1/30/2017

1

MTH 303 _ 2017 WINTER SEMESTER

NOTES FOR FRIDAY 2017 01 27

Characterization of Trees

Proposition 20. For an ***n*-vertex** simple graph ***G*** (with $n \geq 1$), the following are equivalent (and characterize the trees with *n* vertices)

A) ***G*** is connected and has no cycles, i.e. a tree by definition

B) ***G*** is connected and has ***n*-1** edges

C) ***G*** has ***n*-1** edges and no cycles

D) For ***u, v***∈***V***(***G***), ***G*** has exactly one ***u, v***-path

Graph Theory

1

START HERE

Proof of ~~all~~ the result above.
An alternative proof is given in the slides. The proof below is what was given in the lecture.
We will proceed: $A \Rightarrow B \Rightarrow D \Rightarrow$ ~~B~~ A, and consider C later.

$A \Rightarrow B$:- Given that G is connected with no cycles,
RTP: G is connected with $(n-1)$ edges.

By induction on n.
Basis: Certainly true for $n=1,2$.
Inductive step: Suppose the result holds for graphs with $k < n$ vertices ($n \geq 3$) and suppose G has n vertices.
Since G has no cycles, G has a vertex of degree 1. Suppose ~~G has~~ G has m edges.
↳ by Prop. 6

Let u be a vertex of degree 1, i.e. a leaf.
Then, by Lemma 20.2, $G' = G - u$ is again a tree, i.e. it satisfies the inductive hypothesis, having $(n-1)$-vertices.
$\therefore \; e(G') = (n-1)-1$
$= m-1$

$\Rightarrow m = n-1$, as reqd.

——— X ——— X ———

$B \Rightarrow D$: Given that G is connected and has $(n-1)$-edges.
RTP: For $u, v \in V(G)$, G has exactly one u-v-path.

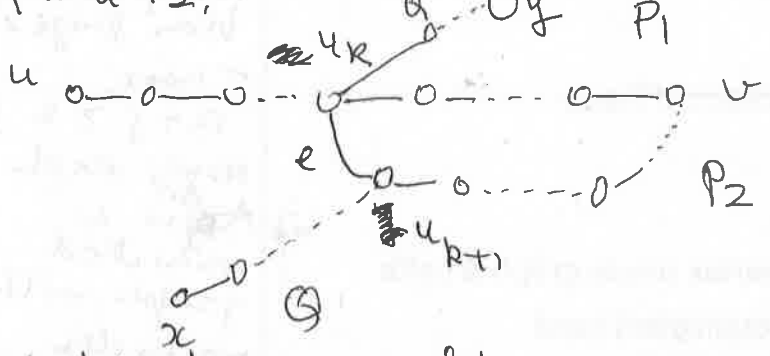
Since G ~~has~~ is connected, G has at least one u,v-path.
To prove uniqueness,

Cont'd from RH column.
$\therefore e(G) = (n_1 - 1) + \cdots + (n_k - 1)$
$= (n_1 + n_2 + \cdots + n_k) - k$
$= n - j - k$
$= n-1$, by hypothesis.
$\therefore j + k = 1$ ②
$\Rightarrow\Leftarrow$

Proof continued from page 2:
Thus, $k + j \geq 2$ ①
Now, each ~~Ak~~ A_i is a connected graph with no cycles.
$\therefore$ ~~Ak~~ A_i satisfies A, and so by B, ~~Ak~~ A_i has ~~nk~~ $(n_i - 1)$ edges.

B ⇒ D (cont'd)

suppose BWOC that there are vertices $u, v \in V(G)$ and two distinct u,v-paths, P_1 and P_2.



Starting from u, let $e = u_k u_{k+1}$ be the first edge on P_2 which is not on P_1.

We claim $G-e$ is connected. To prove the claim, let x,y be any two vertices in $G-e$. Since G is connected, there is an x,y-path in G, say Q. If $e \notin Q$, then Q is an x,y-path in $G-e$. If e lies in Q, then we get an x,y-walk in $G-e$ as follows: Follow Q up to u_{k+1}, then proceed along P_2 to v, return from v to u_k along P_1, and then proceed from u_k to y along Q. By Prop. 2, this walk contains an x,y-path in $G-e$.

This completes the proof of the claim.

Now, $G-e$ has $(n-2)$-edges. ∴ By Prop. 3, it has at least $n-(n-2) = 2$ components, i.e. it is not connected ⇒⇐

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D ⇒ A. Given that for $u,v \in V(G)$, there is a unique u,v-path in G.

RTP: G is connected and has no cycles.

Clearly, G is connected. Suppose BWOC that it has a cycle C_k. Then, for $u,v \in C_k$, there are 2 u,v-paths in C_k and hence in G ⇒⇐

——x——x——

To deal with C, we first show: B ⇒ C.

Given: G is connected and has $n-1$ edges.

RTP: G has $n-1$ edges and no cycles.

Suppose BWOC that G has a cycle C. Then, it has an edge e which is not a cut edge (Prop. 4). ∴ $G-e$ is connected with n vertices and $(n-2)$ edges, contradicting Prop. 3.

Finally, we show C ⇒ A.

Given, G has $n-1$ edges and no cycles, we need to show it is connected. If $n = 1, 2, 3$, the result is obvious, so WMA $n \geq 4$. So let G have k non-trivial components A_1 to A_k with n_k vertices each, and j trivial components.