Notes for Saturday 20150411

Remark: $K_{m,n}$ is Hamiltonian iff $m=n$ $(n \geq 2)$.

Necessity is obvious.

Sufficiency by induction on n.

Base Case: For $n=2$, $K_{2,2} = C_4$ has a Hamiltonian cycle.

IH: Assume ~~result~~ holds for complete bipartite graphs with $<n$ vertices in each partite set, and consider $G = K_{n,n}$.

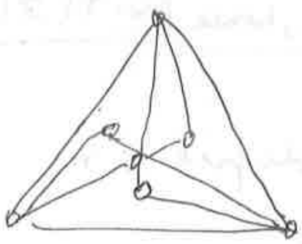
Let $G' = G - \{u_1, v_1\}$ where u_1 and $v_1 \in$ distinct partite sets. Then $G' = K_{n-1,n-1}$, hence has a Hamiltonian cycle, say:

$u_2 \cdots v_k u_2$.

Then in G: $u_2 \cdots v_k u_1 v_1 u_2$ is a Hamiltonian cycle.

Remark: Condition that G be bipartite with equal partite sets and also 2-connected, but not Hamiltonian - see example in book.

Condition of Prop 58 not sufficient:



This graph satisfies the condition, but is not Hamiltonian since all edges incident with the three vertices of degree 2 must be used, but that requires using at least three edges incident with the central vertex.

Proof of Prop. 58: When leaving a component of $G-S$, a Hamiltonian cycle can only pass to S, and the arrivals in S must use distinct vertices in S. Hence, S must have at least as many vertices as $G-S$ has components.

Proof of Prop. 59: NB: the condition $n(G) \geq 3$ is needed, since K_2 satisfies $\delta(G) \geq \frac{n(G)}{2}$, but is not Hamiltonian.

We proceed by contradiction. If there is a non-Hamiltonian graph satisfying the hypotheses, minimum degree is not decreased by adding edges. So we consider a maximal counter-example, i.e. G is non-Hamiltonian, satisfies $\delta(G) \geq \frac{n(G)}{2}$ and G is maximal, i.e. adding an edge creates a Hamiltonian cycle.

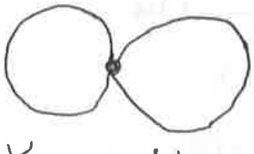
If $uv \notin E(G)$, the maximality of $G \Rightarrow$ G has a Hamiltonian path $v_1 v_2 \ldots v_n$ from $u = v_1$ to $v = v_n$, $\because$ every spanning cycle in $G + uv$ contains this edge. We make a small change in this cycle to avoid using edge uv. If a neighbour of u directly follows a neighbour of v on this path, say $u \leftrightarrow v_{i+1}$ and $v \leftrightarrow v_i$, then $(u, v_{i+1}, v_{i+2}, \ldots, v, v_i, v_{i-1}, \ldots, v_2)$ is a spanning cycle. So we need to show that such a cycle exists, we prove that there is a common index in the sets $S = \{i : u \leftrightarrow v_{i+1}\}$ and $T = \{i : v \leftrightarrow v_i\}$. We have

$$|S \cup T| + |S \cap T| = |S| + |T| = d(u) + d(v) \geq n$$

Neither S nor T contains the index n. Thus $|S \cup T| < n$, hence $|S \cap T| \geq 1$, as reqd.

Example to show why this is best possible:-

$\delta(G) = n-1$, best possible degree $< n$, but is not Hamiltonian.



K_n K_n

Proofs for Lemma 60.1:

We observe that we only used the ~~∃~~ condition $\delta(G) \geq \frac{n}{2}$ to obtain the situation $|S| + |T| \geq n$. Hence, ~~∃~~ the proof would follow if ∃ non-adjacent vertices u, v s.t. $d(u) + d(u) \geq n$. This proves sufficiency, necessity is obvious.

Lemma 60.2: The closure $C(G)$ is well-defined.

Proof: We need to show that the order of iteration doesn't affect the final result.

So suppose $e_1, \ldots, e_r$ and $f_1, \ldots, f_s$ be sequences of edges added in forming $C(G)$, the first yielding G_1 and ~~G_2~~ the second yielding G_2.

If in either sequence non-adjacent vertices acquire degrees summing up to at least $n(G)$, then the edge uv must be added before the sequence ends.

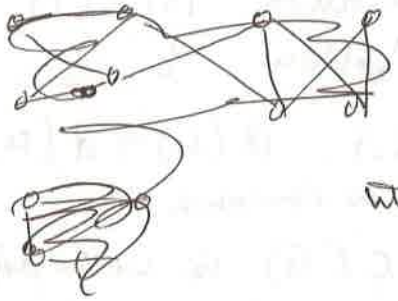
Thus, f_1 being initially addable to G_1 must belong to G_1. Similarly, if $f_1 \ldots f_{i-1} \in E(G_1)$, then f_i becomes addable to G_1 and ∴ belongs to $E(G_1)$.

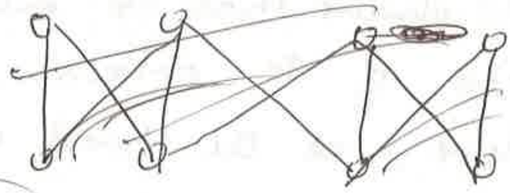
Hence, neither sequence contains a first edge omitted by the other sequence,

i.e. $G_1 \subseteq G_2$ and $G_2 \subseteq G_1$ as required.

Main result follows easily from the two lemmas.

Another example:





~~we see that G is bip~~

There is a complicated example in the text book of a 2-connected bipartite graph with equal partite sets, having no Hamiltonian cycle. But it fails the necessary condition, so obviously it isn't Hamiltonian. Can you find a simpler example.

Another Example: The Petersen Graph is not Hamiltonian.

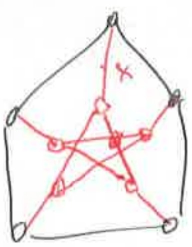
[quite complicated - look up a reference]

→ Note that a spanning cycle must contain an even # of edges joining inner and outer 5-cycles.

→ However, this # can't be two, ∵ in that case the end-points are adjacent, creating a 4-cycle in P $\Rightarrow\Leftarrow$

→ Thus, there must be 4 such edges.

Assume top edge is not chosen:



Assume $u_1 v_1$ is not chosen in C. Then the top two edges in outer cycle must be chosen, neither of the side edges can be chosen and hence the bottom edge must be chosen. The top two edges in inner cycle must be chosen. But this completes a non-spanning cycle which is part of the Hamiltonian cycle - not possible.