3

UNIT

Sequence and Series

CONTENTS

PART-1

Definition of Sequence and Series with Examples, Convergence of Sequence and Series.

CONCEPT OUTLINE

Sequence : An ordered set of real number $a_1, a_2, a_3, \ldots, a_n$ is called a sequence and is denoted by (a_n). If the number of terms is unlimited, then the sequence is said to be an infinite sequence and a_n is its general term.

Series : If $u_1, u_2, u_3, \ldots\ldots, u_n, \ldots$ be an infinite sequence of real numbers, then

$$u_1 + u_2 + u_3 + \ldots.. + u_n + \ldots.. \infty$$

is called an infinite series. An infinite series is denoted by Σu_n and the sum of its first n terms is denoted by s_n.

Convergence, Divergence and Oscillation of a Sequence :

If $\lim_{n\to\infty}(a_n) = l$ is finite and unique, the sequence is said to be convergent.

If $\lim_{n\to\infty}(a_n)$ is infinite $(\pm\infty)$, the sequence is said to be divergent.

If $\lim_{n\to\infty}(a_n)$ is not unique, the sequence is said to be oscillatory.

Convergence, Divergence and Oscillation of a Series : Consider the infinite series $\Sigma u_n = u_1 + u_2 + u_3 + \ldots. + u_n + \ldots.. \infty$ and let the sum of the first n terms be $s_n = u_1 + u_2 + u_3 + \ldots. + u_n$

Clearly, s_n is a function of n and as n increases indefinitely three possibilities arises :

i. If s_n tends to a finite limit as $s_n \to \infty$, the series Σu_n is said to be convergent.

ii. If s_n tends to $\pm\infty$ as $n \to \infty$, the series Σu_n is said to be divergent.

iii. If s_n does not tend to a unique limit as $n \to \infty$, then the series Σu_n is said to be oscillatory or non-convergent.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.1. **Examine the following sequence for convergence :**

i. $a_n = \dfrac{n^2 - 2n}{3n^2 + n}$, **ii.** $a_n = 2^n$ **iii.** $a_n = 3 + (-1)^n$.

Answer

i. $\lim\limits_{n\to\infty}\left(\frac{n^2-2n}{3n^2+n}\right)=\lim\limits_{n\to\infty}\frac{1-2/n}{3+1/n}$ = 1/3 which is finite and unique. Hence the sequence (a_n) is convergent.

ii. $\lim\limits_{n\to\infty}(2^n)=\infty$. Hence the sequence (a_n) is divergent.

iii. $\lim\limits_{n\to\infty}[3+(-1)^n]=3+1=4$, when n is even

$= 3-1=2$, when n is odd

i.e., this sequence doesn't have a unique limit. Hence it oscillates.

Que 3.2. **Examine the following series for convergence :**

i. $1+2+3+ \dots + n + \dots \infty$

ii. $5-4-1+5-4-1+5-4-1+\dots\infty$

Answer

i. Here, $s_n = 1+2+3+\dots+n=\frac{n(n+1)}{2}$

$\therefore \lim\limits_{n\to\infty} s_n = \frac{1}{2}\lim\limits_{n\to\infty} n(n+1)\to\infty$ Hence this series is divergent.

ii. Here, $s_n = 5-4-1+5-4-1+5-4-1+\dots n$ terms

$= 0, 5$ or 1

Clearly in this case, s_n does not tend to a unique limit. Hence the series is oscillatory.

Que 3.3. **Test the following series for convergence :**

i. $\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+\dots\infty$

ii. $1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\dots\infty$

Answer

i. We have $u_n = \frac{2n-1}{n(n+1)(n+2)}=\frac{1}{n^2}\frac{2-1/n}{(1+1/n)(1+2/n)}$

Taking $v_n = 1/n^2$, we have

$$\therefore \lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{2-1/n}{(1+1/n)(1+2/n)}=\frac{2-0}{(1+0)(1+0)}$$

= 2, which is finite and non zero.

Hence, both Σu_n and Σv_n converge or diverge together but $\Sigma v_n = \Sigma 1/n^2$ is known to be convergent. Hence Σu_n is also convergent.

ii. Here $u_n = \frac{n^n}{(n+1)n+1}=\frac{1}{n+1}\cdot\left(\frac{n}{n+1}\right)^2$, ignoring the first term.

Taking $v_n = 1/n$, we have

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)=\lim_{n\to\infty}\frac{n}{n+1}\cdot\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n$$

$$=\lim_{n\to\infty}\left(\frac{1}{1+1/n}\right)\cdot\lim_{n\to\infty}\frac{1}{(1+1/n)^n}=1\cdot\frac{1}{e}\neq 0$$

Now since Σv_n is divergent, therefore Σv_n is also divergent.

Que 3.4. **Determine the nature of the series :**

i. $\frac{\sqrt{2}-1}{3^3-1}+\frac{\sqrt{3}-1}{4^3-1}+\frac{\sqrt{4}-1}{5^3-1}+\dots\infty$ **ii.** $\sum\frac{1}{n}\sin\frac{1}{n}$

Answer

i. We have $u_n = \frac{\sqrt{(n+1)}-1}{(n+2)^3-1}=\frac{\sqrt{n}\left[(1+1/n)-1/\sqrt{n}\right]}{n^3\,[(1+2/n)^3-1/n^3]}$

Taking $v_n = 1/n^{5/2}$, we have

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{\sqrt{[(1+1/n)}-1/\sqrt{n}]}{[(1+2/n)^3-1/n^3]}=1\neq 0$$

Since Σv_n is convergent, therefore Σu_n is also convergent.

ii. Here $u_n = \frac{1}{n}\sin\frac{1}{n}=\frac{1}{n}\left[\frac{1}{n}-\frac{1}{3!\,n^3}+\frac{1}{5!\,n^5}-\dots\right]$

$$=\frac{1}{n^2}\left[1-\frac{1}{3!\,n^2}+\frac{1}{5!\,n^4}-\dots\right]$$

Taking $v_n = 1/n^2$, we have

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\left[1-\frac{1}{3!\,n^2}+\frac{1}{5!\,n^4}-\dots\right]=1\neq 0$$

Since Σv_n is convergent, therefore Σu_n is also convergent.

PART-2

Tests for Convergence of Series (Ratio Test, D'Alembert's Test, Raabe's Test).

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.5. **Discuss in detail about D' Alembert's test or ratio test. Also give its limitations.**

Answer

A. **D'Alembert's Test or Ratio Test :**
In a positive term series Σu_n, if

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lambda, \text{ then the series converges for } \lambda < 1 \text{ and diverges for } \lambda > 1.$$

Case I : When, $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lambda < 1$

By definition of a limit, we can find a positive number r (< 1) such that

$\frac{u_{n+1}}{u_n} < r$ for all $n > m$

Leaving out the first m terms, let the series be $u_1 + u_2 + u_3 + ...$

So that $\frac{u_2}{u_1} < r$, $\frac{u_3}{u_2} < r$, $\frac{u_4}{u_3} < r$,.... and so on. Then $u_1 + u_2 + u_3 + ... \infty$

$$= u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}\frac{u_2}{u_1} + \frac{u_4}{u_3}\frac{u_3}{u_2}\frac{u_2}{u_1} + ... \infty\right)$$

$$< u_1 (1 + r + r^2 + r^3 + ... \infty)$$

$= \frac{u_1}{1-r}$, which is finite quantity. Hence Σu_n is convergent. [$\because r < 1$]

Case II : When, $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lambda > 1$

By definition of limit, we can find m, such that $\frac{u_{n+1}}{u_n} \geq 1$ for all $n \geq m$.

Leaving out the first m terms, let the series be

$u_1 + u_2 + u_3 + ...$ so that $\frac{u_2}{u_1} \geq 1$, $\frac{u_3}{u_2} \geq r$, $\frac{u_4}{u_3} \geq 1$,.... and so on.

$$\therefore \quad u_1 + u_2 + u_3 + u_4 + ... + u_n = u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}\frac{u_2}{u_1} + ...\right)$$

$$\geq u_1 (1 + 1 + 1 + ... \text{ to } n \text{ terms}) = nu_1$$

$\therefore \lim_{n\to\infty} (u_1 + u_2 + ... + u_n) \geq \lim_{n\to\infty} (nu_1)$, which tends to infinity. Hence Σu_n is divergent

B. **Limitations of D'Alembert's Test :**

1. Ratio test fails when $\lambda = 1$.
2. This test makes no reference to the magnitude of u_{n+1}/u_n but concerns only with the limit of this ratio.

Que 3.6. **Test for convergence of the following series :**

i. $\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + ... \infty$

ii. $1 + \frac{2}{5}x + \frac{6}{9}x^2 + \frac{14}{17}x^3 + ... + \frac{2^n - 2}{2^n + 1}x^{n-1} + ...(x > 0)$

Answer

i. We have, $u_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}}$ and $u_{n+1} = \frac{x^{2n}}{(n+2)\sqrt{(n+1)}}$

$$\therefore \quad \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{x^{2n-2}}{(n+1)\sqrt{n}} \frac{(n+2)\sqrt{(n+1)}}{x^{2n}}$$

$$= \lim_{n\to\infty} \left[\frac{n+2}{n+1}\left(\frac{n+1}{n}\right)^{1/2}\right] x^{-2}$$

$$= \lim_{n\to\infty} \left[\frac{1+2/n}{1+1/n}\sqrt{(1+1/n)}\right] x^{-2} = x^{-2}$$

Hence Σu_n converges if $x^{-2} > 1$ *i.e.*, for $x^2 < 1$ and diverges for $x^2 > 1$.

If $x^2 = 1$, then, $u_n = \frac{1}{(n+1)\sqrt{n}} = \frac{1}{n^{3/2}} \frac{1}{1+1/n}$

Taking $v_n = \frac{1}{n^{3/2}}$, we get $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1$, a finite quantity.

$\therefore$ Both Σu_n and Σv_n converge or diverge together. But $\Sigma v_n = \sum \frac{1}{n^{3/2}}$ is a convergent series.

$\therefore$ Σu_n is also convergent. Hence the given series converges if $x^2 \leq 1$ and diverges if $x^2 > 1$.

ii. Here, $$\frac{u_n}{u_{n+1}} = \frac{2^n - 2}{2^n + 1} x^{n-1} \frac{2^{n+1}+1}{2^{n+1}-2} \frac{1}{x^n} = \frac{1 - \frac{2}{2^n}}{1 + \frac{1}{2^n}} \frac{2 + \frac{1}{2^n}}{2 - \frac{2}{2^n}} \frac{1}{x}$$

$$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1-0}{1+0} \frac{2+0}{2-0} \frac{1}{x} = \frac{1}{x}$$

Thus by ratio test, Σu_n converges for $x^{-1} > 1$ *i.e.*, for $x < 1$ and diverges for $x > 1$. But it fails for $x = 1$.

When $x = 1$, $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{2^n - 2}{2^n + 1} = \lim_{n\to\infty} \frac{1 - \frac{1}{2^n}}{1 + \frac{1}{2^n}} = 1 \neq 0$

$\therefore$ Σu_n diverges for $x = 1$. Hence the given series converges for $x < 1$ and diverges for $x \geq 1$.

Que 3.7. **Discuss the convergence of the series.**

$$1 + \frac{2!}{2^2} + \frac{3!}{3^3} + \frac{4!}{4^4} + ... \infty$$

Answer

Given series is

$$\Sigma u_n = \sum_{n=1}^{\infty} \frac{n!}{n^n}$$

Here, $\frac{u_n}{u_{n+1}} = \frac{n!}{n^n} \cdot \frac{(n+1)^{n+1}}{(n+1)!} = \frac{(n+1)^n}{n^n} = \left(1 + \frac{1}{n}\right)^n$

$\therefore$ $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$, which is > 1. Hence the given series is convergent.

Que 3.8. **Examine the convergence of the series :**

$$\frac{x}{1+x} + \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} + ... \infty$$

Answer

Here, $u_n = \frac{x^n}{1+x^n}$ and $u_{n+1} = \frac{x^{n+1}}{1+x^{n+1}}$

$\therefore$ $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \left(\frac{x^n}{x^{n+1}} \cdot \frac{1+x^{n+1}}{1+x^n}\right) = \lim_{n\to\infty} \left(\frac{1+x^{n+1}}{x+x^{n+1}}\right)$

$= \frac{1}{x}$, if $x < 1$ $\qquad [\because x^{n+1} \to 0 \text{ and } n \to \infty]$

Also, $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \left(\frac{1 + 1/x^{n+1}}{1 + x/x^{n+1}}\right) = 1$, if $x > 1$.

By ratio test, Σu_n converges for $x < 1$ and fails for $x \geq 1$.

When $x = 1$, $\Sigma u_n = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + ... \infty$, which is divergent.

Hence the given series converges for $x < 1$ and diverges for $x \geq 1$.

Que 3.9. **Explain Raabe's test in brief.**

Answer

In the positive term series Σu_n, if $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = k$, then the series converges for $k > 1$ and diverges for $k < 1$, but the test fails for $k = 1$, When $k > 1$, choose a number p such that $k > p > 1$, and compare Σu_n with the series $\sum \frac{1}{n^p}$ which is convergent since $p > 1$.

$\therefore$ Σu_n will converge, if from and after some term,

$$\frac{u_n}{u_{n+1}} > \frac{(n+1)^p}{n^p} \text{ or } \left(1 + \frac{1}{n}\right)^p$$

or if, $\frac{u_n}{u_{n+1}} > 1 + \frac{p}{n} + \frac{p(p-1)}{2n^2} + ...$

or if, $n\left(\frac{u_n}{u_{n+1}} - 1\right) > p + \frac{p(p-1)}{2n} + ...$

or if, $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) > \lim_{n\to\infty} \left[p + \frac{p(p-1)}{2n} + ...\right]$

i.e., if $k > p$, which is true. Hence, Σu_n is convergent. The other case when $k < 1$ can be proved similarly.

Que 3.10. **Test for convergence of the following the series :**

i. $\sum \frac{4.7 ... (3n+1)}{1.2 ... n} x^{2n}$ $\qquad$ ii. $\sum \frac{(n!)^2}{(2n)!} x^{2n}$

Answer

i. Here, $\frac{u_n}{u_{n+1}} = \frac{4.7 ...(3n+1)}{1.2....n} x^n \div \frac{4.7.....(3n+4)}{1.2.....(n+1)} x^{n+1} = \frac{n+1}{3n+4} \frac{1}{x}$

$= \left[\frac{1 + 1/n}{3 + 4/n}\right] \frac{1}{x}$

$\therefore$ $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{3x}$

Thus by ratio test, the series converges for $\frac{1}{3x} > 1$, *i.e.* for $x < \frac{1}{3}$ and diverges for $x > \frac{1}{3}$. But it fails for $x = \frac{1}{3}$.

$\therefore$ Let us try the Raabe's test

Now, $\dfrac{u_n}{u_{n+1}} = \left(1+\dfrac{1}{n}\right)\left(1+\dfrac{4}{3n}\right)^{-1}$ [Expand by binomial theorem]

$$= \left(1+\frac{1}{n}\right)\left(1-\frac{4}{3n}+\frac{16}{9n^2}-....\right) = 1-\frac{1}{3n}+\frac{4}{9n^2}+....$$

$$\therefore n\left(\frac{u_n}{u_{n+1}}-1\right) = -\frac{1}{3}+\frac{4}{9n}+....$$

$$\therefore \lim_{n\to\infty} n\left(\frac{u_n}{u_n+1}-1\right) = -\frac{1}{3} \text{ which is} < 1$$

Thus by Raabe's test, the series diverges.
Hence the given series converges for $x < (1/3)$ and diverges for $x \geq (1/3)$.

ii. Here, $\dfrac{u_n}{u_{n+1}} = \left(\dfrac{n!}{(n+1)!}\right)^2 \dfrac{[2(n+1)]!}{(2n)!} \dfrac{x^{2n}}{x^{2(n+1)}}$

$$= \frac{(2n+1)(2n+2)}{(n+1)^2}\frac{1}{x^2} = \frac{2(2n+1)}{n+1}\frac{1}{x^2}$$

$$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{2(2+1/n)}{1+1/n}\frac{1}{x^2} = \frac{4}{x^2}$$

Thus by ratio test, the series converges for $x^2 < 4$ diverges for $x^2 < 4$ and diverges for $x^2 > 4$. But fails for $x^2 = 4$.

When $x^2 = 4$, $n\left(\dfrac{u_n}{u_{n+1}}-1\right) = n\left(\dfrac{2n+1}{2n+2}-1\right) = -\dfrac{n}{2n+2}$

$$\therefore \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = -\frac{1}{2} < 1$$

Thus by Raabe's test, the series diverges.
Hence the given series converges for $x^2 < 4$ and diverges for $x^2 \geq 4$.

PART-3

Fourier Series.

CONCEPT OUTLINE

Fourier Series in the Interval $C < x < C + 2\pi$: The Fourier series for the function $f(x)$ in the interval $C < x < C+ 2\pi$ is given by

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Where a_0, a_n and b_n are called Fourier coefficients, and given as

$$a_0 = \frac{1}{\pi}\int_C^{C+2\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\cos nx\, dx$$

$$b_n = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\sin nx\, dx$$

Fourier Series when Interval is Changed : Fourier series in the interval $C < x < C + 2L$ is given as

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}+\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

Where, $a_0 = \dfrac{1}{L}\displaystyle\int_C^{C+2L} f(x)dx$

$$a_n = \frac{1}{L}\int_C^{C+2L} f(x)\cos\frac{n\pi x}{L}dx$$

and $b_n = \dfrac{1}{L}\displaystyle\int_C^{C+2L} f(x)\sin\frac{n\pi x}{L}dx$

Note :

i. If $C = -L$, then interval is $-L < x < L$ and

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

ii. If $f(x)$ is an odd function then,
$a_n = a_0 = 0$.

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

iii. If $f(x)$ is an even function then,
$b_n = 0$ and $a_0 = \dfrac{2}{L}\displaystyle\int_0^L f(x)dx$,

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx,$$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.11. **Find the Fourier series expansion of the function**

$$f(x) = \begin{cases} -1, & \text{for } -\pi < x < -\pi/2 \\ 0, & \text{for } -\pi/2 < x < \pi/2 \\ 1, & \text{for } \pi/2 < x < \pi \end{cases}$$

Hence deduce that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots.$$

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Answer

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2}(-1)dx + \int_{\pi/2}^{\pi}(1)dx\right]$$

$$= \frac{1}{\pi}\left[-\left(-\frac{\pi}{2}+\pi\right)+\left(\pi-\frac{\pi}{2}\right)\right]$$

$$a_0 = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2} -\cos nx\, dx + \int_{\pi/2}^{\pi} \cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[-\left\{\frac{\sin nx}{n}\right\}_{-\pi}^{-\pi/2} + \left\{\frac{\sin nx}{n}\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin n\pi/2}{n} - \frac{\sin n\pi}{n} + \frac{\sin n\pi}{n} - \frac{\sin n\pi/2}{n}\right]$$

$$a_n = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2} -\sin nx\, dx + \int_{\pi/2}^{\pi} \sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{\cos nx}{n}\right\}_{-\pi}^{-\pi/2} - \left\{\frac{\cos nx}{n}\right\}_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi/2}{n} - \frac{\cos n\pi}{n} - \frac{\cos n\pi}{n} + \frac{\cos n\pi/2}{n}\right]$$

$$b_n = \frac{2}{\pi n}\left[\cos\frac{n\pi}{2} - \cos n\pi\right]$$

Hence required series is,

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(\cos\frac{n\pi}{2} - \cos n\pi\right)\sin nx$$

Putting $x = \pi/2$ in the above series,

$$[f(x)]_{x=\pi/2} = \sum \frac{2}{\pi n}\left(\cos\frac{n\pi}{2} - \cos n\pi\right)\sin\frac{n\pi}{2}$$

$$\frac{0+1}{2} = \frac{2}{\pi}\sum\frac{1}{n}\left(\cos\frac{n\pi}{2} - \cos n\pi\right)\sin\frac{n\pi}{2}$$

Putting $n = 1, 2, 3, 4, \ldots\ldots$

$$\frac{\pi}{4} = \frac{1}{1}\left(\cos\frac{\pi}{2} - \cos\pi\right)\sin\frac{\pi}{2} + 0$$

$$+\frac{1}{3}(\cos\frac{3\pi}{2} - \cos 3\pi)\sin\frac{3\pi}{2} + 0$$

$$+\frac{1}{5}\left(\cos\frac{5\pi}{2} - \cos 5\pi\right)\sin\frac{5\pi}{2} + 0 + \ldots.$$

$$\frac{\pi}{4} = 1 + \frac{1}{3}(-1) + \frac{1}{5}(1) + \frac{1}{7}(-1) + \ldots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots.$$

Que 3.12. **Find the Fourier series to represent the function $f(x)$ given by**

$$f(x) = \begin{cases} \pi x & ; \ 0 \le x \le 1 \\ \pi(2-x) & ; \ 1 \le x \le 2 \end{cases}$$

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Answer

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

Then $\quad a_0 = \int_0^2 f(x)dx = \int_0^1 \pi x\, dx + \int_1^2 \pi(2-x)\, dx$

$$a_0 = \pi\left[\frac{x^2}{2}\right]_0^1 + \pi\left[2x - \frac{x^2}{2}\right]_1^2 = \pi\left(\frac{1}{2}\right) + \pi\left[(4-2) - \left(2 - \frac{1}{2}\right)\right]$$

$$a_0 = \pi$$

$$a_n = \int_0^2 f(x)\cos n\pi x\, dx$$

$$= \int_0^1 \pi x \cos n\pi x\, dx + \int_1^2 \pi(2-x)\cos n\pi x\, dx$$

$$a_n = \left[\pi x \frac{\sin n\pi x}{n\pi} - \pi\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_0^1 + \left[\pi(2-x)\frac{\sin n\pi x}{n\pi} - (-\pi)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$= \left[\frac{\cos n\pi}{n^2\pi} \frac{-1}{n^2\pi}\right] + \left[\frac{-\cos 2n\pi}{n^2\pi} + \frac{\cos n\pi}{n^2\pi}\right]$$

$$= \frac{2}{n^2\pi}[\cos n\pi - 1] = \frac{2}{n^2\pi}[(-1)^n - 1]$$

$$= \begin{cases} 0, & \text{if } n \text{ is even} \\ -\dfrac{4}{\pi n^2}, & \text{if } n \text{ is odd} \end{cases}$$

$$b_n = \int_0^2 f(x)\sin n\pi x\, dx$$

$$= \int_0^1 \pi x \sin n\pi x\, dx + \int_1^2 \pi(2-x)\sin n\pi x\, dx$$

$$= \left[\pi x\left(-\frac{\cos n\pi x}{n\pi}\right) - \pi\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_0^1 + \left[\pi(2-x)\left(\frac{-\cos n\pi x}{n\pi}\right) - (-\pi)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$= \left[-\frac{\cos n\pi}{n}\right] + \left[\frac{\cos n\pi}{n}\right] = 0$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \ldots\ldots\ldots\right)$$

Que 3.13. **Express $f(x) = |x|$; $-\pi < x < \pi$ as Fourier series.**

AKTU 2013-14, Marks 10

Answer

Since $f(-x) = |-x| = |x| = f(x)$

$\therefore$ $f(x)$ is an even function and hence $b_n = 0$

Let $f(x) = |x| = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos nx$

Where $a_0 = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x)\, dx = \dfrac{2}{\pi}\int_0^{\pi}|x|\, dx = \dfrac{2}{\pi}\int_0^{\pi} x\, dx = \dfrac{2}{\pi}\left[\dfrac{x^2}{2}\right]_0^{\pi} = \pi$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx = \frac{2}{\pi}\int_0^{\pi}|x|\cos nx\, dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\, dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi n^2}[(-1)^n - 1]$$

$$a_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{-4}{\pi n^2}, & \text{if } n \text{ is odd} \end{cases}$$

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\ldots\right)$$

Que 3.14. **Expand $f(x) = x \sin x$ as a Fourier series in $0 < x < 2\pi$.**

AKTU 2014-15, Marks 10

Answer

$f(x) = x\sin x$; $0 < x < 2\pi$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\, dx = \frac{1}{\pi}[x(-\cos x) + \sin x]_0^{2\pi} = \frac{1}{\pi}[-2\pi]$$

$$a_0 = -2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos nx\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x[\sin(1+n)x + \sin(1-n)x]\, dx$$

$$= \frac{1}{2\pi}\left[-x\frac{\cos(n+1)x}{n+1} + \frac{\sin(n+1)x}{(n+1)^2} + \frac{x\cos(n-1)x}{n-1} - \frac{\sin(n-1)x}{(n-1)^2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{-2\pi}{n+1} + \frac{2\pi}{n-1}\right] = \frac{1}{n-1} - \frac{1}{n+1}$$

$$a_n = \frac{2}{n^2-1}, n \neq 1$$

When $n = 1$, we have

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos x\, dx = \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x\, dx$$

$$= \frac{1}{2\pi}\left[x\left\{\frac{-\cos 2x}{2}\right\} + \frac{\sin 2x}{4}\right]_0^{2\pi} = \frac{1}{2\pi}\left[\frac{-2\pi}{2}\right]$$

$$a_1 = -\frac{1}{2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin nx \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x[\cos(1-n)x - \cos(1+n)x]dx$$

$$= \frac{1}{2\pi}\left[x\frac{\sin(n-1)x}{n-1} + \frac{\cos(n-1)x}{(n-1)^2} - \frac{x\sin(n+1)x}{(n+1)} - \frac{\cos(n+1)x}{(n+1)^2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right]$$

$$b_n = 0$$

When $n = 1$, we have

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin x \, dx = \frac{1}{2\pi}\int_0^{2\pi} x(1-\cos 2x)\,dx$$

$$= \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - 1.\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi(2\pi) - \frac{4\pi^2}{2} - \frac{1}{4} + \frac{1}{4}\right] = \frac{1}{2\pi}(2\pi^2) = \pi$$

$$\therefore \quad f(x) = \frac{a_0}{2} + a_1\cos x + b_1 \sin x + \sum_{n=2}^{\infty} a_n \cos nx + \sum_{n=2}^{\infty} b_n \sin nx$$

$$= -1 - \frac{1}{2}\cos x + \pi \sin x + \sum_{n=2}^{\infty} \frac{2}{n^2-1}\cos nx$$

Que 3.15. **Find the Fourier series to represent the function $f(x)$ given by**

$$f(x) = \begin{cases} -K & \text{for } -\pi < x < 0 \\ K & \text{for } 0 < x < \pi \end{cases}$$

Hence show that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... = \frac{\pi}{4}$. **AKTU 2015-16, Marks 10**

Answer

$$f(x) = \begin{cases} -K & \text{for } -\pi < x < 0 \\ K & \text{for } 0 < x < \pi \end{cases}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{0}(-K)dx + \frac{1}{\pi}\int_0^{\pi} K\,dx$$

$$a_0 = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} -K\cos nx\,dx + \frac{1}{\pi}\int_0^{\pi} K\cos nx\,dx$$

$$a_n = -\frac{K}{\pi}\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + \frac{K}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi}$$

$$a_n = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-K\sin nx)dx + \frac{1}{\pi}\int_0^{\pi} K\sin nx\,dx$$

$$= \frac{1}{\pi}\left[-K\left(\frac{-\cos nx}{n}\right)\right]_{-\pi}^{0} + \frac{1}{\pi}\left[-K\left(\frac{-\cos nx}{n}\right)\right]_{-\pi}^{0}$$

$$= \frac{K}{\pi}\left[\frac{1}{n} - \frac{(-1)^n}{n} - \frac{(-1)^n}{n} + \frac{1}{n}\right]$$

$$b_n = \frac{K}{\pi}\left[\frac{2}{n} - \frac{2(-1)^n}{n}\right]$$

$$= \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{4K}{n\pi}, & \text{if } n \text{ is odd} \end{cases}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$= b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + ...$$

$$f(x) = \frac{4K}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + ...\right]$$

Now putting $x = \frac{\pi}{2}$

$$f\left(\frac{\pi}{2}\right) = K = \frac{4K}{\pi}\left[1 + \frac{1}{3}(-1) + \frac{1}{5}(1) + \frac{1}{7}(-1) + ...\right]$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...$$

Que 3.16. **Find the Fourier series expansion of the following function of period 2π, defined as**

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 \le x < \pi \end{cases}$$

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...$$

AKTU 2012-13, Marks 10

Answer

Same as Q. 3.15, Page 3–15F, Unit-3, (Putting $K = 1$).

Que 3.17. **Find the Fourier series of**

$$f(x) = x^3 \text{ in } (-\pi, \pi)$$

AKTU 2015-16, Marks 05

Answer

$f(x) = x^3$ is an odd function.

$\therefore$ $a_0 = 0$ and $a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^\pi f(x) \sin nx\, dx = \frac{2}{\pi}\int_0^\pi x^3 \sin nx\, dx$$

$$\left[\int uv = uv_1 - u'v_2 + u''v_3 - u'''v_4 + ...\right]$$

$$= \frac{2}{\pi}\left[x^3\left(\frac{-\cos nx}{n}\right) - 3x^2\left(-\frac{\sin nx}{n^2}\right) + 6x\left(\frac{\cos nx}{n^3}\right) - 6\left(\frac{\sin nx}{n^4}\right)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[-\frac{\pi^3 \cos n\pi}{n} + \frac{6\pi \cos n\pi}{n^3}\right] = 2(-1)^n\left[-\frac{\pi^2}{n} + \frac{6}{n^3}\right]$$

$$\therefore \quad f(x) = x^3 = 2\left[-\left(-\frac{\pi^2}{1} + \frac{6}{1^3}\right)\sin x + \left(-\frac{\pi^2}{2} + \frac{6}{2^3}\right)\sin 2x - \left(-\frac{\pi^2}{3} + \frac{6}{3^3}\right)\sin 3x ...\right]$$

Que 3.18. **Obtain Fourier series for the function**

$$f(x) = \begin{cases} x, & -\pi < x < 0 \\ -x, & 0 < x < \pi \end{cases}$$ **and hence show that** $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8}$

AKTU 2017-18, Marks 07

Answer

Let the Fourier series be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad ...(3.18.1)$$

Now, $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} x\, dx + \int_0^\pi -x\, dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{x^2}{2}\right]_{-\pi}^0 - \left[\frac{x^2}{2}\right]_0^\pi\right\} = \frac{1}{\pi}\left\{0 - \frac{\pi^2}{2} - \frac{\pi^2}{2}\right\} = -\pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} x \cos nx\, dx + \int_0^\pi -x \cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left\{\left[\frac{x \sin nx}{n}\right]_{-\pi}^0 - \int_{-\pi}^0 1\frac{\sin nx}{n}dx + \left[\frac{-x \sin nx}{n}\right]_0^\pi - \int_0^\pi (-1)\frac{\sin nx}{n}dx\right\}$$

$$= \frac{1}{\pi}\left\{\frac{1}{n^2}(\cos nx)_{-\pi}^0 - \frac{1}{n^2}(\cos nx)_0^\pi\right\}$$

$$= \frac{1}{\pi}\left[\left\{\frac{1-(-1)^n}{n^2}\right\} - \left\{\frac{(-1)^n - 1}{n^2}\right\}\right] = \frac{1}{\pi}\left[\frac{2(1-(-1)^n)}{n^2}\right]$$

$$= \frac{2}{\pi n^2}\{1-(-1)^n\}$$

$$= \begin{cases} 0, & \text{if } n \text{ is even.} \\ \dfrac{4}{\pi n^2}, & \text{if } n \text{ is odd.} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} x \sin nx\, dx + \int_0^\pi (-x) \sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\left\{x\left(\frac{-\cos nx}{n}\right)\right\}_{-\pi}^0 - \int_{-\pi}^0 1\left(\frac{-\cos nx}{n}\right)dx + \left\{(-x)\left(\frac{-\cos nx}{n}\right)\right\}_0^\pi - \int_0^\pi (-1)\left(\frac{-\cos nx}{n}\right)dx\right]$$

$$= \frac{1}{\pi}\left[\left\{0-\frac{\pi}{n}\cos n\pi\right\}+\frac{1}{n}\left[\frac{\sin nx}{n}\right]_{-\pi}^{0}+\left\{\frac{\pi(-1)^n}{n}-0\right\}-\frac{1}{n}\left[\frac{\sin nx}{n}\right]_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}(-1)^n+\frac{1}{n}\pi(-1)^n\right]$$

= 0, whatever be the value of n.

Therefore, the Fourier series is

$$f(x) = \frac{-\pi}{2}+\frac{4}{\pi}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+...\right] \quad ...(3.18.2)$$

Since the function $f(x)$ is discontinuous at $x = 0$, by Dirichlet's condition

$$f(0) = \frac{1}{2}[\text{LHL + RHL}] = (1/2)[f(0-0)+f(0+0)] = 0$$

Put $x = 0$ in eq. (3.18.2), we get

$$0 = \frac{-\pi}{2}+\frac{4}{\pi}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+...\right]$$

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+....... = \frac{\pi^2}{8}$$

PART-4

Half Range Fourier Sine and Cosine Series.

CONCEPT OUTLINE

Half Range Series : Half series is found when a periodic function is expanded in half range of its period *i.e.*, to expand $f(x)$ in range $(0, L)$ having a period of $2L$.

A function $f(x)$ defined in the interval $(0, L)$ has two half range series that are called Fourier cosine and Fourier sine series.

Half Range Cosine Series : The half range cosine series is given as

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

Where
$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

Half Range Sine Series : The half range sine series is given as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

Where,
$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.19. **Expand $f(x) = x$ as a half range**

i. Sine series in $0 < x < 2$

ii. Cosine series in $0 < x < 2$.

Answer

i. Let
$$x = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2} \quad ...(3.19.1)$$

Where,
$$b_n = \int_0^2 x\sin\frac{n\pi x}{2}\,dx$$

$$= \left\{x\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right)\right\}_0^2 - \int_0^2\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right)dx$$

$$= -\frac{4}{n\pi}\cos n\pi = -\frac{4}{n\pi}(-1)^n$$

Hence from eq. (3.19.1),

$$x = -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{2}$$

ii. Let
$$x = \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{2} \quad ...(3.19.2)$$

Where,
$$a_0 = \int_0^2 x\,dx = \left(\frac{x^2}{2}\right)_0^2 = 2$$

and
$$a_n = \int_0^2 x\cos\frac{n\pi x}{2}\,dx = \left(x\,\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right)_0^2 - \int_0^2 \frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\,dx$$

$$= -\frac{2}{n\pi}\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) = \frac{4}{n^2\pi^2}(\cos n\pi - 1)$$

Hence,
$$x = 1+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(\cos n\pi - 1)}{n^2}\cos\frac{n\pi x}{2}$$

Que 3.20. **Find the half range cosine series expansion of**

$$f(x) = x - x^2, \quad 0 < x < 1$$

AKTU 2011-12, 2012-13; Marks 05

Answer

$f(x) = x - x^2, \quad 0 < x < 1$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{1}$$

$$a_0 = 2\int_0^1 f(x)dx = 2\int_0^1 (x - x^2)dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 2\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{2}{6} = \frac{1}{3}$$

$$a_n = \frac{2}{1}\int_0^1 f(x)\cos\frac{n\pi x}{1}dx = 2\int_0^1 (x - x^2)\cos n\pi x\, dx$$

$$= 2\left[(x - x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (1 - 2x)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right) + (-2)\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= 2\left[(-1)\frac{\cos n\pi}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right] = 2\left[\frac{(-1)^{n+1}}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right]$$

$$f(x) = \frac{1}{6} + \sum_{n=1}^{\infty}\frac{2}{n^2\pi^2}[(-1)^{n+1} - 1]\cos n\pi x$$

Que 3.21. **Find the Fourier half range sine series for**

$$f(x) = (x + 1) \text{ for } 0 < x < \pi.$$

AKTU 2013-14, Marks 05

Answer

$$f(x) = x + 1$$

$$x + 1 = \sum_{n=1}^{\infty} b_n \sin nx$$

Where, $\quad b_n = \frac{2}{\pi}\int_0^{\pi}(x + 1)\sin nx\, dx$

$$= \frac{2}{\pi}\left[\left[(x + 1)\left(\frac{-\cos nx}{n}\right)\right]_0^{\pi} - \int_0^{\pi}(1)\left(-\frac{\cos nx}{n}\right)dx\right]$$

$$= \frac{2}{\pi}\left[(\pi + 1)\left(\frac{-\cos n\pi}{n}\right) + \frac{\cos 0°}{n}\right] + \left[\frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\left[(\pi + 1)\left(-(-1)^n\right) + 1\right]}{n} + \left[\frac{\sin n\pi - \sin 0°}{n^2}\right]\right]$$

$$= \frac{2}{\pi n}\,[1 - (1 + \pi)(-1)^n]$$

$$= \begin{cases} -\dfrac{2}{n}; \text{ If } n \text{ is even} \\ \dfrac{2}{\pi n}(2 + \pi); \text{ If } n \text{ is odd} \end{cases}$$

Hence Fourier sine series is

$$\therefore \quad f(x) = x + 1 = \frac{2(2 + \pi)}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \text{.........}\right]$$

$$- 2\left[\frac{\sin 2x}{2} + \frac{\sin 4x}{4} + \frac{\sin 6x}{6} + \text{...}\right]$$

Que 3.22. **Find the half range sine expansion of**

$$f(t) = \begin{cases} t & ; \; 0 < t < 2 \\ 4 - t & ; \; 2 < t < 4 \end{cases}$$

AKTU 2014-15, Marks 05

Answer

$$b_n = \frac{1}{2}\int_0^4 f(t)\sin\frac{n\pi t}{4}\,dt$$

$$= \frac{1}{2}\left[\int_0^2 t\sin\left(\frac{n\pi t}{4}\right)dt + \int_2^4 (4 - t)\sin\left(\frac{n\pi t}{4}\right)dt\right]$$

$$= \frac{1}{2}\left[\left\{t\left(-\frac{4}{n\pi}\cos\frac{n\pi t}{4}\right) + \frac{16}{n^2\pi^2}\sin\left(\frac{n\pi t}{4}\right)\right\}_0^2\right.$$

$$\left. + \left\{(4 - t)\left(-\frac{4}{n\pi}\cos\frac{n\pi t}{4}\right) - \frac{16}{n^2\pi^2}\sin\frac{n\pi t}{4}\right\}_2^4\right]$$

$$= \frac{1}{2}\left[-\frac{8}{n\pi}\cos\frac{n\pi}{2} + \frac{16}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) + \frac{8}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{16}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right]$$

$$= \frac{16}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

Hence the Fourier series is,

$$f(x) = \sum_{n=0}^{\infty} \frac{16}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi t}{4}\right)$$

Que 3.23. **Obtain the Fourier expansion of $f(x) = x \sin x$ as cosine series in $(0, \pi)$ and hence show that**

$$\frac{1}{1\times 3} - \frac{1}{3\times 5} + \frac{1}{5\times 7} - \ldots\ldots\ldots = \left(\frac{\pi - 2}{4}\right)$$

AKTU 2016-17, Marks 07

Answer

Let the Fourier series be

$$f(x) = x\sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Now,
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x \, dx = \frac{2}{\pi}\int_0^{\pi} x\sin x \, dx$$

[$\because$ $x \sin x$ is an even function]

Using $\int uv\,dx = uv_1 - u'v_2 + \ldots$, we have

$$= \frac{2}{\pi}[x(-\cos x) + (\sin x)]_0^{\pi}$$

$$= \frac{2}{\pi}(-\pi\cos\pi) = 2$$

And,
$$a_n = \frac{2}{\pi}\int_0^{\pi} x\sin x \cos nx \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x(2\cos nx \sin x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x[\sin(n+1)x - \sin(n-1)x]\,dx$$

[$\because$ $2 \cos A \sin B = \sin (A + B) - \sin (A - B)$]

$$= \frac{1}{\pi}\left[x\left\{\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - \left\{\frac{-\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi\left\{\frac{-\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right\} - \left\{\frac{-\sin(n+1)\pi}{(n+1)^2} + \frac{\sin(n-1)\pi}{(n-1)^2}\right\} - 0\right]$$

$$= \frac{1}{\pi}\left[\pi\left\{\frac{-\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right\} - \{0-0\}\right]$$

$$= \frac{1}{\pi}\left[\pi\left\{\frac{-\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right\}\right], n \neq 1$$

When n is odd, $n \neq 1$, $(n - 1)$ and $(n + 1)$ are even.

$$\therefore \quad a_n = \frac{1}{\pi}\left[\pi\left\{-\frac{1}{n+1} + \frac{1}{n-1}\right\}\right] = \frac{1}{n-1} - \frac{1}{n+1} = \frac{2}{n^2-1}$$

When n is even, $(n - 1)$ and $(n + 1)$ are odd, therefore $\cos (n + 1) \pi$ and $\cos (n - 1) \pi$ are -1.

$$\therefore \quad a_n = -\frac{1}{n-1} + \frac{1}{n+1} = \frac{-2}{n^2-1}$$

When $n = 1$,
$$a_1 = \frac{2}{\pi}\int_0^{\pi} x\sin x \cos x\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - 1.\left(\frac{-\sin 2x}{4}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left\{\frac{-\pi\cos 2\pi}{2}\right\} = -\frac{1}{2}$$

Now the Fourier series is,

$$f(x) = x\sin x = 1 - \frac{1}{2}\cos x - 2\left\{\frac{\cos 2x}{2^2-1} - \frac{\cos 3x}{3^2-1} + \frac{\cos 4x}{4^2-1} - \frac{\cos 5x}{5^2-1}\ldots\ldots\right\} \quad \ldots(3.23.1)$$

Putting $x = \frac{\pi}{2}$ in eq. (3.23.1), we get

$$\frac{\pi}{2}\sin\frac{\pi}{2} = 1 - 2\left(\frac{-1}{2^2-1} + \frac{1}{4^2-1} - \frac{1}{6^2-1} + \ldots\right)$$

$$\frac{\pi}{2} - 1 = 2\left(\frac{1}{3} - \frac{1}{15} + \frac{1}{35} - \ldots\right)$$

$$\frac{\pi - 2}{4} = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7}\ldots\ldots$$

Que 3.24. **Obtain half range cosine series for e^x the function**

$$f(t) = \begin{cases} 2t & , 0 < t < 1 \\ 2(2-t) & , 1 < t < 2 \end{cases}$$

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Answer

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{l}$$

$$a_0 = \frac{2}{l}\int_0^l f(t)\,dt = \frac{2}{2}\left[\int_0^1 2t\,dt + \int_1^2 2(2-t)dt\right]$$

$$= \left[\left(\frac{2t^2}{2}\right)_0^1 + \left(4t - t^2\right)_1^2\right]$$

$$a_0 = [1 + 1] = 2$$

$$a_n = \frac{2}{2}\int_0^l f(t)\cos\frac{n\pi t}{l}\,dt$$

$$= \frac{2}{2}\left[\int_0^1 2t\cos\frac{n\pi t}{2}\,dt + \int_1^2 2(2-t)\cos\frac{n\pi t}{2}\,dt\right]$$

Using integration by parts

$$= \frac{2}{2}\left[\left(2t\frac{2}{n\pi}\sin\frac{n\pi t}{2} + 2\frac{2^2}{n^2\pi^2}\cos\frac{n\pi t}{2}\right)_0^1 + \left(2(2-t)\frac{2}{n\pi}\sin\frac{n\pi t}{2} - 2\frac{2^2}{n^2\pi^2}\cos\frac{n\pi t}{2}\right)_1^2\right]$$

$$= \left[\left(\frac{4}{n\pi}\sin\frac{n\pi}{2} + \frac{8}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)\right) + \left(-\frac{8}{n^2\pi^2}\cos n\pi - \frac{4}{n\pi}\sin\frac{n\pi}{2} + \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2}\right)\right]$$

$$= \frac{16}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{8}{n^2\pi^2} - \frac{8}{n^2\pi^2}\cos n\pi$$

$$= \frac{8}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - 1 - \cos n\pi\right]$$

When n is odd, $\cos\frac{n\pi}{2} = 0$ and $\cos n\pi = -1$

$\therefore$ $\quad a_n = 0 \Rightarrow a_1 = a_3 = a_5 \,..... = 0$

When n is even, $\quad a_2 = \frac{8}{2^2\pi^2}\left[2\cos\frac{2\pi}{2} - 1 - \cos 2\pi\right] = -\frac{8}{\pi^2}$

$$a_4 = \frac{8}{4^2\pi^2}\left[2\cos\frac{4\pi}{2} - 1 - \cos 4\pi\right] = 0$$

$$a_6 = \frac{8}{6^2\pi^2}\left[2\cos\frac{6\pi}{2} - 1 - \cos 6\pi\right] = -\frac{8}{9\pi^2}$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\cos\frac{n\pi t}{l}$$

$$= 1 + \left(-\frac{8}{\pi^2} + 0 - \frac{8}{9\pi^2}\right) = 1 - \frac{8}{\pi^2}\left(1 + \frac{1}{9} + ...\right)$$

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