For ~~Thu~~ Tuesday 20150203

Proof of Hall's Theorem (Thm. 2):–

Necessity is obvious; if for some k ~~subsets~~ sets
$|A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k}| < k$, then we cannot
find k distinct elements $a_1, \ldots, a_k$ s.t.
$a_j \in A_{i_j}$, $j = 1, 2, \ldots, k$.

Sufficiency: By induction on the no. of sets in the family $S = (A_1, A_2, \ldots, A_n)$

Base Case: $n=1$ ~~is~~ obvious, $|A_1| \geq 1$, hence there is an SDR

Inductive Step: Suppose the result holds for all families with $< n$ sets ($n>1$), and

Consider $S = \{A_1, \ldots, A_n\}$, for which Hall's Condition holds.

Case 1: Suppose that for all k, $1 \leq k < n$,

$$|A_{i_1} \cup \cdots \cup A_{i_k}| \geq k+1 \text{, for any choice of the } A_{i_j}.$$

Now, $|A_1| \geq 1$, so select $x_1 \in A_1$.

Put $B_i = A_i - \{x_1\}$ for $i = 2, \ldots, n$, so that $S' = \{B_2, \ldots, B_n\}$ is a family of $(n-1)$ sets. For any k, $1 \leq k \leq n-1$

$$|B_{i_1} \cup \cdots \cup B_{i_k}| = |A_{i_1} - \{x_1\} \cup A_{i_2} - \{x_1\} \cup \cdots \cup \{A_{i_k} - \{x_1\}|$$

$$= |A_{i_1} \cup \cdots \cup A_{i_k}| - |\{x_1\}|$$

$$\geq (k+1) - 1 = k.$$

Hence, the family S' satisfies Hall's Condition, and so by IH, S' has an SDR $= \{x_2, \ldots, x_n\}$.

Since $B_i \subseteq A_i$ for $i = 2, \ldots, n$ and $x_1 \notin B_i$ for any i,
$\{x_1, \ldots, x_n\}$ is an SDR for S.

Hall's Theorem (cont'd) ②

Case 2: For some k, $1 \le k < n$, $\exists$ sets $A_{i_1}, A_{i_2}, \dots, A_{i_k} \in S$ s.t. $|A_{i_1} \cup A_{i_2} \cup \dots \cup A_{i_k}| = k$ ①

For simplicity of indexing we shall assume that these are the sets $A_1, A_2, \dots, A_k$.

Consider the family $S_1 = \{A_1, \dots, A_k)$. Since $k < n$ and S_1 satisfies Hall's Condition, S_1 has an SDR $= \{x_1, \dots, x_k\}$.

Put $B_j = A_j - \bigcup_{i=1}^{k} A_i$ for $j = k+1, \dots, n$.

and consider the family $S_2 = (B_{k+1}, \dots, B_n)$.

Now for any r indices, $1 \le r \le n-k$, $k+1 \le i_1 < \dots < i_r \le n$, we have

$$|B_{i_1} \cup \dots \cup B_{i_r}| = |B_{i_1} \cup \dots \cup B_{i_r}| + |A_1 \cup A_k \cup \dots \cup A_k| - k$$

$$= |B_{i_1} \cup \dots \cup B_{i_r} \cup A_1 \cup \dots \cup A_k| - k,$$ since $A_1 \cup \dots \cup A_k$ is disjoint from all the B's

$$= |A_{i_1} \cup \dots \cup A_{i_r} \cup A_1 \cup \dots \cup A_k| - k$$

$$\ge (r+k) - k,$$ since the A's satisfy Hall's condition.

$$= r$$

Hence, the family S_2 satisfies Hall's Condition and $\therefore$ by IH, has an SDR, say $\{x_{k+1}, \dots, x_n\}$.

Since, $x_j \in B_j \subseteq A_j$ for each $j = k+1, \dots, n$ and the B_j's are disjoint from $A_1, \dots, A_k$, we get $\{x_1, \dots, x_n\}$ as an SDR for S, as desired.