SUBMISSION – 20 minutes – open notes

1. Prove that if G is a connected graph with at least two vertices, then G has a vertex which is not a cut-vertex.
2. Prove or disprove: If G is a connected graph with at least one cut-vertex, and $u, v \in V(G)$ with $d(u,v) = \text{diam}(G)$, then no block of G contains both u and v.

Each question worth 5 marks.

Monday, March 16, 2015

Solution Set

1. Let $u \in V(G)$ and $v \in V(G)$ such that ~~the there $d(u,v)$ is maximum amongst~~ v has maximum distance from u, i.e.

$$d(u,v) = \max\{d(u,x) : x \in V(G)\}$$

We claim that v is not a cut-vertex of G. Suppose BWOC that $G' = G - v$ has more than one component, let K be a component of G' such that $u \notin K$, and suppose $w \in K$.

Then, every path from u to w passes through v, i.e. $d(u,w) > d(u,v)$ $\Rightarrow\Leftarrow$.

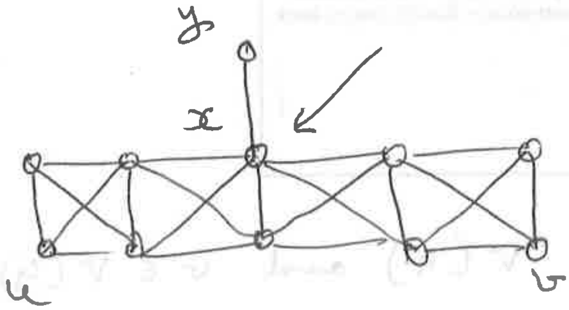
Claim follows.

NB: it is an easy corollary that actually G contains <u>at least two</u> vertices that are not cut-vertices. Let $u, v \in V(G)$ such that $d(u,v) = \text{diam}(G)$. Then, by the proof above, both v and u are not cut-vertices.

(PTO)

2. DISPROVE

As we can infer from the above example, we need a block in which vertices are relatively far from each other, but with one vertex to which all the others in the block are quite close. The ~~example~~ graph below works as a counter-example to the statement:-



Here $d(u,v) = \text{diam}(G) = 4$

x is a cut-vertex

~~G=~~ $K = G - y$ is a block

[the edge xy forms another block]

There could be other counter-examples with the same theme.