MTH303: Homework 2

Assigned on Wednesday 20170125

1. If every vertex of a loopless graph G has degree at least 3, then prove that G has a cycle of even length.
2. Suppose that G is a graph and D is an orientation of G that is strongly connected. Prove that if G has an odd cycle, then D has an odd cycle.
3. Prove or disprove
   1. Every Eulerian bipartite graph has an even number of edges
   2. If D is an orientation of a simple graph with 10 vertices, then the vertices of D cannot have distinct out-degrees
4. Prove that there is an n-vertex tournament with in-degree equal to out-degree at every vertex if and only if n is odd.
5. Create an example of a graph in which every vertex is a king. Can you generalize this into a class of graphs for which every vertex becomes a king?

**Q1 - If every vertex of a loopless graph G has degree at least 3, then prove that G has a cycle of even length.**

**Ans-2** Degree of every vertex >= 3. Let G have n such vertices. This implies Sum of all degrees >= 3n.This implies, 2\*n(e) >= 3n; where n(e) is the number of edges in G n(e) >= 3n/2;

Since n(e) > n-1, we can say that the graph will definitely have at least 1 cycle.

**To prove**: There will be at least 1 cycle of even length.

Consider a vertex v of G belonging to one such cycle. Since it is part of a cycle, it cannot be a cut vertex. It will have at least 3 vertices to which is connected to. Consider any 3 neighbours of v, namely, x,y,z. Removing v from G still keeps the graph connected. (Since v is not a cut vertex). As a result, there is a path from x to y and y to z. Let the length of the minimal path from x to y be k1 and y to z be k2.

**Case 1**: both k1, k2 are odd or both are even. Then there will be a cycle v-x.....y.....z-v of even length

**Case 2**: one of them is odd, say k1. This implies k2 is even. Then the cycle v-y....z-v is of even length.

Hence, there will always be a cycle of even length.

**Q2 - Suppose that G is a graph and D is an orientation of G that is strongly connected. Prove that if G has an odd cycle, then D has an odd cycle.**

**Ans-2** We shall first prove that D has a closed directed walk of odd length. Let v1; v2; vm be a cyclic ordering of vertices which form a cycle in the underlying graph G. Now, (working modulo m), consider each pair of vertices vi, vi+1. If the edge between them is oriented from vi to vi+1, let Wi be the directed path consisting of these two vertices and this one edge. If this edge is oriented from vi+1 to vi , then since D is strongly connected, we may choose Wi to be a directed path from vi+1 to vi. If Wi has even length, then Wi together with the edge (vi+1; vi) is an odd directed cycle and we are done. Thus, we may assume that Wi has odd length. So, in other words, we have constructed for each i a directed walk from vi to vi+1 of odd length. Concatenating these walks (in the obvious manner) yields a closed directed walk of odd length. Now, by minimality, every closed directed walk of odd length contains a directed cycle of odd length.

**Q3 - Prove or disprove**

**a) - Every Eulerian bipartite graph has an even number of edges.**

**Ans-** Eulerian implies it covers all edges and forms a cycle. Since bipartite graph can have no odd cycles. So the number of edges is even.

**b) If D is an orientation of a simple graph with 10 vertices, then the vertices of D cannot have distinct out-degrees.**

**Ans-** Disprove:- Let G be a simple and complete graph with 10 vertices(0,1,2,..9). Then we can have orientation in such a manner that every vertex will have a distinct out degree without causing a cycle. We can have such orientation as **for all i(from 0-9) Vi has outdegree i**….ie, V0 will have outdegree 0, V1 will have 1 and so on.

**Q4 - Prove that there is an n-vertex tournament with in-degree equal to out-degree at every vertex if and only if n is odd.**

**Ans-4** In a tournament, the sum of in-degree and out-degree at every vertex is n-1. If in-degree = out-degree, this implies in-degree = (n-1)/2 = Out-degree. This implies n-1 is even. This implies n must be odd.

**Q5. Create an example of a graph in which every vertex is a king. Can you generalize this into a class of graphs for which every vertex becomes a king?**

**Ans-5 –** Example: G=(V,E); V(G) = {a,b,c}; E(G) = {(a->b), (b->c), (c->a)}

Generalization: An orientation of completely connected (clique) graph G with n(G)=1 or n(G)=3 or n(G)>=5 which is strongly connected**.**