Notes for Thursday 20150416

— See note on last page before going to Theorem 11 proof.

Prop 62 :- Suppose the edges of K_6 are arbitrarily colored red or blue.

Consider a vertex $u \in K_6$.

Since $d(v) = 5$, there must be at least 3 edges of the same color incident to u.

Since we are considering a symmetric situation, wolog the 3 edges are red.

Consider now the 3 end-points, say v_1, v_2, v_3.

If any of the 3 edges joining them are red, say $v_1 v_2$ is red, then $\{u, v_1, v_2\}$ is a red Δ.

OTOH, if all the three edges are blue, then $\{v_1, v_2, v_3\}$ is a blue Δ, and we are done.

NB: In order to ~~complete~~ show that $R(3,3) = 6$, we need to show that there is a coloring of ~~K_5~~ edges of K_5 with no red Δ and no blue Δ. See the diagram:-

Proof of Thm. 11

We will prove the special case $k=2$, i.e. we will consider red-blue colourings of the edges only. Note that for this to be meaningful, $p \geq 2$ and $q \geq 2$, i.e. $p+q \geq 4$.

We also note that if the numbers exist (i.e. are finite), then $R(p,q) = R(q,p)$.

We proceed by induction on $m = p+q$.

Base Case: (i) $R(2,q) = q = R(q,2) \ \forall q \geq 2$

(ii) $R(3,3) \leq 6$.

So suppose the result holds when $p+q < m$, $m \geq 7$ and now suppose $p+q = m$.

If either p or q is 2, we are done.

Else put $n = R(p-1,q) + R(p,q-1)$ and arbitrarily color the edges of K_n red or blue.

Select $v \in K_n$ and divide the remaining vertices of K_n into two sets as follows:

$w \in M_1$ if vw is ~~blue~~ red

$w \in M_2$ if vw is ~~red~~ blue.

Since $n = |M_1| + |M_2| + 1$, either

$|M_1| \geq R(p-1, q)$ or $|M_2| \geq R(p, q-1)$.

In the former case, if M_1 has a blue K_q we are done; if not, then it has a red K_{p-1} which together with v forms a red K_p.

The latter case is analogous.

NB: The above result can be generalized for $k>2$ and also to the general finite version of Ramsey's Theorem. Here we have to do a double induction on k and $\sum p_i$. Details are in the text book.

Proof of Prop. 6.4:- Let there be given a red-blue coloring of K_9. We show that there is either a red K_3 or a blue K_4.

First, observe that every vertex of K_9 cannot be incident with exactly 3 red edges:- if so, then the red-subgraph of K_9 would be a 3-regular graph with 9 vertices $\Rightarrow\Leftarrow$

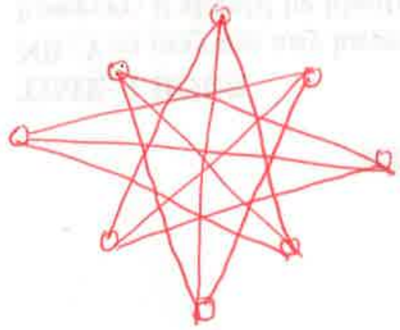
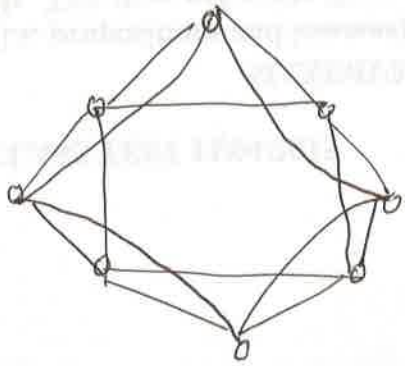
$\therefore$ there are two cases:

Case 1: There exists a vertex v_1 that is incident with (at least) four red edges, say v_1v_2, v_1v_3, ~~v_2~~ v_1v_4, v_1v_5. If any two of the vertices v_2, v_3, v_4, v_5 are joined by a red edge, we get a red Δ. Else, they are all joined by blue edges, and we get a blue K_4.

Case 2. There exists a vertex v_1 that is incident with 6 blue edges.
Let these be v_1v_i, $i = 2, \ldots, 7$.
Since ~~R(3)~~ $R(3,3) = 6$, the subgraph H induced by $\{v_2, \ldots, v_7\}$ is a K_6, which either has a red ~~K~~ K_3 or a blue K_3.
If H has a red K_3, we are done.
If it has ~~a~~ blue K_3, say $\{v_2, v_3, v_4\}$, then $\{v_1, \ldots, v_4\}$ is a blue K_4, as reqd.

$\therefore \quad R(3,4) \leq 9$.

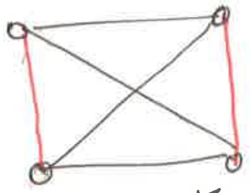
To complete the proof, we require to show a coloring of K_8 which has neither a red K_3 nor a blue K_4. Consider :-

Examples of Graph Ramsey Numbers :-

1. $R(P_3, K_3) = 5$.

First we show $R(P_3, K_3) \geq 5$.
Consider the following coloring of K_4

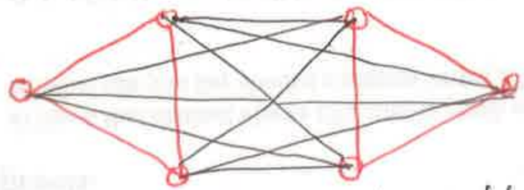


— so $R(P_3, K_3) > 4$

It remains to show $R(P_3, K_3) \leq 5$.
Let a red-blue coloring of K_5 be given.
Consider $v_1 \in K_5$. If v_1 is incident with 2 red edges, then a red P_3 is produced. $\therefore$ v_1 is incident with at most one red edge, so there are 3 blue edges, say v_1v_2, v_1v_3, v_1v_4. If there is a blue edge joining any two of them, we get a blue K_3.
But then otherwise v_2v_3 and v_3v_4 are red, and we get a red P_3.

2. $R(K_{1,3}, K_3) = 7$

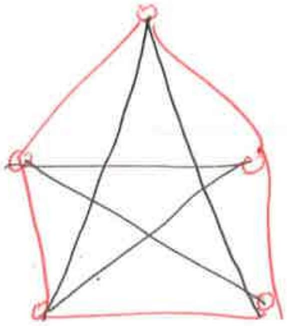
First, we show $R(K_{1,3}, K_3) \geq 7$. Consider the following red-blue coloring of K_6 :-
There is no red $K_{1,3}$
nor blue K_3



For the other part, suppose there is a red-blue coloring of K_7 which has no red $K_{1,3}$ and no blue K_3. Select a vertex v_1. At most two red edges are incident with v_1, and so at least 4 blue edges, say v_1v_i' ($2 \leq i \leq 5$).
If any edge joining vertices in $\{v_2, \ldots, v_5\}$ is blue, then we get a blue K_3 with v_1. So all the edges v_2v_3, v_2v_4, v_2v_5 are colored red, giving a red $K_{1,3} \Rightarrow\Leftarrow$

Another Example:- $R(K_{1,3}, C_4) = 6$.

First, we show $R \geq 6$:- consider the following coloring of K_5 which has no red $K_{1,3}$ nor a blue C_4:-



For the other part, let a red-blue coloring of K_6 be given, where we denote the vertices by v_i, $i = 1, \dots, 6$. Since $R(3,3) = 6$, we have either a red K_3 or a blue K_3. We consider these two cases separately:-

Case 1: There is a red K_3, say v_1, v_2, v_3. If there is no red $K_{1,3}$, then every edge joining a vertex in $\{v_1, v_2, v_3\}$ and a vertex in $\{v_4, v_5, v_6\}$ is blue. So a blue C_4 is produced.

Case 2. There is a blue K_3. Now assume that $\{v_1, v_2, v_3\}$ is a blue K_3. If some vertex in $\{v_4, v_5, v_6\}$ is joined ~~by~~ to two vertices in $\{v_1, v_2, v_3\}$ by blue edges we get a blue C_4. If it is joined to all of them by red edges, then a red $K_{1,3}$ is produced. So every vertex in $\{v_4, v_5, v_6\}$ is joined to one vertex by a blue edge, and two vertices by a red edge. If any of the edges in $\{v_4, v_5, v_6\}$ is red, we get a red $K_{1,3}$. So in a $\{v_4, v_5, v_6\}$ form a blue K_3. Thus any two blue edges that join a vertex in $\{v_1, v_2, v_3\}$ to a vertex in $\{v_4, v_5, v_6\}$ form a blue C_4.

Proof of Prop. 66:-
We will make use of the result that if G is a graph s.t. $\deg v \geq k-1$ for every vertex in G, and T is a tree with k vertices, then G contains a subgraph isomorphic to T (left as an exercise or reference).

We will proceed by induction:-

Step 1. $R(T_m, K_n) \geq (m-1)(n-1)+1 = t+1$, say

Let there be given a red-blue coloring of K_t s.t. the resulting red subgraph is $(n-1)T_{m-1}$, i.e. it consists of $(n-1)$ copies of K_{m-1}. Since each component of ~~the~~ the red subgraph has order $(m-1)$, it contains no connected subgraph of order $>(m-1)$, in particular there is no red tree of order m. The blue subgraph is then the complete $(n-1)$-partite graph where each partite set contains exactly $(m-1)$-vertices. So there is no blue K_n either.

Step 2: $R(T_m, K_n) \leq (m-1)(n-1)+1$.

We proceed by induction on the order of the complete graph K_n. We let $n=2$, and show that $R(T_m, K_2) \leq (m-1)(2-1)+1 = m$.
~~Let there be~~ This follows from ~~$R(m$~~ $R(m,2) = m$
So the basis case holds.

Assume for every tree of order m and an integer $k \geq 2$ that $R(T_m, K_k) \leq (m-1)(k-1)+1$

we show that $R(T_m, K_{k+1}) \leq (m-1)k+1 = t$, say

Let a red-blue coloring be given.

Case 1. There is a vertex v_1 in K_t that is incident with $(m-1)(k-1)+1$ blue edges.

Suppose $v_1 v_i$ is a blue edge for $2 \leq i \leq (m-1)(k-1)+2$.

Consider the subgraph $H = \{ v_i : 2 \leq i \leq (m-1)(k-1)+2 \}$.
$\cong K_{(m-1)(k-1)+1}$.

By IH, H contains either a red T_m or a blue K_k. If the first we are done; if the second, we get a blue K_{k+1} by adjoining v_1,

~~5a~~ 6a

Proof of Prop. 66 [cont'd]

Case 2: Every vertex of K_t is incident with at most $(m-1)(k-1)$ blue edges.

$\therefore$ Every vertex of K_t is incident with at least $(m-1)$ red edges, i.e. the red subgraph has min degree $\geq m-1$.

By the previous result, the red subgraph contains a red T_m, and we are done.

Note: Ramsey's Theorem (finite case) can be regarded as a generalization of the Pigeon-Hole Principle (PHP). PHP corresponds to Ramsey's Theorem for $r=1$, if the "holes" are allowed to have variable capacities; & if $p_i = 2$ for $i = 1, 2, \ldots, k$, then we have the standard PHP, and in this case, the ~~concept~~ Ramsey number is $k+1$.