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Definitions

Graph A Graph G is a triplet consisting of:

- A vertex set $V(G)$
- An edge set $E(G)$
- A relation between an edge and a pair of vertices

Null graph Edge set and vertex set are both empty.

Simple graph has no loops (edges like (u, u)) and no multiple edges between a pair of vertices

Adjacent vertices Endpoint of the same edge. (aka Neighbours)

Incident An edge $e = (u, v)$ is incident upon u and v.

Degree of a vertex is the number of edges incident upon it.

- The maximum degree of G is written as $\Delta(G)$
- The minimum degree of G is written as $\delta(G)$

Adjacency matrix $A(G)$ is $n \times n$ in which $a_{i,j}$ = number of edges in G with endpoints v_i, v_j

Incidence matrix $M(G)$ is $n \times m$ in which $m_{i,j} = 1$ if v_i is an endpoint of e_j, else 0.

Complement of a simple graph G is $\overline{G}$ where:

- $\overline{G}$ is a simple graph.
- $V(\overline{G}) = V(G)$
- $E(\overline{G}) = \{uv \mid uv \notin E(G)\}$

Subgraph of a graph G is H such that:

- $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$
- Assignment of endpoints of edges is same

Induced subgraph H is a subgraph of G s.t. $E(H)$ contains all edges of G whose endpoints lie in $V(H)$

Complete graph K_n is a simple graph whose n vertices are pairwise adjacent

Clique of G is a complete subgraph of G.

Independent set in a graph is a set of pairwise non-adjacent vertices.

Isomorphism from a simple graph G to H is a *bijection* $f : V(G) \to V(H)$ s.t. $uv \in E(G) \iff f(u)f(v) \in E(H)$. If an isomorphism exists we say G is isomorphic to H ($G \cong H$)

Vertex transitive is a graph G if $\forall u, v \in V(G)$ there is an automorphism that maps u to v.

Bipartite is a graph G where vertices can be partitioned into two sets X and Y where both are independent sets.

Complete Bipartite is a graph K_r, s where the two vertices are adjacent $\iff$ they are in different partite sets. And $|V(X)| = r, |V(Y)| = s$.

Chromatic number of a graph G, written $\chi(G)$ is the minimum number of colors needed to label vertices s.t. adjacent vertices recieve different colors.

Walk of length k is a sequence $v_0, e_1, v_1, e_2, \ldots, e_k, v_k$ s.t. $\forall i, e_i = (v_{i-1}, v_i)$

- A *trail* is a walk with no repeated edge.
- A *path* is a walk with no repeated vertex.
- A u, v-walk starts at u and ends at v.
- A walk is *closed* if it has length at least one and equal endpoints.

Path is a sequence of *distinct* vertices s.t. two consecutive vertices are adjacent.

Cycle is a *closed* path.

Connected graph has at least one path between every pair of vertices.

Component of a graph is one of its maximal connected subgraphs.

- A component with no edges is *trivial*
- An *isolated vertex* has degree 0.

Cut-edge or Cut-vertex is an edge or a vertex whose deletion increases the number of components of a graph.

Maximal path in a graph G is a path P which is not contained in any longer path.

Eulerian graph has a closed trail (aka *circuit*) containing all edges.

Even graph has all vertex degrees even.

k-Regular graph has $\Delta(G) = \delta(G) = k$.

Neighborhood of a vertex v, written as $N(v)$, is the set of vertices adjacent to v.

Order and size of a graph are $n(G) = |V(G)|$ and $e(G) = |E(G)|$ respectively.

Degree sequence of a graph is the list of vertex degrees, usually written in non-decreasing order $d_1 \geq \cdots \geq d_n$.

Graphic sequence is a degree sequence of a simple graph.

2-switch is the replacement of edges xy and zw by yz and xw, given that they did not appear in the graph originally.

Directed Graph G is a triplet consisting of:

- A vertex set $V(G)$
- An edge set $E(G)$
- A relation between an edge and an *ordered* pair of vertices

Underlying graph of a digraph D is a graph obtained by treating edges of D as unordered pairs.

Weakly connected digraph is one with a connected underlying graph.

Strongly connected digraph is one where for each ordered pair (u, v) of vertices, there is a path from u to v.

De-Bruijn cycle is a cyclic arrangment of 2^n binary digits, such that the 2^n possible strings of n consecutive digits in the cycle are all distinct.

De-Bruijn digraph D_n has vertices as $(n-1)$ binary tuples, and an edge from u to v if the last $n-2$ digits of u match the first $n-2$ digits of v.

Orientation of a graph is an assignment of a direction to each edge, turning initial graph into a digraph.

Tournament is an orientation of a complete graph.

King of a digraph is a vertex from which every vertex is reachable by a path of length at most 2.

Forest is an acyclic graph.

Tree is a connected acyclic graph.

Leaf is a vertex of degree 1.

Spanning subgraph of G is a subgraph with vertex set $V(G)$. If the subgraph is a tree it is known as a *spanning tree*.

Distance from u to v, written as $d(u, v)$ is the length of the shortest path from u to v. If no such path exists $d(u, v) = \infty$

Diameter of a graph G, $diam\ G = max_{u,v \in V(G)} d(u, v)$

Eccentricity of a vertex u is, $\epsilon(u) = max_{v \in V(G)} d(u, v)$

Radius of a graph is, $rad\ G = min_{u \in V(G)} \epsilon(u)$

Center of a graph is the subgraph induced by the vertices of minimum eccentricity.

Rooted tree is a tree T with a vertex x specified as root. denotation $T(x)$.

Branching is an orientation of a rooted tree $T(x)$ where in-degree of x is 0 and all other vertices have in-degree exactly 1.

Contraction of a graph G, written $G.e$, is removal of an edge $e = (u, v)$ and replacement of u and v with a single vertex whose incident edges are the union of those of u and v except e.

Matching of a graph G is, $M \subseteq E(G)$ s.t. edges in M share no endpoints.

- Vertices incident to edges in M are said to be *saturated* by M.
- A *perfect matching* saturates every vertex in the graph.

Maximal matching M is maximal if every edge not in M is incident to an edge already in M.

M-alternating path is a path that alternates between edges in M and edges not in M. If the endpoints of such a path are unsaturated by M then it is called an *M-augmenting path.*

Transversal of a finite family of finite sets is an injection $f : S \to \cup\{A \mid A \in S\}$ such that $f(A) \in A$.

Vertex cover of a graph G is a set $Q \subseteq V(G)$ that contains at least one endpoint of every edge. Q covers $E(G)$.

Edge cover of a graph G is a set $L \subseteq E(G)$ s.t. every vertex of G is incident to some edge in L.

Independence number of a graph is the size of the maximal possible independent set.

- Independence number: $\alpha(G)$
- Max size of a matching: $\alpha'(G)$
- Max size of a vertex cover $\beta(G)$
- Max size of a edge cover $\beta'(G)$

Weighted matching is a matching in a weighted complete bipartite graph. We extend a general weighted bipartite graph to $K_{n,n}$ form by adding 0-weight edges.

Weighted cover for weighted $K_{n,n}$ is a choice of numerical labels $u_i, \ldots, u_n$ and $v_j, \ldots, v_n$ in $K_{n,n}$, s.t $\forall i, j : u_i + v_j \geq w_{i,j}$

Equality Subgraph $G_{u,v}$ for a cover (u, v) is the spanning subgraph of $K_{n,n}$ having the edges $x_i y_j$ such that $u_i + v_j = w_{i,j}$.

Stable Matching is a matching S of men M, and women W, s.t. $\nexists m \in M, w \in W : m$ prefers w, and w prefers m, over their partners in S.

Separating set or **vertex cut** of a graph G is a set $S \subseteq V(G)$ s.t. $G - S$ has more than one component.

Connectivity of G, written $\kappa(G)$ is the minimum size of a vertex set S s.t. $G - S$ is disconnected or *has only one vertex.*

k-Connected graph has connectivity at-least k.

Disconnecting set or **cut** of a graph G is a set $F \subseteq E(G)$ s.t. $G - F$ has more than one component.

Edge connectivity of G, written $\kappa'(G)$ is the minimum size of a disconnecting set.

Edge cut is an edge set of the form $[S, \overline{S}]$, where S is a non-empty proper subset of $V(G)$ and $\overline{S}$ denotes $V(G) - S$. It consists of all edges between vertices of S and $\overline{S}$.

Bond is a minimal non-empty edge cut.

Block of a graph G is a maximal conected subgraph G that has no cut-vertex.

g-cage is a 3-regular graph of minimum order, with girth $g \geq 3$.

Peterson graph is a simple graph whose vertices are the two-element subsets of a 5-element set, where two vertices are adjacent if the corresponding two-element subsets are disjoint.

- If two vertices are non-adjacent in Peterson graph, they have exactly one common neighbour.
- Peterson graph has girth 5, and is the unique 5-cage.
- Peterson graph has 120 automorphisms.

x, y-Separators Given $x, y \in V(G)$, a set $S \subseteq V(G) - x, y$ is an x, y-separator if $G - S$ has no x, y path. $\kappa(x, y)$ denotes minimum size of the separator set and $\lambda(x, y)$ denotes maximum size of a set of pairwise disjoint x, y paths.

Network is a digraph with non-negative capacity $c(e)$ on each edge e, and a distinguished source vertex s and sink vertex t.

Flow is a function f, that assigns value $f(e)$ to each edge e. Also, $f^+(v)$ is the total flow on edges leaving v, $f^-(v)$ works similarly.

Feasible flow satisfies the capacity constraints $\forall e : 0 \leq f(e) \leq c(e)$ and conservation constraints $\forall v \notin s, t : f^+(v) = f^-(v)$

Flow value $val(f)$ is the net flow into the sink: $val(f) = f^-(t) - f^+(t)$.

f-augmenting path is a source-to-sink path P in the underlying graph G s.t. $\forall e \in E(P)$,

- If P follows e in fwd direction, then $f(e) < c(e)$.
- If P follows e in bwd direction, then $f(e) > 0$.

Tolerance of P is $min_{e \in E(P)} \epsilon(e)$ where $\epsilon(e) = c(e) - f(e)$ for fwd, and $\epsilon(e) = f(e)$ for bwd.

Source-Sink cut $[S, T]$ consists of the edges from a source set S to a sink set T, where S, T partition $V(G)$, with $s \in S, t \in T$.

Capacity of cut is written $cap(S, T)$. It is the total of the capacities on the edges of $[S, T]$.

k-colorable graph G has a labeling $f : V(G) \to S$, where $|S| = k$. Vertices of same color form a *color class.*

k-chromatic graph G has the minimum coloring of size k, or $\chi(G) = k$.

Color-critical graph G has $\chi(H) < \chi(G)$ for every *proper* subgraph H.

Chromatic number of a graph G, written $\omega(G)$, is the maximum size of a clique in G.

Interval graph is a graphical representation of a set of open intervals on a real line. Each interval is a vertex, and two vertices are adjacent $\iff$ corresponding intervals intersect.

Curve is the image of a continuous map from $[0, 1]$ to $\mathbb{R}^2$.

Polygon curve is a curve composed of finitely many line segments.

Drawing of a graph G is a function f defined on $V(G) \cup E(G)$ that assigns each vertex v a point $f(v)$ in the plane and assigns each edge uv a polygonal $f(u), f(v)$-curve.

Crossing is a point in $f(e) \cap f(e')$ that is not a common endpoint.

Planar graph has a drawing without crossings.

Open set in the plane is a set $U \subseteq \mathbb{R}^2$ s.t. $\forall p \in U$, all points in neighbourhood of p belong to U.

Region is an open-set U that contains a polygon u, v-curve for every pair $u, v \in U$.

Face of a plane graph is one of the maximal regions of the plane that contain no point used in the embedding.

Dual graph G^* of a plane graph G is a plane graph s.t. vertices of G^* are faces of G, and there is an edge e^* in G^* for every pair of faces that share a common edge e in G.

Length of a face $l(F_i)$ in a plane graph G is the total length of the closed walk(s) bounding the face F_i.

Maximal planar graph is a simple planar graph that is not a spanning subgraph of another planar graph.

Triangulation is a simple planar graph where every face boundary is a 3-cycle.

Graph subdivison is a graph obtained from G by successive edge subdivisions.

Kuratowski Subgraph is a subdivision of K_5 or $K_{3,3}$.

Minimal nonplanar graph is a nonplanar graph s.t. every proper subgraph is planar.

Convex embedding is a planer embedding in which each face boundary is a convex polygon.

Hamiltonian graph is a graph with a spanning cycle. Also called a *Hamiltonian cycle.*

Hamiltonian Closure denoted by $C(G)$ is the graph with vertex set $V(G)$ obtained by iteratively adding edges joining pair of nonadjacent vertices in G, whose degree sum is at least n, until no such pair remains.

Theorems

Euler's Theorem A Graph G is Eulerian $\iff$ it has at most one non-trivial component and its vertices all have even degree.
Corollaries

1. Let G be a connected graph with exactly two vertices with odd degrees, say u and v. Then G has an eulerian trail that starts at u and ends at v.
2. Every connected non-trivial even graph decomposes into cycles.

Philip Hall's Theorem (*Graph Theoretic*) An X, Y-bigraph G has a matching that saturates $X \iff \forall S \subseteq X, |N(S)| \geq |S|$
(*Set Theoretic*) A finite family of finite sets S has a transversal $\iff$ for each subfamily $W \subseteq S$

$$|W| \leq |\bigcup\{A \mid A \in W\}|$$

Corollaries

1. for $k > 0$, every k-regular bipartite graph has a perfect matching.

Konig-Egervary Theorem If G is a bipartite graph, then the maximum size of a matching in G equals the minimum size of a vertex cover of G.

Menger's Theorem If x, y are vertices of a graph G and xy not an edge, then $\kappa(x, y) = \lambda(x, y)$.

Max-Flow Min-Cut Theorem In every network, the maximum value of a flow is equal to the minimum capacity of a source/sink cut.
Lemmas

- L5.1 If P is an f-augmenting path with tolerance z, then changing flow by $+z$ on edges followed forward by P and by $-z$ on edges followed backward by P produces a feasible flow g with $val(g) = val(f) + z$.

Brook's Theorem If G is a connected graph other than a complete graph or an odd cycle, then $\chi(G) \leq \Delta(G)$.

Euler's Formula If a connected plane graph G has exactly n vertices, e edges, and f faces, then $n - e + f = 2$.

Kuratowski's Theorem A graph G is non-planar $\iff$ G has a Kuratowski subgraph.
or A graph G is planar $\iff$ G does not contain a Kuratowski subgraph.
Tutte's version: If G is a 3-connected graph without Kuratowski subgraphs, then G has a convex embedding in the plane with no three vertices on a line.
Lemmas

- L8.1 If F is the edge set of a face in a planar embedding of G, then G has an embedding with F being the edge set of the unbounded face.
- L8.2 Every minimal non-planar graph is 2-connected.
- L8.3 If G is a graph with fewest edges among all nonplanar graphs without Kuratowski subgraphs, then G is 3-connected.

Heawood's Theorem Every planar graph is 5-colorable.

Appel-Haken Theorem Every planar graph is 4-colorable.

Propositions

1. Let G be a graph. Then, the sum of the degrees of the vertices is twice the number of edges, i.e.
$$\sum_{v \in V(G)} d(v) = 2|E(G)|$$
2. Every u, v-walk contains a u, v-path
3. Every graph with n vertices and k edges has at least $n-k$ components
4. An edge e is a cut-edge $\iff$ e belongs to no cycles
5. A graph with at least two vertices is bipartite $\iff$ it has no odd cycle.
   - L5.1 Every closed odd walk contains an odd cycle
6. If every vertex of graph G has degree at least 2, then G contains a cycle.
7. If G is a simple graph in which every vertex has degree at least k, then G contains a path of length at least k. If $k \geq 2$, then G also contains a cycle of length at least $k+1$
8. If $k > 0$, then a k-regular bipartite graph has the same number of vertices in each partite set.
9. The minimum number of edges in a connected graph with $|V(G)| = n$ is $n-1$
10. If G is a simple n-vertex graph with $\delta(G) \geq \frac{n-1}{2}$, then G is connected.
11. Every loopless graph G has a bipartite sub-graph with at least $\frac{e(G)}{2}$ edges.
12. The maximum number of edges in an n-vertex triangle-free simple graph is $\left\lfloor \frac{n^2}{4} \right\rfloor$
13. The nonnegative integers $d_1, \dots, d_n$ are the vertex degrees of some graph $\iff$ $\sum_i d_i$ is even.
14. For $n > 1$, an integer list d of size n is graphic $\iff$ d' is graphic, where d' is obtained from d by deleting its largest element Δ and subtracting 1 from its Δ next largest elements. The only 1-element graphic sequence is $d_1 = 0$.
15. If G and H are two simple graphs with vertex set V, then $\forall v \in V, d_G(v) = d_H(v)$ $\iff$ there is a sequence of 2-switches that transforms G into H.
16. If G is a digraph with $\delta^+(G) \geq 1$ or $\delta^-(G) \geq 1$, then G contains a cycle.
17. A digraph is Eulerian $\iff$ $\forall v \in V, d^+(v) = d^-(v)$ and the underlying graph has at most one non-trivial component
18. In a digraph D_n constructed for a De-Bruijn cycle of length n:
   - L18.1 D_n is eulerian.
   - L18.2 The labels on the edges in any Eulerian circuit of D_n form a cyclic arrangement in which the $2n$ consecutive segments of length n are disinct.
19. Every tournament has a king.
20. For an n-vertex simple graph G (with $n \geq 1$), the following are equivalent (and characterize a **tree**):
   (a) G is connected and has no cycles
   (b) G is connected and has $n-1$ edges
   (c) G has $n-1$ edges and no cycles
   (d) $\forall u, v \in V(G), G$ has exactly one u, v-path

   Lemmas:
   - L20.1 A tree with at least two vertices has at least two leaves.
   - L20.2 Deleting a leaf from an n-vertex tree produces a tree with $n-1$ vertices.

   Corollaries:
   - Every edge of a tree is a cut-edge
   - Adding one edge to a tree forms exactly one cycle
   - Every connected graph contains a spanning tree
21. If T, T' are spanning trees of a connected graph G and $e \in E(T) - E(T')$, then there is an edge $e' \in E(T') - E(T)$ such that $T - e + e'$ is a spanning tree of G.
22. If G is a simple graph then $diam\ G \geq 3 \implies diam\ \bar(G) \leq 3$
23. The center of a tree is a vertex or an edge.
24. For a set $S \subseteq N$ of size n, there are n^{n-2} trees with vertex set S.
25. Let $\tau(G)$ denote the number of spanning trees of a graph G. If $e \in E(G)$ is not a loop then $\tau(G) = \tau(G-e) + \tau(G.e)$ (where $\tau(G.e)$ are the number of spanning trees that contain e)
26. In a connected weighted graph G, Kruskal's algorithm constructs a minimum spanning tree.
27. Given a (di)graph G and a vertex $u \in V(G)$, Dijkstra's algorithm computes $d(u, z)$ for every $z \in V(G)$.
28. A matching M in a graph G is a maximum matching $\iff$ G has no M-augmenting path.
   - L28.1 Every component of the symmetric difference of two matching is a path or an even cycle.
29. If G is a graph without isolated vertices, then $\alpha'(G) + \beta'(G) = n(G)$
   - L29.1 $S \subseteq V(G)$ is an independent set $\iff$ $\overline{S}$ is a vertex cover. And hence $\alpha(G) + \beta(G) = n(G)$
30. Repeatedly applying the augmenting path algorithm to a bipartite graph produces a matching and a vertex cover of equal size.
31. The Hungarian Algorithm finds a maximum weight matching and a minimum cost cover.
   - L31.1 For a perfect matching M and $cover(u, v)$ in a weighted bipartite graph G, $c(u, v) \geq w(M)$. Also $c(u, v) = w(M)$ $\iff$ M consists of edges $x_i y_j$ such that $u_i + v_j = w_{i,j}$. In this case, M and (u, v) are optimal.
32. The Gale-Shapley Proposal Algorithm produces a stable perfect matching.
33. If G is a simple graph, then $\kappa(G) \leq \kappa'(G) \geq \delta(G)$.
34. If G is a connected graph, then an edge cut F is a bond $\iff$ $G - F$ has exactly two components.
35. Two blocks in a graph share at most one vertex.
36. If G is a 3-regular (simple) graph, then $\kappa(G) = \kappa'(G)$.
37. A graph G having at least three vertices is 2-connected $\iff$ $\forall u, v \in V(G)$ there exist (at least two distinct) internally disjoint u, v-paths in G.
   - L37.1 If T is a spanning tree of a connected graph G grown by DFS from u, then every edge of G not in T consists of two vertices v, w such that v lies on the u, w-path in T.
38. For a graph with at least 3 vertices these are equivalent, and characterize a 2-connected graph:
   (a) G is connected and has no cut-vertex
   (b) $\forall x, y \in V(G)$, there are internally disjoint x, y-paths.
   (c) $\forall x, y \in V(G)$, there is a cycle through x and y.
   (d) $\delta(G) \geq 1$, and every pair of edges in G lies on a common cycle.

   Lemmas and Corollaries:
   - L38.1 If G is a k-connected graph, and G is obtained from G by adding a new vertex y with at least k neighbors in G, then G is k-connected.
   - C38.2 If G is 2-connected then the graph G' obtained by subdividing an edge of G is also 2-connected.
39. For every graph G, $\chi(G) \geq \omega(G)$ and $\chi(G) \geq \frac{n(G)}{\alpha(G)}$.
40. $\chi(G) \leq \Delta(G) + 1$
41. If G has a degree sequence $d_1 \geq \dots \geq d_n$, then $\chi(G) \leq 1 + max_i mind_i, i - 1$.
42. If G is an interval graph, then $\chi(G) = \omega(G)$.
43. If H is a k-critical graph, then $\delta(H) \geq k - 1$.
44. If G is a graph, then $\chi(G) \leq 1 + max\delta(H) : H \subseteq G$.
45. K_5 and $K_{3,3}$ cannot be drawn without crossings.
46. For a plane graph G, $2e(G) = \sum_i l(F_i)$.
47. If G is a simple planar graph with at least three vertices, then $e(G) \leq 3n(G) - 6$. If also G is triangle-free, then $e(G) \leq 2n(G) - 4$.
48. Edges in a plane graph G form a cycle in G $\iff$ the corresponding dual edges form a bond in G^*.
49. The following are equivalent for a plane graph G.
   (a) G is bipartite.
   (b) Every face of G has even length.
   (c) Dual graph G^* is Eulerian.
50. For a simple n-vertex graph G, the following are equivalent:
   (a) G has $3n - 6$ edges.
   (b) G is a triangulation.
   (c) G is a maximal planar graph.
51. Every finite simple planar graph has an embedding in which all edges are straight line segments.
52. Every planar graph is 6-colorable.
53. If G is hamiltonian, then for each non-empty set $S \subseteq V(G)$, the graph $G - S$ has at most $|S|$ components.
54. If G is simple, with at least three vertices, and $\delta(G) \geq n(G)/2$, then G is hamiltonian.
55. A simple n-vertex graph is hamiltonian $\iff$ its closure is hamiltonian.
   - L55.1 If u, v are distinct non-adjacent vertices of G with $d(u) + d(v) \geq n$, then G is hamiltonian $\iff$ $G + uv$ is.
   - L55.2 Closure of G is well defined.
56. Let G be a simple graph with degree sequence $d_1 \geq \dots \geq d_n$, where $n \geq 3$. If $i < n/2$, implies that $d_i > i$ or $d_{n-1} \geq n - I$, then G is Hamiltonian.