Solution

## QUIZ – 20 minutes – open notes

1. Show that adding one edge to a tree produces exactly one cycle.
2. A tournament D is **transitive** if whenever uv and vw are (directed) edges in D, then so is uw. Show that a tournament D is transitive if and only if D has no (strong) cycles. [*Hint: sufficiency is easier to prove.*]

Each question worth 5 marks.

Thursday, February 05, 2015

Let $T$ be a tree with $n$ vertices.

~~Let e be an~~

Construct a graph $G$ by adding an edge $e = uv$, where $u, v \in V(T)$ and $uv \notin E(T)$.

Since $G$ has $n$ vertices and is connected, it is not a tree and hence has a cycle $C$.

Suppose BWOC that $G$ has another cycle $C'$.

Since $C \neq C'$, $\exists$ an edge $e'$ belonging to precisely one of the two cycles, $C$ and $C'$.

WOLOG, $e' \in C'$.

Now consider $G' = G - e'$.

Since $e' \notin C$, the cycle $C$ is contained in $G'$.

~~But now~~

Furthermore, since $e'$ is in a cycle in $G$, $e'$ is not a cut-edge of $G$, i.e. $G'$ is connected.

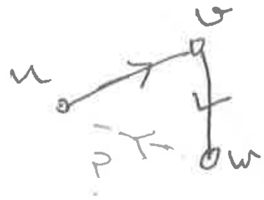
Hence, $G'$ is a connected graph with $(n-1)$ edges, containing a cycle $\Rightarrow\Leftarrow$.

Result follows.

Q2: A Tournament $D$ is transitive if and only if $D$ has no (strong) cycles.

Proof: [$\Leftarrow$] Sufficiency.

Suppose $D$ has no cycles, and suppose $uv$ and $vw$ are two edges in $D$.



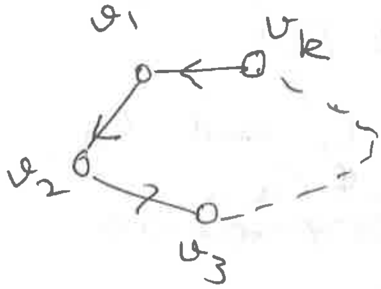

Suppose $wu \in E(D)$

Then we get the cycle $uvw$ in $D$.

$\Rightarrow\Leftarrow$

$\therefore$ $uw \in E(D)$ and so $D$ is transitive.

[$\Rightarrow$] Given that $D$ is transitive, and suppose BWOC that $v_1 v_2 \dots v_k v_1$ is a cycle in $D$.




We see that there are both outgoing and incoming edges from $v_1$ to other vertices of $C$ ( $v_1 v_2$ is outgoing, $v_k v_1$ is incoming). Suppose that $v_{i+1}$ is the first vertex on $C$ s.t. there is an edge $v_{i+1} v_1$ $(2 < i+1 \leq k)$ (1)

Hence ~~$v_1 v_i$~~ $v_1 v_i \in E(D)$.

We get ~~$v_1 v_1 \in E$~~ $v_1 v_i \in E(D)$,

$v_i v_{i+1} \in E(D)$.

By transitivity $v_1 v_{i+1} \in E(D)$ (2),

contradicting (1).

Result follows