph H has a perfect matching. By induction on $\Delta(H)$ this yields a proper $\Delta(H)$-edge coloring. It $\therefore$ suffices to show that for every bipartite graph G there is a k-regular bipartite graph H containing it, where $k = \Delta(G)$.

To do this – first add vertices to the smaller partite set of H, if necessary, to equalize the sizes. If the resulting G' is not regular, then each partite set has a vertex with degree $< k = \Delta(G)$. Add an edge with these two vertices as endpoints. Continue adding ~~edges~~ such edges until the graph becomes regular.

(Prop. 55 – cont'd). This is the required H

The "Fat Triangle" Example :-



The edges are ~~pairwise~~ pairwise intersecting, and hence require distinct.

Thus,

$\chi'(G) = 12$

$\frac{3}{2}\Delta(G) = \frac{3.8}{2} = 12$

$\Delta(G) + \mu(G) = 8 + 4 = 12$

(PTO)

Proof of Vizing's Theorem is involved and will be omitted.
However, we will do some examples.

Example 1: Let $G = K_{2n}$. Then $\chi'(G) = 2n-1 = \Delta(G)$.

Answer: Since the no. of vertices is even, we let $V = \{1, 2, \dots, 2n\}$ be the vertex set, while maximum degree is $2n-1$. We consider the set $S = \{c_i : i = 1, 2, \dots, 2n-1\}$ of colors. We will produce an actual coloring of the edges using each color from S exactly n times by the foll. procedure :- We construct a $(2n-1) \times 2n$ matrix as follows :-

$$\begin{bmatrix} 1 & 2 & 3 & 4 & \cdots & 2n-2 & 2n-1 & 2n \\ 2 & 3 & 4 & 5 & \cdots & 2n-1 & 1 & 2n \\ 3 & 4 & 5 & 6 & \cdots & 1 & 2 & 2n \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 2n-1 & 1 & 2 & 3 & \cdots & 2n-3 & 2n-2 & 2n \end{bmatrix}$$

Take the elements in Row 1 of the matrix and arrange them as pairs $(1, 2n), (2, 2n-1), (3, 2n-2), \dots$ and so on. There will be n such pairs with each such pair corresponding to a unique edge of G. Assign color c_1 to these edges.

Take elements in row 2 as pairs $(2, 2n), (3, 1), (4, 2n-1)$ and so on, and assign color c_2.

Proceed in this way ~~and~~ until elements in all rows are paired and the corresponding edges are colored.

Example 2: Let $G = K_{2n-1}$, i.e. complete graph with $(2n-1)$ vertices.

Then $\chi'(G) = 2n-1 = \cancel{\chi}\ \Delta(G) + 1$.

Proof: Color the edges of K_{2n} with $2n-1$ colors as above. Delete one vertex. Thus, we have an edge-coloring of G with $(2n-1)$-colors. Suppose it is possible to color the edges of this graph with $(2n-2)$-colors. Now, there are $(n-1)(2n-1)$ edges in all in G; dividing by $(2n-2)$ we get:

$$(n-1)(2n-1) = \cancel{n-1} \ (n-1)(2n-2) + \cancel{n+1} \ (n-1)$$

i.e. some color from the the set of $(2n-2)$ colors has to be used on more than ~~$(n-1)$ vertices, i.e. on at least n verti~~ $(n-1)$ edges, i.e. on at least n edges, which would require $2n$ vertices – but there are only $(2n-1)$ vertices available. $\therefore$ $(2n-1)$ colors are required.