SOLUTIONS

(not nec. in same order)

1

MTH303 – MID-SEMESTER EXAMINATION 23/02/2015

TIME: 1 HOUR **MAXIMUM MARKS: 50**

NB: You may use any known result (i.e. propositions and lemmas) without proof; however, it should be identified clearly. Marks will depend on the correctness and completeness of your proofs. All questions have equal marks.

1. Show that the sequence $d_1 \geq d_2 \geq \ldots\ldots \geq d_n > 0$ of positive integers is the degree sequence of a tree with n vertices if and only if $d_1 + d_2 + \ldots\ldots + d_n = 2(n-1)$.

2. The k-cube (or hypercube) is the graph Q_k, whose vertices are ordered k-tuples (for $k \geq 1$) with entries from $\{0,1\}$, in which two vertices are adjacent if and only if they differ in exactly one position. Show that the k-cube is a k-regular bipartite graph, and determine the number of its vertices and edges.

3. Prove or disprove:

a) Any two 3-regular (simple) graphs with 6 vertices must be isomorphic.

b) Every k-regular (simple) bipartite graph ($k \geq 1$) has a perfect matching.

4. Show that a digraph G (V, E) is strongly connected if and only if the following property is satisfied: for every non-empty $X \subset V$, there exists a directed edge from a vertex $x \in X$ to a vertex $y \in V - X$.

5. Find a maximum weight matching and a minimum weight vertex cover for the complete bipartite graph whose weight matrix is given below, using the Hungarian Algorithm, NB: marks will not be awarded unless the steps of the Algorithm are shown with brief explanations.

$$\begin{bmatrix} 6 & 3 & 2 & 1 & 8 \\ 4 & 6 & 5 & 4 & 8 \\ 5 & 7 & 6 & 5 & 8 \\ 9 & 9 & 9 & 5 & 7 \\ 8 & 7 & 9 & 10 & 7 \end{bmatrix}$$

Q2. Let $G = Q_k$ for some fixed but arbitrary $k \geq 1$.

Then $|V(G)| = 2^k$ (1)

If $\bar{x} \in V(G)$ and $\bar{y} \in V(G)$ is adjacent, then $\bar{y}$ differs from $\bar{x}$ in exactly one position. Since $\bar{x}$ has precisely k positions, $d(\bar{x}) = k$, i.e. G is k-regular.

If $\bar{x} = (x_1, x_2, \ldots, x_k)$, then the parity of $\bar{x}$ = parity of $\sum_{i=1}^{k} x_i$ = no. of 1's in $\bar{x}$, either even or odd.

Let X = vertices with ~~odd~~ even parity, Y = vertices with odd parity.

If $\bar{y}$ is adjacent to $\bar{x}$, then the number of 1's in $\bar{y}$ differs from the no. of 1's in $\bar{x}$ by exactly one (either 0 becomes 1 or 1 becomes 0), i.e. the parity changes. Hence, every edge in G has one end-point in X and one in Y, i.e. G is bipartite with partite sets X and Y.

Alternatively, since the parity changes ~~[illegible]~~ with every edge on a path, every cycle in G is even, hence G is bipartite.

By the way, $|X| = |Y|$.

Finally, $|E(G)| = \frac{1}{2}\left[\sum_{v \in V(G)} d(v)\right] = \frac{1}{2} \cdot k \cdot 2^k = k \cdot 2^{k-1}$ (2)

Q 1. ~~[illegible]~~ Yet another instance of TONCAS.

[$\Rightarrow$] Necessity.

Suppose $\exists$ a tree T with n vertices with the given degree sequence: $d_1 \geq d_2 \geq \cdots \geq \cdots \geq d_n > 0$.

Then, $|E(T)| = 2(n-1) = \sum_{i=1}^{n} d_i$ (Prop. 1 and Prop.)

[$\Leftarrow$] Given a sequence of n +ve integers s.t. $d_1 + \cdots + d_n = 2(n-1)$. We will proceed by induction on n. But first we observe:-

(a) $d_i = 1$ for at least one i. For, if not, then $\sum_{i=1}^{n} d_i \geq 2n > 2(n-1)$.

(b) $d_i \geq 2$ for at least one i, provided $n > 2$.

For, if not, $\sum_{i=1}^{n} d_i = n = 2(n-1)$ $\Rightarrow n = 2$.

Base Case: $n = 2$. Then $d_1 + d_2 = 2(2-1) = 2$ $\Rightarrow d_1 = d_2 = 1$, which is the degree sequence of the tree o—o.

So suppose the result holds for all ~~degree~~ sequences of +ve integers with n terms, and suppose $D: d_1 \geq d_2 \geq \cdots \geq d_{n+1} = 2(n+1-1) = 2n$.

Using observations (a) and (b), wma $d_1 \geq 2$ and $d_{n+1} = 1$.

Construct a new sequence $D': d_1', \ldots, d_n'$ s.t.

$d_1' = d_1 - 1 \geq 1$

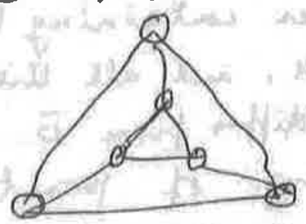
$d_i' = d_i$ for $i = 2, \ldots, n$.

Then, D' is a sequence of +ve terms satisfying $d_1' + \cdots + d_n' = (d_1 - 1) + d_2 + \cdots + d_n = d_1 + d_2 + \cdots + d_n - 1 = 2n - d_{n+1} - 1 = 2n - 2 = 2(n-1)$.

$\exists$ a tree T' which realizes ~~D~~ D' (with re-arrangement of terms if necessary). Let $u' \in V(T')$ with degree d_1'. Construct a new tree T with $n+1$ vertices by adding a new vertex v' whose only edge is to u'. Then T has D as its degree sequence.

Q3. (a) DISPROVE

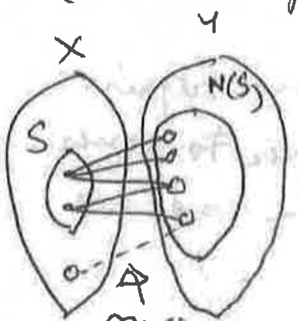
$K_{3,3}$ is a 3-regular graph on 6 vertices as is G:



However, G is not isomorphic to $K_{3,3}$, since G has a Δ and $\therefore$ cannot be bipartite.

3(b) PROVE

We apply Hall's Theorem.
Let G be a k-regular bipartite graph with bipartite sets X and Y, and let $S \subseteq X$, $|S| = m$, say.
Hence, there are mk edges with end-points in $N(S) \subseteq Y$.
But now, each of these end-points has degree precisely k, and there may be some ~~edges with~~ other edges having end-points in $N(S)$, besides those from S (see diagram)



such an edge may certainly exist.

$\therefore |N(S)|k \geq mk$

or $|N(S)| \geq m = |S|$

Since Hall's condition is satisfied, $\exists$ a perfect matching.

Q4. [$\Longrightarrow$] Suppose G is strongly connected, let $X \subset V$ be non-empty. Then, $\exists$ a vertex $u \in X$ and a vertex $v \in Y = V - X$.
Since G is strongly connected, $\exists$ a directed path P: $u = x_1 x_2 \ldots x_k = v$.
Let x_i be the ~~first~~ final vertex on P s.t. $x_i \notin X$
(could be x_k in the worst case). Putting
$x = x_{i-1}$ and $y = x_i$, we are done.

[$\Longleftarrow$] Suppose the given condition holds, and let $u, v \in V(G)$.
If $\exists$ a directed path from u to v, we are done.
So suppose there is no such path.
Let $X = \{x \in V : x$ is the end-vertex in a directed path P in G commencing from $u\}$.
Then, $X \neq \emptyset$ since $u \in X$ and $X \neq V$, since $v \notin X$.
But now, $\exists\, x \in X$ and $y \in Y = V - X$ s.t. there exists a directed edge xy.
But then, there is a directed path commencing at u and ending at $y \Rightarrow y \in X \Rightarrow \Leftarrow$. Result follows.

Q5. Starting with the given matrix of weights, construct the excess matrix and proceed:

$$\begin{array}{c} \\ 8 \\ 8 \\ 8 \\ 9 \\ 10 \end{array}\begin{array}{c} \begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \end{array} \\ \begin{bmatrix} 6 & 3 & 2 & 1 & 8 \\ 4 & 6 & 5 & 4 & 8 \\ 4 & 7 & 6 & 5 & 8 \\ 9 & 9 & 9 & 5 & 7 \\ 8 & 7 & 9 & 10 & 7 \end{bmatrix} \end{array} \xrightarrow[\text{Excess matrix}]{\text{Iteration 1}} \begin{array}{c} \\ 8 \\ 8 \\ 8 \\ 9 \\ 10 \end{array}\begin{array}{c} \begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \end{array} \\ \begin{bmatrix} 2 & 5 & 6 & 7 & 0 \\ 4 & 2 & 3 & 4 & 0 \\ 3 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 4 & 2 \\ 2 & 3 & 1 & 0 & 3 \end{bmatrix} \end{array}$$

0's are covered by R_4, R_5, C_5

$\epsilon = 1$

Iteration 2

$$\begin{array}{c} \\ 7 \\ 7 \\ 7 \\ 9 \\ 10 \end{array}\begin{array}{c} \begin{array}{ccccc} 0 & 0 & 0 & 0 & 1 \end{array} \\ \begin{bmatrix} 1 & 4 & 5 & 6 & 0 \\ 3 & 1 & 2 & 3 & 0 \\ 2 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 4 & 3 \\ 2 & 3 & 1 & 0 & 4 \end{bmatrix} \end{array}$$

0's are covered by R_3, R_4, R_5, C_5

$\epsilon = 1$

Iteration 3

$$\begin{array}{c} \\ 6 \\ 6 \\ 7 \\ 9 \\ 10 \end{array}\begin{array}{c} \begin{array}{ccccc} & & & & 2 \end{array} \\ \begin{bmatrix} 0 & 3 & 4 & 5 & 0 \\ 2 & 0 & 1 & 2 & 0 \\ [illegible] & 0 & [illegible] & [illegible] & 1 \\ 0 & 0 & 0 & 4 & 4 \\ 2 & 3 & 1 & 0 & 5 \end{bmatrix} \end{array}$$

We now require 5 lines to cover the 0's so we have a perfect matching.

A maximum weight matching is:-

$$\left.\begin{array}{l} x_1 \leftrightarrow y_1 \\ x_2 \leftrightarrow y_5 \\ x_3 \leftrightarrow y_2 \\ x_4 \leftrightarrow y_3 \\ x_5 \leftrightarrow y_4 \end{array}\right\}$$

with weight: $6 + 8 + 7 + 9 + 10 = 40$ ①

The minimum weighted cover is the one shown in iteration, with ~~weight~~ minimum cover $= 6 + 6 + 7 + 9 + 10 + 2 = 40$ ②

① and ② are equal as expected.