MTH303: Homework 3

Assigned on Friday 20170203

Ques-1. Let v be a vertex in a connected graph G. Prove that there exists a spanning tree T of G such that the distance of every vertex from v is the same in G and in T.

Ans-1 For each vertex, say u, run the Dijkstra's Algorithm to get the Minimum Distance Spanning tree. This spanning tree will have the same d(u,v) for all vertices v, in G and T.

Ques-2. Given a graph G with distinct edge costs, how many minimum cost spanning trees exist in G?

Ans-2 let T and T' be the two spanning tree we get. Order e in the increasing order of their weights. 'e' be the first edge in T and not in T'. T' will contain an another edge e' which is not in T to compensate for 'e'. But e'>e and hence T' will not be min spanning tree

Ques-3 Let T1 and T2 are two trees on the same vertex set V such that $d_{T1}(v) = d_{T2}(v)$ for all v $\in$ V. Show that T2 can be obtained from T1 by series of 2-switches with each intermediate graph being a tree.

Ans 3. See below.

Ques-4 For n ≥ 4, prove that the minimum number of edges in an n-vertex graph with diameter 2 and maximum degree (n-2) is (2n-4).

Ans-4 Consider a vertex v in graph G with degree n-2. This implies that there exists a vertex u such that u is not a neighbor of v otherwise v will have degree n-1. But it is given that diameter of G is 2. This implies that u is connected to all the neighbors of v otherwise the diameter will be 3 if it is connected to some neighbors of v. Thus, minimum number of edges = edges between v and N(v) + edges between u and N(u) = n-2+n-2 = 2n-4

Hence, proved.

Ques-5 Every tree of even order has exactly one spanning subgraph where every vertex has odd degree.

Ans-5 A spanning *subgraph* is a subgraph of a graph consisting of the same vertex set and a subset of the edge set of the graph, which is not necessarily a tree. A spanning *tree* is a spanning subgraph that is also a tree.

Anyway, for the question, you can use induction on the number of vertices = 2k. For k = 1, we have 2 vertices, and so there is only one possible tree, which is just two vertices connected by 1 edge -- the graph itself is a spanning subgraph with odd vertices.

Suppose for k >= 1, every tree with 2k vertices has a unique spanning subgraph with all odd vertices.

Now since each leaf has degree 1 in the tree, any such spanning subgraph must include all edges incident to leaves.

Consider a tree with 2(k+1) vertices. Consider a longest possible path in this tree. Suppose one endpoint of the path is a vertex v, which would have to be a leaf. Suppose u is the vertex adjacent to v in the path.

Case 1: u has another adjacent leaf, call it w.
T - {v, w} has a unique such spanning subgraph by our assumption (we assumed this for any tree with 2k vertices), so use that spanning subgraph and add edges uv and uw -- the degree of u gains 2, so it stays odd, and v and w only have the 1 incident edge, so their degree will be 1, which is odd, and we have the type of subgraph we need.

Case 2: u has no other adjacent leaf.
Since p is the longest path, this means the degree of u is 2, or else there would be a cycle in the graph. So, we know that T - {u, v} has a unique such spanning subgraph by assumption -- add uv to that subgraph and we have the type of subgraph we need (u + uv + v will be a separate component).

So, by induction, this holds for any number of vertices 2n for a positive integer n.

Ques-6 Every tree T has a vertex v, such that for all e in E(T), the component of T-e that has v has at least ceil(n(T)/2) vertices.

Ans-6 We can prove the above problem by induction on number of vertices k of the tree.

Basis Step: For k=2, it is true as removing the only edge of the tree will divide the tree into two components containing 1 vertex=ceil(2/2) each.
Let us assume it is true for a tree with k vertices. Hence there must exist a vertex v for which the above condition holds true.

Inductive Step: Let us add a new vertex u to the tree. Let this new tree be R. Let us add u to the subtree T' of v containing maximum vertices because for all other sub-trees, v will still hold the property. Removing any edge connecting leaf nodes of any subtree would make n-1 vertices in the component containing v. Hence, assuming extreme case, we consider removing edges connecting v to T'. Since we are assuming the extreme case, number of vertices in subtree T-T' be ceil(k/2) as this is the only case when property will be violated for v. In this case, the neighbor of v in the subtree T', say v', is the vertex which holds the above property as removing the edge from v to v' will make two components containing ceil(k/2)+1 and floor(k/2) vertices. For all other vertices, the property will hold true for v'.

Ques-3 Let T1 and T2 are two trees on the same vertex set V such that $d_{T1}(v) = d_{T2}(v)$ for all $v \in V$. Show that T2 can be obtained from T1 by series of 2-switches with each intermediate graph being a tree.

Answer: We will proceed by induction on $|V| = n$.

Base Case: $n = 2$. There is only one tree in this case, so the result holds trivially.

Inductive Step: Suppose the result holds for all trees on vertex sets of ~~or~~ order $< n$ $(n \geq 3)$, and suppose $|V| = n$.

Since T_1, T_2 are trees, T_1 has a leaf, say u, i.e. ~~D_T~~ $d_{T_1}(u) = 1$.

Hence, $d_{T_2}(u) = 1$ also, i.e. u is a leaf in T_2 also.

Let ~~e = e~~ uw_1 and uw_2 be the edges incident with u in T_1 and T_2 respectively.

Case 1: $w_1 = w_2$.

In this case, the trees $T_1' = T_1 - u$ and $T_2' = T_2 - u$ ~~are~~ have the same vertex set, and $d_{T_1'}(v) = d_{T_2'}(v)$ for all the remaining vertices. Since the order ~~of~~ of the vertex set is now $(n-1)$, by the IH, T_1' can be transformed into T_2' by a series of

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2-switches. Re-attaching the vertex u, we see that T_1 can be transformed into T_2 by a series of 2-switches.

Case 2 :- $w_1 \neq w_2$.

WLOG, $d(w_1) \geq d(w_2)$. Since u is a leaf in both T_1 and T_2, we see that uw_1 is not an edge in T_2 and uw_2 is not an edge in T_2.

Now, $d_{T_2}(w_1) \geq d_{T_2}(w_2)$, and anyway w_2 is adjacent to u in T_2, while w_1 is not adjacent to u in T_2. Hence, $\exists$ a vertex w_3 s.t. ~~w_1w_3~~ w_1w_3 is an edge in T_2 but w_2w_3 is not an edge in T_2.

Hence, we have a 2-switch uw_2 and w_1w_3 to uw_1 and w_2w_3, transforming T_2 to a tree T_3, without changing the degrees of any vertex, i.e. $d_{T_3}(v) = d_{T_2}(v) = d_{T_1}(v)$ for all $v \in V$.

Now, looking at T_3 and T_1, we see that they fall ~~int~~ under case 1. Hence, T_1 can be transformed into T_3 by a series of 2-switches.

By the transitivity and symmetry of transforming by 2-switches, we get that T_1 can be transformed into T_2 by a series of 2-switches.