Mini Project 1
Principles of Numerical Analysis

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Abstract

The task at hand for the mini project was to design a numerical scheme to find points of minimum distance on the curve from a given external point. Such a point can be identified by its property that the derivative at the point of minimum distance is perpendicular to the line segment joining the point of minimum distance and the given external point. The approach used for the problem is to use Gradient Descent to approximate an excellent starting point which is very close to the point of minimum distance, and using that point as the initial guess for the iterative Secant method.

The approach for the problem is motivated from the research paper - A fast natural Newton method

Contents

CHAPTER 1

Problem Statement

Given

- a *simple closed curve* C in $\mathbb{R}^2$ given by C:[0,1]$\rightarrow$ $\mathbb{R}^2$ such that

  C(t) = (X(t),Y(t)) , $0 \leq t \leq 1$, where X,Y : $\mathbb{R} \rightarrow \mathbb{R}$ are infinitely are infinitely differentiable periodic functions with period 1 (for instance, the ellipse given by C(t) = (2cos(2πt), 3sin(2πt)) is one such curve), and
- a point (x0,y0) $\in \mathbb{R}^2$,

  The task was to find the *closest point* (x_c, y_c), that is, find $t_c \in$ [0,1) such that $\sqrt{(X(t_c) - x_o)^2 + (Y(t_c) - y_o)^2}$ =min$_{t\in[0,1)}$ $\sqrt{(X(t) - x_o)^2 + (Y(t) - y_o)^2}$

CHAPTER 2

Methods

2.1 Getting the starting point for iterative approach

The behavior of iterative methods like the Newton's method depends on the initial guess. If you provide a guess that is sufficiently close to a simple root, Newton's method will converge quadratically to the nearby root. However, if your guess is near a critical point of the function, Newton's method will produce a "next guess" that is far away from the initial guess. Further iterations might converge to an arbitrary root, might endlessly cycle in a periodic or aperiodic manner, or might diverge to infinity. So it becomes important to find an initial guess close to the global minima of the distance function from (x_o, y_o).

2.1.1 Gradient Descent

Gradient descent is a first-order iterative optimization algorithm for finding the minimum of a function. To find a minimum of a function using gradient descent, one takes steps proportional to the negative of the gradient (or approximate gradient) of the function at the current point. The loss function used is square of the distance from (x_o, y_o) i.e.

$$dist(t) = (X(t) - x_o)^2 + (Y(t) - y_o)^2 \tag{2.1}$$

Equation depends only on parameter *t*. Multiple random initialization are used for Gradient so that the minimum from all these initializations can be chosen. Multiple initializations ensure that the final value of *t* from gradient descent is not from a local minima, but from the global minima of *dist(t)*.

Update for gradient descent is (*alpha* is the learning rate)

$$t_{n+1} = t_n - \alpha * \nabla dist(t_n) \tag{2.2}$$

2.2 Secant method

In numerical analysis, the secant method is a root-finding algorithm that uses a succession of roots of secant lines to better approximate a root of a function f. The secant method can be thought of as a finite-difference approximation of Newton's method.
As only the first derivative of the curve is given and the calculation of the second derivative using *diff* in MATLAB is computationally expensive, secant method was chosen over newton's method with the value of *t* from gradient descent as the starting value.
Update for secant method is

$$t_{n+1} = t_n - dist(t_n) * (t_n - t_{n-1})/(dist(t_n) - dist(t_{n-1})) \tag{2.3}$$

Multiple initializations are not required for secant method as the initial guess is very close to the root.

CHAPTER 3

Conclusion

This method for ortho projection on the curve is robust as areas where gradient descent and secant methods lack on their own are strengthened by the combination of these algorithms. Gradient descent slows down towards the end (close to the minima) and thus, it requires a large amount of iterations for convergence to the minima. Methods like Newton's method and Secant method converge quicker than gradient descent but are heavily dependent on the initial guess of the value. Therefore, using gradient descent to come up with an initial guess for secant method is a good strategy for optimization.

3.1 Future outlook

Versions of gradient descent like gradient descent with momentum and decaying the learning rate over time can speed up the approximations for gradient descent.

References

[1] Nicolas Le Roux, Andrew Fitzgibbon *A fast natural Newton method*, Microsoft Research.