Proof of Proposition ~~23~~ 24 (Tuesday 20170131)

Defn: A rooted tree $T(x)$ is a tree with a specified vertex x denoted as the root of T.

An orientation of a rooted tree ~~with a~~ in which every vertex but the root ~~vertex~~ which has in-degree 0, has in-degree 1 is called a branching.

There is an evident bijection between x-trees and x-branchings.

Proof: We show that the number of labelled branchings on n ~~xx~~ vertices is n^{n-1}. Cayley's formula then follows directly, because each labelled tree on n vertices gives rise to n labelled branchings, one for each choice of the root vertex.

Consider, first, the number of ways in which a labelled branching can be built up, one edge at a time, starting with the empty graph. In order to end up with a branching, the subgraph constructed at each stage must be a branching forest. Initially, there are n components, each consisting of an isolated vertex. At each stage, the no. of components decreases by one. ~~At each~~

~~I~~ If there are k components, the number of choices for a new edge (u, v) is $n(k-1)$:- any one of the n vertices may play the role of u, but v has to be the root of one of the $(k-1)$ components which do not contain u.

(PTO)

PROP. 24 (continued)

Hence, the total number of ways to construct a branching following this approach is:-

$$\prod_{i=1}^{n-1} n(n-i) = n^{n-1}(n-1)!$$

OTOH, any individual branching on n vertices could have been constructed exactly $(n-1)!$ times by this procedure; once for each of the orders in which the $(n-1)$ edges are selected.

$\therefore$ The number of labelled branchings on n vertices is $\cancel{=}$ $\dfrac{n^{n-1}(n-1)!}{(n-1)!}$

$= n^{n-1}$, as we claimed at the start.