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A famous instance of TONCAS!

Theorem 2 (Philip Hall's Theorem): An X,Y-bigraph G has a matching that saturates X if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$.

- *Remark: This is a famous Theorem, proved by Philip Hall in 1935, frequently known as Hall's Marriage Theorem. It has an equivalent formulation, in terms of finite sets as follows:*
- Let S be a finite family of finite sets, in which the sets may be repeated several times. A transversal, or system of distinct representatives (SDR), for S is an injection $f: S \to \cup\{A: A \in s\}$ such that $f(A) \in A$. Then the above theorem can be stated as follows:
- Theorem 2 (Philip Hall's Theorem): A family S has a transversal if and only if for each subfamily $W \subseteq S$,

$$|W| \leq |\cup\{A: A \in W\}|.$$

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Proof of Hall's Theorem:

Necessity is obvious: If for some k sets $|A_{i_1} \cup A_{i_2} \cup \dots \cup A_{i_k}| < k$, then we cannot find k distinct elements $a_1, a_2, \dots, a_k$ s.t. $a_j \in A_{i_j}$.

Sufficiency: We will proceed by induction on the number of sets in the family $S = \langle A_1, \dots, A_n \rangle$.

Base Case: obviously holds for $n=1$. If $|A_1| \geq 1$, then we have an SDR.

Inductive Step: Suppose the result holds for all families with $< n$ sets $(n > 1)$, and consider $S = \langle A_1, \dots, A_n \rangle$ for which Hall's condition holds.

Case 1: Suppose that for all k, $1 \leq k < n$, the stronger condition $|A_{i_1} \cup \dots \cup A_{i_k}| \geq k+1$ for any choice of the sets A_{i_j}.

Now, $|A_1| \geq 1$, so select $x_1 \in A_1$

Put $B_i = A_i - \{x_1\}$ for $i = 2, \dots, n$, so that $S' = \langle B_2, \dots, B_n \rangle$ is a family of $(n-1)$ sets.

Case 1 (continued).

For any k, $1 \le k \le n-1$,

$|B_{i_1} \cup \cdots \cup B_{i_k}| =$

$| A_{i_1} - \{x_1\} \cup A_{i_2} - \{x_1\} \cup \cdots \cup A_{i_k} - \{x_1\} |$

$= |A_{i_1} \cup \cdots \cup A_{i_k}| - |\{x_1\}|$

$\ge (k+1) - 1 = k$

Hence, the family S' satisfies IH, and so S' has an SDR $= \{x_2, \ldots, x_n\}$.

Since $B_i \subseteq A_i$ for $i = 2, \ldots, n$ and $x_1 \notin B_i$ for any i, $\{x_1, x_2, \ldots, x_n\}$ is an SDR for S.

Case 2: For some k, $1 \le k < n$, there are sets $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$ in S s.t.

$$|A_{i_1} \cup \cdots \cup A_{i_k}| = k \qquad (1)$$

For simplicity of indexing, we shall assume these are the sets $A_1, \ldots, A_k$.

Consider the family $S_1 = \langle A_1, \ldots, A_k \rangle$. Since $k < n$, and S_1 satisfies Hall's condition, S_1 has an SDR, $\{x_1, \ldots, x_k\}$

Now, put $B_j = A_j - \bigcup_{i=1}^{k} A_i$ for $j = k+1, \ldots, n$, and consider the family $S_2 = \langle B_{k+1}, \ldots, B_n \rangle$.

Now for any r indices, $1 \le r \le n-k$, $k+1 \le i_1 < \cdots < i_r \le n$, we have:

$|B_{i_1} \cup \cdots \cup B_{i_r}| =$

$|B_{i_1} \cup \cdots \cup B_{i_r}| + |A_1 \uplus \cdots \uplus A_k| - k$

$= |B_{i_1} \cup \cdots \cup B_{i_r} \cup A_1 \cup \cdots \cup A_k| - k$, since $A_1 \cup \cdots \cup A_k$ is disjoint from all the B's

$= |A_{i_1} \cup \cdots \cup A_{i_r} \cup A_1 \cup \cdots \cup A_k| - k$

$\ge (r+k) - k$, since the A's satisfy Hall's condition

$= r$

Hence, the family S_2 also satisfies Hall's condition, and so by IH has an SDR, say $\{x_{k+1}, \ldots, x_n\}$.

Since $x_j \in B_j \subseteq A_j$ for each $j = k+1, \ldots, n$, and the B_j's are disjoint from $A_1, \ldots, A_k$, we get $\{x_1, \ldots, x_k, x_{k+1}, \ldots, x_n\}$ as an SDR for S, as desired.