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(Q1) Spherical harmonics are a set of functions used to represent functions on the surface of the sphere. They are a higher dimensional analogy of the fourier series which form a complete basis for the set of periodic functions of a single variable. They are the eigenfunctions of the angular part of the Laplacian in 3 dimensions.

$$Y_l^m(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta)\, e^{im\phi}$$

↳ Legendre polynomials

where $P_l^m = \frac{(-1)^m}{2^l\, l!}(1-x^2)^{m/2}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l$

Properties of spherical harmonics:

i) The spherical harmonics are normalized

$$\int_0^{2\pi}\int_0^{\pi} Y_{l,m}^*(\theta,\phi)\, Y_{l,m}(\theta,\phi)\sin\theta\, d\theta\, d\phi = 1$$

ii) Triple product integral rule (see Q5). Rotational invariance.

iii) They are orthogonal

$$\int_0^{2\pi}\int_0^{\pi} Y_{l_1,m_1}^*(\theta,\phi)\, Y_{l_2,m_2}(\theta,\phi)\sin\theta\, d\theta\, d\phi = \delta_1$$

i.e $= 1$ if $l_1 = l_2$, $m_1 = m_2$
else 0.

iv) Completeness property, ie. any well behaved function of $\theta$ and $\phi$ can be written as

$$f(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} a_{lm}\, Y_l^m(\theta,\phi)$$

Applications → Representation of gravitational & magnetic fields of planetary bodies. They also have direct applicability in computer graphics. Light transport involves qts defined over spherical/hemispherical domains.

Q2) Schrodinger's eqn

$$= -\frac{\hbar^2}{2m}\left(\nabla^2 + V\right)\psi = E\psi$$

and as we are only concerned with spherical harmonics the reduced eqn is

$$\left(\sin\theta \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) + \frac{\partial^2}{\partial\phi^2}\right)Y = -\left(l(l+1)\sin^2\theta\right)Y$$

for $\boxed{a)\ Y_{0,0}(\theta,\phi) = \frac{1}{2\sqrt{\pi}}}$

we get LHS = 0 as constant so diff of const = 0

and RHS = 0 $(l=0) \therefore l(l+1)\sin^2\theta = 0$

hence it satisfies schrodinger eqn.

$$E = \frac{\hbar^2 l(l+1)}{2mr} = 0 \quad \boxed{\text{energy} = 0}$$

for angular momentum $= \frac{h}{2\pi}\sqrt{l(l+1)}$

as $l=0$ $\boxed{\text{angular momentum} = 0}$

b) $Y_{2,-1}(\theta,\phi) = \frac{1}{2}\sqrt{\frac{15}{2\pi}}\, e^{-i\phi}\sin\theta\cos\theta$

LHS = $\frac{\partial}{\partial\theta}Y = \sqrt{\frac{15}{8\pi}}\frac{e^{-i\phi}}{2}\left(\frac{\partial}{\partial\theta}\sin 2\theta\right)$

$$= \sqrt{\frac{15}{8\pi}}\frac{e^{-i\phi}}{\cancel{2}}\left(\cos 2\theta\right)\times\cancel{2}$$

$$\frac{\partial}{\partial\theta}\left(\sqrt{\frac{15}{8\pi}}e^{-i\phi}\left(\cos 2\theta \sin\theta\right)\right)$$

$$= \sqrt{\frac{15}{8\pi}}e^{-i\phi}\left(\cos\theta\cos 2\theta + (-)(\sin\theta)(2\sin 2\theta)\right)$$

$$\frac{\partial^2}{\partial\phi^2}Y = \sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta \frac{\partial}{\partial\phi}\left(\frac{\partial}{\partial\phi}e^{-i\phi}\right)$$

$$= \sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta(-i)(-i)\left(e^{-i\phi}\right)$$

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: RHS ③

LHS =

$$\sqrt{\frac{15}{8\pi}}\, e^{-i\phi}(\sin\theta)\left(\cos\theta\cos 2\theta - \frac{\sin 2\theta}{2} - 2\sin\theta\sin 2\theta\right)$$

$$= \sqrt{\frac{15}{8\pi}}\, e^{-i\phi}(\sin\theta)\left(2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta - \sin\theta\cos\theta\right)$$

$$= \sqrt{\frac{15}{8\pi}}\, e^{-i\phi}\left(\cancel{2\cos\theta}(\cancel{\cos}\ \sin\theta)\times(3\sin\theta\sin 2\theta\right).$$

$$\underline{RHS} = (-\cancel{2})(3)\frac{\sin^2\theta}{\cancel{2}}\sqrt{\frac{15}{8\pi}}\, e^{-i\phi}\sin 2\theta$$

$$= -3\times\sqrt{\frac{15}{8\pi}}\, e^{-i\phi}\left(\sin^2\theta\sin 2\theta\right) = LHS)$$

$\boxed{LHS = RHS}$

$$\text{Energy} = (2)(3)\frac{\hbar^2}{2mr^2} = \frac{3\cancel{6}\hbar^2}{\cancel{2}I} = \boxed{\frac{3\hbar^2}{I}}$$

moment of inertia

$$\text{Angular momentum} = \frac{h}{2\pi}\sqrt{l(l+1)} = \boxed{\frac{h\sqrt{6}}{2\pi}}$$

c) $$Y_{3,3} = \left(-\frac{1}{8}\sqrt{\frac{35}{\pi}}\, e^{3i\phi}\sin^3\theta\right)$$

$$\frac{\partial Y}{\partial\theta} = -\frac{1}{8}\sqrt{\frac{35}{\pi}}\, e^{3i\phi}\left(3\sin^2\theta\cos\theta\right)$$

$$\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) = -\frac{3}{8}\sqrt{\frac{35}{\pi}}\, e^{3i\phi}\left(\frac{\partial}{\partial\theta}\sin^3\theta\cos\theta\right)$$

$$= -\frac{3}{8}\sqrt{\frac{35}{\pi}}\, e^{3i\phi}\left(3\sin^2\cos^2\theta - \sin^4\theta\right)$$

$$= -\frac{3}{8}\sqrt{\frac{35}{\pi}}\, e^{3i\phi}\left(3\sin^2\theta - 4\sin^4\theta\right)$$

$$LHS = -\frac{3}{8}\sqrt{\frac{35}{\pi}}\, e^{3i\phi}\left(3\sin^3\theta - 4\sin^5\theta - 3\sin^3\theta\right)$$

$$= \cancel{9Y} - 12Y\sin^2\theta - \cancel{9Y} = -12\left(-\frac{1}{8}\sqrt{\frac{35}{\pi}}\, e^{i3\phi}\sin^5\theta\right)$$

$$RHS = -(3)(3+1)\sin^2\theta\times\left(-\frac{1}{8}\sqrt{\frac{35}{\pi}}\, e^{-i3\phi}\sin^3\theta\right)\right)$$

$\therefore \boxed{LHS = RHS}$ proven.

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Energy =

Energy $= \dfrac{l(l+1)\hbar^2}{2I} = \dfrac{3 \times \cancel{4}^{\,2} \times \hbar^2}{\cancel{2}I}$

$$\boxed{E = \frac{6\hbar^2}{I}}$$

Angular momentum $= \dfrac{h}{2\pi}\sqrt{l(l+1)} = \dfrac{h}{\cancel{2\pi}}\sqrt{(3)(4)}$

$$\boxed{L = \sqrt{3}\,\frac{h}{\pi}}$$

Q3) Confirm that $Y_{3,+3}(\theta,\phi)$ is normalized to 1

sol) for integrating over the sphere

<u>LHS</u> = $\int_0^{2\pi}\int_0^{\pi} Y^{*}_{3,+3}(\theta,\phi)\, Y_{3,+3}(\theta,\phi)\, \sin\theta\, d\theta\, d\phi$

$Y_{3,+3}(\theta,\phi) = -\frac{1}{8}\sqrt{\frac{35}{\pi}}\, e^{3i\phi}\, \sin^3\theta$

$\therefore Y^{*}_{3,+3}(\theta,\phi) = -\frac{1}{8}\sqrt{\frac{35}{\pi}}\, e^{-3i\phi}\, \sin^3\theta$

$\therefore$ LHS $= \int_0^{2\pi}\int_0^{\pi}\left(-\frac{1}{8}\times\sqrt{\frac{35}{\pi}}\right)^2 \left(e^{0}\right)\sin^7\theta\, d\theta\, d\phi$

$= \frac{1}{64}\times\frac{35}{\pi}\int_0^{\pi}\int_0^{2\pi}\left(\sin^7\theta\, d\theta\right) d\phi$

$= \frac{35}{64\pi}\int_0^{\pi}\sin^7\theta\, d\theta \times (2\pi)$

$= \frac{35}{32}\boxed{\int_0^{\pi}\sin^7\theta\, d\theta}$

$\int_0^{\pi}\sin^7\theta\, d\theta = \int_0^{\pi}\sin^6\theta \times \sin\theta\, d\theta = \int_0^{\pi}\left(1-\cos^2\theta\right)^3 \sin\theta\, d\theta$

· Substituting $\cos\theta = t$

$\therefore -\sin\theta\, d\theta = dt$

Limits = $\cos 0 \rightarrow \cos\pi = 1 \rightarrow -1$.

$\therefore \int_1^{-1}\left(1-t^2\right)^3 dt$

$= \int_1^{-1}(-1)\left(1-2t^2+t^4\right)\left(1-t^2\right) dt$

$= -\int_1^{-1} 1-2t^2+t^4-t^2+2t^4-t^6\, dt$

$= -\left(t-\frac{2t^3}{3}+\frac{t^5}{5}-\frac{t^3}{3}+\frac{2t^5}{5}-\frac{t^7}{7}\right)\Big|_{+1}^{-1}$

$= \frac{t^7}{7}-\frac{3t^5}{5}+t^3-t\Big|_1^{-1}$

$= -\frac{1}{7}+\frac{3}{5}-\cancel{1}+\cancel{1}-\left(\frac{1}{7}-\frac{3}{5}+\cancel{1}-\cancel{1}\right) = \frac{6}{5}-\frac{2}{7} = \frac{32}{35}$

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$\therefore$ LHS $= \frac{35}{32} \times \left(\frac{32}{35}\right) = 1 =$ RHS

hence proved that $Y_{3,+3}$ is normalised

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Q4). Evaluate $\int_0^{\pi}\int_0^{2\pi} Y^*_{3,2}(\theta,\phi)\, Y_{3,-2}(\theta,\phi) \sin\theta\, d\theta\, d\phi$

$$= Y_{3,2}(\theta,\phi) = \frac{1}{4}\sqrt{\frac{105}{2\pi}}\, e^{2i\phi} \sin^2\theta \cos\theta$$

$$\therefore Y^*_{3,2}(\theta,\phi) = \frac{1}{4}\sqrt{\frac{105}{2\pi}}\, e^{-2i\phi} \sin^2\theta \cos\theta$$

$$Y_{3,-2}(\theta,\phi) = \frac{1}{4}\sqrt{\frac{105}{2\pi}}\, e^{-2i\phi} \sin^2\theta \cos\theta$$

$\therefore$ LHS $=$

$$\int_0^{2\pi}\int_0^{\pi} \left(\frac{1}{4}\sqrt{\frac{105}{2\pi}}\right)^2 \left(e^{-2i\phi}\right)^2 \left(\sin^4\theta\right)\cos^2\theta \sin\theta\, d\theta\, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi} \left(\frac{1}{16}\times\frac{105}{2\pi}\right) \times \left(e^{-4i\phi} \times \sin^5\theta \cos^2\theta\, d\theta\, d\phi\right)$$

$$= \frac{105}{32\pi} \times \underbrace{\int_0^{2\pi} \sin^5\theta \cos^2\theta\, d\theta}_{A} \times \underbrace{\int_0^{\pi} e^{-4i\phi}\, d\phi}_{B}$$

$$B = \int_0^{\pi} e^{-4i\phi}\, d\phi = \left.\frac{e^{-4i\phi}}{(-4i)}\right|_0^{\pi}$$

$$= \frac{e^{-4i\pi} - e^{0}}{(-4i)} = \frac{\cos(-4\pi) \cancel{\cos 0} - 1}{-4i} = 0$$

Integral $= \boxed{0.}$

Anything multiplied by $0 = \underline{0}$.

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Q5) a) $\int_0^{\pi}\int_0^{2\pi} Y^*_{2,0}(\theta,\phi)\, Y_{1,0}(\theta\phi)\, Y_{3,0}(\theta,\phi) \sin\theta\, d\theta\, d\phi$

↳ from class lectures we know that all triple integral $\int_0^{\pi}\int_0^{2\pi} Y^*_{l'',m_l''}\; Y_{l',m_l'}\; Y_{l,m_l}$ $= 0$ unless $m_l'' = m_l' + m_l$ & $l''$, $l'$ and $l$ can form a triangle

here $m_l'' = 0 = m_l' = 0 = m_l$
but $l'' = 2$, $l' = 1$ & $l = 3$ cannot form a triangle as $l'' + l' = l$ & in a triangle sum of 2 sides is always lesser than the third $\therefore$ $\boxed{\text{integral} = 0}$

b) $\int_0^{\pi}\int_0^{2\pi} Y^*_{2,0}\; Y_{1,0}\; Y_{+2,0} \sin\theta\, d\theta\, d\phi$.

Here too $m_l'' = m_l' = m_l = 0$ satisfying 1st condition

but again a triangle cannot be formed by $l, l', l''$ $\therefore$ $\boxed{\text{integral} = 0}$ as $l + l'' > l'$

now prooving using integration.

a) $Y^*_{2,0} = \frac{1}{4}\sqrt{\frac{5}{\pi}}\left(3\cos^2\theta - 1\right)$

$Y_{1,0} = \frac{1}{2}\sqrt{\frac{3}{\pi}} \cos\theta$

$Y_{3,0} = \frac{1}{4}\sqrt{\frac{7}{\pi}}\left(5\cos^3\theta - 3\cos\theta\right)$

$\int_0^{2\pi}\int_0^{\pi} \left(\frac{1}{4\times 2\times 4}\sqrt{\frac{5\times3\times7}{\pi^3}}\right)\left(3\cos^2\theta - 1\right)\cos\theta\left(5\cos^3\theta - 3\cos\theta\right) \sin\theta\, d\theta\, d\phi$

↳ we do not have dependence over $\phi$ hence simply multiplying by $2\pi$.

$$\Rightarrow\left(\frac{1}{32}\frac{\sqrt{105}\times\cancel{\pi}}{\cancel{\pi}\sqrt{\pi}}\right)\left(\int_0^1 (3t^2-1)2t(5t^3-3t)\cdot(dt)\right)$$

using $\cos\theta = t$

$\therefore -\sin\theta \, d\theta = dt$

limits $= \cos 0 = 1 \quad : \quad \cos 2\pi = 1$

$$-\int_1^0 15t^6 - 14t^4 + 3t^2 \, dt - \int_0^1 15t^6 - 14t^4 + 3t^2 \, dt$$

$$= -\left(\frac{15t^7}{7} - \frac{14t^5}{5} + t^3\right)\Big|_1^0 - \left(\frac{15t^7}{7} - \frac{14t^5}{5} + t^3\right)\Big|_0^1$$

$$= +\frac{15}{7} - \frac{14}{5} + 1 - \frac{15}{7} + \frac{14}{5} - 1 = 0$$

b) $Y_{2,0}(\theta,\phi) = \frac{1}{4}\sqrt{\frac{5}{\pi}}(3\cos^2\theta - 1) = Y^*_{2,0}$

$Y_{1,0}(\theta,\phi) = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta$

$$\therefore \int_0^{2\pi}\int_0^{\pi}\left(\frac{1}{4\times 2\times 4}\sqrt{\frac{5\times 5\times 3}{\pi^3}}\right)\left(3\cos^2\theta - 1\right)^2 \cos\theta \sin\theta \, d\theta \, d\phi.$$

as no $\phi$ term appears multiplying $\pi$

$$\Rightarrow \frac{5}{32\cancel{\pi}}\sqrt{\frac{3}{\pi}} \times \cancel{\pi} \int_1^1 (3t^2-1)^2 t(-dt)$$

$= \boxed{\text{Integral} = 0}$

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