PART-1

Linear Differential Equations of n^{th} Order with Constant Coefficients.

CONCEPT OUTLINE

Differential Equation : An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a differential equation.

For example, $\log\left(\dfrac{dy}{dx}\right) = ax + by$

$$(1 - x^2)(1 - y)dx = xy(1 + y)dy$$

$$\frac{dy}{dx} = \sec(x + y)$$

Order of a Differential Equation : The order of a differential equation is the order of the highest derivative involved in a differential equation.

For example, $\dfrac{d^4x}{dt^4} + \dfrac{d^2x}{dt^2} + \left(\dfrac{dx}{dt}\right)^5 = e^5$ is of 4^{th} order.

Degree of a Differential Equation : The degree of a differential equation is the power of the highest derivative which occurs in it, after the differential equation has been made free from radicals and fractions as far as the derivatives are concerned.

For example, $\dfrac{d^4x}{dt^4} + \dfrac{d^2x}{dt^2} + \left(\dfrac{dx}{dt}\right)^5 = e^t$, is of first degree.

Linear Differential Equation : A linear differential equation is an equation in which the dependent variable and its derivatives appear only in the first degree.

For example, $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 9y = 4x^2 - 7$

The above equation is called a LDE (linear differential equation) with constant coefficients.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.1. **Write the procedure to find complementary function.**

Answer

Following are the steps to find complementary function :

Step I : Put the RHS of the given equation equals to zero. *i.e.*, $f(D)\,y = 0$

Step II : Replace $\frac{d}{dx} \approx D$, $\frac{d^2}{dx^2} \approx D^2$ and so on *i.e.*, convert the given equation in symbolic form.

Step III : Make an auxiliary equation replacing D by m.

e.g., $(D^2 + 4D + 7) = 0$ then its auxiliary equation is

$$m^2 + 4m + 7 = 0$$

Step IV : Find the roots of auxiliary equation (AE), CF will depend upon the type of root.

Case I : If all roots of the AE are real and distinct say $m_1, m_2, \ldots, m_n$

Then, $$\text{CF} = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \ldots + C_n e^{m_n x}$$

Where $C_1, C_2, \ldots, C_n$ are constants.

Case II : If roots of AE are real and equal say

$$m_1 = m_2 \ldots = m_n = m \text{ (say).}$$

Then, $\text{CF} = (C_1 + C_2 x + C_3 x^2 + \ldots + C_n x^n)\, e^{mx}$

If some roots are equal, others are distinct say

$$m_1 = m_2 = m_3 = m$$

and $m_4, m_5, \ldots, m_n$

Then, $\text{CF} = (C_1 + C_2 x + C_3 x^2)\, e^{mx} + C_4 e^{m_4 x} + C_5 e^{m_5 x} + \ldots + C_n e^{m_n x}$

Case III : If the roots of AE are complex say

$$m = \alpha \pm i\beta, \text{ then}$$

$$\text{CF} = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$$

or $$\text{CF} = C_1 e^{\alpha x} e^{i\beta x} + C_2 e^{\alpha x} e^{-i\beta x}$$

$$\text{CF} = C_1 e^{\alpha x} (\cos \beta x + i \sin \beta x) + C_2 e^{\alpha x} (\cos \beta x - i \sin \beta x)$$

$$\text{CF} = e^{\alpha x}(C_1 + C_2) \cos \beta x + i e^{\alpha x}(C_1 - C_2) \sin \beta x$$

$$\text{CF} = e^{\alpha x}[A \cos \beta x + iB \sin \beta x]$$

or changing the constants

$$\text{CF} = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$$

This expression may be written as

$$\text{CF} = C_1 e^{\alpha x}(\cos \beta x + C_2)$$

or $$\text{CF} = C_1 e^{\alpha x}(\sin \beta x + C_2)$$

Case IV : If the AE has irrational roots say

$$= \alpha \pm \sqrt{\beta}, \text{ where } \beta \text{ is positive}$$

Then, $$\text{CF} = e^{\alpha x}(C_1 \cosh \sqrt{\beta}\, x + C_2 \sinh \sqrt{\beta}\, x)$$

Que 1.2. **Explain the method to find out the particular integral when the function in RHS is e^{ax}, $f(a) \neq 0$ and e^{ax}, $f(a) = 0$.**

Answer

A. Case I :

When RHS function is e^{ax}, $f(a) \neq 0$,

Then, $$\text{PI} = \frac{1}{f(D)} e^{ax}$$

Now replace D by a so PI will be,

$$= \frac{e^{ax}}{f(a)}$$

If $f(a) = 0$, it will be a case of failure.

B. Case II :

When RHS of function is e^{ax}, $f(a) = 0$,

Then, $$\text{PI} = \frac{e^{ax}}{f(D)}$$

Now, $$\text{PI} = \frac{x e^{ax}}{f'(D)}$$

Multiply with x and differentiate denominator once.

Again if, $f'(a) = 0$ then, continue to multiply with x and differentiate denominator,

$$\text{PI} = x^n \frac{e^{ax}}{f^n(a)}$$

Que 1.3. **What is the procedure to find particular integral when the RHS function is either sin ax, cos ax while $f(-a^2) \neq 0$, or sin ax, cos ax while $f(-a^2) = 0$?**

Answer

Case I : When function is $\sin ax$ or $\cos ax$ and $f(-a^2) \neq 0$,

$$\text{PI} = \frac{\sin ax}{f(D^2)}$$

or $$\text{PI} = \frac{\cos ax}{f(D^2)}$$

In both cases replace D^2 by $-a^2$ but $f(-a)^2 \neq 0$. If after replacing D^2 by $-a^2$ any term of D exist in denominator then, multiply the operator by its conjugate, again D^2 by $-a^2$. Terms of D in numerator stands for differentiation of function.

Case II : When function is $\sin ax$ or $\cos ax$ and $f(-a^2) = 0$,

$$\text{PI} = \frac{\sin ax}{f(D^2)} = x\frac{\sin ax}{f'(-a^2)}$$

Repeat this step again if $f'(-a^2) = 0$.

Que 1.4. Solve $\frac{d^2y}{dx^2} + 4y = \sin^2 2x$ with conditions $y(0) = 0$, $y'(0) = 0$.

AKTU 2012-13, Marks 05

Answer

$$\frac{d^2y}{dx^2} + 4y = \sin^2 2x$$

$$\frac{d^2y}{dx^2} + 4y = \frac{1}{2} - \frac{\cos 4x}{2} \qquad \left[\because \cos 4x = 1 - 2\sin^2 2x,\ \sin^2 2x = \frac{1-\cos 4x}{2}\right]$$

The auxiliary equation is

$$m^2 + 4 = 0$$

$$m = \pm 2i$$

$$\therefore \quad \text{CF} = C_1 \cos 2x + C_2 \sin 2x$$

$$\text{PI} = \frac{1}{D^2+4}\sin^2 2x = \frac{1}{2}\left[\frac{1}{D^2+4}(1-\cos 4x)\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2+4}(e^{0x}) - \frac{1}{D^2+4}\cos 4x\right]$$

$$= \frac{1}{2}\left[\frac{1}{0+4} - \frac{1}{-16+4}\cos 4x\right]$$

$$= \frac{1}{2}\left[\frac{1}{4} + \frac{1}{12}\cos 4x\right]$$

$$= \frac{1}{8}\left[1 + \frac{1}{3}\cos 4x\right]$$

Now the complete solution is,

$$y = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{8}\left[1 + \frac{1}{3}\cos 4x\right] \quad ...(1.4.1)$$

$$y' = -2C_1 \sin 2x + 2C_2 \cos 2x - \frac{1}{6}\sin 4x \quad ...(1.4.2)$$

Using boundary conditions,

$$y(0) = 0 \text{ and } y'(0) = 0$$

From eq. (1.4.1), we have

$$0 = C_1 + \frac{1}{6} \Rightarrow C_1 = -\frac{1}{6}$$

From eq. (1.4.2), we have

$$0 = 2C_2 \Rightarrow C_2 = 0$$

Putting the value of C_1 and C_2 in eq. (1.4.1), we get

$$y = -\frac{1}{6}\cos 2x + \frac{1}{8}\left[1 + \frac{1}{3}\cos 4x\right]$$

Que 1.5. A function $n(x)$ satisfies the differential equation $\frac{d^2n(x)}{dx^2} - \frac{n(x)}{L^2} = 0$, where L is a constant. The boundary conditions are $n(0) = x$ and $n(\infty) = 0$. Find the solution to this equation.

AKTU 2016-17, Marks 07

Answer

$$\frac{d^2n(x)}{dx^2} - \frac{n(x)}{L^2} = 0$$

The auxiliary equation is

$$m^2 - \frac{1}{L^2} = 0$$

$$m = \pm \frac{1}{L}$$

$$\text{CF} = C_1 e^{-\frac{1}{L}x} + C_2 e^{\frac{1}{L}x}$$

Complete solution,

$$n(x) = \text{CF} + \text{PI}$$

$$n(x) = C_1 e^{-\frac{x}{L}} + C_2 e^{\frac{x}{L}} \qquad (\because \ \text{PI} = 0)$$

Boundary conditions are wrong. So we can't solve it further.

Que 1.6. Solve $\frac{d^2x}{dt^2} + 9x = \cos 3t.$ **AKTU 2013-14, Marks 05**

Answer

$$\frac{d^2x}{dt^2} + 9x = \cos 3t$$

$$(D^2 + 9)\,x = \cos 3t$$

Auxiliary equation : $m^2 + 9 = 0$

$$m^2 = -9 \Rightarrow m = \pm 3i$$

$$\text{CF} = (C_1 \cos 3t + C_2 \sin 3t)$$

$$\text{PI} = \frac{1}{D^2 + 9} \cos 3t$$

$$\text{PI} = t\,\frac{1}{2D} \cos 3t = \frac{t}{2}\left(\frac{\sin 3t}{3}\right) = \frac{t \sin 3t}{6}$$

Complete solution, $x = \text{CF} + \text{PI} = C_1 \cos 3t + C_2 \sin 3t + \frac{t}{6} \sin 3t$

Que 1.7. Find the particular solution of the differential equation $\frac{d^2y}{dx^2} + a^2 y = \sec ax$ **AKTU 2016-17, Marks 07**

Answer

Auxiliary equation is,

$$m^2 + a^2 = 0$$

$$m = \pm ai$$

$$\text{CF} = C_1 \cos ax + C_2 \sin ax$$

$$\text{PI} = \frac{1}{D^2 + a^2} \sec ax$$

$$= \frac{1}{(D^2 - ia)(D + ia)} \sec ax$$

$$= \frac{1}{2ia}\left[\frac{1}{D - ia} - \frac{1}{D + ia}\right] \sec ax$$

$$= \frac{1}{2ia}\left[\frac{1}{(D - ia)} \sec ax - \frac{1}{(D + ia)} \sec ax\right]$$

$$= \frac{1}{2ia}[P_1 - P_2]$$

Where, $P_1 = \frac{1}{D - ia} \sec ax$

$$= e^{iax} \int e^{-iax} \sec ax\, dx$$

$$= e^{iax} \int (\cos ax - i \sin ax) \sec ax\, dx$$

$$= e^{iax} \int (1 - i \tan ax)\, dx$$

$$= e^{iax} \left\{x + i\left(\frac{\log \cos ax}{a}\right)\right\}$$

Similarly, $P_2 = \frac{1}{D + ia} (\sec ax) = e^{-iax} \left\{x - i\left(\frac{\log \cos ax}{a}\right)\right\}$

Replacing i by $-i$)

$$\therefore \quad \text{PI} = \frac{1}{2ia}\left[e^{iax}\left\{x + i\left(\frac{\log \cos ax}{a}\right)\right\} - e^{-iax}\left\{x - i\left(\frac{\log \cos ax}{a}\right)\right\}\right]$$

$$= \frac{1}{2ia}\left[x(e^{iax} - e^{-iax}) + i\left(\frac{\log \cos ax}{a}\right)(e^{iax} + e^{-iax})\right]$$

$$= \frac{1}{2ia}\left[2\,ixa \sin ax + \frac{i}{a} \log \cos ax\ 2 \cos ax\right]$$

$$= \frac{1}{a}\left[x \sin ax + \frac{1}{a} \cos ax \log \cos ax\right]$$

Que 1.8. Solve $(D^2 - 2D + 1)\, y = e^x \sin x$ **AKTU 2016-17, Marks 7.5**

Answer

$$(D^2 - 2D + 1)y = e^x \sin x$$

Auxiliary equation,

$$m^2 - 2m + 1 = 0$$

$$m^2 - m - m + 1 = 0$$
$$m(m-1) - 1(m-1) = 0$$
$$(m-1)^2 = 0$$
$$m = 1, 1$$
$$\text{CF} = (C_1 + C_2 x)e^x$$
$$\text{PI} = \frac{1}{(D^2 - 2D + 1)} e^x \sin x$$
$$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 1} \sin x$$
$$= e^x \frac{1}{(D^2 + 2D + 1 - 2D - 2 + 1)} \sin x = e^x \frac{\sin x}{D^2}$$

Replace D^2 by -1

$$= -e^x \sin x$$

$\therefore$ Complete solution $= \text{CF} + \text{PI}$

$$y = (C_1 + C_2 x)e^x - e^x \sin x$$

Que 1.9. Solve : $(D^2 - 3D + 2)\, y = x^2 + 2x + 1.$

AKTU 2014-15, Marks 05

Answer

$$(D^2 - 3D + 2)\, y = x^2 + 2x + 1$$

Auxiliary equation,

$$m^2 - 3m + 2 = 0$$
$$(m-1)(m-2) = 0$$
$$m = 1, 2$$
$$\text{CF} = C_1 e^x + C_2 e^{2x}$$
$$\text{PI} = \frac{1}{(D^2 - 3D + 2)} (x^2 + 2x + 1)$$
$$= \frac{1}{2}\left[1 + \frac{D^2 - 3D}{2}\right]^{-1} (x^2 + 2x + 1)$$
$$= \frac{1}{2}\left[1 - \frac{D^2}{2} + \frac{3D}{2} + \left(\frac{D^2 - 3D}{2}\right)^2 \ldots\right](x^2 + 2x + 1)$$
$$= \frac{1}{2}\left[1 - \frac{D^2}{2} + \frac{3D}{2} + \frac{9D^2}{4}\right](x^2 + 2x + 1)$$

(Neglecting higher terms)

$$= \frac{1}{2}\left[x^2 + 2x + 1 - \frac{2}{2} + \frac{3}{2}(2x + 2) + \frac{9}{4} \times 2\right]$$
$$= \frac{1}{2}\left[x^2 + 2x + 3x + 3 + \frac{9}{2}\right]$$

$$= \frac{1}{2}\left[x^2 + 5x + \frac{15}{2}\right]$$
$$y = \text{CF} + \text{PI}$$
$$= C_1 e^x + C_2 e^{2x} + \frac{1}{2}\left[x^2 + 5x + \frac{15}{2}\right]$$

Que 1.10. Solve the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \cos x$.

AKTU 2013-14, Marks 05

Answer

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \cos x$$

Auxiliary equation,

$$(m^2 - 2m + 1) = 0$$
$$(m-1)^2 = 0$$
$$m = 1, 1$$
$$\text{CF} = (C_1 + C_2 x)e^x$$
$$\text{PI} = \frac{1}{(D-1)^2} x e^x \cos x = \frac{1}{(D-1)^2} e^x (x \cos x)$$
$$= e^x \frac{1}{(D+1-1)^2} x \cos x$$
$$= e^x \frac{1}{D^2} x \cos x = e^x \frac{1}{D}[x \sin x + \cos x]$$
$$= e^x [-x \cos x + \sin x + \sin x]$$
$$\text{PI} = e^x [-x \cos x + 2 \sin x]$$

Complete solution is given by

$$y = \text{CF} + \text{PI}$$
$$y = (C_1 + C_2 x)\, e^x + e^x (-x \cos x + 2 \sin x)$$

Que 1.11. Solve the following differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^2 e^{-x} \cos x.$$

AKTU 2011-12, Marks 05

Answer

Same as Q. 1.10, Page 1–10F, Unit-1.

(**Answer :** $y = (C_1 + C_2 x)\, e^{-x} + e^{-x}(-x^2 \cos x + 4x \sin x + 6 \cos x)$)

Que 1.12. Solve $(D^2 - 2D + 4)\, y = e^x \cos x + \sin x \cos 3x$.

AKTU 2017-18, Marks 07

Answer

Given equation, $(D^2 - 2D + 4)\, y = e^x \cos x + \sin x \cos 3x$

Auxiliary equation,

$$m^2 - 2m + 4 = 0$$

$$m = \frac{+2 \pm \sqrt{4-16}}{2}$$

$$m = \frac{2 \pm \sqrt{-12}}{2}$$

$$m = 1 \pm i\sqrt{3}$$

Complementary function is

$$\text{CF} = e^x (C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x)$$

Particular integral, PI = $P_1 + P_2$

$$P_1 = e^x \cos x$$

$$= \frac{1}{D^2 - 2D + 4} e^x \cos x$$

$$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 4} \cos x$$

$$= e^x \frac{1}{D^2 + 3} \cos x$$

$$= e^x \frac{1}{-1+3} \cos x$$

$$= e^x \frac{\cos x}{2}$$

$$P_2 = \frac{1}{D^2 - 2D + 4} \sin x \cos 3x$$

$$= \frac{1}{2} \frac{1}{(D^2 - 2D + 4)} 2 \sin x \cos 3x$$

$$= \frac{1}{2} \frac{1}{D^2 - 2D + 4} \{\sin (x + 3x) + \sin (x - 3x)\}$$

$$= \frac{1}{2} \frac{1}{D^2 - 2D + 4} (\sin 4x - \sin 2x)$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - 2D + 4} \sin 4x - \frac{1}{D^2 - 2D + 4} \sin 2x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-(4)^2 - 2D + 4} \sin 4x - \frac{1}{-(2)^2 - 2D + 4} \sin 2x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-12 - 2D} \sin 4x - \frac{1}{-2D} \sin 2x\right]$$

$$= \frac{1}{4}\left[\frac{-1}{D+6} \sin 4x + \frac{1}{D} \sin 2x\right]$$

$$= \frac{1}{4}\left[\frac{-(D-6)}{D^2 - 36} \sin 4x - \frac{\cos 2x}{2}\right]$$

$$= \frac{1}{4}\left[\frac{-(D-6)}{-52} \sin 4x - \frac{\cos 2x}{2}\right]$$

$$= \frac{1}{4}\left[\frac{4 \cos 4x - 6 \sin 4x}{52} - \frac{\cos 2x}{2}\right]$$

$$= \frac{1}{4}\left[\frac{4 \cos 4x - 6 \sin 4x}{52}\right] - \frac{\cos 2x}{8}$$

Complete solution,

$$y = \text{CF} + \text{PI}$$

$$= \text{CF} + P_1 + P_2$$

$$y = e^x \left(C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x\right) + e^x \frac{\cos x}{2} + \frac{1}{4}\left[\frac{4 \cos 4x - 6 \sin 4x}{52}\right] - \frac{\cos 2x}{8}$$

PART-2

Simultaneous Linear Differential Equations.

CONCEPT OUTLINE

Simultaneous Differential Equation : If two or more dependent variables are functions of a single independent variable, the equations which consist of the derivatives of such variables are called simultaneous differential equations.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.13. Solve the simultaneous equation $\frac{dx}{dt} + 5x - 2y = t$, $\frac{dy}{dt} + x + y = 0$ being given $x = 0, y = 0$ when $t = 0$.

AKTU 2014-15, Marks 10

Answer

$$(D+5)x - 2y = t \quad \text{...(1.13.1)}$$
$$x + (D+1)y = 0 \quad \text{...(1.13.2)}$$

On multiplying eq. (1.13.2) by $(D+5)$ and subtracting from eq. (1.13.1), we get

$$(D+1)(D+5)y + 2y = -t$$
$$(D^2 + 6D + 5 + 2)y = -t$$

Auxiliary equation, $m^2 + 6m + 7 = 0$

$$m = \frac{-6 \pm \sqrt{36-28}}{2} \Rightarrow m = -3 \pm \sqrt{2}$$

$$\text{CF} = e^{-3t}(C_1 \cosh \sqrt{2}\,t + C_2 \sinh \sqrt{2}\,t)$$

$$\text{PI} = \frac{1}{D^2 + 6D + 7}(-t)$$

$$= \frac{-1}{7}\left(1 + \frac{D^2+6D}{7}\right)^{-1}(t) = -\frac{1}{7}\left(1 - \frac{6D}{7}\right)t$$

$$\text{PI} = -\frac{1}{7}\left(t - \frac{6}{7}\right)$$

$$y = e^{-3t}(C_1 \cosh \sqrt{2}\,t + C_2 \sinh \sqrt{2}\,t) - \frac{1}{7}\left(t - \frac{6}{7}\right) \quad \text{...(1.13.3)}$$

$$\frac{dy}{dt} = e^{-3t}(-C_1\sqrt{2}\sinh \sqrt{2}\,t + \sqrt{2}\,C_2 \cosh \sqrt{2}\,t)$$
$$- 3e^{-3t}(C_1 \cosh \sqrt{2}\,t + C_2 \sinh \sqrt{2}\,t) - \frac{1}{7}$$

From eq. (1.13.2),

$$x = -\frac{dy}{dt} - y$$

$$x = -e^{-3t}(-C_1\sqrt{2}\sinh \sqrt{2}\,t + \sqrt{2}\,C_2 \cosh \sqrt{2}\,t) + \frac{1}{7}$$
$$+ 3e^{-3t}(C_1 \cosh \sqrt{2}\,t + C_2 \sinh \sqrt{2}\,t)$$
$$- e^{-3t}(C_1 \cosh \sqrt{2}\,t + C_2 \sinh \sqrt{2}\,t) + \frac{1}{7}\left(t - \frac{6}{7}\right)$$

$$x = -e^{-3t}(-C_1\sqrt{2}\sinh \sqrt{2}\,t + \sqrt{2}\,C_2 \cosh \sqrt{2}\,t)$$
$$+ 2e^{-3t}(C_1 \cosh \sqrt{2}\,t + C_2 \sinh \sqrt{2}\,t) + \frac{t}{7} + \frac{1}{49} \quad \text{...(1.13.4)}$$

Boundary conditions

$$x(0) = 0, y(0) = 0$$

From eq. (1.13.3) and eq. (1.13.4), we have

$$0 = C_1 + \frac{6}{7}$$

$$C_1 = -\frac{6}{7}$$

and $$0 = -\sqrt{2}\,C_2 + 2C_1 + \frac{1}{49}$$

$$\sqrt{2}\,C_2 = -\frac{12}{7} + \frac{1}{49}$$

$$\sqrt{2}\,C_2 = -\frac{83}{49}$$

$$C_2 = -\frac{83}{49\sqrt{2}}$$

Now, $$y = e^{-3t}\left[-\frac{6}{7}\cosh \sqrt{2}t - \frac{83}{49\sqrt{2}}\sinh \sqrt{2}t\right] - \frac{1}{7}\left(t - \frac{6}{7}\right)$$

$$x = -e^{-3t}\left(-\frac{6}{7}\sqrt{2}\sinh \sqrt{2}t - \frac{83}{49}\cosh \sqrt{2}t\right)$$
$$+ 2e^{-3t}\left(t - \frac{6}{7}\sqrt{2}t - \frac{83}{49\sqrt{2}}\sinh \sqrt{2}t\right) + \frac{t}{7} + \frac{1}{49}$$

Que 1.14. Solve the following simultaneous equations.

$$\frac{d^2x}{dt^2} + y = \sin t$$

$$\frac{d^2y}{dt^2} + x = \cos t$$

AKTU 2015-16, Marks 10

Answer

Let $\frac{d}{dt} \equiv D$ then the given system of equations become

$$D^2x + y = \sin t \quad \text{...(1.14.1)}$$
$$x + D^2y = \cos t \quad \text{...(1.14.2)}$$

Multiplying eq. (1.14.1) by D^2, we get

$$D^4x + D^2y = -\sin t \quad \text{...(1.14.3)}$$

Subtracting eq. (1.14.2) from eq. (1.14.3), we get

$$(D^4 - 1)\,x = -\sin t - \cos t$$

Auxiliary equation is

$$m^4 - 1 = 0$$

$$(m^2 - 1)(m^2 + 1) = 0$$

$$\Rightarrow \quad m = 1, -1, \pm i$$

$$\therefore \quad \text{CF} = C_1 e^t + C_2 e^{-t} + C_3 \cos t + C_4 \sin t$$

$$\text{PI} = \frac{1}{D^4 - 1}(-\sin t - \cos t)$$

$$= -t \frac{1}{4D^3}(\sin t + \cos t) = \frac{t}{4}(-\cos t + \sin t)$$

$$\therefore \quad x = C_1 e^t + C_2 e^{-t} + C_3 \cos t + C_4 \sin t + \frac{t}{4}(\sin t - \cos t) \quad \text{...(1.14.4)}$$

$$Dx = C_1 e^t + C_2 e^{-t} - C_3 \sin t + C_4 \cos t + \frac{t}{4}(\cos t + \sin t) + \frac{1}{4}(\sin t - \cos t)$$

$$D^2 x = C_1 e^t + C_2 e^{-t} - C_3 \cos t + C_4 \sin t + \frac{t}{4}(-\sin t + \cos t) + \frac{1}{4}(\cos t + \sin t) + \frac{1}{4}(\cos t + \sin t)$$

From eq. (1.14.1), $y = \sin t - \dfrac{d^2x}{dt^2}$

$$y = -C_1 e^t - C_2 e^{-t} + C_3 \cos t + C_4 \sin t + \frac{t}{4}(\sin t - \cos t) + \frac{1}{2}(\sin t - \cos t)$$

Eq. (1.14.4) and eq. (1.14.5), when taken together, give the complete solution of the given system of equations.

Que 1.15. **Solve the following :**

$$\frac{dx}{dt} = 3x + 8y$$

$$\frac{dy}{dt} = -x - 3y \text{ with } x(0) = 6 \text{ and } y(0) = -2$$

AKTU 2013-14, Marks 05

Answer

$$\frac{dx}{dt} = 3x + 8y$$

$$\frac{dy}{dt} = -x - 3y$$

Let $\dfrac{d}{dt} = D$, so the given equation reduces to

$$(D - 3)\,x - 8y = 0 \quad \text{...(1.15.1)}$$

$$x + (D + 3)\,y = 0 \quad \text{...(1.15.2)}$$

Multiply by $(D + 3)$ in eq. (1.15.1) and multiply by 8 in eq. (1.15.2), then add both equations

$$(D^2 - 9 + 8)\,x = 0$$

$$(D^2 - 1)\,x = 0$$

Auxiliary equation is, $m^2 - 1 = 0$

$$m = \pm 1$$

$$\text{CF} = C_1 e^{-t} + C_2 e^t \quad \text{...(1.15.3)}$$

$$\text{PI} = 0$$

$$\therefore \quad x = C_1 e^{-t} + C_2 e^t$$

From eq. (1.15.1),

$$8y(t) = \frac{dx(t)}{dt} - 3x(t)$$

or

$$8y = \frac{dx}{dt} - 3x$$

$$8y = C_1(-1)e^{-t} + C_2 e^t - 3\,[C_1 e^{-t} + C_2 e^t]$$

$$8y = -4C_1 e^{-t} - 2\,C_2 e^t$$

$$y = -0.5\,C_1 e^{-t} - 0.25\,C_2 e^t \quad \text{...(1.15.4)}$$

Apply boundary conduction,

$$x(0) = 6$$

From eq. (1.15.3), $6 = C_1 + C_2$ ...(1.15.5)

From eq. (1.15.4), $y(0) = -2 = -0.5\,C_1 - 0.25\,C_2$ ...(1.15.6)

By solving eq. (1.15.5) and eq. (1.15.6), we get

$$C_1 = 2$$

$$C_2 = 4$$

$$\therefore \quad x = 2e^{-t} + 4e^t$$

$$y = -e^{-t} - e^t$$

Que 1.16. **Solve** $\dfrac{dx}{dt} + 2x + 4y = 1 + 4t; \dfrac{dy}{dt} + x - y = \dfrac{3}{2}t^2$.

AKTU 2012-13, Marks 05

Answer

$$\frac{dx}{dt} + 2x + 4y = 1 + 4t, \quad \frac{dy}{dt} + x - y = \frac{3}{2}t^2$$

Writing D for $\dfrac{d}{dt}$, the given equation becomes

$$(D + 2)\,x + 4y = 1 + 4t \quad \text{...(1.16.1)}$$

$$x + (D - 1)y = \frac{3}{2}t^2 \quad \text{...(1.16.2)}$$

To eliminate y, multiplying eq. (1.16.1) by $(D - 1)$ and multiplying eq. (1.16.2) by 4, then subtracting, we get

$$[(D+2)(D-1)-4]x = (D-1)(1) + 4(D-1)t - 6t^2$$
$$(D^2 + 2D - D - 2 - 4)x = -1 + 4 - 4t - 6t^2$$
$$(D^2 + D - 6)x = 3 - 4t - 6t^2$$

Auxiliary equation is

$$m^2 + m - 6 = 0$$
$$m^2 + 3m - 2m - 6 = 0$$
$$m(m+3) - 2(m+3) = 0$$
$$(m+3)(m-2) = 0 \Rightarrow m = 2, -3$$

$\therefore$ $$\text{CF} = C_1 e^{2t} + C_2 e^{-3t}$$

$$\text{PI} = \frac{1}{(D^2 + D - 6)}(3 - 4t - 6t^2)$$

$$= \frac{3}{(D^2+D-6)} e^{0t} - \frac{4t}{(D^2+D-6)} - \frac{6}{(D^2+D-6)} t^2$$

$$= -\frac{3}{6} + \frac{4}{6} \frac{1}{\left[1+\left(-\frac{D^2}{6}-\frac{D}{6}\right)\right]} t + \frac{6}{6} \frac{1}{\left[1+\left(-\frac{D^2}{6}-\frac{D}{6}\right)\right]} t^2$$

$$= -\frac{3}{6} + \frac{4}{6}\left[1+\left(-\frac{D}{6}-\frac{D^2}{6}\right)\right]^{-1} t + \left[1+\left(-\frac{D}{6}-\frac{D^2}{6}\right)\right]^{-1} t^2$$

$$= -\frac{3}{6} + \frac{4}{6}\left[1+\frac{D}{6}+\frac{D^2}{6}\right] t + \left[1-\left(-\frac{D}{6}-\frac{D^2}{6}\right)+\left(-\frac{D}{6}-\frac{D^2}{6}\right)^2\right] t^2$$

$$= -\frac{3}{6} + \frac{4t}{6} + \frac{4}{36} + t^2 + \frac{2t}{6} + \frac{2}{6} + \frac{2}{36} = t^2 + \frac{6t}{6} + \frac{(-18+4+12+2)}{36}$$

$$\text{PI} = t^2 + t$$

So, $$x = C_1 e^{2t} + C_2 e^{-3t} + t^2 + t$$

Now $$\frac{dx}{dt} = 2C_1 e^{2t} - 3C_2 e^{-3t} + 2t + 1$$

Substituting the values of x and $\frac{dx}{dt}$ in eq. (1.16.1), we get

$$4y = -2C_1 e^{2t} + 3C_2 e^{-3t} - 2t - 1 - 2C_1 e^{2t} - 2C_2 e^{-3t} - 2t^2 - 2t + 1 + 4t$$

$$y = -C_1 e^{2t} + \frac{1}{4} C_2 e^{-3t} - \frac{1}{2} t^2$$

Que 1.17. **Solve the simultaneous differential equations**

$$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = y \text{ and } \frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 25x + 16e^t.$$

AKTU 2017-18, Marks 07

Answer

$$(D^2 - 4D + 4)x - y = 0 \quad \text{...(1.17.1)}$$
$$-25x + (D^2 + 4D + 4)y = 16e^t \quad \text{...(1.17.2)}$$

Multiplying eq. (1.17.1) by $D^2 + 4D + 4$ and adding to eq. (1.17.2), we get

$$(D^2 - 4D + 4)(D^2 + 4D + 4)x - 25y = 16e^t$$
$$(D^4 - 8D - 9)x = 16e^t$$

Auxiliary equation is,

$$m^4 - 8m^2 - 9 = 0$$
$$\Rightarrow (m^2 - 9)(m^2 + 1) = 0 \Rightarrow m = \pm i, \pm 3$$

$\therefore$ $$\text{CF} = C_1 e^{-3t} + C_2 e^{-3t} + C_3 \cos t + C_4 \sin t$$

$$\text{PI} = \frac{1}{D^4 - 8D^2 - 9}(16e^t) = -e^t$$

$\therefore$ $$x = C_1 e^{3t} + C_2 e^{-3t} + C_3 \cos t + C_4 \sin t - e^t \quad \text{...(1.17.3)}$$

$$\frac{dx}{dt} = 3C_1 e^{3t} - 3C_2 e^{-3t} + C_3(-\sin t) + C_4 \cos t - e^t$$

$$\frac{d^2x}{dt^2} = 9C_1 e^{3t} + 9C_2 e^{-3t} - C_3 \cos t - C_4 \sin t - e^t$$

From eq. (1.17.1), $$y = \frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x$$
$$= 9C_1 e^{3t} + 9C_2 e^{-3t} - C_3 \cos t - C_4 \sin t - e^t$$
$$-4(3C_1 e^{3t} - 3C_2 e^{-3t} - C_3 \sin t + C_4 \cos t - e^t)$$
$$+4(C_1 e^{3t} + C_2 e^{-3t} + C_3 \cos t + C_4 \sin t - e^t)$$

$$\Rightarrow y = C_1 e^{3t} + 25C_2 e^{-3t} + (3C_3 - 4C_4)\cos t + (4C_3 + 3C_4)\sin t - e^t \quad \text{...(1.17.4)}$$

Eq. (1.17.3) and eq. (1.17.4) when taken together give the complete solution.

PART-3

Second Order Linear Differential Equations with Variable Coefficients, Solution by Changing Independent Variable, Reduction of Order.

CONCEPT OUTLINE

Second Order Linear Differential Equation : A differential equation of the form $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ is known linear differential equation of second order, where P, Q and R are functions of x alone.

Method of Reduction of Order to Solve Second Order Linear Differential Equation :

Let $y = u$ be a part of the complementary function of the given differential equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad \text{...(1)}$$

Where u is a function of x, then, we have

$$\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = R \quad \text{...(2)}$$

Let $y = uv$, be the complete solution of eq. (1), where v is a function of x.

Differentiating y w.r.t x,

$$\frac{dy}{dx} = u\frac{dv}{dx} + \frac{du}{dx}v$$

Again
$$\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2}$$

Substituting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in eq. (1), we get

$$u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2} + P\left(u\frac{dv}{dx} + v\frac{du}{dx}\right) + Q(uv) = R$$

$$u\frac{d^2v}{dx^2} + \left(2\frac{du}{dx} + Pu\right)\frac{dv}{dx} + \left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right)v = R$$

$$u\frac{d^2v}{dx^2} + \left(2\frac{du}{dx} + Pu\right)\frac{dv}{dx} = R$$

$$\frac{d^2v}{dx^2} + \left(\frac{2}{u}\frac{du}{dx} + P\right)\frac{dv}{dx} = \frac{R}{u} \quad \text{...(3)}$$

Put $\frac{dv}{dx} = p$ then, $\frac{d^2v}{dx^2} = \frac{dp}{dx}$

Now eq. (3) becomes, $\frac{dp}{dx} + \left(\frac{2}{u}\frac{du}{dx} + P\right)P = \frac{R}{u}$ ...(4)

Eq. (4), is a linear differential equation of first order in p and x.

$$\text{IF} = e^{\int\left(\frac{2}{u}\frac{du}{dx} + P\right)dx} = e^{\left(\int\frac{2}{u}du + \int P\,dx\right)} = u^2 e^{\int P\,dx}$$

Solution of eq. (4) is given by

$$pu^2 e^{\int P\,dx} = \int\frac{R}{u}u^2 e^{\int P\,dx}\,dx + C_1$$

Where C_1 is an arbitrary constant of integration.

$$\Rightarrow \quad p = \frac{1}{u^2}e^{-\int P\,dx}\left[\int Ru\,e^{\int P\,dx}\,dx + C_1\right]$$

$$\therefore \quad \frac{dv}{dx} = \frac{1}{u^2}e^{-\int P\,dx}\left[\int Ru\,e^{\int P\,dx}\,dx + C_1\right]$$

Integration yields, $v = \int\frac{1}{u^2}e^{-\int P\,dx}\left[\int Ru\,e^{\int P\,dx}\,dx + C_1\right]dx + C_2$

where C_2 is an arbitrary constant of integration.

Hence the complete solution of eq. (1) is given by,

$$y = uv$$

$$\Rightarrow \quad y = u\int\frac{1}{u^2}e^{-\int P\,dx}\left[\int Ru\,e^{\int P\,dx}\,dx + C_1\right]dx + C_2u$$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.18. Solve $(3x+2)^2\frac{d^2y}{dx^2} - (3x+2)\frac{dy}{dx} - 12y = 6x$.

Answer

$$(3x+2)^2\frac{d^2y}{dx^2} - (3x+2)\frac{dy}{dx} - 12y = 6x$$

Using $3x + 2 = e^t$, $(3x+2)^2\frac{d^2y}{dx^2} = 9D(D-1)y$ and $(3x+2)\frac{dy}{dx} = 3Dy$,

we get

$$9D(D-1)y - 3Dy - 12y = 2(e^t - 2)$$

$$(9D^2 - 9D - 3D - 12)y = 2(e^t - 2)$$

The auxiliary equation is

$$9m^2 - 12m - 12 = 0$$

$$(m-2)\left(m + \frac{2}{3}\right) = 0$$

$$m = 2, -\frac{2}{3}$$

Therefore, the complementary function is

$$\text{CF} = C_1e^{2t} + C_2e^{\frac{-2t}{3}}$$

and
$$\text{PI} = \frac{1}{9D^2 - 12D + 12}2(e^t - 2)$$

$$= 2\left\{\frac{1}{9D^2 - 12D - 12}e^t - 2\frac{e^0}{9D^2 - 12D - 12}\right\}$$

$$= 2\frac{1}{9-12-12}e^z - 4\frac{1}{0-0-12} = \frac{2e^z}{-15}+\frac{1}{3}$$

The solution is $y = \text{CF} + \text{PI}$

$$y = C_1e^{2z} + C_2e^{\frac{-2z}{3}} + \frac{1}{3} - \frac{2}{15}e^z$$

Using, $z = \log(3x+2)$, we get

$$y = C_1e^{2\log(3x+2)} + C_2e^{\frac{-2\log(3x+2)}{3}} + \frac{1}{3} - \frac{2}{15}e^{\log(3x+2)}$$

$$= C_1(3x+2)^2 + C_2(3x+2)^{-2/3} + \frac{1}{3} - \frac{2}{15}(3x+2)$$

C_1 and C_2 are arbitrary constants of integration.

Que 1.19. Solve $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = \frac{12\log x}{x^2}$

Answer

Given equation may be written as

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 12\log x \qquad \text{(Let, } z = \log x\text{)}$$

or $\{D(D-1)+D\}\, y = 12z$

$$D^2y = 12z$$

Auxiliary equation is, $m^2 = 0$

$$m = 0, 0$$

$$\text{CF} = (C_1 + C_2 z)\, e^{0z} = C_1 + C_2 z$$

$$\text{PI} = \frac{1}{D^2}12z = 12\frac{1}{D^2}z = 12\frac{z^3}{6} = 2z^3$$

Complete solution, $y = \text{CF} + \text{PI}$

$$y = C_1 + C_2 z + 2z^3$$

$$y = C_1 + C_2\log x + 2(\log x)^3$$

Que 1.20. **Write the procedure for solving the linear differential equation by changing the independent variable.**

Answer

Let the given differential equation is

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad \text{...(1.20.1)}$$

Let the independent variable be changed from x to z and $z = f(x)$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dz}\frac{dz}{dx}\right)$$

$$= \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\frac{d^2z}{dx^2}$$

Substituting the values of dy/dx and d^2y/dx^2 in eq. (1.20.1), we have

$$\left(\frac{dz}{dx}\right)^2\frac{d^2y}{dz^2} + \left(\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right)\frac{dy}{dz} + Qy = R$$

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1y = R_1 \qquad \text{...(1.20.2)}$$

Where,

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{(dz/dx)^2},$$

$$Q_1 = \frac{Q}{(dz/dx)^2}\quad R_1 = \frac{R}{(dz/dx)^2}$$

P_1, Q_1, and R_1 are functions of x but may be expressed as functions of z by the given relation between z and x.

Here, we choose z to make the coefficient of dy/dx zero, *i.e.*,

$$P_1 = 0$$

and

$$\frac{d^2z}{dx^2} + P\frac{dz}{dx} = 0$$

or

$$\frac{d^2z/dx^2}{dz/dx} = -P$$

Integrating, we get

$$\ln\frac{dz}{dx} = -\int P\,dx$$

$$\frac{dz}{dx} = e^{-\int P dx}$$

Integrating again, we get

$$z = \int e^{-\int P dx}dx$$

Now, eq. (1.20.2) reduces to

$$\frac{d^2y}{dz^2} + Q_1y = R_1$$

Which can be solved easily provided Q_1 comes out to be a constant or a constant multiplied by $1/z^2$. Again if we choose z such that,

$$Q_1 = \frac{Q}{(dz/dx)^2} = a^2 \text{ (Constant)}$$

$$a^2\left(\frac{dz}{dx}\right)^2 = Q$$

$$a\frac{dz}{dx} = \sqrt{Q}$$

$$az = \int \sqrt{Q}\, dx$$

Then eq. (1.20.2) reduces to

$$x\frac{d^2y}{dx^2} + P_1\frac{dy}{dz} + a^2y = R_1$$

Which can be solved easily provided P_1 comes out to be a constant.

Que 1.21. **Solve by changing the independent variable :**

$$\frac{d^2y}{dx^2} + (3\sin x - \cot x)\frac{dy}{dx} + 2y\sin^2 x = e^{-\cos x}\sin^2 x$$

AKTU 2014-15, Marks 05

Answer

$y'' + (3\sin x - \cot x)y' + 2y\sin^2 x = e^{-\cos x}\sin^2 x$

Changing independent variable

$$z = f(x)$$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx},\quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dz}\frac{dz}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dz}\right)\frac{dz}{dx} + \frac{d^2z}{dx^2}$$

$$= \frac{d}{dz}\left(\frac{dy}{dz}\right)\left(\frac{dz}{dx}\right)\left(\frac{dz}{dx}\right) + \frac{d^2z}{dx^2}$$

$$= \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\frac{d^2z}{dx^2}$$

Now from given equation,

$$\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{d^2z}{dx^2} + (3\sin x - \cot x)\frac{dy}{dz}\frac{dz}{dx} + 2y\sin^2 x = e^{-\cos x}\sin^2 x$$

$$\frac{d^2y}{dz^2} + \frac{d^2z}{dx^2} + \frac{(3\sin x - \cot x)\frac{dy}{dz}\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} + \frac{2\sin^2 x}{\left(\frac{dz}{dx}\right)^2}y = e^{-\cos x}\sin^2 x$$

This can be written as

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$$

Where,
$$P_1 = \frac{\frac{d^2y}{dz^2} + (3\sin x - \cot x)\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$$

$$Q_1 = \frac{2\sin^2 x}{(dz/dx)^2},\quad R_1 = \frac{e^{-\cos x}\sin^2 x}{(dz/dx)^2}$$

Choose $Q_1 = 2$, *i.e.*, $2 = \dfrac{2\sin^2 x}{\left(\frac{dz}{dx}\right)^2} \Rightarrow \left(\dfrac{dz}{dx}\right)^2 = \sin^2 x \Rightarrow \dfrac{dz}{dx} = \sin x$

$$z = -\cos x$$

$$\frac{d^2z}{dx^2} = \cos x$$

Now,
$$P_1 = \frac{\cos x + (3\sin x - \cot x)\sin x}{\sin^2 x}$$

$$= \frac{\cos x + 3\sin^2 x - \frac{\cos x}{\sin x}\sin x}{\sin^2 x} = 3$$

$$R_1 = \frac{e^{-\cos x}\sin^2 x}{\sin^2 x} = e^{-\cos x}$$

$$\frac{d^2y}{dz^2} + 3\frac{dy}{dz} + 2y = e^{-\cos x}$$

$$\frac{d^2y}{dz^2} + 3\frac{dy}{dz} + 2y = e^{z}$$

Auxiliary equation is $m^2 + 3m + 2 = 0$

$$m = -1, -2$$

$$CF = C_1e^{-z} + C_2e^{-2z}$$

$$PI = \frac{1}{(D+2)(D+1)}e^z = \frac{1}{D^2+3D+2}e^z$$

Put, $D = -1$

$$= \frac{1}{1+3+2}e^z\ \frac{e^z}{6}$$

$\therefore$ Complete solution $= CF + PI = C_1\dfrac{e^z}{6} + C_2e^{-z} + e^{-z} = C_1e^{-\cos x}$
$+ C_2e^{-\cos x} + e^{-\cos x/6}$

PART-4

Normal Form.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.22. **How can we solve differential equation by removing the first derivative or converting in normal form ?**

Answer

A part of the complementary function is needed to find the complete solution, it is not always possible to find an integral belonging to CF in such cases, we reduce the given equation to the form in which the term containing the first derivative is absent. For this, we shall change the dependent variable in the equation.

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad \text{....(1.22.1)}$$

By putting $y = uv$, where u is some function of x, so that

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

and
$$\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2}$$

On substituting dy/dx and d^2y/dx^2 in terms of u and v in eq. (1.22.1), we get

$$u\frac{d^2v}{dx^2} + \left(Pu + 2\frac{du}{dx}\right)\frac{dv}{dx} + \left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right)v = R$$

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} + \left(\frac{1}{u}\frac{d^2u}{dx^2} + \frac{P}{u}\frac{du}{dx} + Q\right)v = R/u \quad \text{...(1.22.2)}$$

Let us choose u such that,

$$P + \frac{2}{u}\frac{du}{dx} = 0$$

$$\frac{du}{dx} = -\frac{P}{2}u$$

$$\frac{du}{u} = -\frac{P}{2}dx$$

$$\therefore \quad u = e^{-1/2 \int P dx}$$

Now, from eq. (1.22.2), we have

$$\frac{d^2v}{dx^2} + \left[\frac{1}{u}\left(-\frac{u}{2}\frac{dP}{dx} - \frac{P}{2}\frac{du}{dx}\right) + \frac{P}{u}\frac{du}{dx} + Q\right]v = R\, e^{1/2 \int P dx}$$

$$\frac{d^2v}{dx^2} + \left[-\frac{1}{2}\frac{dP}{dx} - \frac{P}{2u}\left(-\frac{P}{2}u\right) + \frac{P}{2}\left(\frac{-P}{2}u\right) + Q\right]v = R\, e^{1/2 \int P dx}$$

$$\frac{d^2v}{dx^2} + \left[Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2\right]v = R\, e^{1/2 \int P dx}$$

or
$$\left.\begin{aligned} &\frac{d^2v}{dx^2} + Xv = Y \\ &\text{Where } X = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2 \\ &\text{and } \quad Y = R\, e^{1/2\int P dx} \end{aligned}\right\} \quad \text{...(1.22.3)}$$

Eq. (1.22.3) may easily be integrated and is known as normal form of eq. (1.22.1).

Que 1.23. **Solve the following equation by reducing into normal form.**

$$\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + (x^2 - 8)y = x^2 e^{-x^2/2}.$$

AKTU 2011-12, Marks 05

OR

Solve the following differential equation by reducing into normal form :

$$y'' + 2xy' + (x^2 - 8)y = x^2 e^{-\frac{1}{2}x^2}.$$

AKTU 2012-13, Marks 05

Answer

$$\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + (x^2 - 8)y = x^2 e^{-x^2/2}$$

On comparison with, $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$, we have

$$P = 2x,\ Q = x^2 - 8,\ R = x^2 e^{-x^2/2}$$

$$v = e^{-\frac{1}{2}\int P dx} = e^{-\frac{1}{2}\int 2x dx} = e^{-\frac{x^2}{2}}$$

We know that, u is given by

$$\frac{d^2u}{dx^2} + Q_1 u = R_1 \quad \text{... (1.23.1)}$$

Where,
$$Q_1 = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4} = x^2 - 8 - \frac{1}{2}(2) - \frac{4x^2}{4}$$

$$Q_1 = -9$$

$$R_1 = \frac{R}{v} = \frac{x^2 e^{-x^2/2}}{e^{-x^2/2}} = x^2$$

On putting the value of Q_1 and R_1 in eq. (1.23.1), we get

$$\frac{d^2u}{dx^2} - 9u = x^2$$

$$(D^2 - 9)u = x^2$$

Auxiliary equation, $m^2 - 9 = 0$

$$m = \pm 3$$

$$CF = C_1 e^{3x} + C_2 e^{-3x}$$

$$PI = \frac{1}{D^2 - 9} x^2 = \frac{1}{9}\left(1 - \frac{D^2}{9}\right)^{-1} x^2 = \frac{1}{9}\left(1 + \frac{D^2}{9}\right) x^2$$

$$PI = -\frac{1}{9}\left(x^2 + \frac{2}{9}\right)$$

Complete solution, $u = CF + PI = C_1 e^{3x} + C_2 e^{-3x} - \frac{1}{9}\left(x^2 + \frac{2}{9}\right)$

Thus $$y = uv = \left[C_1 e^{3x} + C_2 e^{-3x} - \frac{1}{9}\left(x^2 + \frac{2}{9}\right)\right] e^{-\frac{x^2}{2}}$$

Que 1.24. **Using normal form, solve :**

$$\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 1)\, y = -3e^{x^2} \sin 2x$$ **AKTU 2013-14, Marks 05**

Answer

Here, $P = -4x,\ Q = 4x^2 - 1,\ R = -3e^{x^2} \sin 2x$

Let $y = uv$ be the complete solution.

Now, $$u = e^{-\frac{1}{2}\int(-4x)dx} = e^{x^2}$$

$$Q_1 = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}$$

$$= 4x^2 - 1 - \frac{1}{2}(-4) - \frac{1}{4}(16x^2) = 1$$

Also, $$R_1 = \frac{R}{u} = \frac{-3e^{x^2} \sin 2x}{e^{x^2}} = -3 \sin 2x$$

Hence normal form is, $\frac{d^2v}{dx^2} + v = -3 \sin 2x$

Auxiliary equation, $m^2 + 1 = 0 \Rightarrow m = \pm i$

$$CF = C_1 \cos x + C_2 \sin x$$

$$PI = \frac{1}{D^2 + 1}(-3 \sin 2x) = \frac{-3}{(-4+1)} \sin 2x$$

$$PI = \sin 2x$$

Complete solution, $v = CF + PI = C_1 \cos x + C_2 \sin x + \sin 2x$

Hence the complete solution of given differential equation is

$$y = uv = e^{x^2}(C_1 \cos x + C_2 \sin x + \sin 2x)$$

PART-5

Method of Variation of Parameters.

CONCEPT OUTLINE

Method of Variation of Parameters : By this method the general solution is obtained by varying the arbitrary constants of the complementary function that is why the method is known as method of variation of parameters.

Procedure : First find the complementary function of the given differential equation.

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = X$$

Let it is be $CF = Ay_1 + By_2$ ...(1)

So that y_1 and y_2 satisfy given differential equation let us assume

$$PI = u\, y_1 + v\, y_2 \quad ...(2)$$

Where u and v are given by

$$u = \int \frac{-X y_2}{y_1 y_2' - y_2 y_1'} dx$$

and $$v = \int \frac{X y_1}{y_1 y_2' - y_2 y_1'} dx$$

Putting u and v in eq. (2), we can find PI and then complete solution

$$y = CF + PI$$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.25. **Apply method of variation of parameters to solve**

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x$$ **AKTU 2011-12, Marks 10**

Answer

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x$$

$$[D(D-1) + 4D + 2]y = e^{e^t} \qquad [\because x = e^t]$$

$$(D^2 + 3D + 2)y = e^{e^t}$$

Auxiliary equation, $m^2 + 3m + 2 = 0 \Rightarrow m = -1, -2$

$$CF = C_1 e^{-t} + C_2 e^{-2t}$$

$$\text{PI} = \frac{1}{D^2+3D+2} e^{e^z}$$

(Using General method to find PI)

$$= \frac{1}{(D+1)(D+2)} e^{e^z} = \left(\frac{1}{D+1} - \frac{1}{D+2}\right) e^{e^z}$$

$$= \frac{1}{D+1} e^{e^z} - \frac{1}{D+2} e^{e^z}$$

$$= e^{-z}\int e^z e^{e^z}\, dz - e^{-2z}\int e^{2z} e^{e^z}\, dz$$

Let $e^z = t \Rightarrow e^z\, dz = dt$

$$= e^{-z}\int e^t\, dt - e^{-2z}\int t e^t\, dt = e^{-z} e^t - e^{-2z}(te^t - e^t)$$

$$= e^{-z} e^{e^z} - e^{-2z}(e^z e^{e^z} - e^{e^z}) = e^{-2z} e^{e^z}$$

Complete solution, $y = \text{CF} + \text{PI}$

$$y = C_1 e^{-z} + C_2 e^{-2z} + e^{-2z} e^{e^z}$$

$$y = C_1\left(\frac{1}{x}\right) + C_2\left(\frac{1}{x^2}\right) + \left(\frac{1}{x^2}\right) e^x$$

Que 1.26. **Using variation of parameters method, solve**

$$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 12y = 0$$

AKTU 2015-16, Marks 10

Answer

Same as Q. 1.25, Page 1–28F, Unit-1.

(**Answer :** $y = C_1 x_3 + C_2 / x_4$)

Que 1.27. **Apply method of variation of parameters to find the general solution of**

$$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 3x = \frac{e^t}{1+e^t}$$

AKTU 2012-13, Marks 10

Answer

$$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 3x = \frac{e^t}{1+e^t}$$

$$(D^2 - 4D + 3)x = \frac{e^t}{1+e^t}$$

Auxiliary equation, $m^2 - 4m + 3 = 0$

$$m = 1, 3$$

$$\text{CF} = C_1 e^t + C_2 e^{3t}$$

Here, part of CF are $u = e^t$, $v = e^{3t}$. Also, $R = \frac{e^t}{1+e^t}$

Let $x = Ae^t + Be^{3t}$ be the complete solution of the given equation where A and B are suitable function of t.

To determine A and B, we have

$$A = \int\frac{-Rv}{uv_1 - u_1v}\, dt + C_1 = -\int\frac{e^t e^{3t}}{(1+e^t)(3e^{4t} - e^{4t})}\, dt + C_1$$

$$= -\int\frac{e^{4t}}{2(1+e^t)e^{4t}}\, dt + C_1 = -\int\frac{e^{-t}}{2(e^{-t}+1)}\, dt + C_1$$

$$= \frac{1}{2}\ln(e^{-t}+1) + C_1$$

$$B = \int\frac{Ru}{uv_1 - u_1v}\, dt + C_2$$

$$= \int\frac{e^t e^t}{(1+e^t)(3e^{4t} - e^{4t})}\, dt + C_2 = \int\frac{e^{2t}}{2(1+e^t)e^{4t}}\, dt + C_2$$

$$= \frac{1}{2}\int\frac{e^{-2t}}{(1+e^t)}\, dt + C_2 = \frac{1}{2}\int\frac{e^{-3t}}{(e^{-t}+1)}\, dt + C_2$$

$$= -\frac{1}{4}(e^{-t}+1)^2 - \frac{1}{2}\ln(e^{-t}+1) + C_2$$

Hence the complete solution is

$$x = \left[\frac{1}{2}\ln(e^{-t}+1) + C_1\right]e^t + \left[-\frac{1}{4}(e^{-t}+1)^2 \ln(e^{-t}+1) + (e^{-t}+1) + C_2\right]e^{3t}$$

Que 1.28. **Solve by method of variation of parameters for the differential equation :**

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \left(\frac{e^{3x}}{x^2}\right)$$

AKTU 2016-17, Marks 07

Answer

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \left(\frac{e^{3x}}{x^2}\right)$$

Auxiliary equation,

$$m^2 - 6m + 9 = 0$$

$$(m-3)^2 = 0$$

$$m = 3, 3$$

So, $\text{CF} = (C_1 + C_2x)e^{3x}$

Here $u = e^{3x}$ and $v = x e^{3x}$ are two parts of CF

Also, $R = \frac{e^{3x}}{x^2}$

Let the complete solution be

$$y = A\,e^{3x} + Bx\,e^{3x}$$

To determine the values of A and B, we have

$$A = \int -\frac{Rv}{uv_1 - u_1v}\,dx + C_1$$

$$A = \int -\frac{\left(\dfrac{e^{3x}}{x^2}\right)xe^{3x}}{e^{3x}(e^{3x} + 3x\,e^{3x}) - xe^{3x}\,3e^{3x}}\,dx + C_1$$

$$A = -\int \frac{e^{6x}/x}{e^{6x}}\,dx + C_1$$

$$A = -\int \frac{1}{x}\,dx + C_1$$

$$A = -\log x + C_1$$

$$B = \int \frac{Ru}{uv_1 - u_1v}\,dx + C_2$$

$$B = \int \frac{\dfrac{e^{3x}}{x^2}\,e^{3x}}{e^{3x}(e^{3x} + 3x\,e^{3x}) - 3e^{3x}\,xe^{3x}}\,dx + C_2$$

$$B = \int \frac{1}{x^2}\,dx + C_2$$

$$B = -\frac{1}{x} + C_2$$

Hence the complete solution is

$$y = (-\log x + C_1)\,e^{3x} + \left(-\frac{1}{x} + C_2\right)x\,e^{3x}$$

Que 1.29. **Use variation of parameters method to solve the differential equation $x^2y'' + xy' - y = x^2e^x$.**

AKTU 2017-18, Marks 07

Answer

$$x^2y'' + xy' - y = x^2e^x. \qquad \text{...(1.29.1)}$$

$$y'' + \frac{y'}{x} - \frac{y}{x^2} = e^x \qquad \text{...(1.29.2)}$$

Here, $R = e^x$

Consider the equation $y'' + \dfrac{y'}{x} - \dfrac{y}{x^2} = 0$ for finding parts of CF

Put $x = e^z$ so that $z = \log x$

So, $[D(D-1) + D - 1]\,y = 0$

$$(D^2 - 1)\,y = 0 \qquad \text{...(1.29.3)}$$

Auxiliary equation, $m^2 - 1 = 0 \Rightarrow m = \pm 1$

$\therefore$ $$\text{CF} = C_1e^x + C_2e^{-x} = C_1x + C_2\frac{1}{x}$$

Hence parts of CF are x and $\dfrac{1}{x}$

Let $u = x$ and $v = \dfrac{1}{x}$

Let $y = Ax + \dfrac{B}{x}$ be the complete solution, where A and B are some suitable functions of x. A and B are determined as follows :

$$A = -\int \frac{Rv}{uv_1 - u_1v}\,dx + C_1$$

$$= -\int \frac{e^x\,\dfrac{1}{x}}{x\left(\dfrac{-1}{x^2}\right) - 1\left(\dfrac{1}{x}\right)}\,dx + C_1$$

$$= -\int \frac{e^x\,\dfrac{1}{x}}{\left(\dfrac{-2}{x}\right)}\,dx + C_1 = \frac{1}{2}e^x + C_1$$

and $$B = \int \frac{Ru}{uv_1 - u_1v}\,dx + C_2 = \int \frac{e^x x}{x\left(\dfrac{-1}{x^2}\right) - \left(\dfrac{1}{x}\right)}\,dx + C_2$$

$$= \int \frac{e^x x}{\left(\dfrac{-2}{x}\right)}\,dx + C_2 = -\frac{1}{2}\int x^2e^x\,dx + C_2$$

$$= -\frac{1}{2}\left[x^2\,e^x - \int 2x\,e^x\,dx\right] + C_2 = -\frac{1}{2}[x^2 - 2(x-1)\,e^x] + C_2$$

$$= -\frac{1}{2}x^2e^x + (x-1)\,e^x\,C_2$$

Hence the complete solution is given by

$$y = Ax + \frac{B}{x} = \left(\frac{1}{2}e^x + C_1\right)x + \left[-\frac{1}{2}x^2\,e^x + (x-1)\,e^x + C_2\right]\frac{1}{x}$$

$$y = C_1x + \frac{C_2}{x} + \left(1 - \frac{1}{x}\right)e^x$$

PART-6

Cauchy Euler Equation.

CONCEPT OUTLINE

Cauchy-Euler Equation : An equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2}y}{dx^{n-2}} + \dots\dots + a_{n-1} x \frac{dy}{dx} + a_n y = Q$$

Where a_i's are constants and Q is a function of x, called Cauchy's homogeneous linear equation. Such equations can be reduced to linear differential equations with constant coefficients by the substitution

$$x = e^t \quad \text{or} \quad z = \log x$$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.30. Solve : $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = (\log x)\sin(\log x)$.

UPTU 2014-15, Marks 05

Answer

$$x^2 y'' + xy' + y = (\log x)\sin(\log x)$$

This is the Cauchy Euler equation.

Put $x = e^t$, $t = \log x$, $x^2 y'' = D(D-1)y$, and we get $xy' = Dy$

$$[D(D-1) + D + 1]y = t \sin t$$

$$[D^2 - D + D + 1]y = t \sin t$$

$$(D^2 + 1)y = t \sin t$$

Auxiliary equation, $m^2 + 1 = 0$, $m = \pm i$

$$\text{CF} = C_1 \cos t + C_2 \sin t$$

$$\text{PI} = \frac{1}{D^2+1} t \sin t$$

$$= \text{Imaginary part of } \frac{1}{D^2+1} e^{it} \sin t$$

Put $D = D + i$,

$$= \text{Imaginary part of } e^{it} \frac{1}{(D+i)^2+1} \sin t$$

$$= \text{Imaginary part of } e^{it} \frac{1}{D^2 - 1 + 2Di + 1} \sin t$$

$$= \text{Imaginary part of } e^{it} \frac{1}{D^2 + 2Di} \sin t$$

Put $D^2 = -1$,

$$= \text{Imaginary part of } e^{it} \frac{1}{2Di - 1} \sin t$$

$$= \text{Imaginary part of } e^{it} \frac{2Di+1}{(2Di+1)(2Di-1)} \sin t$$

$$= \text{Imaginary part of } e^{it} \frac{(2Di+1)}{-4D^2-1} \sin t$$

$$= \text{Imaginary part of } e^{it} \frac{(1+2Di)}{3} \sin t$$

$$= \text{Imaginary part of } \frac{1}{3}(\cos t + i \sin t)(\sin t - 2i \cos t)$$

$$= \frac{1}{3}(\sin^2 t - 2\cos^2 t)$$

$$\text{PI} = \frac{1}{3}(\sin^2 t - 2\cos^2 t)$$

Complete solution, $y = \text{CF} + \text{PI} = C_1 \cos t + C_2 \sin t + \dfrac{1}{3}(\sin^2 t - 2\cos^2 t)$

Where, $t = \log x$

$$y = C_1 \cos(\log x) + C_2 \sin(\log x) + \frac{1}{3}[\sin^2(\log x) - 2\cos^2(\log x)]$$

PART-7

Series Solution (Frobenius Method).

CONCEPT OUTLINE

Frobenius Method : Following are the steps of solving differential equation with the help of frobenius method :

1. Assume $y = a_0 x_m + a_1 x^{m+1} + a_2 x^{m+2} + \dots$ ...(1)
2. Substitute from eq. (1) for y, $\dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}$ in given equation
3. Equate to zero the coefficient of lowest power of x. This gives a quadratic equation in m which is known as the Indicial equation.
4. Equate to zero, the coefficients of other powers of x to find $a_1, a_2, a_3, \dots$ in terms of a_0.
5. Substitute the values of $a_1, a_2, a_3, \dots$ in eq. (1) to get the series solution of the given equation having a_0 as arbitrary constant. Obviously, this is not the complete solution of given equation since the complete solution must have two independent arbitrary constants.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.31. Find the series solution of the following differential equation.

$$2x(1-x)\frac{d^2y}{dx^2}+(1-x)\frac{dy}{dx}+3y=0$$

AKTU 2015-16, Marks 10

Answer

$$2x(1-x)y''+(1-x)y'+3y=0 \quad ...(1.31.1)$$

Dividing eq. (1.31.1) by $2x(1-x)$, we get

$$y''+\frac{1}{2x}y'+\frac{3}{2x(1-x)}y=0 \quad ...(1.31.2)$$

Comparing eq. (1.31.2) with $y''+P(x)y'+Q(x)y=0$, we get

$$P(x)=\frac{1}{2x} \text{ and } Q(x)=\frac{3}{2x(1-x)}$$

Here $P(x)$ and $Q(x)$ both are non-analytic at $x=0$. But $xP(x)=\frac{1}{2}$ and

$x^2Q(x)=\frac{3x}{(1-x)}$ are analytic therefore $x=0$ is a regular singular point.

Let the solution of the given differential equation is

$$y=\sum_{k=0}^{\infty}a_kx^{m+k}$$

$$y'=\sum_{k=0}^{\infty}a_k(m+k)x^{m+k-1}$$

$$y''=\sum_{k=0}^{\infty}a_k(m+k)(m+k-1)x^{m+k-2}$$

Putting all these values in given differential equation and collecting the like terms, we get

$$\sum_{k=0}^{\infty}a_k(m+k+1)(-2m-2k+3)x^{m+k}+\sum_{k=0}^{\infty}a_k(m+k)(2m+2k-1)x^{m+k-2}=0 \quad ...(1.31.3)$$

Equating the coefficient of lowest degree term x^{m-2} to zero.

$$a_0m(2m-1)=0$$

$\because$ $a_0\neq 0$

$\therefore$ $m=0,\frac{1}{2}$

Roots are different and not differing by an integer. The general term is obtained by replacing k by $k+1$ in second summation of eq. (1.31.3).

$a_k(m+k+1)(-2m-2k+3)+a_{k+1}(m+k+1)(2m+2k+1)=0$

$\therefore$ $$a_{k+1}=\frac{-(m+k+1)(-2m-2k+3)}{(m+k+1)(2m+2k+1)}a_k$$

Thus, $$a_{k+1}=\frac{2m+2k-3}{2m+2k+1}a_k$$

Putting $k=0,1,2$.........

$$a_1=\frac{2m-3}{2m+1}a_0$$

$$a_2=\frac{(2m-1)}{(2m+3)}a_1$$

$$a_3=\frac{(2m+1)}{(2m+5)}a_2$$

$$a_4=\frac{(2m+3)}{(2m+7)}a_3$$

$$a_5=\frac{(2m+5)}{(2m+9)}a_4$$

At $m=0$, $a_1=-3a_0, a_2=a_0, a_3=\frac{1}{5}a_0, a_4=\frac{3}{35}a_0, a_5=\frac{1}{21}a_0$

$$y_1=y_{m=0}=x^m(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+....)$$

$$=x^0a_0\left(1-3x+x^2+\frac{1}{5}x^3+\frac{3}{35}x^4+\frac{1}{21}x^5+....\right)$$

$$y_1=a_0\left(1-3x+\frac{3x^2}{1.3}+\frac{3}{3.5}x^3+\frac{3}{5.7}x^4+\frac{3}{7.9}x^5+.....\right)$$

At $m=1/2$, $a_1=-a_0, a_2=0, a_3=0, a_4=a_5=a_6=............=0$

$$y_2=(y)_{m=1/2}=x^{1/2}a_0(1-x+0......)$$

$$y_2=\sqrt{x}\,a_0(1-x)$$

General solution is $y=Ay_1+By_2$

$$y=A\left(1-3x+\frac{3}{1.3}x^2+\frac{3}{3.5}x^3+\frac{3}{5.7}x^4+\frac{3}{7.9}x^5+.....\right)+B\sqrt{x}(1-x)$$

Que 1.32. Solve in series : $2x^2y''+x(2x+1)y'-y=0$.

AKTU 2014-15, Marks 10

Answer

$$2x^2y''+x(2x+1)y'-y=0 \quad ...(1.32.1)$$

$x=0$ is a regular singular point.

Let, $$y = \sum_{k=0}^{\infty} a_k x^{m+k}$$

$$y' = \sum_{k=0}^{\infty} a_k (m + k)\, x^{m+k-1}$$

$$y'' = \sum_{k=0}^{\infty} a_k (m + k)\, x^{m+k-2} (m + k - 1)$$

Putting the value of y, y' and y'' in eq. (1.32.1), we get

$$2\sum_{k=0}^{\infty} a_k (m+k)(m+k-1)\, x^{m+k} + 2\sum_{k=0}^{\infty} a_k (m+k)\, x^{m+k+1} + \sum_{k=0}^{\infty} a_k (m+k) x^{(m+k)} - \sum_{k=0}^{\infty} a_k\, x^{m+k} = 0$$

$$2\sum_{k=0}^{\infty} a_k (m+k)\, x^{m+k+1} + \sum_{k=0}^{\infty} a_k\, [(m+k)\,(2m+2k-2+1)-1] x^{m+k} = 0$$

$$2\sum_{k=0}^{\infty} a_k (m+k)\, x^{m+k+1} + \sum_{k=0}^{\infty} a_k (m+k-1)(2m+2k+1) x^{m+k} = 0$$

Equating the lowest degree term to zero by putting $k = 0$ in second summation,

$a_0\,(m-1)\,(2m+1) = 0$

$a_0 \neq 0$

$$m = 1, -\frac{1}{2}$$

Roots are different and their difference is not an integer.

Thus, $$y = C_1 (y)_{m=1} + C_2 (y)_{m} = \frac{-1}{2}$$

Equating the general terms,

$2a_k\,(m+k) + a_{k+1}\,(m+k)\,(2m+2k+3) = 0$

$$a_{k+1} = \frac{-2a_k}{(2m+2k+3)}$$

Putting $k = 0, 1, 2, \ldots$

$$a_1 = \frac{-2a_0}{2m+3}$$

$$a_2 = \frac{-2a_1}{(2m+5)}$$

$$a_3 = \frac{-2a_2}{(2m+7)} \text{ and so on}$$

At $m = 1$,

$$a_1 = \frac{-2a_0}{5}$$

$$a_2 = \frac{-2}{7}\left(\frac{-2a_0}{5}\right) = \frac{4a_0}{35}$$

$$a_3 = \frac{-2}{9}\left(\frac{4a_0}{35}\right) = \frac{-8a_0}{5.7.9}$$

$$a_4 = \frac{16a_0}{5.7.9.11}$$

At $m = -\frac{1}{2}$,

$$a_1 = \frac{-2a_0}{2} = -a_0$$

$$a_2 = \frac{-2}{4}(-a_0) = \frac{a_0}{2}$$

$$a_3 = \frac{-2}{6}\left(\frac{a_0}{2}\right) = \frac{-a_0}{6}$$

$$a_4 = \frac{-2}{8}\left(\frac{-a_0}{6}\right) = \frac{a_0}{24}$$

Thus, $$y = C_1 x\, a_0 \left[1 - \frac{2}{5}x + \frac{4}{35}x^2 - \frac{8}{5.7.9}x^3 + \frac{16}{5.7.9.11}x^4 \ldots\right] + C_2 x^{-1/2} a_0 \left[1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 \ldots\right]$$

Que 1.33. **Use Frobenius series method to find the series solution of $(1-x^2)y'' - xy' + 4y = 0$**

AKTU 2011-12, Marks 10

Answer

$$(1-x^2)y'' - xy' + 4y = 0$$

Let $$x + 1 = t$$

$$t(2-t)y'' - (t-1)y' + 4y = 0 \qquad \ldots(1.33.1)$$

Dividing eq. (1.33.1) by $t(2-t)$, we get

$$y'' - \frac{(t-1)}{t(2-t)}y' + \frac{4}{t(2-t)}y = 0 \qquad \ldots(1.33.2)$$

Comparing eq. (1.33.2) with $y'' + P(t)\,y' + Q(t)\,y = 0$

$$P(t) = \frac{-(t-1)}{t(2-t)} \text{ and } Q(t) = \frac{4}{t(2-t)}$$

$t = 0$ is a singular point for the given differential equation.

Let, $$y = \sum_{k=0}^{\infty} a_k t^{m+k} \text{ is a solution}$$

$$y' = \sum a_k (m+k) t^{m+k-1}$$

$$y'' = \sum a_k (m+k)(m+k-1) t^{m+k-2}$$

From eq. (1.33.1),

$$t(2-t)\sum a_k (m+k)(m+k-1)\, t^{m+k-2} - (t-1)\sum a_k (m+k)\, t^{m+k-1} + 4\sum a_k t^{m+k} = 0$$

$$2\sum a_k (m+k)(m+k-1)\, t^{m+k-1} - \sum a_k (m+k)(m+k-1) t^{m+k}$$

$$-\sum a_k(m+k)\,t^{m+k} + \sum a_k(m+k)\,t^{m+k-1} + 4\sum a_k\,t^{m+k} = 0$$

$$\sum a_k(m+k)(2m+2k-2+1)\,t^{m+k-1} - \sum a_k[(m+k)(m+k)-4]t^{m+k} = 0$$

$$\sum a_k(m+k)(2m+2k-1)\,t^{m+k-1} - \sum a_k(m+k+2)(m+k-2)t^{m+k} = 0 \quad \text{...(1.33.3)}$$

Putting $k = 0$ in lowest degree term, t^{m-1}

$$a_0 m(2m-1) = 0$$

$\because$ $a_0 \neq 0$

$\therefore$ $m = 0, 1/2$

Putting $k = k + 1$ in first summation and $k = k$ in second summation of eq. (1.33.3)

$$a_{k+1}(m+k+1)(2m+2k+1) - a_k(m+k+2)(m+k-2) = 0$$

$$a_{k+1} = \frac{(m+k+2)(m+k-2)}{(m+k+1)(2m+2k+1)}a_k$$

Putting $k = 0, 1, 2, 3, \ldots$

$$a_1 = \frac{(m+2)(m-2)}{(m+1)(2m+1)}a_0,\; a_2 = \frac{(m+3)(m-1)}{(m+2)(2m+3)}a_1,\; a_3 = \frac{(m+4)m}{(m+3)(2m+5)}a_2$$

At $m = 0$,

$$a_1 = \frac{-4}{1}a_0 = -4a_0$$

$$a_2 = \frac{-3}{6}a_1 = 2a_0$$

$$a_3 = 0$$

At $m = 1/2$,

$$a_1 = -\frac{5}{4}a_0$$

$$a_2 = \frac{7}{32}a_0$$

$$a_3 = \frac{3}{128}a_0$$

Thus,

$$y = C_1(y)_{m=0} + C_2(y)_{m=1/2}$$

$$y = C_1\left[\sum_{k=0}^{\infty} a_k t^{m+k}\right]_{m=0} + C_2\left[\sum_{k=0}^{\infty} a_k t^{m+k}\right]_{m=\frac{1}{2}}$$

$$= C_1[a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots] + C_2\left[a_0(t)^{\frac{1}{2}} + a_1(t)^{\frac{3}{2}} + a_2(t)^{\frac{5}{2}} + a_3(t)^{\frac{7}{2}} + \ldots\right]$$

$$= C_1[a_0 + (-4a_0)(1+x) + 2a_0(1+x)^2 + 0] +$$

$$C_2\left[a_0(1+x)^{1/2} + a_0\left(\frac{-5}{4}\right)(1+x)^{3/2} + \frac{7}{32}a_0(1+x)^{5/2} + \frac{3}{128}a_0(1+x)^{7/2} + \ldots\right]$$

$$= C_1 a_0[1 - 4 - 4x + 2 + 2x^2 + 4x] + C_2 a_0(1+x)^{1/2}$$

$$\left[1 - \frac{5}{4}(1+x) + \frac{7}{32}(1+x)^2 + \frac{3}{128}(1+x)^3 + \ldots\right]$$

$$= C_1 a_0[1 + 2x^2] + C_2 a_0(1+x)^{1/2}$$

$$\left[1 - \frac{5}{4}(1+x) + \frac{7}{32}(1+x)^2 + \frac{3}{128}(1+x)^3 + \ldots\right]$$

Que 1.34. **Find the Frobenius series solution of the following differential equation about $x = 0$.**

$$\mathbf{2x^2y'' + 7x(x+1)y' - 3y = 0.}$$

AKTU 2012-13, Marks 10

Answer

$$2x^2y'' + 7x(x+1)y' - 3y = 0 \quad \text{...(1.34.1)}$$

$x = 0$ is a regular singular point.

Let,

$$y = \sum_{k=0}^{\infty} a_k x^{m+k}$$

$$y' = \sum_{k=0}^{\infty} a_k(m+k)\,x^{m+k-1}$$

$$y'' = \sum_{k=0}^{\infty} a_k(m+k)\,x^{m+k-2}(m+k-1)$$

Putting the value of y, y' and y'' in eq. (1.34.1), we have

$$2x^2\sum_{k=0}^{\infty} a_k(m+k)(m+k-1)\,x^{m+k-2} + 7x^2\sum_{k=0}^{\infty} a_k(m+k)\,x^{m+k-1}$$

$$+7x\sum_{k=0}^{\infty} a_k(m+k)x^{m+k-1} - 3\sum_{k=0}^{\infty} a_k\,x^{m+k} = 0$$

$$\sum_{k=0}^{\infty} a_k(m+k)(2m+2k-2+7)x^{m+k} - 3\sum_{k=0}^{\infty} a_k x^{m+k} + 7\sum_{k=0}^{\infty} a_k(m+k)x^{m+k+1} = 0$$

$$\sum_{k=0}^{\infty} a_k(m+k)(2m+2k+5)\,x^{m+k} - 3\sum_{k=0}^{\infty} a_k x^{m+k} + 7\sum_{k=0}^{\infty} a_k(m+k)x^{m+k+1} = 0$$

Equating the lowest degree term to zero by putting $k = 0$ in first summation,

$$a_0\,m(2m+5) - 3 = 0$$

$\because$ $a_0 \neq 0$

$\therefore$ $2m^2 + 5m - 3 = 0$

$$m = -3, \frac{1}{2}$$

Roots are different and their difference is not an integer,

Thus, $y = C_1(y)_{m=-3} + C_2(y)_{m=1/2}$

Equating the general terms,

$$a_{k+1}[(m+k+1)(2m+2k+2+5)-3] + 7a_k(m+k) = 0$$

$$a_{k+1} = \frac{-7a_k(m+k)}{[(m+k+1)(2m+2k+7)-3]}$$

Putting $k = 0, 1, 2, \ldots$

$$a_1 = \frac{-7a_0\,m}{[(m+1)(2m+7)-3]}$$

$$a_2 = \frac{-7a_0(m+1)}{[(m+2)(2m+9)-3]}$$

$$= \frac{49\,a_0\,m(m+1)}{[(m+1)(2m+7)-3][(m+2)(2m+9)-3]}$$

$$a_3 = \frac{-7a_2(m+2)}{[(m+3)(2m+11)-3]}$$

$$= \frac{-343\,a_0\,m(m+1)(m+2)}{[(m+1)(2m+7)-3][(m+2)(2m+9)-3][(m+3)(2m+11)-3]}$$

At $m = \frac{1}{2}$,

$$a_1 = \frac{-7a_0}{18}$$

$$a_2 = \frac{-7\,a_1 \times (3/2)}{[(5/2)\times 10 - 3]} = \frac{49\,a_0}{264}$$

$$a_3 = \frac{-7\,a_2 \times (5/2)}{[(7/2)\times 12 - 3} = -\frac{1215\,a_0}{20592}$$

At $m = -3$,

$$a_1 = \frac{-21a_0}{5}$$

$$a_2 = \frac{-7\,a_1 \times (-2)}{[(-1)\times 3 - 3]} = \frac{49\,a_0}{5}$$

$$a_3 = 0$$

Thus,

$$y = C_1(y)_{m=-3} + C_2(y)_{m=1/2}$$

$$y = C_1[a_0x^{-3}x^0 + a_1x^{-3}x^1 + a_2x^{-3}x^2 + a_3x^{-3}x^3 \ldots] + C_2[a_0x^{1/2}x^0 + a_1x^{1/2}x^1 + a_2x^{1/2}x^2 + a_3x^{1/2}x^3 \ldots]$$

$$y = C_1\,a_0\,x^{-3}\left[1 - \frac{21}{5}x + \frac{49}{5}x^2 + \ldots\right] + C_2\,a_0\,x^{1/2}\left[1 - \frac{7}{18}x + \frac{49}{264}x^2 - \frac{1215}{20592}x^3 \ldots\right]$$

Que 1.35. **Find the series solution by Forbenius method for the differential equation $(1 - x^2)y'' - 2xy' + 20y = 0$**

AKTU 2016-17, Marks 07

Answer

Same as Q. 1.33, Page 1–38F, Unit-1.

Answer : $y = [A + B\log(x+1)]\left(1 - 10t + \frac{45}{2}(x+1)^2 t^2 + \left(\frac{-35}{2}(x+1)^3 + \ldots\right)\right)$

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