Notes for Tuesday 20150406

Proposition 52:— If G is a planar graph, then $\chi(G) \le 6$.

Lemma (this is actually a problem in the textbook):
If G is planar, then $\delta(G) \le 5$.

Proof: We use Proposition 50.

Now, if $\bar{d}$ = average degree in G, then $\delta \le \bar{d}$.

Hence, $\delta \le \bar{d} = \dfrac{\sum_{v \in V(G)} d(v)}{|V(G)|} = \dfrac{2|E(G)|}{|V(G)|} \le \dfrac{2(3|V(G)|-6)}{|V(G)|}$

$= 6 - \dfrac{12}{|V(G)|} < 6$, as reqd.

Proof of Main Proposition 52:—

By induction on $n = |V(G)|$

Base Case: The result is obviously true for $n \le 6$.

IH: Suppose result holds for n, and suppose $|V(G)| = n+1$.

Now, $v \in V(G)$ with $d(v) \le 5$.

Consider $G' = G - v$. Then G' is planar, with n vertices.

Hence, by IH, it is 6-colorable.

Since v has at most 5 neighbours, we can always give it the 6th color to get a proper 6-coloring of G.

Proof of the 5-color Theorem (Thm. 59):—

We will follow essentially the same logic as above; however, the difficulty comes if v has 5 neighbours which all have different colors! Anyway, we proceed by induction on n.

Base Case: Obviously, ~~t~~ result holds if $n \le 5$.

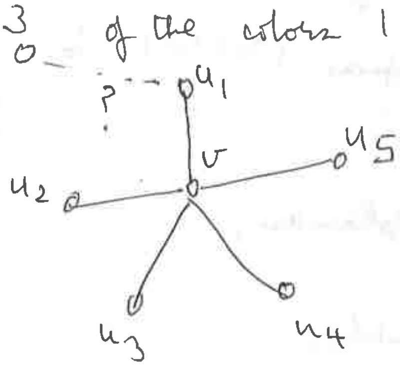
IH: Suppose result holds ~~for a~~ for all graphs with n vertices, and suppose G has $n+1$ vertices.

(PTO)

Let $v \in G$ with $d(v) \le 5$. Let $G' = G - v$, so by induction G' is 5-colorable, so we properly color G' with 5 colors. Now, if $N(v)$ ~~has~~ uses ≤ 4 different colors, we are done.

So we are reduced to the case where $d(v) = 5$ and the 5 neighbours of v have 5 different colors. Since we can't extend this coloring to G, we have to do some recoloring.

Let us have a crossing-free embedding of G — every vertex of G (except v) is colored by one of the colors 1 to 5.



Let u_1 to u_5 be the 5 neighbours of v in anti-clockwise order, and wolog, let them have colors 1 to 5 respectively.

We want to change the color of one of v's neighbours, say of u_1 from 1 to 3 & now we can color v ~~to~~ with u_1 and we are done. However, u_1 might have a neighbour with color 3, that would lead to a problem.

Let $H_{1,3}$ be the subgraph of G induced by vertices of color 1 and 3. Notice, that if in one component of $H_{1,3}$ we exchange colors 1 and 3, the coloring of G' would remain proper (since v is not yet colored).

We therefore exchange the colors 1 and 3 in the component which contains u_1. So it would be possible to color v with color 1. The problem however is that u_3 could have been in the same component of $H_{1,3}$ as u_1, so now it has color 3. If they are in separate components, we are done (since then u_3 retains color 3, and color 1 is available for v).

So we are left with the case in which u_1 and u_3 are in the same component of $H_{1,3}$.

So now we try to recolor vertex 2 with color 4. Proceeding in the same way, we are forced to the situation that u_2 and u_4 are in the same component of $H_{2,4}$.

We claim this cannot happen. Now, since u_1 and u_3 are in the same component of $H_{1,3}$, there exists a path P from u_1 to u_3 in $H_{1,3}$. Similarly, there is a path Q in $H_{2,4}$ from u_2 to u_4. Clearly, P and Q have no vertices in common. Furthermore, path P along with v forms a cycle, which becomes a simple closed curve in the plane. However vertices u_2 and u_4 are on different sides of this closed curve, i.e. the path Q must cross this curve $\Rightarrow\Leftarrow$ $\therefore$ u_2, u_4 are in different components, and we are done.