MTH303 – CLASS TEST 13/02/2015

TIME: 1 HOUR **MAXIMUM MARKS: 50**

NB: You may use any known result (i.e. propositions and lemmas) without proof; however, it should be identified clearly. Marks will depend on the correctness and completeness of your proofs. All questions have equal marks.

1. Are the following degree sequences graphic? Justify your answers.
   a. 55432221
   b. 55542111

2. If a graph has at least one cycle, its girth is the length of its shortest cycle. A graph with no cycle has infinite girth. Let G be a graph with girth 4 in which every vertex has degree k. Prove that G has at least 2k vertices. Determine all such graphs with exactly 2k vertices.

3. Let G be a connected simple graph and let e be a cut-edge. Show that G – e has precisely two components.

4. Prove or disprove:
   a) If the center of a simple graph G is either a single vertex or an edge, then G is a tree.
   b) If a tournament has a directed circuit, then it must have a directed triangle.

5. Show that if T is a tree with n vertices, and G is a simple graph with δ (G) ≥ n – 1, then T is isomorp to a subgraph of G.

Solutions Below
(NB: other methods / proofs ~~are also~~ may also ~~app~~ be applicable.)

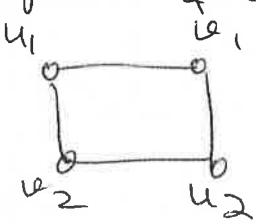
Q 1. We apply the recursive condition for graphic sequences (Proposition 14):-

(a) 55432221 ⟶ 4321121 ⟶ 4322111 ⟶ 211011
⟶ 211110 ⟶ 00110 ⟶ 11000 which is realizable by the graph G' = o—o o o o
Hence, G is realizable - YES

(b) 55542111 ⟶ 4431011 ⟶ 4431110 ⟶
320010 ⟶ 321000.
This is not realizable since there must be at least Δ vertices ~~after the lead~~ with non-zero degree following the initial vertex of degree Δ - NO

(PTO)

Q2. By hypothesis, G contains a cycle C_4 (see diagram)



[NB: this is also $K_{2,2}$ and represents the smallest G ~~...~~ satisfying hypothesis]

Put $X = \{u_1, u_2\}$ and

$Y = \{v_1, v_2\}$

Now, u_1 is adjacent to $k-2$ additional vertices, say $v_3, \ldots, v_k$ which we adjoin to X, getting ~~X~~ $\{v_1, v_2, \ldots, v_k\} = Y$.

There cannot be an edge between any v_i and v_j, since then we get a Δ $v_i u_1 v_j v_i$.

~~By~~ Hence, all the $k-2$ vertices adjacent to v_1, other than u_1 and u_2, are distinct from the v_i's, and we may place them in X, getting

$\{u_1, u_2, \ldots, u_k\} = X$.

Thus, we require at least $2k$ distinct vertices to satisfy the hypothesis.

Furthermore, $K_{k,k}$ with exactly $2k$ vertices satisfies the hypothesis, and is the only such graph with exactly $2k$ vertices.

Q3. ~~Let~~ [The following proof is more detailed than others, but is instructive, since it proceeds directly from definitions.]

Let G be a connected simple graph, and let $e = uv$ be a cut edge.

Let $C_1, C_2, \ldots, C_k$ be the components

Q3. (Continued)

Claim 1: u and v must be in distinct components of G'

Suppose not, and let $P = v u_1 u_2 \ldots u_m$ be a vu-path in G'. Then $P' = Pv$ is a cycle in G, containing e. But this contradicts Proposition

This proves Claim 1.

WMA $u \in C_1$, $v \in C_2$.

Claim 2: if $x \in$ ~~V(G)~~ $V(G')$ then either $x \in C_1$ or $x \in C_2$.

Suppose $x \notin C_1$. Now, $\exists$ an xv path in G, since G is connected, say P

If $e \notin P$, then P is an xv-path in G'.

If $e \in P$, then $P = x u_1 u_2 \ldots u v$, since P being a path, v cannot be repeated in it.

But then $x u_1 u_2 \ldots u$ is an xu-path in G', contrary to assumption.

$\therefore$ $e \notin P$, and so P is an xv-path in G'.

This proves Claim 2.

Since every x in G' belongs to C_1 or C_2,

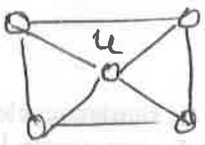
$k = 2$ as required.

Q 4 (a) Disprove.

Recall that the eccentricity of a vertex $u \in V(G)$ is given by $E(u) = \max\{d(u,v) : v \in V(G)\}$ for connected graphs.

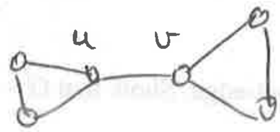
The center of G is then the induced subgraph of ~~verts~~ vertices of min eccentricity.

Consider $G_1 =$

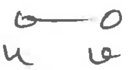


Here, centre of $G_1 = \{u\}$ but G_1 is not a tree.

Similarly, for $G_2 =$

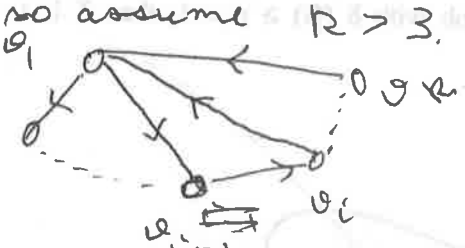


centre of G_2 is



, but G_2 is not a tree.

(b) Prove: Let $C = v_1 v_2 \ldots v_k v_1$ be a directed circuit in the tournament T. If $k=3$, we are done, so assume $k > 3$.



Let v_i be the first vertex on C s.t. the orientation of the edge joining v_1 and v_i is $v_i v_1$. If there is no such i, then $v_1 v_{k-1} v_k v_1$ is a directed Δ.

But now, $v_1 v_{i-1} v_i v_1$ is a directed Δ.

Q 5. Prove by induction on n.

Base Case: $n=2$: then $T = $ o—o and if G is any graph with $\delta(G) \geq n-1 = 2-1 = 1$, then G has as an edge e, which is isomorphic to T.

Inductive Step: suppose the result holds for all trees with n vertices, let T be a tree with $n+1$ vertices, and let G be a graph with $\delta(G) \geq (n+1)-1 = n$.

Now, if u is a leaf of T, then $T_1 = T - u$ is a tree with n vertices, and clearly G satisfies $\delta(G) \geq n-1$.

Hence, $\exists$ an isomorphism

$$\varphi : T_1 \longrightarrow G.$$

Let v be the vertex in T adjacent to u, so $v \in T_1$.

Put $\varphi(v) = v' \in G$.

Now ~~$\delta(v)$~~ $d(v') \geq n$, whereas $\varphi(T_1)$ contains only $(n-1)$ vertices other than v' itself. $\therefore v'$ is adjacent to some vertex $u' \in G$, $u' \notin \varphi(T_1)$. Define a new ~~isomorphism~~ mapping $\varphi' : T \to G$ by

$$\varphi'(x) = \varphi(x) \text{ if } x \in T_1$$
$$= u' \text{ if } x = u.$$

Then, φ' is an isomorphism ~~iso~~ into G since it is injective and preserves adjacencies.