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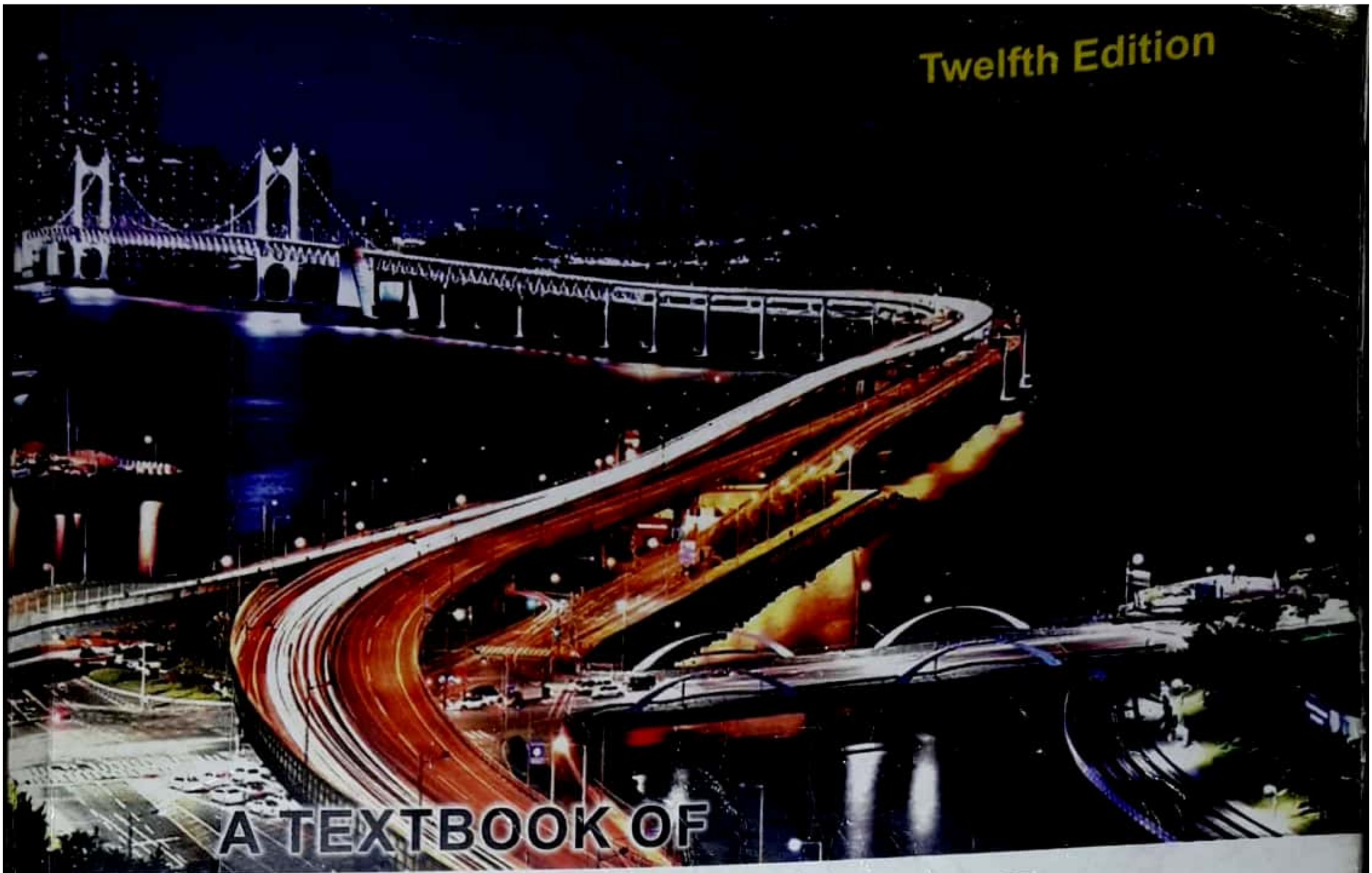

A TEXTBOOK OF

ENGINEERING MATHEMATICS

Strictly as per the New AICTE Model Syllabus (KAS-203)

For DR. A.P.J. Abdul Kalam Technical University, Lucknow

SEMESTER-II

N.P. Bali

Dr. Manish Goyal

A TEXTBOOK OF

ENGINEERING MATHEMATICS

For

B.TECH. I YEAR, ***SEMESTER II***

According to the New AICTE Model Syllabus (KAS-203) of

Dr. A.P.J. ABDUL KALAM TECHNICAL UNIVERSITY, LUCKNOW

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SYLLABUS

DR. A.P.J. ABDUL KALAM TECHNICAL UNIVERSITY, LUCKNOW

(B.Tech. Semester-II)

ENGINEERING MATHEMATICS—II

(KAS-203)

(Common to all B. Tech. Courses except B. Tech. Biotechnology and Agricultural Engineering)

MODULE 1: Ordinary Differential Equation of Higher Order **[10]**

Linear differential equation of nth order with constant coeffcients, Simultaneous linear differential equations, Second order linear differential equations with variable coefficients, Solution by changing independent variable, Reduction of order, Normal form, Method of variation of parameters, Cauchy-Euler equation, Series solutions (Frobenius Method).

MODULE 2: Multivariable Calculus-II **[08]**

Improper integrals, Beta & Gama function and their properties, Dirichlet's integral and its applications, Application of definite integrals to evaluate surface areas and volume of revolutions.

MODULE 3: Sequences and Series **[08]**

Definition of Sequence and series with examples, Convergence of sequence and series, Tests for convergence of series, (Ratio test, D' Alembert's test, Raabe's test). Fourier series, Half range Fourier sine and cosine series.

MODULE 4: Complex Variable – Differentiation **[08]**

Limit, Continuity and differentiability, Functions of complex variable, Analytic functions, Cauchy–Riemann equations (Cartesian and Polar form) Harmonic function, Method to find Analytic functions, Conformal mapping, Mobius transformation and their properties.

MODULE 5: Complex Variable – Integration **[08]**

Complex integrals, Contour integrals, Cauchy–Goursat theorem, Cauchy integral formula, Taylor's series, Laurent's series, Liouvilles's theorem, Singularities, Classification of Singularities, zeros of analytic functions, Residues, Methods of finding residues, Cauchy Residue theorem, Evaluation of real integrals of the type $\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$ and $\int_{-\infty}^{\infty} f(x)dx$.

MODULE 1

Ordinary Differential Equations of Higher Order

1.1 DEFINITIONS

1.1.1 Differential Equation

A **differential equation** is an equation involving differentials or differential coefficients.

Or

An equation involving the dependent variable, independent variable and the differential coefficient (or coefficients) of the dependent variable with respect to the independent variable (or variables) is known as a differential equation. For example:

$$\frac{dy}{dx} = \cot x \qquad \ldots(1) \qquad\qquad \frac{d^2y}{dx^2} + y = 0 \qquad \ldots(2)$$

$$y = x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^3 \qquad \ldots(3) \qquad\qquad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z \qquad \ldots(4)$$

$$\frac{d^2y}{dx^2} + \sqrt{1+\left(\frac{dy}{dx}\right)^3} = 0 \qquad \ldots(5) \qquad\qquad \rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}} \qquad \ldots(6)$$

are all differential equations.

1.1.2 Ordinary Differential Equation

A differential equation which involves only one independent variable is called an ordinary differential equation *e.g.*, All equations excluding (4) given above are ordinary differential equations.

1.1.3 Partial Differential Equation

A differential equation which involves two or more independent variables and partial derivatives with respect to them is called partial differential equation.

e.g., Equation (4) given above is a partial differential equation.

1.1.4 Order of a Differential Equation

The **order** of a differential equation is the order of the highest ordered derivative occurring in the differential equation. *e.g.*, Eqns. (1), (2) and (5) are of orders 1, 2 and 2 respectively.

1.1.5 Degree of a Differential Equation

The **degree** of a differential equation is the degree of the highest ordered derivative present in the differential equation when it is made free from radical signs and fractional powers. *e.g.*, Degree of equation (1) is 1.

Now consider equation (5)

$$\frac{d^2y}{dx^2} + \sqrt{1 + \left(\frac{dy}{dx}\right)^3} = 0$$

It involves radical sign. So to find the degree, we shall remove the radical sign. To achieve the purpose, squaring, we get

$$\left(\frac{d^2y}{dx^2}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^3$$

Clearly its degree is 2.

Again, consider equation (6)

$$\rho = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}.$$

It involves fractional powers. So to find the degree, we shall remove the fractional power. To achieve the purpose, squaring, we get

$$\rho^2 = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^3}{\left(\frac{d^2y}{dx^2}\right)^2}$$

Clearly its degree is 2.

Note. Equations of degree higher than one are called non-linear equations.

1.1.6 Solution of a Differential Equation

The solution is one which satisfies. A solution (or integral) of a differential equation is a relation, free from derivatives, between the variables which satisfies the given equation. It is also called **primitive** because the differential equation can be regarded as a relation derived from it.

Thus if $y = f(x)$ is the solution, then by replacing y and its derivatives with respect to x, the given differential equation will reduce to an identity.

e.g., $y = c_1 \cos x + c_2 \sin x$ is the solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$.

1.1.7 General Solution

The **general** (or **complete**) **solution** of a differential equation is the solution in which the number of arbitrary constants is equal to the order of the differential equation.

Thus, $y = c_1 \cos x + c_2 \sin x$ (involving two arbitrary constants c_1, c_2) is the general solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$ of second order.

1.1.8 Particular Solution

A **particular solution** of a differential equation is the solution which is obtained from its general solution by giving particular values to the arbitrary constants.

For example, $y = c_1e^x + c_2e^{-x}$ is the general solution of the differential equation $\frac{d^2y}{dx^2} - y = 0$, whereas $y = e^x - e^{-x}$ or $y = e^x$ are its particular solutions.

The solution of a differential equation of n^{th} order is its particular solution if it contains less than n arbitrary constants.

1.2 ARBITRARY CONSTANTS

In order to find out the number of arbitrary constants in the most general solution of a differential equation, we shall study how a differential equation is formed if the solution is given.

Let the solution be $f(x, y, a) = 0$ ...(1)

where 'a' is an arbitrary constant.

Differentiating equation (1) w.r.t. x, we get

$$\phi\left(x, y, \frac{dy}{dx}, a\right) = 0 \quad \text{...(2)}$$

Eliminating 'a' between eqns. (1) and (2), we get

$$\psi\left(x, y, \frac{dy}{dx}\right) = 0 \quad \text{...(3)}$$

which is a differential equation of I order.

Thus, the general solution of a differential equation of I order contains one arbitrary constant.

Again, let the solution be

$$f(x, y, a, b) = 0 \quad \text{...(1)}$$

where 'a' and 'b' are two arbitrary constants.

Differentiating equation (1) w.r.t. x, we get

$$\phi_1\left(x, y, \frac{dy}{dx}, a, b\right) = 0 \quad \text{...(2)}$$

Again, Differentiation w.r.t. x yields

$$\phi_2\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, a, b\right) = 0 \quad \text{...(3)}$$

Eliminating a and b from eqns. (1), (2) and (3), we get

$$\psi\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0 \quad \text{...(4)}$$

which is a differential equation of II order.

Thus, the general solution of a differential equation of II order contains two arbitrary constants and so on.

Therefore, it is concluded that the number of arbitrary constants in the general solution of an ordinary differential equation is equal to the order of the differential equation.

1.3 FORMATION OF A DIFFERENTIAL EQUATION

Differential equations are formed by elimination of arbitrary constants. To eliminate two arbitrary constants, we require two more equations besides the given relation, leading us to second order derivatives and hence a differential equation of the second order. Elimination of n arbitrary constants leads us to n^{th} order derivatives and hence a differential equation of the n^{th} order.

Example. *Eliminate the arbitrary constants A and B from the equation* $y = e^x$ *(A cos x + B sin x) and obtain the differential equation.*

Sol. We have the relation.

$$y = e^x (\mathrm{A} \cos x + \mathrm{B} \sin x) \qquad \ldots(1)$$

Differentiating equation (1) w.r.t. x, we have

$$\frac{dy}{dx} = e^x (\mathrm{A} \cos x + \mathrm{B} \sin x) + e^x (-\mathrm{A} \sin x + \mathrm{B} \cos x)$$

$$= y + e^x (-\mathrm{A} \sin x + \mathrm{B} \cos x) \qquad \ldots(2)$$

Differentiating again w.r.t. x, we have

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + e^x (-\mathrm{A} \sin x + \mathrm{B} \cos x) + e^x (-\mathrm{A} \cos x - \mathrm{B} \sin x)$$

$$= \frac{dy}{dx} + \left(\frac{dy}{dx} - y\right) - y \qquad \text{[Using (1) and (2)]}$$

or

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

which is the required differential equation.

1.4 LINEARLY DEPENDENT AND INDEPENDENT SOLUTIONS

Consider a second order differential equation

$$a_0(x) \frac{d^2y}{dx^2} + a_1(x) \frac{dy}{dx} + a_2(x)\, y = 0 \qquad \ldots(1)$$

where $a_0(x)$, $a_1(x)$ and $a_2(x)$ are continuous on an interval (a, b) and $a_0(x) \neq 0 \;\forall\; x \in (a, b)$. Then two solutions $y_1(x)$ and $y_2(x)$ of equation (1) are said to be linearly dependent if $\exists$ two constants c_1 and c_2, not both zero, such that

$$c_1y_1 + c_2y_2 = 0 \quad \forall\, x \in (a, b)$$

Two solutions $y_1(x)$ and $y_2(x)$ are said to be linearly independent if they are not linearly dependent.

In other words, two solutions $y_1(x)$ and $y_2(x)$ are said to be linearly independent if

$$c_1y_1 + c_2y_2 = 0$$

$$\Rightarrow \quad c_1 = 0 \text{ and } c_2 = 0\,;\, x \in (a, b).$$

1.5 THE WRONSKIAN OR WRONSKI DETERMINANT

I.M. Hone (1778–1853) was a polish mathematician who changed his name to Wronski.

Let $y_1(x)$ and $y_2(x)$ be two solutions of the second order differential equation

$$a_0(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x)\,y=0.$$

Then the Wronskian of $y_1(x)$ and $y_2(x)$ is given by

$$W(y_1, y_2) \quad \text{or} \quad W(x)=\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x)\end{vmatrix}$$

Again,

Let $\quad y_1(x), y_2(x), \ldots\ldots, y_n(x)$ be n solutions of

$$\frac{d^ny}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+\ldots\ldots+P_ny=0$$

Then, Wronskian of $y_1(x), y_2(x), \ldots\ldots, y_n(x)$ is denoted by $W(y_1, y_2, \ldots\ldots, y_n)$ or $W(x)$ and defined by the determinant

$$W(x)=\begin{vmatrix} y_1 & y_2 & \ldots\ldots & y_n \\ y_1' & y_2' & \ldots\ldots & y_n' \\ y_1'' & y_2'' & \ldots\ldots & y_n'' \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots\ldots & y_n^{(n-1)}\end{vmatrix}.$$

1.6 SOME IMPORTANT THEOREMS

(*i*) If $y_1(x)$ and $y_2(x)$ are any two solutions of $a_0(x)\dfrac{d^2y}{dx^2}+a_1(x)\dfrac{dy}{dx}+a_2(x)\,y=0$, then the linear combination $c_1y_1(x)+c_2y_2(x)$, where c_1 and c_2 are constants is also a solution of the given equation.

(*ii*) Two solutions $y_1(x)$ and $y_2(x)$ of the equation

$$a_0(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x)\,y=0\ ;\ a_0(x)\neq 0,\ x\in(a,b)$$

are linearly dependent iff their Wronskian is identically zero.

Corollary. Solutions are linearly independent iff their Wronskian is not zero at some point $x_0 \in (a, b)$.

ILLUSTRATIVE EXAMPLES

Example 1. *Show that $y_1(x) = \sin x$ and $y_2(x) = \sin x - \cos x$ are linearly independent solutions of $y'' + y = 0$. Determine the constants c_1 and c_2 so that*

$$\sin x + 3\cos x \equiv c_1y_1(x) + c_2y_2(x).$$

Sol. Given equation is $\qquad y''+y=0 \qquad \ldots(1)$

Here, $\qquad y_1(x) = \sin x$

$\therefore \qquad y_1'(x) = \cos x \quad$ and $\quad y_1''(x) = -\sin x$

Since, $\quad y_1''(x)+y_1(x) = -\sin x+\sin x = 0$ hence $y_1(x)$ is a solution of (1).

Similarly, we can show that $y_2(x)$ is also a solution of (1).

Now, the Wronskian of $y_1(x)$ and $y_2(x)$ is given by

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} \sin x & \sin x - \cos x \\ \cos x & \cos x + \sin x \end{vmatrix}$$

$$= \sin x\,(\cos x + \sin x) - \cos x(\sin x - \cos x) = 1 \neq 0$$

which shows that $y_1(x)$ and $y_2(x)$ are linearly independent solutions of (1).

Given that $\sin x + 3\cos x \equiv c_1 y_1(x) + c_2 y_2(x)$

$$\equiv c_1 \sin x + c_2 (\sin x - \cos x)$$

Comparing the coefficients of $\sin x$ and $\cos x$ on both sides, we get

$$c_1 + c_2 = 1 \quad \text{and} \quad -c_2 = 3$$

so that,

$$c_1 = 4, \; c_2 = -3.$$

Example 2. *Prove that the functions 1, x, x^2 are linearly independent. Hence form the differential equation whose roots are 1, x, x^2.*

Sol. Let $y_1(x) = 1, \; y_2(x) = x$ and $y_3(x) = x^2$.

Then the Wronskian $W(x)$ of y_1, y_2, y_3 is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix}$$

or

$$W(x) = 2 \neq 0 \quad \forall\, x \in (-\infty, \infty)$$

$\therefore$ y_1, y_2 and y_3 are linearly independent.

The general solution of the required differential equation may be written as

$$y = c_1 y_1 + c_2 y_2 + c_3 y_3 = c_1 + c_2 x + c_3 x^2 \quad \ldots(1)$$

where c_1, c_2, c_3 are arbitrary constants.

Differentiating (1), we get $\quad y' = c_2 + 2c_3 x \quad \ldots(2)$

Differentiating again, we get $\quad y'' = 2c_3 \quad \ldots(3)$

Differentiating (3), we get $\quad y''' = 0$ or $\dfrac{d^3 y}{dx^3} = 0 \quad \ldots(4)$

Since equation (4) is free from arbitrary constants c_1, c_2 and c_3, hence equation (4) is the required differential equation.

Example 3. *Determine the differential equation whose set of independent solutions is $\{e^x, xe^x, x^2e^x\}$.* (A.K.T.U. 2018)

Sol. Let the general solution of the required differential equation be

$$y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x \quad \ldots(1)$$

Differentiating (1) w.r.t. x, we get

$$y' = c_1 e^x + c_2 (x + 1) e^x + c_3 (x^2 + 2x) e^x \quad \ldots(2)$$

From (1) and (2), we get

$$y = y' - c_2 e^x - 2c_3 x e^x \quad \ldots(3)$$

Differentiating (3) w.r.t. x, we get

$$y' = y'' - c_2 e^x - 2c_3 (x + 1) e^x \quad \ldots(4)$$

From (3) and (4), we get

$$y = y' + y' - y'' + 2c_3 e^x = 2y' - y'' + 2c_3 e^x \quad \ldots(5)$$

Differentiating (5) w.r.t. x, we get

$$y' = 2y'' - y''' + 2c_3e^x \quad \text{...(6)}$$

From (5) and (6), we get $\quad y = 2y' - y'' + y' - 2y'' + y'''$

$$\Rightarrow \quad y''' - 3y'' + 3y' - y = 0$$

which is the required differential equation.

Example 4. *If* $y = y_1(x)$ *and* $y = y_2(x)$ *are two solutions of the equation* $\dfrac{d^2y}{dx^2} + P(x)\dfrac{dy}{dx}$ $+ Q(x)\, y = 0$ *then*

show that $\quad y_1\left(\dfrac{dy_2}{dx}\right) - y_2\left(\dfrac{dy_1}{dx}\right) = ce^{-\int Pdx}$, where c is constant. (A.K.T.U. 2017)

Sol. Since y_1 and y_2 are solutions of $y'' + Py' + Qy = 0$, therefore we have

$$y_1'' + Py_1' + Qy_1 = 0 \quad \text{...(1)}$$

and

$$y_2'' + Py_2' + Qy_2 = 0 \quad \text{...(2)}$$

From equations (1) and (2),

$$\frac{y''_1 + Py'_1}{y_1} = \frac{y''_2 + Py'_2}{y_2} \quad \text{| Eliminating Q}$$

$$\Rightarrow \quad \frac{\dfrac{d}{dx}(y_1' + Py_1) - y_1\dfrac{dP}{dx}}{y_1} = \frac{\dfrac{d}{dx}(y_2' + Py_2) - y_2\dfrac{dP}{dx}}{y_2}$$

$$\Rightarrow \quad y_2\frac{d}{dx}(y_1' + Py_1) = y_1\frac{d}{dx}(y_2' + Py_2) \quad \text{| Cross-multiplying}$$

Integration w.r.t. x yields,

$$\int y_2\frac{d}{dx}(y'_1 + Py_1)\,dx = \int y_1\frac{d}{dx}(y'_2 + Py_2)dx$$

$$\Rightarrow \quad y_2(y_1' + Py_1) - \int y'_2(y'_1 + Py_1)\,dx = y_1(y_2' + Py_2) - \int y'_2(y'_2 + Py_2)\,dx$$

$$\Rightarrow \quad y_2y'_1 - \int Py_1y'_1dx = y_1y'_2 - \int Py'_1y_2dx$$

$$\Rightarrow \quad y_1y'_2 - y_2y'_1 = -\int P(y_1y'_2 - y'_1y_2)dx$$

$$\frac{\dfrac{d}{dx}(y_1y'_2 - y_2y'_1)}{y_1y'_2 - y'_1y_2} = -P \quad \text{| Differentiating both sides w.r.t. } x$$

Integration w.r.t. x yields,

$$\log(y_1y_2' - y_1'y_2) - \log c = -\int Pdx$$

or,

$$y_1y_2' - y_1'y_2 = ce^{-\int Pdx}$$

or,

$$y_1\left(\frac{dy_2}{dx}\right) - y_2\left(\frac{dy_1}{dx}\right) = ce^{-\int Pdx}$$

where c is constant.

1.7 DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AND FIRST DEGREE

A differential equation of the form $\frac{dy}{dx} = f(x, y)$ or $Mdx + Ndy = 0$, where M, N are functions of x and y, is called a differential equation of the first order and first degree.

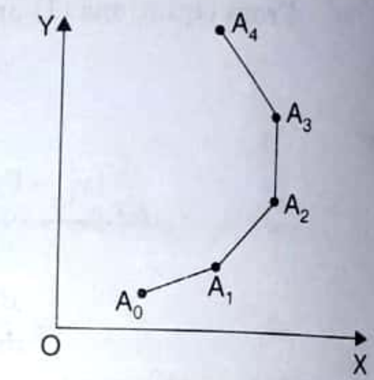
1.7.1 Geometrical Interpretation

Let $$f\left(x, y, \frac{dy}{dx}\right) = 0 \quad ...(1)$$

be a differential equation of the first order and first degree.

We know that the direction of a curve at a particular point is determined by drawing a tangent line at that point, *i.e.*, its slope is given by $\frac{dy}{dx}$ at that particular point.

Let $A_0(x_0, y_0)$ be any point in the plane. Let $m_0 = \frac{dy_0}{dx_0}$ be the slope of the curve at A_0 derived from (1). Take a neighbouring point $A_1(x_1, y_1)$ such that the slope of A_0A_1 is m_0. Let $m_1 = \frac{dy_1}{dx_1}$ be the slope of the curve at A_1 derived from (1). Take a neighbouring point $A_2(x_2, y_2)$ such that the slope of A_1A_2 is m_1. Continuing like this, we get a succession of points. If the points are taken sufficiently close to each other, they approximate a smooth curve $C : y = \phi(x)$ which is a solution of (1) corresponding to the initial point $A_0(x_0, y_0)$. Any point on C and the slope of the tangent at that point satisfy (1). If the moving point starts at any other point, not on C and moves as before, it will describe another curve. The equation of each such curve is a *particular solution* of the differential equation (1). The equation of the system of all such curves is the general solution of equation (1).



1.8 SOLUTION OF DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE

As we cannot integrate every function so it is not possible to solve every differential equation. Even the differential equation of first order and first degree cannot be solved in every case. They can be solved, however, if they belong to standard forms. Here, we will discuss only few standard forms which are necessary for the understanding of topics ahead.

1.9 VARIABLES SEPARABLE FORM

If it is possible to write a differential equation of first order and first degree in the form
$$f_1(x)\,dx = f_2(y)\,dy$$
we say that the variables are separable. Such equations can be solved immediately by integration and the solution is given by
$$\int f_1(x)\,dx = \int f_2(y)\,dy + c$$
where c is an arbitrary constant of integration.

1.9.1 Steps for Solution

1. Separate the variables as $f_1(x)\,dx = f_2(y)\,dy$.
2. Integrate both sides as $\int f_1(x)\,dx = \int f_2(y)\,dy$.
3. Add an arbitrary constant to any of the sides.

1.9.2 Differential Equations of the Form $\dfrac{dy}{dx} = f(ax + by + c)$ ...(1)

can be reduced to a form in which the variables are separable by the substitution $ax + by + c = t$

so that $$a + b\frac{dy}{dx} = \frac{dt}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{b}\left(\frac{dt}{dx} - a\right)$$

$\therefore$ Equation (1) becomes $$\frac{1}{b}\left(\frac{dt}{dx} - a\right) = f(t) \quad \text{or} \quad \frac{dt}{dx} = a + bf(t)$$

or $$\frac{dt}{a + bf(t)} = dx$$

After integrating both sides, t is to be replaced by its value.

Example 5. *Solve:* $\dfrac{dy}{dx} = \dfrac{1 + y^2}{1 + x^2}$.

Sol. Separating the variables, we get

$$\frac{dy}{1 + y^2} = \frac{dx}{1 + x^2}$$

Integrating both sides, we get

$$\tan^{-1} y = \tan^{-1} x + \tan^{-1} c$$

$$\Rightarrow \quad \tan^{-1} y - \tan^{-1} x = \tan^{-1} c$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{y - x}{1 + xy}\right) = \tan^{-1} c$$

$$\Rightarrow \quad \frac{y - x}{1 + xy} = c.$$

where c is an arbitrary constant.

Example 6. *Solve:* $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$.

Sol. Separating the variables, we get

$$\frac{\sec^2 x}{\tan x}\,dx + \frac{\sec^2 y}{\tan y}\,dy = 0$$

Integration yields

$$\log \tan x + \log \tan y = \log c$$

$$\Rightarrow \quad \log \tan x \tan y = \log c$$

$$\Rightarrow \quad \tan x \tan y = c.$$

where c is an arbitrary constant.

Example 7. *Solve:* $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}.$

Sol. Multiplying both sides by e^y, we get

$$e^y \frac{dy}{dx} = e^x + x^2 \quad \Rightarrow \quad e^y\, dy = (e^x + x^2)\, dx$$

Integration gives

$$e^y = e^x + \frac{x^3}{3} + c,$$ where c is an arbitrary constant of integration.

Example 8. *Solve:* $x\, y\, y' = \left(\frac{1+y^2}{1+x^2}\right)(1 + x + x^2).$

Sol. We have, $xy \frac{dy}{dx} = \frac{1+y^2}{1+x^2}\,(1 + x + x^2)$

Separating the variables,

$$\frac{y\, dy}{1+y^2} = \frac{(1+x+x^2)\, dx}{x(1+x^2)} = \left(\frac{1}{x} + \frac{1}{1+x^2}\right) dx$$

Integrating both sides,

$$\frac{1}{2}\int \frac{2y}{1+y^2}\, dy = \int \left(\frac{1}{x} + \frac{1}{1+x^2}\right) dx$$

$$\Rightarrow \quad \frac{1}{2}\log(1 + y^2) = \log x + \tan^{-1} x + c$$

where c is an arbitrary constant of integration.

Example 9. *Solve:* $3e^x \tan y\, dx + (1 + e^x) \sec^2 y\, dy = 0;\ y(0) = \frac{\pi}{4}.$

Sol. The given equation is

$$\frac{3e^x}{1+e^x}\, dx + \frac{\sec^2 y}{\tan y}\, dy = 0$$

Integrating, we have

$$3 \log(1 + e^x) + \log \tan y = \log c$$

$$\Rightarrow \quad \log(1 + e^x)^3 \tan y = \log c$$

$$\Rightarrow \quad (1 + e^x)^3 \tan y = c \qquad \ldots(1)$$

which is the general solution of the given equation.

Since $y = \frac{\pi}{4}$ when $x = 0$, we have from equation (1),

$$(1 + 1)^3 \times 1 = c \quad \Rightarrow \quad c = 8$$

$\therefore$ The required particular solution is

$$(1 + e^x)^3 \tan y = 8.$$

Example 10. *Solve:* $\frac{dy}{dx} = \cos(x+y) + \sin(x+y)$.

D

Sol. Let $x + y = v$, then

$$1+\frac{dy}{dx}=\frac{dv}{dx} \Rightarrow \frac{dy}{dx}=\frac{dv}{dx}-1$$

$\therefore$ From the given equation

$$\frac{dv}{dx}-1 = \cos v + \sin v,$$

$$\Rightarrow \frac{dv}{1+\cos v+\sin v} = dx \qquad | \text{ Separating the variables}$$

Integration yields

$$\int \frac{dv}{1+\cos v+\sin v} = \int dx + c$$

$$\Rightarrow \int \frac{dv}{1+\frac{1-\tan^2 v/2}{1+\tan^2 v/2}+\frac{2\tan v/2}{1+\tan^2 v/2}} = x + c$$

$$\Rightarrow \int \frac{\sec^2 v/2}{2(1+\tan v/2)} dv = x + c$$

$$\Rightarrow \log(1+\tan v/2) = x + c$$

$$\Rightarrow \log\left\{1+\tan\left(\frac{x+y}{2}\right)\right\} = x + c$$

where c is an arbitrary constant of integration.

1.10 LINEAR DIFFERENTIAL EQUATIONS

A differential equation is said to be linear if the dependent variable and its derivative occur only in the first degree and are not multiplied together.

The general form of a linear differential equation of the first order is $\frac{dy}{dx} + Py = Q$ ...(1)

where P and Q are functions of x only or constants.

Equation (1) is also known as *Leibnitz's linear equation**.

To solve it, we multiply both sides by $e^{\int P dx}$ and get

$$\frac{dy}{dx} e^{\int P dx} + y(e^{\int P dx} P) = Qe^{\int P dx}$$

or

$$\frac{d}{dx}(ye^{\int P dx}) = Qe^{\int P dx}$$

Integrating both sides, we get

$$ye^{\int P dx} = \int Qe^{\int P dx} dx + c$$

*Gottfried wilhelm Leibnitz (1646–1716) was a German mathematician.

Note 1. In the general form of a linear differential equation, the coefficient of $\frac{dy}{dx}$ is unity.

Note 2. The factor $e^{\int P dx}$ on multiplying by which the L.H.S. of (1) becomes the differential coefficient of a single function is called the integrating factor (briefly written as I.F.) of (1).

Thus I.F. $= e^{\int P dx}$ and the solution is

$$\boxed{y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + c.}$$

Note 3. Sometimes a differential equation takes linear form if we regard x as dependent variable and y as independent variable. The equation can then be put as $\frac{dx}{dy} + Px = Q$, where P, Q are functions of y only or constants. The integrating factor in this case is $e^{\int P dy}$ and the solution is

$$\boxed{x\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dy + c.}$$

Example 11. *Solve:* $\frac{dy}{dx} + \frac{3x^2}{1+x^3}\,y = \frac{\sin^2 x}{1+x^3}$.

Sol. Comparing with $\frac{dy}{dx} + Py = Q$, we get

$$P = \frac{3x^2}{1+x^3}, \quad Q = \frac{\sin^2 x}{1+x^3}$$

$$\text{I.F.} = e^{\int P dx} = e^{\int \frac{3x^2}{1+x^3} dx} = e^{\log(1+x^3)} = 1 + x^3$$

Hence, solution is given by

$$y(1+x^3) = \int \frac{\sin^2 x}{1+x^3}(1+x^3)\,dx + c$$

$$= \frac{1}{2}\int (1-\cos 2x)\,dx + c = \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + c$$

$$\Rightarrow \quad y(1+x^3) = \frac{x}{2} - \frac{\sin 2x}{4} + c$$

where c is an arbitrary constant of integration.

Example 12. *Solve:* $\frac{dy}{dx} + \frac{3y}{x} = \frac{1}{x^4}$.

Sol. Comparing with $\frac{dy}{dx} + Py = Q$, we get $P = \frac{3}{x}$ and $Q = \frac{1}{x^4}$

$$\text{I.F.} = e^{\int P\,dx} = e^{3\int \frac{1}{x} dx} = e^{3 \log x} = x^3$$

$\therefore$ The solution is

$$y(\text{I.F.}) = \int Q(\text{I.F.})\, dx + c$$

$$\Rightarrow \quad yx^3 = \int \frac{1}{x^4}(x^3)\, dx + c$$

$$\Rightarrow \quad yx^3 = \log x + c$$

where c is an arbitrary constant of integration.

Example 13. *Solve:* $\cos^2 x \dfrac{dy}{dx} + y = \tan x.$

Sol. Dividing throughout by $\cos^2 x$, we get

$$\frac{dy}{dx} + \sec^2 x \,.\, y = \tan x \sec^2 x$$

Comparing with $\dfrac{dy}{dx} + Py = Q$, we get, $P = \sec^2 x$, $\quad Q = \tan x \sec^2 x$

$$\therefore \quad \text{I.F.} = e^{\int P dx} = e^{\int \sec^2 x\, dx} = e^{\tan x}$$

Hence the solution is given by

$$y \,.\, e^{\tan x} = \int e^{\tan x} \,.\, \tan x \sec^2 x\, dx + c \quad \big|\ c \text{ is an arbitrary constant}$$

$$= \int e^t \,.\, t\, dt + c \qquad \begin{array}{|l} \text{Put } \tan x = t \\ \therefore\ \sec^2 x\, dx = dt \end{array}$$

$$= t\, e^t - e^t + c$$

$$= \tan x \,.\, e^{\tan x} - e^{\tan x} + c$$

$$\Rightarrow \quad y = \tan x - 1 + c\, e^{-\tan x}.$$

Example 14. *Solve the equation* $x\dfrac{dy}{dx} - \dfrac{1}{2}y = x + 1$ *and prove that the only solution for which* x *and* y *can attain the value unity is given by* $y = 2x + \sqrt{x} - 2$. [G.B.T.U. (C.O.) 2011]

Sol. We have,

$$\frac{dy}{dx} - \frac{1}{2x}y = 1 + \frac{1}{x} \qquad \text{...(1)}$$

$$\text{I.F.} = e^{-\int \frac{1}{2x} dx} = e^{-\frac{1}{2}\log x} = \frac{1}{\sqrt{x}}$$

Solution to equation (1) is

$$y\left(\frac{1}{\sqrt{x}}\right) = \int \left(1 + \frac{1}{x}\right)\frac{1}{\sqrt{x}}\, dx + c = 2\sqrt{x} - \frac{2}{\sqrt{x}} + c$$

$$\Rightarrow \quad y = 2x - 2 + c\sqrt{x} \qquad \text{...(2)}$$

Applying the condition $y = 1$ when $x = 1$ in equation (2),

$$1 = 2 - 2 + c \quad \Rightarrow \quad \boxed{c = 1}$$

$\therefore$ From (2), we get

$$y = 2x + \sqrt{x} - 2.$$

Example 15. *Solve:* $(1 + y^2)\, dx = (\tan^{-1} y - x)\, dy$.

Sol. The given equation can be written as

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2}$$

which is of the form $\frac{dx}{dy} + \text{P}x = \text{Q}$

Here,
$$\text{P} = \frac{1}{1+y^2},\ \text{Q} = \frac{\tan^{-1} y}{1+y^2}$$

$$\text{I.F.} = e^{\int \text{P}dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

$\therefore$ The solution is

$$x\,(\text{I.F.}) = \int \text{Q}\,(\text{I.F.})\, dy + c$$

or
$$xe^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2} e^{\tan^{-1} y}\, dy + c = \int te^t\, dt + c, \text{ where } t = \tan^{-1} y$$

$$= te^t - e^t + c = (\tan^{-1} y - 1)\, e^{\tan^{-1} y} + c$$

1.11 LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

The equation of the form

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots + a_n y = \text{Q} \qquad \ldots(1)$$

where $a_0, a_1, a_2, \ldots\ldots, a_n$ are all constants and Q is a function of x alone is called a linear differential equation of n^{th} order with constant coefficients.

1.12 THE OPERATOR D

The part $\frac{d}{dx}$ of the symbol $\frac{dy}{dx}$ may be regarded as an operator such that when it operates on y, the result is the derivative of y. Similarly, $\frac{d^2}{dx^2}, \frac{d^3}{dx^3}, \ldots\ldots, \frac{d^n}{dx^n}$ may be regarded as operators.

For brevity, we write $\frac{d}{dx} \equiv \text{D}, \frac{d^2}{dx^2} \equiv \text{D}^2, \ldots\ldots, \frac{d^n}{dx^n} \equiv \text{D}^n$

Thus, the symbol D is a **differential operator** or simply an **operator**.

Written in symbolic form, equation (1) becomes

$$(a_0\text{D}^n + a_1\text{D}^{n-1} + a_2\text{D}^{n-2} + \ldots\ldots + a_{n-1}\,\text{D} + a_n)y = \text{Q}$$

or
$$\boxed{f(\text{D})y = \text{Q}}$$

The operator D can be treated as an algebraic quantity. Thus,

$$D(u + v) = Du + Dv, \quad D(\lambda u) = \lambda\, Du \quad \text{and} \quad D^pD^q\, u = D^qD^p\, u = D^{p+q}\, u$$

The polynomial $f(D)$ can be factorised by ordinary rules of algebra and the factors may be written in any order.

1.13 THEOREMS

Theorem 1. *If $y = y_1$, $y = y_2$,......, $y = y_n$ are n linearly independent solutions of the differential equation*

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n)y = 0$$

then $u = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$ is also its solution, where c_1, c_2, ..., c_n are arbitrary constants.

Theorem 2. *If $y = u$ is the complete solution of the equation $f(D)y = 0$ and $y = v$ is a particular solution (containing no arbitrary constants) of the equation $f(D)y = Q$, then the complete solution of the equation*

$$f(D)y = Q \text{ is } y = u + v.$$

Note 1. The part $y = u$ is called the **complementary function (C.F.)** and the part $y = v$ is called the **particular integral (P.I.)** of the equation $f(D)\, y = Q$.

Note 2. The complete solution is $\boxed{\mathbf{y = C.F. + P.I.}}$

Thus in order to solve the equation $f(D)\, y = Q$, we first find the C.F. i.e., the complete solution of equation $f(D)\, y = 0$ and then the P.I. i.e., a particular integral (solution) of equation $f(D)\, y = Q$.

1.14 COMPLEMENTARY FUNCTION (C.F.)

Consider the differential equation

$$f(D)y = Q \qquad \ldots(1)$$

Complementary function is actually the solution of the given differential equation (1) when its right hand side member *i.e.*, Q is replaced by zero. To find C.F., we first find auxiliary equation.

1.15 AUXILIARY EQUATION (A.E.)

Consider the differential equation $(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots + a_n)y = 0$ ...(1)

Let $y = e^{mx}$ be a solution of (i), then

$$Dy = me^{mx},\ D^2y = m^2e^{mx},\ \ldots\ldots,\ D^{n-2}y = m^{n-2}e^{mx},\ D^{n-1}y = m^{n-1}e^{mx},\ D^ny = m^ne^{mx}$$

Substituting the values of y, Dy, D^2y, ..., D^ny in (1), we get

$$(m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots\cdots + a_n)\, e^{mx} = 0$$

or

$$m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots\cdots + a_n = 0, \text{ since } e^{mx} \neq 0 \qquad \ldots(2)$$

Thus $y = e^{mx}$ will be a solution of equation (1) if m satisfies equation (2).

Equation (2) is called the auxiliary equation for the differential equation (1).

1.15.1 Definition

The equation obtained by equating to zero the symbolic coefficient of y is called the **auxiliary equation,** briefly written as **A.E.**

1.15.2 Steps for Finding Auxiliary Equation

Step 1. Replace y by 1

Step 2. Replace $\dfrac{dy}{dx}$ by m

Step 3. Replace $\dfrac{d^2y}{dx^2}$ by m^2 and so on replace $\dfrac{d^ny}{dx^n}$ by m^n

Step 4. By doing so, we get an algebraic equation in m of degree n called auxiliary equation.

1.16 RULES FOR FINDING THE COMPLEMENTARY FUNCTION

Consider the equation

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots + a_n)y = 0 \quad \ldots(1)$$

where all the a_i's are constant.

Its auxiliary equation is

$$m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots\ldots + a_n = 0 \quad \ldots(2)$$

It is an algebraic equation in m of degree n. So it will give n values of m on solving.

Let $m = m_1, m_2, m_3, \ldots\ldots, m_n$ be the roots of the A.E. The C.F. of equation (1) depends upon the nature of roots of the A.E. The following cases arise.

Case I. *When the roots of auxiliary equation are real and distinct*

Equation (1) is equivalent to

$$(D - m_1)(D - m_2) \ldots\ldots (D - m_n)y = 0 \quad \ldots(3)$$

Equation (3) will be satisfied by the solutions of the equations

$$(D - m_1)y = 0,\ (D - m_2)y = 0,\ \ldots\ldots,\ (D - m_n)y = 0$$

Now, consider the equation $(D - m_1)y = 0$, *i.e.*, $\dfrac{dy}{dx} - m_1 y = 0$

It is a linear equation and I.F. $= e^{\int -m_1\,dx} = e^{-m_1 x}$

$\therefore$ Its solution is $y \cdot e^{-m_1 x} = \int 0 \cdot e^{-m_1 x}\,dx + c_1$ or $y = c_1 e^{m_1 x}$

Similarly, the solution of $(D - m_2)y = 0$ is $y = c_2 e^{m_2 x}$

....

the solution of $(D - m_n)y = 0$ is $y = c_n e^{m_n x}$

$$\therefore \quad \boxed{\text{C.F.} = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots\ldots + c_n e^{m_n x}} \quad \ldots(4)$$

Case II. *When the roots of auxiliary equation are equal*

(a) When two roots of auxiliary equation are equal

Let

$$m_1 = m_2$$

Solution of eqn. (3) is (as in case I)

$$y = \text{C.F.} + \text{P.I.}$$

$$= c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots\ldots + c_n e^{m_n x} + 0 \qquad | \because \text{P.I.} = 0 \text{ as } Q = 0$$

$$= (c_1 + c_2)e^{m_1 x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x} \qquad | \text{ Here } m_1 = m_2$$

$$= c\, e^{m_1 x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

It contains $(n-1)$ arbitrary constants and is, therefore, not the complete solution of equation (1).

The part of C.F. corresponding to the repeated root is the complete solution of

$$(D - m_1)(D - m_1)y = 0$$

$$\Rightarrow \quad (D - m_1)v = 0 \qquad \text{| Putting } (D - m_1)y = v$$

$$\Rightarrow \quad \frac{dv}{dx} - m_1 v = 0$$

Its solution is $\quad v = c_2 e^{m_1 x}$

$$\therefore \quad (D - m_1)y = c_2 e^{m_1 x}$$

$$\Rightarrow \quad \frac{dy}{dx} - m_1 y = c_2 e^{m_1 x}, \quad \text{which is a linear equation.}$$

$$\therefore \quad \text{I.F.} = e^{-m_1 x}$$

Its solution is

$$ye^{-m_1 x} = \int c_2 e^{m_1 x} . e^{-m_1 x}\, dx + c_1 = c_2 x + c_1$$

$$\Rightarrow \quad y = (c_2 x + c_1)\, e^{m_1 x}$$

$$\therefore \quad \text{Part of C.F.} = (c_1 + c_2 x)\, e^{m_1 x}$$

Hence, $\boxed{\text{complete C.F.} = (c_1 + c_2 x + c_3 x^2)\, e^{m_1 x} + c_4\, e^{m_4 x} + \ldots\ldots + c_n\, e^{m_n x}}$

(b) If however, three roots of the auxiliary equation are equal say $m_1 = m_2 = m_3$, then proceeding as above,

$$\boxed{\text{C.F.} = (c_1 + c_2 x + c_3 x^2)\, e^{m_1 x} + c_4\, e^{m_4 x} + \ldots\ldots + c_n\, e^{m_n x}}$$

Case III. *When two roots of auxiliary equation are imaginary* and distinct

Let $\quad m_1 = \alpha + i\beta \quad$ and $\quad m_2 = \alpha - i\beta$, then from (4),

$$\text{C.F.} = c_1 e^{(\alpha + i\beta)x} + c_2\, e^{(\alpha - i\beta)x} + c_3\, e^{m_3 x} + \ldots\ldots + c_n\, e^{m_n x}$$

$$= e^{\alpha x}\, (c_1 e^{i\beta x} + c_2 e^{-i\beta x}) + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

$$= e^{\alpha x}\, [c_1\, (\cos \beta x + i \sin \beta x) + c_2\, (\cos \beta x - i \sin \beta x)] + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

$$= e^{\alpha x}\, [(c_1 + c_2) \cos \beta x + i\, (c_1 - c_2) \sin \beta x] + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

$$\text{C.F.} = e^{\alpha x}\, (C_1 \cos \beta x + C_2 \sin \beta x) + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

[taking $c_1 + c_2 = C_1$, $i(c_1 - c_2) = C_2$]

Case IV. *When roots of auxiliary equation are repeated imaginary*

Let $\quad m_1 = m_2 = \alpha + i\beta \quad$ and $\quad m_3 = m_4 = \alpha - i\beta \quad$ then by case II,

$$\text{C.F.} = e^{\alpha x}\, [(c_1 + c_2 x) \cos \beta x + (c_3 + c_4\, x) \sin \beta x] + c_5\, e^{m_5 x} + \ldots\ldots + c_n\, e^{m_n x}$$

Case V. *When roots of auxiliary equation are irrational* and distinct

Let $m_1 = \alpha + \sqrt{\beta}$ and $m_2 = \alpha - \sqrt{\beta}$ then

C.F. of eqn. (1) is given by

$$\text{C.F.} = e^{\alpha x}\,(c_1 \cosh \sqrt{\beta}x + c_2 \sinh \sqrt{\beta}x) + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

Case VI. *When roots of auxiliary equation are repeated irrational*

Let $m_1 = m_2 = \alpha + \sqrt{\beta}$ and $m_3 = m_4 = \alpha - \sqrt{\beta}$ then by case II,

$$\text{C.F.} = e^{\alpha x}\,[(c_1 + c_2 x)\cosh \sqrt{\beta}x + (c_3 + c_4 x)\sinh \sqrt{\beta}x\,] + c_5 e^{m_5 x} + c_6 e^{m_6 x} + \ldots\ldots + c_n e^{m_n x}$$

Example 16. *Solve:* $\dfrac{d^3y}{dx^3} - 7\dfrac{dy}{dx} - 6y = 0.$

Sol. The auxiliary equation is

$$m^3 - 7m - 6 = 0$$

$$\Rightarrow \quad (m + 1)(m + 2)(m - 3) = 0 \quad \Rightarrow \quad m = -1, -2, 3$$

The roots are real and distinct

$\therefore$ Complementary Function (C.F.) $= c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x}$

Particular Integral (P.I.) $= 0$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x}$$

where c_1, c_2 and c_3 are arbitrary constants of integration.

Example 17. *Solve:* $(D^3 - 3D^2 + 4)\,y = 0$, *where* $D \equiv \dfrac{d}{dx}$.

Sol. The auxiliary equation is

$$m^3 - 3m^2 + 4 = 0$$

$$\Rightarrow \quad (m + 1)(m - 2)^2 = 0 \quad \Rightarrow \quad m = -1, 2, 2$$

$$\therefore \quad \text{C.F.} = c_1 e^{-x} + (c_2 + c_3 x)\,e^{2x}$$

$$\text{P.I.} = 0$$

$\therefore$ The complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{-x} + (c_2 + c_3 x)\,e^{2x}$$

where c_1, c_2 and c_3 are arbitrary constants of integration.

Example 18. *Solve:* $(D^4 - n^4)y = 0$, *where* $D \equiv \dfrac{d}{dx}$.

Sol. The auxiliary equation is

$$m^4 - n^4 = 0$$

$$\Rightarrow \quad (m^2 - n^2)(m^2 + n^2) = 0$$

$$\Rightarrow \quad m = \pm n, \pm ni$$

$$\therefore \quad \text{C.F.} = c_1 e^{nx} + c_2 e^{-nx} + e^{0x}(c_3 \cos nx + c_4 \sin nx)$$

$$= c_1 e^{nx} + c_2 e^{-nx} + c_3 \cos nx + c_4 \sin nx$$

$$\text{P.I.} = 0$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1e^{nx} + c_2e^{-nx} + c_3 \cos nx + c_4 \sin nx$$

where c_1, c_2, c_3 and c_4 are arbitrary constants of integration.

Example 19. *Solve:* $\dfrac{d^4y}{dx^4} + 13\dfrac{d^2y}{dx^2} + 36y = 0.$

Sol. The auxiliary equation is

$$m^4 + 13m^2 + 36 = 0$$

$$\Rightarrow \quad (m^2 + 9)(m^2 + 4) = 0 \quad \Rightarrow \quad m = \pm 3i, \pm 2i$$

$$\therefore \quad \text{C.F.} = e^{0x}(c_1 \cos 3x + c_2 \sin 3x) + e^{0x}(c_3 \cos 2x + c_4 \sin 2x)$$

$$= c_1 \cos 3x + c_2 \sin 3x + c_3 \cos 2x + c_4 \sin 2x$$

$$\text{P.I.} = 0$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 \cos 3x + c_2 \sin 3x + c_3 \cos 2x + c_4 \sin 2x$$

where c_1, c_2, c_3 and c_4 are arbitrary constants of integration.

Example 20. *Solve:* $(D^2 - 2D + 4)^2 y = 0;\ D \equiv \dfrac{d}{dx}.$

Sol. The auxiliary equation is

$$(m^2 - 2m + 4)^2 = 0$$

$$m = \frac{2 \pm \sqrt{4 - 16}}{2} \text{ (twice)} = 1 \pm \sqrt{3}\,i,\ 1 \pm \sqrt{3}\,i$$

The roots are repeated imaginary

$$\therefore \quad \text{C.F.} = e^x[(c_1 + c_2x) \cos \sqrt{3}x + (c_3 + c_4x) \sin \sqrt{3}x]$$

$$\text{P.I.} = 0$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = e^x[(c_1 + c_2x) \cos \sqrt{3}x + (c_3 + c_4x) \sin \sqrt{3}x]$$

where c_1, c_2, c_3 and c_4 are arbitrary constants of integration.

Example 21. *Solve:* $\dfrac{d^4y}{dx^4} - 4\dfrac{d^3y}{dx^3} + 8\dfrac{d^2y}{dx^2} - 8\dfrac{dy}{dx} + 4y = 0.$

Sol. The auxiliary equation is

$$m^4 - 4m^3 + 8m^2 - 8m + 4 = 0$$

$$\Rightarrow \quad (m^2 - 2m + 2)^2 = 0$$

$$\Rightarrow \quad m = \frac{2 \pm \sqrt{4 - 8}}{2} \text{ (twice)} = \frac{2 \pm 2i}{2} \text{ (twice)} = 1 \pm i,\ 1 \pm i$$

$$\therefore \quad \text{C.F.} = e^x[(c_1 + c_2x) \cos x + (c_3 + c_4x) \sin x]$$

$$\text{P.I.} = 0$$

The complete solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$\Rightarrow \quad y = e^x[(c_1 + c_2x) \cos x + (c_3 + c_4x) \sin x]$$

where c_1, c_2, c_3 and c_4 are arbitrary constants of integration.

Example 22. *Solve:* $\dfrac{d^4y}{dx^4} + m^4 y = 0.$

Sol. The auxiliary equation is

$$M^4 + m^4 = 0$$

$$\Rightarrow \quad M^4 + m^4 + 2M^2m^2 = 2\,M^2m^2$$

$$\Rightarrow \quad (M^2 + m^2)^2 = 2\,M^2m^2$$

$$\Rightarrow \quad M^2 + m^2 = \pm\sqrt{2}\,M\,m.$$

Case I. *Taking (+)ve sign:*

$$M^2 + m^2 - \sqrt{2}\,M\,m = 0$$

$$\therefore \quad M = \frac{\sqrt{2}m \pm \sqrt{2m^2 - 4m^2}}{2} = \frac{\sqrt{2}m \pm i\sqrt{2}m}{2} = \frac{m}{\sqrt{2}} \pm i\,\frac{m}{\sqrt{2}}.$$

Case II. *Taking (–)ve sign:*

$$M^2 + m^2 + \sqrt{2}\,Mm = 0$$

$$\therefore \quad M = \frac{-\sqrt{2}m \pm \sqrt{2m^2 - 4m^2}}{2} = \frac{-\sqrt{2}m \pm i\sqrt{2}m}{2} = -\frac{m}{\sqrt{2}} \pm i\,\frac{m}{\sqrt{2}}$$

$$\therefore \quad M = \frac{m}{\sqrt{2}} \pm i\,\frac{m}{\sqrt{2}}\,;\, -\frac{m}{\sqrt{2}} \pm i\,\frac{m}{\sqrt{2}}$$

$$\text{C.F.} = e^{\frac{m}{\sqrt{2}}x}\left(c_1 \cos\frac{m}{\sqrt{2}}x + c_2 \sin\frac{m}{\sqrt{2}}x\right) + e^{-\frac{m}{\sqrt{2}}x}\left(c_3 \cos\frac{m}{\sqrt{2}}x + c_4 \sin\frac{m}{\sqrt{2}}x\right)$$

$$\text{P.I.} = 0$$

$\therefore$ Complete solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$= e^{\frac{m}{\sqrt{2}}x}\left(c_1 \cos\frac{m}{\sqrt{2}}x + c_2 \sin\frac{m}{\sqrt{2}}x\right) + e^{-\frac{m}{\sqrt{2}}x}\left(c_3 \cos\frac{m}{\sqrt{2}}x + c_4 \sin\frac{m}{\sqrt{2}}x\right).$$

where c_1, c_2, c_3, c_4 are arbitrary constants of integration.

Example 23. *Solve the differential equation:* $(D^2 + 1)^3 (D^2 + D + 1)^2 y = 0,\ D \equiv \dfrac{d}{dx}.$

Sol. Auxiliary equation is

$$(m^2 + 1)^3 (m^2 + m + 1)^2 = 0$$

$$\Rightarrow \quad (m^2 + 1)^3 = 0 \text{ gives } m = \pm i, \pm i, \pm i$$

and $\quad (m^2 + m + 1)^2 = 0$ gives $\quad m = \dfrac{-1 \pm \sqrt{3}i}{2}, \dfrac{-1 \pm \sqrt{3}i}{2}$

Hence, $\quad \text{C.F.} = e^{0x}[(c_1 + c_2x + c_3x^2)\cos x + (c_4 + c_5x + c_6x^2)\sin x]$

$$+ e^{-x/2}\left[(c_7 + c_8x)\cos\frac{\sqrt{3}}{2}x + (c_9 + c_{10}x)\sin\frac{\sqrt{3}}{2}x\right]$$

P.I. = 0

Therefore the complete solution is

$$y = \text{C.F.} + \text{P.I.}$$
$$= (c_1 + c_2x + c_3x^2)\cos x + (c_4 + c_5 x + c_6x^2)\sin x + e^{-x/2}\left\{(c_7 + c_8x)\cos\frac{\sqrt{3}}{2}x + (c_9 + c_{10}x)\sin\frac{\sqrt{3}}{2}x\right\}$$

where $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9$ and c_{10} are arbitrary constants of integration.

Example 24. *Solve the differential equation* $\dfrac{d^2i}{dt^2} + \dfrac{R}{L}\dfrac{di}{dt} + \dfrac{i}{LC} = 0,$ *where* $R^2C = 4L$ *and R, C, L are constants.* [G.B.T.U. (SUM) 2010]

Sol. The given equation is

$$\left(D^2 + \frac{R}{L}D + \frac{1}{LC}\right)i = 0, \quad \text{where } D \equiv \frac{d}{dt}$$

Auxiliary equation is

$$m^2 + \frac{R}{L}m + \frac{1}{LC} = 0$$

$$\Rightarrow \qquad m = \frac{-\dfrac{R}{L} \pm \sqrt{\dfrac{R^2}{L^2} - \dfrac{4}{LC}}}{2}$$

$$= -\frac{R}{2L} \pm \frac{1}{2}\sqrt{\frac{4}{LC} - \frac{4}{LC}} \qquad \left| \because R^2 = \frac{4L}{C}\right.$$

$$= -\frac{R}{2L}, -\frac{R}{2L}$$

Hence, $\qquad \text{C.F.} = (c_1 + c_2t)\, e^{-\frac{R}{2L}t}$

P.I. = 0

$\therefore$ The complete solution is

$$i = \text{C.F.} + \text{P.I.} = (c_1 + c_2t)\, e^{-\frac{R}{2L}t}$$

where c_1 and c_2 are arbitrary constants of integration.

Example 25. *Solve the differential equation:*

$$\frac{d^2y}{dx^2} + y = 0\ ;\ \textit{given that } y(0) = 2 \textit{ and } y\left(\frac{\pi}{2}\right) = -2.$$

Sol. The auxiliary equation is

$$m^2 + 1 = 0 \quad \Rightarrow \quad m = \pm i$$

$\therefore \qquad \text{C.F.} = c_1 \cos x + c_2 \sin x$

P.I. = 0

Example 22. *Solve:* $\dfrac{d^4y}{dx^4} + m^4 y = 0.$

Sol. The auxiliary equation is

$$M^4 + m^4 = 0$$

$$\Rightarrow \quad M^4 + m^4 + 2M^2m^2 = 2\,M^2m^2$$

$$\Rightarrow \quad (M^2 + m^2)^2 = 2\,M^2m^2$$

$$\Rightarrow \quad M^2 + m^2 = \pm\sqrt{2}\,M\,m.$$

Case I. *Taking (+)ve sign:*

$$M^2 + m^2 - \sqrt{2}\,M\,m = 0$$

$$\therefore \quad M = \frac{\sqrt{2}m \pm \sqrt{2m^2 - 4m^2}}{2} = \frac{\sqrt{2}m \pm i\sqrt{2}m}{2} = \frac{m}{\sqrt{2}} \pm i\frac{m}{\sqrt{2}}.$$

Case II. *Taking (–)ve sign:*

$$M^2 + m^2 + \sqrt{2}\,Mm = 0$$

$$\therefore \quad M = \frac{-\sqrt{2}m \pm \sqrt{2m^2 - 4m^2}}{2} = \frac{-\sqrt{2}m \pm i\sqrt{2}m}{2} = -\frac{m}{\sqrt{2}} \pm i\frac{m}{\sqrt{2}}$$

$$\therefore \quad M = \frac{m}{\sqrt{2}} \pm i\frac{m}{\sqrt{2}}\,;\, -\frac{m}{\sqrt{2}} \pm i\frac{m}{\sqrt{2}}$$

$$\text{C.F.} = e^{\frac{m}{\sqrt{2}}x}\left(c_1\cos\frac{m}{\sqrt{2}}x + c_2\sin\frac{m}{\sqrt{2}}x\right) + e^{-\frac{m}{\sqrt{2}}x}\left(c_3\cos\frac{m}{\sqrt{2}}x + c_4\sin\frac{m}{\sqrt{2}}x\right)$$

$$\text{P.I.} = 0$$

$\therefore$ Complete solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$= e^{\frac{m}{\sqrt{2}}x}\left(c_1\cos\frac{m}{\sqrt{2}}x + c_2\sin\frac{m}{\sqrt{2}}x\right) + e^{-\frac{m}{\sqrt{2}}x}\left(c_3\cos\frac{m}{\sqrt{2}}x + c_4\sin\frac{m}{\sqrt{2}}x\right).$$

where c_1, c_2, c_3, c_4 are arbitrary constants of integration.

Example 23. *Solve the differential equation:* $(D^2 + 1)^3\,(D^2 + D + 1)^2\,y = 0,\ D \equiv \dfrac{d}{dx}.$

Sol. Auxiliary equation is

$$(m^2 + 1)^3\,(m^2 + m + 1)^2 = 0$$

$$\Rightarrow \quad (m^2 + 1)^3 = 0 \text{ gives } m = \pm i, \pm i, \pm i$$

and $\quad (m^2 + m + 1)^2 = 0$ gives $\quad m = \dfrac{-1 \pm \sqrt{3}i}{2}, \dfrac{-1 \pm \sqrt{3}i}{2}$

Hence,

$$\text{C.F.} = e^{0x}\left[(c_1 + c_2x + c_3x^2)\cos x + (c_4 + c_5x + c_6x^2)\sin x\right]$$

$$+ e^{-x/2}\left[(c_7 + c_8x)\cos\frac{\sqrt{3}}{2}x + (c_9 + c_{10}x)\sin\frac{\sqrt{3}}{2}x\right]$$

$$\text{P.I.} = 0$$

Therefore the complete solution is

$$y = \text{C.F.} + \text{P.I.}$$
$$= (c_1 + c_2x + c_3x^2)\cos x + (c_4 + c_5 x + c_6x^2)\sin x$$
$$+ e^{-x/2}\left\{(c_7 + c_8x)\cos\frac{\sqrt{3}}{2}x + (c_9 + c_{10}x)\sin\frac{\sqrt{3}}{2}x\right\}$$

where $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9$ and c_{10} are arbitrary constants of integration.

Example 24. *Solve the differential equation* $\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0,$

where $R^2C = 4L$ *and R, C, L are constants.* [G.B.T.U. (SUM) 2010]

Sol. The given equation is

$$\left(D^2 + \frac{R}{L}D + \frac{1}{LC}\right)i = 0, \quad \text{where } D \equiv \frac{d}{dt}$$

Auxiliary equation is

$$m^2 + \frac{R}{L}m + \frac{1}{LC} = 0$$

$$\Rightarrow \quad m = \frac{-\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}}{2}$$

$$= -\frac{R}{2L} \pm \frac{1}{2}\sqrt{\frac{4}{LC} - \frac{4}{LC}} \qquad \because R^2 = \frac{4L}{C}$$

$$= -\frac{R}{2L}, -\frac{R}{2L}$$

Hence, $\quad \text{C.F.} = (c_1 + c_2t)\, e^{-\frac{R}{2L}t}$

$\quad \text{P.I.} = 0$

$\therefore$ The complete solution is

$$i = \text{C.F.} + \text{P.I.} = (c_1 + c_2t)\, e^{-\frac{R}{2L}t}$$

where c_1 and c_2 are arbitrary constants of integration.

Example 25. *Solve the differential equation:*

$$\frac{d^2y}{dx^2} + y = 0 \text{ ; given that } y(0) = 2 \text{ and } y\left(\frac{\pi}{2}\right) = -2.$$

Sol. The auxiliary equation is

$$m^2 + 1 = 0 \quad \Rightarrow \quad m = \pm i$$

$\therefore \quad \text{C.F.} = c_1\cos x + c_2\sin x$

$\quad \text{P.I.} = 0$

Hence the general solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 \cos x + c_2 \sin x$$

Applying the condition $y(0) = 2$, we get $\quad 2 = c_1 \quad$ ...(1)

Applying the condition $y\left(\frac{\pi}{2}\right) = -2$, we get $\quad -2 = c_2$

Hence from (1), the particular solution is

$$y = 2(\cos x - \sin x)$$

TEST YOUR KNOWLEDGE

Solve the differential equations (1–16):

1. $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0$
2. $\frac{d^2y}{dx^2} + (a+b)\frac{dy}{dx} + aby = 0$
3. $\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = 0$
4. $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = 0$
5. $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0$
6. $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - \frac{dy}{dx} + y = 0$
7. $(D^4 - D^3 - 9D^2 - 11D - 4)y = 0$
8. $\frac{d^4y}{dx^4} + 8\frac{d^2y}{dx^2} + 16y = 0$
9. $\frac{d^5y}{dx^5} - \frac{d^3y}{dx^3} = 0$
10. $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 8y = 0$
11. $(D^2 + 1)^2 (D - 1)\, y = 0$
12. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$
13. $(D^6 - 1)\, y = 0$
14. $(D^6 + 1)\, y = 0$
15. $\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0$, given that when $t = 0$, $x = 0$ and $\frac{dx}{dt} = 0$.
16. $\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 8y = 0$ under the conditions $y(0) = 0$, $y'(0) = 0$ and $y''(0) = 2$.
17. A function $n(x)$ satisfies the differential equation $\frac{d^2n(x)}{dx^2} - \frac{n(x)}{L^2} = 0$, where L is a constant. The boundary conditions are $n(0) = \infty$, and $n(\infty) = 0$. Find the solution to this equation. (A.K.T.U. 2017)

Answers

1. $y = c_1e^{3x} + c_2e^{4x}$
2. $y = c_1e^{-ax} + c_2e^{-bx}$
3. $y = c_1e^{-x} + c_2e^{-2x} + c_3e^{-3x}$
4. $x = (c_1 + c_2t)e^{-3t}$
5. $y = (c_1 + c_2x + c_3x^2)e^x$
6. $y = (c_1 + c_2x)\, e^x + c_3\, e^{-x}$
7. $y = e^{-x}(c_1 + c_2x + c_3x^2) + c_4e^{4x}$
8. $y = (c_1 + c_2x) \cos 2x + (c_3 + c_4x) \sin 2x$
9. $y = c_1 + c_2x + c_3x^2 + c_4e^{-x} + c_5e^x$
10. $y = c_1e^{2x} + c_2 \cos 2x + c_3 \sin 2x$
11. $y = (c_1 + c_2x) \cos x + (c_3 + c_4x) \sin x + c_5e^x$
12. $y = e^{2x}(c_1 \cosh\sqrt{3}\,x + c_2 \sinh\sqrt{3}\,x)$

13. $y = c_1e^x + c_2e^{-x} + e^{x/2}\left(c_3 \cos \frac{\sqrt{3}}{2}x + c_4 \sin \frac{\sqrt{3}}{2}x\right) + e^{-x/2}\left(c_5 \cos \frac{\sqrt{3}}{2}x + c_6 \sin \frac{\sqrt{3}}{2}x\right)$

14. $y = c_1 \cos x + c_2 \sin x + e^{\sqrt{3}x/2}\left(c_3 \cos \frac{x}{2} + c_4 \sin \frac{x}{2}\right) + e^{-\sqrt{3}x/2}\left(c_5 \cos \frac{x}{2} + c_6 \sin \frac{x}{2}\right).$

15. $x = 0$ **16.** $y = x^2e^{-2x}$. **17.** $n(x) = \infty e^{-\frac{x}{L}}$

1.17 THE INVERSE OPERATOR $\dfrac{1}{f(D)}$

$\frac{1}{f(\text{D})}$ Q is that function of x, *free from arbitrary constants,* which when operated upon by f(D) gives Q.

Thus $$f(\text{D})\left\{\frac{1}{f(\text{D})}\text{Q}\right\} = \text{Q}$$

$\therefore$ f(D) and $\frac{1}{f(\text{D})}$ are inverse operators.

Note 1. $\frac{1}{\text{D}}\, Q = \int Q\, dx.$

Note 2. $\frac{1}{\text{D} - a}\, Q = e^{ax} \int Qe^{-ax}\, dx.$

1.18 RULES FOR FINDING THE PARTICULAR INTEGRAL (P.I.)

Consider the differential equation, $(\text{D}^n + a_1\text{D}^{n-1} + a_2\text{D}^{n-2} + \cdots\cdots + a_{n-1}\text{D} + a_n)y = \text{Q}$

It can be written as $f(\text{D})y = \text{Q}$

$$\therefore \quad \text{P.I.} = \frac{1}{f(\text{D})}\text{Q}.$$

1.18.1 Case I. *When* $Q = e^{ax}$ *(or* e^{ax+b}*)*

Since
$$\text{D}\, e^{ax} = a\, e^{ax}$$
$$\text{D}^2\, e^{ax} = a^2\, e^{ax}$$
$$\dots\dots\dots\dots$$
$$\dots\dots\dots\dots$$
$$\text{D}^{n-1}\, e^{ax} = a^{n-1}\, e^{ax}$$
$$\text{D}^n e^{ax} = a^n\, e^{ax}$$

$$\therefore \quad (\text{D}^n + a_1\text{D}^{n-1} + \cdots\cdots + a_{n-1}\text{D} + a_n)e^{ax} = (a^n + a_1a^{n-1} + \cdots\cdots + a_{n-1}\,a + a_n)e^{ax}$$

or
$$f(\text{D})\, e^{ax} = f(a)\, e^{ax}$$

Operating on both sides by $\frac{1}{f(\text{D})}$;

$$\frac{1}{f(\text{D})}[f(\text{D})\, e^{ax}] = \frac{1}{f(\text{D})}[f(a)\, e^{ax}]$$

or
$$e^{ax} = f(a)\,\frac{1}{f(D)}\,e^{ax}$$

Dividing both sides by $f(a)$, $\dfrac{1}{f(a)}\,e^{ax} = \dfrac{1}{f(D)}\,e^{ax}$, provided $f(a) \neq 0$

Hence $\boxed{\dfrac{1}{f(D)}\,e^{ax} = \dfrac{1}{f(a)}\,e^{ax}\text{, provided } f(a) \neq 0.}$

Case of failure: If $f(a) = 0$, the above method fails.

Since $f(a) = 0$, $D = a$ is a root of $f(D) = 0$

$\therefore$ $D - a$ is a factor of $f(D)$.

Let
$$f(D) = (D - a)\,\phi(D), \text{ where } \phi(a) \neq 0$$

Then
$$\frac{1}{f(D)}\,e^{ax} = \frac{1}{(D-a)\,\phi(D)}\,e^{ax} = \frac{1}{D-a}\cdot\frac{1}{\phi(a)}\,e^{ax} \quad \text{...(1)}$$
$$= \frac{1}{\phi(a)}\cdot\frac{1}{D-a}\,e^{ax} = \frac{1}{\phi(a)}\,e^{ax}\int e^{ax}\,.\,e^{-ax}\,dx \quad \text{[by Note 2]}$$
$$= \frac{1}{\phi(a)}\,e^{ax}\int 1\,dx = x\,.\,\frac{1}{\phi(a)}\,e^{ax} \quad \text{...(2)}$$

Differentiating both sides of (1) w.r.t. D, we get
$$f'(D) = (D - a)\,\phi'(D) + \phi(D)$$
$\Rightarrow$
$$f'(a) = \phi(a)$$

$\therefore$ From (2), we have $\boldsymbol{\dfrac{1}{f(D)}\,e^{ax} = x\,.\,\dfrac{1}{f'(a)}\,e^{ax}}$ **provided** $\boldsymbol{f'(a) \neq 0}$

Another case of failure:

If $f'(a) = 0$, then $\dfrac{1}{f(D)}\,e^{ax} = x^2\,.\,\dfrac{1}{f''(a)}\,e^{ax}$, provided $f''(a) \neq 0$ and so on.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the P.I. of $(4D^2 + 4D - 3)y = e^{2x}$.*

Sol.
$$\text{P.I.} = \frac{1}{4D^2 + 4D - 3}\,e^{2x} = \frac{1}{4(2)^2 + 4(2) - 3}\,e^{2x} \quad \text{(Replacing D by 2)}$$
$$= \frac{1}{21}\,e^{2x}.$$

Example 2. *Find the P.I. of $(D^3 - 3D^2 + 4)y = e^{2x}$.*

Sol.
$$\text{P.I.} = \frac{1}{D^3 - 3D^2 + 4}\,e^{2x}.$$

Here the denominator vanishes when D is replaced by 2. It is a case of failure. We multiply the numerator by x and differentiate the denominator w.r.t. D.

$\therefore$
$$\text{P.I.} = x\,.\,\frac{1}{3D^2 - 6D}\,e^{2x}$$

It is again a case of failure. We multiply the numerator by x and differentiate the denominator w.r.t. D.

$$\therefore \qquad \text{P.I.} = x^2 \cdot \frac{1}{6D-6} e^{2x} = x^2 \cdot \frac{1}{6(2)-6} e^{2x} = \frac{x^2}{6} \cdot e^{2x}.$$

Example 3. *Find the P.I. of* $(D + 1)^3 y = e^{-x}$.

Sol.
$$\text{P.I.} = \frac{1}{(D+1)^3} e^{-x} = x \cdot \frac{1}{3(D+1)^2} e^{-x} \qquad | \text{ Case of failure}$$

$$= x^2 \cdot \frac{1}{3.2(D+1)} e^{-x} \qquad | \text{ Again case of failure}$$

$$= x^3 \cdot \frac{1}{3.2.1} e^{-x} = \frac{x^3}{6} e^{-x}.$$

Example 4. *Solve:*

(i) $(D^3 - 2D^2 + 4D - 8)\, y = 8$

(ii) $(D - 2)^3\, y = 17\, e^{2x}$. (M.T.U. 2011)

Sol. *(i)* Auxiliary equation is

$$m^3 - 2m^2 + 4m - 8 = 0$$

$$\Rightarrow \quad (m^2 + 4)(m - 2) = 0$$

$$\Rightarrow \quad m = 2, \pm 2i$$

$$\therefore \qquad \text{C.F.} = c_1 e^{2x} + c_2 \cos 2x + c_3 \sin 2x$$

$$\text{P.I.} = \frac{1}{D^3 - 2D^2 + 4D - 8} (8\, e^{0x}) \qquad | \because e^{0x} = 1$$

$$= \frac{1}{(0)^3 - 2(0)^2 + 4(0) - 8} (8e^{0x}) = -1$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{2x} + c_2 \cos 2x + c_3 \sin 2x - 1$$

where c_1, c_2 and c_3 are arbitrary constants of integration.

(ii) Auxiliary equation is

$$(m - 2)^3 = 0 \;\Rightarrow\; m = 2, 2, 2$$

$$\therefore \qquad \text{C.F.} = (c_1 + c_2 x + c_3 x^2)\, e^{2x}$$

$$\text{P.I.} = \frac{1}{(D-2)^3} 17e^{2x} \qquad | \text{ Case of failure}$$

$$= 17x \cdot \left[\frac{1}{3(D-2)^2} e^{2x}\right] \qquad | \text{ Again case of failure}$$

$$= \frac{17}{3} x^2 \cdot \left[\frac{1}{2(D-2)} e^{2x}\right] \qquad | \text{ Again a case of failure}$$

$$= \frac{17}{6} x^3 e^{2x}$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = (c_1 + c_2x + c_3x^2)\, e^{2x} + \frac{17}{6} x^3 e^{2x}$$

where c_1, c_2 and c_3 are arbitrary constants of integration.

Example 5. *Solve:* $2\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} - 2y = e^x$.

Sol. The auxiliary equation is

$$2m^3 - m^2 + 4m - 2 = 0$$

$$\Rightarrow \quad (2m - 1)(m^2 + 2) = 0$$

$$\Rightarrow \quad m = \frac{1}{2}, \pm \sqrt{2}\, i$$

$$\therefore \quad \text{C.F.} = c_1 e^{x/2} + c_2 \cos \sqrt{2}\, x + c_3 \sin \sqrt{2}\, x$$

$$\text{P.I.} = \frac{1}{2D^3 - D^2 + 4D - 2} e^x = \frac{1}{2(1)^3 - (1)^2 + 4(1) - 2} e^x = \frac{1}{3} e^x$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{x/2} + c_2 \cos \sqrt{2}\, x + c_3 \sin \sqrt{2}\, x + \frac{1}{3} e^x$$

where c_1, c_2 and c_3 are arbitrary constants of integration.

Example 6. *Solve:* $(D^3 - 6D^2 + 11D - 6)y = e^{-2x} + e^{-3x}$.

Sol. Auxiliary equation is

$$m^3 - 6m^2 + 11m - 6 = 0$$

or $\quad (m - 1)(m - 2)(m - 3) = 0$

whence $\quad m = 1, 2, 3$

$$\therefore \quad \text{C.F.} = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$$

$$\text{P.I.} = \frac{1}{D^3 - 6D^2 + 11D - 6} (e^{-2x} + e^{-3x})$$

$$= \frac{1}{D^3 - 6D^2 + 11D - 6} e^{-2x} + \frac{1}{D^3 - 6D^2 + 11D - 6} e^{-3x}$$

$$= \frac{1}{(-2)^3 - 6(-2)^2 + 11(-2) - 6} e^{-2x} + \frac{1}{(-3)^3 - 6(-3)^2 + 11(-3) - 6} e^{-3x}$$

$$= -\frac{1}{60} e^{-2x} - \frac{1}{120} e^{-3x} = -\frac{1}{120} (2e^{-2x} + e^{-3x})$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^x + c_2 e^{2x} + c_3 e^{3x} - \frac{1}{120} (2e^{-2x} + e^{-3x})$$

where c_1, c_2 and c_3 are arbitrary constants of integration.

Example 7. *Solve:* $(D^2 - a^2)y = e^{ax} - e^{-ax}$.

Sol. Auxiliary equation is

$$m^2 - a^2 = 0$$

$$\Rightarrow \qquad m = \pm a$$

$$\therefore \qquad \text{C.F.} = c_1 e^{ax} + c_2 e^{-ax}$$

$$\text{P.I.} = \frac{1}{D^2 - a^2}(e^{ax} - e^{-ax}) = \frac{1}{D^2 - a^2}(e^{ax}) - \frac{1}{D^2 - a^2}(e^{-ax})$$

$$= x \cdot \frac{1}{2D}(e^{ax}) - x \cdot \frac{1}{2D}(e^{-ax}) = \frac{x}{2} \cdot \frac{e^{ax}}{a} - \frac{x}{2}\left(\frac{e^{-ax}}{-a}\right)$$

$$= \frac{x}{2}\left(\frac{e^{ax} + e^{-ax}}{a}\right) = \frac{x}{a}\cosh ax$$

The complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{ax} + c_2 e^{-ax} + \frac{x}{a}\cosh ax$$

where c_1 and c_2 are arbitrary constants of integration.

Example 8. *Solve:* $(D^2 + D + 1)\, y = (1 + e^x)^2$.

Sol. Auxiliary equation is

$$m^2 + m + 1 = 0$$

$$\Rightarrow \qquad m = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\, i$$

$$\text{C.F.} = e^{-\frac{1}{2}x}\left(c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x\right)$$

$$\text{P.I.} = \frac{1}{D^2 + D + 1}(1 + e^x)^2 = \frac{1}{D^2 + D + 1}(1 + e^{2x} + 2e^x)$$

$$= \frac{1}{D^2 + D + 1}(e^{0x}) + \frac{1}{D^2 + D + 1}(e^{2x}) + \frac{1}{D^2 + D + 1}(2e^x)$$

$$= \frac{1}{(0)^2 + (0) + 1}e^{0x} + \frac{1}{(2)^2 + (2) + 1}e^{2x} + \frac{2}{(1)^2 + (1) + 1}e^x = 1 + \frac{1}{7}e^{2x} + \frac{2}{3}e^x$$

Hence complete solution is

$$y = \text{C.F.} + \text{P.I.} = e^{-\frac{x}{2}}\left(c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x\right) + 1 + \frac{1}{7}e^{2x} + \frac{2}{3}e^x$$

where c_1 and c_2 are arbitrary constants of integration.

Example 9. *Solve :* $(D + 2)(D - 1)^2\, y = e^{-2x} + 2 \sinh x$.

Sol. Auxiliary equation is

$$(m + 2)(m - 1)^2 = 0 \quad \Rightarrow \quad m = -2, 1, 1$$

$$\therefore \qquad \text{C.F.} = c_1 e^{-2x} + (c_2 + c_3 x)\, e^x$$

$$\text{P.I.} = \frac{1}{(D + 2)(D - 1)^2}(e^{-2x} + 2 \sinh x)$$

$$= \frac{1}{(D + 2)(D - 1)^2}(e^{-2x} + e^x - e^{-x}) \qquad \left[\because \sinh x = \frac{e^x - e^{-x}}{2}\right]$$

Now, $\frac{1}{(D+2)(D-1)^2} e^{-2x} = \frac{1}{D+2}\left[\frac{1}{(D-1)^2} e^{-2x}\right] = \frac{1}{D+2}\left[\frac{1}{(-2-1)^2} e^{-2x}\right]$

$$= \frac{1}{9} \cdot \frac{1}{D+2} e^{-2x}$$

| Case of failure

$$= \frac{x}{9} e^{-2x}$$

$$\frac{1}{(D+2)(D-1)^2} e^{x} = \frac{1}{(D-1)^2}\left[\frac{1}{D+2} e^{x}\right] = \frac{1}{(D-1)^2}\left[\frac{1}{1+2} e^{x}\right]$$

$$= \frac{1}{3} \cdot \frac{1}{(D-1)^2} e^{x}$$

| Case of failure

$$= \frac{1}{3} \cdot x \frac{1}{2(D-1)} e^{x}$$

| Case of failure

$$= \frac{1}{3} \cdot x^2 \cdot \frac{1}{2} e^{x} = \frac{1}{6} x^2 e^{x}$$

$$\frac{1}{(D+2)(D-1)^2} e^{-x} = \frac{1}{(-1+2)(-1-1)^2} e^{-x} = \frac{1}{4} e^{-x}$$

$\therefore$ $$\text{P.I.} = \frac{x}{9} e^{-2x} + \frac{x^2}{6} e^{x} + \frac{1}{4} e^{-x}$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{-2x} + (c_2 + c_3 x) e^{x} + \frac{x}{9} e^{-2x} + \frac{x^2}{6} e^{x} + \frac{1}{4} e^{-x}.$$

where c_1, c_2 and c_3 are arbitrary constants of integration.

Example 10. *Solve the differential equation*

$$\frac{d^3 y}{dx^3} - 3\frac{d^2 y}{dx^2} + 3\frac{dy}{dx} - y = e^{x} + 2.$$

Sol. The given equation is

$$(D^3 - 3D^2 + 3D - 1)y = e^{x} + 2$$

or $$(D-1)^3 y = e^{x} + 2$$

Auxiliary equation is

$$(m-1)^3 = 0 \quad \Rightarrow \quad m = 1, 1, 1$$

$\therefore$ $$\text{C.F.} = (c_1 + c_2 x + c_3 x^2) e^{x}$$

$$\text{P.I.} = \frac{1}{(D-1)^3}(e^{x} + 2) = \frac{1}{(D-1)^3} e^{x} + \frac{1}{(D-1)^3}(2e^{0x})$$

$$= x \cdot \frac{1}{3(D-1)^2} e^{x} + \frac{1}{(0-1)^3}(2e^{0x}) = x^2 \cdot \frac{1}{3.2(D-1)}(e^{x}) - 2$$

$$= x^3 \cdot \frac{1}{3 \cdot 2 \cdot 1}(e^{x}) - 2 = \frac{x^3}{6} e^{x} - 2$$

$\therefore$ The complete solution is

$$y = \text{C.F.} + \text{P.I.} = (c_1 + c_2 x + c_3 x^2) e^{x} + \frac{x^3}{6} e^{x} - 2$$

where c_1, c_2 and c_3 are arbitrary constants of integration.

TEST YOUR KNOWLEDGE

Solve the following differential equations:

1. $\dfrac{d^3y}{dx^3} + y = 3 + 5e^x$

2. $\dfrac{d^2y}{dx^2} - 4y = (1 + e^x)^2$

3. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = -2\cosh x$

4. $(2D + 1)^2 y = 4e^{-x/2}$

5. $(D^2 - 2kD + k^2)\, y = e^{kx}$

6. $\dfrac{d^3y}{dx^3} + 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + y = e^{-x}$

7. $(D + 2)(D - 1)^3 y = e^x$

8. $\dfrac{d^2y}{dx^2} + 31\dfrac{dy}{dx} + 240y = 272\, e^{-x}$

9. $\dfrac{d^2y}{dx^2} + 2p\dfrac{dy}{dx} + (p^2 + q^2)\, y = e^{2x}$

10. $(D^4 + D^3 + D^2 - D - 2)\, y = e^x$

11. $\dfrac{d^3y}{dx^3} + y = 3 + e^{-x} + 5e^{2x}$

12. $y'' + 4y' + 13y = 18e^{-2x};\ y(0) = 0,\ y'(0) = 9.$

Answers

1. $y = c_1 e^{-x} + e^{\frac{1}{2}x}\left(c_2 \cos \dfrac{\sqrt{3}}{2}x + c_3 \sin \dfrac{\sqrt{3}}{2}x\right) + 3 + \dfrac{5}{2}e^x$

2. $y = c_1 e^{2x} + c_2 e^{-2x} - \dfrac{1}{4} - \dfrac{2}{3}e^x + \dfrac{1}{4}xe^{2x}$

3. $y = e^{-2x}(c_1 \cos x + c_2 \sin x) - \dfrac{1}{10}e^x - \dfrac{1}{2}e^{-x}$

4. $y = \left(c_1 + c_2 x + \dfrac{x^2}{2}\right)e^{-x/2}$

5. $y = (c_1 + c_2 x)\, e^{kx} + \dfrac{x^2}{2}e^{kx}$

6. $y = (c_1 + c_2 x + c_3 x^2)\, e^{-x} + e^{-x} \cdot \dfrac{x^3}{6}$

7. $y = (c_1 + c_2 x + c_3 x^2)\ e^x + c_4\, e^{-2x} + \dfrac{x^3 e^x}{18}$

8. $y = c_1 e^{-15x} + c_2 e^{-16x} + \dfrac{136}{105}e^{-x}$

9. $y = e^{-px}(c_1 \cos qx + c_2 \sin qx) + \dfrac{e^{2x}}{(2 + p)^2 + q^2}$

10. $y = c_1 e^x + c_2 e^{-x} + e^{-x/2}\left[c_3 \cos \dfrac{\sqrt{7}}{2}x + c_4 \sin \dfrac{\sqrt{7}}{2}x\right] + \dfrac{1}{8}xe^x$

11. $y = c_1 e^{-x} + e^{x/2}\left(c_2 \cos \dfrac{\sqrt{3}}{2}x + c_3 \sin \dfrac{\sqrt{3}}{2}x\right) + 3 + \dfrac{5}{9}e^{2x} + \dfrac{1}{3}xe^{-x}$

12. $y = e^{-2x}(-2\cos 3x + 3\sin 3x + 2).$

1.18.2 Case II. *When Q = sin (ax + b) or cos (ax + b)*

$$D \sin(ax + b) = a\cos(ax + b)$$

$$\mathbf{D^2 \sin(ax + b) = (-a^2)\sin(ax + b)}$$

$$D^3 \sin(ax + b) = -a^3 \cos(ax + b)$$

$$D^4 \sin(ax + b) = a^4 \sin(ax + b)$$

or $$(\mathbf{D}^2)^2 \sin (ax + b) = (-a^2)^2 \sin (ax + b)$$

....

....

In general, $$(D^2)^n \sin (ax + b) = (-a^2)^n \sin (ax + b)$$

$\therefore$ $$f(D^2) \sin (ax + b) = f(-a^2) \sin (ax + b)$$

Operating on both sides by $\dfrac{1}{f(D^2)}$,

$$\frac{1}{f(D^2)} \{f(D^2) \sin (ax + b)\} = \frac{1}{f(D^2)} \{f(-a^2) \sin (ax + b)\}$$

or $$\sin (ax + b) = f(-a^2) \frac{1}{f(D^2)} \sin (ax + b).$$

Dividing both sides by $f(-a^2)$,

$$\frac{1}{f(\mathbf{D}^2)} \sin (ax + b) = \frac{1}{f(-a^2)} \sin (ax + b), \text{ provided } f(-a^2) \neq 0$$

Similarly, $$\frac{1}{f(\mathbf{D}^2)} \cos (ax + b) = \frac{1}{f(-a^2)} \cos (ax + b), \text{ provided } f(-a^2) \neq 0$$

Steps: When $\mathbf{Q} = \sin (ax + b)$ or $\cos (ax + b)$,

1. Replace D^2 by $-a^2$,
   D^4 by a^4,
   D^6 by $-a^6$,
   D^8 by a^8 and so on.
2. By doing so, following possibilities arise:
   (a) If denominator reduces to a constant, it will be final step in finding P.I.
   (b) If denominator reduces into D only, we are then only to integrate the given function Q once.
   (c) If denominator reduces to a factor of the form $\alpha D + \beta$ then we operate by its conjugate $\alpha D - \beta$ on both numerator and denominator from left hand side such as

$$\frac{\alpha D - \beta}{\alpha D - \beta} \cdot \left[\frac{1}{(\alpha D + \beta)} \sin (ax + b)\right]$$

By doing so, denominator will become $\alpha^2 D^2 - \beta^2$ which in turn reduces to a constant by replacing D^2 by $-a^2$.

Now, we operate $\sin (ax + b)$ by $(\alpha D - \beta)$ and consequently, find the required particular integral.

Case of failure: If $f(-a^2) = 0$, the above method fails. Then we proceed as follows :

$$\frac{1}{f(\mathbf{D}^2)} \cos (ax + b) = x \cdot \frac{1}{f'(\mathbf{D}^2)} \cos (ax + b), \text{ provided } f'(-a^2) \neq 0$$

$$\frac{1}{f(\mathrm{D}^2)} \sin (ax + b) = x \cdot \frac{1}{f'(\mathrm{D}^2)} \sin (ax + b), \text{ provided } f'(-a^2) \neq 0$$

If $f'(-a^2) = 0$, then

$$\frac{1}{f(\mathrm{D}^2)} \sin (ax + b) = x^2 \cdot \frac{1}{f''(\mathrm{D}^2)} \sin (ax + b), \text{ provided } f''(-a^2) \neq 0$$

$$\frac{1}{f(\mathrm{D}^2)} \cos (ax + b) = x^2 \cdot \frac{1}{f''(\mathrm{D}^2)} \cos (ax + b), \text{ provided } f''(-a^2) \neq 0$$

and so on.

Steps:

1. When $f(-a^2) = 0$, we differentiate the denominator w.r.t. D and multiply the expression by x simultaneously in the same step.
2. When $f'(-a^2) = 0$ (*i.e.*, step 1 fails) we again differentiate the reduced denominator in D w.r.t. D and again multiply the remaining expression by x simultaneously.
3. If there is another case of failure, above process is to be repeated again and again until we reach a constant in the denominator or any other possibility(ies) which we have discussed before in the same article.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the following differential equation:*

$(D^2 + 4)y = \sin 3x + \cos 2x.$

Sol. Auxiliary equation is

$$m^2 + 4 = 0 \quad \Rightarrow \quad m = \pm 2i$$

$$\therefore \quad \text{C.F.} = c_1 \cos 2x + c_2 \sin 2x$$

$$\text{P.I.} = \frac{1}{\mathrm{D}^2 + 4} (\sin 3x) + \frac{1}{\mathrm{D}^2 + 4} (\cos 2x)$$

$$= \frac{1}{-(3)^2 + 4} \sin 3x + x \cdot \frac{1}{2\mathrm{D}} (\cos 2x)$$

$$= -\frac{1}{5} \sin 3x + \frac{x}{2}\left(\frac{\sin 2x}{2}\right) = -\frac{1}{5} \sin 3x + \frac{x}{4} \sin 2x$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{5} \sin 3x + \frac{x}{4} \sin 2x$$

where c_1 and c_2 are arbitrary constants of integration.

Example 2. *Find the P.I. of* $(D^3 + 1)y = \sin(2x + 1)$.

Sol. $$\text{P.I.} = \frac{1}{D^3+1}\sin(2x+1) = \frac{1}{D(-2^2)+1}\sin(2x+1) \quad [\text{Putting } D^2 = -2^2]$$

$$= \frac{1}{1-4D}\sin(2x+1)$$

Operating N′ and D′ by (1 + 4D)

$$= \frac{1+4D}{(1+4D)(1-4D)}\sin(2x+1) = \frac{1+4D}{1-16D^2}\sin(2x+1)$$

$$= \frac{1+4D}{1-16(-2^2)}\sin(2x+1) \quad [\text{Putting } D^2 = -2^2]$$

$$= \frac{1}{65}[\sin(2x+1) + 4D\sin(2x+1)]$$

$$= \frac{1}{65}[\sin(2x+1) + 8\cos(2x+1)] \quad \left[\because D \equiv \frac{d}{dx}\right]$$

Example 3. *Solve the following differential equations:*

(*i*) $\frac{d^2y}{dx^2} + a^2 y = \sin ax$ (*ii*) $(D^2 + 4)y = \cos^2 x$.

Sol. (*i*) The auxiliary equation is

$$m^2 + a^2 = 0 \Rightarrow m = \pm ai$$

$$\therefore \quad \text{C.F.} = c_1 \cos ax + c_2 \sin ax$$

$$\text{P.I.} = \frac{1}{D^2+a^2}(\sin ax) = x \cdot \frac{1}{2D}\sin ax$$

$$= \frac{x}{2}\left[\frac{-\cos ax}{a}\right] = -\frac{x}{2a}\cos ax$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1\cos ax + c_2 \sin ax - \frac{x}{2a}\cos ax$$

where c_1 and c_2 are arbitrary constants of integration.

(*ii*) The auxiliary equation is

$$m^2 + 4 = 0 \Rightarrow m = \pm 2i$$

$$\therefore \quad \text{C.F.} = c_1 \cos 2x + c_2 \sin 2x$$

$$\text{P.I.} = \frac{1}{D^2+4}\cos^2 x = \frac{1}{2}\left[\frac{1}{D^2+4}(1+\cos 2x)\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2+4}(e^{0x}) + \frac{1}{D^2+4}(\cos 2x)\right]$$

$$= \frac{1}{2}\left[\frac{1}{4} + x \cdot \frac{1}{2D}(\cos 2x)\right]$$

$$= \frac{1}{2}\left[\frac{1}{4} + \frac{x}{4}\sin 2x\right] = \frac{1}{8}(1 + x\sin 2x)$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{8}(1 + x \sin 2x)$$

where c_1 and c_2 are arbitrary constants of integration.

Example 4. *Solve*: $\dfrac{d^4y}{dx^4} - m^4 y = \cos mx.$

Sol. Auxiliary equation is

$$M^4 - m^4 = 0$$

$$(M^2 - m^2)(M^2 + m^2) = 0$$

$$M = \pm m, \pm mi$$

$$\therefore \quad \text{C.F.} = c_1 e^{mx} + c_2 e^{-mx} + c_3 \cos mx + c_4 \sin mx$$

$$\text{P.I.} = \frac{1}{D^4 - m^4}(\cos mx) = x \cdot \frac{1}{4D^3}\cos mx$$

$$= \frac{x}{4} \cdot \frac{1}{D^2}\left(\frac{-\sin mx}{m}\right) = -\frac{x}{4m^2}\left(\frac{\sin mx}{m}\right) = -\frac{x}{4m^3}\sin mx$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{mx} + c_2 e^{-mx} + c_3 \cos mx + c_4 \sin mx - \frac{x}{4m^3}\sin mx$$

where c_1, c_2, c_3 and c_4 are arbitrary constants of integration.

Example 5. *Solve*: $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} - 2y = e^x + \cos x.$

Sol. Auxiliary equation is

$$m^3 - 3m^2 + 4m - 2 = 0$$

$$\Rightarrow \quad (m^2 - 2m + 2)(m - 1) = 0 \quad \Rightarrow \quad m = 1, 1 \pm i$$

$$\therefore \quad \text{C.F.} = c_1 e^x + e^x(c_2 \cos x + c_3 \sin x)$$

$$\text{P.I.} = \frac{1}{(D^3 - 3D^2 + 4D - 2)}e^x + \frac{1}{(D^3 - 3D^2 + 4D - 2)}\cos x$$

$$= x \cdot \frac{1}{3D^2 - 6D + 4}(e^x) + \frac{1}{(-D + 3 + 4D - 2)}(\cos x)$$

$$= x \cdot \frac{1}{(7-6)} e^x + \frac{1}{3D+1} (\cos x) = x e^x + \frac{3D-1}{9D^2-1} (\cos x)$$

$$= xe^x - \frac{1}{10} (-3 \sin x - \cos x)$$

$\therefore$ Complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1e^x + e^x (c_2 \cos x + c_3 \sin x) + xe^x + \frac{1}{10} (3 \sin x + \cos x)$$

where c_1, c_2, c_3 are arbitrary constants of integration.

Example 6. *Solve:* $(D^2 - 4D + 1)y = \cos x \cos 2x + \sin^2 x$.

Sol. Auxiliary equation is

$$m^2 - 4m + 1 = 0$$

$$\Rightarrow \quad m = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$$

$\therefore$ $\text{C.F.} = e^{2x} (c_1 \cosh \sqrt{3}\, x + c_2 \sinh \sqrt{3}\, x)$

$$\text{P.I.} = \frac{1}{D^2-4D+1} (\cos x \cos 2x) + \frac{1}{D^2-4D+1} (\sin^2 x)$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4D+1} (\cos 3x) + \frac{1}{D^2-4D+1} (\cos x)\right] + \frac{1}{D^2-4D+1}\left(\frac{1-\cos 2x}{2}\right)$$

$$= \frac{1}{2} (P_1 + P_2) + P_3 \qquad \text{...(1)}$$

where, $$P_1 = \frac{1}{D^2-4D+1} (\cos 3x)$$

$$= \frac{1}{-9-4D+1} (\cos 3x) = -\frac{1}{4(D+2)} \cos 3x$$

$$= -\frac{1}{4} \frac{D-2}{(D^2-4)} \cos 3x = -\frac{1}{4} \frac{(D-2)}{(-9-4)} \cos 3x = \frac{1}{52} (-3 \sin 3x - 2 \cos 3x)$$

$$P_2 = \frac{1}{D^2-4D+1} (\cos x) = \frac{1}{-1-4D+1} \cos x = -\frac{1}{4} \sin x$$

$$P_3 = \frac{1}{2}\left[\frac{1}{D^2-4D+1} (1) - \frac{1}{D^2-4D+1} (\cos 2x)\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4D+1} (e^{0x}) - \frac{1}{-4-4D+1} (\cos 2x)\right]$$

$$= \frac{1}{2}\left[\frac{1}{(0)^2-4(0)+1} (e^{0x}) + \frac{1}{4D+3} (\cos 2x)\right]$$

$$= \frac{1}{2}\left[1 + \frac{4D-3}{16D^2-9}(\cos 2x)\right] = \frac{1}{2}\left[1 + \frac{4D-3}{(-64-9)}(\cos 2x)\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{73}(-8\sin 2x - 3\cos 2x)\right] = \frac{1}{2}\left[1 + \frac{1}{73}(8\sin 2x + 3\cos 2x)\right]$$

$\therefore$ From (1),

$$\text{P.I.} = -\frac{1}{104}(3\sin 3x + 2\cos 3x) - \frac{1}{8}\sin x + \frac{1}{2} + \frac{1}{146}(8\sin 2x + 3\cos 2x)$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$= e^{2x}(c_1 \cosh \sqrt{3}\,x + c_2 \sinh \sqrt{3}\,x) + \frac{1}{2} - \frac{1}{8}\sin x$$

$$- \frac{1}{104}(3\sin 3x + 2\cos 3x) + \frac{1}{146}(8\sin 2x + 3\cos 2x)$$

where c_1 and c_2 are arbitrary constants of integration.

Example 7. *Solve:* $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y + 37\sin 3x = 0$ *and find the value of y when* $x = \frac{\pi}{2}$ *being given that* $y = 3$, $\frac{dy}{dx} = 0$ *when* $x = 0$. (G.B.T.U. 2011)

Sol. We have,

$$(D^2 + 2D + 10)\,y = -37\sin 3x$$

Auxiliary equation is

$$m^2 + 2m + 10 = 0$$

$$\Rightarrow \quad m = -1 \pm 3i$$

$$\therefore \quad \text{C.F.} = e^{-x}(c_1\cos 3x + c_2 \sin 3x)$$

$$\text{P.I.} = \frac{1}{D^2 + 2D + 10}(-37\sin 3x)$$

$$= (-37)\frac{1}{2D+1}(\sin 3x) \qquad | \text{ Replacing } D^2 \text{ by } -9$$

$$= (-37)\frac{2D-1}{4D^2-1}(\sin 3x) \qquad | \text{ Operating by } 2D-1$$

$$= (-37)\frac{(2D-1)}{(-37)}(\sin 3x) \qquad | \; D^2 = -9$$

$$= 6\cos 3x - \sin 3x$$

Hence the general solution is

$$y = \text{C.F.} + \text{P.I.} = e^{-x}(c_1\cos 3x + c_2\sin 3x) + 6\cos 3x - \sin 3x \qquad \ldots(1)$$

Applying the condition $y(0) = 3$ in (1),

$$3 = c_1 + 6 \;\Rightarrow\; c_1 = -3$$

From (1), $$\frac{dy}{dx} = e^{-x}(-3c_1 \sin 3x + 3c_2 \cos 3x) - e^{-x}(c_1 \cos 3x + c_2 \sin 3x) - 18 \sin 3x - 3 \cos 3x \quad ...(2)$$

Applying the condition $\frac{dy}{dx} = 0$ when $x = 0$ in (2),

$$0 = 3c_2 - c_1 - 3$$

$$\Rightarrow \quad 0 = 3c_2 \qquad (\because \; c_1 = -3)$$

$$\Rightarrow \quad c_2 = 0$$

Substituting the values of c_1 and c_2 in equation (1), we get

$$y = (6 - 3e^{-x}) \cos 3x - \sin 3x$$

when $x = \frac{\pi}{2}$, $$y = -\sin \frac{3\pi}{2} = 1.$$

TEST YOUR KNOWLEDGE

Solve the following differential equations:

1. $\frac{d^3y}{dx^3} + a^2 \frac{dy}{dx} = \sin ax$
2. $(D^2 - 4D + 3)y = \sin 3x \cos 2x$
3. $\frac{d^2y}{dx^2} + 4y = e^x + \sin 2x$
4. (i) $(D^2 + 9)\, y = \cos 2x + \sin 2x$ (ii) $(D^2 + 5D - 6)\, y = \sin 3x + \cos 2x$ [U.P.T.U. 2010]
5. $\frac{d^2y}{dx^2} + 2k \frac{dy}{dx} + k^2 y = a \cos px$ [U.P.T.U. 2015]
6. $(D^2 - 8D + 9)\, y = 40 \sin 5x$
7. $(D^2 - 4D + 4)y = e^{-4x} + 5 \cos 3x$
8. $(D^4 + 2D^3 - 3D^2)y = 3e^{2x} + 4 \sin x$
9. $(D^2 - 4D - 5)y = e^{2x} + 3 \cos (4x + 3)$
10. $(D^2 + 5D - 6)y = \sin 4x \sin x$
11. (i) $(D^2 + 4)y = \cos x \cos 3x$ (ii) $(D^4 + 2D^2n^2 + n^4)y = \cos mx \; ; \; m \neq n$.
12. $\frac{d^2y}{dx^2} + 4y = \sin^2 2x$ with conditions $y(0) = 0,\ y'(0) = 0$. [G.B.T.U. 2013]

Answers

1. $y = c_1 + c_2 \cos ax + c_3 \sin ax - \frac{x}{2a^2} \sin ax$
2. $y = c_1 e^x + c_2 e^{3x} + \frac{1}{884}(10 \cos 5x - 11 \sin 5x) + \frac{1}{20}(\sin x + 2 \cos x)$
3. $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{5} e^x - \frac{x}{4} \cos 2x$
4. (i) $y = c_1 \cos 3x + c_2 \sin 3x + \frac{1}{5}(\cos 2x + \sin 2x)$

   (ii) $y = c_1 e^x + c_2 e^{-6x} - \frac{1}{30}(\cos 3x + \sin 3x) + \frac{1}{20}(\sin 2x - \cos 2x)$

5. $y = (c_1 + c_2x)\, e^{-kx} + \dfrac{a\,[(k^2 - p^2)\cos px + 2kp \sin px]}{(k^2 + p^2)^2}$

6. $y = e^{4x}\,(c_1 \cosh \sqrt{7}\,x + c_2 \sinh \sqrt{7}\,x) + \dfrac{25}{29}\cos 5x - \dfrac{10}{29}\sin 5x$

7. $y = (c_1 + c_2x)\, e^{2x} + \dfrac{1}{36} e^{-4x} - \dfrac{5}{169}(12 \sin 3x + 5 \cos 3x)$

8. $y = c_1 + c_2x + c_3e^x + c_4\, e^{-3x} + \dfrac{3}{20}\, e^{2x} + \dfrac{2}{5}(\cos x + 2 \sin x)$

9. $y = c_1\, e^{-x} + c_2e^{5x} - \dfrac{1}{9} e^{2x} - \dfrac{3}{697}[16 \sin (4x + 3) + 21 \cos (4x + 3)]$

10. $y = c_1e^x + c_2e^{-6x} + \dfrac{1}{2}\left[\dfrac{\sin 3x - \cos 3x}{30} + \dfrac{31 \cos 5x - 25 \sin 5x}{1586}\right]$

11. (i) $y = c_1 \cos 2x + c_2 \sin 2x + \dfrac{x}{8} \sin 2x - \dfrac{1}{24} \cos 4x.$

(ii) $y = (c_1 + c_2\, x)\, (c_3 \cos nx + c_4 \sin nx) + \dfrac{1}{(n^2 - m^2)^2} \cos mx.$

12. $y = c_1 \cos 2x + c_2 \sin 2x + \dfrac{1}{8}\left(1 + \dfrac{1}{3} \cos 4x\right)$

1.18.3 Case III. *When $Q = x^m$, m being a positive integer.*

Here $\qquad \text{P.I.} = \dfrac{1}{f(D)}\, x^m$

Steps:

1. Take out the lowest degree term from $f(D)$ to make the first term unity (so that Binomial Theorem for a negative index is applicable). The remaining factor will be of the form $1 + \phi(D)$ or $1 - \phi(D)$.
2. Take this factor in the numerator. It takes the form $[1 + \phi(D)]^{-1}$ or $[1 - \phi(D)]^{-1}$.
3. Expand it in ascending powers of D as far as the term containing D^m, since $D^{m+1}(x^m) = 0$, $D^{m+2}(x^m) = 0$ and so on.
4. Operate on x^m term by term.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the P.I. of $(D^2 + 5D + 4)y = x^2 + 7x + 9$.*

Sol. $\text{P.I.} = \dfrac{1}{D^2 + 5D + 4}(x^2 + 7x + 9) = \dfrac{1}{4\left(1 + \dfrac{5D}{4} + \dfrac{D^2}{4}\right)}(x^2 + 7x + 9)$

$$= \frac{1}{4}\left[1 + \left(\frac{5D}{4} + \frac{D^2}{4}\right)\right]^{-1}(x^2 + 7x + 9)$$

$$= \frac{1}{4}\left[1 - \left(\frac{5D}{4} + \frac{D^2}{4}\right) + \left(\frac{5D}{4} + \frac{D^2}{4}\right)^2 - \ldots\ldots\right](x^2 + 7x + 9)$$

$$= \frac{1}{4}\left(1 - \frac{5D}{4} - \frac{D^2}{4} + \frac{25D^2}{16} + \ldots\ldots\right)(x^2 + 7x + 9)$$

$$= \frac{1}{4}\left(1 - \frac{5D}{4} + \frac{21D^2}{16}\right)(x^2 + 7x + 9) \qquad | \text{ Leaving higher powers of D}$$

$$= \frac{1}{4}\left[(x^2 + 7x + 9) - \frac{5}{4}D(x^2 + 7x + 9) + \frac{21}{16}D^2(x^2 + 7x + 9)\right]$$

$$= \frac{1}{4}\left[(x^2 + 7x + 9) - \frac{5}{4}(2x + 7) + \frac{21}{16}(2)\right] = \frac{1}{4}\left(x^2 + \frac{9}{2}x + \frac{23}{8}\right).$$

Example 2. *Find the P.I. of $y'' - 6y' + 9y = 2x^2 - x + 3$.*

Sol. $$\text{P.I.} = \frac{1}{D^2 - 6D + 9}(2x^2 - x + 3) = \frac{1}{(D-3)^2}(2x^2 - x + 3)$$

$$= \frac{1}{9}\left(1 - \frac{D}{3}\right)^{-2}(2x^2 - x + 3) = \frac{1}{9}\left(1 + \frac{2D}{3} + \frac{3D^2}{9} + \ldots\ldots\right)(2x^2 - x + 3)$$

$$= \frac{1}{9}\left(1 + \frac{2D}{3} + \frac{D^2}{3}\right)(2x^2 - x + 3) \qquad | \text{ Leaving higher powers of D}$$

$$= \frac{1}{9}\left[2x^2 - x + 3 + \frac{2}{3}(4x - 1) + \frac{1}{3}(4)\right] = \frac{1}{9}\left[2x^2 + \frac{5}{3}x + \frac{11}{3}\right].$$

Example 3. *Solve*: $(D^3 - D^2 - 6D)\,y = 1 + x^2$.

Sol. Auxiliary equation is

$$m^3 - m^2 - 6m = 0$$

$$\Rightarrow \quad m(m - 3)(m + 2) = 0 \quad \Rightarrow \quad m = 0, -2, 3$$

$$\therefore \quad \text{C.F.} = c_1 + c_2e^{-2x} + c_3e^{3x}$$

$$\text{P.I.} = \frac{1}{D^3 - D^2 - 6D}(1 + x^2) = \frac{1}{-6D - D^2 + D^3}(1 + x^2)$$

$$= -\frac{1}{6D\left\{1 + \left(\frac{D - D^2}{6}\right)\right\}}(1 + x^2) = -\frac{1}{6D}\left[1 + \left(\frac{D - D^2}{6}\right)\right]^{-1}(1 + x^2)$$

$$= -\frac{1}{6D}\left[1 - \left(\frac{D - D^2}{6}\right) + \left(\frac{D - D^2}{6}\right)^2 - \ldots\ldots\right] - (1 + x^2)$$

$$= -\frac{1}{6D}\left[1 - \frac{D}{6} + \frac{D^2}{6} + \frac{D^2}{36}\right](1 + x^2) = -\frac{1}{6D}\left[1 + x^2 - \frac{1}{6}(2x) + \frac{7}{36}(2)\right]$$

$$= -\frac{1}{6D}\left(1+x^2-\frac{x}{3}+\frac{7}{18}\right) = -\frac{1}{6D}\left(x^2-\frac{x}{3}+\frac{25}{18}\right)$$

$$= -\frac{1}{6}\left(\frac{x^3}{3}-\frac{x^2}{6}+\frac{25}{18}x\right) = -\frac{x}{18}\left(x^2-\frac{x}{2}+\frac{25}{6}\right)$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 + c_2e^{-2x} + c_3e^{3x} - \frac{x}{18}\left(x^2-\frac{x}{2}+\frac{25}{6}\right)$$

where c_1, c_2 and c_3 are arbitrary constants of integration.

Example 4. *Solve* $(D-2)^2y = 8(e^{2x} + \sin 2x + x^2)$.

Sol. Auxiliary equation is

$$(m-2)^2 = 0 \quad \Rightarrow \quad m = 2, 2$$

$$\therefore \quad \text{C.F.} = (c_1 + c_2x)\, e^{2x} \quad \text{—— (1)}$$

$$\text{P.I.} = \frac{1}{(D-2)^2}\,[8(e^{2x} + \sin 2x + x^2)]$$

$$= 8\left[\frac{1}{(D-2)^2}e^{2x} + \frac{1}{(D-2)^2}\sin 2x + \frac{1}{(D-2)^2}x^2\right]$$

Now, $\frac{1}{(D-2)^2}e^{2x} = x \cdot \frac{1}{2(D-2)}e^{2x}$ | Case of failure

$= x^2 \cdot \frac{1}{2}e^{2x}$ | Case of failure

$\frac{1}{(D-2)^2}\sin 2x = \frac{1}{D^2-4D+4}\sin 2x = \frac{1}{-2^2-4D+4}\sin 2x$ [Putting $D^2 = -2^2$]

$$= -\frac{1}{4D}\sin 2x = -\frac{1}{4}\int \sin 2x\,dx = -\frac{1}{4}\left(-\frac{\cos 2x}{2}\right) = \frac{1}{8}\cos 2x$$

$$\frac{1}{(D-2)^2}x^2 = \frac{1}{(2-D)^2}x^2 = \frac{1}{4\left(1-\frac{D}{2}\right)^2}x^2 = \frac{1}{4}\left(1-\frac{D}{2}\right)^{-2}x^2$$

$$= \frac{1}{4}\left[1 + D + \frac{3}{4}D^2 + \ldots\ldots\right]x^2 = \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right)$$

$$\therefore \quad \text{P.I.} = 8\left[\frac{x^2}{2}e^{2x} + \frac{1}{8}\cos 2x + \frac{1}{4}\left(x^2+2x+\frac{3}{2}\right)\right]$$

$$= 4x^2e^{2x} + \cos 2x + 2x^2 + 4x + 3 \quad \text{—— (2)}$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = (c_1 + c_2x)e^{2x} + 4x^2e^{2x} + \cos 2x + 2x^2 + 4x + 3$$

where c_1 and c_2 are arbitrary constants of integration.

Example 5. *If* $\dfrac{d^2x}{dt^2}+\dfrac{g}{b}(x-a)=0$ *; a, b and g are positive numbers and* $x=a'$, $\dfrac{dx}{dt}=0$ *when* $t=0$*, show that* $x=a+(a'-a)\cos\sqrt{\dfrac{g}{b}}\,t.$

Sol. We have $\quad \dfrac{d^2x}{dt^2}+\dfrac{g}{b}x=\dfrac{ag}{b}$

Auxiliary equation is $\quad m^2+\dfrac{g}{b}=0 \;\Rightarrow\; m=\pm\sqrt{\dfrac{g}{b}}\,i$

$$\therefore \qquad \text{C.F.}=c_1\cos\sqrt{\frac{g}{b}}\,t+c_2\sin\sqrt{\frac{g}{b}}\,t$$

$$\text{P.I.}=\frac{1}{D^2+\frac{g}{b}}\left(\frac{ag}{b}\right)=\frac{ag}{b}\cdot\frac{1}{\frac{g}{b}\left(1+\frac{bD^2}{g}\right)}(1)=a\left(1+\frac{bD^2}{g}\right)^{-1}(1)=a$$

$\therefore$ General solution is

$$x=\text{C.F.}+\text{P.I.}$$

$$\Rightarrow \qquad x=c_1\cos\sqrt{\frac{g}{b}}\,t+c_2\sin\sqrt{\frac{g}{b}}\,t+a \qquad \text{...(1)}$$

At $t=0$, $x=a'$

$\therefore$ From (1), $\quad a'=c_1+a$

$$\Rightarrow \qquad c_1=a'-a \qquad \text{...(2)}$$

Now, $$\frac{dx}{dt}=\sqrt{\frac{g}{b}}\left(-c_1\sin\sqrt{\frac{g}{b}}\,t+c_2\cos\sqrt{\frac{g}{b}}\,t\right) \qquad \text{...(3)}$$

At $\quad t=0,\quad \dfrac{dx}{dt}=0$

$\therefore$ From (3), $\quad 0=\sqrt{\dfrac{g}{b}}\,.\,c_2$

$$\Rightarrow \qquad c_2=0 \qquad \text{...(4)}$$

$\therefore$ From (1), (2) and (4), the complete solution is

$$x=(a'-a)\cos\sqrt{\frac{g}{b}}\,t+a$$

Example 6. *Find the solution of the equation* $(D^2-1)y=1$ *which vanishes when* $x=0$ *and tends to a finite limit as* $x\to-\infty$ *and D stands for* $\dfrac{d}{dx}$.

Sol. We have $(D^2-1)y=1$

Auxiliary equation is

$$m^2-1=0 \;\Rightarrow\; m=\pm 1$$

$$\text{C.F.} = c_1e^x + c_2e^{-x}$$

$\therefore$

$$\text{P.I.} = \frac{1}{D^2 - 1}(1) = -(1 - D^2)^{-1}(1) = -(1 + D^2 + \dots)(1) = -1$$

$\therefore$ General solution is

$$y = c_1e^x + c_2e^{-x} - 1 \qquad \dots(1)$$

when $x = 0, y = 0$

$\therefore$ From (1), $\quad 0 = c_1 + c_2 - 1 \Rightarrow c_1 + c_2 = 1 \qquad \dots(2)$

Also, y tends to a finite limit as $x \to -\infty$

This condition will be satisfied only when $c_2 = 0$

$\therefore$ From (2), $\quad c_1 = 1$

Hence from (1), Particular solution is $y = e^x - 1$.

TEST YOUR KNOWLEDGE

Solve the following differential equations:

1. $\dfrac{d^2y}{dx^2} - 4y = x^2$
2. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 3y = \cos x + x^2$
3. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = x + \sin x$
4. $\dfrac{d^3y}{dx^3} + 2\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = e^{2x} + x^2 + x$
5. $(D^5 - D)y = 12e^x + 8\sin x - 2x$
6. $\dfrac{d^2y}{dx^2} + y = e^{2x} + \cosh 2x + x^3$ *(U.P.T.U. 2014)*
7. $(D^3 + 8)y = x^4 + 2x + 1$
8. $(D^2 + 2D + 1)y = 2x + x^2$
9. $(D^2 + D - 6)y = x$
10. $(D^3 + 3D^2 + 2D)y = x^2$
11. $(D^6 - D^4)y = x^2$
12. $(D^2 - 1)y = 2x^4 - 3x + 1$
13. $(D^2 - 4D + 4)y = x^2 + e^x + \cos 2x$
14. $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + 4y = x^2 + e^x.$
15. (*i*) $(D^2 + 2D + 1)y = x^2 + x + 1$ (*ii*) $(D^2 - 3D + 2)y = x^2 + 2x + 1$ *(U.P.T.U. 2015)* *(A.K.T.U. 2016)*
16. $(D^3 - 1)y = 3x^4 - 2x^3$

Answers

1. $y = c_1e^{2x} + c_2e^{-2x} - \dfrac{1}{4}\left(x^2 + \dfrac{1}{2}\right)$
2. $y = e^x(c_1\cos\sqrt{2}\,x + c_2\sin\sqrt{2}\,x) + \dfrac{1}{4}(\cos x - \sin x) + \dfrac{1}{27}(9x^2 + 12x + 2)$
3. $y = c_1e^x + c_2e^{-2x} - \dfrac{1}{10}(\cos x + 3\sin x) - \dfrac{1}{4}(2x + 1)$
4. $y = c_1 + (c_2 + c_3x)e^{-x} + \dfrac{1}{3}x^3 - \dfrac{3}{2}x^2 + 4x + \dfrac{1}{18}e^{2x}$
5. $y = c_1 + (c_2 + 3x)e^x + c_3e^{-x} + c_4\cos x + c_5\sin x + x^2 + 2x\sin x$
6. $y = c_1\cos x + c_2\sin x + \dfrac{1}{5}e^{2x} + \dfrac{1}{5}\cosh 2x + x^3 - 6x$

7. $y = c_1 e^{-2x} + e^{x} (c_2 \cos\sqrt{3}\,x + c_3 \sin\sqrt{3}\,x) + \frac{1}{8}(x^4 - x + 1)$

8. $y = (c_1 + c_2 x) e^{-x} + x^2 - 2x + 2$ 9. $y = c_1 e^{-3x} + c_2 e^{2x} - \frac{1}{6}\left(x + \frac{1}{6}\right)$

10. $y = c_1 + c_2 e^{-x} + c_3 e^{-2x} + \frac{x}{12}(2x^2 - 9x + 21)$ 11. $y = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 e^{x} + c_6 e^{-x} - \frac{x^4}{12} - \frac{x^6}{360}$

12. $y = c_1 e^{x} + c_2 e^{-x} - 2x^4 - 24x^2 + 3x - 49$ 13. $y = (c_1 + c_2 x) e^{2x} + \frac{1}{8}(2x^2 + 4x + 3) - \frac{1}{8}\sin 2x + e^{x}$

14. $y = e^{x/2}\left(c_1 \cos\frac{\sqrt{15}}{2}x + c_2 \sin\frac{\sqrt{15}}{2}x\right) + \frac{1}{4}\left(e^{x} + x^2 + \frac{x}{2} - \frac{3}{8}\right)$

15. (i) $y = (c_1 + c_2 x) e^{-x} + x^2 - 3x + 5$ (ii) $y = c_1 e^{x} + c_2 e^{2x} + \frac{1}{2}\left(x^2 + 5x + \frac{15}{2}\right)$

16. $y = c_1 e^{x} + e^{-\frac{x}{2}}\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right) - [3x^4 - 2x^3 + 72x - 12]$.

1.18.4 Case IV. *When $Q = e^{ax} V$, where V is a function of x*

Let u be a function of x, then by successive differentiation, we have

$$D(e^{ax} u) = e^{ax} Du + a\, e^{ax} u = e^{ax} (D + a)u$$

$$D^2 (e^{ax} u) = D[e^{ax} (D + a) u] = e^{ax} (D^2 + aD) u + ae^{ax} (D + a)u$$

$$= e^{ax} (D^2 + 2aD + a^2) u = e^{ax} (D + a)^2 u$$

Similarly, $D^3 (e^{ax} u) = e^{ax} (D + a)^3 u$

In general, $D^n (e^{ax} u) = e^{ax} (D + a)^n u$

$\therefore$ $f(D) (e^{ax} u) = e^{ax} f(D + a)u$

Operating on both sides by $\frac{1}{f(D)}$,

$$\frac{1}{f(D)} [f(D) (e^{ax} u)] = \frac{1}{f(D)} [e^{ax} f(D + a)u]$$

$$\Rightarrow \quad e^{ax} u = \frac{1}{f(D)} [(e^{ax} f(D + a) u] \quad ...(1)$$

Now let $f(D + a) u = V$, *i.e.*, $u = \frac{1}{f(D+a)} V$

$\therefore$ From (1) we have $e^{ax} \frac{1}{f(D+a)} V = \frac{1}{f(D)} (e^{ax} V)$

or
$$\boxed{\frac{1}{f(D)} (e^{ax} V) = e^{ax} \frac{1}{f(D+a)} V}$$

Thus e^{ax} which is on the right of $\frac{1}{f(D)}$ may be taken out to the left provided D is replaced by D + a.

ILLUSTRATIVE EXAMPLES

Example 1. *Obtain the general solution of the differential equation*

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 12y = (x-1)\,e^{2x}.$$

Sol. The given equation is

$$(D^2 + 4D - 12)\,y = (x-1)\,e^{2x} \qquad \ldots(1)$$

Auxiliary equation is

$$m^2 + 4m - 12 = 0$$

$$\Rightarrow \quad (m-2)(m+6) = 0$$

$$\Rightarrow \quad m = 2, -6$$

$$\therefore \quad \text{C.F.} = c_1e^{2x} + c_2e^{-6x}$$

$$\text{P.I.} = \frac{1}{D^2 + 4D - 12}(x-1)\,e^{2x}$$

$$= e^{2x}\cdot\frac{1}{\{(D+2)^2 + 4(D+2) - 12\}}(x-1)$$

$$= e^{2x}\cdot\frac{1}{D^2 + 8D}(x-1) = e^{2x}\cdot\frac{1}{8D}\left(1+\frac{D}{8}\right)^{-1}(x-1)$$

$$= e^{2x}\cdot\frac{1}{8D}\left(1-\frac{D}{8}\right)(x-1) \qquad | \text{ Leaving higher power terms}$$

$$= e^{2x}\cdot\frac{1}{8D}\left(x-1-\frac{1}{8}\right) = e^{2x}\cdot\frac{1}{8D}\left(x-\frac{9}{8}\right) = \frac{e^{2x}}{8}\left(\frac{x^2}{2}-\frac{9x}{8}\right)$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1e^{2x} + c_2e^{-6x} + e^{2x}\left(\frac{x^2}{16}-\frac{9x}{64}\right)$$

where c_1 and c_2 are arbitrary constants of integration.

Example 2. *Find the complete solution of* $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = xe^{3x} + \sin 2x.$

[G.B.T.U. (SUM) 2010]

Sol. Auxiliary equation is

$$m^2 - 3m + 2 = 0$$

$$\Rightarrow \quad (m-1)(m-2) = 0 \quad \Rightarrow \quad m = 1, 2$$

$$\therefore \quad \text{C.F.} = c_1e^{x} + c_2e^{2x}$$

$$\text{P.I.} = \frac{1}{D^2 - 3D + 2}(x\,e^{3x} + \sin 2x)$$

$$= \frac{1}{D^2 - 3D + 2}(e^{3x}\,.\,x) + \frac{1}{D^2 - 3D + 2}(\sin 2x)$$

$$= e^{3x}\,.\,\frac{1}{\{(D+3)^2 - 3(D+3) + 2\}}(x) + \frac{1}{-4-3D+2}(\sin 2x)$$

$$= e^{3x} \cdot \frac{1}{D^2 + 3D + 2}(x) + \frac{1}{-3D-2}(\sin 2x)$$

$$= e^{3x} \cdot \frac{1}{2\left[1+\left(\frac{3D+D^2}{2}\right)\right]}(x) - \frac{(3D-2)}{9D^2-4}(\sin 2x)$$

$$= \frac{e^{3x}}{2} \cdot \left[1+\left(\frac{3D+D^2}{2}\right)\right]^{-1}(x) - \frac{(3D-2)}{(-40)} \sin 2x$$

$$= \frac{e^{3x}}{2}\left(1-\frac{3D}{2}\right)(x) + \frac{1}{40}(6 \cos 2x - 2 \sin 2x)$$

$$= \frac{e^{3x}}{2}\left(x-\frac{3}{2}\right) + \frac{1}{20}(3 \cos 2x - \sin 2x)$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^x + c_2 e^{2x} + \frac{e^{3x}}{2}\left(x-\frac{3}{2}\right) + \frac{1}{20}(3 \cos 2x - \sin 2x)$$

where c_1 and c_2 are arbitrary constants of integration.

Example 3. *Find the P.I. of* $(D^2 - 3D + 2)y = 2e^x \cos \frac{x}{2}$.

Sol.
$$\text{P.I.} = \frac{1}{D^2 - 3D + 2}\left(2e^x \cos \frac{x}{2}\right)$$

$$= 2e^x \cdot \frac{1}{\{(D+1)^2 - 3(D+1) + 2\}} \cos \frac{x}{2} = 2e^x \cdot \frac{1}{D^2 - D} \cos \frac{x}{2}$$

$$= 2e^x \cdot \frac{1}{-\frac{1}{4} - D} \cos \frac{x}{2} = -2e^x \left[\frac{\left(\frac{1}{4} - D\right)}{\left(\frac{1}{4} - D\right)\left(\frac{1}{4} + D\right)} \cos \frac{x}{2}\right]$$

$$= -2e^x \frac{\left(\frac{1}{4} - D\right)}{\frac{1}{16} - D^2} \cos \frac{x}{2} = -2e^x \cdot \frac{\frac{1}{4} - D}{\left(\frac{1}{16} + \frac{1}{4}\right)} \cos \frac{x}{2}$$

$$= -\frac{32}{5} e^x \left(\frac{1}{4} \cos \frac{x}{2} + \frac{1}{2} \sin \frac{x}{2}\right) = -\frac{16}{5} e^x \left(\sin \frac{x}{2} + \frac{1}{2} \cos \frac{x}{2}\right).$$

Example 4. *Obtain the general solution of the differential equation*

$$y'' - 2y' + 2y = x + e^x \cos x.$$

Sol. The given equation is

$$(D^2 - 2D + 2)y = x + e^x \cos x.$$

Auxiliary equation is,

$$m^2 - 2m + 2 = 0$$

$$\Rightarrow \quad m = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

$\therefore$ C.F. $= e^x (c_1 \cos x + c_2 \sin x)$

$$\text{P.I.} = \frac{1}{D^2 - 2D + 2}(x) + \frac{1}{D^2 - 2D + 2}(e^x \cos x)$$

$$= \frac{1}{2 - 2D + D^2}(x) + e^x \cdot \frac{1}{(D+1)^2 - 2(D+1) + 2}(\cos x)$$

$$= \frac{1}{2}\left[1 - \left(\frac{2D - D^2}{2}\right)\right]^{-1}(x) + e^x \cdot \frac{1}{D^2 + 1}\cos x$$

$$= \frac{1}{2}\left[1 + \left(\frac{2D - D^2}{2}\right)\right](x) + e^x \cdot x \cdot \frac{1}{2D}\cos x \qquad | \text{ Case of failure}$$

$$= \frac{1}{2}[1 + D](x) + e^x \cdot \frac{x}{2}\sin x \qquad | \text{ Leaving higher powers}$$

$$= \frac{1}{2}(x + 1) + \frac{xe^x}{2}\sin x$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = e^x (c_1 \cos x + c_2 \sin x) + \frac{1}{2}(x + 1) + \frac{xe^x}{2}\sin x$$

where c_1 and c_2 are arbitrary constants of integration.

Example 5. *Solve:* $\dfrac{d^2y}{dx^2} - 4y = x \sinh x$.

Sol. Given equation is

$$(D^2 - 4)y = x \sinh x$$

Auxiliary equation is

$$m^2 - 4 = 0 \text{ so that } m = \pm 2$$

$\therefore$ C.F. $= c_1 e^{2x} + c_2 e^{-2x}$

$$\text{P.I.} = \frac{1}{D^2 - 4} x \sinh x = \frac{1}{D^2 - 4} x \left(\frac{e^x - e^{-x}}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - 4} e^x \cdot x - \frac{1}{D^2 - 4} e^{-x} \cdot x\right] = \frac{1}{2}\left[e^x \frac{1}{(D+1)^2 - 4} x - e^{-x} \frac{1}{(D-1)^2 - 4} x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{D^2 + 2D - 3} x - e^{-x} \frac{1}{D^2 - 2D - 3} x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{-3\left(1 - \frac{2D}{3} - \frac{D^2}{3}\right)} x - e^{-x} \frac{1}{-3\left(1 + \frac{2D}{3} - \frac{D^2}{3}\right)} x\right]$$

$$= -\frac{1}{6}\left[e^x \left\{1 - \left(\frac{2D}{3} + \frac{D^2}{3}\right)\right\}^{-1} x - e^{-x} \left\{1 + \left(\frac{2D}{3} - \frac{D^2}{3}\right)\right\}^{-1} x\right].$$

$$= -\frac{1}{6}\left[e^x\left(1+\frac{2D}{3}\ldots\ldots\right)x - e^{-x}\left(1-\frac{2D}{3}\ldots\ldots\right)x\right] = -\frac{1}{6}\left[e^x\left(x+\frac{2}{3}\right) - e^{-x}\left(x-\frac{2}{3}\right)\right]$$

$$= -\frac{x}{3}\left(\frac{e^x - e^{-x}}{2}\right) - \frac{2}{9}\left(\frac{e^x + e^{-x}}{2}\right) = -\frac{x}{3}\sinh x - \frac{2}{9}\cosh x$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{2x} + c_2 e^{-2x} - \frac{x}{3}\sinh x - \frac{2}{9}\cosh x$$

where c_1 and c_2 are arbitrary constants of integration.

Example 6. *Solve:* $\dfrac{d^4 y}{dx^4} - y = \cos x \cosh x$.

Sol. Given equation is $(D^4 - 1)y = \cos x \cosh x$.

Auxiliary equation is

$$m^4 - 1 = 0 \quad \text{or} \quad (m^2 - 1)(m^2 + 1) = 0 \quad \text{so that } m = \pm 1, \pm i$$

$$\therefore \quad \text{C.F.} = c_1 e^x + c_2 e^{-x} + e^{0x}(c_3 \cos x + c_4 \sin x)$$

$$= c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$$

$$\text{P.I.} = \frac{1}{D^4 - 1}\cos x \cosh x = \frac{1}{D^4 - 1}\cos x\left(\frac{e^x + e^{-x}}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^4 - 1}e^x \cos x + \frac{1}{D^4 - 1}e^{-x}\cos x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{(D+1)^4 - 1}\cos x + e^{-x}\frac{1}{(D-1)^4 - 1}\cos x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{D^4 + 4D^3 + 6D^2 + 4D}\cos x + e^{-x}\frac{1}{D^4 - 4D^3 + 6D^2 - 4D}\cos x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{(-1^2)^2 + 4D(-1^2) + 6(-1^2) + 4D}\cos x \right.$$

$$\left. + e^{-x}\frac{1}{(-1^2)^2 - 4D(-1^2) + 6(-1^2) - 4D}\cos x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{(-5)}\cos x + e^{-x}\frac{1}{(-5)}\cos x\right] = -\frac{1}{5}\left(\frac{e^x + e^{-x}}{2}\right)\cos x = -\frac{1}{5}\cosh x \cos x$$

Hence complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x - \frac{1}{5}\cos x \cosh x$$

where c_1, c_2, c_3 and c_4 are arbitrary constants of integration.

Example 7. (*i*) *Solve:* $(D^2 - 2D + 1)y = x e^x \sin x$

(*ii*) *Solve:* $\dfrac{d^2 y}{dx^2} - 2\dfrac{dy}{dx} + y = xe^x \cos x$.

Sol. (*i*) Auxiliary equation is

$$m^2 - 2m + 1 = 0 \quad \Rightarrow \quad m = 1, 1$$

$$\text{C.F.} = (c_1 + c_2 x)e^x$$

$\therefore$

$$\text{P.I.} = \frac{1}{D^2 - 2D + 1}(x\, e^x \sin x) = \frac{1}{(D-1)^2}(x\, e^x \sin x)$$

$$= e^x \cdot \frac{1}{(D+1-1)^2}(x \sin x) = e^x \cdot \frac{1}{D^2}(x \sin x)$$

$$= e^x \cdot \frac{1}{D}(-x \cos x + \sin x) = -e^x (x \sin x + 2\cos x)$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = (c_1 + c_2 x)\, e^x - e^x (x \sin x + 2 \cos x)$$

where c_1 and c_2 are arbitrary constants of integration.

(*ii*) Auxiliary equation is

$$m^2 - 2m + 1 = 0 \quad \Rightarrow \quad m = 1, 1$$

$$\text{C.F.} = (c_1 + c_2\, x)e^x$$

$\therefore$

$$\text{P.I.} = \frac{1}{D^2 - 2D + 1}(x\, e^x \cos x) = \frac{1}{(D-1)^2}(x\, e^x \cos x)$$

$$= e^x \cdot \frac{1}{(D+1-1)^2}(x \cos x) = e^x \cdot \frac{1}{D^2}(x \cos x)$$

$$= e^x \cdot \frac{1}{D}(x \sin x + \cos x) = e^x (-x \cos x + 2 \sin x)$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = (c_1 + c_2\, x)\, e^x + e^x (-x \cos x + 2 \sin x)$$

where c_1 and c_2 are arbitrary constants of integration.

Example 8. *Solve:* $(D^4 + 6D^3 + 11D^2 + 6D)y = 20\, e^{-2x} \sin x$.

Sol. Auxiliary equation is

$$m^4 + 6m^3 + 11m^2 + 6m = 0$$

$$\Rightarrow \quad m(m^3 + 6m^2 + 11m + 6) = 0$$

$$\Rightarrow \quad m = 0, -1, -2, -3$$

$$\therefore \quad \text{C.F.} = c_1 + c_2 e^{-x} + c_3 e^{-2x} + c_4\, e^{-3x}$$

$$\text{P.I.} = \frac{1}{D^4 + 6D^3 + 11D^2 + 6D}(20\, e^{-2x} \sin x)$$

$$= \frac{1}{D(D+1)(D+2)(D+3)}(20\, e^{-2x} \sin x)$$

$$= 20\, e^{-2x} \cdot \frac{1}{(D-2)(D-1)\, D(D+1)}(\sin x)$$

$$= 20\, e^{-2x} \cdot \frac{1}{D^4 - 2D^3 - D^2 + 2D}(\sin x)$$

$$= 20\, e^{-2x} \cdot \frac{1}{2+4D} \sin x = 10\, e^{-2x} \frac{1-2D}{1-4D^2} \sin x$$

$$= 2\, e^{-2x} (\sin x - 2\cos x)$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 + c_2 e^{-x} + c_3 e^{-2x} + c_4 e^{-3x} + 2e^{-2x}(\sin x - 2\cos x)$$

where c_1, c_2, c_3 and c_4 are arbitrary constants of integration.

Example 9. *Solve the differential equation*

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = \frac{e^{-x}}{x+2}.$$

Sol. Auxiliary equation is

$$m^2 + 2m + 1 = 0$$

$$\Rightarrow \quad (m+1)^2 = 0 \quad \Rightarrow \quad m = -1, -1$$

$$\therefore \quad \text{C.F.} = (c_1 + c_2 x)\, e^{-x}$$

$$\text{P.I.} = \frac{1}{(D+1)^2}\left(\frac{e^{-x}}{x+2}\right) = e^{-x} \cdot \frac{1}{(D-1+1)^2}\left(\frac{1}{x+2}\right)$$

$$= e^{-x} \frac{1}{D^2}\left(\frac{1}{x+2}\right) = e^{-x} \cdot \frac{1}{D} \log(x+2)$$

$$= e^{-x}\left[\log(x+2) \cdot x - \int \frac{1}{x+2} \cdot x\, dx\right] = e^{-x}\left[x\log(x+2) - \int\left(1 - \frac{2}{x+2}\right)dx\right]$$

$$= e^{-x}\,[x\log(x+2) - x + 2\log(x+2)] = e^{-x}\,[(x+2)\log(x+2) - x]$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = (c_1 + c_2 x)e^{-x} + e^{-x}\,\{(x+2)\log(x+2) - x\}$$

where c_1 and c_2 are arbitrary constants of integration.

Example 10. *Solve:* $(D^2 + 2D + 1)y = x \cos x.$

Sol. Auxiliary equation is

$$m^2 + 2m + 1 = 0$$

$$\Rightarrow \quad m = -1, -1$$

$$\text{C.F.} = (c_1 + c_2 x)\, e^{-x}$$

$$\text{P.I.} = \frac{1}{D^2+2D+1}(x\cos x) = \text{Real part of } \frac{1}{D^2+2D+1}(xe^{ix})$$

$$= \text{R.P. of } e^{ix} \cdot \frac{1}{(D+i)^2 + 2(D+i) + 1}(x) = \text{R.P. of } e^{ix} \cdot \frac{1}{D^2 + 2D(1+i) + 2i}(x)$$

$$= \text{R.P. of } \frac{e^{ix}}{2i}\left[1 + \frac{1+i}{i}D + \frac{D^2}{2i}\right]^{-1}(x)$$

$$= \text{R.P. of } \frac{e^{ix}}{2i}\left(1 - \frac{1+i}{i}D\right)x \qquad | \text{ Leaving higher powers}$$

$$= \text{R.P. of } \frac{e^{ix}}{2i}\left(x - \frac{1+i}{i}\right)$$

$$= \text{R.P. of } \frac{1}{2}(\cos x + i\sin x)(-ix + 1 + i) = \frac{1}{2}\cos x + \frac{1}{2}(x-1)\sin x$$

$\therefore$ The complete solution is given by

$$y = \text{C.F.} + \text{P.I.} = (c_1 + c_2x)\, e^{-x} + \frac{1}{2}\cos x + \frac{1}{2}(x-1)\sin x$$

where c_1 and c_2 are the arbitrary constants of integration.

Example 11. *Solve the following differential equation:*

$$(D^2 - 4D + 4)\, y = 8x^2 e^{2x} \sin 2x.$$

Sol. The auxiliary equation is

$$m^2 - 4m + 4 = 0 \quad \Rightarrow \quad m = 2, 2$$

$\therefore$ $\text{C.F.} = (c_1 + c_2x)\, e^{2x}$

$$\text{P.I.} = \frac{1}{(D-2)^2}(8x^2e^{2x}\sin 2x) = 8\,e^{2x} \cdot \frac{1}{(D+2-2)^2}(x^2 \sin 2x)$$

$$= 8\,e^{2x} \cdot \frac{1}{D^2}(x^2 \sin 2x) = 8\,e^{2x} \cdot \frac{1}{D}\int x^2 \sin 2x\, dx$$

$$= 8\,e^{2x} \cdot \frac{1}{D}\left[x^2 \cdot \left(-\frac{\cos 2x}{2}\right) - \int 2x \cdot \left(\frac{-\cos 2x}{2}\right) dx\right]$$

$$= 8\,e^{2x} \cdot \frac{1}{D}\left[\frac{-x^2}{2}\cos 2x + x \cdot \frac{\sin 2x}{2} - \int 1 \cdot \frac{\sin 2x}{2}\, dx\right]$$

$$= 8\,e^{2x} \cdot \frac{1}{D}\left[\frac{-x^2}{2}\cos 2x + \frac{x \sin 2x}{2} + \frac{\cos 2x}{4}\right]$$

$$= 8\,e^{2x} \cdot \left[\left(\frac{-x^2}{2}\right)\frac{\sin 2x}{2} - \int (-x)\frac{\sin 2x}{2}\, dx + \int x\, \frac{\sin 2x}{2}\, dx + \frac{\sin 2x}{8}\right]$$

$$= 8\,e^{2x}\left[\frac{-x^2}{4}\sin 2x + \frac{\sin 2x}{8} + \int x \sin 2x\, dx\right]$$

$$= 8\,e^{2x}\left[\left(\frac{1}{8} - \frac{x^2}{4}\right)\sin 2x + x \cdot \left(\frac{-\cos 2x}{2}\right) - \int 1 \cdot \left(\frac{-\cos 2x}{2}\right) dx\right]$$

$$= 8\,e^{2x}\left[\left(\frac{1}{8} - \frac{x^2}{4}\right)\sin 2x - \frac{x}{2}\cos 2x + \frac{\sin 2x}{4}\right]$$

$$= 8\,e^{2x}\left[\left(\frac{3}{8} - \frac{x^2}{4}\right)\sin 2x - \frac{x}{2}\cos 2x\right]$$

$$= e^{2x}\left[(3 - 2x^2)\sin 2x - 4x\cos 2x\right]$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = (c_1 + c_2x)\, e^{2x} + e^{2x}[(3 - 2x^2)\sin 2x - 4x\cos 2x]$$

where c_1 and c_2 are arbitrary constants of integration.

Example 12. *A body executes damped forced vibrations given by the equation*

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + b^2x = e^{-kt}\sin \omega t.$$

Solve the equation for both the cases, when $\omega^2 \neq b^2 - k^2$ *and* $\omega^2 = b^2 - k^2$.

Sol. Auxiliary equation is

$$m^2 + 2km + b^2 = 0 \quad \Rightarrow \quad m = -k \pm \sqrt{k^2 - b^2}$$

For damped force vibrations, $k^2 < b^2$

$$\therefore \qquad m = -k \pm i\sqrt{b^2 - k^2}$$

Case I. *When* $\omega^2 \neq b^2 - k^2$

$$\therefore \qquad \text{C.F.} = e^{-kt}\{c_1 \cos \sqrt{b^2 - k^2}\, t + c_2 \sin \sqrt{b^2 - k^2}\, t\}$$

$$\text{P.I.} = \frac{1}{D^2 + 2kD + b^2}(e^{-kt}\sin \omega t)$$

$$= e^{-kt}\cdot\frac{1}{(D-k)^2 + 2k(D-k) + b^2}\sin \omega t$$

$$= e^{-kt}\cdot\frac{1}{D^2 + (b^2 - k^2)}\sin \omega t = e^{-kt}\cdot\frac{1}{-\omega^2 + b^2 - k^2}\sin \omega t$$

$$= \frac{e^{-kt}}{b^2 - k^2 - \omega^2}\sin \omega t$$

Hence complete solution is

$$x = e^{-kt}\{c_1 \cos \sqrt{b^2 - k^2}\, t + c_2 \sin \sqrt{b^2 - k^2}\, t\} + \frac{e^{-kt}}{b^2 - k^2 - \omega^2}\sin \omega t$$

where c_1 and c_2 are arbitrary constants of integration.

Case II. *When* $\omega^2 = b^2 - k^2$

$$\text{C.F.} = e^{-kt}(c_1 \cos \omega t + c_2 \sin \omega t)$$

$$\text{P.I.} = \frac{1}{D^2 + 2kD + b^2}(e^{-kt}\sin \omega t)$$

$$= e^{-kt}\cdot\frac{1}{(D-k)^2 + 2k(D-k) + b^2}\sin \omega t$$

$$= e^{-kt}\cdot\frac{1}{D^2 + b^2 - k^2}\sin \omega t$$

$$= e^{-kt}\cdot t\cdot\frac{1}{2D}\sin \omega t \qquad | \text{ Case of failure}$$

$$= \frac{te^{-kt}}{2}\left(\frac{-\cos \omega t}{\omega}\right) = -\frac{t}{2\omega}e^{-kt}\cos \omega t$$

Hence complete solution is

$$x = \text{C.F.} + \text{P.I.} = e^{-kt}(c_1 \cos \omega t + c_2 \sin \omega t) - \frac{t}{2\omega}e^{-kt}\cos \omega t$$

where c_1 and c_2 are arbitrary constants of integration.

TEST YOUR KNOWLEDGE

Solve the following differential equations:

1. $(D-a)^2 y = e^{ax} f''(x)$
2. $(D^2 - 2D)y = e^x \sin x$
3. (i) $(D^2 - 4D + 4)y = e^x \cos x$ (ii) $(D^2 - 2D + 1)y = e^x \sin x$ (A.K.T.U. 2016, 2017)
4. (i) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 4y = e^{2x} \cos x$ (ii) $(D^3 - 7D - 6)y = (x+1)\,e^{2x}$
5. (i) $(D^2 - 2D + 5)y = e^{2x} \sin x$ (ii) $(D^2 - 2D + 5)y = e^{2x} \cos x$ (U.P.T.U. 2013)
6. (i) $\dfrac{d^2y}{dx^2} + y = e^{-x} + \cos x + x^3 + e^x \sin x$ (ii) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = x^2 e^{3x}$ (U.P.T.U. 2014)
7. (i) $(D^2 - 3D + 2)\,y = xe^x + \sin 2x$ (ii) $(D^2 - 1)\,y = xe^x + \cos^2 x$
8. $(D-1)^2 (D^2+1)^2 y = \sin^2 \dfrac{x}{2} + e^x + x$
9. $(D^2 - 6D + 13)y = 8e^{3x} \sin 4x + 2^x$
10. $(D^2 + 4)y = e^x \sin^2 x$
11. $(D^3 + 2D^2 + D)y = x^2 e^{2x} + \sin^2 x$
12. $(D^2 - 4D + 3)y = e^x \cos 2x + \cos 3x$
13. $(D^2 + 4D + 8)y = 12e^{-2x} \sin x \sin 3x$
14. $\dfrac{d^2y}{dx^2} + 2y = x^2 e^{3x} + e^x \cos 2x$ (U.K.T.U. 2012)
15. $(D-1)^2 y = e^x \sec^2 x \tan x$
16. $(D^2 - 2D + 1)\,y = x \sin x$
17. $(D^2 - 1)\,y = x^2 \cos x$
18. $(D^2 - 1)\,y = x \sin x + x^2 e^x$ (A.K.T.U. 2018)
19. $(D^2 - 2D + 4)y = e^x \cos x + \sin x \cos 3x.$ (G.B.T.U. 2012)
20. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = x^2 e^{-x} \cos x$

Answers

1. $y = e^{ax}\,[c_1 + c_2 x + f(x)]$
2. $y = c_1 + c_2 e^{2x} - \dfrac{1}{2} e^x \sin x$
3. (i) $y = (c_1 + c_2 x)\,e^{2x} - \dfrac{e^x}{2} \sin x$ (ii) $y = (c_1 + c_2 x)\,e^x - e^x \sin x$
4. (i) $y = e^x(c_1 \cos \sqrt{3}\,x + c_2 \sin \sqrt{3}\,x) + \dfrac{1}{13} e^{2x}\,(2 \sin x + 3 \cos x)$

   (ii) $y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x} - \dfrac{e^{2x}}{144}(12x + 17)$
5. (i) $y = e^x\,(c_1 \cos 2x + c_2 \sin 2x) + \dfrac{e^{2x}}{10}\,(2 \sin x - \cos x)$

   (ii) $y = e^x\,(c_1 \cos 2x + c_2 \sin 2x) + \dfrac{e^{2x}}{10}\,(\sin x + 2 \cos x)$
6. (i) $y = c_1 \cos x + c_2 \sin x + \dfrac{1}{2} e^{-x} + \dfrac{1}{2} x \sin x + x^3 - 6x - \dfrac{1}{5} e^x\,(2 \cos x - \sin x)$

   (ii) $y = (c_1 + c_2 x)\,e^x + \dfrac{e^{3x}}{4}\left(x^2 - 2x + \dfrac{3}{2}\right)$

7. (i) $y = c_1 e^x + c_2 e^{2x} - e^x \left(\frac{x^2}{2} + x\right) + \frac{1}{20}(3\cos 2x - \sin 2x)$

(ii) $y = c_1 e^x + c_2 e^{-x} + \frac{1}{4} e^x (x^2 - x) - \frac{1}{2} - \frac{1}{10}\cos 2x$

8. $y = (c_1 + c_2 x)e^x + (c_3 + c_4 x)\cos x + (c_5 + c_6 x)\sin x + \frac{1}{2} - \frac{x^2}{32}\sin x + \frac{x^2}{8} e^x + x + 2$

9. $y = e^{3x}(c_1 \cos 2x + c_2 \sin 2x) - \frac{2}{3} e^{3x}\sin 4x + \frac{2^x}{(\log 2)^2 - 6\log 2 + 13}$

10. $y = c_1 \cos 2x + c_2 \sin 2x + \frac{e^x}{10} - \frac{e^x}{34}(4\sin 2x + \cos 2x)$

11. $y = c_1 + (c_2 + c_3 x)e^{-x} + \frac{e^{2x}}{18}\left(x^2 - \frac{7}{3}x + \frac{11}{6}\right) + \frac{1}{100}(3\sin 2x + 4\cos 2x) + \frac{x}{2}$

12. $y = c_1 e^x + c_2 e^{3x} - \frac{1}{8} e^x(\sin 2x + \cos 2x) - \frac{1}{30}(2\sin 3x + \cos 3x)$

13. $y = e^{-2x}(c_1 \cos 2x + c_2 \sin 2x) + \frac{1}{2} e^{-2x}(3x\sin 2x + \cos 4x)$

14. $y = c_1 \cos\sqrt{2}\,x + c_2 \sin\sqrt{2}\,x + \frac{e^{3x}}{11}\left(x^2 - \frac{12}{11}x + \frac{50}{121}\right) + \frac{e^x}{17}(4\sin 2x - \cos 2x)$

15. $y = e^x\left(c_1 + c_2 x + \frac{1}{2}\tan x\right)$

16. $y = (c_1 + c_2 x)e^x + \frac{1}{2}[(x+1)\cos x - \sin x]$

17. $y = c_1 e^x + c_2 e^{-x} + x\sin x + \left(\frac{1 - x^2}{2}\right)\cos x.$

18. $y = c_1 e^x + c_2 e^{-x} - \frac{1}{2}(x\sin x + \cos x) + \frac{x e^x}{12}(2x^2 - 3x + 3)$

19. $y = e^x(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x) + \frac{e^x \cos x}{2} - \frac{1}{8}\cos 2x + \frac{1}{104}(2\cos 4x - 3\sin 4x)$

20. $y = (c_1 + c_2 x)e^{-x} + e^{-x}(-x^2 \cos x + 4x\sin x + 6\cos x).$

1.18.5 Case V. *When Q is any other function of x*

Resolve $f(D)$ into linear factors.

Let $f(D) \equiv (D - m_1)(D - m_2) \ldots\ldots (D - m_n)$

$$\text{Then P.I.} = \frac{1}{f(D)} Q = \frac{1}{(D - m_1)(D - m_2)\ldots\ldots(D - m_n)} Q$$

$$= \left(\frac{A_1}{D - m_1} + \frac{A_2}{D - m_2} + \ldots\ldots + \frac{A_n}{D - m_n}\right) Q \qquad \text{(Partial Fractions)}$$

$$= A_1 \frac{1}{D - m_1} Q + A_2 \frac{1}{D - m_2} Q + \ldots\ldots + A_n \frac{1}{D - m_n} Q$$

$$= A_1 e^{m_1 x}\int Q e^{-m_1 x}\,dx + A_2 e^{m_2 x}\int Q e^{-m_2 x}\,dx + \ldots\ldots + A_n e^{m_n x}\int Q e^{-m_n x}\,dx.$$

Remark. Remember the following formulae:

(i) $\frac{1}{D-\alpha} Q = e^{\alpha x} \int e^{-\alpha x} Q\, dx$

(ii) $\frac{1}{D+\alpha} Q = e^{-\alpha x} \int e^{\alpha x} Q\, dx$

ILLUSTRATIVE EXAMPLES

Example 1. *Find the complete solution of $(D^2 + a^2)y = \sec ax$.* [A.K.T.U. 2011, 2017]

Sol. Auxiliary equation is

$$m^2 + a^2 = 0 \quad \Rightarrow \quad m = \pm ai$$

$$\therefore \quad \text{C.F.} = c_1 \cos ax + c_2 \sin ax$$

$$\text{P.I.} = \frac{1}{D^2 + a^2} \sec ax$$

$$= \frac{1}{(D - ia)(D + ia)} \sec ax = \frac{1}{2ia}\left[\frac{1}{D - ia} - \frac{1}{D + ia}\right] \sec ax$$

$$= \frac{1}{2ia}\left[\frac{1}{D - ia} (\sec ax) - \frac{1}{D + ia} (\sec ax)\right] = \frac{1}{2ia} (P_1 - P_2)$$

where $\quad P_1 = \frac{1}{D - ia} (\sec ax)$

$$= e^{iax} \int e^{-iax} \sec ax\, dx = e^{iax} \int (\cos ax - i \sin ax) \sec ax\, dx$$

$$= e^{iax} \int (1 - i \tan ax)\, dx = e^{iax} \left\{x + i\left(\frac{\log \cos ax}{a}\right)\right\}$$

$$P_2 = \frac{1}{D + ia} (\sec ax) = e^{-iax} \left\{x - i\left(\frac{\log \cos ax}{a}\right)\right\} \qquad | \text{ Replacing } i \text{ by } -i$$

$$\therefore \quad \text{P.I.} = \frac{1}{2ia}\left[e^{iax}\left\{x + i\left(\frac{\log \cos ax}{a}\right)\right\} - e^{-iax}\left\{x - i\left(\frac{\log \cos ax}{a}\right)\right\}\right]$$

$$= \frac{1}{2ia}\left[x(e^{iax} - e^{-iax}) + i\left(\frac{\log \cos ax}{a}\right)(e^{iax} + e^{-iax})\right]$$

$$= \frac{1}{2ia}\left[2ix \sin ax + \frac{i}{a} (\log \cos ax)(2 \cos ax)\right] = \frac{1}{a}\left[x \sin ax + \frac{1}{a} \cos ax \log \cos ax\right]$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 \cos ax + c_2 \sin ax + \frac{1}{a}\left(x \sin ax + \frac{1}{a} \cos ax \log \cos ax\right)$$

where c_1 and c_2 are arbitrary constants of integration.

Example 2. *Find the general solution of the differential equation*

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}.$$

Sol. Auxiliary equation is

$$m^2 + 3m + 2 = 0$$

$$\Rightarrow \quad (m+1)(m+2) = 0 \quad \Rightarrow \quad m = -1, -2$$

$$\therefore \quad \text{C.F.} = c_1 e^{-x} + c_2 e^{-2x}$$

$$\text{P.I.} = \frac{1}{(D+1)(D+2)}\, e^{e^x} = \left(\frac{1}{D+1} - \frac{1}{D+2}\right) e^{e^x}$$

$$= \frac{1}{D+1}(e^{e^x}) - \frac{1}{D+2}(e^{e^x}) = P_1 - P_2$$

where $\quad P_1 = \frac{1}{D+1}(e^{e^x}) = e^{-x}\int e^x e^{e^x}\, dx = e^{-x}\int e^z\, dz \qquad |$ Put $e^x = z \;\therefore\; e^x\, dx = dz$

$$= e^{-x} e^{e^x}$$

$$P_2 = \frac{1}{D+2}(e^{e^x}) = e^{-2x}\int e^{2x} e^{e^x}\, dx = e^{-2x}\int z e^z\, dz, \qquad \text{where } e^x = z \;\therefore\; e^x\, dx = dz$$

$$= e^{-2x}(z-1)\, e^z = e^{-2x}(e^x - 1)\, e^{e^x} = (e^{-x} - e^{-2x})\, e^{e^x}$$

$$\therefore \quad \text{P.I.} = e^{-x} e^{e^x} - (e^{-x} - e^{-2x})\, e^{e^x} = e^{-2x} e^{e^x}$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x} e^{e^x}$$

where c_1 and c_2 are arbitrary constants of integration.

Example 3. *Solve:* $\dfrac{d^2y}{dx^2} + y = \text{cosec } x.$

Sol. Auxiliary equation is

$$m^2 + 1 = 0 \quad \Rightarrow \quad m = \pm i$$

$$\therefore \quad \text{C.F.} = c_1 \cos x + c_2 \sin x$$

$$\text{P.I.} = \frac{1}{D^2+1}\, \text{cosec } x = \frac{1}{(D+i)(D-i)}\, \text{cosec } x$$

$$= \frac{1}{2i}\left(\frac{1}{D-i} - \frac{1}{D+i}\right) \text{cosec } x \qquad \text{(Partial Fractions)}$$

$$= \frac{1}{2i}\left(\frac{1}{D-i}\, \text{cosec } x - \frac{1}{D+i}\, \text{cosec } x\right)$$

Now $\dfrac{1}{D-i}\, \text{cosec } x = e^{ix}\displaystyle\int \text{cosec } x\, e^{-ix}\, dx$

$$= e^{ix}\int \text{cosec } x\, (\cos x - i \sin x)\, dx = e^{ix}\int (\cot x - i)\, dx$$

$$= e^{ix}(\log \sin x - ix)$$

Changing i to $-i$, we have $\dfrac{1}{D+i}\, \text{cosec } x = e^{-ix}(\log \sin x + ix)$

$$\therefore \quad \text{P.I.} = \frac{1}{2i}\left[e^{ix}(\log \sin x - ix) - e^{-ix}(\log \sin x + ix)\right]$$

$$= \log \sin x \left(\frac{e^{ix} - e^{-ix}}{2i}\right) - x\left(\frac{e^{ix} + e^{-ix}}{2}\right)$$

$$= (\log \sin x) \,.\, \sin x - x \cos x$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 \cos x + c_2 \sin x + \sin x \log \sin x - x \cos x$$

where c_1 and c_2 are arbitrary constants of integration.

Example 4. *Solve:* $\dfrac{d^2y}{dx^2} + a^2y = \tan ax.$

Sol. Auxiliary equation is $m^2 + a^2 = 0 \Rightarrow m = \pm ia$

$\therefore$ $$\text{C.F.} = c_1 \cos ax + c_2 \sin ax$$

$$\text{P.I.} = \frac{1}{D^2 + a^2} \tan ax = \frac{1}{(D + ia)(D - ia)} \tan ax$$

$$= \frac{1}{2ia}\left[\frac{1}{D - ia} - \frac{1}{D + ia}\right] \tan ax \qquad \text{(Partial Fractions)}$$

$$= \frac{1}{2ia}\left[\frac{1}{D - ia} \tan ax - \frac{1}{D + ia} \tan ax\right]$$

Now, $$\frac{1}{D - ia} \tan ax = e^{iax} \int \tan ax \,.\, e^{-iax}\, dx$$

$$= e^{iax} \int \tan ax\,(\cos ax - i \sin ax) = e^{iax} \int \left(\sin ax - i\frac{\sin^2 ax}{\cos ax}\right) dx$$

$$= e^{iax} \int \left(\sin ax - i\frac{1 - \cos^2 ax}{\cos ax}\right) dx = e^{iax} \int [\sin ax - i(\sec ax - \cos ax)]\, dx$$

$$= e^{iax}\left[-\frac{\cos ax}{a} - \frac{i}{a}\log(\sec ax + \tan ax) + i\frac{\sin ax}{a}\right]$$

$$= -\frac{1}{a} e^{iax} [(\cos ax - i \sin ax) + i \log(\sec ax + \tan ax)]$$

$$= -\frac{1}{a} e^{iax} [e^{-iax} + i \log(\sec ax + \tan ax)] = -\frac{1}{a} [1 + ie^{iax} \log(\sec ax + \tan ax)]$$

Changing i to $-i$, we have $\dfrac{1}{D + ia} \tan ax = -\dfrac{1}{a}[1 - ie^{-iax} \log(\sec ax + \tan ax)]$

$$\therefore \quad \text{P.I.} = \frac{1}{2ia}\left[-\frac{1}{a}\{1 + ie^{iax} \log(\sec ax + \tan ax)\} + \frac{1}{a}\{1 - ie^{-iax} \log(\sec ax + \tan ax)\}\right]$$

$$= -\frac{1}{a^2} \log(\sec ax + \tan ax)\left(\frac{e^{iax} + e^{-iax}}{2}\right) = -\frac{1}{a^2} \log(\sec ax + \tan ax) \,.\, \cos ax$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 \cos ax + c_2 \sin ax - \frac{1}{a^2} \cos ax \log(\sec ax + \tan ax)$$

where c_1 and c_2 are arbitrary constants of integration.

Example 5. *Solve the differential equation:* $\dfrac{d^2y}{dx^2} + y = x - \cot x.$

Sol. Auxiliary equation is

$$m^2 + 1 = 0 \qquad \Rightarrow \quad m = \pm i$$

$$\therefore \qquad \text{C.F.} = c_1 \cos x + c_2 \sin x$$

$$\text{P.I.} = \frac{1}{D^2+1}(x - \cot x) = \frac{1}{D^2+1}(x) - \frac{1}{(D-i)(D+i)} \cot x$$

$$= (1 + D^2)^{-1}(x) - \frac{1}{2i}\left[\frac{1}{D-i} - \frac{1}{D+i}\right] \cot x$$

$$= (1 - D^2)(x) - \frac{1}{2i}\left[\frac{1}{D-i}(\cot x) - \frac{1}{D+i}(\cot x)\right]$$

$$= x - \frac{1}{2i}(P_1 - P_2) \qquad \ldots(1)$$

Now,

$$P_1 = \frac{1}{D-i} \cot x = e^{ix} \int e^{-ix} \cot x \, dx$$

$$= e^{ix} \int (\cos x - i \sin x) \cot x \, dx = e^{ix} \int \left(\frac{\cos^2 x}{\sin x} - i \cos x\right) dx$$

$$= e^{ix} \int (\operatorname{cosec} x - \sin x - i \cos x) \, dx$$

$$= e^{ix} [\log(\operatorname{cosec} x - \cot x) + \cos x - i \sin x]$$

$$= e^{ix} [\log(\operatorname{cosec} x - \cot x) + e^{-ix}]$$

$$P_2 = \frac{1}{D+i} \cot x = e^{-ix} [\log(\operatorname{cosec} x - \cot x) + e^{ix}]$$

$\therefore$ From (1),

$$\text{P.I.} = x - \frac{1}{2i} [e^{ix} \{\log(\operatorname{cosec} x - \cot x) + e^{-ix}\} - e^{-ix} \{\log(\operatorname{cosec} x - \cot x) + e^{ix}\}]$$

$$= x - \frac{1}{2i} [(e^{ix} - e^{-ix}) \log(\operatorname{cosec} x - \cot x)]$$

$$= x - \sin x \log(\operatorname{cosec} x - \cot x)$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 \cos x + c_2 \sin x + x - \sin x \log(\operatorname{cosec} x - \cot x)$$

where c_1 and c_2 are arbitrary constants of integration.

TEST YOUR KNOWLEDGE

Solve the following differential equations:

1. $(D^2 + 4)y = \tan 2x$
2. $(D^2 + 1)y = \sec x$
3. $(D^2 + a^2)y = \operatorname{cosec} ax$
4. $(D^2 + 2D + 2)y = e^{-x} \sec^3 x$ (A.K.T.U. 2017)
5. $(D^2 - 2D + 2)y = e^x \tan x$.

Answers

1. $y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4} \cos 2x \log (\sec 2x + \tan 2x)$
2. $y = c_1 \cos x + c_2 \sin x + x \sin x + \cos x \log \cos x$
3. $y = c_1 \cos ax + c_2 \sin ax + \frac{1}{a^2} \sin ax \log \sin ax - \frac{x}{a} \cos ax$
4. $y = e^{-x}\left(c_1 \cos x + c_2 \sin x + \frac{\sin x \tan x}{2}\right)$
5. $y = e^x (c_1 \cos x + c_2 \sin x) - e^x \cos x \log (\sec x + \tan x)$.

1.19 HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS (EULER-CAUCHY EQUATIONS)

An equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots + a_{n-1} x \frac{dy}{dx} + a_n y = Q \qquad \ldots(1)$$

where a_i's are constants and Q is a function of x, is called Cauchy's homogeneous linear equation.

Such equations can be reduced to linear differential equations with constant coefficients by the substitution

$$x = e^z \quad \text{or} \quad z = \log x$$

so that $$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz} \cdot \frac{1}{x} \quad \text{or} \quad x \frac{dy}{dx} = \frac{dy}{dz} = \text{D}y, \text{ where D} \equiv \frac{d}{dz}$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x} \cdot \frac{d^2 y}{dz^2} \cdot \frac{dz}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2 y}{dz^2} \qquad \left(\because \frac{dz}{dx} = \frac{1}{x}\right)$$

or $$x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dz^2} - \frac{dy}{dz} = \text{D}^2 y - \text{D}y = \text{D}(\text{D}-1)y$$

Similarly, $$x^3 \frac{d^3 y}{dx^3} = \text{D}(\text{D}-1)(\text{D}-2)y \text{ and so on.}$$

Substituting these values is equation (1), we get a linear differential equation with constant coefficients, which can be solved by the methods already discussed.

1.19.1 Steps for Solution

1. Put $x = e^z$ so that $z = \log x$ and Let $\text{D} \equiv \frac{d}{dz}$

2. Replace $x \frac{d}{dx}$ by D,

$x^2 \frac{d^2}{dx^2}$ by D(D – 1)

$x^3 \frac{d^3}{dx^3}$ by D(D – 1) (D – 2) and so on.

3. By doing so, this type of equation reduces to linear differential equation with constant coefficients which is then solved as before.

1.20 LEGENDRE'S LINEAR DIFFERENTIAL EQUATION

An equation of the form

$$(a+bx)^n \frac{d^n y}{dx^n} + a_1(a+bx)^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots + a_{n-1}(a+bx)\frac{dy}{dx} + a_n y = Q \quad \ldots(1)$$

where a_i's are constants and Q is a function of x, is called Legendre's linear differential equation.

Such equations can be reduced to linear differential equations with constant co-efficients by the substitution

$$a + bx = e^z \text{ i.e. } z = \log(a+bx) \text{ so that } \frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{b}{a+bx}\cdot\frac{dy}{dz}$$

or $$(a+bx)\frac{dy}{dx} = b\frac{dy}{dz} = b\,\text{D}y, \text{ where D} \equiv \frac{d}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{b}{a+bx}\frac{dy}{dz}\right) = -\frac{b^2}{(a+bx)^2}\cdot\frac{dy}{dz} + \frac{b}{a+bx}\frac{d^2y}{dz^2}\cdot\frac{dy}{dx}$$

$$= -\frac{b^2}{(a+bx)^2}\frac{dy}{dz} + \frac{b}{a+bx}\frac{d^2y}{dz^2}\cdot\frac{b}{a+bx} = \frac{b^2}{(a+bx)^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

or $$(a+bx)^2\frac{d^2y}{dx^2} = b^2(\text{D}^2y - \text{D}y) = b^2\,\text{D}(\text{D}-1)y$$

Similarly, $(a+bx)^3 \dfrac{d^3y}{dx^3} = b^3\,\text{D}(\text{D}-1)(\text{D}-2)y.$

Substituting these values in equation (i), we get a linear differential equation with constant coefficients, which can be solved by the methods already discussed.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve:* $x^3\dfrac{d^3y}{dx^3} + 2x^2\dfrac{d^2y}{dx^2} + 2y = 10\left(x + \dfrac{1}{x}\right)$.

Sol. Put $x = e^z$ so that $z = \log x$ and let $\text{D} \equiv \dfrac{d}{dz}$ then the given differential equation reduces to

$$[\text{D}(\text{D}-1)(\text{D}-2) + 2\text{D}(\text{D}-1) + 2]y = 10(e^z + e^{-z})$$

or $$(\text{D}^3 - \text{D}^2 + 2)y = 10\,(e^z + e^{-z})$$

which is a linear equation with constant coefficients.

Its Auxiliary equation is

$$m^3 - m^2 + 2 = 0 \quad \text{or} \quad (m+1)(m^2 - 2m + 2) = 0$$

$\therefore \quad m = -1, \dfrac{2 \pm \sqrt{4-8}}{2} = -1, 1 \pm i$

$\therefore \quad \text{C.F.} = c_1 e^{-z} + e^z (c_1 \cos z + c_3 \sin z) = \dfrac{c_1}{x} + x\,[c_2 \cos(\log x) + c_3 \sin(\log x)]$

$$\text{P.I.} = 10 \frac{1}{D^3 - D^2 + 2}(e^z + e^{-z}) = 10\left(\frac{1}{D^3 - D^2 + 2} e^z + \frac{1}{D^3 - D^2 + 2} e^{-z}\right)$$

$$= 10\left(\frac{1}{1^3 - 1^2 + 2} e^z + z \cdot \frac{1}{3D^2 - 2D} e^{-z}\right) = 10\left(\frac{1}{2} e^z + z \cdot \frac{1}{3(-1)^2 - 2(-1)} e^{-z}\right)$$

$$= 5e^z + 2ze^{-z} = 5x + \frac{2}{x} \log x$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = \frac{c_1}{x} + x\,[c_2 \cos(\log x) + c_3 \sin(\log x)] + 5x + \frac{2}{x} \log x$$

where c_1, c_2 and c_3 are arbitrary constants of integration.

Example 2. *Solve:* $x^3 \dfrac{d^3 y}{dx^3} + 3x^2 \dfrac{d^2 y}{dx^2} + x \dfrac{dy}{dx} + y = x + \log x.$

Sol. Put $x = e^z$ so that $z = \log x$ and let $D \equiv \dfrac{d}{dz}$ then the given differential equation reduces to

$$[D(D-1)(D-2) + 3D(D-1) + D + 1]\, y = e^z + z$$

$$\Rightarrow \quad (D^3 + 1)y = e^z + z$$

Auxiliary equation is

$$m^3 + 1 = 0$$

$$\Rightarrow \quad (m+1)(m^2 - m + 1) = 0 \quad \Rightarrow \quad m = -1, \frac{1 \pm \sqrt{3}i}{2}$$

$$\therefore \quad \text{C.F.} = c_1 e^{-z} + e^{z/2}\left(c_2 \cos \frac{\sqrt{3}}{2} z + c_3 \sin \frac{\sqrt{3}}{2} z\right)$$

$$\text{P.I.} = \frac{1}{D^3 + 1}(e^z + z) = \frac{1}{D^3 + 1}(e^z) + \frac{1}{1 + D^3}(z)$$

$$= \frac{e^z}{2} + (1 + D^3)^{-1}(z) = \frac{e^z}{2} + (1 - D^3)(z) \qquad | \text{ Leaving higher terms}$$

$$= \frac{e^z}{2} + z$$

$\therefore$ The complete solution is

$$y = c_1 e^{-z} + e^{z/2}\left(c_2 \cos \frac{\sqrt{3}}{2} z + c_3 \sin \frac{\sqrt{3}}{2} z\right) + \frac{e^z}{2} + z$$

$$\therefore \quad y = \frac{c_1}{x} + \sqrt{x}\left[c_2 \cos \frac{\sqrt{3}}{2}(\log x) + c_3 \sin \frac{\sqrt{3}}{2}(\log x)\right] + \frac{x}{2} + \log x$$

where c_1, c_2 and c_3 are the arbitrary constants of integration.

Example 3. *Solve:* $x^2 \dfrac{d^2y}{dx^2} + 4x \dfrac{dy}{dx} + 2y = e^x$. (M.T.U. 2013)

Sol. Put $x = e^z$ so that $z = \log x$ and let $D \equiv \dfrac{d}{dz}$ then the given differential equation reduces to

$$\{D(D-1) + 4D + 2\}y = e^{e^z}$$

$$(D^2 + 3D + 2)y = e^{e^z}$$

Auxiliary equation is

$$m^2 + 3m + 2 = 0$$

$$\Rightarrow \quad (m+1)(m+2) = 0 \quad \Rightarrow \quad m = -1, -2$$

$$\therefore \quad \text{C.F.} = c_1 e^{-z} + c_2 e^{-2z}$$

$$\text{P.I.} = \frac{1}{D^2 + 3D + 2}(e^{e^z}) = \left(\frac{1}{D+1} - \frac{1}{D+2}\right) e^{e^z}$$

$$= \frac{1}{D+1}(e^{e^z}) - \frac{1}{D+2} e^{e^z} = e^{-z} \int e^z \,.\, e^{e^z}\, dz - e^{-2z} \int e^{2z} e^{e^z}\, dz$$

$$= e^{-z} e^{e^z} - e^{-2z}(e^z - 1) e^{e^z} = e^{-2z} e^{e^z}$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{-z} + c_2 e^{-2z} + e^{-2z} e^{e^z} = \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{1}{x^2} e^x$$

where c_1 and c_2 are arbitrary constants of integration.

Example 4. *By reducing to homogeneous, solve the differential equation*

$$(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos\{\log(1+x)\}.$$

Sol. Put $1 + x = e^z$ so that $z = \log(1+x)$ and let $D \equiv \dfrac{d}{dz}$ then the given differential equation reduces to

$$\{D(D-1) + D + 1\}y = 4\cos z$$

$$\Rightarrow \quad (D^2 + 1)y = 4\cos z$$

Auxiliary equation is

$$m^2 + 1 = 0 \quad \Rightarrow \quad m = \pm i$$

$$\therefore \quad \text{C.F.} = c_1 \cos z + c_2 \sin z$$

$$\text{P.I.} = \frac{1}{D^2 + 1}(4\cos z) = 4z \,.\, \frac{1}{2D} \cos z = 2z \sin z$$

Hence the complete solution is

$$y = c_1 \cos z + c_2 \sin z + 2z \sin z$$

$$= c_1 \cos\{\log(1+x)\} + c_2 \sin\{\log(1+x)\} + 2\log(1+x)\sin\{\log(1+x)\}$$

where c_1 and c_2 are arbitrary constants of integration.

Example 5. *Solve:* $(x+1)^2 \dfrac{d^2y}{dx^2} + (x+1)\dfrac{dy}{dx} = (2x+3)(2x+4).$ [A.K.T.U. 2011]

Sol. Put $x + 1 = e^z$ so that $z = \log(x+1)$ and let $D \equiv \dfrac{d}{dz}$ then the given differential equation reduces to

$$[D(D-1) + D]y = (2e^z + 1)(2e^z + 2)$$

$$D^2y = 4e^{2z} + 6e^z + 2$$

Auxiliary equation is

$$m^2 = 0 \Rightarrow m = 0, 0$$

$$\therefore \quad \text{C.F.} = c_1 + c_2 z$$

$$\text{P.I.} = \frac{1}{D^2}(4e^{2z} + 6e^z + 2) = e^{2z} + 6e^z + z^2$$

Hence complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 + c_2 z + e^{2z} + 6e^z + z^2$$

$$= c_1 + c_2 \log(x+1) + (x+1)^2 + 6(x+1) + [\log(x+1)]^2$$

where c_1 and c_2 are arbitrary constants of integration.

TEST YOUR KNOWLEDGE

Solve:

1. $\dfrac{d^3y}{dx^3} - \dfrac{4}{x}\dfrac{d^2y}{dx^2} + \dfrac{5}{x^2}\dfrac{dy}{dx} - \dfrac{2y}{x^3} = 1$

2. $x^2 \dfrac{d^2y}{dx^2} - 2y = x^2 + \dfrac{1}{x}$

3. $x^2 \dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} - 20y = (x+1)^2$

4. $x^2 \dfrac{d^3y}{dx^3} - 4x\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} = 4$

5. (*i*) $x^4 \dfrac{d^3y}{dx^3} + 2x^3 \dfrac{d^2y}{dx^2} - x^2\dfrac{dy}{dx} + xy = 1$ (*ii*) $x^2 \dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + 6y = x^2$

6. (*i*) $x^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = 2\log x$ (*ii*) $x^2 \dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + 6y = x$

7. (*i*) $x^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - 3y = x^2 \log x$ (*ii*) $x^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + 2y = x\log x$

8. (*i*) $x^2y'' + xy' - y = x^3 e^x$. (*A.K.T.U. 2016*) (*ii*) $\left(\dfrac{d}{dx} + \dfrac{1}{x}\right)^2 y = x^{-4}$. (*M.T.U. 2012*)

9. $x^2 \dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} - 12y = x^3 \log x$

10. $(x^2D^2 - xD + 4)y = \cos(\log x) + x\sin(\log x)$

11. (*i*) $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 5y = \sin(\log x)$ (*ii*) $x^2 \dfrac{d^2y}{dx^2} + 5x\dfrac{dy}{dx} + 4y = x\log x$

12. (i) $x^2 \dfrac{d^2y}{dx^2} - 2x \dfrac{dy}{dx} + 2y = x^{-1}$ (ii) $x^3 y''' + xy' - y = 3x^4$

13. (i) $x^3 \dfrac{d^3y}{dx^3} + 2x^2 \dfrac{d^2y}{dx^2} + 3x \dfrac{dy}{dx} - 3y = x + x^2$ (ii) $x^2 \dfrac{d^2y}{dx^2} + x \dfrac{dy}{dx} - y = x^m$

14. (i) $x^4 \dfrac{d^4y}{dx^4} + 6x^3 \dfrac{d^3y}{dx^3} + 9x^2 \dfrac{d^2y}{dx^2} + 3x \dfrac{dy}{dx} + y = (1 + \log x)^2$

(ii) $[x^2D^2 - (2m - 1)xD + (m^2 + n^2)]y = n^2x^m \log x$

15. $\dfrac{d^2y}{dx^2} + \dfrac{1}{x} \cdot \dfrac{dy}{dx} = \dfrac{12 \log x}{x^2}$

16. $(x + a)^2 \dfrac{d^2y}{dx^2} - 4(x + a) \dfrac{dy}{dx} + 6y = x$

17. $(1 + 2x)^2 \dfrac{d^2y}{dx^2} - 6(1 + 2x) \dfrac{dy}{dx} + 16y = 8(1 + 2x)^2$

18. $(2x + 3)^2 \dfrac{d^2y}{dx^2} - 2(2x + 3) \dfrac{dy}{dx} - 12y = 6x$

19. $x^2 \dfrac{d^2y}{dx^2} + 3x \dfrac{dy}{dx} + y = \dfrac{1}{(1-x)^2}$

20. (i) $x^2 \dfrac{d^2y}{dx^2} - 3x \dfrac{dy}{dx} + y = \dfrac{(\log x) \sin (\log x) + 1}{x}$

(ii) $x^2 \dfrac{d^2y}{dx^2} + x \dfrac{dy}{dx} + y = (\log x) \sin (\log x)$

(U.P.T.U. 2015)

Answers

1. $y = c_1x^2 + x^{5/2} (c_2x^{\sqrt{21}/2} + c_3x^{-\sqrt{21}/2}) - \dfrac{1}{5} x^3$

2. $y = c_1x^2 + \dfrac{c_2}{x} + \dfrac{1}{3}\left(x^2 + \dfrac{1}{x}\right) \log x$

3. $y = c_1x^{-5} + c_2x^{-4} - \dfrac{x^2}{14} - \dfrac{x}{9} - \dfrac{1}{20}$

4. $y = c_1 + c_2x^3 + c_3x^4 + \dfrac{2}{3} x$

5. (i) $y = (c_1 + c_2 \log x)x + c_2x^{-1} + \dfrac{1}{4x} \log x$ (ii) $y = c_1x^2 + c_2x^3 - x^2 \log x$

6. (i) $y = x (c_1 + c_2 \log x) + 2 \log x + 4$ (ii) $y = c_1x^2 + c_2x^3 + \dfrac{x}{2}$

7. (i) $y = c_1x^3 + \dfrac{c_2}{x} - \dfrac{x^2}{3}\left(\log x + \dfrac{2}{3}\right)$ (ii) $y = x [c_1 \cos (\log x) + c_2 \sin (\log x)] + x \log x$

8. (i) $y = c_1x + c_2 \cdot \dfrac{1}{x} + \left(x - 3 + \dfrac{3}{x}\right) e^x$ (ii) $y = x^{-1/2}\left[c_1 \cos \dfrac{\sqrt{3}}{2} (\log x) + c_2 \sin \dfrac{\sqrt{3}}{2} (\log x)\right] + \dfrac{1}{3} x^{-2}$

9. $y = c_1x^3 + c_2x^{-4} + \dfrac{x^3}{98} \log x (7 \log x - 2)$

10. $y = x [c_1 \cos (\sqrt{3} \log x) + c_2 \sin (\sqrt{3} \log x)] + \dfrac{3}{13} \cos (\log x) - \dfrac{2}{13} \sin (\log x) + \dfrac{1}{2} x \sin (\log x)$

11. (i) $y = x^2 [c_1 \cos (\log x) + c_2 \sin (\log x)] + \dfrac{1}{8} [\sin (\log x) + \cos (\log x)]$

(ii) $y = \left(\dfrac{c_1 + c_2 \log x}{x^2}\right) + \dfrac{x}{9}\left(\log x - \dfrac{2}{3}\right)$

12. (i) $y = c_1x + c_2x^2 + \frac{1}{6x}$ (ii) $y = x[c_1 + c_2 \log x + c_3 (\log x)^2] + \frac{1}{9} x^4$

13. (i) $y = c_1x + c_2 \cos(\sqrt{3} \log x) + c_3 \sin(\sqrt{3} \log x) + \frac{1}{7} x^2 + \frac{1}{4} x \log x$

(ii) $y = c_1x + \frac{c_2}{x} + \frac{x^m}{m^2 - 1}$

14. (i) $y = (c_1 + c_2 \log x) \cos(\log x) + (c_3 + c_4 \log x) \sin(\log x) + (\log x)^2 + 2 \log x - 3$

(ii) $y = x^m [c_1 \cos(n \log x) + c_2 \sin(n \log x)] + x^m \log x$

15. $y = c_1 \log x + c_2 + 2 (\log x)^3$

16. $y = c_1 (x + a)^2 + c_2 (x + a)^3 + \frac{1}{2} (x + a) - \frac{1}{6} a$

17. $y = (1 + 2x)^2 [c_1 + c_2 \log(1 + 2x) + \{\log(1 + 2x)\}^2]$

18. $y = c_1(2x + 3)^{-1} + c_2 (2x + 3)^3 - \frac{3}{4} (2x + 3) + 3$

19. $y = \frac{1}{x}(c_1 + c_2 \log x) + \frac{1}{x} \log\left(\frac{x}{1-x}\right)$

20. (i) $y = x^2 [c_1 \cosh(\sqrt{3} \log x) + c_2 \sinh(\sqrt{3} \log x)] + \frac{1}{61x} [\log x \{5 \sin(\log x) + 6 \cos(\log x)\}$

$+ \frac{2}{61} \{27 \sin(\log x) + 191 \cos(\log x)\}] + \frac{1}{6x}$.

(ii) $y = c_1 \cos(\log x) + c_2 \sin(\log x) + \frac{\log x}{4} [\sin(\log x) - (\log x) \cos(\log x)]$.

1.21 SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS

Now we discuss differential equations in which there is one independent variable and two or more than two dependent variables. Such equations are called *simultaneous linear equations*. To solve such equations completely, we must have as many simultaneous equations as the number of dependent variables.

Let x, y be the two dependent variables and t be the independent variable. Consider the simultaneous equations

$$f_1(D)x + f_2(D)y = T_1 \qquad ...(1)$$

and

$$\phi_1(D)x + \phi_2(D)y = T_2 \qquad ...(2)$$

where $D \equiv \frac{d}{dt}$ and T_1, T_2 are functions of t.

To eliminate y, operating on both sides of (1) by $\phi_2(D)$ and on both sides of (2) by $f_2(D)$ and subtracting,

we get $$[f_1(D)\,\phi_2(D) - \phi_1(D)\,f_2(D)]x = \phi_2(D)T_1 - f_2(D)T_2 \quad \text{or} \quad f(D)x = T$$

which is a linear equation in x and t and can be solved by the methods already discussed. Substituting the value of x in either (1) or (2), we get the value of y.

Note. We can also eliminate x to get a linear equation in y and t.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the following simultaneous differential equations* [M.T.U. 2011]

$$\frac{dx}{dt} = 3x + 2y, \quad \frac{dy}{dt} = 5x + 3y.$$

Sol. Let $\frac{d}{dt} \equiv D$ then the given system of equations become

$$(D - 3)x - 2y = 0 \quad ...(1)$$

$$-5x + (D - 3)y = 0 \quad ...(2)$$

Operating eqn. (1) by (D – 3) and multiplying eqn. (2) by 2 then adding, we get

$$[(D - 3)^2 - 10]\, x = 0$$

$$(D^2 - 6D - 1)\, x = 0$$

Auxiliary equation is

$$m^2 - 6m - 1 = 0$$

$$\Rightarrow \quad m = \frac{6 \pm \sqrt{36 + 4}}{2} = 3 \pm \sqrt{10}$$

$$\text{C.F.} = e^{3t}(c_1 \cosh \sqrt{10}\, t + c_2 \sinh \sqrt{10}\, t)$$

$$\text{P.I.} = 0$$

$$\therefore \quad x = e^{3t}\,(c_1 \cosh \sqrt{10}\, t + c_2 \sinh \sqrt{10}\, t) \quad ...(3)$$

From (1),

$$2y = \frac{dx}{dt} - 3x$$

$$= e^{3t}\sqrt{10}\,(c_1 \sinh \sqrt{10}\, t + c_2 \cosh \sqrt{10}\, t)$$

$$+ 3e^{3t}\,(c_1 \cosh \sqrt{10}\, t + c_2 \sinh \sqrt{10}\, t)$$

$$- 3e^{3t}\,(c_1 \cosh \sqrt{10}\, t + c_2 \sinh \sqrt{10}\, t)$$

$$\Rightarrow \quad y = \frac{\sqrt{10}}{2} e^{3t}\,(c_1 \sinh \sqrt{10}\, t + c_2 \cosh \sqrt{10}\, t) \quad ...(4)$$

Equations (3) and (4), when taken together, give the complete solution. c_1 and c_2 are arbitrary constants of integration.

Example 2. *Solve the following simultaneous differential equations*

$$\frac{dx}{dt} = -wy, \quad \frac{dy}{dt} = wx.$$

Also show that the point (x, y) lies on a circle.

Sol. Let $\frac{d}{dt} \equiv D$ then the given system of equations becomes

$$Dx + wy = 0 \quad ...(1)$$

$$-wx + Dy = 0 \quad ...(2)$$

Operating eqn. (1) by D and multiplying eqn. (2) by w then subtracting (2) from (1), we get

$$(D^2 + w^2)\, x = 0$$

Auxiliary equation is

$$m^2 + w^2 = 0$$

$$\Rightarrow \qquad m = \pm\, wi$$

$$\therefore \qquad \text{C.F.} = c_1 \cos wt + c_2 \sin wt$$

$$\text{P.I.} = 0$$

$$\therefore \qquad x = \text{C.F.} + \text{P.I.} = c_1 \cos wt + c_2 \sin wt \qquad ...(3)$$

From (3), $$\frac{dx}{dt} = -\, wc_1 \sin wt + wc_2 \cos wt$$

From (1), $$wy = -\, Dx = w\,(c_1 \sin wt - c_2 \cos wt)$$

$$\therefore \qquad y = c_1 \sin wt - c_2 \cos wt \qquad ...(4)$$

Eliminating w between eqns. (3) and (4), we get

$$x^2 + y^2 = c_1^2 + c_2^2 \qquad ...(5)$$

which is a circle with centre (0, 0) and radius $\sqrt{c_1^2 + c_2^2}$.

Eqn. (5) shows that the point (x, y) lies on a circle.

Example 3. *Solve the following simultaneous differential equations:*

$$\frac{dx}{dt} + 5x - 2y = t, \quad \frac{dy}{dt} + 2x + y = 0$$

given that $x = y = 0$ when $t = 0$. (U.P.T.U. 2015)

Sol. Let $\frac{d}{dt} \equiv D$ then the given system of equations becomes

$$(D + 5)\, x - 2y = t \qquad ...(1)$$

$$2x + (D + 1)\, y = 0 \qquad ...(2)$$

Operating (1) by (D + 1), we get

$$(D^2 + 6D + 5)\, x - 2\,(D + 1)\, y = 1 + t \qquad ...(3)$$

Multiplying (2) by 2, we get

$$4x + 2\,(D + 1)\, y = 0 \qquad ...(4)$$

Adding (3) and (4), we get

$$(D^2 + 6D + 9)\, x = 1 + t$$

Auxiliary equation is

$$m^2 + 6m + 9 = 0 \quad \Rightarrow \quad m = -\,3, -\,3$$

$$\therefore \qquad \text{C.F.} = (c_1 + c_2 t)\, e^{-3t}$$

$$\text{P.I.} = \frac{1}{(D+3)^2}(1+t) = \frac{1}{9}\left(1+\frac{D}{3}\right)^{-2}(1+t)$$

$$= \frac{1}{9}\left(1-\frac{2D}{3}\right)(1+t) = \frac{1}{9}\left(1+t-\frac{2}{3}\right) = \frac{1}{9}\left(t+\frac{1}{3}\right)$$

$$\therefore \qquad x = \text{C.F.} + \text{P.I.} = (c_1 + c_2 t)\, e^{-3t} + \frac{1}{9}\left(t + \frac{1}{3}\right) \qquad \text{...(5)}$$

From (5), $$\frac{dx}{dt} = -3\,(c_1 + c_2 t)\, e^{-3t} + c_2 e^{-3t} + \frac{1}{9}$$

From (1), $$2y = \frac{dx}{dt} + 5x - t$$

$$= -3\,(c_1 + c_2 t)\, e^{-3t} + c_2 e^{-3t} + \frac{1}{9} + 5\,(c_1 + c_2 t)\, e^{-3t} + \frac{5}{9}\left(t + \frac{1}{3}\right) - t$$

$$= 2\,(c_1 + c_2 t)\, e^{-3t} + c_2 e^{-3t} - \frac{4}{9}t + \frac{8}{27}$$

$$y = (c_1 + c_2 t)\, e^{-3t} + \frac{c_2}{2} e^{-3t} - \frac{2}{9}t + \frac{4}{27} \qquad \text{...(6)}$$

Eqns. (5) and (6), when taken together, give the general solution.

Applying condition $x(0) = 0$ in (5), we get

$$0 = c_1 + \frac{1}{27} \quad \Rightarrow \quad c_1 = -\frac{1}{27}$$

Applying condition $y(0) = 0$ in (6), we get

$$0 = c_1 + \frac{c_2}{2} + \frac{4}{27} = \frac{c_2}{2} + \frac{1}{9}$$

$$\Rightarrow \qquad c_2 = -\frac{2}{9}$$

Hence the required particular solution is given by

$$x = -\frac{1}{27}(1 + 6t)\, e^{-3t} + \frac{1}{9}\left(t + \frac{1}{3}\right)$$

and $$y = -\frac{2}{27}(2 + 3t)\, e^{-3t} - \frac{2}{9}t + \frac{4}{27}$$

Example 4. *Solve:* $\dfrac{d^2x}{dt^2} + y = \sin t,\ \dfrac{d^2y}{dt^2} + x = \cos t.$ *(A.K.T.U. 2016)*

Sol. Let $\dfrac{d}{dt} \equiv \text{D}$ then the given system of equations become

$$\text{D}^2 x + y = \sin t \qquad \text{...(1)}$$

$$x + \text{D}^2 y = \cos t \qquad \text{...(2)}$$

Operating eqn. (1) by D^2, we get

$$\text{D}^4 x + \text{D}^2 y = -\sin t \qquad \text{...(3)}$$

Subtracting (2) from (3), we get

$$(\text{D}^4 - 1)\, x = -\sin t - \cos t$$

Auxiliary equation is

$$m^4 - 1 = 0$$

$$(m^2 - 1)(m^2 + 1) = 0$$

$$m = 1, -1, \pm i$$

$\Rightarrow$ $$\text{C.F.} = c_1e^t + c_2e^{-t} + c_3 \cos t + c_4 \sin t$$

$\therefore$ $$\text{P.I.} = \frac{1}{D^4 - 1}(-\sin t - \cos t)$$

$$= -t.\frac{1}{4D^3}(\sin t + \cos t) = \frac{t}{4}(-\cos t + \sin t)$$

$\therefore$ $$x = c_1e^t + c_2e^{-t} + c_3 \cos t + c_4 \sin t + \frac{t}{4}(\sin t - \cos t) \quad ...(4)$$

$$Dx = c_1e^t - c_2e^{-t} - c_3 \sin t + c_4 \cos t + \frac{t}{4}(\cos t + \sin t) + \frac{1}{4}(\sin t - \cos t)$$

$$D^2x = c_1e^t + c_2e^{-t} - c_3 \cos t - c_4 \sin t + \frac{t}{4}(-\sin t + \cos t) + \frac{1}{4}(\cos t + \sin t) + \frac{1}{4}(\cos t + \sin t)$$

From (1), $$y = \sin t - \frac{d^2x}{dt^2}$$

$$y = -c_1e^t - c_2e^{-t} + c_3 \cos t + c_4 \sin t + \frac{t}{4}(\sin t - \cos t) + \frac{1}{2}(\sin t - \cos t) \quad ...(5)$$

Equations (4) and (5), when taken together, give the complete solution of the given system of equations.

Example 5. *Solve:* $\frac{dx}{dt} + 4x + 3y = t,\ \frac{dy}{dt} + 2x + 5y = e^t.$ (U.P.T.U. 2015)

Sol. Writing D for $\frac{d}{dt}$, the given equations become

$$(D + 4)x + 3y = t \quad ...(1)$$

and $$2x + (D + 5)y = e^t \quad ...(2)$$

To eliminate y, operating (1) by $(D + 5)$ and multiplying (2) by 3 then subtracting, we get

$$[(D + 4)(D + 5) - 6]x = (D + 5)t - 3e^t$$

or $$(D^2 + 9D + 14)x = 1 + 5t - 3e^t$$

Auxiliary equation is

$$m^2 + 9m + 14 = 0 \Rightarrow m = -2, -7$$

$\therefore$ $$\text{C.F.} = c_1e^{-2t} + c_2e^{-7t}$$

$$\text{P.I.} = \frac{1}{D^2 + 9D + 14}(1 + 5t - 3e^t)$$

$$= \frac{1}{D^2 + 9D + 14}e^{0t} + 5\frac{1}{D^2 + 9D + 14}t - 3\frac{1}{D^2 + 9D + 14}e^t$$

$$= \frac{1}{0^2 + 9(0) + 14}e^{0t} + 5.\frac{1}{14\left(1 + \frac{9D}{14} + \frac{D^2}{14}\right)}t - 3\frac{1}{1^2 + 9(1) + 14}e^t$$

$$= \frac{1}{14} + \frac{5}{14}\left[1 + \left(\frac{9D}{14} + \frac{D^2}{14}\right)\right]^{-1}t - \frac{1}{8}e^t = \frac{1}{14} + \frac{5}{14}\left[1 - \left(\frac{9D}{14} + \frac{D^2}{14}\right) + \ldots\ldots\right]t - \frac{1}{8}e^t$$

$$= \frac{1}{14} + \frac{5}{14}\left(t - \frac{9}{14}\right) - \frac{1}{8}e^t = \frac{5}{14}t - \frac{31}{196} - \frac{1}{8}e^t$$

$$\therefore \quad x = c_1e^{-2t} + c_2e^{-7t} + \frac{5}{14}t - \frac{31}{196} - \frac{1}{8}e^t$$

Now, $\frac{dx}{dt} = -2c_1e^{-2t} - 7c_2e^{-7t} + \frac{5}{14} - \frac{1}{8}e^t$

Substituting the values of x and $\frac{dx}{dt}$ in (1), we have

$$3y = t - \frac{dx}{dt} - 4x$$

$$\Rightarrow \quad 3y = t + 2c_1e^{-2t} + 7c_2e^{-7t} - \frac{5}{14} + \frac{1}{8}e^t - 4c_1e^{-2t} - 4c_2e^{-7t} - \frac{10}{7}t + \frac{31}{49} + \frac{1}{2}e^t$$

$$\therefore \quad y = \frac{1}{3}\left[-2c_1e^{-2t} + 3c_2e^{-7t} - \frac{3}{7}t + \frac{27}{98} + \frac{5}{8}e^t\right]$$

Hence,
$$x = c_1e^{-2t} + c_2e^{-7t} + \frac{5}{14}t - \frac{31}{196} - \frac{1}{8}e^t$$
$$y = -\frac{2}{3}c_1e^{-2t} + c_2e^{-7t} - \frac{1}{7}t + \frac{9}{98} + \frac{5}{24}e^t.$$

Example 6. *Solve the simultaneous differential equations*

$$\frac{dx}{dt} + \frac{dy}{dt} + 3x = \sin t \quad \text{and} \quad \frac{dx}{dt} + y - x = \cos t.$$

Sol. Let $\frac{d}{dt} \equiv D$ then the given system of equations reduces to

$$(D + 3)x + Dy = \sin t \quad \text{...(1)}$$
$$(D - 1)x + y = \cos t \quad \text{...(2)}$$

Operating (2) by D and then subtracting from (1), we get

$$[(D + 3) - D(D - 1)]x = \sin t + \sin t = 2\sin t$$

$$\Rightarrow \quad (D^2 - 2D - 3)x = -2\sin t$$

Auxiliary equation is

$$m^2 - 2m - 3 = 0 \Rightarrow m = -1, 3$$

$$\therefore \quad \text{C.F.} = c_1e^{-t} + c_2e^{3t}$$

$$\text{P.I.} = \frac{1}{D^2 - 2D - 3}(-2\sin t) = \frac{-2}{-2D - 4}\sin t$$

$$= \frac{1}{D+2}\sin t = \frac{D-2}{D^2 - 4}\sin t = -\frac{1}{5}(\cos t - 2\sin t)$$

$$\therefore \quad x = c_1e^{-t} + c_2e^{3t} - \frac{1}{5}(\cos t - 2\sin t) \quad \text{...(3)}$$

But $$y = x + \cos t - \frac{dx}{dt} = c_1e^{-t} + c_2e^{3t} - \frac{1}{5}(\cos t - 2\sin t)$$

$$+ \cos t - \left[-c_1e^{-t} + 3c_2e^{3t} + \frac{1}{5}\sin t + \frac{2}{5}\cos t\right]$$

$$= 2c_1e^{-t} - 2c_2e^{3t} + \frac{2}{5}\cos t + \frac{1}{5}\sin t \qquad ...(4)$$

Eqns. (3) and (4) when taken together give the complete solution.

Example 7. *Solve the simultaneous differential equations*

$$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = y \quad \textit{and} \quad \frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 25x + 16e^t. \qquad \text{(A.K.T.U. 2018)}$$

Sol. Let $\frac{d}{dt} \equiv \text{D}$ then the given system of equations is

$$(\text{D}^2 - 4\text{D} + 4)x - y = 0 \qquad ...(1)$$

$$-25x + (\text{D}^2 + 4\text{D} + 4)y = 16e^t \qquad ...(2)$$

Operating (1) by $\text{D}^2 + 4\text{D} + 4$ and adding to (2), we get

$$(\text{D}^2 - 4\text{D} + 4)(\text{D}^2 + 4\text{D} + 4)x - 25x = 16e^t$$

$$\Rightarrow \qquad (\text{D}^4 - 8\text{D}^2 - 9)x = 16e^t$$

Auxiliary equation is

$$m^4 - 8m^2 - 9 = 0$$

$$\Rightarrow \qquad (m^2 - 9)(m^2 + 1) = 0 \quad \Rightarrow \quad m = \pm i, \pm 3$$

$$\therefore \qquad \text{C.F.} = c_1e^{3t} + c_2e^{-3t} + c_3\cos t + c_4\sin t$$

$$\text{P.I.} = \frac{1}{\text{D}^4 - 8\text{D}^2 + 9}(16e^t) = 8e^t$$

$$\therefore \qquad x = c_1e^{3t} + c_2e^{-3t} + c_3\cos t + c_4\sin t + 8e^t \qquad ...(3)$$

$$\frac{dx}{dt} = 3c_1e^{3t} - 3c_2e^{-3t} + c_3(-\sin t) + c_4\cos t + 8e^t$$

$$\frac{d^2x}{dt^2} = 9c_1e^{3t} + 9c_2e^{-3t} - c_3\cos t - c_4\sin t + 8e^t$$

From (1), $$y = \frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x$$

$$= 9c_1e^{3t} + 9c_2e^{-3t} - c_3\cos t - c_4\sin t + 8e^t$$

$$- 4(3c_1e^{3t} - 3c_2e^{-3t} - c_3\sin t + c_4\cos t + 8e^t)$$

$$+ 4(c_1e^{3t} + c_2e^{-3t} - c_3\cos t + c_4\sin t + 8e^t)$$

$$\Rightarrow \qquad y = c_1e^{3t} + 25c_2e^{-3t} + (3c_3 - 4c_4)\cos t + (4c_3 + 3c_4)\sin t + 8e^t \quad ...(4)$$

Eqns. (3) and (4) when taken together give the complete solution.

Example 8. *Solve:* $\dfrac{d^2x}{dt^2} + 4x + 5y = t^2$; $\dfrac{d^2y}{dt^2} + 5x + 4y = t + 1$.

Sol. Writing D for $\dfrac{d}{dt}$, the given equations become

$$(D^2 + 4)x + 5y = t^2 \qquad ...(1)$$

and

$$5x + (D^2 + 4)y = t + 1 \qquad ...(2)$$

To eliminate y, operating (1) by $(D^2 + 4)$ and multiplying (2) by 5 then subtracting, we get

$$[(D^2 + 4)^2 - 25]x = (D^2 + 4)t^2 - 5(t + 1)$$

or

$$(D^4 + 8D^2 - 9)x = 2 + 4t^2 - 5t - 5 = 4t^2 - 5t - 3$$

Auxiliary equation is

$$m^4 + 8m^2 - 9 = 0$$

or

$$(m^2 + 9)(m^2 - 1) = 0 \quad \therefore \quad m = \pm 1, \pm 3i$$

$$\therefore \quad \text{C.F.} = c_1e^t + c_2e^{-t} + c_3 \cos 3t + c_4 \sin 3t$$

$$\text{P.I.} = \frac{1}{D^4 + 8D^2 - 9}(4t^2 - 5t - 3) = \frac{1}{-9\left(1 - \frac{8D^2}{9} - \frac{D^4}{9}\right)}(4t^2 - 5t - 3)$$

$$= -\frac{1}{9}\left[1 - \left(\frac{8D^2}{9} + \frac{D^4}{9}\right)\right]^{-1}(4t^2 - 5t - 3)$$

$$= -\frac{1}{9}\left[1 + \left(\frac{8D^2}{9} + \frac{D^4}{9}\right) + \ldots\ldots\right](4t^2 - 5t - 3)$$

$$= -\frac{1}{9}\left[4t^2 - 5t - 3 + \frac{8}{9}(8)\right] = -\frac{1}{9}\left(4t^2 - 5t + \frac{37}{9}\right)$$

$$\therefore \quad x = c_1e^t + c_2e^{-t} + c_3 \cos 3t + c_4 \sin 3t - \frac{4}{9}t^2 + \frac{5}{9}t - \frac{37}{81}$$

Now,

$$\frac{dx}{dt} = c_1e^t - c_2e^{-t} - 3c_3 \sin 3t + 3c_4 \cos 3t - \frac{8}{9}t + \frac{5}{9}$$

$$\frac{d^2x}{dt^2} = c_1e^t + c_2e^{-t} - 9c_3 \cos 3t - 9c_4 \sin 3t - \frac{8}{9}$$

Substituting the values of x and $\dfrac{d^2x}{dt^2}$ in (1), we have

$$5y = t^2 - 4x - \frac{d^2x}{dt^2}$$

$$= t^2 - 4c_1e^t - 4c_2e^{-t} - 4c_3 \cos 3t - 4c_4 \sin 3t$$

$$+ \frac{16}{9}t^2 - \frac{20}{9}t + \frac{148}{81} - c_1e^t - c_2e^{-t} + 9c_3 \cos 3t + 9c_4 \sin 3t + \frac{8}{9}$$

$$\therefore \quad y = \frac{1}{5}\left[-5c_1e^t - 5c_2e^{-t} + 5c_3 \cos 3t + 5c_4 \sin 3t + \frac{25}{9}t^2 - \frac{20}{9}t + \frac{220}{81}\right]$$

Hence $x = c_1e^t + c_2e^{-t} + c_3 \cos 3t + c_4 \sin 3t - \frac{1}{9}\left(4t^2 - 5t + \frac{37}{9}\right)$

$$y = -c_1e^t - c_2e^{-t} + c_3 \cos 3t + c_4 \sin 3t + \frac{1}{9}\left(5t^2 - 4t + \frac{44}{9}\right).$$

Example 9. *Solve:* $\frac{d^2x}{dt^2} + \frac{dy}{dt} + 3x = e^{-t}$

$$\frac{d^2y}{dt^2} - 4\frac{dx}{dt} + 3y = \sin 2t.$$ [M.T.U. (SUM) 2011]

Sol. Let $D \equiv \frac{d}{dt}$ then we have

$$(D^2 + 3)x + Dy = e^{-t} \quad ...(1)$$

$$-4Dx + (D^2 + 3)y = \sin 2t \quad ...(2)$$

Operating (1) by $(D^2 + 3)$ and (2) by D then subtracting, we get

$$[(D^2 + 3)^2 + 4D^2]\, x = 4e^{-t} - 2 \cos 2t$$

$$(D^4 + 10D^2 + 9)\, x = 4e^{-t} - 2 \cos 2t$$

Auxiliary equation is

$$m^4 + 10m^2 + 9 = 0 \Rightarrow m = \pm i, \pm 3i$$

$$\text{C.F.} = c_1 \cos t + c_2 \sin t + c_3 \cos 3t + c_4 \sin 3t$$

$$\text{P.I.} = \frac{1}{D^4 + 10D^2 + 9}(4e^{-t}) - \frac{1}{D^4 + 10D^2 + 9}(2 \cos 2t)$$

$$= \frac{1}{1 + 10 + 9}(4e^{-t}) - \frac{1}{16 - 40 + 9}(2 \cos 2t) = \frac{1}{5}e^{-t} + \frac{2}{15}\cos 2t$$

$$\therefore \quad x = c_1 \cos t + c_2 \sin t + c_3 \cos 3t + c_4 \sin 3t + \frac{1}{5}e^{-t} + \frac{2}{15}\cos 2t \quad ...(3)$$

Again operating (1) by 4D and (2) by $(D^2 + 3)$ then adding, we get

$$[(D^2 + 3)^2 + 4D^2]\, y = -4e^{-t} - \sin 2t$$

$$(D^4 + 10D^2 + 9)\, y = -4e^{-t} - \sin 2t$$

Auxiliary equation is

$$m^4 + 10m^2 + 9 = 0 \Rightarrow m = \pm i, \pm 3i$$

$$\text{C.F.} = c_5 \cos t + c_6 \sin t + c_7 \cos 3t + c_8 \sin 3t$$

$$\text{P.I.} = \frac{1}{D^4 + 10D^2 + 9}(-4e^{-t}) - \frac{1}{D^4 + 10D^2 + 9}(\sin 2t)$$

$$= -\frac{1}{5}e^{-t} + \frac{1}{15}\sin 2t$$

$$\therefore \quad y = c_5 \cos t + c_6 \sin t + c_7 \cos 3t + c_8 \sin 3t - \frac{1}{5}e^{-t} + \frac{1}{15}\sin 2t \quad ...(4)$$

Equations (3) and (4), when taken together, give the complete solution.

Example 10. *Solve the simultaneous equations:*

$$\frac{dx}{dt} + \frac{dy}{dt} - 2y = 2\cos t - 7\sin t$$

$$\frac{dx}{dt} - \frac{dy}{dt} + 2x = 4\cos t - 3\sin t. \quad \text{(U.K.T.U. 2011)}$$

Sol. The given equations may be written as

$$Dx + (D-2)y = 2\cos t - 7\sin t \quad ...(1)$$

$$(D+2)x - Dy = 4\cos t - 3\sin t \quad ...(2)$$

Operating (1) by D and (2) by (D – 2), we get

$$D^2x + D(D-2)y = -2\sin t - 7\cos t$$

$$(D^2-4)x - D(D-2)y = -4\sin t - 8\cos t - 3\cos t + 6\sin t$$

Adding, we get

$$(2D^2-4)x = -18\cos t$$

$$\Rightarrow \quad (D^2-2)x = -9\cos t$$

Auxiliary equation is

$$m^2 - 2 = 0 \Rightarrow m = \pm\sqrt{2}$$

$$\text{C.F.} = c_1\cosh\sqrt{2}t + c_2\sinh\sqrt{2}t$$

$$\text{P.I.} = \frac{1}{D^2-2}(-9\cos t) = 3\cos t$$

$$\therefore \quad x = \text{C.F.} + \text{P.I.} = c_1\cosh\sqrt{2}t + c_2\sinh\sqrt{2}t + 3\cos t \quad ...(3)$$

Again, operating (1) by (D + 2) and (2) by D, we get

$$(D+2)Dx + (D^2-4)y = -2\sin t - 7\cos t + 4\cos t - 14\sin t$$

$$D(D+2)x - D^2y = -4\sin t - 3\cos t$$

Subtracting, we get

$$(2D^2-4)y = -12\sin t$$

$$(D^2-2)y = -6\sin t$$

$$\text{C.F.} = c_3\cosh\sqrt{2}t + c_4\sinh\sqrt{2}t$$

$$\text{P.I.} = \frac{1}{D^2-2}(-6\sin t) = 2\sin t$$

$$\therefore \quad y = \text{C.F.} + \text{P.I.} = c_3\cosh\sqrt{2}t + c_4\sinh\sqrt{2}t + 2\sin t \quad ...(4)$$

Eqns. (3) and (4), when taken together, give the complete solution.

TEST YOUR KNOWLEDGE

Solve the following systems of simultaneous differential equations:

1. $\frac{dx}{dt} + 7x - y = 0, \ \frac{dy}{dt} + 2x + 5y = 0$ [G.B.T.U. 2010]

2. $\frac{dx}{dt} + x - 2y = 0, \ \frac{dy}{dt} + x + 4y = 0 ; x(0) = y(0) = 1$ (U.P.T.U. 2015)

3. $\frac{dx}{dt} - 7x + y = 0, \ \frac{dy}{dt} - 2x - 5y = 0$

4. $\frac{dx}{dt} = 3x + 8y, \ \frac{dy}{dt} = -x - 3y; x(0) = 6, y(0) = -2$ (G.B.T.U. 2010)

5. $\frac{dx}{dt} - y = t,\ \frac{dy}{dt} + x = 1$

6. $\frac{dx}{dt} = y + 1,\ \frac{dy}{dt} = x + 1.$

7. $\frac{dx}{dt} + y = \sin t,\ \frac{dx}{dt} + x = \cos t$; given that $x = 2$ and $y = 0$ when $t = 0$.

8. $\frac{dx}{dt} - y = e^t,\ \frac{dy}{dt} + x = \sin t;\ x(0) = 1,\ y(0) = 0$ [G.B.T.U. 2010, 2011]

9. $\frac{dx}{dt} + 5x + y = e^t,\ \frac{dy}{dt} + x + 5y = e^{5t}.$

10. $\frac{dx}{dt} + 2x - 3y = t,\ \frac{dy}{dt} - 3x + 2y = e^{2t}$ (U.P.T.U. 2013)

11. (i) $\frac{d^2x}{dt^2} + m^2y = 0,\ \frac{d^2y}{dt^2} - m^2x = 0$ (ii) $\frac{d^2x}{dt^2} - 3x - 4y = 0,\ \frac{d^2y}{dt^2} + x + y = 0$

12. $\frac{dx}{dt} + 2x + 4y = 1 + 4t,\ \frac{dy}{dt} + x - y = \frac{3}{2}t^2$ (G.B.T.U. 2013)

13. $\frac{d^2x}{dt^2} + 16x - 6\frac{dy}{dt} = 0,\ 6\frac{dx}{dt} + \frac{d^2y}{dt^2} + 16y = 0$

14. $(D - 1)x + Dy = 2t + 1,\ (2D + 1)x + 2Dy = t$

15. $(D^2 - 1)x + 8Dy = 16e^t$ and $Dx + 3(D^2 + 1)y = 0$

16. $\frac{dx}{dt} + \frac{dy}{dt} + 2x + y = 0,\ \frac{dy}{dt} + 5x + 3y = 0$ [G.B.T.U. (C.O.) 2011]

17. $\frac{dx}{dt} = -4(x + y),\ \frac{dx}{dt} + 4\frac{dy}{dt} = -4y$ with conditions $x(0) = 1,\ y(0) = 0.$ (U.P.T.U. 2014)

18. $\frac{dx}{dt} + \frac{2}{t}(x - y) = 1,\ \frac{dy}{dt} + \frac{1}{t}(x + 5y) = t.$

Answers

1. $x = e^{-6t}(A\cos t + B\sin t),\ y = e^{-6t}[(A + B)\cos t - (A - B)\sin t]$
2. $x = 4e^{-2t} - 3e^{-3t},\ y = -2e^{-2t} + 3e^{-3t}$
3. $x = e^{6t}(c_1\cos t + c_2\sin t),\ y = e^{6t}[(c_1 + c_2)\sin t + (c_1 - c_2)\cos t]$
4. $x = 4e^t + 2e^{-t},\ y = -e^t - e^{-t}$
5. $x = c_1\cos t + c_2\sin t + 2,\ y = -c_1\sin t + c_2\cos t - t$
6. $x = c_1e^t + c_2e^{-t} - 1,\ y = c_1e^t - c_2e^{-t} - 1$ 7. $x = e^t + e^{-t},\ y = e^{-t} - e^t + \sin t$
8. $x = 2\sin t + \frac{3}{2}\cos t + \frac{t}{2}\cos t - \frac{1}{2}e^t,\ y = \frac{1}{2}\cos t - \frac{3}{2}\sin t + \frac{t}{2}\sin t - \frac{1}{2}e^t$
9. $x = c_1e^{-6t} + c_2e^{-4t} + \frac{6e^t}{35} - \frac{e^{5t}}{99},\ y = c_1e^{-6t} - c_2e^{-4t} - \frac{1}{35}e^t + \frac{10}{99}e^{5t}.$
10. $x = c_1e^{-5t} + c_2e^t + \frac{3}{7}e^{2t} - \frac{2}{5}t - \frac{13}{25},\ y = c_1e^{-5t} + c_2e^t + \frac{4}{7}e^{2t} - \frac{3}{5}t - \frac{12}{25}$
11. (i) $x = e^{mt/\sqrt{2}}\left(c_1\cos\frac{mt}{\sqrt{2}} + c_2\sin\frac{mt}{\sqrt{2}}\right) + e^{-mt/\sqrt{2}}\left(c_3\cos\frac{mt}{\sqrt{2}} + c_4\sin\frac{mt}{\sqrt{2}}\right)$;

$y = e^{mt/\sqrt{2}}\left(c_1\sin\frac{mt}{\sqrt{2}} - c_2\cos\frac{mt}{\sqrt{2}}\right) + e^{-mt/\sqrt{2}}\left(c_4\cos\frac{mt}{\sqrt{2}} - c_3\sin\frac{mt}{\sqrt{2}}\right)$

(ii) $x = (c_1 + c_2t)e^{-t} + (c_3 + c_4t)e^t,\ y = -\frac{1}{2}[c_1 + c_2(1 + t)]e^{-t} + \frac{1}{2}[c_4(1 - t) - c_3]e^t$

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12. $x = c_1 e^{2t} + c_2 e^{-3t} + t + t^2,\ y = -c_1 e^{2t} + \frac{c_2}{4} e^{-3t} - \frac{t^2}{2}$

13. $x = c_1 \cos 2t + c_2 \sin 2t - c_3 \cos 8t + c_4 \sin 8t,\ y = c_1 \sin 2t + c_2 \cos 2t - c_3 \sin 8t + c_4 \cos 8t$

14. $x = -t - \frac{2}{3},\ y = \frac{1}{2} t^2 + \frac{4}{3} t + c$

15. $y = c_1 \cos \frac{t}{\sqrt{3}} + c_2 \sin \frac{t}{\sqrt{3}} + c_3 \cosh \sqrt{3}\, t + c_4 \sinh \sqrt{3}\, t + 2e^t$

$x = \sqrt{3}\, c_1 \sin \frac{t}{\sqrt{3}} - \sqrt{3}\, c_2 \cos \frac{t}{\sqrt{3}} - 3\sqrt{3}\, c_3 \sinh \sqrt{3}\, t - 3\sqrt{3}\, c_4 \cosh \sqrt{3}\, t - 6e^t - 3t.$

16. $x = \left(\frac{c_1 - 3c_2}{5}\right) \sin t - \left(\frac{c_2 + 3c_1}{5}\right) \cos t,\ y = c_1 \cos t + c_2 \sin t$

17. $x = (1 - 2t)e^{-2t},\ y = te^{-2t}$

18. $x = At^{-4} + Bt^{-3} + \frac{t^2}{15} + \frac{3t}{10}$; $y = -At^{-4} - \frac{1}{2} Bt^{-3} + \frac{2t^2}{15} - \frac{t}{20}.$

1.22 LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER

A differential equation of the form $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ is known as linear differential equation of second order, where P, Q and R are functions of x alone.

Note. Coefficient of $\frac{d^2y}{dx^2}$ must always be 1.

A linear differential equation of second order can be solved by changing dependent and independent variables as well as by the method of variation of parameters.

Above methods are illustrated below with some suitable and important examples.

1.22.1 Method 1. To Find the Complete Solution of $y'' + Py' + Qy = R$ when Part of Complementary Function is Known (*Method of reduction of order*)

Let $y = u$ be a part of the complementary function of the given differential equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad ...(1)$$

where u is a function of x. Then, we have

$$\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0 \qquad ...(2)$$

Let $y = uv$ be the complete solution of equation (1), where v is a function of x.

Differentiating y w.r.t. x,

$$\frac{dy}{dx} = u\frac{dv}{dx} + \frac{du}{dx} \cdot v$$

Again,
$$\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{du}{dx} \cdot \frac{dv}{dx} + v\frac{d^2u}{dx^2}$$

Substituting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in equation (1), we get

$$u\frac{d^2v}{dx^2}+2\frac{du}{dx}\frac{dv}{dx}+v\frac{d^2u}{dx^2}+\mathrm{P}\left(u\frac{dv}{dx}+v\frac{du}{dx}\right)+\mathrm{Q}(uv)=\mathrm{R}$$

$$\Rightarrow \quad u\frac{d^2v}{dx^2}+\left(2\frac{du}{dx}+\mathrm{P}u\right)\frac{dv}{dx}+\left(\frac{d^2u}{dx^2}+\mathrm{P}\frac{du}{dx}+\mathrm{Q}u\right)v=\mathrm{R}$$

$$\Rightarrow \quad u\frac{d^2v}{dx^2}+\left(2\frac{du}{dx}+\mathrm{P}u\right)\frac{dv}{dx}=\mathrm{R} \qquad | \text{ Using (2)}$$

$$\Rightarrow \quad \frac{d^2v}{dx^2}+\left(\frac{2}{u}\frac{du}{dx}+\mathrm{P}\right)\frac{dv}{dx}=\frac{\mathrm{R}}{u} \qquad \text{...(3)}$$

Put $\frac{dv}{dx}=p$ then, $\frac{d^2v}{dx^2}=\frac{dp}{dx}$

Now (3) becomes, $\frac{dp}{dx}+\left(\frac{2}{u}\frac{du}{dx}+\mathrm{P}\right)p=\frac{\mathrm{R}}{u}$ ...(4)

Equation (4) is a linear differential equation of I order in p and x.

$$\text{I.F.}=e^{\int\left(\frac{2}{u}\frac{du}{dx}+\mathrm{P}\right)dx}=e^{\left(\int\frac{2}{u}du+\int \mathrm{P}\,dx\right)}=u^2e^{\int \mathrm{P}\,dx}$$

Solution of (4) is given by

$$pu^2e^{\int \mathrm{P}\,dx}=\int\frac{\mathrm{R}}{u}u^2e^{\int \mathrm{P}\,dx}dx+c_1$$

where c_1 is an arbitrary constant of integration.

$$\Rightarrow \quad p=\frac{1}{u^2}e^{-\int \mathrm{P}\,dx}\left[\int \mathrm{R}u\,e^{\int \mathrm{P}\,dx}dx+c_1\right]$$

$$\therefore \quad \frac{dv}{dx}=\frac{1}{u^2}e^{-\int \mathrm{P}\,dx}\left[\int \mathrm{R}u\,e^{\int \mathrm{P}\,dx}dx+c_1\right]$$

Integration yields, $v=\int\frac{1}{u^2}e^{-\int \mathrm{P}\,dx}\left[\int \mathrm{R}u\,e^{\int \mathrm{P}\,dx}dx+c_1\right]dx+c_2$

where c_2 is an arbitrary constant of integration.

Hence the complete solution of (1) is given by,

$$y=uv$$

$$\Rightarrow \quad y=u\int\frac{1}{u^2}e^{-\int \mathrm{P}\,dx}\left[\int \mathrm{R}u\,e^{\int \mathrm{P}\,dx}dx+c_1\right]dx+c_2u$$

To find out the part of C.F. of the linear differential equation of II order given by

$$\frac{d^2y}{dx^2}+\mathrm{P}\frac{dy}{dx}+\mathrm{Q}y=\mathrm{R}.$$

Remember:

Condition	*Part of C.F.*
(*i*) $1+\frac{\mathrm{P}}{a}+\frac{\mathrm{Q}}{a^2}=0$	e^{ax}
(*ii*) $1+\mathrm{P}+\mathrm{Q}=0$	e^x
(*iii*) $1-\mathrm{P}+\mathrm{Q}=0$	e^{-x}
(*iv*) $m(m-1)+\mathrm{P}mx+\mathrm{Q}x^2=0$	x^m
(*v*) $\mathrm{P}+\mathrm{Q}x=0$	x
(*vi*) $2+2\mathrm{P}x+\mathrm{Q}x^2=0$	x^2

Proof. (*i*) Let $y = e^{ax}$ be a part of C.F. of the equation $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R.$

Then, $a^2e^{ax} + P\,ae^{ax} + Qe^{ax} = 0$

$\Rightarrow \quad a^2 + Pa + Q = 0$

$\Rightarrow \quad 1 + \frac{P}{a} + \frac{Q}{a^2} = 0.$

(*iv*) Let $y = x^m$ be a part of C.F of the equation $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ then,

$$m(m-1)\,x^{m-2} + P\,mx^{m-1} + Qx^m = 0$$

$\Rightarrow \quad m(m-1) + P\,mx + Qx^2 = 0$

Rest all the parts are the deductions from proofs (*i*) and (*iv*).

Steps for solution:

1. Make the coefficient of $\frac{d^2y}{dx^2}$ as 1 if it is not so.
2. Compare the given differential equation with the standard form $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ and hence find P, Q and R which are functions of x only.
3. Apply the mentioned conditions on P and Q. If any one of the conditions is satisfied, write down the corresponding part of C.F. as u.
4. Let $y = uv$ be the complete solution of the given differential equation, where v is, another function of x, to be determined.
5. Find y' and y'' from y and substitute the required values in the given equation.
6. Arrange the expression in decreasing order of the derivatives of v w.r.t. x.
7. Check whether the coefficient of v is zero ? If not, we are somewhere wrong. It is the time to recheck.
8. Put $\frac{dv}{dx} = p$ so that $\frac{d^2v}{dx^2} = \frac{dp}{dx}$. Then we get a I order differential equation in p and x, which we solve by the methods discussed before.
9. Integrating $\frac{dv}{dx}$ w.r.t. x, we get v in terms of x.
10. At last $y = uv$ will be the complete solution of the given differential equation having two arbitrary constants.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve:* $\sin^2 x\,\frac{d^2y}{dx^2} = 2y$, *given that* $y = \cot x$ *is a solution of it.*

Sol. $$\frac{d^2y}{dx^2} - 2(\text{cosec}^2 x)y = 0 \qquad ...(1)$$

Here, a part of C.F. = $\cot x$

Let $y = v \cot x$ be the complete solution of (1)

$$\therefore \quad \frac{dy}{dx} = v\,(-\operatorname{cosec}^2 x) + \cot x \,\frac{dv}{dx}$$

$$\frac{d^2y}{dx^2} = v\,(2 \operatorname{cosec}^2 x \cot x) - 2 \operatorname{cosec}^2 x \,\frac{dv}{dx} + \cot x \,\frac{d^2v}{dx^2}$$

Substituting the values of y, $\frac{d^2y}{dx^2}$, we get

$$\cot x \,\frac{d^2v}{dx^2} - 2 \operatorname{cosec}^2 x \,\frac{dv}{dx} = 0$$

$$\Rightarrow \quad \cot x \,\frac{dp}{dx} - 2 \operatorname{cosec}^2 x \,.\, p = 0, \quad \text{where } p = \frac{dv}{dx}$$

$$\Rightarrow \quad \frac{dp}{p} = 2\,\frac{\operatorname{cosec}^2 x}{\cot x}\, dx$$

Integration yields,

$$\log\, p = -\,2 \log \cot x + \log c_1 \qquad | \; c_1 \text{ is arbitrary constant}$$

$$\Rightarrow \quad \log p + \log \cot^2 x = \log c_1$$

$$\Rightarrow \quad p = c_1 \tan^2 x$$

$$\therefore \quad \frac{dv}{dx} = c_1\,(\sec^2 x - 1)$$

Integrating with respect to x, we get

$$v = c_1\,(\tan x - x) + c_2$$

where c_2 is an arbitrary constant of integration.

Hence the complete solution is given by

$$y = v \cot x = [c_1\,(\tan x - x) + c_2] \cot x$$

$$y = c_1(1 - x \cot x) + c_2 \cot x.$$

Example 2. *Solve:* $\frac{d^2y}{dx^2} - \cot x\,\frac{dy}{dx} - (1 - \cot x)y = e^x \sin x.$

Sol. Comparing with the standard form, we get

$$\mathrm{P} = -\cot x,\ \mathrm{Q} = -\,(1 - \cot x),\ \mathrm{R} = e^x \sin x$$

$$1 + \mathrm{P} + \mathrm{Q} = 1 - 1 + \cot x - \cot x = 0$$

$\therefore$ A part of C.F. $= e^x$

Let $y = v\,e^x$ be the complete solution of given equation, then

$$\frac{dy}{dx} = v\,e^x + e^x\,\frac{dv}{dx}$$

$$\frac{d^2y}{dx^2} = v\,e^x + 2e^x\,\frac{dv}{dx} + e^x\,\frac{d^2v}{dx^2}$$

Substituting for y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in given equation, we get

$$\frac{d^2v}{dx^2} + (2 - \cot x)\frac{dv}{dx} = \sin x$$

$$\Rightarrow \quad \frac{dp}{dx} + (2 - \cot x)p = \sin x \qquad ...(1) \quad \text{where } p = \frac{dv}{dx}.$$

This is a linear differential equation of I order in p and x.

$$\text{I.F.} = e^{\int (2-\cot x)\,dx} = \frac{e^{2x}}{\sin x}$$

Solution of (1) is, $\quad p\,\frac{e^{2x}}{\sin x} = \int \sin x \cdot \frac{e^{2x}}{\sin x}\,dx + c_1 = \frac{e^{2x}}{2} + c_1$

where c_1 is an arbitrary constant of integration.

$$p = \frac{1}{2}\sin x + c_1 e^{-2x}\sin x$$

$$\frac{dv}{dx} = \frac{1}{2}\sin x + c_1 e^{-2x}\sin x$$

Integrating, we get $\quad v = -\frac{1}{2}\cos x - \frac{1}{5}c_1 e^{-2x}(\cos x + 2\sin x) + c_2$

Hence the complete solution is given by,

$$y = v e^x = \left[-\frac{1}{2}\cos x - \frac{1}{5}c_1 e^{-2x}(\cos x + 2\sin x) + c_2\right]e^x.$$

Example 3. *Solve:* $(1 - x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x(1 - x^2)^{3/2}$.

Sol. Comparing with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$, we get

$$P = \frac{x}{1 - x^2},\; Q = -\frac{1}{1 - x^2},\; R = x\sqrt{1 - x^2}$$

$\because \quad P + Qx = 0$

$\therefore \quad$ A part of C.F. $= x$

Let $y = vx$ be the complete solution of the given differential equation

$$\therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{and} \quad \frac{d^2y}{dx^2} = x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}$$

Substituting the values, we get

$$\frac{d^2v}{dx^2} + \frac{2 - x^2}{x(1 - x^2)}\frac{dv}{dx} = (1 - x^2)^{1/2}$$

$$\Rightarrow \quad \frac{dp}{dx} + \frac{(2 - x^2)}{x(1 - x^2)}p = (1 - x^2)^{1/2} \qquad ...(1) \qquad \text{where } p = \frac{dv}{dx}.$$

$$\text{I.F.} = e^{\int \frac{(2 - x^2)}{x(1 - x^2)}dx} = e^{\int\left(\frac{2}{x} + \frac{x}{1 - x^2}\right)dx} = \frac{x^2}{\sqrt{1 - x^2}}$$

Solution of (1) is, $$p \cdot \frac{x^2}{\sqrt{1-x^2}} = \int \sqrt{1-x^2} \cdot \frac{x^2}{\sqrt{1-x^2}}\, dx + c_1$$

$$\Rightarrow \quad p\, \frac{x^2}{\sqrt{1-x^2}} = \frac{x^3}{3} + c_1 \qquad | \; c_1 \text{ is arbitrary constant}$$

$$\Rightarrow \quad p = \frac{x}{3}\sqrt{1-x^2} + c_1 \frac{\sqrt{1-x^2}}{x^2}$$

$$\frac{dv}{dx} = \frac{x}{3}\sqrt{1-x^2} + c_1 \frac{\sqrt{1-x^2}}{x^2}$$

Integrating, we get $$v = \frac{1}{3}\int x\sqrt{1-x^2}\, dx + c_1 \int \frac{\sqrt{1-x^2}}{x^2}\, dx + c_2$$

$$= -\frac{1}{9}(1-x^2)^{3/2} + c_1\left[\frac{-\sqrt{1-x^2}}{x} - \sin^{-1} x\right] + c_2$$

where c_2 is also an arbitrary constant of integration.

Hence the complete solution is given by,

$$y = vx = \frac{-x(1-x^2)^{3/2}}{9} - c_1\left[\sqrt{1-x^2} + x\sin^{-1} x\right] + c_2 x.$$

Example 4. *Solve:* $(x \sin x + \cos x)\dfrac{d^2y}{dx^2} - x\cos x\dfrac{dy}{dx} + y\cos x = 0$ *of which* $y = x$ *is a solution.*

Sol. $$\frac{d^2y}{dx^2} - \left(\frac{x\cos x}{x\sin x + \cos x}\right)\frac{dy}{dx} + \left(\frac{\cos x}{x\sin x + \cos x}\right)y = 0 \qquad ...(1)$$

Here a part of C.F. $= x$ | Given

Let $y = vx$ be the complete solution of equation (1)

$$\therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\frac{d^2y}{dx^2} = x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}$$

Substituting the values of y, $\dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2}$ in (1), we get

$$\frac{d^2v}{dx^2} + \left(\frac{2}{x} - \frac{x\cos x}{x\sin x + \cos x}\right)\frac{dv}{dx} = 0$$

$$\Rightarrow \quad \frac{dp}{dx} + \left(\frac{2}{x} - \frac{x\cos x}{x\sin x + \cos x}\right)p = 0, \qquad \text{where } p = \frac{dv}{dx}$$

$$\Rightarrow \quad \frac{dp}{p} + \left(\frac{2}{x} - \frac{x\cos x}{x\sin x + \cos x}\right)dx = 0$$

Integration yields, $px^2 = c_1 (x \sin x + \cos x)$

or
$$\frac{dv}{dx} = c_1 \left(\frac{\sin x}{x} + \frac{\cos x}{x^2} \right)$$

Again integration yields,
$$v = -c_1 \frac{\cos x}{x} + c_2$$

Hence the complete solution is given by
$$y = vx = -c_1 \cos x + c_2 x$$

where c_1 and c_2 are the arbitrary constants of integration.

TEST YOUR KNOWLEDGE

Solve the following differential equations:

1. $x \dfrac{d^2y}{dx^2} - (2x-1)\dfrac{dy}{dx} + (x-1)y = 0$

2. $x \dfrac{d^2y}{dx^2} - (3+x)\dfrac{dy}{dx} + 3y = 0$

3. $x^2 \dfrac{d^2y}{dx^2} - (x^2 + 2x)\dfrac{dy}{dx} + (x+2)y = x^3 e^x$ of which $y = x$ is a solution.

4. $(x+2)\dfrac{d^2y}{dx^2} - (2x+5)\dfrac{dy}{dx} + 2y = (x+1)e^x$

5. $y'' - 4xy' + (4x^2 - 2)y = 0$ given that $y = e^{x^2}$ is a solution.

6. Solve : $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = 0$, given that $x + \dfrac{1}{x}$ is one integral.

Answers

1. $y = (c_1 \log x + c_2) e^x$

2. $y = -c_1(x^3 + 3x^2 + 6x + 6) + c_2 e^x$

3. $y = x(x-1)e^x + c_1 x e^x + c_2 x$

4. $y = -e^x - \dfrac{1}{4} c_1 (2x+5) + c_2 e^{2x}$

5. $y = e^{x^2}(c_1 x + c_2)$

6. $y = \dfrac{A}{x} + c_2\left(x + \dfrac{1}{x}\right)$.

1.22.2 Method 2. To Find the Complete Solution of $y'' + Py' + Qy = R$ when it is Reduced to Normal Form (*Removal of first derivative*)

When the part of C.F. can not be determined by the previous method, we reduce the given differential equation in **normal form** by eliminating the term in which there exists first derivative of the dependent variable.

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad ...(1)$$

Let $y = uv$ be the complete solution of eqn. (1), where u and v are the functions of x.

$$\therefore \qquad \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

and
$$\frac{d^2y}{dx^2} = v\frac{d^2u}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2}$$

Substituting the values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in eqn. (1), we get

$$\frac{d^2v}{dx^2}+\left(\frac{2}{u}\frac{du}{dx}+\mathrm{P}\right)\frac{dv}{dx}+v\left(\frac{1}{u}\frac{d^2u}{dx^2}+\frac{\mathrm{P}}{u}\frac{du}{dx}+\mathrm{Q}\right)=\frac{\mathrm{R}}{u} \qquad ...(2)$$

Let us choose u such that $\frac{2}{u}\frac{du}{dx}+\mathrm{P}=0$ ...(3)

which on solving gives,

$$u=e^{-\int \frac{\mathrm{P}}{2}dx} \qquad ...(4)$$

From (3), $$\frac{du}{dx}=-\frac{\mathrm{P}u}{2}$$

Differentiating, we get $$\frac{d^2u}{dx^2}=-\frac{1}{2}\left[\mathrm{P}\left(\frac{du}{dx}\right)+\frac{d\mathrm{P}}{dx}(u)\right]$$

$$=-\frac{1}{2}\left[\mathrm{P}\left(\frac{-\mathrm{P}u}{2}\right)+u\frac{d\mathrm{P}}{dx}\right]=\frac{\mathrm{P}^2u}{4}-\frac{u}{2}\frac{d\mathrm{P}}{dx}$$

$$\text{Coefficient of } v=\frac{1}{u}\frac{d^2u}{dx^2}+\frac{\mathrm{P}}{u}\frac{du}{dx}+\mathrm{Q}=\frac{1}{u}\left[\frac{\mathrm{P}^2u}{4}-\frac{u}{2}\frac{d\mathrm{P}}{dx}\right]+\frac{\mathrm{P}}{u}\left(\frac{-\mathrm{P}u}{2}\right)+\mathrm{Q}$$

$$=\mathrm{Q}-\frac{1}{2}\frac{d\mathrm{P}}{dx}-\frac{\mathrm{P}^2}{4}=\mathrm{I}\text{ (say)}$$

$$\text{RHS}=\frac{\mathrm{R}}{u}=\mathrm{R}e^{\frac{1}{2}\int \mathrm{P}\,dx}=\mathrm{S}\text{ (say)}$$

Then (2) becomes, $$\frac{d^2v}{dx^2}+\mathrm{I}v=\mathrm{S} \qquad ...(5)$$

This is known as the normal form of equation (1).

Solving (5), we get v in terms of x. Ultimately, $y = uv$ is the complete solution.

Steps for solution:

1. Make the coefficient of $\frac{d^2y}{dx^2}$ as 1 if it is not so.
2. Compare with standard form $y'' + \mathrm{P}y' + \mathrm{Q}y = \mathrm{R}$ to get P, Q and R.
3. Let $y = uv$ be the complete solution of the given equation.
4. Find $u=e^{-\frac{1}{2}\int \mathrm{P}\,dx}$, $\mathrm{I}=\mathrm{Q}-\frac{1}{2}\frac{d\mathrm{P}}{dx}-\frac{\mathrm{P}^2}{4}$ and $\mathrm{S}=\frac{\mathrm{R}}{u}$.
5. Check whether I is a constant or $\frac{\text{constant}}{x^2}$? If I is not any one of them, the method is not applicable.
6. If I is a constant, we get a linear differential equation of II order with constant coefficients while if I is $\frac{\text{constant}}{x^2}$, we get a homogeneous linear differential equation with variable coefficients.

7. Normal form is given by $\frac{d^2v}{dx^2} + Iv = S$ which we solve for v.

8. $y = uv$ will be the complete solution of the given differential equation.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve:* $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 1)y = -3e^{x^2} \sin 2x.$

Sol. Here, $P = -4x$, $Q = 4x^2 - 1$, $R = -3e^{x^2} \sin 2x$

Let $y = uv$ be the complete solution.

Now, $$u = e^{-\frac{1}{2}\int (-4x)\, dx} = e^{x^2}$$

$$I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4} = 4x^2 - 1 - \frac{1}{2}(-4) - \frac{1}{4}(16x^2) = 1.$$

Also, $$S = \frac{R}{u} = \frac{-3e^{x^2} \sin 2x}{e^{x^2}} = -3 \sin 2x$$

Hence normal form is,

$$\frac{d^2v}{dx^2} + v = -3 \sin 2x$$

Auxiliary equation is

$$m^2 + 1 = 0 \Rightarrow m = \pm i$$

$$\text{C.F.} = c_1 \cos x + c_2 \sin x$$

where c_1 and c_2 are arbitrary constants of integration.

$$\text{P.I.} = \frac{1}{D^2 + 1}(-3 \sin 2x) = \frac{-3}{(-4+1)} \sin 2x = \sin 2x$$

$\therefore$ Solution is, $v = c_1 \cos x + c_2 \sin x + \sin 2x$

Hence the complete solution of given differential equation is

$$y = uv = e^{x^2}(c_1 \cos x + c_2 \sin x + \sin 2x).$$

Example 2. *Solve:* $\frac{d^2y}{dx^2} + \frac{1}{x^{1/3}}\frac{dy}{dx} + \left(\frac{1}{4x^{2/3}} - \frac{1}{6x^{4/3}} - \frac{6}{x^2}\right)y = 0.$

Sol. Comparing with the standard form, we get

$$P = x^{-1/3},\ Q = \frac{1}{4}x^{-2/3} - \frac{1}{6}x^{-4/3} - 6x^{-2},\ R = 0$$

Let $y = uv$ be the complete solution of given equation.

Now, $$u = e^{-\frac{1}{2}\int P\,dx} = e^{-\frac{1}{2}\int x^{-1/3}\,dx} = e^{-\frac{3}{4}x^{2/3}}$$

$$I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}$$

$$= \frac{1}{4x^{2/3}} - \frac{1}{6x^{4/3}} - \frac{6}{x^2} - \frac{1}{2}\left(-\frac{1}{3}x^{-4/3}\right) - \frac{1}{4}\cdot\frac{1}{x^{2/3}}$$

$\Rightarrow$ $$I = -\frac{6}{x^2}$$

Also, $$S = \frac{R}{u} = 0$$

$\therefore$ Normal form is,

$$\frac{d^2v}{dx^2} - \frac{6}{x^2}v = 0$$

$\Rightarrow$ $$x^2\frac{d^2v}{dx^2} - 6v = 0 \qquad \ldots(1)$$

Put $x = e^z$ so that $z = \log x$ and let $D \equiv \frac{d}{dz}$ then eqn. (1) becomes,

$$[D(D-1) - 6]\,v = 0$$

$\Rightarrow$ $$(D^2 - D - 6)\,v = 0 \qquad \ldots(2)$$

Solution of eqn. (2) is, $v = c_1e^{3z} + c_2\,e^{-2z} = c_1x^3 + c_2x^{-2}$

where c_1 and c_2 are arbitrary constants of integration.

$\therefore$ Complete solution of given differential equation is

$$y = e^{-\frac{3}{4}x^{2/3}}\left(c_1\,x^3 + \frac{c_2}{x^2}\right).$$

TEST YOUR KNOWLEDGE

Solve the following differential equations:

1. $x\frac{d}{dx}\left(x\frac{dy}{dx} - y\right) - 2x\frac{dy}{dx} + 2y + x^2y = 0$

2. $\frac{d}{dx}\left(\cos^2 x\frac{dy}{dx}\right) + y\cos^2 x = 0$

3. $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} = n^2y$

4. $\frac{d^2y}{dx^2} - 2\tan x\frac{dy}{dx} + 5y = e^x\sec x$ (U.P.T.U. 2015)

5. $\left(\frac{d^2y}{dx^2} + y\right)\cot x + 2\left(\frac{dy}{dx} + y\tan x\right) = \sec x$

6. $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 3)y = e^{x^2}$

7. $\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (x^2 + 2)y = e^{\frac{1}{2}(x^2 + 2x)}$ [G.B.T.U. (C.O.) 2011]

(G.B.T.U. 2012, 2013)

8. $\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} + (x^2 - 8)\,y = x^2\,e^{-x^2/2}$.

Answers

1. $y = x(c_1 \cos x + c_2 \sin x)$

2. $y = \sec x\,(c_1 \cos \sqrt{2}\,x + c_2 \sin \sqrt{2}\,x)$

3. $y = \dfrac{1}{x}\,(c_1 e^{nx} + c_2 e^{-nx})$

4. $y = \sec x\left[c_1 \cos \sqrt{6}x + c_2 \sin \sqrt{6}x + \dfrac{e^x}{7}\right]$

5. $y = \dfrac{1}{2}\,(\sin x - x \cos x) + c_1\,x \cos x + c_2 \cos x$

6. $y = e^{x^2}\,(c_1 e^x + c_2 e^{-x} - 1)$

7. $y = e^{x^2/2}\,(c_1 \cos \sqrt{3}\,x + c_2 \sin \sqrt{3}\,x) + \dfrac{1}{4}e^{\frac{1}{2}(x^2 + 2x)}$

8. $y = e^{-\frac{x^2}{2}}\left[c_1\,e^{3x} + c_2\,e^{-3x} - \dfrac{1}{9}\left(x^2 + \dfrac{2}{9}\right)\right]$

1.22.3 Method 3 To Find the Complete Solution of $y'' + Py' + Qy = R$ by Changing the Independent Variable

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad \text{...(1)}$$

Let us relate x and z by the relation,

$$z = f(x) \qquad \text{...(2)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} \qquad \text{...(3)}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dz}\cdot\frac{dz}{dx}\right)$$

$$= \frac{dy}{dz}\cdot\frac{d^2z}{dx^2} + \frac{dz}{dx}\cdot\frac{d}{dz}\left(\frac{dy}{dz}\right)\cdot\frac{dz}{dx} = \frac{dy}{dz}\cdot\frac{d^2z}{dx^2} + \left(\frac{dz}{dx}\right)^2\frac{d^2y}{dz^2} \qquad \text{...(4)}$$

Substituting in (1), we get

$$\frac{dy}{dz}\cdot\frac{d^2z}{dx^2} + \left(\frac{dz}{dx}\right)^2\frac{d^2y}{dz^2} + P\,\frac{dy}{dz}\cdot\frac{dz}{dx} + Qy = R$$

$$\Rightarrow \qquad \frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1y = R_1 \qquad \text{...(5)}$$

where $P_1 = \dfrac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2}$, $Q_1 = \dfrac{Q}{\left(\dfrac{dz}{dx}\right)^2}$, $R_1 = \dfrac{R}{\left(\dfrac{dz}{dx}\right)^2}$

Here P_1, Q_1, R_1 are functions of x which can be transformed into functions of z using the relation $z = f(x)$.

Choose z such that Q_1 = constant = a^2 (say)

$$\Rightarrow \qquad \frac{Q}{\left(\frac{dz}{dx}\right)^2} = a^2 \quad \Rightarrow \quad a\frac{dz}{dx} = \sqrt{Q}$$

$$\Rightarrow \qquad dz = \frac{\sqrt{Q}}{a}\,dx$$

Integration yields, $\quad z = \int \frac{\sqrt{Q}}{a}\,dx$

If this value of z makes P_1 as constant then equation (5) can be solved.

Steps for solution:

1. Make the coefficient of $\frac{d^2y}{dx^2}$ as 1 if it is not so.
2. Compare the equation with standard form $y'' + Py' + Qy = R$ and get P, Q and R.
3. Choose z such that $\left(\frac{dz}{dx}\right)^2 = Q$

   Here Q is taken in such a way that it remains the whole square of a function without surd and its negative sign is ignored.
4. Find $\frac{dz}{dx}$ hence obtain z (on integration) and $\frac{d^2z}{dx^2}$ (on differentiation).
5. Find P_1, Q_1 and R_1 by the formulae

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}, \quad Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}, \quad R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}.$$

6. Reduced equation is $\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$ which we solve for y in terms of z.
7. We write the complete solution as y in terms of x by replacing the value of z in terms of x.

ILLUSTRATIVE EXAMPLES

Example 1. *By changing the independent variable, solve the differential equation*

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} + 4x^2y = x^4. \qquad \text{(U.P.T.U. 2015)}$$

Sol.
$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} + 4x^2y = x^4 \qquad ...(1)$$

Here, $\quad P = -\frac{1}{x},\ Q = 4x^2,\ R = x^4$

Choose z such that $\left(\frac{dz}{dx}\right)^2 = 4x^2$

$\Rightarrow$ $$\frac{dz}{dx} = 2x \quad \ldots(2)$$

$$z = x^2 \text{ (on integrating)} \quad \ldots(3)$$

From (2), $$\frac{d^2z}{dx^2} = 2 \quad \text{| Differentiating (2) w.r.t. } x$$

$\therefore$ $$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{2 - \frac{1}{x}(2x)}{4x^2} = 0$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{4x^2}{4x^2} = 1$$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{x^4}{4x^2} = \frac{x^2}{4}$$

Reduced equation is,

$$\frac{d^2y}{dz^2} + y = \frac{z}{4} \quad | \because \; z = x^2 \text{ from (3)}$$

Auxiliary equation is,

$$m^2 + 1 = 0 \quad \Rightarrow \quad m = \pm i$$

$$\text{C.F.} = c_1 \cos z + c_2 \sin z$$

$$\text{P.I.} = \frac{1}{D^2 + 1}\left(\frac{z}{4}\right) = (1 + D^2)^{-1}\left(\frac{z}{4}\right)$$

$$= (1 - D^2 + \ldots\ldots)\left(\frac{z}{4}\right) = \frac{z}{4} \quad \text{| Leaving higher powers}$$

$\therefore$ Solution is $$y = c_1 \cos z + c_2 \sin z + \frac{z}{4}$$

Complete solution is given by

$$y = c_1 \cos(x^2) + c_2 \sin(x^2) + \frac{x^2}{4}.$$

Example 2. *Solve:* $(1 + x)^2 \frac{d^2y}{dx^2} + (1 + x)\frac{dy}{dx} + y = 4 \cos \log (1 + x).$

Choose z such that $\left(\frac{dz}{dx}\right)^2 = 4x^2$

$$\Rightarrow \qquad \frac{dz}{dx} = 2x \qquad \text{...(2)}$$

$$z = x^2 \text{ (on integrating)} \qquad \text{...(3)}$$

From (2), $\frac{d^2z}{dx^2} = 2$ | Differentiating (2) w.r.t. x

$$\therefore \qquad P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{2 - \frac{1}{x}(2x)}{4x^2} = 0$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{4x^2}{4x^2} = 1$$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{x^4}{4x^2} = \frac{x^2}{4}$$

Reduced equation is,

$$\frac{d^2y}{dz^2} + y = \frac{z}{4} \qquad | \because \; z = x^2 \text{ from (3)}$$

Auxiliary equation is,

$$m^2 + 1 = 0 \qquad \Rightarrow \qquad m = \pm i$$

$$\text{C.F.} = c_1 \cos z + c_2 \sin z$$

$$\text{P.I.} = \frac{1}{D^2 + 1}\left(\frac{z}{4}\right) = (1 + D^2)^{-1}\left(\frac{z}{4}\right)$$

$$= (1 - D^2 + \ldots\ldots)\left(\frac{z}{4}\right) = \frac{z}{4} \qquad | \text{ Leaving higher powers}$$

$\therefore$ Solution is $\quad y = c_1 \cos z + c_2 \sin z + \frac{z}{4}$

Complete solution is given by

$$y = c_1 \cos (x^2) + c_2 \sin (x^2) + \frac{x^2}{4}.$$

Example 2. *Solve:* $(1 + x)^2 \frac{d^2y}{dx^2} + (1 + x)\frac{dy}{dx} + y = 4 \cos \log (1 + x)$.

Sol. $$\frac{d^2y}{dx^2} + \frac{1}{1+x}\frac{dy}{dx} + \frac{y}{(1+x)^2} = \frac{4}{(1+x)^2}\cos\log(1+x) \quad ...(1)$$

Choose z such that,

$$\left(\frac{dz}{dx}\right)^2 = \frac{1}{(1+x)^2}$$

$$\Rightarrow \quad \frac{dz}{dx} = \frac{1}{1+x} \quad ...(2)$$

Integration yields, $z = \log(1+x)$ ...(3)

From (2), $$\frac{d^2z}{dx^2} = -\frac{1}{(1+x)^2}$$

$$\therefore \quad P_1 = \frac{-\frac{1}{(1+x)^2} + \frac{1}{1+x}\cdot\frac{1}{1+x}}{\frac{1}{(1+x)^2}} = 0$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = 1$$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = 4\cos\log(1+x) = 4\cos z \quad | \text{ From (3)}$$

Reduced equation is

$$\frac{d^2y}{dz^2} + y = 4\cos z$$

Auxiliary equation is $m^2 + 1 = 0 \quad \Rightarrow \quad m = \pm i$

$$\text{C.F.} = c_1\cos z + c_2\sin z$$

$$\text{P.I.} = \frac{1}{D^2+1}(4\cos z) = 4\cdot\frac{z}{2}\sin z = 2z\sin z$$

Complete solution is

$$y = c_1\cos z + c_2\sin z + 2z\sin z$$

$$y = c_1\cos\log(1+x) + c_2\sin\log(1+x) + 2\log(1+x)\sin\log(1+x).$$

Example 3. *Solve by changing the independent variable :*

$$\frac{d^2y}{dx^2} + (3\sin x - \cot x)\frac{dy}{dx} + 2y\sin^2 x = e^{-\cos x}\sin^2 x. \quad \text{(U.P.T.U. 2015)}$$

Sol. Choose z such that

$$\left(\frac{dz}{dx}\right)^2 = \sin^2 x \Rightarrow \quad \frac{dz}{dx} = \sin x \quad ...(1)$$

...(2)

Integration yields, $z = -\cos x$

From (1), $$\frac{d^2z}{dx^2} = \cos x$$

$$\therefore \quad P_1 = \frac{\cos x + (3\sin x - \cot x)\sin x}{\sin^2 x} = 3$$

$$Q_1 = \frac{2\sin^2 x}{\sin^2 x} = 2$$

$$R_1 = \frac{e^{-\cos x} . \sin^2 x}{\sin^2 x} = e^{-\cos x} = e^z$$

Reduced equation is, $\dfrac{d^2y}{dz^2} + 3\dfrac{dy}{dz} + 2y = e^z$

Auxiliary equation is $m^2 + 3m + 2 = 0 \Rightarrow m = -1, -2$

$$\therefore \quad \text{C.F.} = c_1 e^{-z} + c_2 e^{-2z}$$

$$\text{P.I.} = \frac{1}{D^2 + 3D + 2}(e^z) = \frac{e^z}{6}$$

Complete solution is, $y = c_1 e^{-z} + c_2 e^{-2z} + \dfrac{e^z}{6}$

$$\Rightarrow \quad y = c_1 e^{\cos x} + c_2 e^{2\cos x} + \frac{e^{-\cos x}}{6}$$

where c_1 and c_2 are arbitrary constants of integration.

Example 4. *Solve by changing the independent variable*

$$(1 + x^2)^2 \frac{d^2y}{dx^2} + 2x(1 + x^2)\frac{dy}{dx} + 4y = 0.$$

Sol. The given equation is

$$\frac{d^2y}{dx^2} + \frac{2x}{1+x^2}\frac{dy}{dx} + \frac{4}{(1+x^2)^2}y = 0 \quad \text{...(1)}$$

Here, $P = \dfrac{2x}{1+x^2}$, $Q = \dfrac{4}{(1+x^2)^2}$, $R = 0$

Choose z such that

$$\left(\frac{dz}{dx}\right)^2 = \frac{4}{(1+x^2)^2}$$

$$\Rightarrow \quad \frac{dz}{dx} = \frac{2}{1+x^2} \quad \text{...(2)}$$

Integration yields, $z = 2\tan^{-1} x$

From (2), $\dfrac{d^2z}{dx^2} = \dfrac{-4x}{(1+x^2)^2}$

$$\therefore \quad P_1 = \frac{\dfrac{-4x}{(1+x^2)^2} + \dfrac{2x}{1+x^2} \cdot \dfrac{2}{1+x^2}}{\dfrac{4}{(1+x^2)^2}} = 0$$

$$Q_1 = \frac{\dfrac{4}{(1+x^2)^2}}{\dfrac{4}{(1+x^2)^2}} = 1, \quad R_1 = \frac{0}{\left\{4/(1+x^2)^2\right\}} = 0$$

Reduced equation is $\dfrac{d^2y}{dz^2} + y = 0$

Auxiliary equation is $m^2 + 1 = 0 \Rightarrow m = \pm i$

$$\text{C.F.} = c_1 \cos z + c_2 \sin z$$

$$\text{P.I.} = 0$$

$\therefore$ $y = c_1 \cos z + c_2 \sin z$

Complete solution is $y = c_1 \cos (2 \tan^{-1} x) + c_2 \sin (2 \tan^{-1} x)$

where c_1 and c_2 are arbitrary constants of integration.

Example 5. *Solve by changing the independent variable*

$$\cos x \frac{d^2y}{dx^2} + \sin x \frac{dy}{dx} - 2y \cos^3 x = 2 \cos^5 x.$$

Sol. The given equation is

$$\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} - 2y \cos^2 x = 2 \cos^4 x \qquad ...(1)$$

Choose z such that

$$\left(\frac{dz}{dx}\right)^2 = \cos^2 x \Rightarrow \frac{dz}{dx} = \cos x \qquad ...(2)$$

Integration yields, $z = \sin x$

From (1), $\dfrac{d^2z}{dx^2} = -\sin x$

$\therefore$

$$P_1 = \frac{-\sin x + \tan x \cos x}{\cos^2 x} = 0$$

$$Q_1 = \frac{-2\cos^2 x}{\cos^2 x} = -2$$

$$R_1 = \frac{2\cos^4 x}{\cos^2 x} = 2 \cos^2 x = 2(1 - z^2)$$

Reduced equation is

$$\frac{d^2y}{dz^2} - 2y = 2(1 - z^2)$$

Auxiliary equation is

$$m^2 - 2 = 0 \Rightarrow m = \pm \sqrt{2}$$

$\therefore$ $\text{C.F.} = c_1 \cosh \sqrt{2} z + c_2 \sinh \sqrt{2} z$

$$\text{P.I.} = \frac{1}{D^2 - 2} [2(1 - z^2)] = \frac{2}{-2 + D^2} (1 - z^2) = -\left(1 - \frac{D^2}{2}\right)^{-1} (1 - z^2)$$

$$= -\left(1 + \frac{D^2}{2}\right)(1 - z^2) = -\left[1 - z^2 + \frac{1}{2}(-2)\right] = -[1 - z^2 - 1] = z^2$$

$$\therefore \qquad y = c_1 \cosh \sqrt{2}\, z + c_2 \sinh \sqrt{2}\, z + z^2$$

Complete solution is

$$y = c_1 \cosh\sqrt{2}\ (\sin x) + c_2 \sinh \sqrt{2}\ (\sin x) + \sin^2 x$$

where c_1 and c_2 are arbitrary constants of integration.

TEST YOUR KNOWLEDGE

Solve the following differential equations by changing the independent variable:

1. $\dfrac{d^2y}{dx^2} + \cot x \dfrac{dy}{dx} + 4y \operatorname{cosec}^2 x = 0$ (*U.K.T.U. 2012*)
2. $x \dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 4x^3 y = 8x^3 \sin x^2$
3. $x \dfrac{d^2y}{dx^2} + (4x^2 - 1) \dfrac{dy}{dx} + 4x^3 y = 2x^3$ (*M.T.U. 2013*)
4. $x^6 \dfrac{d^2y}{dx^2} + 3x^5 \dfrac{dy}{dx} + a^2 y = \dfrac{1}{x^2}$ (*U.P.T.U. 2014*)
5. $\dfrac{d^2y}{dx^2} - \cot x \dfrac{dy}{dx} - y \sin^2 x = 0$
6. $x^4 \dfrac{d^2y}{dx^2} + 2x^3 \dfrac{dy}{dx} + n^2 y = 0$ (*U.K.T.U. 2011*)
7. $\dfrac{d^2y}{dx^2} + \tan x \dfrac{dy}{dx} + y \cos^2 x = 0$
8. $\dfrac{d^2y}{dx^2} - \cot x \dfrac{dy}{dx} - y \sin^2 x = \cos x - \cos^3 x.$ (*U.P.T.U. 2013*)

Answers

1. $y = c_1 \cos \left(2 \log \tan \dfrac{x}{2}\right) + c_2 \sin \left(2 \log \tan \dfrac{x}{2}\right)$
2. $y = c_1 e^{x^2} + c_2 e^{-x^2} - \sin x^2$
3. $y = (c_1 + c_2 x^2)\, e^{-x^2} + \dfrac{1}{2}$
4. $y = c_1 \cos \left(\dfrac{a}{2x^2}\right) + c_2 \sin \left(\dfrac{a}{2x^2}\right) + \dfrac{1}{a^2 x^2}$
5. $y = c_1 e^{-\cos x} + c_2 e^{\cos x}$
6. $y = c_1 \cos \left(-\dfrac{n}{x}\right) + c_2 \sin \left(-\dfrac{n}{x}\right)$
7. $y = c_1 \sin (\sin x) + c_2 \cos (\sin x)$
8. $y = c_1 e^{-\cos x} + c_2 e^{\cos x} - \cos x.$

1.22.4 Method 4. To Find the Complete Solution of $y'' + Py' + Qy = R$ by the Method of Variation of Parameters

$$\frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = R \qquad ...(1)$$

Let the complementary function of (1) be

$$y = Au + Bv \qquad ...(2)$$

where u and v are the functions of x and A, B are constants.

$\because$ u and v are parts of C.F.

$$\therefore \qquad \frac{d^2u}{dx^2} + P \frac{du}{dx} + Qu = 0 \qquad ...(3)$$

and

$$\frac{d^2v}{dx^2} + P \frac{dv}{dx} + Qv = 0 \qquad ...(4)$$

Let the complete solution of (1) be

$$y = Au + Bv \qquad \text{...(5)}$$

where A and B are not constants but suitable functions of x to be so chosen that (5) satisfies (1). Now,

$$y_1 = Au_1 + Bv_1 + A_1u + B_1v$$

$$\Rightarrow \qquad y_1 = Au_1 + Bv_1 + (A_1u + B_1v) \qquad \text{...(6)}$$

Let us choose A and B such that

$$A_1u + B_1v = 0 \qquad \text{...(7)}$$

Now (6) becomes, $\quad y_1 = Au_1 + Bv_1 \qquad \text{...(8)}$

$$\therefore \qquad y_2 = A_1u_1 + Au_2 + B_1v_1 + Bv_2 \qquad \text{...(9)}$$

Substituting the values of y, y_1, y_2 from (5), (8) and (9) in (1) respectively, we get

$$(A_1u_1 + Au_2 + B_1v_1 + Bv_2) + P(Au_1 + Bv_1) + Q(Au + Bv) = R$$

$$\Rightarrow \qquad A_1u_1 + B_1v_1 + A(u_2 + Pu_2 + Qu) + B(v_2 + Pv_1 + Qv) = R$$

$$\Rightarrow \qquad A_1u_1 + B_1v_1 = R \qquad \text{...(10)}$$

| Using (3) and (4)

Solving (7) and (10) for A_1 and B_1, we get

$$A_1u + B_1v = 0$$

$$A_1u_1 + B_1v_1 - R = 0$$

$$\Rightarrow \qquad \frac{A_1}{-Rv} = \frac{B_1}{Ru} = \frac{1}{uv_1 - u_1v}$$

$$\Rightarrow \qquad A_1 = \frac{-Rv}{uv_1 - u_1v} = \phi(x) \qquad | \text{ say} \qquad \text{...(11)}$$

$$B_1 = \frac{Ru}{uv_1 - u_1v} = \psi(x) \qquad | \text{ say} \qquad \text{...(12)}$$

Integrating (11), we get $A = \int \phi(x)\,dx + a$ ...(13)

where a is an arbitrary constant of integration.

Integrating (12), we get $B = \int \psi(x)\,dx + b$ ...(14)

where b is also an arbitrary constant of integration.

Putting the above values in (5), we get

$$y = \left[\int \phi(x)\,dx + a\right] u + \left[\int \psi(x)\,dx + b\right] v$$

$$\Rightarrow \qquad y = u\int \phi(x)\,dx + v\int \psi(x)\,dx + au + bv$$

This gives the complete solution of (1).

Steps for solution:

1. Find out the parts of C.F.
2. Let them be u and v.
3. Consider $y = Au + Bv$ as the complete solution of equation given.

4. A and B are determined by the formulae

$$A = \int \frac{-Rv}{uv_1 - u_1 v} dx + c_1 \quad \text{and} \quad B = \int \frac{Ru}{uv_1 - u_1 v} dx + c_2$$

where c_1 and c_2 are the arbitrary constants of integration.

5. Write $y = Au + Bv$ as the complete solution.

Note. A and B can also be obtained by the following formulae

$$A = \int \frac{-Rv}{W} dx + a \quad \text{and} \quad B = \int \frac{Ru}{W} dx + b \quad \text{where } W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = uv_1 - u_1 v$$

Extension. The above method can be extended to linear equations of any order.

Consider, $$\frac{d^3y}{dx^3} + P\frac{d^2y}{dx^2} + Q\frac{dy}{dx} + Ry = S \quad ...(1)$$

Let $y = u, y = v, y = w$ be the solutions of the given equation when $S = 0$.

Let the complete solution of (1) be

$y = Au + Bv + Cw$ where A, B, C are some suitable functions of x.

The conditions to determine A, B, C are

$$A_1 u + B_1 v + C_1 w = 0 \quad ...(2)$$

$$A_1 u_1 + B_1 v_1 + C_1 w_1 = 0 \quad ...(3)$$

and $$A_1 u_2 + B_1 v_2 + C_1 w_2 = S \quad ...(4)$$

Solving (2), (3) and (4), we get A_1, B_1 and C_1 which by integration will give A, B and C.

As the solution is obtained by varying the arbitrary constants of the complementary function, the above method is known as that of **Variation of Parameters.**

ILLUSTRATIVE EXAMPLES

Example 1. *Solve by the method of variation of parameters:*

$$\frac{d^2y}{dx^2} + a^2 y = \sec ax. \qquad \text{(U.P.T.U. 2013, 2014, 2015)}$$

Sol. Here, $u = \cos ax$, $v = \sin ax$ are two parts of C.F.

Also, $R = \sec ax$.

Let the complete solution be

$$y = A \cos ax + B \sin ax \quad ...(1)$$

where A and B are suitable functions of x.

To determine the values of A and B, we have

$$A = \int \frac{-Rv}{uv_1 - u_1 v} dx + c_1$$

$$= \int \frac{-\sec ax \,.\, \sin ax}{\{\cos ax \,.\, a \cos ax - (-a \sin ax) \sin ax\}} dx + c_1$$

$$= -\int \frac{\tan ax}{a}\, dx + c_1$$

$$\Rightarrow \qquad A = \frac{1}{a^2} \log \cos ax + c_1$$

where c_1 is an arbitrary constant of integration.

$$B = \int \frac{Ru}{uv_1 - u_1 v}\, dx + c_2$$

$$= \int \frac{\sec ax \,.\, \cos ax}{\{\cos ax \,.\, a \cos ax - (-a \sin ax)\ \sin ax\}}\, dx + c_2$$

$$= \frac{1}{a}\int dx + c_2 = \frac{x}{a} + c_2$$

where c_2 is an arbitrary constant of integration.

Hence the complete solution is given by

$$y = A \cos ax + B \sin ax$$

$$= \left(\frac{\log \cos ax}{a^2} + c_1\right) \cos ax + \left(\frac{x}{a} + c_2\right) \sin ax.$$

Example 2. *Solve by method of variation of parameters:*

$$\frac{d^2 y}{dx^2} - 2\frac{dy}{dx} = e^x \sin x.$$

Sol. Parts of C.F. are 1 and e^{2x}.

Let $\quad u = 1, v = e^{2x} \quad$ Also, $\quad R = e^x \sin x$

Let $\quad y = A + Be^{2x}$ be the complete solution

where A and B are suitable functions of x determined by

$$A = \int \frac{-Rv}{uv_1 - u_1 v}\, dx + c_1 = -\int \frac{e^x \sin x \,.\, e^{2x}}{1.\, 2\, e^{2x}}\, dx + c_1$$

$$= -\frac{1}{2}\int e^x \sin x\, dx + c_1 = -\frac{1}{2}\left[\frac{e^x}{1+1}(\sin x - \cos x)\right] + c_1$$

$$= -\frac{e^x}{4}(\sin x - \cos x) + c_1$$

where c_1 is an arbitrary constant of integration.

$$B = \int \frac{Ru}{uv_1 - u_1 v}\, dx + c_2 = \int \frac{e^x \sin x \,.\, 1}{1.\, 2e^{2x}}\, dx + c_2$$

$$= \frac{1}{2}\int e^{-x} \sin x\, dx + c_2 = \frac{1}{2}\left[\frac{e^{-x}}{1+1}(-\sin x - \cos x)\right] + c_2$$

$$= -\frac{e^{-x}}{4}(\sin x + \cos x) + c_2$$

where c_2 is an arbitrary constant of integration.

The complete solution is

$$y = A + B e^{2x}$$

$$= \frac{e^x}{4}(\cos x - \sin x) + c_1 + \left[-\frac{e^{-x}}{4}(\sin x + \cos x) + c_2 \right] e^{2x}$$

$$= \frac{e^x}{4}(\cos x - \sin x) + c_1 - \frac{e^x}{4}(\sin x + \cos x) + c_2 e^{2x}$$

$$\Rightarrow \quad y = c_1 + c_2 e^{2x} - \frac{e^x}{2} \sin x$$

Example 3. *Apply the method of variation of parameters to solve the ordinary differential equations:*

(*i*) $\frac{d^2 y}{dx^2} + y = \tan x$ (U.P.T.U. 2015; U.K.T.U. 2011)

(*ii*) $(D^2 - 1)y = 2(1 - e^{-2x})^{-1/2}$. [M.T.U. 2011]

Sol. (*i*) Parts of C.F. are $u = \cos x$ and $v = \sin x$

Let $y = A \cos x + B \sin x$ be the complete solution of the given equation where A and B are determined as:

$$A = -\int \frac{Rv}{uv_1 - u_1 v} dx + c_1$$

$$= -\int \frac{\tan x . \sin x}{\cos^2 x + \sin^2 x} dx + c_1 \qquad | \because R = \tan x$$

$$= -\int \frac{1 - \cos^2 x}{\cos x} dx + c_1 = -\int (\sec x - \cos x)\, dx + c_1$$

$$\Rightarrow \quad A = \sin x - \log(\sec x + \tan x) + c_1$$

and

$$B = \int \frac{Ru}{uv_1 - u_1 v} dx + c_2 = \int \frac{\tan x . \cos x}{1} dx + c_2$$

$$\Rightarrow \quad B = -\cos x + c_2$$

Hence the complete solution is

$$y = A \cos x + B \sin x$$

$$= [\sin x - \log(\sec x + \tan x) + c_1] \cos x + (-\cos x + c_2) \sin x$$

$$\Rightarrow \quad y = c_1 \cos x + c_2 \sin x - \cos x \log(\sec x + \tan x)$$

where c_1 and c_2 are arbitrary constants of integration.

(*ii*) Parts of C.F. are $u = e^x$ and $v = e^{-x}$

Let $y = A e^x + B e^{-x}$ be the complete solution of the given equation where A and B are determined as:

$$A = -\int \frac{Rv}{uv_1 - u_1 v} dx + c_1$$

$$= -\int \frac{2(1-e^{-2x})^{-1/2} \cdot e^{-x}}{e^x(-e^{-x}) - e^x \cdot e^{-x}}\, dx + c_1 = -2\int \frac{e^{-x}}{-2\sqrt{1-e^{-2x}}}\, dx + c_1$$

$$= \int \frac{e^{-x}}{\sqrt{1-e^{-2x}}}\, dx + c_1 \qquad \Big|\ \text{Put } e^{-x} = t \quad \therefore \quad e^{-x}\,dx = -dt$$

$$= -\int \frac{dt}{\sqrt{1-t^2}} + c_1$$

$\Rightarrow$ $$A = -\sin^{-1}(e^{-x}) + c_1$$

and $$B = \int \frac{Ru}{uv_1 - u_1v}\, dx + c_2$$

$$= \int \frac{2(1-e^{-2x})^{-1/2} \cdot e^x}{(-2)}\, dx + c_2 = -\int \frac{e^x}{\sqrt{1-e^{-2x}}}\, dx + c_2$$

$$= -\int \frac{e^{2x}}{\sqrt{e^{2x}-1}}\, dx + c_2 \qquad \Big|\ \text{Put } e^{2x} = t \Rightarrow e^{2x}\,dx = \frac{dt}{2}$$

$$= -\frac{1}{2}\int \frac{dt}{\sqrt{t-1}} + c_2 = -\frac{1}{2}\frac{(t-1)^{1/2}}{(1/2)} + c_2$$

$\Rightarrow$ $$B = -(e^{2x}-1)^{1/2} + c_2$$

Hence the complete solution is

$$y = A e^x + B e^{-x}$$

$$y = [-\sin^{-1}(e^{-x}) + c_1]\, e^x + [-(e^{2x}-1)^{1/2} + c_2]\, e^{-x}$$

$\Rightarrow$ $$y = c_1 e^x + c_2 e^{-x} - e^x \sin^{-1}(e^{-x}) - e^{-x}(e^{2x}-1)^{1/2}$$

where c_1 and c_2 are arbitrary constants of integration.

Example 4. *Solve by the method of variation of parameters:*

$$\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}. \qquad \text{[G.B.T.U. (SUM) 2010]}$$

Sol. Here $u = e^x$, $v = e^{-x}$ which are parts of C.F. Also, $R = \dfrac{2}{1+e^x}$

Let $y = Ae^x + Be^{-x}$ be the complete solution of given equation, where A and B are determined as:

$$A = \int \frac{-Rv}{uv_1 - u_1v}\, dx + c_1 = \int \frac{-2 \cdot e^{-x}}{(1+e^x)\{e^x(-e^{-x}) - e^x \cdot e^{-x}\}}\, dx + c_1$$

$$= -2\int \frac{e^{-x}}{(1+e^x)(-2)}\, dx + c_1 = \int \frac{e^{-x}}{1+e^x}\, dx + c_1$$

$$= \int \frac{e^{-2x}}{e^{-x}+1}\, dx + c_1 = \log\left(\frac{1+e^x}{e^x}\right) - e^{-x} + c_1$$

$$B = \int \frac{Ru}{uv_1 - u_1v}\, dx + c_2 = \int \frac{2e^x}{(1+e^x)\{e^x(-e^{-x}) - e^x \cdot e^{-x}\}}\, dx + c_2$$

$$= -\int \frac{e^x}{1+e^x}\,dx + c_2 = -\log(1+e^x) + c_2$$

Hence complete solution is

$$y = \left[\log\left(\frac{1+e^x}{e^x}\right) - e^{-x} + c_1\right] e^x + [-\log(1+e^x) + c_2]\, e^{-x} \qquad | \because \; y = Au + Bv$$

where c_1 and c_2 are arbitrary constants of integration.

Example 5. *Solve by the method of variation of parameters:*

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \frac{e^x}{1+e^x}.$$ [M.T.U. (SUM) 2011]

Sol. Here, parts of C.F. are $u = e^x$, $v = e^{2x}$. Also, $R = \dfrac{e^x}{1+e^x}$

Let $y = Ae^x + Be^{2x}$ be the complete solution of the given equation where A and B are suitable functions of x.

To determine A and B, we have

$$A = \int \frac{-Rv}{uv_1 - u_1v}\,dx + c_1 = -\int \frac{e^x \cdot e^{2x}}{(1+e^x)(2e^{3x} - e^{3x})}\,dx + c_1$$

$$= -\int \frac{1}{1+e^x}\,dx + c_1 = -\int \frac{e^{-x}}{e^{-x}+1}\,dx + c_1 = \log(e^{-x}+1) + c_1$$

$$B = \int \frac{Ru}{uv_1 - u_1v}\,dx + c_2 = \int \frac{e^x \cdot e^x}{(1+e^x)\,e^{3x}}\,dx + c_2$$

$$= \int \frac{e^{-x}}{1+e^x}\,dx + c_2 = \int \frac{e^{-2x}}{e^{-x}+1}\,dx + c_2 = \log(1+e^{-x}) - (1+e^{-x}) + c_2$$

Hence the complete solution is

$$y = [\log(e^{-x}+1) + c_1]\, e^x + [\log(1+e^{-x}) - (1+e^{-x}) + c_2]\, e^{2x}$$

Example 6. *Solve by method of variation of parameters:*

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{-x}\log x.$$

Sol. Parts of C.F. are $u = e^{-x}$, $v = xe^{-x}$ and $R = e^{-x}\log x$

Let $y = Ae^{-x} + Bxe^{-x}$ be the complete solution where A and B are some suitable functions of x. To determine A and B, we have

$$A = -\int \frac{Rv}{uv_1 - u_1v}\,dx + c_1 = -\int \frac{e^{-x}\log x \cdot xe^{-x}}{e^{-x}(e^{-x} - xe^{-x}) + xe^{-2x}}\,dx + c_1$$

$$= -\int x \log x \, dx + c_1 = -\frac{x^2}{2}\log x + \frac{x^2}{4} + c_1$$

$$B = \int \frac{Ru}{uv_1 - u_1 v}\, dx + c_2 = \int \frac{e^{-x}\log x \,.\, e^{-x}}{e^{-2x}}\, dx + c_2$$

$$= \int \log x \, dx + c_2 = x \log x - x + c_2$$

Hence the complete solution is

$$y = Ae^{-x} + Bxe^{-x} = \left(-\frac{x^2}{2}\log x + \frac{x^2}{4} + c_1\right)e^{-x} + (x \log x - x + c_2)\, xe^{-x}$$

Example 7. *Use the variation of parameter method to solve the differential equation* $x^2y'' + xy' - y = x^2e^x$. (A.K.T.U. 2018)

Sol. The given equation is

$$x^2y'' + xy' - y = x^2e^x \quad ...(1)$$

$$\Rightarrow \quad y'' + \frac{y'}{x} - \frac{y}{x^2} = e^x \quad ...(2)$$

Here, $R = e^x$

Consider the equation $y'' + \dfrac{y'}{x} - \dfrac{y}{x^2} = 0$ for finding parts of C.F.

Put $x = e^z$ so that $z = \log x$ and Let $D \equiv \dfrac{d}{dz}$ then the above equation reduces to

$$[D(D-1) + D - 1]\, y = 0$$

$$\Rightarrow \quad (D^2 - 1)\, y = 0 \quad ...(3)$$

Auxiliary equation is $\quad m^2 - 1 = 0 \quad \Rightarrow \quad m = \pm 1$

$$\therefore \quad \text{C.F.} = c_1e^z + c_2e^{-z} = c_1x + c_2 \,.\, \frac{1}{x}$$

Hence parts of C.F. are x and $\dfrac{1}{x}$

Let $\quad u = x \quad$ and $\quad v = \dfrac{1}{x}$

Let $y = Ax + \dfrac{B}{x}$ be the complete solution, where A and B are some suitable functions of x. A and B are determined as follows:

$$A = -\int \frac{Rv}{uv_1 - u_1v}\, dx + c_1 = -\int \frac{e^x \,.\, \frac{1}{x}}{x\,.\left(\frac{-1}{x^2}\right) - 1\,.\left(\frac{1}{x}\right)}\, dx + c_1$$

$$= -\int \frac{e^x \cdot \frac{1}{x}}{\left(\frac{-2}{x}\right)} dx + c_1 = \frac{1}{2} e^x + c_1$$

and
$$B = \int \frac{Ru}{uv_1 - u_1 v} dx + c_2 = \int \frac{e^x \cdot x}{x\left(\frac{-1}{x^2}\right) - 1\left(\frac{1}{x}\right)} dx + c_2$$

$$= \int \frac{e^x \cdot x}{\left(\frac{-2}{x}\right)} dx + c_2 = -\frac{1}{2}\int x^2 e^x \, dx + c_2$$

$$= -\frac{1}{2}\left[x^2 e^x - \int 2x\, e^x \, dx\right] + c_2 = -\frac{1}{2}[x^2 e^x - 2(x-1)e^x] + c_2$$

$$= -\frac{1}{2} x^2 e^x + (x-1)e^x + c_2$$

Hence the complete solution is given by

$$y = Ax + \frac{B}{x} = \left(\frac{1}{2}e^x + c_1\right)x + \left[-\frac{1}{2}x^2 e^x + (x-1)e^x + c_2\right] \cdot \frac{1}{x}$$

$$\Rightarrow \qquad y = c_1 x + \frac{c_2}{x} + \left(1 - \frac{1}{x}\right)e^x$$

where c_1 and c_2 are arbitrary constants of integration.

Example 8. *Using variation of parameters method, solve:*

$$x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 12y = x^3 \log x.$$

Sol. Consider the equation

$$x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 12y = 0 \text{ for finding parts of C.F.}$$

Put $x = e^z$ so that $z = \log x$ and Let $D \equiv \frac{d}{dz}$ then the given equation reduces to

$$[D(D-1) + 2D - 12]\, y = 0$$

$$\Rightarrow \qquad (D^2 + D - 12)\, y = 0$$

Auxiliary equation is

$$m^2 + m - 12 = 0 \quad \Rightarrow \quad m = 3, -4$$

$$\therefore \qquad \text{C.F.} = c_1 e^{3z} + c_2 e^{-4z} = c_1 x^3 + c_2 x^{-4}$$

Hence, parts of C.F. are x^3 and x^{-4}

Let $u = x^3$ and $v = x^{-4}$. Also, $R = x \log x$

Let $y = Au + Bv$ be the complete solution, where A and B are some suitable functions of x. A and B are determined as follows:

$$A = -\int \frac{Rv}{uv_1 - u_1 v} dx + c_1 = -\int \frac{x \log x \cdot x^{-4}}{x^3 \cdot (-4x^{-5}) - 3x^2(x^{-4})} dx + c_1$$

$$= -\int \frac{x^{-3}\log x}{-7x^{-2}}\,dx + c_1 = \frac{1}{7}\int \frac{\log x}{x}\,dx + c_1 = \frac{1}{14}(\log x)^2 + c_1$$

and
$$B = \int \frac{Ru}{uv_1 - u_1v}\,dx + c_2 = \int \frac{x\log x \,.\, x^3}{-7x^{-2}}\,dx + c_2$$

$$= -\frac{1}{7}\int x^6 \log x \; dx + c_2 = -\frac{1}{7}\left[\log x \,.\, \frac{x^7}{7} - \int \frac{1}{x}\cdot\frac{x^7}{7}\,dx\right] + c_2$$

$$= -\frac{1}{7}\left[\frac{x^7\log x}{7} - \frac{1}{7}\left(\frac{x^7}{7}\right)\right] + c_2 = \frac{x^7}{49}\left(\frac{1}{7} - \log x\right) + c_2$$

Hence the complete solution is given by

$$y = Ax^3 + Bx^{-4} = \left[\frac{1}{14}(\log x)^2 + c_1\right]x^3 + \left[\frac{x^7}{49}\left(\frac{1}{7} - \log x\right) + c_2\right]x^{-4}$$

where c_1 and c_2 are arbitrary constants of integration.

Example 9. *By the method of variation of parameters, solve the differential equation*

$$\frac{d^2y}{dx^2} + (1 - \cot x)\frac{dy}{dx} - y\cot x = \sin^2 x.$$

Sol. Take $\quad y'' + (1 - \cot x)\,y' - y\cot x = 0 \qquad \text{...(1)}$

Obviously, $y = e^{-x}$ is a part of C.F.

Let $y = ve^{-x}$ be the complete solution of (1)

$$\therefore \quad \frac{dy}{dx} = e^{-x}\frac{dv}{dx} - ve^{-x}$$

$$\therefore \quad \frac{d^2y}{dx^2} = ve^{-x} - 2e^{-x}\frac{dv}{dx} + e^{-x}\frac{d^2v}{dx^2}$$

Substituting above values in equation (1), we get

$$\frac{d^2v}{dx^2} - (1 + \cot x)\frac{dv}{dx} = 0$$

$$\Rightarrow \quad \frac{dp}{dx} - (1 + \cot x)\,p = 0, \quad \text{where } p = \frac{dv}{dx}.$$

Its solution is given by

$$p = c_1 e^x \sin x$$

$$\therefore \quad \frac{dv}{dx} = c_1 e^x \sin x$$

Integration gives, $\quad v = c_1\dfrac{e^x}{2}(\sin x - \cos x) + c_2$

Hence complete solution of (1) is

$$y = ve^{-x} = \frac{c_1}{2}(\sin x - \cos x) + c_2 e^{-x} \qquad \text{...(2)}$$

Hence $u = e^{-x}$ and $v = \sin x - \cos x$ are the two parts of C.F.

To determine the values of A and B, we have

$$A = \int \frac{-\sin^2 x\,(\sin x - \cos x)}{e^{-x}(\cos x + \sin x) + e^{-x}(\sin x - \cos x)}\,dx + c_1$$

$$= -\int \frac{\sin^2 x\,(\sin x - \cos x)}{2e^{-x}\sin x}\,dx + c_1 = \frac{1}{2}\int e^x \sin x\,(\cos x - \sin x)\,dx + c_1$$

$$= \frac{e^x}{20}(3\sin 2x - \cos 2x - 5) + c_1$$

where c_1 is an arbitrary constant of integration.

$$B = \int \frac{\sin^2 x\, e^{-x}}{e^{-x}(\cos x + \sin x) + e^{-x}(\sin x - \cos x)}\,dx + c_2$$

$$= \frac{1}{2}\int \sin x\,dx + c_2 = \frac{-\cos x}{2} + c_2$$

where c_2 is an arbitrary constant of integration.

Hence the complete solution of the given differential equation is given by

$$y = Ae^{-x} + B(\sin x - \cos x)$$

$$= \left[\frac{e^x}{20}(3\sin 2x - \cos 2x - 5) + c_1\right]e^{-x} + \left(-\frac{\cos x}{2} + c_2\right)(\sin x - \cos x).$$

Example 10. *Apply the method of variation of parameters to solve*

$$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = e^{2x}.$$

Sol. Here, $u = e^x$, $v = e^{2x}$, $w = e^{3x}$ which are parts of C.F.

Let $y = Ae^x + Be^{2x} + Ce^{3x}$ be the complete solution of the given equation, where A, B and C are the suitable functions of x.

To determine the values of A, B and C, we have the equations

$$A_1(e^x) + B_1(e^{2x}) + C_1(e^{3x}) = 0 \quad \text{...(1)}$$

$$A_1(e^x) + B_1(2e^{2x}) + C_1(3e^{3x}) = 0 \quad \text{...(2)}$$

$$A_1(e^x) + B_1(4e^{2x}) + C_1(9e^{3x}) = e^{2x} \quad \text{...(3)}$$

From (1) and (2), $\dfrac{A_1}{e^{5x}} = \dfrac{B_1}{-2e^{4x}} = \dfrac{C_1}{e^{3x}} = \lambda$ (say)

Substituting the values of A_1, B_1, C_1 in (3), we get

$$e^{2x} = \lambda(e^{6x} - 8e^{6x} + 9e^{6x}) = 2\lambda e^{6x}$$

$\Rightarrow$

$$\lambda = \frac{1}{2}e^{-4x}$$

$\therefore$

$$A_1 = \frac{1}{2}e^x,\ B_1 = -1,\ C_1 = \frac{1}{2}e^{-x}$$

Integrating, $A = \frac{1}{2}e^x + a,\quad B = -x + b,\quad C = -\frac{1}{2}e^{-x} + c$

$\therefore$ The complete solution is

$$y = \frac{1}{2}e^{2x} + ae^x - xe^{2x} + be^{2x} - \frac{1}{2}e^{2x} + ce^{3x} = ae^x + be^{2x} + ce^{3x} - xe^{2x}$$

where a, b and c are arbitrary constants of integration.

TEST YOUR KNOWLEDGE

Solve the following differential equations by the method of variation of parameters:

1. $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$
2. $y_2 + 4y = 4\tan 2x$
3. $\frac{d^2y}{dx^2} + y = x$
4. $\frac{d^2y}{dx^2} + y = \sec x \tan x$
5. $y_2 - 3y_1 + 2y = e^{2x} + x^2$ (*U.P.T.U. 2014*)
6. $(D^2 + 1)y = \tan^2 x$
7. (*i*) $(D^2 + 1)y = \operatorname{cosec} x \cot x$ (*ii*) $x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 12y = 0$ (*A.K.T.U. 2016*)
8. (*i*) $\frac{d^2y}{dx^2} + y = x\cos x$ (*ii*) $\frac{d^2y}{dx^2} + y = x\sin x$
9. (*i*) $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = \sin(\log x)$ (*ii*) $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = x\log x$
10. (*i*) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = e^x\tan x$ (*ii*) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x\log x$
11. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \sin e^{-x}$ (*M.T.U. 2012*)
12. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 3x\,e^{-x}$
13. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \frac{e^{3x}}{x^2}$ (*A.K.T.U. 2017*)
14. $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x$ (*G.B.T.U. 2012*)
15. $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = x^2 + \frac{1}{x^2}$
16. $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 9y = 48x^5$. [*G.B.T.U. (C.O.) 2010*]
17. $\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 3x = \frac{e^t}{1+e^t}$ (*G.B.T.U. 2013*)

Answers

1. $y = (a - x)\cos x + (b + \log \sin x)\sin x$
2. $y = c_1\cos 2x + b\sin 2x - \cos 2x \log\tan\left(\frac{\pi}{4} + x\right)$
3. $y = c_1\cos x + c_2\sin x + x$
4. $y = c_1\cos x + c_2\sin x + x\cos x + \sin x\log\sec x - \sin x$

5. $y = c_1 e^x + c_2 e^{2x} + xe^{2x} + \frac{3}{2}x + \frac{7}{4} + \frac{1}{2}x^2 - e^{2x}$

6. $y = c_1 \cos x + c_2 \sin x - \cos x (\sec x + \cos x) + \sin x \log (\sec x + \tan x) + \sin^2 x.$

7. (i) $y = c_1 \cos x + c_2 \sin x - \cos x \log \sin x - x \sin x - \sin x \cot x$

(ii) $y = c_1 x^3 + c_2 x^{-4}$

8. (i) $y = c_1 \cos x + \left(c_2 - \frac{1}{8}\right) \sin x + \frac{x^2}{4} \sin x + \frac{x}{4} \cos x$

(ii) $y = c_1 \cos x + c_2 \sin x + \frac{x}{4} \sin x - \frac{x^2}{4} \cos x$

9. (i) $y = c_1 x^2 + c_2 x^3 + \frac{1}{10} (\sin \log x + \cos \log x)$ (ii) $y = c_1 x \log x + c_2 x + \frac{1}{6} x (\log x)^3$

10. (i) $y = e^x (c_1 \cos x + c_2 \sin x) - e^x \cos x \log (\sec x + \tan x)$

(ii) $y = (c_1 + c_2 x) e^x + x^2 e^x \left(\frac{1}{2} \log x - \frac{3}{4}\right)$

11. $y = c_1 e^x + c_2 e^{2x} - e^{2x} \sin e^{-x}$

12. $y = c_1 e^{-2x} + c_2 x e^{-2x} + \frac{1}{2} x^3 e^{-2x}$

13. $y = (c_1 x + c_2) e^{3x} - e^{3x} \log x$

14. $y = (e^x + c_1) \frac{1}{x} + [(1 - x) e^x + c_2] \frac{1}{x^2}$

15. $y = \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{1}{12} x^2 - \frac{1}{x^2} \log x$

16. $y = (4x^2 + c_1) x^3 + (c_2 - x^8) x^{-3}.$

17. $x = \left[\frac{1}{2} \log (e^{-t} + 1) + c_1\right] e^t + \left[-\frac{1}{4} (e^{-t} + 1)^2 - \frac{1}{2} \log (e^{-t} + 1) + (e^{-t} + 1) + c_2\right] e^{3t}$

1.23 SERIES SOLUTIONS

The solution of ordinary linear differential equations of second order with variable coefficients in the form of an infinite convergent series is called *solution in series* or *integration in series.*

The series solution of certain differential equations give rise to *special functions* such as Bessel's function, Legendre's polynomials, Laguerre's polynomial, Hermite's polynomial, Chebyshev polynomials. These special functions have wide applications in engineering.

1.24 DEFINITIONS

1.24.1 Power Series

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \ldots$$

is called a power series in ascending powers of $x - x_0$.

In particular, a power series in ascending powers of x is an infinite series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$

e.g.,
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

1.24.2 Analytic Function

A function $f(x)$ defined on an interval containing the point $x = x_0$ is called analytic at x_0 if its Taylor series $\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$ exists and converges to $f(x)$ for all x in the interval of convergence of Taylor's series.

Note 1. A rational function is analytic except at those values of x at which its denominator is zero. *e.g.*, Rational function $\frac{x}{x^2 - 5x + 6}$ is analytic everywhere except at $x = 2$ and $x = 3$.

Note 2. All polynomial functions e^x, $\sin x$, $\cos x$, $\sinh x$ and $\cosh x$ are analytic everywhere.

1.24.3 Ordinary Point (*M.T.U. 2013*)

A point $x = x_0$ is called an ordinary point of the equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\, y = 0 \qquad \ldots(1)$$

if both the functions $P(x)$ and $Q(x)$ are analytic at $x = x_0$.

1.24.4 Regular and Irregular Singular Points (*M.T.U. 2013*)

If the point $x = x_0$ is not an ordinary point of the differential equation (1), then it is called a singular point of equation (1). There are two types of singular points:

(*i*) Regular singular point.

(*ii*) Irregular singular point.

A singular point $x = x_0$ of the differential equation (1) is called a regular singular point of (1) if both $(x - x_0)\, P(x)$ and $(x - x_0)^2\, Q(x)$ are analytic at $x = x_0$.

A singular point which is not regular is called an irregular singular point.

Remark 1. When $x = 0$ is an ordinary point of equation (1), its every solution can be expressed as a series of the form

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots = \sum_{n=0}^{\infty} a_n x^n .$$

Remark 2. When $x = 0$ is a regular singular point of equation (1), at least one of its solution can be expressed as

$$y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \ldots = x^m (a_0 + a_1 x + a_2 x^2 + \ldots) = \sum_{n=0}^{\infty} a_n x^{m+n}$$

where m may be a positive or negative integer or a fraction.

Remark 3. If $x = 0$ is an irregular singular point of equation (1), then discussion of solution of the equation is beyond the scope of this book.

In particular, a power series in ascending powers of x is an infinite series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$

e.g.,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

1.24.2 Analytic Function

A function $f(x)$ defined on an interval containing the point $x = x_0$ is called analytic at x_0 if its Taylor series $\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$ exists and converges to $f(x)$ for all x in the interval of convergence of Taylor's series.

Note 1. A rational function is analytic except at those values of x at which its denominator is zero. *e.g.,* Rational function $\frac{x}{x^2 - 5x + 6}$ is analytic everywhere except at $x = 2$ and $x = 3$.

Note 2. All polynomial functions e^x, $\sin x$, $\cos x$, $\sinh x$ and $\cosh x$ are analytic everywhere.

1.24.3 Ordinary Point

(M.T.U. 2013)

A point $x = x_0$ is called an ordinary point of the equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0 \qquad \ldots(1)$$

if both the functions $P(x)$ and $Q(x)$ are analytic at $x = x_0$.

1.24.4 Regular and Irregular Singular Points

(M.T.U. 2013)

If the point $x = x_0$ is not an ordinary point of the differential equation (1), then it is called a singular point of equation (1). There are two types of singular points:

(*i*) Regular singular point.

(*ii*) Irregular singular point.

A singular point $x = x_0$ of the differential equation (1) is called a regular singular point of (1) if both $(x - x_0)\,P(x)$ and $(x - x_0)^2\,Q(x)$ are analytic at $x = x_0$.

A singular point which is not regular is called an irregular singular point.

Remark 1. When $x = 0$ is an ordinary point of equation (1), its every solution can be expressed as a series of the form

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots = \sum_{n=0}^{\infty} a_n x^n.$$

Remark 2. When $x = 0$ is a regular singular point of equation (1), at least one of its solution can be expressed as

$$y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \ldots = x^m (a_0 + a_1 x + a_2 x^2 + \ldots) = \sum_{n=0}^{\infty} a_n x^{m+n}$$

where m may be a positive or negative integer or a fraction.

Remark 3. If $x = 0$ is an irregular singular point of equation (1), then discussion of solution of the equation is beyond the scope of this book.

1.25 SOME IMPORTANT DIFFERENTIAL EQUATIONS

(*i*) **Legendre's differential equation**

The differential equation of the form

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)\,y = 0$$

or

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\} + n(n+1)\,y = 0$$

is called Legendre's differential equation, where n is a real number.

(*ii*) **Bessel's differential equation**

The differential equation of the form

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)\,y = 0$$

is called Bessel's differential equation of order n, where n is a positive constant.

(*iii*) **Chebyshev's differential equation**

The differential equation of the form

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2y = 0$$

is called Chebyshev's differential equation.

1.26 POWER SERIES SOLUTION, WHEN $x = 0$ IS AN ORDINARY POINT OF THE EQUATION $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0$

Steps for solution:

1. Assume its solution to be of the form $y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$ ...(1)
2. Find $\frac{dy}{dx}$ (or y') and $\frac{d^2y}{dx^2}$ (or y'') from y.
3. Substitute the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given differential equation.
4. Equate to zero the coefficients of various powers of x and find $a_2, a_3, a_4, a_5, \ldots$ in terms of a_0 and a_1.
5. Equate to zero, the coefficient of x^n. The relation so obtained is called the *recurrence relation*. It helps us in finding the values of other constants easily.
6. Give different values to n in the recurrence relation to determine various a_i's in terms of a_0 and a_1.
7. Substitute the values of $a_2, a_3, a_4, \ldots$ in assumed solution (1) above to get the series solution of the given equation having a_0 and a_1 as arbitrary constants.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve in series the differential equation*

$$\frac{d^2y}{dx^2} + xy = 0. \qquad \text{[G.B.T.U. (SUM) 2010]}$$

Sol. Comparing the given equation with the form $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0$,

we get $\quad P(x) = 0,\ Q(x) = x$

At $x = 0$, Both $P(x)$ and $Q(x)$ are analytic, hence $x = 0$ is an *ordinary point.*

Assume its solution to be

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots \qquad \ldots(1)$$

Then, $\quad \frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots + n\,a_nx^{n-1} + \ldots$

and $\quad \frac{d^2y}{dx^2} = 2.1\,a_2 + 3.2\,a_3x + 4.3\,a_4x^2 + 5.4\,a_5x^3 + \ldots + n(n-1)\,a_nx^{n-2} + \ldots$

Substituting these values in the given differential equation, we get

$$[2.1.\,a_2 + 3.2\,.\,a_3x + 4.3\,.\,a_4x^2 + 5.4\,.\,a_5x^3 + \ldots + n(n-1)\,a_nx^{n-2} + \ldots]$$
$$+ x\,[a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots] = 0$$

$$2.1\,.\,a_2 + (3.2\,a_3 + a_0)\,x + (4.3\,.\,a_4 + a_1)x^2 + (5.4\,.\,a_5 + a_2)x^3 + \ldots$$
$$+ \{(n+2)\,(n+1)\,a_{n+2} + a_{n-1}\}x^n + \ldots = 0$$

Equating to zero, the various powers of x as,

Coefficient of $x^0 = 0$

$\Rightarrow \quad 2.1\,.\,a_2 = 0 \quad \Rightarrow \quad \boxed{a_2 = 0}$

Coefficient of $x = 0$

$\Rightarrow \quad 3.2\,.\,a_3 + a_0 = 0$

$\Rightarrow \quad a_3 = -\frac{a_0}{3.2} \quad \Rightarrow \quad \boxed{a_3 = -\frac{a_0}{3!}}$

Coefficient of $x^2 = 0$

$\Rightarrow \quad 4.3\,.\,a_4 + a_1 = 0$

$\Rightarrow \quad a_4 = -\frac{a_1}{4.3} \quad$ or $\quad \boxed{a_4 = -\frac{2a_1}{4!}}$

Coefficient of $x^3 = 0$

$\Rightarrow \quad 5.4\,.\,a_5 + a_2 = 0$

$\Rightarrow \quad a_5 = -\frac{a_2}{5.4} \quad$ or $\quad \boxed{a_5 = 0}$

Coefficient of $x^4 = 0$

$\Rightarrow \quad 6.5\,.\,a_6 + a_3 = 0$

$\Rightarrow \quad a_6 = -\frac{a_3}{6.5} = \frac{a_0}{6.5.3!} \quad$ or $\quad \boxed{a_6 = \frac{4a_0}{6!}}$

and so on.

Coefficient of $x^n = 0$

$$\Rightarrow \quad (n+2)(n+1)a_{n+2} + a_{n-1} = 0$$

$$\Rightarrow \quad \boxed{a_{n+2} = -\frac{a_{n-1}}{(n+2)(n+1)}}$$

which is the recurrence relation.

Putting $n = 5, 6, 7, \ldots$, successively in recurrence relation, we obtain

$$a_7 = \frac{5.2a_1}{7!}, a_8 = 0, a_9 = \frac{-7.4}{9!} a_0 \text{ and so on.}$$

Substituting these values in (1), we get

$$y = a_0 + a_1 x - \frac{a_0}{3!}x^3 - \frac{2a_1}{4!}x^4 + \frac{4a_0}{6!}x^6 + \frac{5.2\,a_1}{7!}x^7 - \frac{7.4}{9!}a_0 x^9 + \ldots$$

$$\Rightarrow \quad y = a_0\left[1 - \frac{x^3}{3!} + \frac{1.4}{6!}x^6 - \frac{1.4.7}{9!}x^9 + \ldots\right] + a_1\left[x - \frac{2}{4!}x^4 + \frac{2.5}{7!}x^7 - \ldots\right]$$

where a_0 and a_1 are constants.

Example 2. *Solve in series the differential equation*

$$(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0 \text{ about the point } x = 0.$$

Sol. Comparing the given differential equation with the form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0, \text{ we get}$$

$$P(x) = \frac{x}{1+x^2} \quad \text{and} \quad Q(x) = \frac{-1}{1+x^2}.$$

Both $P(x)$ and $Q(x)$ are analytic at $x = 0$

$\therefore$ $x = 0$ is an *ordinary point* of the given differential equation.

Assume the solution to be

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots \qquad \ldots(1)$$

Then,
$$\frac{dy}{dx} = a_1 + 2a_2 x + 3a_3 x^2 + \ldots + na_n x^{n-1} + \ldots$$

and
$$\frac{d^2y}{dx^2} = 2.1.\,a_2 + 3.2.\,a_3 x + \ldots + n(n-1)a_n\,x^{n-2} + \ldots$$

Substituting these values in given equation, we get

$$(1+x^2)\,[2.1.\,a_2 + 3.2.\,a_3 x + 4.3.a_4 x^2 + \ldots + n(n-1)\,a_n x^{n-2} + \ldots]$$
$$+ x\,[a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots + n\,a_n x^{n-1} + \ldots]$$
$$- [a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots] = 0$$

Coefficient of $x^0 = 0$

$$\Rightarrow \quad 2.1.\,a_2 - a_0 = 0 \quad \Rightarrow \quad \boxed{a_2 = \frac{a_0}{2}}$$

Coefficient of $x = 0$

$\Rightarrow \quad 3.2\, a_3 + a_1 - a_1 = 0 \qquad \Rightarrow \boxed{a_3 = 0}$

Coefficient of $x^2 = 0$

$\Rightarrow \quad 2.1 \,.\, a_2 + 4.3.a_4 + 2a_2 - a_2 = 0$

$\Rightarrow \quad 4.3\, a_4 + 3a_2 = 0$

$$\Rightarrow \quad a_4 = -\frac{a_2}{4} = -\frac{a_0}{8} \quad \text{or} \quad \boxed{a_4 = -\frac{a_0}{8}}$$

Coefficient of $x^3 = 0$

$\Rightarrow \quad 5.4.\, a_5 + 3.2.\, a_3 + 3a_3 - a_3 = 0$

$\Rightarrow \quad 20a_5 + 8a_3 = 0 \qquad \Rightarrow \boxed{a_5 = 0}$

Coefficient of $x^4 = 0$

$\Rightarrow \quad 6.5 \,.\, a_6 + 4.3 \,.\, a_4 + 4a_4 - a_4 = 0$

$\Rightarrow \quad 30a_6 + 15a_4 = 0$

$$\Rightarrow \quad a_6 = -\frac{a_4}{2} = \frac{a_0}{16} \quad \text{or} \quad \boxed{a_6 = \frac{a_0}{16}}$$

Similarly, $a_7 = 0, a_9 = 0, a_{11} = 0$ and so on.

Also, Coefficient of $x^n = 0$

$$(n + 2)(n + 1)\, a_{n+2} + n(n - 1)a_n + na_n - a_n = 0$$

$$\Rightarrow \quad a_{n+2} = -\left(\frac{n-1}{n+2}\right) a_n \qquad | \because \quad n + 1 \neq 0$$

Putting $n = 6, 8, 10, \ldots$, we get

$$a_8 = -\frac{5}{8} a_6 = -\frac{5a_0}{128}$$

$$a_{10} = -\frac{7}{10} a_8 = \frac{7a_0}{256} \text{ and so on.}$$

Substituting these values in (1), we get

$$y = a_0 + a_1 x + \frac{a_0}{2} x^2 - \frac{a_0}{8} x^4 + \frac{a_0}{16} x^6 - \frac{5a_0}{128} x^8 + \frac{7a_0}{256} x^{10} - \ldots$$

$$\Rightarrow \quad y = a_0 \left(1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16} - \frac{5x^8}{128} + \frac{7x^{10}}{256} - \ldots\right) + a_1 x$$

where a_0 and a_1 are constants.

Example 3. *Solve in series Chebyshev's differential equation (when n = 2)*

Or

Solve: $(1 - x^2)y'' - xy' + 4y = 0$ in series. (U.P.T.U. 2014)

Sol. Comparing the given differential equation with the form

$y'' + P(x)\, y' + Q(x)\, y = 0$, we get

$$P(x) = \frac{-x}{1-x^2},\ Q(x) = \frac{4}{1-x^2}$$

Since both P(x) and Q(x) are analytic at $x = 0$, hence $x = 0$ is an *ordinary point* of the given equation.

Assume the solution to be

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots \quad \ldots(1)$$

Then,
$$y' = a_1 + 2.a_2 x + 3.a_3 x^2 + \ldots + n a_n x^{n-1} + \ldots$$

and
$$y'' = 2.1.\, a_2 + 3.2.a_3 x + \ldots + n(n-1) a_n x^{n-2} + \ldots$$

Substituting these values in given equation, we get

$$(1 - x^2)\,[2.1.\, a_2 + 3.2.\, a_3 x + 4.3.\, a_4 x^2 + \ldots + n(n-1)\, a_n x^{n-2} + \ldots]$$

$$- x\,[a_1 + 2a_2 x + 3a_3 x^2 + \ldots + n a_n x^{n-1} + \ldots] + 4\,[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots] = 0$$

Coefficient of $x^0 = 0$

$$\Rightarrow \quad 2.1.\, a_2 + 4a_0 = 0 \quad \Rightarrow \quad \boxed{a_2 = -2a_0}$$

Coefficient of $x = 0$

$$\Rightarrow \quad 3.2\, a_3 - a_1 + 4a_1 = 0 \quad \Rightarrow \quad \boxed{a_3 = -\frac{a_1}{2}}$$

Coefficient of $x^2 = 0$

$$\Rightarrow \quad 4.3.\, a_4 - 2.1\, a_2 - 2a_2 + 4a_2 = 0 \quad \Rightarrow \quad \boxed{a_4 = 0}$$

Coefficient of $x^3 = 0$

$$\Rightarrow \quad 5.4\, a_5 - 3.2\, a_3 - 3a_3 + 4a_3 = 0$$

$$\Rightarrow \quad a_5 = \frac{a_3}{4} = \frac{1}{4}\left(\frac{-a_1}{2}\right) = -\frac{a_1}{8}$$

$$\Rightarrow \quad \boxed{a_5 = -\frac{a_1}{8}} \text{ and so on.}$$

Substituting these values in assumed solution (1), we get

$$y = a_0 + a_1 x - 2a_0 x^2 - \frac{a_1}{2} x^3 - \frac{a_1}{8} x^5 + \ldots$$

$$\Rightarrow \quad y = a_0 (1 - 2x^2) + a_1 x \left(1 - \frac{x^2}{2} - \frac{x^4}{8} - \ldots\right)$$

where a_0 and a_1 are constants.

Example 4. *Find the power series solution of the following differential equation about $x = 0$*

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0.$$

(U.P.T.U. 2013)

Sol. Comparing the given differential equation with the form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \text{ we get}$$

$$P(x) = \frac{-2x}{1-x^2}, \quad Q(x) = \frac{2}{1-x^2}$$

Since both P(x) and Q(x) are analytic at $x = 0$, hence $x = 0$ is an *ordinary point* of the given equation.

Assume the solution to be

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots \quad \ldots(1)$$

Then, $y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots + n\,a_n\,x^{n-1} + \ldots$

and $y'' = 2.1.\ a_2 + 3.2.\ a_3\,x + 4.3.\ a_4x^2 + \ldots + n(n-1)a_n\,x^{n-2} + \ldots$

Substituting these values in given equation, we get

$$(1-x^2)\,[2.1.\ a_2 + 3.2.\ a_3x + 4.3.\ a_4x^2 + \ldots + n(n-1)\,a_nx^{n-2} + \ldots]$$
$$-\,2x\,[a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots + n\,a_n\,x^{n-1} + \ldots]$$
$$+\,2\,[a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_n\,x^n + \ldots] = 0$$

Coefficient of $x^0 = 0$

$$\Rightarrow \quad 2.1.\ a_2 + 2a_0 = 0 \quad \Rightarrow \quad \boxed{a_2 = -\,a_0}$$

Coefficient of $x = 0$

$$\Rightarrow \quad 3.2.\ a_3 - 2a_1 + 2a_1 = 0 \quad \Rightarrow \quad \boxed{a_3 = 0}$$

Coefficient of $x^2 = 0$

$$\Rightarrow \quad 4.3.a_4 - 2.1.\ a_2 - 4a_2 + 2a_2 = 0$$

$$\Rightarrow \quad 12a_4 - 4a_2 = 0 \quad \Rightarrow \quad a_4 = \frac{a_2}{3} = -\frac{a_0}{3} \quad \Rightarrow \quad \boxed{a_4 = -\frac{a_0}{3}}$$

Coefficient of $x^3 = 0$

$$\Rightarrow \quad 5.4.a_5 - 3.2.\ a_3 - 6a_3 + 2a_3 = 0$$

$$\Rightarrow \quad 20a_5 - 10a_3 = 0 \quad \Rightarrow \quad \boxed{a_5 = 0}$$

Coefficient of $x^4 = 0$

$$\Rightarrow \quad 6.5.a_6 - 4.3.\ a_4 - 8a_4 + 2a_4 = 0$$

$$\Rightarrow \quad 30a_6 - 18a_4 = 0 \quad \Rightarrow \quad a_6 = \frac{3}{5}a_4 \quad \Rightarrow \quad \boxed{a_6 = -\frac{a_0}{5}}$$

Also, $a_7 = 0$, $a_9 = 0$ and so on.

Substituting these values in assumed solution (1), we get

$$y = a_0 + a_1x - a_0x^2 - \frac{a_0}{3}x^4 - \frac{a_0}{5}x^6 - \ldots\ldots$$

$$\Rightarrow \quad y = a_0\left(1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} - \ldots\right) + a_1x$$

where a_0 and a_1 are constants.

Example 5. *Solve in series the Legendre's differential equation*

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + p(p+1)y = 0. \qquad \text{[G.B.T.U. (C.O.) 2010]}$$

Sol. Here, $\quad P(x) = \dfrac{-2x}{1-x^2},\ Q(x) = \dfrac{p(p+1)}{1-x^2}$

Since both P(x) and Q(x) are analytic at $x = 0$ $\therefore$ $x = 0$ is an *ordinary point* of the given differential equation.

Let the solution be $\quad y = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_n\,x^n + \ldots\ldots = \sum_{n=0}^{\infty} a_n\,x^n \qquad \ldots(1)$

$$\therefore \quad \frac{dy}{dx} = \sum_{n=0}^{\infty} n\,a_n\,x^{n-1} \qquad \ldots(2)$$

$$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} n(n-1)\, a_n\, x^{n-2} \qquad ...(3)$$

Substituting the above values in the given equation, we get

$$(1-x^2)\sum_{n=0}^{\infty} n(n-1)\, a_n\, x^{n-2} - 2x\sum_{n=0}^{\infty} n\, a_n\, x^{n-1} + p(p+1)\sum_{n=0}^{\infty} a_n\, x^n = 0$$

$$\Rightarrow \qquad \sum_{n=0}^{\infty} n(n-1)\, a_n\, x^{n-2} - \sum_{n=0}^{\infty} a_n[n(n-1) + 2n - p(p+1)]x^n = 0$$

$$\Rightarrow \qquad \sum_{n=0}^{\infty} n(n-1)\, a_n\, x^{n-2} - \sum_{n=0}^{\infty} a_n(n-p)\,(n+p+1)x^n = 0$$

This is an identity in x.

Coefficient of $x^n = 0$

$$\Rightarrow \qquad (n+2)\,(n+1)\, a_{n+2} - (n-p)\,(n+p+1)\, a_n = 0$$

$$\therefore \qquad a_{n+2} = \frac{(n-p)(n+p+1)}{(n+2)\,(n+1)}\, a_n$$

Putting $n = 0, 2, 4,$ ..... etc., we get

$$a_2 = \frac{-\,p(p+1)}{2.1}\, a_0$$

$$a_4 = \frac{(2-p)(3+p)}{4.3}\, a_2 = \frac{(p-2)(p)(p+1)(p+3)}{4!}\, a_0 \text{ etc.}$$

Again, putting $n = 1, 3, 5,$ ...... etc., we get

$$a_3 = \frac{(1-p)(p+2)}{3.2}\, a_1 = -\frac{(p-1)\,(p+2)}{3!}\, a_1$$

$$a_5 = \frac{(3-p)(p+4)}{5.4}\, a_3 = \frac{(p-3)(p-1)\,(p+2)(p+4)}{5!}\, a_1 \text{ etc.}$$

Substituting these values in eqn. (1), we get

$$y = a_0\left[1 - \frac{p(p+1)}{2!}x^2 + \frac{(p-2)\,p\,(p+1)(p+3)}{4!}x^4 - ......\right]$$

$$+\, a_1\left[x - \frac{(p-1)(p+2)}{3!}x^3 + \frac{(p-3)(p-1)(p+2)(p+4)}{5!}x^5 + ...\right]$$

Note. Above method is an *aliter* to the method of solution in series discussed before and preferred when, we get the recurrence relation in between a_n and a_{n+2}.

Example 6. *Solve the differential equation* $y'' + (x-1)^2\, y' - 4(x-1)\, y = 0$ *in series about the ordinary point* $x = 1$.

Sol. Put $x = t + 1$ (or $x - 1 = t$)

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt} \qquad \left(\because\ \frac{dt}{dx} = 1\right)$$

$$\Rightarrow \qquad \frac{d}{dx} \equiv \frac{d}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2}$$

$\therefore$ The given equation becomes,

$$\frac{d^2y}{dt^2} + t^2y' - 4ty = 0 \qquad ...(1)$$

Now, $t = 0$ is an *ordinary point*.

Assume the solution to be | given

$$y = a_0 + a_1t + a_2t^2 + a_3t^3 + ... + a_nt^n + ... \qquad ...(2)$$

then $\quad y' = a_1 + 2a_2t + 3a_3t^2 + ... + n\,a_n\,t^{n-1} + ...$

and $\quad y'' = 2a_2 + 3.2.\,a_3t + ... + n\,(n-1)\,a_n\,t^{n-2} + ...$

Substituting these values in eqn. (1), we get

$$[2a_2 + 3.2.\,a_3t + 4.3.\,a_4\,t^2 + ... + n(n-1)\,a_n\,t^{n-2} + ...]$$
$$+ t^2\,[a_1 + 2a_2t + 3a_3\,t^2 + 4a_4t^3 + ... + n\,a_nt^{n-1} + ...]$$
$$- 4t\,[a_0 + a_1t + a_2t^2 + a_3t^3 + ... + a_nt^n + ...] = 0$$

Coefficient of $t^0 = 0$

$\Rightarrow \quad 2a_2 = 0 \quad \Rightarrow \quad \boxed{a_2 = 0}$

Coefficient of $t = 0$

$\Rightarrow \quad 3.2.\,a_3 - 4a_0 = 0 \quad \Rightarrow \quad \boxed{a_3 = \frac{2a_0}{3}}$

Coefficient of $t^2 = 0$

$\Rightarrow \quad 4.3.\,a_4 + a_1 - 4a_1 = 0$

$\Rightarrow \quad 12a_4 = 3a_1 \quad \Rightarrow \quad \boxed{a_4 = \frac{a_1}{4}}$

Coefficient of $t^3 = 0$

$\Rightarrow \quad 5.4.\,a_5 + 2a_2 - 4a_2 = 0 \quad \Rightarrow \quad \boxed{a_5 = 0}$

Coefficient of $t^4 = 0$

$\Rightarrow \quad 6.5.\,a_6 + 3a_3 - 4a_3 = 0$

$$a_6 = \frac{a_3}{6.5} = \frac{2a_0}{6.5.3} \quad \Rightarrow \quad \boxed{a_6 = \frac{a_0}{45}}$$

Now, $\quad$ Coefficient of $t^n = 0$

$\Rightarrow \quad (n+2)\,(n+1)\,a_{n+2} + (n-1)\,a_{n-1} - 4a_{n-1} = 0$

$$\Rightarrow \quad a_{n+2} = -\frac{(n-5)}{(n+2)(n+1)}\,a_{n-1}$$

Putting $n = 5, 6, 7, 8, ...$, we get

$$a_7 = 0$$

$$a_8 = \frac{-1}{8.7}\,a_5 = 0$$

$$a_9 = \frac{-2}{9.8}\,a_6 = \frac{-2}{9.8}\,\frac{a_0}{45} = -\frac{a_0}{1620}$$

and so on.

Substituting these values in (2), we get

$$y = a_0 + a_1 t + \frac{2}{3} a_0 t^3 + \frac{a_1}{4} t^4 + \frac{a_0}{45} t^6 - \frac{a_0}{1620} t^9 + \dots$$

$$= a_0 \left(1 + \frac{2}{3} t^3 + \frac{1}{45} t^6 - \frac{1}{1620} t^9 + \dots\right) + a_1 \left(t + \frac{t^4}{4}\right)$$

$$\Rightarrow \quad y = a_0 \left[1 + \frac{2}{3}(x-1)^3 + \frac{1}{45}(x-1)^6 - \frac{1}{1620}(x-1)^9 + \dots\right] + a_1 \left[(x-1) + \frac{(x-1)^4}{4}\right]$$

where a_0 and a_1 are constants.

TEST YOUR KNOWLEDGE

Solve the following equations in series: [Dashes denote differentiation w.r.t. x]

1. $\frac{d^2y}{dx^2} - y = 0$

2. $y'' + x^2 y = 0$

3. (i) $y'' + xy' + y = 0$ (ii) $y'' - xy' + y = 0$

4. (i) $y'' - xy' + x^2 y = 0$ (ii) $y'' + xy' + x^2 y = 0$

5. $(1 - x^2) y'' + 2xy' + y = 0$

6. $(2 + x^2) y'' + xy' + (1 + x) y = 0$

7. (i) $(x^2 + 1) y'' + xy' - xy = 0$ (ii) $(1 - x^2) y'' - 2xy' + 20y = 0$. (A.K.T.U. 2017)

8. (i) $(x^2 - 1) y'' + 4xy' + 2y = 0$ (ii) $(x^2 - 1) y'' + xy' - y = 0$

9. (i) $y'' + xy' + (x^2 + 2) y = 0$ (U.P.T.U. 2014) (ii) $(x^2 - 1) y'' + 3xy' + xy = 0$; $y(0) = 4$, $y'(0) = 6$.

10. (i) $y'' - xy' + 2y = 0$ near $x = 1$ (ii) $y'' + (x - 3) y' + y = 0$ near $x = 2$.

Answers

1. $y = a_0 \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots\right) + a_1 \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right) = a_0 \cosh x + a_1 \sinh x$

2. $y = a_0 \left(1 - \frac{x^4}{3.4} + \frac{x^8}{3.4.7.8} - \dots\right) + a_1 \left(x - \frac{x^5}{4.5} + \frac{x^9}{4.5.8.9} - \dots\right)$

3. (i) $y = a_0 \left(1 - \frac{x^2}{2} + \frac{x^4}{2.4} - \frac{x^6}{2.4.6} + \dots\right) + a_1 \left(x - \frac{x^3}{3} + \frac{x^5}{3.5} - \frac{x^7}{3.5.7} + \dots\right)$

(ii) $y = a_0 \left(1 - \frac{x^2}{2!} - \frac{x^4}{4!} - \frac{3}{6!} x^6 - \frac{3.5}{8!} x^8 + \dots\right) + a_1 x$

4. (i) $y = a_0 \left(1 - \frac{x^4}{12} - \frac{x^6}{90} - \dots\right) + a_1 \left(x + \frac{x^3}{6} - \frac{x^5}{40} - \frac{x^7}{144} + \dots\right)$

(ii) $y = a_0 \left(1 - \frac{x^4}{12} + \frac{x^6}{90} - \dots\right) + a_1 \left(x - \frac{x^3}{6} - \frac{x^5}{40} - \dots\right)$

5. $y = a_0 \left(1 - \frac{x^2}{2} + \frac{x^4}{8} + \dots\right) + a_1 \left(x - \frac{x^3}{2} + \frac{x^5}{40} + \dots\right)$

6. $y = a_0 \left(1 - \frac{x^2}{4} - \frac{x^3}{12} + \frac{5x^4}{96} + \dots\right) + a_1 \left(x - \frac{x^3}{6} - \frac{x^4}{24} + \dots\right)$

7. (i) $y = a_0\left(1+\frac{x^3}{6}-\frac{3x^5}{40}+\dots\right)+a_1\left(x-\frac{x^3}{6}+\frac{x^4}{12}+\frac{3x^5}{40}-\dots\right)$

(ii) $y = a_0\left[1-\frac{4\cdot 5}{2!}x^2+\frac{2\cdot 4\cdot 5\cdot 7}{4!}x^4-\dots\right]+a_1\left[x-\frac{3\cdot 6}{3!}x^3+\frac{1\cdot 3\cdot 6\cdot 8}{5!}x^5+\dots\right]$.

8. (i) $y = a_0(1+x^2+x^4+\dots)+a_1(x+x^3+x^5+\dots)$ (ii) $y = a_0\left(1+\frac{x^2}{2}+\frac{x^4}{4}+\dots\right)+a_1x$

9. (i) $y = c_0\left(1-x^2+\frac{x^4}{4}+\dots\right)+c_1\left(x-\frac{x^3}{2}+\frac{3}{40}x^5-\dots\right)$

(ii) $y = 4+6x+\frac{11}{3}x^3+\frac{1}{2}x^4+\frac{11}{4}x^5+\dots$

10. (i) $y = a_0\left[1-(x-1)^2-\frac{1}{3}(x-1)^3-\dots\right]+a_1\left[(x-1)+\frac{1}{2}(x-1)^2-\dots\right]$

(ii) $y = a_0\left[1-\frac{1}{2}(x-2)^2-\frac{1}{6}(x-2)^3-\frac{1}{12}(x-2)^4+\dots\right]$

$+a_1\left[(x-2)+\frac{1}{2}(x-2)^2-\frac{1}{6}(x-2)^3-\frac{1}{6}(x-2)^4+\dots\right]$

1.27 FROBENIUS METHOD: SERIES SOLUTION WHEN $X = 0$ IS A REGULAR SINGULAR POINT OF THE DIFFERENTIAL EQUATION

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)\,y=0$$

Steps for solution:

1. Assume $y = a_0x^m+a_1x^{m+1}+a_2x^{m+2}+\dots$ ...(1)

2. Substitute from (1) for y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ in given equation.

3. Equate to zero the coefficient of *lowest power* of x. This gives a quadratic equation in m which is known as the *Indicial equation.*

4. Equate to zero, the coefficients of other powers of x to find $a_1, a_2, a_3, \dots$ interms of a_0.

5. Substitute the values of $a_1, a_2, a_3,\dots$in (1) to get the series solution of the given equation having a_0 as arbitrary constant. Obviously, this is not the complete solution of given equation since the complete solution must have two independent arbitrary constants.

The method of complete solution depends on the nature of roots of the indicial equation.

1.27.1 Case I. *When Roots are distinct and do not differ by an integer*

e.g.,
$$m_1=\frac{1}{2},\ m_2=1$$

Let m_1 and m_2 be the roots then complete solution is

$$\boxed{y = c_1\,(y)_{m_1} + c_2\,(y)_{m_2}}$$

ILLUSTRATIVE EXAMPLES

Example 1. *Solve in series the differential equation*

$$2x\,(1-x)\,\frac{d^2y}{dx^2} + (5-7x)\,\frac{dy}{dx} - 3y = 0.$$

Sol. Comparing the given equation with

$$\frac{d^2y}{dx^2} + P(x)\,\frac{dy}{dx} + Q(x)y = 0, \text{ we get}$$

$$P(x) = \frac{5-7x}{2x(1-x)},\; Q(x) = \frac{-3}{2x\,(1-x)}$$

At $x = 0$, Both $P(x)$ and $Q(x)$ are not analytic, hence $x = 0$ is a *singular point.*

Now,
$$x\,P(x) = \frac{5-7x}{2(1-x)}$$
$$x^2\,Q(x) = \frac{-3x}{2(1-x)}$$

At $x = 0$, both $x\,P(x)$ and $x^2\,Q(x)$ are analytic, hence $x = 0$ is a *regular singular* point.

Let us assume

$$y = a_0\,x^m + a_1x^{m+1} + a_2\,x^{m+2} + a_3x^{m+3} + \ldots \qquad \ldots(1)$$

Then, $y' = ma_0x^{m-1} + (m+1)\,a_1\,x^m + (m+2)\,a_2\,x^{m+1} + (m+3)\,a_3\,x^{m+2} + \ldots$

and $y' = m(m-1)\,a_0\,x^{m-2} + (m+1)\,m\,a_1\,x^{m-1} + (m+2)\,(m+1)\,a_2\,x^m + (m+3)\,(m+2)\,a_3\,x^{m+1} + \ldots$

Substituting these values in given equation, we get

$$2x\,(1-x)\,[m(m-1)\,a_0\,x^{m-2} + (m+1)\,ma_1\,x^{m-1} + (m+2)\,(m+1)\,a_2\,x^m + (m+3)\,(m+2)\,a_3\,x^{m+1} + \ldots]$$
$$+ (5-7x)\,[m\,a_0\,x^{m-1} + (m+1)\,a_1\,x^m + (m+2)\,a_2\,x^{m+1} + (m+3)\,a_3\,x^{m+2} + \ldots]$$
$$- 3\,[a_0\,x^m + a_1\,x^{m+1} + a_2\,x^{m+2} + a_3\,x^{m+3} + \ldots] = 0$$

Now, coefficient of lowest power of $x = 0$

$\Rightarrow$ Coefficient of $x^{m-1} = 0$

$\Rightarrow$ $2m\,(m-1)\,a_0 + 5m\,a_0 = 0$

$\Rightarrow$ $(2m^2 + 3m)\,a_0 = 0$

$\Rightarrow$ $2m^2 + 3m = 0$ $\qquad (\because\; a_0 \neq 0)$

This is called indicial equation

$$m(2m+3) = 0$$

$\Rightarrow$ $\boxed{m = 0, -3/2}$

Roots are distinct and do not differ by an integer.

Now, Coefficient of $x^m = 0$

$\Rightarrow$ $$2(m+1)\,m\,a_1 - 2m\,(m-1)\,a_0 + 5(m+1)\,a_1 - 7ma_0 - 3a_0 = 0$$

$\Rightarrow$ $$(m+1)(2m+5)\,a_1 = (2m^2 - 2m + 7m + 3)\,a_0$$

$$a_1 = \frac{(m+1)(2m+3)}{(m+1)(2m+5)}\,a_0$$

$\Rightarrow$ $$\boxed{a_1 = \frac{2m+3}{2m+5}\,a_0}$$

Coefficient of $x^{m+1} = 0$

$\Rightarrow$ $$2(m+2)(m+1)\,a_2 - 2(m+1)\,m\,a_1 + 5(m+2)\,a_2 - 7(m+1)\,a_1 - 3a_1 = 0$$

$\Rightarrow$ $$(m+2)(2m+7)\,a_2 = (2m^2 + 2m + 7m + 7 + 3)\,a_1$$
$$= (2m^2 + 9m + 10)\,a_1 = (2m+5)(m+2)\,a_1$$

$\Rightarrow$ $$a_2 = \frac{2m+5}{2m+7}\,a_1 = \frac{2m+5}{2m+7}\cdot\frac{2m+3}{2m+5}\,a_0$$

$\Rightarrow$ $$\boxed{a_2 = \frac{2m+3}{2m+7}\,a_0}$$

Similarly, $$a_3 = \frac{2m+7}{2m+9}\,a_2 = \frac{2m+7}{2m+9}\cdot\frac{2m+3}{2m+7}\,a_0$$

$\Rightarrow$ $$\boxed{a_3 = \frac{2m+3}{2m+9}\,a_0}$$

and so on.

Hence, from (1),

$$y = x^m\left[a_0 + \frac{2m+3}{2m+5}\,a_0\,x + \frac{2m+3}{2m+7}\,a_0\,x^2 + \frac{2m+3}{2m+9}\,a_0\,x^3 + \ldots\right]$$

$\Rightarrow$ $$y = a_0\,x^m\left[1 + \left(\frac{2m+3}{2m+5}\right)x + \left(\frac{2m+3}{2m+7}\right)x^2 + \left(\frac{2m+3}{2m+9}\right)x^3 + \ldots\right] \quad \ldots(2)$$

Now, $$y_1 = (y)_{m=0}$$

$$\boxed{y_1 = a_0\left[1 + \frac{3}{5}x + \frac{3}{7}x^2 + \frac{3}{9}x^3 + \ldots\right]} \quad \ldots(3)$$

Also, $$y_2 = (y)_{m=-3/2} = a_0\,x^{-3/2}\,(1 + 0\,.\,x + 0\,.\,x^2 + 0\,.\,x^3 + \ldots)$$

$$\boxed{y_2 = a_0\,x^{-3/2}} \quad \ldots(4)$$

Hence the complete solution is given by

$$y = c_1 y_1 + c_2 y_2 = c_1\,a_0\left(1 + \frac{3}{5}x + \frac{3}{7}x^2 + \frac{3}{9}x^3 + \ldots\right) + c_2 a_0\,x^{-3/2}$$

$\Rightarrow$ $$y = \mathrm{A}\left(1 + \frac{3}{5}x + \frac{3}{7}x^2 + \frac{3}{9}x^3 + \ldots\right) + \mathrm{B}x^{-3/2}$$

where A and B are constants.

Example 2. *Solve in series the differential equation* $2x^2 \dfrac{d^2y}{dx^2} + (2x^2 - x)\dfrac{dy}{dx} + y = 0.$

Sol. Comparing the given equation with $\dfrac{d^2y}{dx^2} + P(x)\dfrac{dy}{dx} + Q(x)\,y = 0$, we get

$$P(x) = \frac{2x^2 - x}{2x^2} = 1 - \frac{1}{2x} \quad \text{and} \quad Q(x) = \frac{1}{2x^2}$$

At $x = 0$, Both $P(x)$ and $Q(x)$ are not analytic, hence $x = 0$ is a *singular point.*

Now, $\qquad x\,P(x) = x - \dfrac{1}{2}$ and $x^2\,Q(x) = \dfrac{1}{2}$

Since both $x\,P(x)$ and $x^2\,Q(x)$ are analytic at $x = 0$, hence $x = 0$ is a *regular singular point.*

Let us assume

$$y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + \ldots \qquad \ldots(1)$$

Then, $\quad y' = m\,a_0 x^{m-1} + (m+1)\,a_1 x^m + (m+2)\,a_2 x^{m+1} + (m+3)\,a_3 x^{m+2} + \ldots$

and $\quad y'' = m(m-1)\,a_0 x^{m-2} + (m+1)\,m\,a_1 x^{m-1} + (m+2)(m+1)\,a_2 x^m + (m+3)(m+2)\,a_3 x^{m+1} + \ldots$

Substituting these values in given equation, we get

$$2x^2\,[m(m-1)\,a_0 x^{m-2} + (m+1)m\,a_1 x^{m-1} + (m+2)(m+1)\,a_2 x^m + (m+3)(m+2)\,a_3 x^{m+1} + \ldots] + (2x^2 - x)\,[m\,a_0 x^{m-1} + (m+1)\,a_1 x^m + (m+2)\,a_2 x^{m+1} + (m+3)\,a_3 x^{m+2} + \ldots] + [a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + \ldots] = 0$$

Now, Coeff. of lowest power of $x = 0$ *i.e.,* Coeff. of $x^m = 0$

$$2m\,(m-1)\,a_0 - m\,a_0 + a_0 = 0$$

$$\Rightarrow \quad (2m^2 - 3m + 1)\,a_0 = 0$$

$$\Rightarrow \quad (2m-1)(m-1) = 0 \text{ (since } a_0 \neq 0)$$

which is indicial equation.

Its roots are $\boxed{m = 1, \frac{1}{2}}$

Roots are distinct and donot differ by an integer.

Now, $\quad$ Coefficient of $x^{m+1} = 0$

$$\Rightarrow \quad 2m\,(m+1)\,a_1 + 2m\,a_0 - (m+1)\,a_1 + a_1 = 0$$

$$\Rightarrow \quad (2m^2 + m)\,a_1 + 2m\,a_0 = 0$$

$$\Rightarrow \quad \boxed{a_1 = -\frac{2}{2m+1}\,a_0} \qquad | \because \; m \neq 0$$

Coefficient of $x^{m+2} = 0$

$$\Rightarrow \quad 2(m+2)(m+1)\,a_2 + 2(m+1)\,a_1 - (m+2)\,a_2 + a_2 = 0$$

$$\Rightarrow \quad (2m^2 + 5m + 3)\,a_2 + 2(m+1)\,a_1 = 0$$

$$\Rightarrow \quad (2m+3)(m+1)\,a_2 + 2(m+1)\,a_1 = 0$$

$$\Rightarrow \quad a_2 = \frac{-2}{2m+3} a_1 = \frac{(-2)}{2m+3} \cdot \frac{(-2)}{2m+2} a_0$$

$$\Rightarrow \quad \boxed{a_2 = \frac{4}{(2m+1)(2m+3)} a_0}$$

Similarly, we can find

$$a_3 = \frac{-8}{(2m+1)(2m+3)(2m+5)} a_0$$

$$a_4 = \frac{16}{(2m+1)(2m+3)(2m+5)(2m+7)} a_0$$

and so on.

$$\therefore \quad y = a_0 x^m \left[1 - \frac{2}{2m+1} x + \frac{4}{(2m+1)(2m+3)} x^2 - \frac{8}{(2m+1)(2m+3)(2m+5)} x^3 + \ldots\right] \quad \ldots(2)$$

Now, $y_1 = (y)_{m=1}$

$$y_1 = a_0 x \left[1 - \frac{2}{3} x + \frac{4}{3.5} x^2 - \frac{8}{3.5.7} x^3 + \ldots\right]$$

or
$$\boxed{y_1 = a_0 x \left(1 - \frac{2}{3} x + \frac{2^2}{3.5} x^2 - \frac{2^3}{3.5.7} x^3 + \ldots\right)} \quad \ldots(3)$$

and $y_2 = (y)_{m=1/2}$

$$\boxed{y_2 = a_0 x^{1/2} \left[1 - x + \frac{1}{2} x^2 - \frac{1}{6} x^3 + \ldots\right]} \quad \ldots(4)$$

Hence the complete solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 a_0 x \left(1 - \frac{2}{3} x + \frac{2^2}{3.5} x^2 - \frac{2^3}{3.5.7} x^3 + \ldots\right) + c_2 a_0 \sqrt{x} \left(1 - x + \frac{1}{2} x^2 - \frac{1}{6} x^3 + \ldots\right)$$

$$\Rightarrow \quad \boxed{y = Ax \left(1 - \frac{2}{3} x + \frac{2^2}{3.5} x^2 - \frac{2^3}{3.5.7} x^3 + \ldots\right) + B\sqrt{x} \left(1 - x + \frac{1}{2} x^2 - \frac{1}{6} x^3 + \ldots\right)}$$

where A and B are constants.

TEST YOUR KNOWLEDGE

Solve in series:

1. $9x(1-x) \dfrac{d^2y}{dx^2} - 12 \dfrac{dy}{dx} + 4y = 0$
2. $x(2+x^2) \dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 6xy = 0$
3. $3x \dfrac{d^2y}{dx^2} + 2 \dfrac{dy}{dx} + y = 0$
4. $2x^2 \dfrac{d^2y}{dx^2} - x \dfrac{dy}{dx} + (1-x^2) y = 0$ [*G.B.T.U. (C.O.) 2011*]

5. $2x^2 y'' + xy' - (x+1)y = 0$ (A.K.T.U. 2015, 2017)

6. $2x(1-x)\dfrac{d^2y}{dx^2} + (1-x)\dfrac{dy}{dx} + 3y = 0$ (A.K.T.U. 2016, 2018)

7. $2x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + (x-5)y = 0$

8. $y'' + \dfrac{1}{4x}y' + \dfrac{1}{8x^2}y = 0$

9. $2x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + (x^2+1)y = 0$

10. $4x\dfrac{d^2y}{dx^2} + 2(1-x)\dfrac{dy}{dx} - y = 0.$

11. $2x^2y'' + 7x(x+1)y' - 3y = 0$ (G.B.T.U. 2013)

12. $2x^2y'' + x(2x+1)y' - y = 0$ (U.P.T.U. 2015)

Answers

1. $y = A\left(1+\frac{1}{3}x+\frac{1.4}{3.6}x^2+\frac{1.4.7}{3.6.9}x^3+\ldots\right)+Bx^{7/3}\left(1+\frac{8}{10}x+\frac{8.11}{10.13}x^2+\frac{8.11.14}{10.13.16}x^3+\ldots\right)$

2. $y = A\left(1+3x^2+\frac{3}{5}x^4-\frac{1}{15}x^6+\ldots\right)+Bx^{3/2}\left(1+\frac{3}{8}x^2-\frac{3.1}{8.16}x^4+\frac{5.3.1}{8.16.24}x^6-\ldots\right)$

3. $y = A\left(1-\frac{x}{2}+\frac{x^2}{20}-\frac{x^3}{480}+\ldots\right)+Bx^{1/3}\left(1-\frac{x}{4}+\frac{x^2}{56}-\frac{x^3}{1680}+\ldots\right)$

4. $y = Ax\left(1+\frac{x^2}{2.5}+\frac{x^4}{2.4.5.9}+\ldots\right)+Bx^{1/2}\left(1+\frac{x^2}{2.3}+\frac{x^4}{2.4.3.7}+\ldots\right)$

5. $y = Ax\left(1+\frac{1}{5}x+\frac{1}{70}x^2+\ldots\right)+Bx^{-1/2}\left(1-x-\frac{1}{2}x^2+\ldots\right)$

6. $y = A\left(1-3x+\frac{3x^2}{1.3}+\frac{3x^3}{3.5}+\frac{3x^4}{5.7}+\ldots\right)+B\sqrt{x}\,(1-x)$

7. $y = c_1x^{5/2}\left(1-\frac{x}{9}+\frac{x^2}{198}-\frac{x^3}{7722}+\ldots\right)+c_2x^{-1}\left(1+\frac{x}{5}+\frac{x^2}{30}+\frac{x^3}{90}+\ldots\right)$

8. $y = A\sqrt{x}+Bx^{1/4}$

9. $y = Ax\left(1-\frac{x^2}{10}+\frac{x^4}{360}-\ldots\right)+Bx^{1/2}\left(1-\frac{x^2}{6}+\frac{x^4}{168}-\ldots\right)$

10. $y = A\left(1+\frac{x}{2.1!}+\frac{x^2}{2^2.2!}+\frac{x^3}{2^3.3!}+\ldots\right)+B\sqrt{x}\left(1+\frac{x}{1.3}+\frac{x^2}{1.3.5}+\frac{x^3}{1.3.5.7}+\ldots\right).$

11. $y = A\sqrt{x}\left(1-\frac{7}{18}x+\frac{49}{264}x^2\ldots\right)+Bx^{-3}\left(1-\frac{21}{5}x+\frac{49}{5}x^2\ldots\right)$

12. $y = Ax^{-1/2}\left(1-x+\frac{x^2}{2}-\frac{x^3}{6}+\ldots.\right)+Bx\left(1-\frac{2x}{5}+\frac{4x^2}{35}-\frac{8x^3}{315}+\ldots.\right)$

1.27.2 Case II. *When Roots are Equal e.g.,* $m_1 = m_2 = 0$

Complete solution is

$$\boxed{y = c_1 (y)_{m_1} + c_2 \left(\frac{\partial y}{\partial m}\right)_{m_1}}$$

ILLUSTRATIVE EXAMPLES

Example 1. *Solve in series:* $x(x-1)\dfrac{d^2y}{dx^2} + (3x-1)\dfrac{dy}{dx} + y = 0.$

Sol. Comparing the given equation with

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0, \text{ we get}$$

$$P(x) = \frac{3x-1}{x(x-1)} \text{ and } Q(x) = \frac{1}{x(x-1)}$$

At $x = 0$, Both $P(x)$ and $Q(x)$ are not analytic, hence $x = 0$ is a *singular point.*

Now, $\quad x\,P(x) = \dfrac{3x-1}{x-1} \quad$ and $\quad x^2\,Q(x) = \dfrac{x}{x-1}$

Both $x\,P(x)$ and $x^2\,Q(x)$ are analytic at $x = 0$, hence $x = 0$ is a *regular singular point.*

Let us assume

$$y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + \ldots \quad \ldots(1)$$

Then, $\quad y' = m\,a_0\,x^{m-1} + (m+1)\,a_1\,x^m + (m+2)\,a_2\,x^{m+1} + (m+3)\,a_3\,x^{m+2} + \ldots$

and $\quad y'' = m(m-1)\,a_0\,x^{m-2} + (m+1)m\,a_1\,x^{m-1} + (m+2)(m+1)\,a_2\,x^m + (m+3)(m+2)\,a_3\,x^{m+1} + \ldots$

Substituting these values in given equation, we get

$$x(x-1)[m(m-1)\,a_0\,x^{m-2} + (m+1)\,m\,a_1\,x^{m-1} + (m+2)(m+1)\,a_0\,x^m + (m+3)(m+2)\,a_3\,x^{m+1} + \ldots]$$
$$+ (3x-1)[m\,a_0\,x^{m-1} + (m+1)\,a_1\,x^m + (m+2)\,a_2\,x^{m+1} + (m+3)\,a_3\,x^{m+2} + \ldots]$$
$$+ [a_0\,x^m + a_1\,x^{m+1} + a_2\,x^{m+2} + a_3\,x^{m+3} + \ldots] = 0$$

Now, Coefficient of lowest power of $x = 0$

$\Rightarrow \quad$ Coefficient of $x^{m-1} = 0$

$\Rightarrow \quad -m(m-1)\,a_0 - m\,a_0 = 0 \quad \Rightarrow \quad -m^2\,a_0 = 0$

$m^2 = 0 \qquad (\because \; a_0 \neq 0)$

which is *Indicial equation*

Its roots are $\boxed{m = 0, 0}$

Roots are equal.

Now, Coefficient of $x^m = 0$

$\Rightarrow \quad m(m-1)\,a_0 - (m+1)\,m\,a_1 + 3m\,a_0 - (m+1)\,a_1 + a_0 = 0$

$\Rightarrow \quad (m+1)^2 a_0 - (m+1)^2\,a_1 = 0$

$\Rightarrow \quad \boxed{a_1 = a_0} \qquad (\because \; m \neq -1)$

Coefficient of $x^{m+1} = 0$

$\Rightarrow \quad (m+1)\,m\,a_1 - (m+2)(m+1)\,a_2 + 3(m+1)\,a_1 - (m+2)a_2 + a_1 = 0$

$\Rightarrow \quad (m+2)^2 a_1 - (m+2)^2 a_2 = 0$

$\Rightarrow \quad a_2 = a_1 \qquad (\because \; m \neq -2)$

$\Rightarrow \quad \boxed{a_2 = a_0}$

Similarly, we can show that

$$a_3 = a_0$$

$$a_4 = a_0 \quad \text{and so on.}$$

$\therefore \quad y = a_0 x^m (1 + x + x^2 + x^3 + \ldots)$ | From (1)

Now, $\quad y_1 = (y)_{m=0} = a_0 x^0 (1 + x + x^2 + x^3 + \ldots) = a_0 (1 + x + x^2 + x^3 + \ldots)$

$$y_2 = \left(\frac{\partial y}{\partial m}\right)_{m=0} = [a_0 (1 + x + x^2 + x^3 + \ldots)\, x^m \log x]_{m=0}$$

$$= a_0 \log x (1 + x + x^2 + x^3 + \ldots)$$

Hence the complete solution is given by

$$y = c_1 y_1 + c_2 y_2 = c_1 a_0 (1 + x + x^2 + x^3 + \ldots) + c_2 a_0 \log x (1 + x + x^2 + x^3 + \ldots)$$

$$y = (\mathrm{A} + \mathrm{B} \log x)(1 + x + x^2 + x^3 + \ldots)$$

where A and B are constants.

Example 2. *Solve in series the differential equation:* $x\dfrac{d^2 y}{dx^2} + \dfrac{dy}{dx} - y = 0$.

Sol. Comparing with the equation

$$\frac{d^2 y}{dx^2} + \mathrm{P}(x)\frac{dy}{dx} + \mathrm{Q}(x)\, y = 0, \text{ we get}$$

$$\mathrm{P}(x) = \frac{1}{x} \text{ and } \mathrm{Q}(x) = -\frac{1}{x}$$

Since at $x = 0$, both P(x) and Q(x) are not analytic $\therefore \; x = 0$ is a *singular point.*

Also, $\quad x\,\mathrm{P}(x) = 1 \quad$ and $\quad x^2\,\mathrm{Q}(x) = -x$

Both x P(x) and x^2 Q(x) are analytic at $x = 0$ $\quad \therefore \; x = 0$ is a *regular singular point.*

Let us assume

$$y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + \ldots \qquad \ldots(1)$$

Then, $\quad y' = m\,a_0 x^{m-1} + (m+1)\,a_1 x^m + (m+2)\,a_2 x^{m+1} + (m+3)\,a_3 x^{m+2} + \ldots$

and $\quad y'' = m(m-1)\,a_0 x^{m-2} + (m+1)\,m\,a_1 x^{m-1} + (m+2)(m+1)\,a_2 x^m + (m+3)(m+2)\,a_3 x^{m+1} + \ldots$

Substituting these values in the given equation, we get

$$x\,[m(m-1)\,a_0 x^{m-2} + (m+1)m\,a_1 x^{m-1} + (m+2)(m+1)\,a_2 x^m + (m+3)(m+2)\,a_3 x^{m+1} + \ldots]$$
$$+ [m\,a_0 x^{m-1} + (m+1)\,a_1 x^m + (m+2)\,a_2 x^{m+1} + (m+3)\,a_3 x^{m+2} + \ldots]$$
$$- [a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + \ldots] = 0$$

Now, $\quad$ Coefficient of $x^{m-1} = 0$

$\Rightarrow \quad m(m-1)\,a_0 + m a_0 = 0$

$\Rightarrow \quad m^2 a_0 = 0 \;\Rightarrow\; m^2 = 0 \qquad (\because \; a_0 \neq 0)$

which is Indicial equation.

Its roots are $\boxed{m = 0, 0}$ *which are equal.*

Coefficient of $x^m = 0$

$$\Rightarrow \quad (m+1)\,ma_1 + (m+1)a_1 - a_0 = 0 \quad \Rightarrow \quad (m+1)^2 a_1 = a_0$$

$$\Rightarrow \quad \boxed{a_1 = \frac{a_0}{(m+1)^2}}$$

Coefficient of $x^{m+1} = 0$

$$\Rightarrow \quad (m+2)(m+1)\,a_2 + (m+2)\,a_2 - a_1 = 0 \quad \Rightarrow \quad (m+2)^2 a_2 = a_1$$

$$\Rightarrow \quad a_2 = \frac{a_1}{(m+2)^2} \quad \Rightarrow \quad \boxed{a_2 = \frac{a_0}{(m+1)^2\,(m+2)^2}}$$

Similarly, $\quad a_3 = \dfrac{a_0}{(m+1)^2\,(m+2)^2\,(m+3)^2}$ and so on.

$$\therefore \quad \text{From (1), } y = a_0 x^m \left[1 + \frac{x}{(m+1)^2} + \frac{x^2}{(m+1)^2\,(m+2)^2} + \frac{x^3}{(m+1)^2\,(m+2)^2\,(m+3)^2} + \ldots\right] \quad \ldots(2)$$

$$\text{Now,} \quad y_1 = (y)_{m=0} = a_0 \left[1 + x + \frac{x^2}{(2\,!)^2} + \frac{x^3}{(3\,!)^2} + \ldots\right] \quad \ldots(3)$$

To get the second independent solution, differentiate (1) partially w.r.t. m.

$$\frac{\partial y}{\partial m} = a_0 x^m \log x \left[1 + \frac{x}{(m+1)^2} + \frac{x^2}{(m+1)^2\,(m+2)^2} + \frac{x^3}{(m+1)^2\,(m+2)^2\,(m+3)^2} + \ldots\right]$$
$$+ a_0 x^m \left[-\frac{2x}{(m+1)^3} - \frac{2}{(m+1)^2\,(m+2)^2}\left\{\frac{1}{m+1} + \frac{1}{m+2}\right\}x^2 \right.$$
$$\left. - \frac{2}{(m+1)^2\,(m+2)^2\,(m+3)^2}\left\{\frac{1}{m+1} + \frac{1}{m+2} + \frac{1}{m+3}\right\}x^3 - \ldots\right]$$

$$\text{The second solution is } y_2 = \left(\frac{\partial y}{\partial m}\right)_{m=0} = a_0 \log x \left[1 + x + \frac{x^2}{(2\,!)^2} + \frac{x^3}{(3\,!)^2} + \ldots\right]$$
$$- 2a_0 \left[x + \frac{1}{(2\,!)^2}\left(1 + \frac{1}{2}\right)x^2 + \frac{1}{(3\,!)^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right)x^3 + \ldots\right]$$
$$= y_1 \log x - 2a_0 \left[x + \frac{1}{(2\,!)^2} + \left(1 + \frac{1}{2}\right)x^2 + \frac{1}{(3\,!)^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right)x^3 + \ldots\right]$$

Hence the complete solution is

$$y = c_1 y_1 + c_2 y_2 = (c_1 a_0 + c_2 a_0 \log x)\left[1 + x + \frac{x^2}{(2\,!)^2} + \frac{x^3}{(3\,!)^2} + \ldots\right]$$
$$- 2c_2 a_0 \left[x + \frac{1}{(2\,!)^2}\left(1 + \frac{1}{2}\right)x^2 + \frac{1}{(3\,!)^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right)x^3 + \ldots\right]$$

$$\Rightarrow \qquad y = (A + B \log x)\left[1 + x + \frac{x^2}{(2!)^2} + \frac{x^3}{(3!)^2} + \ldots\right]$$

$$- 2B\left[x + \frac{1}{(2!)^2}\left(1 + \frac{1}{2}\right)x^2 + \frac{1}{(3!)^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right)x^3 + \ldots\right]$$

where $c_1 a_0 = A$, $c_2 a_0 = B$.

TEST YOUR KNOWLEDGE

Solve in series:

1. (i) $xy'' + (1 + x)y' + 2y = 0$ (ii) $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - xy = 0$ *(M.T.U. 2012)*
2. $x^2\dfrac{d^2y}{dx^2} + x(x - 1)\dfrac{dy}{dx} + (1 - x)y = 0$ *(M.T.U. 2011)*
3. $(x - x^2)\dfrac{d^2y}{dx^2} + (1 - 5x)\dfrac{dy}{dx} - 4y = 0$ **4.** $(x - x^2)y'' + (1 - x)y' - y = 0$
5. $x^2y'' - x(1 + x)y' + y = 0$ **6.** $xy'' + y' + x^2y = 0$ *(M.T.U. 2013)*
7. $xy'' + y' + xy = 0.$ *(Bessel's equation of order zero)*

Answers

1. (i) $y = A\left(1 - 2x + \dfrac{3}{2!}x^2 - \dfrac{4}{3!}x^3 + \ldots\right) + B\left[y_1 \log x + a_0\left(3x - \dfrac{13}{4}x^2 + \ldots\right)\right]$

   (ii) $y = (A + B\log x)\left(1 + \dfrac{x^2}{2^2} + \dfrac{x^4}{2^2 . 4^2} + \ldots\right) - B\left(\dfrac{x^2}{2^2} + \dfrac{3x^4}{2 . 4^3} + \ldots\right)$

2. $y = Ax + B\left[x\log x - x + \dfrac{x^2}{4} - \ldots\right]$

3. $y = A(1^2 + 2^2x + 3^2x^2 + 4^2x^3 + \ldots) + B[y_1 \log x - 2a_0(1.2x + 2.3x^2 + 3.4x^3 + \ldots)]$

4. $y = A\left(1 + x + \dfrac{2}{4}x^2 + \dfrac{2.5}{4.9}x^3 + \ldots\right) + B\left[y_1\log x + a_0\left(-2x - x^2 - \dfrac{14}{27}x^3 - \ldots\right)\right]$

5. $y = Ax\left(1 + x + \dfrac{1}{2}x^2 + \dfrac{1}{2.3}x^3 + \ldots\right) + B\left[y_1\log x + a_0x^2\left(-1 - \dfrac{3}{4}x + \ldots\right)\right]$

6. $y = A\left[1 - \dfrac{x^3}{3^2} + \dfrac{x^6}{3^4(2!)^2} - \dfrac{x^9}{3^6(3!)^2} + \ldots\right] + B\left[y_1\log x + 2a_0\left\{\dfrac{x^3}{3^3} - \dfrac{1}{3^5(2!)^2}\left(1 + \dfrac{1}{2}\right)x^6 + \ldots\right\}\right]$

7. $y = A\left(1 - \dfrac{x^2}{2^2} + \dfrac{x^4}{2^2 . 4^2} - \dfrac{x^6}{2^2 . 4^2 . 6^2} + \ldots\right) + B\left[y_1\log x + a_0\left\{\dfrac{x^2}{2^2} - \dfrac{1}{2^2 . 4^2}\left(1 + \dfrac{1}{2}\right)x^4 + \dfrac{1}{2^2 . 4^2 . 6^2}\left(1 + \dfrac{1}{2} + \dfrac{1}{3}\right)x^6 - \ldots\right\}\right]$

1.27.3 Case III. *When Roots are Distinct, Differ by Integer and Making a Coefficient of y Infinite*

Let m_1 and m_2 be the roots such that $m_1 > m_2$.

In this case, if some of the coefficients of y become infinite when $m = m_2$, we modify the form of y by replacing a_0 by $b_0\,(m - m_2)$.

Complete solution is

$$\boxed{y = c_1\,(y)_{m_1} + c_2 \left(\frac{\partial y}{\partial m}\right)_{m_2}}$$

Remark. We can also obtain two independent solutions by putting $m = m_2$ (value of m for which some coefficients of y become infinite) in modified form of y and $\frac{\partial y}{\partial m}$. The result of putting $m = m_1$ in y will give a numerical multiple of that obtained by putting $m = m_2$.

ILLUSTRATIVE EXAMPLES

Example 1. *Obtain the series solution of the Bessel's equation of order two*

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - 4)\,y = 0 \quad \textit{near } x = 0.$$

Sol. Comparing the given equation with the form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \text{ we get}$$

$$P(x) = \frac{1}{x} \quad \text{and} \quad Q(x) = \frac{x^2-4}{x^2} = 1 - \frac{4}{x^2}$$

At $x = 0$, both P(x) and Q(x) are not analytic.

Therefore $x = 0$ is a *singular point.*

Also, $x\,P(x) = 1$ and $x^2\,Q(x) = x^2 - 4$

Both x P(x) and x^2 Q(x) are analytic at $x = 0$

$\therefore$ $x = 0$ is a *regular singular point.*

Let us assume,

$$y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + a_3x^{m+3} + \ldots \quad \ldots(1)$$

Then,
$$\frac{dy}{dx} = m\,a_0x^{m-1} + (m+1)\,a_1x^m + (m+2)\,a_2x^{m+1} + (m+3)\,a_3x^{m+2} + \ldots$$

and
$$\frac{d^2y}{dx^2} = m(m-1)\,a_0x^{m-2} + (m+1)\,ma_1x^{m-1} + (m+2)(m+1)\,a_2x^m + (m+3)(m+2)\,a_3x^{m+1} + \ldots$$

Substituting these values in the given equation, we get

$$\begin{aligned} x^2\,[m(m-1)\,a_0x^{m-2} &+ (m+1)\,ma_1x^{m-1} + (m+2)(m+1)\,a_2x^m + (m+3)(m+2)\,a_3x^{m+1} + \ldots] \\ &+ x\,[m\,a_0x^{m-1} + (m+1)\,a_1x^m + (m+2)\,a_2x^{m+1} + (m+3)\,a_3x^{m+2} + \ldots] \\ &+ (x^2-4)\,[a_0x^m + a_1x^{m+1} + a_2x^{m+2} + a_3x^{m+3} + \ldots] = 0 \end{aligned}$$

Now, Coefficient of lowest power of $x = 0$

$\Rightarrow$ Coefficient of $x^m = 0$

$\Rightarrow$ $m(m-1)a_0 + m\,a_0 - 4a_0 = 0 \quad \Rightarrow \quad (m^2 - 4)\,a_0 = 0$

$\Rightarrow$ $m^2 - 4 = 0$ *(Indicial equation)* $\quad | \because a_0 \neq 0$

$$\boxed{m = -2, 2}$$

Roots are distinct and differ by integer.

Now, Coefficient of $x^{m+1} = 0$

$(m+1)m\,a_1 + (m+1)\,a_1 - 4a_1 = 0$

$\Rightarrow$ $(m^2 + 2m - 3)\,a_1 = 0 \quad \Rightarrow \quad (m+3)(m-1)\,a_1 = 0$

$\Rightarrow$ $\boxed{a_1 = 0}$ $\quad$ | Since $m \neq 1$, and $m \neq -3$

Coefficient of $x^{m+2} = 0$

$\Rightarrow$ $(m+2)(m+1)\,a_2 + (m+2)\,a_2 + a_0 - 4a_2 = 0$

$\Rightarrow$ $(m^2 + 4m)\,a_2 + a_0 = 0$

$\Rightarrow$ $$\boxed{a_2 = \frac{-a_0}{m(m+4)}}$$

Coefficient of $x^{m+3} = 0$

$\Rightarrow$ $(m+3)(m+2)\,a_3 + (m+3)\,a_3 + a_1 - 4a_3 = 0$

$\Rightarrow$ $(m+1)(m+5)\,a_3 = -a_1$

$\Rightarrow$ $\boxed{a_3 = 0}$ $\quad | \because a_1 = 0$

Also, coefficient of $x^{m+4} = 0$

$(m+2)(m+6)\,a_4 + a_2 = 0$

$\Rightarrow$ $$a_4 = \frac{-a_2}{(m+2)(m+6)} = \frac{a_0}{m(m+2)(m+4)(m+6)}$$

$\therefore$ $$\boxed{a_4 = \frac{a_0}{m(m+2)(m+4)(m+6)}}$$

Similarly, $a_5 = a_7 = a_9 = \ldots = 0$

$$a_6 = \frac{-a_0}{m(m+2)(m+4)^2(m+6)(m+8)} \text{ etc.}$$

Substituting above obtained values in assumed y given by eqn. (1), we get

$$y = a_0 x^m \left[1 - \frac{x^2}{m(m+4)} + \frac{x^4}{m(m+2)(m+4)(m+6)} - \frac{x^6}{m(m+2)(m+4)^2(m+6)(m+8)} + \ldots\right] \quad \ldots(2)$$

Putting $m = 2$ (the greater of the two roots) in (2), the first solution is

$$y_1 = a_0 x^2 \left(1 - \frac{x^2}{2.6} + \frac{x^4}{2.4.6.8} - \frac{x^6}{2.4.6^2.8.10} + \dots\right)$$

If we put $m = -2$ in (1), the coefficients become infinite due to the presence of the factor $(m+2)$ in the denominator. To overcome this difficulty, let $a_0 = b_0(m+2)$ so that

$$y = b_0 x^m \left[(m+2) - \frac{(m+2)x^2}{m(m+4)} + \frac{x^4}{m(m+4)(m+6)} - \frac{x^6}{m(m+4)^2(m+6)(m+8)} + \dots\right]$$

Differentiating partially w.r.t. m, we get

$$\frac{\partial y}{\partial m} = b_0 x^m \log x \left[(m+2) - \frac{(m+2)x^2}{m(m+4)} + \frac{x^4}{m(m+4)(m+6)} - \dots\right]$$

$$+ b_0 x^m \left[1 - \frac{(m+2)}{m(m+4)}\left\{\frac{1}{m+2} - \frac{1}{m} - \frac{1}{m+4}\right\} x^2\right.$$

$$\left. + \frac{1}{m(m+4)(m+6)}\left\{-\frac{1}{m} - \frac{1}{m+4} - \frac{1}{m+6}\right\} x^4 \dots\right]$$

The second solution is $y_2 = \left(\dfrac{\partial y}{\partial m}\right)_{m=-2}$

$$= b_0 x^{-2} \log x \left[\frac{x^4}{(-2)(2)(4)} - \frac{x^6}{(-2)(2)^2(4)(6)} \dots\right]$$

$$+ b_0 x^{-2}\left[1 - \frac{x^2}{(-2)(2)} + \frac{1}{(-2)(2)(4)}\left(\frac{1}{2} - \frac{1}{2} - \frac{1}{4}\right) x^4 \dots\right]$$

$$= b_0 x^2 \log x \left[-\frac{1}{2^2.4} + \frac{x^2}{2^3.4.6} \dots\right] + b_0 x^{-2}\left[1 + \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} + \dots\right]$$

Hence the complete solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= Ax^2\left[\left(1 - \frac{x^2}{2.6} + \frac{x^4}{2.4.6.8} - \frac{x^6}{2.4.6^2.8.10} + \dots\right)\right] + B\left[x^2 \log x\left(-\frac{1}{2^2.4} + \frac{x^2}{2^3.4.6} \dots\right)\right.$$

$$\left. + x^{-2}\left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} + \dots\right)\right]$$

where $A = c_1 a_0$, $B = c_2 b_0$.

Example 2. *Solve in series the differential equation* $x^2 \dfrac{d^2y}{dx^2} + 5x\dfrac{dy}{dx} + x^2 y = 0$.

Sol. Comparing the given equation with the form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0, \text{ we get}$$

$$P(x) = \frac{5}{x}, \; Q(x) = 1$$

At $x = 0$, since $P(x)$ is not analytic $\therefore$ $x = 0$ is a *singular point.*

Also, $x\,P(x) = 5$

$x^2\,Q(x) = 0$

Since both $x\,P(x)$ and $x^2\,Q(x)$ are analytic at $x = 0$ $\therefore$ $x = 0$ is a *regular singular point.*

Let us assume

$$y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + \ldots \quad \ldots(1)$$

$$\therefore \quad \frac{dy}{dx} = m\,a_0 x^{m-1} + (m+1)\,a_1 x^m + (m+2)\,a_2 x^{m+1} + \ldots \quad \ldots(2)$$

and $$\frac{d^2y}{dx^2} = m(m-1)\,a_0 x^{m-2} + (m+1)\,m\,a_1 x^{m-1} + (m+2)(m+1)\,a_2 x^m + \ldots \quad \ldots(3)$$

Substituting the above values in given equation, we get

$$x^2\,[m(m-1)\,a_0 x^{m-2} + (m+1)\,ma_1 x^{m-1} + (m+2)(m+1)\,a_2 x^m + \ldots]$$
$$+ 5x\,[ma_0 x^{m-1} + (m+1)\,a_1 x^m + (m+2)\,a_2 x^{m+1} + \ldots]$$
$$+ x^2\,[a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \ldots] = 0 \quad \ldots(4)$$

Equating the coefficient of lowest power of x to zero, we get

$$m(m-1)\,a_0 + 5ma_0 = 0 \qquad [\text{Coeff. of } x^m = 0]$$

$$\Rightarrow \quad (m^2 + 4m)\,a_0 = 0$$

$$\Rightarrow \quad m(m+4) = 0 \qquad \text{(Indicial equation)} \qquad (\because\ a_0 \neq 0)$$

$$\Rightarrow \quad \boxed{m = 0, -4}$$

Hence *the roots are distinct and differing by an integer*. Equating to zero, the coefficients of successive powers of x, we get

Coefficient of $x^{m+1} = 0$

$$(m+1)\,m\,a_1 + 5(m+1)\,a_1 = 0$$

$$\Rightarrow \quad (m+5)(m+1)a_1 = 0 \;\Rightarrow\; \boxed{a_1 = 0} \quad \ldots(5) \qquad |\because\ m \neq -5, -1$$

Coefficient of $x^{m+2} = 0$

$$(m+2)(m+1)\,a_2 + 5(m+2)a_2 + a_0 = 0$$

$$(m+2)(m+6)\,a_2 + a_0 = 0$$

$$\boxed{a_2 = \frac{-a_0}{(m+2)(m+6)}} \quad \ldots(6)$$

Again, Coefficient of $x^{m+3} = 0$

$$(m+3)(m+2)\,a_3 + 5(m+3)\,a_3 + a_1 = 0$$

$$(m+3)(m+7)\,a_3 + a_1 = 0$$

$$\Rightarrow \quad a_3 = \frac{-a_1}{(m+3)(m+7)}$$

$$\Rightarrow \quad \boxed{a_3 = 0} \quad \ldots(7)$$

Similarly,

$$a_5 = a_7 = a_9 = \ldots = 0$$

Now, Coefficient of $x^{m+4} = 0$

$$(m+4)(m+3)a_4 + 5(m+4)a_4 + a_2 = 0$$

$$\Rightarrow \qquad (m+4)(m+8)a_4 = -a_2$$

$$a_4 = \frac{-a_2}{(m+4)(m+8)} = \frac{a_0}{(m+2)(m+4)(m+6)(m+8)} \text{ etc.} \qquad \text{...(8)}$$

These give $y = a_0 x^m \left[1 - \frac{x^2}{(m+2)(m+6)} + \frac{x^4}{(m+2)(m+4)(m+6)(m+8)} - \ldots\right]$ ...(9)

Putting $m = 0$ in (9), we get

$$y_1 = (y)_{m=0} = a_0\left[1 - \frac{x^2}{2.6} + \frac{x^4}{2.4.6.8} - \ldots\right] \qquad \text{...(10)}$$

If we put $m = -4$ in the series given by eqn. (9), the coefficients become infinite. To avoid this difficulty, we put $a_0 = b_0 (m+4)$, so that

$$y = b_0 x^m \left[(m+4) - \frac{(m+4)x^2}{(m+2)(m+6)} + \frac{x^4}{(m+2)(m+6)(m+8)} - \ldots\right] \qquad \text{...(11)}$$

Now, $\frac{\partial y}{\partial m} = y \log x + b_0 x^m \left[1 + \frac{m^2+8m+20}{(m^2+8m+12)^2} x^2 - \frac{(3m^2+32m+76)}{(m^3+16m^2+76m+96)^2} x^4 + \ldots\right]$

Second solution is given by

$$y_2 = \left(\frac{\partial y}{\partial m}\right)_{m=-4} = (y)_{m=-4} \log x + b_0 x^{-4}\left(1 + \frac{x^2}{4} - \frac{x^4}{4} + \ldots\right)$$

$$= b_0 x^{-4} \log x \left[0 - 0 + \frac{x^4}{(-2)(2)(4)} - \frac{x^6}{16} + \ldots\right] + b_0 x^{-4}\left(1 + \frac{x^2}{4} - \frac{x^4}{4} + \ldots\right)$$

$$= b_0 x^{-4} \log x \left(\frac{-x^4}{16} - \frac{x^6}{16} - \ldots\right) + b_0 x^{-4}\left(1 + \frac{x^2}{4} - \frac{x^4}{4} + \ldots\right)$$

Hence the complete solution is given by

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 a_0 \left(1 - \frac{x^2}{12} + \frac{x^4}{384} - \ldots\right) + c_2 b_0 x^{-4} \log x \left(-\frac{x^4}{16} - \frac{x^6}{16} - \ldots\right) + c_2 b_0 x^{-4}\left(1 + \frac{x^2}{4} - \frac{x^4}{4} + \ldots\right)$$

$$\therefore \; y = A\left(1 - \frac{x^2}{12} + \frac{x^4}{384} - \ldots\right) + Bx^{-4}\left(1 + \frac{x^2}{4} - \frac{x^4}{4} + \ldots\right) - B \log x \left(\frac{1}{16} + \frac{x^2}{16} + \ldots\right)$$

where $A = c_1 a_0$ and $B = c_2 b_0$.

TEST YOUR KNOWLEDGE

Solve in series:

1. $x(1-x)\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} - y = 0$

2. $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (x^2-1)y = 0$ [*M.T.U. (SUM) 2011*]

(*Bessel's equation of order one*)

3. $(x + x^2 + x^3)\dfrac{d^2y}{dx^2} + 3x^2\dfrac{dy}{dx} - 2y = 0$ (*U.P.T.U. 2014*)

4. $x(1-x)\dfrac{d^2y}{dx^2} - (1+3x)\dfrac{dy}{dx} + y = 0$.

Answers

1. $y = (A + B\log x)(x + 2x^2 + 3x^3 + 4x^4 + \ldots) + B(1 + x + x^2 + x^3 + \ldots)$

2. $y = Ax\left(1 - \dfrac{x^2}{2.4} + \dfrac{x^4}{2.4^2.6} - \ldots\right) + Bx^{-1}\log x\left(-\dfrac{x^2}{2} + \dfrac{x^4}{2^2.4} - \ldots\right) + Bx^{-1}\left[1 + \dfrac{x^2}{2^2} - \dfrac{3}{2^2.2^3}x^4 + \ldots\right]$

3. $y = Ax\left[1 + x - \dfrac{1}{2}x^2 - \dfrac{1}{2}x^3 + \ldots\right] + B\log x(2x + 2x^2 - x^3 + \ldots) + B(1 - x - 5x^2 - x^3 + \ldots)$

4. $y = (A + B\log x)(1.2x^2 + 2.3x^3 + 3.4x^4 + \ldots) + B(-1 + x + 5x^2 + 11x^3 + \ldots)$.

1.27.4 Case IV. *When Roots are Distinct, Differ by Integer and Making One or More Coefficients Indeterminate*

Let the roots be m_1 and m_2. If one of the coefficients (suppose a_1) become indeterminate when $m = m_2$, the *complete solution is given by putting* $m = m_2$ *in* y which then contains two arbitrary constants.

Note. The result contained by putting $m = m_1$ in y merely gives a numerical multiple of one of the series contained in the first solution. Hence we reject the solution obtained by putting $m = m_1$.

Example. *Solve in series the differential equation:* $xy'' + 2y' + xy = 0$.

Sol. Comparing the given equation with the form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \text{ we get}$$

$$P(x) = \frac{2}{x} \quad \text{and} \quad Q(x) = 1$$

At $x = 0$, $P(x)$ is not analytic $\therefore$ $x = 0$ is a *singular point.*

Also, $xP(x) = 2$ and $x^2Q(x) = x^2$

At $x = 0$, since $xP(x)$ and $x^2Q(x)$ are analytic $\therefore$ $x = 0$ is a *regular singular point.*

Let us assume

$$y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + a_3x^{m+3} + \ldots \quad \ldots(1)$$

Then,
$$\frac{dy}{dx} = ma_0x^{m+1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + (m+3)a_3x^{m+2} + \ldots$$

and
$$\frac{d^2y}{dx^2} = m(m-1)a_0x^{m-2} + (m+1)m\,a_1x^{m-1} + (m+2)(m+1)a_2x^m + (m+3)(m+2)a_3x^{m+1} + \ldots$$

Substituting these values in the given equation, we get

$$x\,[m(m-1)\,a_0\,x^{m-2} + (m+1)\,m\,a_1\,x^{m-1} + (m+2)(m+1)\,a_2 x^m + (m+3)(m+2)\,a_3\,x^{m+1} + \ldots] + 2\,[m\,a_0\,x^{m-1} + (m+1)\,a_1\,x^m + (m+2)\,a_2\,x^{m+1} + \ldots] + x\,[a_0\,x^m + a_1\,x^{m+1} + a_2\,x^{m+2} + a_3\,x^{m+3} + \ldots] = 0$$

Now, Coefficient of $x^{m-1} = 0$

$\Rightarrow$ $m(m-1)\,a_0 + 2m\,a_0 = 0$

$(m^2 + m)\,a_0 = 0$

$\Rightarrow$ $m^2 + m = 0$ (Indicial equation) $\mid \because\ a_0 \neq 0$

$\Rightarrow$ $\boxed{m = 0, -1}$

Hence *roots are distinct and differ by an integer.*

Coefficient of $x^m = 0$

$\Rightarrow$ $(m+1)\,m\,a_1 + 2(m+1)\,a_1 = 0$

$\Rightarrow$ $(m+1)(m+2)a_1 = 0$

$\Rightarrow$ $(m+1)a_1 = 0$ $\mid \because\ m+2 \neq 0$

Since $m+1$ may be zero, hence a_1 *is arbitrary* (or takes the form $\frac{0}{0}$). In other words, a_1 becomes indeterminate.

Hence the solution will contain a_0 and a_1 as arbitrary constants. The complete solution will be given by putting $m = -1$ in y.

Now, Coefficient of $x^{m+1} = 0$

$\Rightarrow$ $(m+2)(m+1)\,a_2 + 2(m+2)\,a_2 + a_0 = 0$

$\Rightarrow$ $(m+2)(m+3)\,a_2 + a_0 = 0$

$$\boxed{a_2 = \frac{-a_0}{(m+2)(m+3)}}$$

Coefficient of $x^{m+2} = 0$

$\Rightarrow$ $(m+3)(m+2)\,a_3 + 2(m+3)\,a_3 + a_1 = 0$

$(m+3)(m+4)\,a_3 + a_4 = 0$

$$\boxed{a_3 = \frac{-a_1}{(m+3)(m+4)}}$$

Coefficient of $x^{m+3} = 0$

$\Rightarrow$ $(m+4)(m+3)\,a_4 + 2(m+4)\,a_4 + a_2 = 0$

$\Rightarrow$ $(m+4)(m+5)\,a_4 = -a_2$

$\Rightarrow$ $$a_4 = \frac{-a_2}{(m+4)(m+5)}$$

$\Rightarrow$ $$\boxed{a_4 = \frac{a_0}{(m+2)(m+3)(m+4)(m+5)}}$$

Coefficient of $x^{m+4} = 0$

$$(m+5)(m+4)a_5 + 2(m+5)a_5 + a_3 = 0$$

$$(m+5)(m+6)a_5 = -a_3$$

$$\boxed{a_5 = \frac{a_1}{(m+3)(m+4)(m+5)(m+6)}}$$

and so on.

Substituting these values in eqn. (1), we get

$$y = x^m\left[a_0 + a_1 x - \frac{a_0}{(m+2)(m+3)}x^2 - \frac{a_1}{(m+3)(m+4)}x^3 + \frac{a_0}{(m+2)(m+3)(m+4)(m+5)}x^4 + \frac{a_1}{(m+3)(m+4)(m+5)(m+6)}x^5 + \ldots\right]$$

$$y = x^m\left[a_0\left\{1 - \frac{x^2}{(m+2)(m+3)} + \frac{x^4}{(m+2)(m+3)(m+4)(m+5)} - \ldots\right\} + a_1\left\{x - \frac{x^3}{(m+3)(m+4)} + \frac{x^5}{(m+3)(m+4)(m+5)(m+6)} - \ldots\right\}\right]$$

Now, $\quad (y)_{m=-1} = x^{-1}\left[a_0\left(1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \ldots\right) + a_1\left(x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \ldots\right)\right]$

$$= x^{-1}[a_0 \cos x + a_1 \sin x]$$

Hence complete solution is given by

$$y = (y)_{m=-1}$$

$$\Rightarrow \quad y = \frac{1}{x}(a_0 \cos x + a_1 \sin x).$$

Note. All those problems, in which $x = 0$ was an ordinary point of $y'' + P(x)y' + Q(x)y = 0$, can also be solved by Frobenius method as given in *Art. 2.5.4* and explained in above illustrative example.

TEST YOUR KNOWLEDGE

Solve in series:

1. $x^2\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + (x^2+2)y = 0$

2. $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + 4y = 0$ *(A.K.T.U. 2014)*

3. $(1-x^2)\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + n(n+1)y = 0.$

Answers

1. $y = x^{-2}(a_0 \cos x + a_1 \sin x)$

2. $y = a_0(1-2x^2) + a_1\left(x - \dfrac{x^3}{2} - \dfrac{x^5}{8} + \dfrac{x^7}{16} - \ldots\right)$

3. $y = a_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - \ldots\right]$

$$+ a_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 + \ldots\right].$$

ASSIGNMENT-I

(2 Marks Questions for Section-A)

1. (i) Find the general solution of $(2D + 1)^2 y = 0$ where $D = \frac{d}{dt}$ *(U.P.T.U. 2014)*

(ii) Solve: $(3D - 1)^2 y = 0$ where $D = \frac{d}{dz}$ *(U.P.T.U. 2015)*

(iii) Solve: $(2D -)^3 y = 0$ *(U.P.T.U. 2015)*

2. (i) Find the particular integral of $\frac{d^2y}{dx^2} - y = x^2$. *(U.P.T.U. 2012)*

(ii) Find the P.I. of $D^2y = x^2 + 2x + 1$ *(U.P.T.U. 2015)*

3. (i) Find the particular integral of $(D^3 + 2D^2 - D - 2)y = e^x$ *(U.P.T.U. 2013)*

(ii) Find the particular integral of $(D^2 - 4D + 2)y = e^{-2x}$ *(U.P.T.U. 2015)*

4. (i) Solve: $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$ *(U.P.T.U. 2013)*

(ii) Solve: $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$. *(U.P.T.U. 2014)*

5. (i) Find the particular integral of $(D^2 - 2D + 4)y = \cos 2x$ *(U.P.T.U. 2015)*

(ii) Find the particular integral of $(D^2 - 4D + 4)y = \cos 2x$ *(U.P.T.U. 2014)*

6. Solve : $(D^4 - 6D^3 + 12D^2 - 8D)y = 0$; $D \equiv \frac{d}{dx}$.

7. Solve : $(D^2 - 2D + 5)^2 y = 0$; $D \equiv \frac{d}{dx}$.

8. Find the particular integral of $(D^2 + a^2)\, y = \sin ax,\ a \neq 0$. *[G.B.T.U. (SUM) 2010]*

9. From a differerntial equation whose general solution is $y = e^{-x}(c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x) + c_3 e^{2x}$.

10. Solve : $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{3x}$.

11. Find the P.I. of $(D^3 - D)y = e^x + e^{-x}$.

12. Find the P.I. of $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4$.

13. Solve: $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 4y = 0$ (G.B.T.U. 2013)

14. Solve: $y'' + y = 0; y(0) = 1, y(\pi/2) = 2$.

15. Find the differential equation which represents the family of straight lines passing through the origin?

16. From a differential equation which satisfies relation $y = A\cos(mt - \alpha)$

17. Find the parts of C.F. of the differential eqn. $x^2y'' + xy' - y = 0$.

18. Solve: $\dot{x}(t) = y,\ \dot{y}(t) = -x;\ x(0) = 0, y(0) = 0$. (M.T.U. 2013)

19. What is the order of the differential equation: $x^2\left(\frac{dy}{dx}\right)^2 + xy\frac{dy}{dx} - 6y^2 = 0$.

[G.B.T.U. (AG) SUM 2010]

20. Find the complementary function of $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$. [M.T.U. (B. Pharm) 2011]

21. Find the P.I. of $(D^2 - 2D + 1)y = \sin x$. [M.T.U. (B. Pharm) 2011]

22. Solve: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$. [M.T.U. (B. Pharm) 2011]

23. Find the particular integral of the differential equ. $(D^2 + D)y = x^2 + 2x + 1$. [M.T.U. (SUM) 2011]

24. A condenser of capacity C is charged through an inductance L and resistance R in serirs where charge q satisfies the equation

$$L\frac{d^2q}{dt^2} + A\frac{dq}{dt} + Bq = 0. \text{ Here find A and B.}$$ [M.T.U. (SUM) 2011]

25. Find the roots of the auxiliary equation of the differential equation $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 9y = 4e^{3t}$.

(A.K.T.U. 2016)

26. Solve: $x\,dy - y\,dx = 0$

27. Find the integrating factor of $\frac{dy}{dx} - \frac{y}{x+1} = e^{3x}(x+1)$.

28. Solve: $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx} + 4y = 0$.

29. Find the P.I. of $(D^2 - 4)y = \sin 2x$.

30. Find the P.I. of $(D^2 + 6D + 5)y = 4e^{-x}$.

31. Find the C.F. of the differential eqn. $x^2y'' - xy' + y = \log x$.

32. Solve: $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$.

33. Find the P.I. of $\frac{d^2y}{dx^2} + y = \cosh 3x$.

34. Find the differential eqn. whose auxiliary eqn. has the roots 0, –1 and –1.

35. In the eqn. $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$, if we obtain $1 + \frac{P}{2} + \frac{Q}{4} = 0$ then find one part of C.F.

36. Find the order and degree of following differential equations. Also explain your answer.

(i) $\left(\frac{d^3y}{dx^3}\right)^4 - 6x^2\left(\frac{dy}{dx}\right)^2 + e^x = \sin xy$ (ii) $\left(\frac{d^3y}{dx^3}\right)^4 - 6x^2\left(\frac{dy}{dx}\right)^8 = 0$ (G.B.T.U. 2010)

(iii) $\frac{d^2y}{dx^2} + \sqrt{1+\left(\frac{dy}{dx}\right)^3} = 0$ (iv) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = \frac{d^2y}{dx^2}$

(A.K.T.U. 2016)

37. Solve: $\frac{dy}{dx} = \frac{1}{2}(y^2 - 1)$; $y(0) = 2$.

38. Solve: $\frac{d^5y}{dx^5} - \frac{d^3y}{dx^3} = 0$.

39. Solve: $\frac{d^2y}{dx^2} = \sec^2 y \tan y$; $y(0) = 0$, $\frac{dy}{dx} = 1$ at $x = 0$.

(U.P.T.U. 2015)

40. Solve: $(2D - 1)^2 y = 0$ where $D = \frac{d}{dt}$.

(U.P.T.U. 2014)

41. (i) Form a differential equation if its general solution is $y = Ae^x + Be^{-x}$. [M.T.U. (B. Pharm.) 2011]

(ii) Form a differential equation whose solution is $y = a \sin x$.

42. Solve: $x^2\frac{dy}{dx} = 2$.

43. Solve: $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$.

[G.B.T.U. (SUM) 2010]

44. Solve: $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} - 6y = 0$.

[G.B.T.U. (B. Pharm) SUM 2010]

45. Solve: $\frac{d^2y}{dx^2} = 0$.

[G.B.T.U. (B. T) SUM 2010]

46. Solve: $\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = 0$.

47. State the criterion for linearly independent solutions of the homogenous linear nth order differerential equation. (M.T.U. 2013)

(G.B.T.U. 2011)

48. Find the P.I. of $(D^2 - 4D + 4)\, y = \sin 2x$.

(G.B.T.U. 2013)

49. Find the particular integral of $\frac{d^2y}{dx^2} = x^2 + 2x - 1$.

(M.T.U. 2011)

50. Find the differential equation whose solution is $xy = ae^x + be^{-x} + x^2$.

(A.K.T.U. 2018)

51. Solve: $(D + 1)^3\, y = 2\, e^{-x}$.

52. Show that the differential equation $ydx - 2xdy = 0$ represents a family of parabolas. (A.K.T.U. 2017)

53. Find the particular integral of $(D - a)^2 y = e^{ax} f''(x)$. (A.K.T.U. 2017)

54. Solve the differential equation $\frac{d^2y}{dx^2} = -12x^2 + 24x - 20$ with the condition $x = 0, y = 5$ and $x = 2, y = 21$ and hence find the value of y at $x = 1$. (A.K.T.U. 2017)

55. For a differential equation $\frac{d^2y}{dx^2} + 2\alpha\frac{dy}{dx} + y = 0$, find the value of α for which the differential equation characteristic equation has equal number of roots. (A.K.T.U. 2017)

56. Classify the singular points of the differential equation $x^2\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = 0$ where a and b are constants. (A.K.T.U. 2012)

57. What are the roots of the indicial equation for the power series solution of the differential equation $2x^2y'' + xy' + (x^2 - 3)y = 0$. (A.K.T.U. 2011)

58. Explain ordinary and singular points of a differential equation of the form:

$$a_2(x)y'' + a_1(x)y' + a_0(x)\, y = 0. \qquad \text{(A.K.T.U. 2013)}$$

Answers

1. (i) $y = (c_1 + c_2t)\, e^{-t/2}$ (ii) $y = (c_1 + c_2z)e^{z/3}$ (iii) $y = (c_1 + c_2x + c_3x^2)\, e^{x/2}$

2. (i) $-(x^2 + 2)$ (ii) $\frac{x^4}{12} + \frac{x^3}{3} + \frac{x^2}{2}$

3. (i) $\frac{x}{6}e^x$ (ii) $\frac{1}{14}e^{-2x}$

4. (i) $y = e^{-x/2}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right)$ (ii) $y = c_1 + (c_2 + c_3x)\, e^{-x}$

5. (i) $-\frac{1}{4}\sin 2x$ (ii) $-\frac{1}{8}\sin 2x$

6. $y = c_1 + (c_2 + c_3x + c_4x^2)\, e^{2x}$ **7.** $y = e^x\, [(c_1 + c_2x)\cos 2x + (c_3 + c_4x)\sin 2x]$

8. $-\frac{x}{2a}\cos ax$ **9.** $\frac{d^3y}{dx^3} - 8y = 0$ **10.** $y = c_1e^x + c_2e^{2x} + \frac{1}{2}e^{3x}$

11. $\frac{1}{2}x(e^x + e^{-x})$ **12.** $\frac{x^3}{3} + 4x$

13. $y = e^{\frac{3}{2}t}\left(c_1 \cos\frac{\sqrt{7}}{2}t + c_2 \sin\frac{\sqrt{7}}{2}t\right)$ **14.** $y = \cos x + 2\sin x$

15. $y\,dx - xdy = 0$ **16.** $\frac{d^2y}{dt^2} = -m^2y$ **17.** $x, \frac{1}{x}$

18. $x = 0, y = 0$

19. 1

20. $e^x (c_1 \cos x + c_2 \sin x)$

21. $\frac{1}{2} \cos x$

22. $y = (c_1 + c_2 x)e^x$

23. $\frac{x^3}{3} + x$

24. $A = R, B = C^{-1}$

25. 3, 3

26. $y = cx$

27. $\frac{1}{x+1}$

28. $y = c_1 e^{-x} + (c_2 + c_3 x) e^{2x}$

29. $-\frac{1}{8} \sin 2x$

30. xe^{-x}

31. C.F. $y = c_1 x + c_2 x \log x$

32. $y = c_1 + c_2 \log x$

33. $\frac{1}{10} \cosh 3x$

34. $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

35. e^{2x}

36. (*i*) 3, 4 (*ii*) 3, 4 (*iii*) 2, 2 (*iv*) 2, 2

37. $y = \frac{3 + e^t}{3 - e^t}$

38. $y = c_1 + c_2 x + c_3 x^2 + c_4 e^x + c_5 e^{-x}$

39. $y = \sin^{-1} x$

40. $y = (c_1 + c_2 t) e^{t/2}$

41. (*i*) $\frac{d^2y}{dx^2} - y = 0$ (*ii*) $\frac{dy}{dx} = y \cot x$

42. $y = -\frac{2}{x} c_1$

43. $y = c_1 e^{-x} + c_2 e^{-2x}$

44. $y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x}$

45. $y = c_1 + c_2 x$

46. $y = (e_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x$

47. $W(y_1, y_2, ..., y_n) \neq 0$

48. $\frac{1}{8} \cos 2x$

49. $\frac{x^4}{12} + \frac{x^3}{3} - \frac{x^2}{2}$

50. $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy = 2 - x^2$

51. $y = (c_1 + c_2 x + c_3 x^2) e^{-x} + \frac{x^3}{3} e^{-x}$.

53. P.I. $= e^{ax} f(x)$.

54. $y(1) = 18$.

55. $\alpha = \pm 1$.

56. $x = 0$ is a regular singular point.

57. $-1, \frac{3}{2}$.

MODULE 2

Multivariable Calculus – II

2.1 FINITE AND INFINITE INTERVALS

An interval is said to be finite or infinite according as its length is finite or infinite. Thus the intervals $[a, b]$, $[a, b)$, $(a, b]$, (a, b), each with length $(b - a)$, are finite (or bounded) if both a and b are finite. The intervals $[a, \infty)$, (a, ∞), $(-\infty, b]$, $(-\infty, b)$ and $(-\infty, \infty)$ are infinite (or unbounded) intervals.

2.2 BOUNDED FUNCTION

A function f is said to be bounded if its range is bounded. Thus, $f : [a, b] \to R$ is bounded if there exist two real numbers m and M, $(m \le \text{M})$ such that

$$m \le f(x) \le \text{M} \ \forall \ x \in [a, b]$$

f is also bounded if there exists a positive real number K such that

$$|f(x)| \le \text{K} \ \forall \ x \in [a, b].$$

2.3 PROPER INTEGRAL

The definite integral $\int_a^b f(x)\,dx$ is called a proper integral if

(*i*) The interval of integration $[a, b]$ is finite (or bounded)

(*ii*) The integrand f is bounded on $[a, b]$

If F(x) is an indefinite integral of $f(x)$, then $\int_a^b f(x)\,dx = \text{F}(b) - \text{F}(a)$.

2.4 IMPROPER INTEGRAL

The definite integral $\int_a^b f(x)\,dx$ is called an improper integral if either or both the above conditions are not satisfied. Thus $\int_a^b f(x)\,dx$ is an improper integral if either the interval of integration $[a, b]$ is not finite or f is not bounded on $[a, b]$ or neither the interval $[a, b]$ is finite nor f is bounded over it.

(*i*) In the definite integral $\int_a^b f(x)\,dx$, if either a or b or both a and b are infinite so that the interval of integration is unbounded but f is bounded then $\int_a^b f(x)\,dx$ is called an **improper integral of the first kind**.

For example, $\int_1^{\infty} \frac{dx}{\sqrt{x}}$, $\int_{-\infty}^{\infty} e^{2x}\, dx$, $\int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 2}$ are improper integrals of the first kind.

(ii) In the definite integral $\int_a^b f(x)\, dx$, if both a and b are finite so that the interval of integration is finite but f has one or more points of infinite discontinuity *i.e.*, f is not bounded on $[a, b]$, then $\int_a^b f(x)\, dx$ is called an **improper integral of the second kind.**

For example, $\int_0^1 \frac{dx}{x^2}$, $\int_1^2 \frac{dx}{2-x}$, $\int_1^4 \frac{dx}{(x-1)(4-x)}$ are improper integrals of the second kind.

(iii) In the definite integral $\int_a^b f(x)\, dx$, if the interval of integration is unbounded (so that a or b or both are infinite) and f is also unbounded, then $\int_a^b f(x)\, dx$ is called an **improper integral of the third kind**.

For example, $\int_0^{\infty} \frac{e^{-x}}{\sqrt{x}}\, dx$ is an improper integral of the third kind.

2.5 IMPROPER INTEGRAL AS THE LIMIT OF A PROPER INTEGRAL

(a) *When the improper integral is of the first kind, either a or b or both a and b are infinite but f is bounded. We define*

(i) $\int_a^{\infty} f(x)\, dx = \lim_{t \to \infty} \int_a^t f(x)\, dx, (t > a)$

The improper integral $\int_a^{\infty} f(x)\, dx$ is said to be **convergent** if the limit on the right hand side exists finitely and the integral is said to be **divergent** if the limit is $+\infty$ or $-\infty$.

If the integral is neither convergent not divergent, then it is said to be **oscillating**.

(ii) $\int_{-\infty}^{b} f(x)\, dx = \lim_{t \to -\infty} \int_t^b f(x)\, dx$, $(t < b)$

The improper integral $\int_{-\infty}^{b} f(x)\, dx$ is said to be convergent if the limit on the right hand side exists finitely and the integral is said to be divergent if the limits is $+\infty$ or $-\infty$.

(iii) $\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{c} f(x)\, dx + \int_c^{\infty} f(x)\, dx$ where c is any real number

$$= \lim_{t_1 \to -\infty} \int_{t_1}^{c} f(x)\, dx + \lim_{t_2 \to \infty} \int_c^{t_2} f(x)\, dx$$

The improper integral $\int_{-\infty}^{\infty} f(x)\, dx$ is said to be convergent if both the limits on the right hand side exist finitely and independent of each other, otherwise it is said to be divergent.

Note: $\int_{-\infty}^{\infty} f(x)\, dx \neq \lim_{t \to \infty} \left[\int_{-t}^{c} f(x)\, dx + \int_c^{t} f(x)\, dx \right]$

(b) *When the improper integral is of the second kind, both a and b are finite but f has one (or more) points of infinite discontinuity on [a, b].*

(i) If $f(x)$ becomes infinite at $x = b$ only, we define $\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0+} \int_a^{b-\varepsilon} f(x)\,dx$

The improper integral $\int_a^b f(x)\,dx$ is said to be convergent if the limit on the right hand side exists finitely and the integral is said to be divergent if the limit is $+\infty$ or $-\infty$.

(ii) If $f(x)$ becomes infinite at $x = a$ only, we define $\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0+} \int_{a+\varepsilon}^{b} f(x)\,dx$

The improper integral $\int_a^b f(x)\,dx$ converges if the limit on the right hand side exists finitely, otherwise it is said to be divergent.

(iii) If $f(x)$ becomes infinite at $x = c$ only where $a < c < b$, we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$= \lim_{\varepsilon_1 \to 0+} \int_a^{c-\varepsilon_1} f(x)\,dx + \lim_{\varepsilon_2 \to 0+} \int_{c+\varepsilon_2}^{b} f(x)dx$$

The improper integral $\int_a^b f(x)\,dx$ is said to be convergent if both the limits on the right hand side exist finitely and independent of each other, otherwise it is said to be divergent.

Note:

1. If f has infinite discontinuity at an end point of the interval of integration, then the point of discontinuity is approached from within the interval.
   Thus if the interval of integration is $[a, b]$ and
   (i) f has infinite discontinuity at 'a', we consider $[a + \varepsilon, b]$ as $\varepsilon \to 0+$.
   (ii) f has infinite discontinuity at 'b', we consider $[a, b - \varepsilon]$ as $\varepsilon \to 0+$.
2. A proper integral is always convergent.
3. If $\int_a^b f(x)\,dx$ is convergent, then
   (i) $\int_a^b kf(x)\,dx$ is convergent, $k \in R$, (ii) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
   where $a < c < b$ and each integral on right hand side is convergent.
4. For any c between a and b, *i.e.*, $a < c < b$, we have

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

If $\int_c^b f(x)\,dx$ is a proper integral, then the two integrals $\int_a^b f(x)\,dx$ and $\int_a^c f(x)\,dx$ converge or diverge together. Thus while testing the integral $\int_a^b f(x)\,dx$ for convergence at a, it may be replaced by $\int_a^c f(x)\,dx$ for any convenient c such that $a < c < b$.

ILLUSTRATIVE EXAMPLES

Example 1. *Examine the convergence of the improper integrals:*

(i) $\int_1^\infty \frac{1}{x} dx$ (ii) $\int_1^\infty \frac{dx}{\sqrt{x}}$

(iii) $\int_1^\infty \frac{dx}{x^{3/2}}$ (iv) $\int_0^\infty \frac{dx}{1+x^2}$

Sol. (i) By definition, $\int_1^\infty \frac{dx}{x} = \lim_{t\to\infty} \int_1^t \frac{dx}{x} = \lim_{t\to\infty} [\log x]_1^t = \lim_{t\to\infty} \log t = \infty$

$\Rightarrow \int_0^\infty \frac{dx}{x}$ is divergent.

(ii) By definition, $\int_1^\infty \frac{dx}{\sqrt{x}} = \lim_{t\to\infty} \int_1^t \frac{dx}{\sqrt{x}} = \lim_{t\to\infty} \int_1^t x^{-1/2}\, dx$

$$= \lim_{t\to\infty} \left[2\sqrt{x}\right]_1^t = \lim_{t\to\infty} (2\sqrt{t} - 2) = \infty$$

$\Rightarrow \int_1^\infty \frac{dx}{\sqrt{x}}$ is divergent.

(iii) By definition, $\int_1^\infty \frac{dx}{x^{3/2}} = \lim_{t\to\infty} \int_1^t x^{-3/2}\, dx = \lim_{t\to\infty} \left[\frac{x^{-1/2}}{-\frac{1}{2}}\right]_1^t$

$$= \lim_{t\to\infty} \left[\frac{-2}{\sqrt{x}}\right]_1^t = \lim_{t\to\infty} \left(\frac{-2}{\sqrt{t}} + 2\right)$$

$= 0 + 2 = 2$ which is finite.

$\Rightarrow \int_1^\infty \frac{dx}{x^{3/2}}$ is convergent and its value is 2.

(iv) By definition, $\int_0^\infty \frac{dx}{1+x^2} = \lim_{t\to\infty} \int_0^t \frac{dx}{1+x^2} = \lim_{t\to\infty} \left[\tan^{-1} x\right]_0^t$

$= \lim_{t\to\infty} \left(\tan^{-1} t - \tan^{-1} 0\right) = \frac{\pi}{2}$, which is finite.

$\Rightarrow \int_1^\infty \frac{dx}{1+x^2}$ is convergent and its value is $\frac{\pi}{2}$.

Example 2. *Examine for convergence the improper integrals:*

(i) $\int_0^\infty e^{-mx}\, dx \ (m > 0)$ (ii) $\int_a^\infty \frac{x}{1+x^2} dx$ (iii) $\int_0^\infty \sin x\, dx$

(iv) $\int_0^\infty \frac{dx}{(1+x)^3}$ (v) $\int_0^\infty \frac{dx}{x^2 + 4a^2}$

Sol. (*i*) By definition, $\int_0^{\infty} e^{-mx}dx = \lim_{t \to \infty} \int_0^t e^{-mx}dx = \lim_{t \to \infty}\left[\frac{e^{-mx}}{-m}\right]_0^t$

$$= \lim_{t \to \infty} -\frac{1}{m}\left(e^{-mt} - 1\right)$$

$$= -\frac{1}{m}(0-1) = \frac{1}{m}, \text{ which is finite.}$$

$\Rightarrow \quad \int_0^{\infty} e^{-mx}\, dx$ is convergent and its value is $\frac{1}{m}$.

(*ii*) By definition, $\int_a^{\infty} \frac{x}{1+x^2} = \lim_{t \to \infty} \int_a^t \frac{x}{1+x^2}dx$

$$= \lim_{t \to \infty}\left[\frac{1}{2}\log\left(1+x^2\right)\right]_a^t$$

$$= \lim_{t \to \infty} \frac{1}{2}\left[\log\left(1+t^2\right) - \log\left(1+a^2\right)\right] = \infty.$$

$\Rightarrow \quad \int_0^{\infty} \frac{x}{1+x^2}dx$ is divergent.

(*iii*) $\int_0^{\infty} \sin x\, dx = \lim_{t \to \infty} \int_0^t \sin x\, dx = \lim_{t \to \infty}[-\cos x]_0^t = \lim_{t \to \infty}(1 - \cos t)$

which does not exist uniquely since $\cos t$ oscillates between -1 and $+1$ when $t \to \infty$.

$\Rightarrow \quad \int_0^{\infty} \sin x\, dx$ oscillates.

(*iv*) $\int_0^{\infty} \frac{dx}{(1+x)^3} = \lim_{t \to \infty} \int_0^t (1+x)^{-3}dx = \lim_{t \to \infty}\left[\frac{(1+x)^{-2}}{-2}\right]_0^t$

$$= \lim_{t \to \infty} -\frac{1}{2}\left[\frac{1}{(1+t)^2} - 1\right] = -\frac{1}{2}(0-1) = \frac{1}{2} \text{ which is finite.}$$

$\Rightarrow \quad \int_0^{\infty} \frac{dx}{(1+x)^3}$ is convergent and its value is $\frac{1}{2}$.

(*v*) $\int_0^{\infty} \frac{dx}{x^2+4a^2} = \lim_{t \to \infty} \int_0^t \frac{dx}{x^2+(2a)^2} = \lim_{t \to \infty}\left[\frac{1}{2a}\tan^{-1}\frac{x}{2a}\right]_0^t$

$$= \lim_{t \to \infty} \frac{1}{2a}\left[\tan^{-1}\frac{t}{2a} - \tan^{-1} 0\right]$$

$$= \frac{1}{2a}\left[\frac{\pi}{2}\right] = \frac{\pi}{4a} \text{ which is finite.}$$

$\Rightarrow \quad \int_0^{\infty} \frac{dx}{x^2+4a^2}$ is convergent and its value is $\frac{\pi}{4a}$

Example 3. *Examine for convergence the improper integrals:*

(i) $\int_3^\infty \frac{dx}{(x-2)^2}$ (ii) $\int_{\sqrt{2}}^\infty \frac{dx}{x\sqrt{x^2-1}}$

(iii) $\int_2^\infty \frac{2x^2}{x^4-1}\,dx$ (iv) $\int_1^\infty \frac{x}{(1+2x)^3}\,dx$

Sol. (i)
$$\int_3^\infty \frac{dx}{(x-2)^2} = \lim_{t\to\infty}\int_3^t (x-2)^{-2}\,dx$$
$$= \lim_{t\to\infty}\left[\frac{(x-2)^{-1}}{-1}\right]_3^t = \lim_{t\to\infty} -\left[\frac{1}{t-2}-1\right]$$
$$= -(0-1) = 1, \text{ which is finite.}$$

$\Rightarrow$ $\int_3^\infty \frac{dx}{(x-2)^2}$ is convergent and its value is 1.

(ii)
$$\int_{\sqrt{2}}^\infty \frac{dx}{x\sqrt{x^2-1}} = \lim_{t\to\infty}\int_{\sqrt{2}}^t \frac{dx}{x\sqrt{x^2-1}} = \lim_{t\to\infty}\left[\sec^{-1}x\right]_{\sqrt{2}}^t$$
$$= \lim_{t\to\infty}\left(\sec^{-1}t - \sec^{-1}\sqrt{2}\right)$$
$$= \frac{\pi}{2}-\frac{\pi}{4} = \frac{\pi}{4} \text{ which is finite.}$$

$\Rightarrow$ $\int_{\sqrt{2}}^\infty \frac{dx}{x\sqrt{x^2-1}}$ is convergent and its value is $\frac{\pi}{4}$.

(iii)
$$\int_2^\infty \frac{2x^2}{x^4-1}\,dx = \lim_{t\to\infty}\int_2^t \frac{(x^2+1)+(x^2-1)}{(x^2+1)(x^2-1)}\,dx$$
$$= \lim_{t\to\infty}\int_2^t\left(\frac{1}{x^2-1}+\frac{1}{x^2+1}\right)dx$$
$$= \lim_{t\to\infty}\left[\frac{1}{2}\log\frac{x-1}{x+1}+\tan^{-1}x\right]_2^t$$
$$= \lim_{t\to\infty}\left[\frac{1}{2}\log\frac{t-1}{t+1}+\tan^{-1}t-\frac{1}{2}\log\frac{1}{3}-\tan^{-1}2\right]$$
$$= \frac{1}{2}\lim_{t\to\infty}\log\frac{1-\frac{1}{t}}{1+\frac{1}{t}}+\frac{\pi}{2}+\frac{1}{2}\log 3-\tan^{-1}2$$
$$= \frac{1}{2}\log 1+\frac{\pi}{2}+\frac{1}{2}\log 3-\tan^{-1}2$$
$$= \frac{\pi}{2}+\frac{1}{2}\log 3-\tan^{-1}2 \text{ which is finite.}$$

$\Rightarrow$ $\int_2^\infty \frac{2x^2}{x^4-1}\,dx$ is convergent and its value is $\frac{\pi}{2}+\frac{1}{2}\log 3-\tan^{-1}2$.

(iv) $$\int_1^\infty \frac{x}{(1+2x)^3}\,dx = \lim_{t\to\infty}\int_1^t \frac{x}{(1+2x)^3}\,dx$$

$$= \lim_{t\to\infty}\int_1^t \frac{\frac{1}{2}(1+2x)-\frac{1}{2}}{(1+2x)^3}\,dx$$

$$= \lim_{t\to\infty}\int_1^t \left[\frac{1}{2}(1+2x)^{-2}-\frac{1}{2}(1+2x)^{-3}\right]dx$$

$$= \lim_{t\to\infty}\left[\frac{-1}{4(1+2x)}+\frac{1}{8(1+2x)^2}\right]_1^t$$

$$= \lim_{t\to\infty}\left[\frac{-1}{4(1+2t)}+\frac{1}{8(1+2t)^2}+\frac{1}{12}-\frac{1}{72}\right]$$

$$= 0+0+\frac{1}{12}-\frac{1}{72}=\frac{5}{72} \text{ which is finite.}$$

$\Rightarrow \int_1^\infty \frac{x}{(1+2x)^3}\,dx$ is convergent and its value is $\frac{5}{72}$.

Example 4. *Examine for convergence the integrals:*

(i) $\int_1^\infty xe^{-x}\,dx$ (ii) $\int_0^\infty x^2 e^{-x}\,dx$ (iii) $\int_0^\infty xe^{-x^2}\,dx$

(iv) $\int_0^\infty x^3 e^{-x^2}\,dx$ (v) $\int_0^\infty x\sin x\,dx$.

Sol. (i) $$\int_1^\infty xe^{-x}\,dx = \lim_{t\to\infty}\int_1^t xe^{-x}\,dx \quad \text{(Integrating by parts)}$$

$$= \lim_{t\to\infty}\left[-xe^{-x}-e^{-x}\right]_1^t$$

$$= \lim_{t\to\infty}\left(-te^{-t}-e^{-t}+e^{-1}+e^{-1}\right)$$

$$= \lim_{t\to\infty}\left(\frac{-t}{e^t}\right)-\lim_{t\to\infty}e^{-t}+\frac{2}{e}$$

(Applying L' Hospital's Rule to first limit)

$$= \lim_{t\to\infty}\left(\frac{-1}{e^t}\right)-0+\frac{2}{e}=0+\frac{2}{e}=\frac{2}{e} \text{ which is finite.}$$

$\Rightarrow \int_1^\infty xe^{-x}\,dx$ is convergent and its value is $\frac{2}{e}$.

(ii) $$\int_0^\infty x^2e^{-x}\,dx = \lim_{t\to\infty}\int_0^t x^2e^{-x}\,dx \quad \text{(Integrating by parts)}$$

$$= \lim_{t\to\infty}\left[-x^2e^{-x}-2xe^{-x}-2e^{-x}\right]_0^t$$

$$= \lim_{t \to \infty} \left(-t^2e^{-t} - 2te^{-t} - 2e^{-t} + 2\right)$$

$$= \lim_{t \to \infty} \left(\frac{-t^2}{e^t}\right) - 2 \lim_{t \to \infty} \left(\frac{t}{e^t}\right) - 0 + 2$$

(Applying L' Hospital's rule)

$$= \lim_{t \to \infty} \left(\frac{-2t}{e^t}\right) - 2 \lim_{t \to \infty} \left(\frac{1}{e^t}\right) + 2$$

(Again applying L' Hospital's rule to first limit)

$$= \lim_{t \to \infty} \left(\frac{-2}{e^t}\right) - 2 \times 0 + 2 = 0 + 2 = 2 \text{ which is finite.}$$

$\Rightarrow \int_1^\infty x^2 e^{-x} dx$ is convergent and its value is 2.

(*iii*) $$\int_0^\infty xe^{-x^2} dx = \lim_{t \to \infty} \int_0^t xe^{-x^2} dx$$

Put $x^2 = z$ so that $2xdx = dz$ or $xdx = \frac{1}{2}dz$

When $x = 0, z = 0$; when $x = t, z = t^2$

$$\therefore \quad \int_0^\infty xe^{-x^2} dx = \lim_{t \to \infty} \int_0^{t^2} \frac{1}{2} e^{-z} dz = \lim_{t \to \infty} \left[-\frac{1}{2} e^{-z}\right]_0^{t^2}$$

$$= \lim_{t \to \infty} -\frac{1}{2}\left(e^{-t^2} - 1\right) = -\frac{1}{2}(0 - 1) = \frac{1}{2} \text{ which is finite.}$$

$\Rightarrow \int_0^\infty x e^{-x^2} dx$ is convergent and its value is $\frac{1}{2}$.

(*iv*) $$\int_0^\infty x^3e^{-x^2} dx = \lim_{t \to \infty} \int_0^t x \cdot x^2 e^{-x^2} dx$$

Put $x^2 = z$ so that $2x\, dx = dz$

When $x = 0, z = 0$; when $x = t, z = t^2$

$$\therefore \quad \int_0^\infty x^3 e^{-x^2} dx = \lim_{t \to \infty} \int_0^{t^2} \frac{1}{2} ze^{-z} dz \qquad \text{(Integrating by parts)}$$

$$= \lim_{t \to \infty} \frac{1}{2}\left[-ze^{-z}\, e^{-z}\right]_0^{t^2}$$

$$= \lim_{t \to \infty} \frac{1}{2}\left[-t^2 e^{-t^2} - e^{-t^2} + 1\right] = -\frac{1}{2} \lim_{t \to \infty} \left(\frac{t^2}{e^{t^2}}\right) - 0 + \frac{1}{2}$$

(Applying L' Hospital's rule)

$$= -\frac{1}{2} \lim_{t \to \infty} \left(\frac{2t}{2t\, e^{t^2}}\right) + \frac{1}{2} = -\frac{1}{2} \lim_{t \to \infty} \left(-\frac{1}{e^{t^2}}\right) + \frac{1}{2}$$

$$= 0 + \frac{1}{2} = \frac{1}{2}, \text{ which is finite.}$$

$\Rightarrow \quad \int_0^\infty x^3 e^{-x^2}\, dx$ is convergent and its value is $\frac{1}{2}$.

(v) $$\int_0^\infty x \sin x\, dx = \lim_{t\to\infty} \int_0^t x \sin x\, dx \qquad \text{(Integrating by parts)}$$

$$= \lim_{t\to\infty} [-x\cos x + \sin x]_0^t = \lim_{t\to\infty} (-t\cos t + \sin t)$$

which oscillates between $-\infty$ and $+\infty$ since $\cos t$ oscillates between -1 and $+1$ as $t \to \infty$.

$\Rightarrow \quad \int_0^\infty x \sin x\, dx$ is not convergent. (In fact, it oscillates infinitely.)

Example 5. *Examine for convergence the integrals:*

(i) $\int_1^\infty \frac{dx}{(1+x)\sqrt{x}}$ (ii) $\int_2^\infty \frac{dx}{x \log x}$

(iii) $\int_0^\infty e^{-x} \sin x\, dx$ (iv) $\int_0^\infty e^{-ax} \cos bx\, dx$.

Sol. (i) $$\int_1^\infty \frac{dx}{(1+x)\sqrt{x}} = \lim_{t\to\infty} \int_1^t \frac{dx}{(1+x)\sqrt{x}}$$

Put $\sqrt{x} = z$ so that $\frac{1}{2\sqrt{x}} dx = dz$

When $x = 1, z = 1$; when $x = t, z = \sqrt{t}$

$$\therefore \quad \int_1^\infty \frac{dx}{(1+x)\sqrt{x}} = \lim_{t\to\infty} \int_1^{\sqrt{t}} \frac{2\,dz}{1+z^2} = \lim_{t\to\infty} \left[2\tan^{-1} z\right]_1^{\sqrt{t}}$$

$$= \lim_{t\to\infty} 2\left[\tan^{-1}\sqrt{t} - \tan^{-1} 1\right]$$

$$= 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi}{2}, \text{ which is finite.}$$

$\Rightarrow \quad \int_1^\infty \frac{dx}{(1+x)\sqrt{x}}$ is convergent and its value is $\frac{\pi}{2}$.

(ii) $$\int_2^\infty \frac{dx}{x \log x} = \lim_{t\to\infty} \int_2^t \frac{1/x}{\log x}\, dx$$

$$= \lim_{t\to\infty} [\log(\log x)]_2^t$$

$$= \lim_{t\to\infty} [\log(\log t) - \log(\log 2)] = \infty.$$

$\Rightarrow \quad \int_2^\infty \frac{dx}{x \log x}$ is divergent.

(iii) $$\int_0^\infty e^{-x} \sin x\, dx = \lim_{t\to\infty} \int_0^t e^{-x} \sin x\, dx$$

$$= \lim_{t\to\infty}\left[\frac{e^{-x}}{(-1)^2+1^2}(-1\sin x - 1\cos x)\right]_0^t$$

$$\left[\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

$$= \lim_{t\to\infty}\left[-\frac{1}{2}e^{-x}(\sin x+\cos x)\right]_0^t$$

$$= \lim_{t\to\infty} -\frac{1}{2}\left[e^{-t}(\sin t+\cos t)-1\right]$$

$$= -\frac{1}{2}\left[(0\times \text{a finite quantity})-1\right] = \frac{1}{2} \text{ which is finite.}$$

$\int_0^\infty e^{-x}\sin x$ is convergent and its value is $\frac{1}{2}$.

(iv) $$\int_0^\infty e^{-ax}\cos bx\,dx = \lim_{t\to\infty}\int_0^t e^{-ax}\cos bx\,dx$$

$$= \lim_{t\to\infty}\left[\frac{e^{-ax}}{(-a)^2+b^2}(-a\cos bx + b\sin bx)\right]_0^t$$

$$= \lim_{t\to\infty}\frac{1}{a^2+b^2}\left[e^{-at}(-a\cos bt+b\sin bt)+a\right]$$

$$= \frac{1}{a^2+b^2}\left[(0\times \text{a finite quantity})+a\right]$$

$$= \frac{a}{a^2+b^2} \text{ which is finite.}$$

$\Rightarrow$ $\int_0^\infty e^{-ax}\cos bx\,dx$ is convergent and its value is $\frac{a}{a^2+b^2}$.

Example 6. *Examine the convergence of the integrals:*

(i) $\int_1^\infty \frac{dx}{x(x+1)}$ (ii) $\int_1^\infty \frac{dx}{x^2(x+1)}$

(iii) $\int_1^\infty \frac{\tan^{-1}x}{x^2}dx$ (iv) $\int_0^\infty e^{-\sqrt{x}}\,dx$.

Sol. (*i*) $$\int_1^\infty \frac{dx}{x(x+1)} = \lim_{t\to\infty}\int_1^t \frac{dx}{x(x+1)}$$

$$= \lim_{t\to\infty}\int_1^t\left(\frac{1}{x}-\frac{1}{x+1}\right)dx \qquad \text{[Partial Fractions]}$$

$$= \lim_{t\to\infty}\left[\log x - \log(x+1)\right]_1^t = \lim_{t\to\infty}\left[\log\frac{x}{x+1}\right]_1^t$$

$$= \lim_{t \to \infty} \left[\log \frac{t}{t+1} - \log \frac{1}{2} \right] = \lim_{t \to \infty} \left[\log \frac{1}{1 + \frac{1}{t}} \right] + \log 2$$

$$= \log 1 + \log 2 = \log 2, \text{ which is finite.}$$

$\Rightarrow \quad \int_1^{\infty} \frac{dx}{x(x+1)}$ is convergent and its value is log 2.

(*ii*)
$$\int_1^{\infty} \frac{dx}{x^2(x+1)} = \lim_{t \to \infty} \int_1^t \frac{dx}{x^2(x+1)} = \lim_{t \to \infty} \int_1^t \left(-\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1} \right) dx$$

[Partial Fractions]

$$= \lim_{t \to \infty} \left[-\log x - \frac{1}{x} + \log(x+1) \right]_1^t = \lim_{t \to \infty} \left[\log \frac{x+1}{x} - \frac{1}{x} \right]_1^t$$

$$= \lim_{t \to \infty} \left[\log\left(1 + \frac{1}{t}\right) - \frac{1}{t} - \log 2 + 1 \right]$$

$$= \log 1 - 0 - \log 2 + 1 = 1 - \log 2 \text{ which is finite.}$$

$\Rightarrow \quad \int_1^{\infty} \frac{dx}{x^2(x+1)}$ is convergent and its value is $1 - \log 2$.

(*iii*)
$$\int_1^{\infty} \frac{\tan^{-1} x}{x^2}\, dx = \lim_{t \to \infty} \int_1^t \frac{\tan^{-1} x}{x^2}\, dx$$

$\frac{P}{b} = \frac{x}{1}$ Put $x = \tan\theta$, so that $dx = \sec^2\theta\, d\theta$ $\qquad \tan^{-1}\tan\theta = \theta$

$\frac{P}{1} = \frac{x}{\sqrt{1+x^2}}$

$$\int \frac{\tan^{-1} x}{x^2}\, dx = \int \frac{\theta}{\tan^2\theta} \sec^2\theta\, d\theta$$

$$= \int \theta \operatorname{cosec}^2\theta\, d\theta = \theta(-\cot\theta) - \int 1\,(-\cot\theta\, d\theta)$$

$$= -\theta\cot\theta + \log\sin\theta$$

$$= \frac{-\tan^{-1} x}{x} + \log \frac{x}{\sqrt{1+x^2}}$$

$$\therefore \quad \int_1^{\infty} \frac{\tan^{-1} x}{x^2}\, dx = \lim_{t \to \infty} \left[-\frac{\tan^{-1} x}{x} + \log \frac{x}{\sqrt{1+x^2}} \right]_1^t$$

$$= \lim_{t \to \infty} \left[-\frac{\tan^{-1} t}{t} + \log \frac{t}{\sqrt{1+t^2}} + \tan^{-1} 1 - \log \frac{1}{\sqrt{2}} \right]$$

$$= 0 + \lim_{t \to \infty} \log \frac{1}{\sqrt{\frac{1}{t^2} + 1}} + \frac{\pi}{4} + \frac{1}{2}\log 2$$

$$= \log 1 + \frac{\pi}{4} + \frac{1}{2}\log 2 = \frac{\pi}{4} + \frac{1}{2}\log 2 \text{ which is finite.}$$

$\Rightarrow \quad \int_0^{\infty} \frac{\tan^{-1} x}{x^2}\, dx$ is convergent and its value is $\frac{\pi}{4} + \frac{1}{2}\log 2$.

(iv) $$\int_0^\infty e^{-\sqrt{x}}\, dx = \lim_{t\to\infty}\int_0^t e^{-\sqrt{x}}\, dx$$

Put $\sqrt{x} = z$, *i.e.*, $x = z^2$ so that $dx = 2z\, dz$

When $x = 0, z = 0$; when $x = t, z = \sqrt{t}$.

$\therefore$ $$\int_0^\infty e^{-\sqrt{x}}\, dx = \lim_{t\to\infty}\int_0^{\sqrt{t}} 2ze^{-z}\, dz$$ [Integrating by parts]

$$= \lim_{t\to\infty} 2\left[-ze^{-z} - e^{-z}\right]_0^{\sqrt{t}} = \lim_{t\to\infty} -2\left[\sqrt{t}\, e^{-\sqrt{t}} + e^{-\sqrt{t}} - 1\right]$$

$$= \lim_{t\to\infty}\left(\frac{-2\sqrt{t}}{e^{\sqrt{t}}}\right) - 0 + 2 \quad \text{(Applying L' Hospital's Rule)}$$

$$= \lim_{t\to\infty}\left(\frac{\dfrac{-1}{\sqrt{t}}}{e^{\sqrt{t}}\cdot\dfrac{1}{2\sqrt{t}}}\right) + 2 = \lim_{t\to\infty}\left(\frac{-2}{e^{\sqrt{t}}}\right) + 2$$

$= 0 + 2 = 2$ which is finite.

$\Rightarrow$ $\int_0^\infty e^{-\sqrt{x}}\, dx$ is convergent and its value is 2.

Example 7. *Examine the convergence of the integrals:*

(i) $\int_{-\infty}^0 e^{2x}\, dx$ (ii) $\int_{-\infty}^0 \dfrac{dx}{p^2 + q^2x^2}$

(iii) $\int_{-\infty}^0 e^{-x}\, dx$ (iv) $\int_{-\infty}^0 \sinh x\, dx$

Sol. (i) $$\int_{-\infty}^0 e^{2x}\, dx = \lim_{t\to-\infty}\int_t^0 e^{2x}\, dx = \lim_{t\to-\infty}\left[\frac{e^{2x}}{2}\right]_t^0$$

$$= \lim_{t\to-\infty}\frac{1}{2}\left(1 - e^{2t}\right) = \frac{1}{2}(1-0) = \frac{1}{2} \text{ which is finite.}$$

$\Rightarrow$ $\int_{-\infty}^0 e^{2x}\, dx$ is convergent and its value is $\dfrac{1}{2}$.

(ii) $$\int_{-\infty}^0 \frac{dx}{p^2+q^2x^2} = \lim_{t\to-\infty}\int_t^0 \frac{dx}{\left(\dfrac{p^2}{q^2}+x^2\right)} = \lim_{t\to-\infty}\left[\frac{1}{q^2}\cdot\frac{1}{p/q}\tan^{-1}\frac{x}{p/q}\right]_t^0$$

$$= \lim_{t\to-\infty}\frac{1}{pq}\left[0 - \tan^{-1}\frac{qt}{p}\right]$$

$$= -\frac{1}{pq}\left(-\frac{\pi}{2}\right) = \frac{\pi}{2pq}, \text{ which is finite.}$$

$\Rightarrow$ $\int_{-\infty}^0 \dfrac{dx}{p^2+q^2x^2}$ is convergent and its value is $\dfrac{\pi}{2pq}$.

(*iii*) $$\int_{-\infty}^{0} e^{-x}\,dx = \lim_{t\to-\infty}\int_{t}^{0} e^{-x}\,dx = \lim_{t\to-\infty}\left[-e^{-x}\right]_t^0$$

$$= \lim_{t\to\infty}(-1+e^{-t}) = -1+\infty = \infty$$

$\Rightarrow$ $\int_{-\infty}^{0} e^{-x}\,dx$ is diverges to $+\infty$.

(*iv*) $$\int_{-\infty}^{0} \sinh x\,dx = \lim_{t\to-\infty}\int_{t}^{0} \frac{e^{x}-e^{-x}}{2}\,dx = \lim_{t\to-\infty}\left[\frac{1}{2}(e^{x}+e^{-x})\right]_t^0$$

$$= \lim_{t\to-\infty}\left[1-\frac{1}{2}(e^{t}+e^{-t})\right] = 1-\frac{1}{2}(0+\infty) = -\infty$$

$\Rightarrow$ $\int_{-\infty}^{0} \sinh x\,dx$ is diverges to $-\infty$.

Example 8. *Examine the convergence of the integrals:*

(*i*) $\int_{-\infty}^{\infty} e^{-x}\,dx$ (*ii*) $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ (*iii*) $\int_{-\infty}^{\infty} \frac{dx}{e^{x}+e^{-x}}$

Sol. (*i*) $$\int_{-\infty}^{\infty} e^{-x}\,dx = \int_{-\infty}^{0} e^{-x}\,dx + \int_{0}^{\infty} e^{-x}\,dx$$

$$= \lim_{t_1\to-\infty}\int_{t_1}^{0} e^{-x}dx + 1\lim_{t_2\to\infty}\int_{t0}^{t_2} e^{-x}\,dx$$

$$= \lim_{t_1\to-\infty}\left[-e^{-x}\right]_{t_1}^{0} + \lim_{t_2\to\infty}\left[-e^{-x}\right]_0^{t_2}$$

$$= \lim_{t_1\to-\infty}\left(-1+e^{-t_1}\right) + \lim_{t_2\to\infty}(-e^{-t_2}+1)$$

$$= (-1+\infty)+(0+1) = \infty$$

$\Rightarrow$ $\int_{-\infty}^{\infty} e^{-x}\,dx$ is diverges to ∞.

(*ii*) $$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}$$

$$= \lim_{t_1\to-\infty}\int_{t_1}^{\infty} \frac{dx}{1+x^2} + \lim_{t_2\to\infty}\int_{0}^{t_2} \frac{dx}{1+x^2}$$

$$= \lim_{t_1\to-\infty}\left[\tan^{-1}x\right]_{t_1}^{0} + \lim_{t_2\to\infty}\left[\tan^{-1}x\right]_0^{t_2}$$

$$= \lim_{t_1\to-\infty}\left[-\tan^{-1}t_1\right] + \lim_{t_2\to\infty}\left[\tan^{-1}t_2\right]$$

$$= -\left(-\frac{\pi}{2}\right)+\frac{\pi}{2} = \pi \text{ which is finite.}$$

$\Rightarrow$ $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ is convergent and its value is π.

(*iii*) $$\int_{-\infty}^{\infty} \frac{dx}{e^{x}+e^{-x}} = \int_{-\infty}^{0} \frac{dx}{e^{x}+e^{-x}} + \int_{0}^{\infty} \frac{dx}{e^{x}+e^{-x}}$$

$$= \lim_{t_1 \to -\infty} \int_{t_1}^{0} \frac{e^x}{e^{2x}+1}\,dx + \lim_{t_2 \to \infty} \int_{0}^{t_2} \frac{e^x}{e^{2x}+1}\,dx$$

Now, $$\int \frac{e^x}{e^{2x}+1}\,dx = \int \frac{dz}{z^2+1} \text{ where } z = e^x$$

$$= \tan^{-1} z = \tan^{-1} e^x$$

$$\therefore \quad \int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}} = \lim_{t_1 \to -\infty} \left[\tan^{-1} e^x\right]_{t_1}^{0} + \lim_{t_2 \to \infty} \left[\tan^{-1} e^x\right]_{0}^{t_2}$$

$$= \lim_{t_1 \to -\infty} \left[\tan^{-1} 1 - \tan^{-1} e^{t_1}\right] + \lim_{t_2 \to \infty} \left[\tan^{-1} e^{t_2} - \tan^{-1} 1\right]$$

$$= \left(\frac{\pi}{4} - \tan^{-1} 0\right) + \left(\tan^{-1} \infty - \frac{\pi}{4}\right) = \frac{\pi}{2}, \text{ which is finite.}$$

$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}}$ is convergent and its value is $\frac{\pi}{2}$.

Example 9. *Test the convergence of the integrals:*

(*i*) $\int_0^1 \frac{dx}{\sqrt{x}}$ (*ii*) $\int_0^1 \frac{dx}{x^2}$ (*iii*) $\int_1^2 \frac{x}{\sqrt{x-1}}\,dx$.

Sol. (*i*) 0 is the only point of infinite discontinuity of the integrand on [0, 1].

$$\therefore \quad \int_0^1 \frac{dx}{\sqrt{x}} = \lim_{\varepsilon \to 0+} \int_{0+\varepsilon}^{1} x^{-1/2}\,dx$$

$$= \lim_{\varepsilon \to 0+} \left[2\sqrt{x}\right]_{\varepsilon}^{1} \lim_{\varepsilon \to 0+} 2(1 - \sqrt{\varepsilon}) = 2, \text{ which is finite.}$$

$\Rightarrow \int_0^1 \frac{dx}{\sqrt{x}}$ is convergent and its value is 2.

(*ii*) 0 is the only point of infinite discontinuity of the integrand on [0, 1].

$$\therefore \quad \int_0^1 \frac{dx}{x^2} = \lim_{\varepsilon \to 0+} \int_{0+\varepsilon}^{1} x^{-2}\,dx = \lim_{\varepsilon \to 0+} \left[-\frac{1}{x}\right]_{\varepsilon}^{1} = \lim_{\varepsilon \to 0+} \left(-1 + \frac{1}{\varepsilon}\right) = \infty$$

$\Rightarrow \int_0^1 \frac{dx}{x^2}$ diverges to ∞.

(*iii*) 1 is the only point of infinite discontinuity of the integrand on [1, 2].

$$\therefore \quad \int_1^2 \frac{x}{\sqrt{x-1}}\,dx = \lim_{\varepsilon \to 0+} \int_{1+\varepsilon}^{2} \frac{(x-1)+1}{\sqrt{x-1}}\,dx$$

$$= \lim_{\varepsilon \to 0+} \int_{1+\varepsilon}^{2} \left(\sqrt{x-1} + \frac{1}{\sqrt{x-1}}\right) dx$$

$$= \lim_{\varepsilon \to 0+} \left[\frac{2}{3}(x-1)^{3/2} + 2\sqrt{x-1}\right]_{1+\varepsilon}^{2}$$

$$= \lim_{\varepsilon \to 0+} \left[\frac{2}{3} + 2 - \frac{2}{3}\varepsilon^{3/2} - 2\sqrt{\varepsilon}\right] = \frac{8}{3}, \text{ which is finite.}$$

$\Rightarrow \int_1^2 \frac{x}{\sqrt{x-1}}\,dx$ is convergent and its value is $\frac{8}{3}$.

(iii) $$\int_{-\infty}^{0} e^{-x}\, dx = \lim_{t \to -\infty} \int_{t}^{0} e^{-x}\, dx = \lim_{t \to -\infty} \left[-e^{-x}\right]_{t}^{0}$$

$$= \lim_{t \to \infty} (-1 + e^{-t}) = -1 + \infty = \infty$$

$\Rightarrow \int_{-\infty}^{0} e^{-x}\, dx$ is diverges to $+\infty$.

(iv) $$\int_{-\infty}^{0} \sinh x\, dx = \lim_{t \to -\infty} \int_{t}^{0} \frac{e^{x} - e^{-x}}{2}\, dx = \lim_{t \to -\infty} \left[\frac{1}{2}(e^{x} + e^{-x})\right]_{t}^{0}$$

$$= \lim_{t \to -\infty} \left[1 - \frac{1}{2}(e^{t} + e^{-t})\right] = 1 - \frac{1}{2}(0 + \infty) = -\infty$$

$\Rightarrow \int_{-\infty}^{0} \sinh x\, dx$ is diverges to $-\infty$.

Example 8. *Examine the convergence of the integrals:*

(i) $\int_{-\infty}^{\infty} e^{-x}\, dx$ (ii) $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ (iii) $\int_{-\infty}^{\infty} \frac{dx}{e^{x} + e^{-x}}$

Sol. (i) $$\int_{-\infty}^{\infty} e^{-x}\, dx = \int_{-\infty}^{0} e^{-x}\, dx + \int_{0}^{\infty} e^{-x}\, dx$$

$$= \lim_{t_1 \to -\infty} \int_{t_1}^{0} e^{-x} dx + 1 \lim_{t_2 \to \infty} \int_{t0}^{t_2} e^{-x}\, dx$$

$$= \lim_{t_1 \to -\infty} \left[-e^{-x}\right]_{t_1}^{0} + \lim_{t_2 \to \infty} \left[-e^{-x}\right]_{0}^{t_2}$$

$$= \lim_{t_1 \to -\infty} \left(-1 + e^{-t_1}\right) + \lim_{t_2 \to \infty} (-e^{-t_2} + 1)$$

$$= (-1 + \infty) + (0 + 1) = \infty$$

$\Rightarrow \int_{-\infty}^{\infty} e^{-x}\, dx$ is diverges to ∞.

(ii) $$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}$$

$$= \lim_{t_1 \to -\infty} \int_{t_1}^{\infty} \frac{dx}{1+x^2} + \lim_{t_2 \to \infty} \int_{0}^{t_2} \frac{dx}{1+x^2}$$

$$= \lim_{t_1 \to -\infty} [\tan^{-1} x]_{t_1}^{0} + \lim_{t_2 \to \infty} \left[\tan^{-1} x\right]_{0}^{t_2}$$

$$= \lim_{t_1 \to -\infty} \left[-\tan^{-1} t_1\right] + \lim_{t_2 \to \infty} \left[\tan^{-1} t_2\right]$$

$$= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi \text{ which is finite.}$$

$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ is convergent and its value is π.

(iii) $$\int_{-\infty}^{\infty} \frac{dx}{e^{x} + e^{-x}} = \int_{-\infty}^{0} \frac{dx}{e^{x} + e^{-x}} + \int_{0}^{\infty} \frac{dx}{e^{x} + e^{-x}}$$

$$= \lim_{t_1 \to -\infty} \int_{t_1}^{0} \frac{e^x}{e^{2x}+1}\,dx + \lim_{t_2 \to \infty} \int_{0}^{t_2} \frac{e^x}{e^{2x}+1}\,dx$$

Now, $$\int \frac{e^x}{e^{2x}+1}\,dx = \int \frac{dz}{z^2+1} \text{ where } z = e^x$$
$$= \tan^{-1} z = \tan^{-1} e^x$$

$$\therefore \quad \int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}} = \lim_{t_1 \to -\infty} \left[\tan^{-1} e^x\right]_{t_1}^{0} + \lim_{t_2 \to \infty} \left[\tan^{-1} e^x\right]_{0}^{t_2}$$

$$= \lim_{t_1 \to -\infty} \left[\tan^{-1} 1 - \tan^{-1} e^{t_1}\right] + \lim_{t_2 \to \infty} \left[\tan^{-1} e^{t_2} - \tan^{-1} 1\right]$$

$$= \left(\frac{\pi}{4} - \tan^{-1} 0\right) + \left(\tan^{-1} \infty - \frac{\pi}{4}\right) = \frac{\pi}{2}, \text{ which is finite.}$$

$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}}$ is convergent and its value is $\frac{\pi}{2}$.

Example 9. *Test the convergence of the integrals:*

(*i*) $\int_0^1 \frac{dx}{\sqrt{x}}$ (*ii*) $\int_0^1 \frac{dx}{x^2}$ (*iii*) $\int_1^2 \frac{x}{\sqrt{x-1}}\,dx$.

Sol. (*i*) 0 is the only point of infinite discontinuity of the integrand on [0, 1].

$$\therefore \quad \int_0^1 \frac{dx}{\sqrt{x}} = \lim_{\varepsilon \to 0+} \int_{0+\varepsilon}^{1} x^{-1/2}\,dx$$

$$= \lim_{\varepsilon \to 0+} \left[2\sqrt{x}\right]_{\varepsilon}^{1} \lim_{\varepsilon \to 0+} 2\left(1 - \sqrt{\varepsilon}\right) = 2, \text{ which is finite.}$$

$\Rightarrow \int_0^1 \frac{dx}{\sqrt{x}}$ is convergent and its value is 2.

(*ii*) 0 is the only point of infinite discontinuity of the integrand on [0, 1].

$$\therefore \quad \int_0^1 \frac{dx}{x^2} = \lim_{\varepsilon \to 0+} \int_{0+\varepsilon}^{1} x^{-2}\,dx = \lim_{\varepsilon \to 0+} \left[-\frac{1}{x}\right]_{\varepsilon}^{1} = \lim_{\varepsilon \to 0+} \left(-1 + \frac{1}{\varepsilon}\right) = \infty$$

$\Rightarrow \int_0^1 \frac{dx}{x^2}$ diverges to ∞.

(*iii*) 1 is the only point of infinite discontinuity of the integrand on [1, 2].

$$\therefore \quad \int_1^2 \frac{x}{\sqrt{x-1}}\,dx = \lim_{\varepsilon \to 0+} \int_{1+\varepsilon}^{2} \frac{(x-1)+1}{\sqrt{x-1}}\,dx$$

$$= \lim_{\varepsilon \to 0+} \int_{1+\varepsilon}^{2} \left(\sqrt{x-1} + \frac{1}{\sqrt{x-1}}\right) dx$$

$$= \lim_{\varepsilon \to 0+} \left[\frac{2}{3}(x-1)^{3/2} + 2\sqrt{x-1}\right]_{1+\varepsilon}^{2}$$

$$= \lim_{\varepsilon \to 0+} \left[\frac{2}{3} + 2 - \frac{2}{3}\varepsilon^{3/2} - 2\sqrt{\varepsilon}\right] = \frac{8}{3}, \text{ which is finite.}$$

$\Rightarrow \int_1^2 \frac{x}{\sqrt{x-1}}\,dx$ is convergent and its value is $\frac{8}{3}$.

Example 10. *Examine the convergence of the integrals:*

(*i*) $\int_0^1 \log x \, dx$ (*ii*) $\int_0^{1/e} \frac{dx}{x(\log x)^2}$ (*iii*) $\int_1^2 \frac{dx}{x \log x}$.

Sol. (*i*) 0 is the only point of infinite discontinuity of the integrand on [0, 1].

$$\therefore \quad \int_0^1 \log x \, dx = \lim_{\varepsilon \to 0+} \int_{0+\varepsilon}^1 (\log x) \cdot 1 \, dx \qquad \text{[Integrating by parts]}$$

$$= \lim_{\varepsilon \to 0+} [x \log x - x]_\varepsilon^1 = \lim_{\varepsilon \to 0+} (-1 - \varepsilon \log \varepsilon + \varepsilon)$$

$$= -1 \text{ which is finite.} \qquad \left[\because \lim_{x \to 0} x^n \log x = 0, n > 0\right]$$

$\Rightarrow \quad \int_0^1 \log x \, dx$ is convergent and its value is -1.

(*ii*) Since $\lim_{x \to 0} x(\log x)^n = 0$, $n > 0$, therefore, 0 is only point of infinite discontinuity of the integrand on $\left[0, \frac{1}{e}\right]$.

$$\therefore \quad \int_0^{1/e} \frac{dx}{x(\log x)^2} = \lim_{\varepsilon \to 0+} \int_{0+\varepsilon}^{1/e} (\log x)^{-2} \cdot \frac{1}{x} \, dx = \lim_{\varepsilon \to 0+} \left[\frac{(\log x)^{-1}}{-1}\right]_\varepsilon^{1/e}$$

$$= \lim_{\varepsilon \to 0+} -\left[\frac{1}{\log \frac{1}{e}} - \frac{1}{\log \varepsilon}\right]$$

$$= -[-1 - 0] = 1 \text{ which is finite.}$$

$\therefore \quad \int_0^{1/e} \frac{dx}{x(\log x)^2}$ is convergent and its value is 1.

(*iii*) 1 is the only point of infinite discontinuity of the integrand on [1, 2].

$$\therefore \quad \int_1^2 \frac{dx}{x \log x} = \lim_{\varepsilon \to 0+} \int_{1+\varepsilon}^2 \frac{1/x}{\log x} \, dx = \lim_{\varepsilon \to 0+} \left[\log (\log x)\right]_{1+\varepsilon}^2$$

$$= \lim_{\varepsilon \to 0+} [\log \log 2 - \log \log (1+\varepsilon)]$$

$$= \log \log 2 - \log 0 = \log \log 2 - (-\infty) = \infty$$

$\Rightarrow \quad \int_1^2 \frac{dx}{x \log x}$ diverges to ∞.

Example 11. *Examine the convergence of the integrals:*

(*i*) $\int_0^a \frac{dx}{\sqrt{a - x}}$ (*ii*) $\int_0^2 \frac{dx}{\sqrt{4 - x^2}}$ (*iii*) $\int_0^{\pi/2} \frac{\cos x}{\sqrt{1 - \sin x}} dx$

Sol. (*i*) a is the only point of infinite discontinuity of the integrand on $[0, a]$.

$$\therefore \quad \int_0^a \frac{dx}{\sqrt{a - x}} = \lim_{\varepsilon \to 0+} \int_0^{a - \varepsilon} (a - x)^{-1/2} \, dx$$

$$= \lim_{\varepsilon \to 0+} \left[-2\sqrt{a-x}\right]_0^{a-\varepsilon} \lim_{\varepsilon \to 0+} -2\left[\sqrt{\varepsilon} - \sqrt{a}\right]$$

$$= 2\sqrt{a} \text{ which is finite.}$$

$\therefore \int_0^a \frac{dx}{\sqrt{a-x}}$ is convergent and its value is $2\sqrt{a}$.

(*ii*) 2 is the only point of infinite discontinuity of the integrand on [0, 2].

$$\therefore \quad \int_0^2 \frac{dx}{\sqrt{4-x^2}} = \lim_{\varepsilon \to 0+} \int_0^{2-\varepsilon} \frac{dx}{\sqrt{4-x^2}} = \lim_{\varepsilon \to 0+} \left[\sin^{-1}\frac{x}{2}\right]_0^{2-\varepsilon}$$

$$= \lim_{\varepsilon \to 0+} \left[\sin^{-1}\frac{2-\varepsilon}{2} - \sin^{-1} 0\right]$$

$$= \sin^{-1} 1 - 0 = \frac{\pi}{2} \text{ which is finite.}$$

$\Rightarrow \int_0^2 \frac{dx}{\sqrt{4-x^2}}$ diverges to $\frac{\pi}{2}$.

(*iii*) $\frac{\pi}{2}$ is the only point of infinite discontinuity of the integrand on $\left[0, \frac{\pi}{2}\right]$.

$$\therefore \quad \int_0^{\pi/2} \frac{\cos x}{\sqrt{1-\sin x}}\, dx = \lim_{\varepsilon \to 0+} \int_0^{\frac{\pi}{2}-\varepsilon} -(1-\sin x)^{-1/2}(-\cos x)\, dx$$

$$= \lim_{\varepsilon \to 0+} \left[-2\sqrt{1-\sin x}\right]_0^{\frac{2}{\pi}-\varepsilon}$$

$$= \lim_{\varepsilon \to 0+} -2\left[\sqrt{1-\sin\left(\frac{\pi}{2}-\varepsilon\right)} - 1\right]$$

$$= -2\left[\sqrt{1-\sin\frac{\pi}{2}} - 1\right] = 2$$

$\Rightarrow \int_0^{\pi/2} \frac{\cos x}{\sqrt{1-\sin x}}\, dx$ converges to 2.

Example 12. *Examine the convergence of the integrals:*

(*i*) $\int_{-1}^{1} \frac{dx}{x^2}$ (*ii*) $\int_a^{3a} \frac{dx}{(x-2a)^2}$.

Sol. (*i*) The integrand becomes infinite at $x = 0$ and $-1 < 0 < 1$

$$\therefore \quad \int_{-1}^{1} \frac{dx}{x^2} = \int_{-1}^{0} \frac{dx}{x^2} + \int_0^1 \frac{dx}{x^2}$$

$$= \lim_{\varepsilon_1 \to 0+} \int_1^{0-\varepsilon_1} \frac{dx}{x^2} + \lim_{\varepsilon_2 \to 0+} \int_{0+\varepsilon_2}^{1} \frac{dx}{x^2}$$

so that 0 enclosed within $(-\varepsilon_1, \varepsilon_2)$ is excluded.

$$= \lim_{\varepsilon_1 \to 0+} \left[-\frac{1}{x}\right]_{-1}^{-\varepsilon_2} + \lim_{\varepsilon_2 \to 0+} \left[-\frac{1}{x}\right]_{\varepsilon_2}^{1}$$

$$= \lim_{\varepsilon_2 \to 0+}\left(\frac{1}{\varepsilon_1} - 1\right) + \lim_{\varepsilon_2 \to 0+}\left(-1 + \frac{1}{\varepsilon_2}\right)$$

$$= (\infty - 1) + (-1 + \infty) = \infty$$

$\Rightarrow \quad \int_{-1}^{1} \frac{dx}{x^2}$ diverges to $+\infty$.

(*ii*) The integrand becomes infinite at $x = 2a$ and $a < 2a < 3a$.

$$\therefore \quad \int_a^{3a} \frac{dx}{(x-2a)^2} = \int_a^{2a} \frac{dx}{(x-2a)^2} + \int_{2a}^{3a} \frac{dx}{(x-2a)^2}$$

$$= \lim_{\varepsilon_1 \to 0+} \int_a^{2a-\varepsilon_1} \frac{dx}{(x-2a)^2} + \lim_{\varepsilon_2 \to 0+} \int_{2a+\varepsilon_2}^{3a} \frac{dx}{(x-2a)^2}$$

$$= \lim_{\varepsilon_1 \to 0+}\left[\frac{-1}{x-2a}\right]_a^{2a-\varepsilon_1} + \lim_{\varepsilon_1 \to 0+}\left[\frac{-1}{x-2a}\right]_{2a+\varepsilon_2}^{3a}$$

$$= \lim_{\varepsilon_1 \to 0+}\left(\frac{1}{\varepsilon_1} - \frac{1}{a}\right) + \lim_{\varepsilon_2 \to 0+}\left(-\frac{1}{a} + \frac{1}{\varepsilon_2}\right)$$

$$= \left(\infty - \frac{1}{a}\right) + \left(-\frac{1}{a} + \infty\right) = \infty$$

$\Rightarrow \quad \int_a^{3a} \frac{dx}{(x-2a)^2}$ diverges to ∞.

Example 13. *Examine the convergence of the integrals:*

(*i*) $\int_0^4 \frac{dx}{x(4-x)}$ (*ii*) $\int_{-a}^{a} \frac{x}{\sqrt{a^2 - x^2}}\, dx$

(*iii*) $\int_0^{\pi} \frac{dx}{\sin x}$ (*iv*) $\int_0^{\pi} \frac{dx}{1 + \cos x}$.

Sol. (*i*) Both the end points 0 and 4 are points of infinite discontinuity of the integrand on [0, 4].

$$\therefore \quad \int_0^4 \frac{dx}{x(4-x)} = \int_0^1 \frac{dx}{x(4-x)} + \int_1^4 \frac{dx}{x(4-x)}$$

$$= \lim_{\varepsilon_1 \to 0+} \int_{0+\varepsilon_1}^{1} \frac{1}{4}\left(\frac{1}{x} + \frac{1}{4-x}\right)dx + \lim_{\varepsilon_2 \to 0+} \int_1^{4-\varepsilon_2} \frac{1}{4}\left(\frac{1}{x} + \frac{1}{4-x}\right)dx$$

$$= \lim_{\varepsilon_1 \to 0+}\left[\frac{1}{4}\log\frac{x}{4-x}\right]_{\varepsilon_1}^{1} + \lim_{\varepsilon_2 \to 0+}\left[\frac{1}{4}\log\frac{x}{4-x}\right]_1^{4-\varepsilon_2}$$

$$= \lim_{\varepsilon_1 \to 0+} \frac{1}{4}\left(\log\frac{1}{3} - \log\frac{\varepsilon_1}{4-\varepsilon_1}\right) + \lim_{\varepsilon_2 \to 0+} \frac{1}{4}\left(\log\frac{4-\varepsilon_2}{\varepsilon_2} - \log\frac{1}{3}\right)$$

$$= \frac{1}{4}\left[\log\frac{1}{3} - (-\infty)\right] + \frac{1}{4}\left[\infty - \log\frac{1}{3}\right] = \infty$$

$\Rightarrow \int_0^4 \frac{dx}{x(4-x)}$ diverges to ∞.

(*ii*) Both the end points $-a$ and a are points of infinite discontinuity of the integrand on $[-a, a]$.

$$\therefore \quad \int_{-a}^{a} \frac{x}{\sqrt{a^2 - x^2}}\,dx = \int_{-a}^{0} \frac{x}{\sqrt{a^2-x^2}}\,dx + \int_0^a \frac{x}{\sqrt{a^2-x^2}}\,dx$$

$$= \lim_{\varepsilon_1 \to 0} \int_{-a+\varepsilon_1}^{0} -\frac{1}{2}(a^2 - x^2)^{-1/2}(-2x)\,dx$$

$$+ \lim_{\varepsilon_2 \to 0+} \int_0^{a-\varepsilon_2} -\frac{1}{2}(a^2 - x^2)^{-1/2}(-2x)\,dx$$

$$= \lim_{\varepsilon_1 \to 0+}\left[-\sqrt{a^2-x^2}\right]_{-a+\varepsilon_1}^{0} + \lim_{\varepsilon_2 \to 0+}\left[-\sqrt{a^2-x^2}\right]_0^{a-\varepsilon_2}$$

$$= \lim_{\varepsilon_1 \to 0+}\left[-a + \sqrt{\varepsilon_1(2a-\varepsilon_1)}\right] + \lim_{\varepsilon_2 \to 0+}\left[-\sqrt{\varepsilon_2(2a-\varepsilon_2)} + a\right]$$

$$= -a + a = 0$$

$\therefore \int_{-a}^{a} \frac{x}{\sqrt{a^2-x^2}}\,dx$ converges to 0.

(*iii*) Both the end points 0 and π are points of infinite discontinuity of the integrand on $[0, \pi]$.

$$\therefore \quad \int_0^{\pi} \frac{dx}{\sin x} = \int_0^{\pi/2} \operatorname{cosec} x\,dx + \int_{\pi/2}^{\pi} \operatorname{cosec} x\,dx$$

$$= \lim_{\varepsilon_1 \to 0+} \int_{0+\varepsilon_1}^{\pi/2} \operatorname{cosec} x\,dx + \lim_{\varepsilon_2 \to 0+} \int_{\pi/2}^{\pi-\varepsilon_2} \operatorname{cosec} x\,dx$$

$$= \lim_{\varepsilon_1 \to 0+}\left[\log\tan\frac{x}{2}\right]_{\varepsilon_1}^{\pi/2} + \lim_{\varepsilon_2 \to 0+}\left[\log\tan\frac{x}{2}\right]_{\pi/2}^{\pi-\varepsilon_2}$$

$$= \lim_{\varepsilon_1 \to 0+}\left[\log\tan\frac{\pi}{4} - \log\tan\frac{\varepsilon_1}{2}\right] + \lim_{\varepsilon_2 \to 0+}\left[\log\tan\left(\frac{\pi}{2} - \frac{\varepsilon_2}{2}\right) - \log\tan\frac{\pi}{4}\right]$$

$$= 0 - (-\infty) + \infty - 0 = \infty$$

$\Rightarrow \int_0^{\pi} \frac{dx}{\sin x}$ diverges to ∞.

(*iv*) π is the only point of infinite discontinuity of the integrand on $[0, \pi]$.

$$\therefore \quad \int_0^{\pi} \frac{dx}{1+\cos x} = \lim_{\varepsilon \to 0+} \int_0^{\pi-\varepsilon} \frac{dx}{2\cos^2 x/2} = \lim_{\varepsilon \to 0+} \int_0^{\pi-\varepsilon} \frac{1}{2}\sec^2\frac{x}{2}\,dx$$

$$= \lim_{\varepsilon \to 0+}\left[\tan\frac{x}{2}\right]_0^{\pi-\varepsilon} = \lim_{\varepsilon \to 0+} \tan\left[\frac{\pi}{2} - \frac{\varepsilon}{2}\right] = \infty$$

$\Rightarrow \int_0^{\pi} \frac{dx}{1+\cos x}$ diverges to ∞.

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Example 14. (*i*) *The improper integral* $\int_a^b \frac{dx}{(x-a)^n}$ *is convergent if and only if* $n < 1$.

(*ii*) *The improper integral* $\int_a^b \frac{dx}{(b-x)^n}$ *is convergent if and only if* $n < 1$.

Sol. (*i*) If $n \le 0$, the integral $\int_a^b \frac{dx}{(x-a)^n}$ is proper.

If $n > 0$, the integral is improper and a is the only point of infinite discontinuity of the integrand on $[a, b]$.

Case I. *When* $n = 1$

$$\int_a^b \frac{dx}{(x-a)^n} = \int_a^b \frac{dx}{x-a} = \lim_{\varepsilon \to 0+} \int_{a+\varepsilon}^b \frac{dx}{x-a}$$

$$= \lim_{\varepsilon \to 0+} [\log (x-a)]_{a+\varepsilon}^b = \lim_{\varepsilon \to 0+} [\log (b-a) - \log \varepsilon]$$

$$= \log (b-a) - (-\infty) = \infty$$

$\Rightarrow \quad \int_a^b \frac{dx}{(x-a)^n}$ diverges if $n = 1$.

Case II. *When* $n \ne 1$

$$\int_a^b \frac{dx}{(x-a)^n} = \lim_{\varepsilon \to 0+} \int_{a+\varepsilon}^b (x-a)^{-n}\, dx = \lim_{\varepsilon \to 0+} \left[\frac{(x-a)^{1-n}}{1-n}\right]_{a+\varepsilon}^b$$

$$= \lim_{\varepsilon \to 0+} \frac{1}{1-n}\left[(b-a)^{1-n} - \varepsilon^{1-n}\right].$$

Sub-Case I. *When* $n > 1$. *So that* $n - 1 > 0$

$$\int_a^b \frac{dx}{(x-a)^n} = \lim_{\varepsilon \to 0+} \frac{1}{1-n}\left[\frac{1}{\varepsilon^{n-1}} - \frac{1}{(b-a)^{n-1}}\right]$$

$$= \frac{1}{1-n}\left[\infty - \frac{1}{(b-a)^{n-1}}\right] = \infty$$

$\Rightarrow \quad \int_a^b \frac{dx}{(x-a)^n}$ diverges if $n > 1$.

Sub-Case II. *When* $0 < n < 1$ *so that* $1 - n > 0$

$$\int_a^b \frac{dx}{(x-a)^n} = \lim_{\varepsilon \to 0+} \frac{1}{1-n}\left[(b-a)^{1-n} - \varepsilon^{1-n}\right]$$

$$= \frac{(b-a)^{1-n}}{1-n} \text{ which is finite.}$$

$\Rightarrow \quad \int_a^b \frac{dx}{(x-a)^n}$ converges if $n < 1$.

Hence $\int_a^b \frac{dx}{(x-a)^n}$ is convergent if and only if $n < 1$.

(*ii*) If $n \le 0$, the integral $\int_a^b \frac{dx}{(b-x)^n}$ is proper.

If $n > 0$, the integral is improper and b is the only point of infinite discontinuity of the integrand on $[a, b]$.

Case I. *When $n = 1$*

$$\int_a^b \frac{dx}{(b-x)^n} = \int_a^b \frac{dx}{b-x} = \lim_{\varepsilon \to 0+} \int_a^{b-\varepsilon} \frac{dx}{b-x} \lim_{\varepsilon \to 0+} \left[\frac{\log(b-x)}{-1}\right]_a^{b-\varepsilon}$$

$$= \lim_{\varepsilon \to 0+} \left[-\log \varepsilon + \log(b-a)\right] = -(-\infty) + \log(b-a) = \infty$$

$\Rightarrow \quad \int_a^b \frac{dx}{(b-x)^n}$ diverges if $n = 1$.

Case II. *When $n \neq 1$*

$$\int_a^b \frac{dx}{(b-x)^n} = \lim_{\varepsilon \to 0+} \int_a^{b-\varepsilon} (b-x)^{-n}\, dx$$

$$= \lim_{\varepsilon \to 0+} \left[\frac{(b-x)^{1-n}}{(1-n)(-1)}\right]_a^{b-\varepsilon} = \lim_{\varepsilon \to 0+} \frac{1}{n-1}\left[\varepsilon^{1-n} - (b-a)^{1-n}\right].$$

Sub-Case I. *When $n > 1$ so that $n - 1 > 0$*

$$\Rightarrow \quad \int_a^b \frac{dx}{(b-x)^n} = \lim_{\varepsilon \to 0+} \frac{1}{n-1}\left[\frac{1}{\varepsilon^{n-1}} - \frac{1}{(b-a)^{n-1}}\right]$$

$$= \frac{1}{(n-1)}\left[\infty - \frac{1}{(b-a)^{n-1}}\right] = \infty$$

$\Rightarrow \quad \int_a^b \frac{dx}{(b-x)^n}$ diverges if $n > 1$.

Sub-Case II. *When $0 < n < 1$ so that $1 - n > 0$*

$$\int_a^b \frac{dx}{(b-x)^n} = \lim_{\varepsilon \to 0+} \frac{1}{1-n}\left[(b-a)^{1-n} - \varepsilon^{1-n}\right]$$

$$= \frac{(b-a)^{1-n}}{1-n} \text{ which is finite.}$$

$\Rightarrow \quad \int_a^b \frac{dx}{(b-x)^n}$ converges if $n < 1$.

Hence, $\int_a^b \frac{dx}{(b-x)^n}$ is convergent if and only if $n < 1$.

Example 15. *The improper integral* $\int_a^\infty \frac{dx}{x^n}$, *(a > 0) is convergent if and only if* $n > 1$.

Sol.
$$\int_a^\infty \frac{dx}{x^n} = \lim_{t \to \infty} \int_a^t \frac{dx}{x^n}$$

Case I. *When* $n = 1$

$$\int_a^\infty \frac{dx}{x^n} = \lim_{t \to \infty} \int_a^t \frac{dx}{x} = \lim_{t \to \infty} [\log x]_a^t = \lim_{t \to \infty} [\log t - \log a] = \infty$$

$\therefore$ $\int_a^\infty \frac{dx}{x^n}$ diverges if $n = 1$.

Case II. *When* $n \neq 1$

$$\int_a^\infty \frac{dx}{x^n} = \lim_{t \to \infty} \int_a^t x^{-n}\, dx = \lim_{t \to \infty} \left[\frac{x^{1-n}}{1-n}\right]_a^t$$

Sub-Case I. *When* $n < 1$, $1 - n > 0$

$$\therefore \quad \int_a^\infty \frac{dx}{x^n} = \lim_{t \to \infty} \frac{1}{1-n}\left[t^{1-n} - a^{1-n}\right] = \infty$$

$\therefore$ $\int_a^\infty \frac{dx}{x^n}$ diverges if $n < 1$.

Sub-Case II. *When* $n > 1$, $n - 1 > 0$

$$\int_a^\infty \frac{dx}{x^n} = \lim_{t \to \infty} -\frac{1}{n-1}\left[\frac{1}{x^{n-1}}\right]_a^t$$

$$= \lim_{t \to \infty} -\frac{1}{n-1}\left[\frac{1}{t^{n-1}} - \frac{1}{a^{n-1}}\right] = \frac{1}{(n-1)a^{n-1}}, \text{ which is finite.}$$

$$\therefore \quad = \int_a^\infty \frac{dx}{x^n} \text{ converges if } n > 1.$$

Hence, $\int_a^\infty \frac{dx}{x^n}$ converges if $n > 1$.

TEST YOUR KNOWLEDGE

1. Examine the convergence of the improper integrals:

(*i*) $\int_0^\infty e^{2x} dx$ (*ii*) $\int_0^\infty \frac{dx}{(1+x)^{2/3}}$ (*iii*) $\int_1^\infty \frac{x}{(1+x)^3} dx$

(*iv*) $\int_{-\infty}^0 \cosh x\, dx$ (*v*) $\int_{-\infty}^1 \frac{dx}{1+x^2}$ (*vi*) $\int_1^2 \frac{dx}{\sqrt{x-1}}$

(vii) $\int_1^2 \frac{dx}{x\sqrt{x^2-1}}$ (viii) $\int_2^3 \frac{x-1}{\sqrt{x-2}}\,dx$ (ix) $\int_0^e \frac{dx}{x(\log x)^3}$

(x) $\int_1^2 \frac{dx}{2-x}$ (xi) $\int_0^1 \frac{dx}{x^2-1}$ (xii) $\int_0^{2a} \frac{dx}{(x-a)^2}$

(xiii) $\int_0^2 \frac{dx}{2x-x^2}$.

Answers

1. (i) Divergent (ii) Divergent (iii) Converges to $\frac{3}{8}$

(iv) Diverges to ∞ (v) Converges to $\frac{3\pi}{4}$ (vi) Converges to 2

(vii) Convergent (viii) Converges to $\frac{8}{3}$ (ix) Converges to $-\frac{1}{2}$

(x) Diverges to ∞ (xi) Diverges to $-\infty$ (xii) Diverges to ∞

(xiii) Diverges to ∞

2.6 GAMMA FUNCTION [A.K.T.U. 2018]

If n is positive, then the definite integral $\int_0^\infty e^{-x}\, x^{n-1}\, dx$, which is a function of n, is called the Gamma function (or Eulerian integral of second kind) and is denoted by $\Gamma(n)$. Thus

$$\boxed{\Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1}\, dx,\ n > 0.}$$

In particular, $\Gamma(1) = \int_0^\infty e^{-x}\, dx = \Big| -e^{-x} \Big|_0^\infty = 1.$

2.7 REDUCTION FORMULA FOR g(n)

$$\Gamma(n+1) = \int_0^\infty e^{-x}\, x^n\, dx$$

Integrating by parts, we have

$$\Gamma(n+1) = \Big| -x^n e^{-x} \Big|_0^\infty + n\int_0^\infty x^{n-1} e^{-x}\, dx = n\int_0^\infty e^{-x} x^{n-1}\, dx \qquad \left[\because\ \underset{x\to\infty}{\text{Lt}} \frac{x^n}{e^x} = 0\right]$$

$$= n\Gamma(n)$$

$\therefore$ $\boxed{\Gamma(n+1) = n\Gamma(n)}$ which is the reduction or recurrence formula for $\Gamma(n)$.

Note 1. If n is a positive integer, then by repeated application of above formula, we get

$$\begin{aligned}\Gamma(n+1) &= n\Gamma(n)\\ &= n(n-1)\,\Gamma(n-1)\\ &= n(n-1)(n-2)\,\Gamma(n-2)\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &= n(n-1)(n-2)\ldots\ldots 3\cdot 2\cdot 1\,\Gamma(1)\\ &= n\,!, \quad \text{since } \Gamma(1)=1\end{aligned}$$

Hence $\boxed{\Gamma(n+1) = n\,!}$ **when n is a positive integer.**

Note 2. If n is a positive fraction, then by repeated application of above formula, we get

$$\Gamma(n) = (n-1)(n-2) \times \text{go on decreasing by } 1 \ldots\ldots$$

the series of factors being continued so long as the factors remain positive, multiplied by Γ (last factor).

Thus $\Gamma\left(\frac{11}{4}\right) = \frac{7}{4}\Gamma\left(\frac{7}{4}\right) = \frac{7}{4}\cdot\frac{3}{4}\Gamma\left(\frac{3}{4}\right)$

The value of $\Gamma\left(\frac{3}{4}\right)$ can be obtained from the table of gamma functions.

Note 3. $\Gamma(n+1) = n\,\Gamma(n)$

$$\begin{aligned}\Rightarrow \quad \Gamma(n) &= \frac{\Gamma(n+1)}{n}, n \neq 0\\ &= \frac{(n+1)\Gamma(n+1)}{n(n+1)} = \frac{\Gamma(n+2)}{n(n+1)}, n \neq 0, -1\\ &= \frac{(n+2)\Gamma(n+2)}{n(n+1)(n+2)} = \frac{\Gamma(n+3)}{n(n+1)(n+2)}, n \neq 0, -1, -2\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &= \frac{\Gamma(n+k+1)}{n(n+1)(n+2)\ldots(n+k)}, n \neq 0, -1, -2\ldots, -k\end{aligned}$$

This result defines $\Gamma(n)$ for $n < 0$, k being the least positive integer such that $n + k + 1 > 0$.

For example, to evaluate $\Gamma(-3.4)$

$$n + k + 1 > 0 \quad \Rightarrow \quad -3.4 + k + 1 > 0 \quad \Rightarrow \quad k > 2.4$$

We choose $k = 3$

$$\therefore \quad \Gamma(-3.4) = \frac{\Gamma(-3.4+3+1)}{(-3.4)(-2.4)(-1.4)(-.4)} = \frac{\Gamma(.6)}{(3.4)(2.4)(1.4)(.4)}$$

The value of $\Gamma(.6)$ can be obtained from the table of gamma functions.

Also we observe that $\Gamma(n)$ is infinite when $n = 0$ or a negative integer.

2.7.1 Value of $\Gamma\left(\frac{1}{2}\right)$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t}\, t^{-1/2}\, dt$$

Putting $t = x^2$ so that $dt = 2xdx$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-x^2}\cdot\frac{1}{x}\cdot 2xdx = 2\int_0^\infty e^{-x^2}dx \qquad \ldots(1)$$

Writing y for x, we have $\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-y^2}\,dy$ ...(2)

Now we use the following result from double integrals:

If $f(x)$ and $g(y)$ are functions of x and y only, and the limits of integration are constants, then the double integral can be represented as a product of two integrals. Thus

$$\int_c^d \int_a^b f(x)\,g(y)\,dx\,dy = \int_a^b f(x)\,dx \cdot \int_c^d g(y)\,dy$$

$\therefore$ From (1) and (2), we have

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^\infty e^{-x^2}\,dx.\int_0^\infty e^{-y^2}\,dy = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

Changing to polar co-ordinates with $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$; the region of integration in this integral is the complete positive quadrant, to cover which, r must vary from 0 to ∞ and θ from 0 to $\frac{\pi}{2}$.

$$\therefore \quad \left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\,r\,dr\,d\theta = 4\int_0^{\pi/2}\left[-\frac{1}{2}e^{-r^2}\right]_0^\infty d\theta = 2\int_0^{\pi/2} d\theta = \pi$$

Hence, $\boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.}$

2.8 TRANSFORMATIONS OF GAMMA FUNCTIONS

(1) $\boxed{\Gamma(n) = k^n\int_0^\infty e^{-kx}\,x^{n-1}\,dx}$ [A.K.T.U. 2016]

Proof. We have, $\Gamma(n) = \int_0^\infty e^{-x}\,x^{n-1}\,dx$

Put $x = ky$

$\therefore$ $dx = k\,dy$

$$= \int_0^\infty e^{-ky}\,k^{n-1}\,y^{n-1}\,k\,dy = k^n\int_0^\infty e^{-ky}\,y^{n-1}\,dy$$

$$\Rightarrow \quad \Gamma(n) = k^n\int_0^\infty e^{-kx}\,x^{n-1}\,dx$$

(2) $\boxed{\Gamma(n) = \int_0^1\left[\log\left(\frac{1}{x}\right)\right]^{n-1} dx\ ; \quad n > 0}$

Proof. We have, $\Gamma(n) = \int_0^\infty e^{-x}\cdot x^{n-1}\,dx$

Put $e^{-x} = y$

$\therefore$ $-x = \log y \ \Rightarrow\ x = \log\left(\frac{1}{y}\right)$ and $dx = -\frac{dy}{y}$

$$\therefore \quad \Gamma(n) = -\int_1^0 y\left(\log\frac{1}{y}\right)^{n-1}\frac{dy}{y} = \int_0^1\left(\log\frac{1}{y}\right)^{n-1} dy$$

Hence, $$\Gamma(n) = \int_0^1 \left[\log\left(\frac{1}{x}\right)\right]^{n-1} dx.$$

(3) $$\boxed{\Gamma(n) = \frac{1}{n}\int_0^\infty e^{-x^{1/n}}\, dx}$$

Proof. We have $\Gamma(n) = \int_0^\infty e^{-x} \cdot x^{n-1}\, dx$

Put $x^n = y \quad \Rightarrow \quad x = y^{1/n}$

$\therefore$ $$dx = \frac{1}{n} y^{(1/n)-1} dy$$

Now, $$\Gamma(n) = \int_0^\infty e^{-y^{1/n}}\, y^{\frac{n-1}{n}} \cdot \frac{1}{n} y^{\frac{1}{n}-1}\, dy = \frac{1}{n}\int_0^\infty e^{-y^{1/n}} dy$$

or $$\Gamma(n) = \frac{1}{n}\int_0^\infty e^{-x^{1/n}} dx.$$

2.8.1 Deduction. Put $n = \frac{1}{2}$ in (3), we get

$$\Gamma(1/2) = 2\int_0^\infty e^{-x^2} dx = 2\left(\frac{\sqrt{\pi}}{2}\right) = \sqrt{\pi}.$$

2.9 BETA FUNCTION

[A.K.T.U. 2018]

If m, n are positive, then the definite integral $\int_0^1 x^{m-1}(1-x)^{n-1}\, dx$, which is a function of m and n, is called the Beta Function (or Eulerian integral of first kind) and is denoted by $\beta(m, n)$. Thus,

$$\boxed{\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx,\ m > 0,\ n > 0.}$$

2.10 SYMMETRY OF BETA FUNCTION i.e., $\beta(m, n) = \beta(n, m)$

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx,\ m > 0,\ n > 0$$

Since $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$

$\therefore$ $$\beta(m, n) = \int_0^1 (1-x)^{m-1}\,[1-(1-x)]^{n-1}\, dx = \int_0^1 x^{n-1}\,(1-x)^{m-1}\, dx = \beta(n, m)$$

Hence, $$\boxed{\beta(m, n) = \beta(n, m).}$$

2.11 TRANSFORMATIONS OF BETA FUNCTION

(1) $$\boxed{\beta(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx}$$

Proof. $\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ | Put $x = \frac{1}{1+y}$ $\therefore$ $dx = -\frac{1}{(1+y)^2}\,dy$

$$= \int_\infty^0 \frac{1}{(1+y)^{m-1}}\left(1-\frac{1}{1+y}\right)^{n-1}\left\{\frac{-1}{(1+y)^2}\right\}dy$$

$$= \int_0^\infty \frac{y^{n-1}}{(1+y)^{m-1}(1+y)^{n-1}(1+y)^2}\,dy = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,dy$$

$$\beta(m, n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$\therefore$ $$\beta(n, m) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

But $\beta(m, n) = \beta(n, m)$

$\therefore$ $$\beta(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

(2) $$\boxed{\beta(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta}$$

Proof. $\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ | Put $x = \sin^2\theta$ $\therefore$ $dx = 2\sin\theta\cos\theta\,d\theta$

$$= \int_0^{\pi/2} \sin^{2m-2}\theta\,(1-\sin^2\theta)^{n-1}\cdot 2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta.$$

2.12 RELATION BETWEEN BETA AND GAMMA FUNCTIONS

$$\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$ [A.K.T.U. 2018; U.K.T.U. 2010]

Proof. From (1) of Art. 2.8, we have

$$\Gamma(n) = k^n \int_0^\infty e^{-kx}\,x^{n-1}\,dx$$

$$= z^n \int_0^\infty e^{-zx}\,x^{n-1}\,dx \qquad \text{(Replace } k \text{ by } z\text{)}$$

Multiplying both sides by $e^{-z}z^{m-1}$, we get

$$\Gamma(n)\cdot e^{-z}z^{m-1} = \int_0^\infty z^n\cdot e^{-zx}\cdot x^{n-1}\cdot e^{-z}\cdot z^{m-1}\,dx = \int_0^\infty z^{n+m-1}e^{-z(1+x)}x^{n-1}\,dx$$

Integrating both sides w.r.t. z from 0 to ∞, we get

$$\Gamma(n) \int_0^\infty e^{-z} z^{m-1}\, dz = \int_0^\infty x^{n-1} \left\{ \int_0^\infty e^{-z(1+x)}\, z^{m+n-1}\, dz \right\} dx$$

$$\Rightarrow \qquad \Gamma n\, \Gamma m = \int_0^\infty x^{n-1} \left\{ \int_0^\infty e^{-y} \cdot \frac{y^{m+n-1}}{(1+x)^{m+n-1}} \frac{dy}{(1+x)} \right\} dx$$

where $z(1+x) = y$ so that $dz = \frac{dy}{1+x}$

$$= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} \left\{ \int_0^\infty e^{-y}\, y^{m+n-1}\, dy \right\} dx$$

$$= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx\, \Gamma(m+n) = \Gamma(m+n)\, \beta(m, n)$$

$$\therefore \qquad \boxed{\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}}$$

Aliter:

We know that $\quad \Gamma(m) = \int_0^\infty e^{-t}\, t^{m-1}\, dt$

Putting $t = x^2$ so that $dt = 2x\, dx$

$$\Gamma(m) = 2 \int_0^\infty e^{-x^2} x^{2m-1}\, dx \qquad \text{...(1)}$$

Similarly, $\quad \Gamma(n) = 2 \int_0^\infty e^{-y^2}\, y^{2n-1}\, dy$

$$\Gamma(m)\, \Gamma(n) = 4 \int_0^\infty e^{-x^2} x^{2m-1}\, dx \cdot \int_0^\infty e^{-y^2}\, y^{2n-1}\, dy$$

$$= 4 \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1}\, dx\, dy$$

Changing to polar co-ordinates, we have

$$\Gamma(m)\, \Gamma(n) = 4 \int_0^{\pi/2} \int_0^\infty e^{-r^2}\, r^{2(m+n)-1} \cos^{2m-1}\theta \sin^{2n-1}\theta\, dr\, d\theta$$

$$= 4 \int_0^\infty e^{-r^2}\, r^{2(m+n)-1}\, dr \cdot \int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta \qquad \text{...(2)}$$

$$= \left[2 \int_0^\infty e^{-r^2}\, r^{2(m+n)-1}\, dr \right] \cdot \left[2 \int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta \right]$$

$$= \Gamma(m+n)\, \beta(m, n) \qquad | \text{ Using (2) of 2.11}$$

$$\text{Hence,} \qquad \boxed{\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}}$$

2.13 TO EVALUATE $\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta$ (U.K.T.U. 2011)

From (2) of Art. 2.11, we have

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{1}{2}\beta(m, n) \quad \text{...(1)}$$

Using the relation $\beta(m, n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$, we get from (1)

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{\Gamma(m)\,\Gamma(n)}{2\,\Gamma(m+n)} \quad \text{...(2)}$$

Replacing m by $\dfrac{m+1}{2}$ and n by $\dfrac{n+1}{2}$ in (2), we get

$$\boxed{\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}}$$

Cor. 1 Putting $m = n = 0$, we have

$$\frac{(\Gamma\frac{1}{2})^2}{2\,\Gamma(1)} = \int_0^{\pi/2} d\theta = \frac{\pi}{2} \quad \Rightarrow \quad \Gamma(\tfrac{1}{2}) = \sqrt{\pi} \qquad | \text{ since } \Gamma 1 = 1$$

Cor. 2 Putting $n = 0$, we get

$$\int_0^{\pi/2} \sin^m\theta \, d\theta = \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)} \cdot \frac{\sqrt{\pi}}{2} \qquad \Big| \because \ \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

and similarly, putting $m = 0$, we get

$$\int_0^{\pi/2} \cos^n\theta \, d\theta = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \cdot \frac{\sqrt{\pi}}{2}$$

2.13.1 Deductions

Using $\int_0^{\infty} \dfrac{x^{n-1}}{1+x}\,dx = \dfrac{\pi}{\sin n\pi}$, where $0 < n < 1$, we can deduce the following important results.

(1) $$\boxed{\Gamma n \Gamma(1-n) = \frac{\pi}{\sin n\pi}}$$ (G.B.T.U. 2010)

Proof. We have,

$$\beta(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$$\Rightarrow \quad \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}}\,dx = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

Setting $m + n = 1$ so that $m = 1 - n$, we get

$$\int_0^\infty \frac{x^{n-1}}{1+x}\,dx = \frac{\Gamma(1-n)\,\Gamma(n)}{\Gamma(1)}$$

$$\Rightarrow \qquad \frac{\pi}{\sin n\pi} = \Gamma(n)\,\Gamma(1-n) \qquad | \because \Gamma(1) = 1$$

(2) $\boxed{\Gamma(\tfrac{1}{2}) = \sqrt{\pi}}$

Proof. We have

$$\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

As a particular case, put $n = \frac{1}{2}$, we get

$$\Gamma(\tfrac{1}{2})\,\Gamma(\tfrac{1}{2}) = \frac{\pi}{\sin\dfrac{\pi}{2}}$$

$$[\Gamma(\tfrac{1}{2})]^2 = \pi \qquad \Rightarrow \qquad \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

(3) $\boxed{\Gamma(\tfrac{1}{4})\,\Gamma(\tfrac{3}{4}) = \pi\sqrt{2}}$

Proof. Putting $n = \frac{1}{4}$ in result (1), we obtain

$$\Gamma(\tfrac{1}{4})\,\Gamma(1-\tfrac{1}{4}) = \frac{\pi}{\sin\dfrac{\pi}{4}}$$

$$\Rightarrow \qquad \Gamma(\tfrac{1}{4})\,\Gamma(\tfrac{3}{4}) = \frac{\pi}{\left(\dfrac{1}{\sqrt{2}}\right)} \quad \text{or} \quad \Gamma(\tfrac{1}{4})\,\Gamma(\tfrac{3}{4}) = \pi\sqrt{2}$$

Note. Similarly $\Gamma(1/3)\,\Gamma(2/3) = \dfrac{\pi}{\sin \pi/3} = \dfrac{2\pi}{\sqrt{3}}$.

2.14 DUPLICATION FORMULA

$\boldsymbol{\Gamma(m)\,\Gamma\left(m + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{(2)^{2m-1}}\,\Gamma(2m)}$ **where m is positive.** (*M.T.U. 2013; G.B.T.U. 2013*)

We have already established

$$\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta = \frac{\Gamma(m)\,\Gamma(n)}{2\,\Gamma(m+n)} \qquad \text{...(1)}$$

Putting $2n - 1 = 0$ or $n = \dfrac{1}{2}$ in (1), we obtain

$$\int_0^{\pi/2} \sin^{2m-1}\theta\,d\theta = \frac{\Gamma(m)\sqrt{\pi}}{2\,\Gamma\left(m+\dfrac{1}{2}\right)} \qquad \text{...(2)} \quad [\because \Gamma(1/2) = \sqrt{\pi}]$$

Again putting $n = m$ in equation (1), we obtain

$$\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2m-1}\theta\,d\theta = \frac{(\Gamma m)^2}{2\,\Gamma(2m)}$$

i.e.,
$$\frac{1}{2^{2m-1}}\int_0^{\pi/2}(2\sin\theta\cos\theta)^{2m-1}\,d\theta=\frac{(\Gamma m)^2}{2\,\Gamma(2m)}$$

i.e.,
$$\frac{1}{2^{2m}}\int_0^{\pi/2}(\sin 2\theta)^{2m-1}\cdot 2d\theta=\frac{(\Gamma m)^2}{2\,\Gamma(2m)}$$

Putting $2\theta=\phi$ so that $2\,d\theta=d\phi$, this reduces to

$$\frac{1}{2^{2m}}\int_0^{\pi}\sin^{2m-1}\phi\,d\phi=\frac{(\Gamma\, m)^2}{2\,\Gamma\,(2m)}$$

i.e.,
$$\frac{2}{2^{2m}}\int_0^{\pi/2}\sin^{2m-1}\phi\,d\phi=\frac{(\Gamma m)^2}{2\,\Gamma(2m)}$$

Replacing ϕ by θ, we finally obtain

$$\int_0^{\pi/2}\sin^{2m-1}\theta\,d\theta=\frac{2^{2m-1}\,(\Gamma m)^2}{2\,\Gamma(2m)} \quad \ldots(3)$$

From (2) and (3), we get

$$\frac{\Gamma(m)\sqrt{\pi}}{2\,\Gamma\left(m+\dfrac{1}{2}\right)}=\frac{2^{2m-1}\,(\Gamma m)^2}{2\,\Gamma(2m)}$$

$$\Rightarrow \quad \boxed{\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma(2m)}$$

2.15 TO SHOW THAT

$$\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right)=\frac{(2\pi)^{\left(\frac{n-1}{2}\right)}}{n^{1/2}}$$

where n is a positive integer greater than one.

Let
$$P=\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\Gamma\left(\frac{n-2}{n}\right)\Gamma\left(\frac{n-1}{n}\right)$$
$$=\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\Gamma\left(1-\frac{2}{n}\right)\Gamma\left(1-\frac{1}{n}\right) \quad \ldots(1)$$

Writing the value of P in the reverse order, we have

$$P=\Gamma\left(1-\frac{1}{n}\right)\Gamma\left(1-\frac{2}{n}\right)\ldots\Gamma\left(\frac{3}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{1}{n}\right) \quad \ldots(2)$$

Multiplying (1) and (2), we get

$$P^2=\left(\Gamma(1/n)\,\Gamma\left(1-\frac{1}{n}\right)\right)\left(\Gamma(2/n)\,\Gamma\left(1-\frac{2}{n}\right)\right)\ldots\left(\Gamma\left(1-\frac{2}{n}\right)\Gamma\,(2/n)\right)\left(\Gamma\left(1-\frac{1}{n}\right)\Gamma\,(1/n)\right)$$

$$P^2=\frac{\pi}{\sin\left(\dfrac{\pi}{n}\right)}\cdot\frac{\pi}{\sin\left(\dfrac{2\pi}{n}\right)}\cdot\frac{\pi}{\sin\left(\dfrac{3\pi}{n}\right)}\cdots\frac{\pi}{\sin\dfrac{(n-1)\,\pi}{n}} \qquad \because\ \Gamma(n)\,\Gamma(1-n)=\frac{\pi}{\sin n\pi}$$

$$\Rightarrow \quad P^2=\frac{\pi^{n-1}}{\sin\left(\dfrac{\pi}{n}\right)\sin\left(\dfrac{2\pi}{n}\right)\sin\left(\dfrac{3\pi}{n}\right)\ldots\sin\left\{\dfrac{(n-1)\,\pi}{n}\right\}} \quad \ldots(3)$$

But from Trigonometry, we know that

$$\frac{\sin n\theta}{\sin\theta} = 2^{n-1}\sin\left(\theta+\frac{\pi}{n}\right)\sin\left(\theta+\frac{2\pi}{n}\right)\ldots\sin\left\{\theta+\frac{(n-1)\pi}{n}\right\}$$

Take Limit as $\theta \to 0$, ...(4)

$$\underset{\theta\to 0}{\text{Lt}}\ \frac{\sin n\theta}{\sin\theta} = \underset{\theta\to 0}{\text{Lt}}\left(n\cdot\frac{\sin n\theta}{n\theta}\cdot\frac{\theta}{\sin\theta}\right) = n$$

$\therefore$
$$n = 2^{n-1}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\sin\frac{3\pi}{n}\ldots\sin\frac{(n-1)\pi}{n} \qquad |\text{ From (4)}$$

Substituting this in equation (3), we obtain

$$P^2 = \frac{\pi^{n-1}}{\left(\dfrac{n}{2^{n-1}}\right)} = \frac{(2\pi)^{n-1}}{n}$$

$\therefore$
$$P = \frac{(2\pi)^{(n-1)/2}}{n^{1/2}}$$

or
$$\boxed{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right) = \frac{(2\pi)^{(n-1)/2}}{n^{1/2}}}$$

2.16 TO SHOW THAT

(i) $\displaystyle\int_0^\infty e^{-ax}x^{n-1}\cos bx\,dx = \frac{\Gamma(n)\cos n\theta}{(a^2+b^2)^{n/2}}$

(ii) $\displaystyle\int_0^\infty e^{-ax}x^{n-1}\sin bx\,dx = \frac{\Gamma(n)\sin n\theta}{(a^2+b^2)^{n/2}}$, where $\theta = \tan^{-1}\left(\dfrac{b}{a}\right)$

We know that $\displaystyle\int_0^\infty e^{-ax}\cdot x^{n-1}\,dx = \frac{\Gamma(n)}{a^n}$, where a, n are (+)ve.

Put $ax = z$ so that $dx = \dfrac{dz}{a}$

$\therefore$
$$\int_0^\infty e^{-ax}x^{n-1}dx = \int_0^\infty e^{-z}\left(\frac{z}{a}\right)^{n-1}\cdot\frac{dz}{a} = \frac{1}{a^n}\int_0^\infty e^{-z}z^{n-1}\,dz = \frac{\Gamma(n)}{a^n}.$$

2.16.1 Deduction

Replacing a by $(a + ib)$, we have

$$\int_0^\infty e^{-(a+ib)x}x^{n-1}\,dx = \frac{\Gamma(n)}{(a+ib)^n}$$

Now $e^{-(a+ib)x} = e^{-ax}\cdot e^{-ibx} = e^{-ax}(\cos bx - i\sin bx)$ ...(1)

Putting $a = r\cos\theta$ and $b = r\sin\theta$ so that $r^2 = a^2 + b^2$ and $\theta = \tan^{-1}\dfrac{b}{a}$

$$(a+ib)^n = (r\cos\theta + ir\sin\theta)^n = r^n(\cos\theta + i\sin\theta)^n$$
$$= r^n(\cos n\theta + i\sin n\theta)$$

[De Moivre's Theorem]

$\therefore$ From (1), we have

$$\int_0^\infty e^{-ax}(\cos bx - i\sin bx)\, x^{n-1}\, dx = \frac{\Gamma(n)}{r^n(\cos n\theta + i\sin n\theta)}$$

$$= \frac{\Gamma(n)}{r^n}(\cos n\theta + i\sin n\theta)^{-1} = \frac{\Gamma(n)}{r^n}(\cos n\theta - i\sin n\theta)$$

Now equating real and imaginary parts on the two sides, we get

$$\int_0^\infty e^{-ax}\, x^{n-1}\cos bx\, dx = \frac{\Gamma(n)}{r^n}\cos n\theta$$

and
$$\int_0^\infty e^{-ax}\, x^{n-1}\sin bx\, dx = \frac{\Gamma(n)}{r^n}\sin n\theta$$

where $r^2 = a^2 + b^2$ and $\theta = \tan^{-1}\dfrac{b}{a}$.

ILLUSTRATIVE EXAMPLES

Example 1. *Using Beta and Gamma functions, evaluate:*

(*i*) $\int_0^\infty x^{1/4}\, e^{-\sqrt{x}}\, dx$ (*ii*) $\int_0^1 \left(\frac{x^3}{1-x^3}\right)^{1/2} dx$ (A.K.T.U. 2014, 2018)

(*iii*) $\int_0^1 x^5(1-x^3)^{10}\, dx$

Sol. (*i*) Let $\quad I = \int_0^\infty x^{1/4}\, e^{-\sqrt{x}}\, dx \quad$ ...(1)

Put $\sqrt{x} = y \Rightarrow x = y^2$ so that $dx = 2y\, dy$ then equation (1) becomes

$$I = \int_0^\infty y^{1/2}e^{-y}\cdot 2y\, dy = 2\int_0^\infty e^{-y}\, y^{3/2}\, dy$$

$$= 2\int_0^\infty e^{-y}\, y^{(5/2)-1} dy = 2\,\Gamma(5/2) \qquad | \text{ By definition}$$

$$= 2\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi} = \frac{3}{2}\sqrt{\pi} \qquad | \because \Gamma(n+1) = n\Gamma(n)$$

(*ii*) Let $\quad I = \int_0^1 x^{3/2}(1-x^3)^{-1/2}\, dx \quad$ ...(1)

Put $x^3 = y \Rightarrow x = y^{1/3}$ so that $dx = \frac{1}{3}y^{-2/3}\, dy$ then equation (1) becomes

$$I = \int_0^1 y^{1/2}(1-y)^{-1/2}\cdot\frac{1}{3}y^{-2/3}\, dy = \frac{1}{3}\int_0^1 y^{-1/6}(1-y)^{-1/2}\, dy$$

$$= \frac{1}{3}\int_0^1 y^{\left(\frac{5}{6}\right)-1}(1-y)^{\left(\frac{1}{2}\right)-1} dy = \frac{1}{3}\beta\left(\frac{5}{6},\frac{1}{2}\right)$$

$$= \frac{1}{3}\frac{\Gamma(5/6)\,\Gamma(1/2)}{\Gamma(4/3)} \qquad \Big| \because \beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$$= \frac{\sqrt{\pi}}{3} \cdot \frac{\Gamma(5/6)}{\frac{1}{3}\Gamma(1/3)} = \sqrt{\pi} \cdot \frac{\Gamma(5/6)\,\Gamma(1/6)\,\Gamma(2/3)}{\Gamma(1/6)\,\Gamma(1/3)\,\Gamma(2/3)} \qquad \left| \because \Gamma(n+1) = n\Gamma n \right.$$

$$= \sqrt{\pi} \cdot \frac{\Gamma(2/3)}{\Gamma(1/6)} \cdot \frac{\pi}{\sin\frac{\pi}{6}} \cdot \frac{\sin\frac{\pi}{3}}{\pi} \qquad \left| \because \Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi} \right.$$

$$= \sqrt{3\pi}\,\frac{\Gamma(2/3)}{\Gamma(1/6)}$$

(*iii*) Let $$\mathrm{I} = \int_0^1 x^5(1-x^3)^{10}\,dx \qquad \ldots(1)$$

Put $x^3 = y \;\Rightarrow\; x = y^{1/3}$ so that $dx = \frac{1}{3}y^{-2/3}dy$ then equation (1) becomes

$$\mathrm{I} = \int_0^1 y^{5/3}(1-y)^{10} \cdot \frac{1}{3}y^{-2/3}dy$$

$$= \frac{1}{3}\int_0^1 y(1-y)^{10}\,dy = \frac{1}{3}\beta(2, 11) = \frac{1}{3}\,\frac{\Gamma 2\,\Gamma(11)}{\Gamma(13)} = \frac{1}{3} \cdot \frac{1}{12 \cdot 11} = \frac{1}{396}$$

Example 2. *Prove that:*

(*i*) $\beta(l, m) \cdot \beta(l+m, n) \cdot \beta(l+m+n, p) = \dfrac{\Gamma l\,\Gamma m\,\Gamma n\,\Gamma p}{\Gamma(l+m+n+p)}$

(*ii*) $\displaystyle\int_0^\infty \frac{x^c}{c^x}\,dx = \frac{\Gamma(c+1)}{(\log c)^{c+1}}\,;\ c > 1$

(*iii*) $\beta(m, m) = 2^{1-2m}\,\beta(m, 1/2)$

(*iv*) $\displaystyle\int_0^{\pi/2} \tan^n x\,dx = \frac{\pi}{2}\sec\frac{n\pi}{2}$

Sol. (*i*) $$\text{LHS} = \beta(l, m) \cdot \beta(l+m, n) \cdot \beta(l+m+n, p)$$

$$= \frac{\Gamma l\,\Gamma m}{\Gamma(l+m)} \cdot \frac{\Gamma(l+m)\,.\,\Gamma n}{\Gamma(l+m+n)} \cdot \frac{\Gamma l+m+n\,\Gamma p}{\Gamma(l+m+n+p)}$$

$$= \frac{\Gamma l\,\Gamma m\,\Gamma n\,\Gamma p}{\Gamma(l+m+n+p)} = \text{RHS}$$

(*ii*) Let $$\mathrm{I} = \int_0^\infty \frac{x^c}{c^x}\,dx = \int_0^\infty e^{-x\log c}\,x^c\,dx \qquad \ldots(1)$$

Put $x\log c = y \;\Rightarrow\; x = \dfrac{y}{\log c}$ so that $dx = \dfrac{dy}{\log c}$ then equation (1) becomes

$$\mathrm{I} = \int_0^\infty e^{-y}\left(\frac{y}{\log c}\right)^c \frac{dy}{\log c} = \frac{1}{(\log c)^{c+1}}\int_0^\infty e^{-y}\,y^c\,dy$$

$$= \frac{\Gamma(c+1)}{(\log c)^{c+1}}\ ;\ c > 1$$

(iii) $$\beta(m, 1/2) = \frac{\Gamma m\, \Gamma(1/2)}{\Gamma(m+1/2)} \qquad \Big|\ \because\ \beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$$= \frac{(\Gamma m)^2 \sqrt{\pi}}{\Gamma m\, \Gamma\left(m+\frac{1}{2}\right)} = \frac{\Gamma(m)^2 \sqrt{\pi}}{\left(\frac{\sqrt{\pi}}{2^{2m-1}}\right)\Gamma 2m} \qquad \text{| By Duplication formula}$$

$$= 2^{2m-1}\frac{\Gamma m\, \Gamma m}{\Gamma(2m)} = 2^{2m-1}\beta(m, m)$$

$$\Rightarrow \qquad \beta(m, m) = 2^{1-2m}\beta(m, 1/2)$$

(iv) Let $$I = \int_0^{\pi/2} \tan^n x\, dx = \int_0^{\pi/2} \sin^n x \cos^{-n} x\, dx$$

$$= \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{-n+1}{2}\right)}{2\,\Gamma\left(\frac{n-n+2}{2}\right)} = \frac{1}{2}\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(1-\left(\frac{n+1}{2}\right)\right)$$

$$= \frac{1}{2}\cdot\frac{\pi}{\sin\left(\frac{n+1}{2}\right)\pi} = \frac{\pi}{2\cos\frac{n\pi}{2}} = \frac{\pi}{2}\sec\frac{n\pi}{2}$$

Example 3. *Evaluate:*

(i) $\displaystyle\int_0^{\infty} \frac{x^8(1-x^6)}{(1+x)^{24}}\, dx$ (M.T.U. 2013) (ii) $\displaystyle\int_0^{\infty} \frac{x^4(1+x^5)}{(1+x)^{15}}\, dx$.

Sol. (i) $$I = \int_0^{\infty} \frac{x^8}{(1+x)^{24}}\, dx - \int_0^{\infty} \frac{x^{14}}{(1+x)^{24}}\, dx = \int_0^{\infty} \frac{x^{9-1}}{(1+x)^{9+15}}\, dx - \int_0^{\infty} \frac{x^{15-1}}{(1+x)^{15+9}}\, dx$$

$$= \beta(9, 15) - \beta(15, 9) = 0 \qquad \big|\ \because\ \beta(m, n) = \beta(n, m)$$

(ii) $$I = \int_0^{\infty} \frac{x^4}{(1+x)^{15}}\, dx + \int_0^{\infty} \frac{x^9}{(1+x)^{15}}\, dx = \int_0^{\infty} \frac{x^{5-1}}{(1+x)^{5+10}}\, dx + \int_0^{\infty} \frac{x^{10-1}}{(1+x)^{10+5}}\, dx$$

$$= \beta(5, 10) + \beta(10, 5) = 2\beta(5, 10) = 2\,\frac{\Gamma(5)\,\Gamma(10)}{\Gamma(15)} \qquad \Big|\ \because\ \beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$$= \frac{1}{5005}. \qquad \text{| On simplification}$$

Example 4. *Evaluate:*

(i) $\displaystyle\int_0^2 x(8-x^3)^{1/3}\, dx$ (ii) $\displaystyle\int_0^{\infty} \frac{dx}{1+x^4}$ (iii) $\displaystyle\int_0^1 \frac{dx}{\sqrt{1+x^4}}$ (G.B.T.U. 2011)

(G.B.T.U. 2012)

Sol. (i) Putting $x^3 = 8y$ or $x = 2y^{1/3}$ so that $dx = \frac{2}{3}y^{-2/3}\, dy$, we get

$$= \frac{1}{\Gamma(m+n+1)}[m\,\Gamma m\,\Gamma n + \Gamma m \cdot n\,\Gamma(n)] \qquad | \because \;\Gamma(n+1) = n\,\Gamma(n)$$

$$= \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)\,(m+n)}(m+n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)} = \beta(m, n).$$

Example 8. *Prove that* $\int_a^b (x-a)^m (b-x)^n\, dx = (b-a)^{m+n+1}\,\beta(m+1, n+1)$

Sol. Let $\qquad I = \int_a^b (x-a)^m (b-x)^n\, dx \qquad \text{...(1)}$

Put $x = a + (b-a)z \Rightarrow x - a = (b-a)z$ and $b - x = (b-a)(1-z)$ so that $dx = (b-a)\,dz$ then (1) becomes

$$I = \int_0^1 (b-a)^m z^m \cdot (b-a)^n (1-z)^n (b-a)\, dz$$

$$= (b-a)^{m+n+1} \int_0^1 z^m (1-z)^n\, dz$$

$$= (b-a)^{m+n+1}\,\beta(m+1, n+1)$$

Example 9. *Prove the following results:*

(i) $\int_0^{\pi/2} \sqrt{\tan\theta}\; d\theta = \int_0^{\pi/2} \sqrt{\cot\theta}\; d\theta = \frac{\pi}{\sqrt{2}}$ (U.K.T.U. 2011)

(ii) $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \times \int_0^{\pi/2} \sqrt{\sin\theta}\; d\theta = \pi.$ (U.P.T.U. 2014)

Sol. (i) $\int_0^{\pi/2} \sqrt{\tan\theta}\; d\theta$

$$= \int_0^{\pi/2} \sqrt{\cot\theta}\; d\theta \qquad \text{...(1)} \;\Big|\; \because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$= \int_0^{\pi/2} \cos^{1/2}\theta \sin^{-1/2}\theta\; d\theta$$

$$= \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)}{2\,\Gamma\left(\frac{-\frac{1}{2}+\frac{1}{2}+2}{2}\right)} = \frac{\Gamma(\frac{1}{4})\,\Gamma(\frac{3}{4})}{2\,\Gamma(1)} = \frac{1}{2}\Gamma(\tfrac{1}{4})\,\Gamma(1-\tfrac{1}{4})$$

$$= \frac{1}{2}\,\frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{\sqrt{2}} \qquad \text{...(2)} \;\Big|\; \because \Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

(ii) LHS
$$= \int_0^{\pi/2} \sin^{-1/2}\theta \cos^0\theta\; d\theta \times \int_0^{\pi/2} \sin^{1/2}\theta\cos^0\theta\; d\theta$$

$$= \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\,\Gamma\left(\frac{-\frac{1}{2}+0+2}{2}\right)} \times \frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\,\Gamma\left(\frac{\frac{1}{2}+0+2}{2}\right)} = \frac{\Gamma(\frac{1}{4})\,\Gamma(\frac{1}{2})}{2\,\Gamma(\frac{3}{4})} \times \frac{\Gamma(\frac{3}{4})\,\Gamma(\frac{1}{2})}{2\,\Gamma(\frac{5}{4})}$$

$$= \frac{\Gamma(\frac{1}{4})\sqrt{\pi}}{4} \times \frac{\sqrt{\pi}}{\frac{1}{4}\Gamma(\frac{1}{4})} = \pi = \text{RHS} \qquad \Big|\; \because \Gamma(n+1) = n\,\Gamma(n) \text{ and } \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

Example 10. *Evaluate:*

(*i*) $\int_0^\infty \cos x^2\, dx$ (*ii*) $\int_{-\infty}^\infty \cos \frac{\pi}{2} x^2\, dx$ (*iii*) $\int_0^1 \log \Gamma(x)\, dx$.

Sol. (*i*) We know that

$$\int_0^\infty e^{-ax} \cdot x^{n-1} \cos bx\, dx = \frac{\Gamma(n) \cos n\theta}{(a^2+b^2)^{n/2}} \text{ where } \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

Put $a = 0$, $\int_0^\infty x^{n-1} \cos bx\, dx = \frac{\Gamma(n)}{b^n} \cos \frac{n\pi}{2}$

Put $x^n = z$ so that $x^{n-1}\, dx = \frac{dz}{n}$ and $x = z^{1/n}$

then, $$\int_0^\infty \cos bz^{1/n}\, dz = \frac{n\,\Gamma(n)}{b^n} \cos \frac{n\pi}{2}$$

or $$\int_0^\infty \cos (bx^{1/n})\, dx = \frac{\Gamma(n+1)}{b^n} \cos \frac{n\pi}{2} \quad ...(1)$$

Here $b = 1$, $n = \frac{1}{2}$

$$\therefore \quad \int_0^\infty \cos x^2\, dx = \Gamma(\tfrac{3}{2}) \cos \frac{\pi}{4} = \frac{\sqrt{\pi}}{2} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

(*ii*) $$I = \int_{-\infty}^\infty \cos \frac{\pi x^2}{2}\, dx = 2\int_0^\infty \cos \frac{\pi x^2}{2}\, dx \quad ...(2)$$

Putting $b = \frac{\pi}{2}$ and $n = \frac{1}{2}$ in equation (1), we get

$$\int_0^\infty \cos\left(\frac{\pi}{2} x^2\right) dx = \frac{\Gamma\left(\frac{3}{2}\right)}{\left(\frac{\pi}{2}\right)^{1/2}} \cos \frac{\pi}{4}$$

$$\therefore \text{ From (2), } I = 2\frac{\Gamma\left(\frac{3}{2}\right)}{\left(\frac{\pi}{2}\right)^{1/2}} \cos \frac{\pi}{4} = 2 \cdot \frac{1}{2}\sqrt{\pi} \cdot \frac{\sqrt{2}}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{2}} = 1.$$

(*iii*) Let $$I = \int_0^1 \log \Gamma(x)\, dx \quad ...(1)$$

$$= \int_0^1 \log \Gamma(1-x)\, dx \quad ...(2)$$

Adding (1) and (2),

$$2I = \int_0^1 (\log \Gamma(x) + \log \Gamma(1-x))\, dx$$

$$= \int_0^1 \log (\Gamma(x)\,\Gamma(1-x))\, dx = \int_0^1 \log\left(\frac{\pi}{\sin \pi x}\right) dx$$

$$= \int_0^1 (\log \pi - \log \sin \pi x)\, dx = \int_0^1 \log \pi\, dx - \int_0^1 \log \sin \pi x\, dx$$

$$= I_1 - I_2 \qquad \ldots(3)$$

where $\quad I_1 = \int_0^1 \log \pi\, dx = \log \pi$

$$I_2 = \int_0^1 \log \sin \pi x\, dx = \int_0^\pi \log \sin t \left(\frac{dt}{\pi}\right) \qquad \Bigg|\ \text{Put } \pi x = t \Rightarrow dx = \frac{1}{\pi}\, dt$$

$$= \frac{1}{\pi} \cdot 2 \int_0^{\pi/2} \log \sin t\, dt = \frac{2}{\pi}\left(-\frac{\pi}{2} \log 2\right) = -\log 2$$

From (3), $\quad 2I = \log \pi + \log 2 = \log 2\pi$

$$I = \frac{1}{2} \log 2\pi.$$

Example 11. *Prove that:*

(a) $\iint_D x^{l-1} y^{m-1} dxdy = \dfrac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m+1)}\, a^{l+m}$, *where D is the domain* $x \geq 0,\ y \geq 0$ *and* $x + y \leq a.$

(b) Establish Dirichlet's integral: $\iiint_V x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \dfrac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}$ *where V is the region* $x \geq 0,\ y \geq 0,\ z \geq 0$ *and* $x + y + z \leq 1.$

Sol. (*a*) Putting $x = a$X and $y = a$Y, the given integral reduces to

$$I = \iint_{D'} (aX)^{l-1} (aY)^{m-1} a^2\, dXdY$$

where D′ is the domain $X \geq 0$, $Y \geq 0$ and $X + Y \leq 1$

$$I = a^{l+m} \int_0^1 \int_0^{1-X} X^{l-1}\, Y^{m-1}\, dYdX$$

$$= a^{l+m} \int_0^1 X^{l-1} \left| \frac{Y^m}{m} \right|_0^{1-X} dX = \frac{a^{l+m}}{m} \int_0^1 X^{l-1} (1-X)^m\, dX$$

$$= \frac{a^{l+m}}{m} \beta(l, m+1) = \frac{a^{l+m}}{m} \cdot \frac{\Gamma(l)\,\Gamma(m+1)}{\Gamma(l+m+1)} = a^{l+m}\, \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m+1)}$$

(*b*) Taking $y + z \leq 1 - x = a$ (say), the given integral

$$I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x^{l-1} y^{m-1} z^{n-1}\, dzdydx$$

$$= \int_0^1 x^{l-1} \left[\int_0^a \int_0^{a-y} y^{m-1} z^{n-1}\, dz\, dy \right] dx$$

$$= \int_0^1 x^{l-1} \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)}\, a^{m+n}\, dx \qquad \text{[by } (a)\text{]}$$

$$= \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)} \int_0^1 x^{l-1} (1-x)^{m+n}\, dx, \text{ since } a = 1 - x$$

$$= \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)} B(l, m+n+1) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)} \cdot \frac{\Gamma(l)\,\Gamma(m+n+1)}{\Gamma(l+m+n+1)}$$

$$= \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}.$$

Example 12. *The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the axes in A, B and C. Apply Dirichlet's integral to find the volume of the tetrahedron OABC. Also find its mass if the density at any point is kxyz.* (A.K.T.U. 2012, 2018)

Sol. Put $\frac{x}{a}=u, \frac{y}{b}=v, \frac{z}{c}=w$, then $u \ge 0, v \ge 0, w \ge 0$ and $u+v+w \le 1$.

Also, $dx = a\,du$, $dy = b\,dv$, $dz = c\,dw$.

$$\text{Volume OABC} = \iiint_D dx\,dy\,dz$$

$$= \iiint_{D'} abc\,du\,dv\,dw, \quad \text{where } u+v+w \le 1$$

$$= abc \iiint_{D'} u^{1-1}\,v^{1-1}\,w^{1-1}\,du\,dv\,dw$$

$$= abc\,\frac{\Gamma(1)\,\Gamma(1)\,\Gamma(1)}{\Gamma(1+1+1+1)} = \frac{abc}{3!} = \frac{abc}{6}.$$

$$\text{Mass} = \iiint_D kxyz\,dx\,dy\,dz = \iiint_{D'} k(au)(bv)(cw)\,abc\,du\,dv\,dw$$

$$= ka^2b^2c^2 \iiint_{D'} u^{2-1}v^{2-1}w^{2-1}\,du\,dv\,dw$$

$$= ka^2b^2c^2\,\frac{\Gamma(2)\,\Gamma(2)\,\Gamma(2)}{\Gamma(2+2+2+1)} = ka^2b^2c^2\,\frac{1!\,1!\,1!}{6!} = \frac{ka^2b^2c^2}{720}.$$

Example 13. *Evaluate $I = \iiint_V x^{\alpha-1}\,y^{\beta-1}\,z^{\gamma-1}\,dx\,dy\,dz$ where v is the region in the first octant bounded by sphere $x^2+y^2+z^2=1$ and the coordinate planes.*

Sol. Let $x^2 = u \Rightarrow x = \sqrt{u} \quad \therefore\ dx = \frac{1}{2\sqrt{u}}du$

$y^2 = v \Rightarrow y = \sqrt{v} \quad \therefore\ dy = \frac{1}{2\sqrt{v}}dv$

$z^2 = w \Rightarrow z = \sqrt{w} \quad \therefore\ dz = \frac{1}{2\sqrt{w}}dw$

Then, $u+v+w = 1$. Also, $u \ge 0, v \ge 0, w \ge 0$.

$$I = \iiint_V (\sqrt{u})^{\alpha-1}\,(\sqrt{v})^{\beta-1}\,(\sqrt{w})^{\gamma-1}\,\frac{du}{2\sqrt{u}} \cdot \frac{dv}{2\sqrt{v}} \cdot \frac{dw}{2\sqrt{w}}$$

$$= \frac{1}{8} \iiint u^{(\alpha/2)-1}\,v^{(\beta/2)-1}\,w^{(\gamma/2)-1}\,du\,dv\,dw$$

$$= \frac{1}{8}\,\frac{\Gamma(\alpha/2)\,\Gamma(\beta/2)\,\Gamma(\gamma/2)}{\Gamma((\alpha/2)+(\beta/2)+(\gamma/2)+1)}.$$

Example 14. *Show that if l, m, n are all positive,*

$$\iiint x^{l-1} y^{m-1} z^{n-1} dx\, dy\, dz = \frac{a^l b^m c^n}{8} \cdot \frac{\Gamma(l/2)\,\Gamma(m/2)\,\Gamma(n/2)}{\Gamma(l/2 + m/2 + n/2 + 1)},$$ *where the triple integral is taken throughout the part of ellipsoid* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ *which lies in the positive octant.*

Sol. Put $\left(\frac{x}{a}\right)^2 = u, \left(\frac{y}{b}\right)^2 = v, \left(\frac{z}{c}\right)^2 = w$ so that $dx = \frac{a\, du}{2\sqrt{u}}$ etc.

$$\therefore \quad \iiint x^{l-1} y^{m-1} z^{n-1} dx\, dy\, dz$$

$$= \iiint a^{l-1}(\sqrt{u})^{l-1} b^{m-1}(\sqrt{v})^{m-1} c^{n-1} (\sqrt{w})^{n-1} \frac{abc\, du\, dv\, dw}{8\sqrt{u}\sqrt{v}\sqrt{w}}$$

$$= \frac{a^l b^m c^n}{8} \iiint u^{\frac{l}{2}-1} v^{\frac{m}{2}-1} w^{\frac{n}{2}-1} du\, dv\, dw \qquad \text{subject to } u + v + w = 1$$

$$= \frac{a^l b^m c^n}{8} \frac{\Gamma(l/2)\,\Gamma(m/2)\,\Gamma(n/2)}{\Gamma\frac{l}{2} + \frac{m}{2} + \frac{n}{2} + 1}.$$

Example 15. *Evaluate the integral* $\iiint x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$

where x, y, z are all positive but limited by the condition $\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q + \left(\frac{z}{c}\right)^r \le 1.$

[G.B.T.U. 2011]

Sol. Let $\left(\frac{x}{a}\right)^p = u \quad \Rightarrow \quad x = au^{1/p} \quad \therefore \quad dx = \frac{a}{p} u^{\frac{1}{p}-1} du$

$\left(\frac{y}{b}\right)^q = v \quad \Rightarrow \quad y = bv^{1/q} \quad \therefore \quad dy = \frac{b}{q} v^{\frac{1}{q}-1} dv$

$\left(\frac{z}{c}\right)^r = w \quad \Rightarrow \quad z = cw^{1/r} \quad \therefore \quad dz = \frac{c}{r} w^{\frac{1}{r}-1} dw$

Then, $u + v + w \le 1$. Also $u \ge 0, v \ge 0, w \ge 0$ since x, y, z are all positive.

$$I = \iiint a^{l-1} u^{\frac{l-1}{p}} b^{m-1} v^{\frac{m-1}{q}} c^{n-1} w^{\frac{n-1}{r}} \frac{abc}{pqr} u^{\frac{1}{p}-1} v^{\frac{1}{q}-1} w^{\frac{1}{r}-1} du\, dv\, dw$$

$$= \frac{a^l b^m c^n}{pqr} \iiint u^{\frac{l}{p}-1} v^{\frac{m}{q}-1} w^{\frac{n}{r}-1} du\, dv\, dw$$

$$= \frac{a^l b^m c^n}{pqr} \frac{\Gamma\left(\frac{l}{p}\right)\Gamma\left(\frac{m}{q}\right)\Gamma\left(\frac{n}{r}\right)}{\Gamma\left(\frac{l}{p} + \frac{m}{q} + \frac{n}{r} + 1\right)}$$

Example 16. *Find the volume of the solid bounded by the co-ordinate planes and the surface* $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}+\sqrt{\frac{z}{c}}=1$.

Sol. Put $\sqrt{\frac{x}{a}}=u, \sqrt{\frac{y}{b}}=v, \sqrt{\frac{z}{c}}=w$ then $u \geq 0, v \geq 0, w \geq 0$ and $u+v+w=1$

Also, $dx = 2au\,du$, $dy = 2bv\,dv$, $dz = 2cw\,dw$

Required volume $= \iiint_D dx\,dy\,dz$

$$= \iiint_{D'} 8\,abc\,uvw\,du\,dv\,dw\,, \qquad \text{where } u+v+w=1$$

$$= 8\,abc \iiint_{D'} u^{2-1}\,v^{2-1}\,w^{2-1}\,du\,dv\,dw$$

$$= 8\,abc\,\frac{\Gamma 2\,\Gamma 2\,\Gamma 2}{\Gamma(2+2+2+1)} = 8\,abc \cdot \frac{1\cdot 1\cdot 1}{\Gamma(7)} = \frac{abc}{90}.$$

Example 17. *Apply Dirichlet's integral to find the volume and the mass contained in the solid region in the first octant of the ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ *if the density at any point is* $\rho(x,y,z)=kxyz$.

(U.P.T.U. 2015)

Sol. Put $\frac{x^2}{a^2}=u, \frac{y^2}{b^2}=v, \frac{z^2}{c^2}=w$ then $u \geq 0, v \geq 0, w \geq 0$ and $u+v+w=1$

Also, $dx = \frac{a}{2\sqrt{u}}\,du, dy = \frac{b}{2\sqrt{v}}\,dv, dz = \frac{c}{2\sqrt{w}}\,dw$

Required Volume $= \iiint_D dx\,dy\,dz = \iiint_{D'} \frac{abc}{8\sqrt{u}\,\sqrt{v}\,\sqrt{w}}\,du\,dv\,dw$

$$= \frac{abc}{8}\iiint_{D'} u^{\frac{1}{2}-1}\,v^{\frac{1}{2}-1}\,w^{\frac{1}{2}-1}\,du\,dv\,dw$$

$$= \frac{abc}{8}\,\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{2}\right)} = \frac{abc}{8}\cdot\frac{\pi\sqrt{\pi}}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}$$

$$= \frac{\pi\,abc}{6}$$

and Required Mass $= \iiint_D k\,xyz\,dx\,dy\,dz$

$$= \iiint_{D'} k\cdot a\sqrt{u}\cdot b\sqrt{v}\cdot c\sqrt{w}\cdot\frac{abc}{8\sqrt{u}\,\sqrt{v}\,\sqrt{w}}\,du\,dv\,dw$$

$$= k\,\frac{a^2b^2c^2}{8}\iiint_{D'} du\,dv\,dw = k\,\frac{a^2b^2c^2}{8}\iiint_{D'} u^{1-1}\,v^{1-1}\,w^{1-1}\,du\,dv\,dw$$

$$= k\,\frac{a^2b^2c^2}{8}\,\frac{\Gamma(1)\,\Gamma(1)\,\Gamma(1)}{\Gamma(4)} = k\,\frac{a^2b^2c^2}{48}.$$

Example 18. *Find the volume of the solid surrounded by the surface*

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} + \left(\frac{z}{c}\right)^{2/3} = 1.$$

Sol. Let $\left(\frac{x}{a}\right)^{2/3} = u \quad \Rightarrow \quad x = au^{3/2} \quad \therefore \quad dx = \frac{3a}{2}u^{1/2}\,du$

$\left(\frac{y}{b}\right)^{2/3} = v \quad \Rightarrow \quad y = bv^{3/2} \quad \therefore \quad dy = \frac{3b}{2}v^{1/2}\,dv$

$\left(\frac{z}{c}\right)^{2/3} = w \quad \Rightarrow \quad z = cw^{3/2} \quad \therefore \quad dz = \frac{3c}{2}w^{1/2}\,dw$

For the positive octant,

$$x \ge 0 \Rightarrow au^{3/2} \ge 0 \Rightarrow u \ge 0$$
$$y \ge 0 \Rightarrow bv^{3/2} \ge 0 \Rightarrow v \ge 0$$
$$z \ge 0 \Rightarrow cw^{3/2} \ge 0 \Rightarrow w \ge 0$$

Then, we have $u + v + w = 1$, $u \ge 0$, $v \ge 0$, $w \ge 0$.

Required volume $= 8\iiint dx\,dy\,dz$

$$= 8\iiint \frac{3a}{2}u^{1/2}\cdot\frac{3b}{2}v^{1/2}\cdot\frac{3c}{2}w^{1/2}\,du\,dv\,dw$$

$$= 27\,abc\iiint u^{\frac{3}{2}-1}\,v^{\frac{3}{2}-1}\,w^{\frac{3}{2}-1}\,du\,dv\,dw$$

$$= 27\,abc\cdot\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{11}{2}\right)} = \frac{4\pi abc}{35}$$

Example 19. *Show that the area bounded by the curve $x^n + y^n = a^n$ and the co-ordinate axes in the first quadrant is* $\dfrac{a^2\,\Gamma\left(\frac{1}{n}\right)^2}{2n\,\Gamma\left(\frac{2}{n}\right)}$.

Sol. Required area $\quad A = \iint_D dx\,dy \quad$ ...(1)

Let, $\left(\frac{x}{a}\right)^n = u$ so that $x = au^{1/n} \quad \therefore \quad dx = \frac{a}{n}u^{\frac{1}{n}-1}\,du$

and $\left(\frac{y}{a}\right)^n = v$ so that $y = av^{1/n} \quad \therefore \quad dy = \frac{a}{n}v^{\frac{1}{n}-1}\,dv$

Then, $u = 0, v = 0$ and $u + v = 1$

From (1),
$$A = \frac{a^2}{n^2}\iint_{D'} u^{\frac{1}{n}-1} v^{\frac{1}{n}-1}\, du\, dv = \frac{a^2}{n^2}\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{2}{n}+1\right)}$$

$$\Rightarrow \qquad A = \frac{a^2}{n^2}\cdot\frac{1}{\left(\frac{2}{n}\right)}\cdot\frac{\Gamma\left(\frac{1}{n}\right)^2}{\Gamma\left(\frac{2}{n}\right)} = \frac{a^2}{2n}\frac{\Gamma\left(\frac{1}{n}\right)^2}{\Gamma\left(\frac{2}{n}\right)}.$$

Example 20. *Find the area and the mass contained in the first quadrant enclosed by the curve* $\left(\frac{x}{a}\right)^\alpha + \left(\frac{y}{b}\right)^\beta = 1$ *where* $\alpha > 0$, $\beta > 0$ *given that density at any point* $\rho(x, y)$ *is* $k\sqrt{xy}$.

Sol. The area A of the plane region is $A = \iint_D dx\, dy$ ...(1)

Let $\left(\frac{x}{a}\right)^\alpha = u$ so that $x = au^{1/\alpha}$ $\quad\therefore\quad dx = \frac{a}{\alpha}u^{\frac{1}{\alpha}-1}\, du$

and $\left(\frac{y}{b}\right)^\beta = v$ so that $y = bv^{1/\beta}$ $\quad\therefore\quad dy = \frac{b}{\beta}v^{\frac{1}{\beta}-1}\, dv$

Then, $u > 0, v > 0$ and $u + v = 1$.

From (1),
$$A = \frac{ab}{\alpha\beta}\iint_{D'} u^{\frac{1}{\alpha}-1} v^{\frac{1}{\beta}-1}\, du\, dv = \frac{ab}{\alpha\beta}\frac{\Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(\frac{1}{\beta}\right)}{\Gamma\left(\frac{1}{\alpha}+\frac{1}{\beta}+1\right)}$$

Now, the total mass M contained in the plane region A is
$$M = \iint_D \rho(x, y)\, dx\, dy = k\iint_D \sqrt{xy}\, dx\, dy$$
$$= k\iint_{D'} \sqrt{a}\, u^{\frac{1}{2\alpha}}\sqrt{b}\, v^{\frac{1}{2\beta}}\cdot\frac{a}{\alpha}u^{\frac{1}{\alpha}-1}\cdot\frac{b}{\beta}v^{\frac{1}{\beta}-1}\, du\, dv$$
$$= k\frac{(ab)^{3/2}}{\alpha\beta}\iint_{D'} u^{\frac{3}{2\alpha}-1} v^{\frac{3}{2\beta}-1}\, du\, dv$$
$$= k\frac{(ab)^{3/2}}{\alpha\beta}\frac{\Gamma(3/2\alpha)\,\Gamma(3/2\beta)}{\Gamma\left(\frac{3}{2\alpha}+\frac{3}{2\beta}+1\right)}$$

TEST YOUR KNOWLEDGE

Prove that (1–22):

1. (i) $\int_0^\infty x^2 e^{-x^2}\, dx = \frac{\sqrt{\pi}}{4}$. (ii) $\int_0^\infty \sqrt{x}\, e^{-x^3}\, dx = \frac{\sqrt{\pi}}{3}$. (iii) $\int_0^\infty e^{-x^2} x^{2n-1} dx = \frac{1}{2}\Gamma(n)$.

2. (i) $\int_0^1 x^5(1-x^3)^3 dx = \frac{1}{60}$ (ii) $\int_0^1 x^3(1-x)^{4/3}\, dx = \frac{243}{7280}$

3. $\int_0^2 (8-x^3)^{-1/3}\, dx = \frac{1}{3}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{2\pi}{3\sqrt{3}}$.

4. (i) $\int_0^\infty 4x^4 e^{-x^4} dx = \Gamma\left(\frac{5}{4}\right)$ (ii) $\int_0^\infty x^6 e^{-2x} dx = \frac{45}{8}$ (iii) $\int_0^1 \sqrt{\frac{1-x}{x}}\, dx = \frac{\pi}{2}$

5. (i) $\int_0^\infty \frac{e^{-st}}{\sqrt{t}}\, dt = \sqrt{\frac{\pi}{s}}, s > 0$ (ii) $\int_0^1 \frac{dx}{\sqrt{-\log x}} = \sqrt{\pi}$.

   (iii) $\int_0^1 \left(\sqrt{\frac{x^3}{1-x^3}}\right)^{1/2} dx = \frac{\Gamma\frac{7}{12}\Gamma\frac{3}{4}}{\Gamma\frac{1}{3}}$

6. (i) $\Gamma(.1)\,\Gamma(.2)\,\Gamma(.3)\ldots\Gamma(.9) = \frac{(2\pi)^{9/2}}{\sqrt{10}}$

   (ii) $\Gamma\left(\frac{3}{2}-p\right)\Gamma\left(\frac{3}{2}+p\right) = \left(\frac{1}{4}-p^2\right)\pi \sec p\pi,\ -1 < 2p < 1$

7. (i) $\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\sqrt{\pi}}{n}\frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}$ (ii) $\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{4\sqrt{2\pi}}\left(\Gamma\frac{1}{4}\right)^2$.

8. (i) $\int_0^\infty \frac{x\, dx}{1+x^6} = \frac{\pi}{3\sqrt{3}}$ (ii) $\int_0^\infty \frac{x^2\, dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$

   (iii) $\int_0^2 \frac{x^2}{\sqrt{2-x}}\, dx = \frac{64\sqrt{2}}{15}$ (iv) $\int_0^3 \frac{dx}{\sqrt{3x-x^2}} = \pi$.

9. (i) $\int_0^\infty \sqrt{y}\, e^{-y^2} dy \times \int_0^\infty \frac{e^{-y^2}}{\sqrt{y}} dx = \frac{\pi}{2\sqrt{2}}$. (ii) $\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}} dx \times \int_0^\infty x^2 e^{-x^4} dx = \frac{\pi}{4\sqrt{2}}$.

10. (i) $\int_0^\infty x^2 e^{-x^4} dx \cdot \int_0^\infty e^{-x^4} dx = \frac{\pi}{8\sqrt{2}}$ (ii) $\int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}} \cdot \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{\pi}{4\sqrt{2}}$.

11. (i) $\int_0^{\pi/2} \sin^3 x \cos^{5/2} x\, dx = \frac{8}{77}$ (ii) $\int_0^a x^{n-1}(a-x)^{m-1}\, dx = a^{m+n-1}\beta(m,n)$

12. (i) $\int_0^1 \frac{dx}{\sqrt{1-x^6}} = \frac{\sqrt{3}}{2}\int_0^1 \frac{dx}{\sqrt{1-x^3}} = \frac{(\Gamma 1/3)^3}{(2)^{7/3}\pi}$ (ii) $\int_0^1 \frac{x\, dx}{\sqrt{1-x^5}} = \frac{1}{5}\beta\left(\frac{2}{5},\frac{1}{2}\right)$.

13. (i) $\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{2}{3}\right)} = (2)^{1/3}\sqrt{\pi}$ (A.K.T.U. 2017) (ii) $\Gamma\left(-\frac{3}{2}\right) = \frac{4}{3}\sqrt{\pi}$ (G.B.T.U. 2013)

    (iii) $\Gamma\left(\frac{1}{6}\right) = 2^{-1/3}\pi^{-1/2}\sqrt{3}\left(\Gamma\frac{1}{3}\right)^2$ (iv) $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$.

14. $\int_0^1 x^m (\log x)^n \, dx = \dfrac{(-1)^n n!}{(m+1)^{n+1}}$, where n is a positive integer and $m > -1$.

15. $\int_0^\infty \int_0^\infty e^{-(ax^2+by^2)} x^{2m-1} y^{2n-1} \, dx\, dy = \dfrac{\Gamma(m)\Gamma(n)}{4a^m b^n}$, where a, b, m, n are positive.

16. (i) $\beta(m,m) \cdot \beta\left(m+\dfrac{1}{2}, m+\dfrac{1}{2}\right) = \dfrac{\pi m^{-1}}{(2)^{4m-1}}$ (ii) $\beta(m,n) = a^m b^n \int_0^\infty \dfrac{x^{m-1}}{(ax+b)^{m+n}} dx$

(U.P.T.U. 2015)

17. $\dfrac{B(p,q+1)}{q} = \dfrac{B(p+1,q)}{p} = \dfrac{B(p,q)}{p+q}, (p > 0, q > 0)$ (U.P.T.U. 2015; U.K.T.U. 2012)

18. $\dfrac{B(m+2,n-2)}{B(m,n)} = \dfrac{m(m+1)}{(n-1)(n-2)}$.

19. $2^n \Gamma\left(n+\dfrac{1}{2}\right) = 1 \cdot 3 \cdot 5 \ldots\ldots (2n-1)\sqrt{\pi}$

20. $\int_0^1 \dfrac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}} dx = \dfrac{B(m,n)}{a^n (a+b)^m}$. $\left[\textbf{Hint.} \text{ Put } \dfrac{x}{a+bx} = \dfrac{z}{a+b \cdot 1}.\right]$

21. $\int_0^{\pi/2} \dfrac{d\theta}{(a\cos^4\theta + b\sin^4\theta)^{1/2}} = \dfrac{(\Gamma \frac{1}{4})^2}{4(ab)^{1/4}\sqrt{\pi}}$ [**Hint.** Put $\tan\theta = t$, then $bt^4 = az$.]

22. $\int_0^\infty x^n e^{-a^2x^2} dx = \dfrac{1}{2a^{n+1}} \Gamma\left(\dfrac{n+1}{2}\right), n > 1$. Deduce that $\int_{-\infty}^{\infty} e^{-a^2x^2} dx = \dfrac{\sqrt{\pi}}{a}$.

23. Assuming $\Gamma n\, \Gamma(1-n) = \pi \operatorname{cosec} n\pi$, $0 < n < 1$, show that $\int_0^\infty \dfrac{x^{p-1}}{1+x} dx = \dfrac{\pi}{\sin p\pi}$; $0 < p < 1$.

(G.B.T.U. 2010)

24. Show that $\iint x^{m-1} y^{n-1} \, dx\, dy$ over the positive octant of the ellipse $\left(\dfrac{x}{a}\right)^2 + \left(\dfrac{y}{b}\right)^2 = 1$ is $\dfrac{a^m b^n}{2n} \beta\left(\dfrac{m}{2}, \dfrac{n}{2}+1\right)$.

25. Find the volume of the solid bounded by co-ordinate planes and the surface

$$\left(\frac{x}{a}\right)^{2n} + \left(\frac{y}{b}\right)^{2n} + \left(\frac{z}{c}\right)^{2n} = 1, n \text{ being a positive integer.}$$

26. (i) Find the volume of the solid bounded by the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

(ii) Find the volume contained in the solid region in the first octant of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2}$ $= 1$. (U.P.T.U. 2014)

(iii) Evaluate $\iiint xyz \, dx\, dy\, dz$ for all positive value of variables of ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

27. (i) Find the mass of the region bounded by ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ if the density varies as the square of the distance from the centre. [**Hint.** $\rho = k(x^2+y^2+z^2)$] (G.B.T.U. 2010)

(ii) Find the mass of a solid $\left(\dfrac{x}{a}\right)^p + \left(\dfrac{y}{b}\right)^q + \left(\dfrac{z}{c}\right)^r = 1$, the density at any point being $\rho = kx^{l-1} y^{m-1} z^{n-1}$ where x, y, z are all positive. (A.K.T.U. 2016)

28. Evaluate $\iiint_V (ax^2 + by^2 + cz^2)\, dx\, dy\, dz$ where V is the region bounded by $x^2 + y^2 + z^2 \le 1$. (G.B.T.U. 2013)

29. Compute $\iiint_v x^2\, dx\, dy\, dz$ over valume of tetraheron bounded by $x = 0, y = 0, z = 0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. (A.K.T.U. 2017)

30. Evaluate $\iiint_v x^2 yz\, dx\, dy\, dz$ throughout the volume bounded by planes $x = 0, y = 0, z = 0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. (A.K.T.U. 2017)

Answers

25. $\frac{abc}{12n^2} \frac{(\Gamma\frac{1}{2n})^3}{\Gamma(\frac{3}{2n})}$ **26.** (i) $\frac{4}{3}\pi\, abc$ (ii) $\frac{\pi\, abc}{6}$ (iii) $\frac{a^2b^2c^2}{48}$

27. (i) $\frac{k\pi\, abc}{16}(a^2 + b^2 + c^2)$ (ii) $\frac{ka^l b^m c^n}{pqr} \frac{(l/p)!\,(m/q)!\,(n/r)!}{\left(\frac{l}{p} + \frac{m}{q} + \frac{n}{r} + 1\right)!}$

28. $\frac{\pi(a + b + c)}{30}$ **29.** $\frac{a^3bc}{60}$ **30.** $\frac{a^3b^2c^2}{2520}$

2.17 LIOUVILLE'S EXTENSION OF DIRICHLET THEOREM

If the variables x, y, z are all positive such that $h_1 < (x + y + z) < h_2$ then

$$\iiint f(x + y + z) x^{l-1} y^{m-1} z^{n-1} dx\, dy\, dz = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l + m + n)} \int_{h_1}^{h_2} f(u)\, u^{l+m+n-1}\, du.$$

Proof. Let, $I = \iiint x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$ under the condition $x + y + z \le u$ then

$$I = \left(\frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l + m + n + 1)}\right) u^{l+m+n} \quad ...(1) \mid \text{By Dirichlet's Theorem}$$

If $x + y + z \le u + \delta u$, then

$$I = (u + \delta u)^{l+m+n} \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l + m + n + 1)} \quad ...(2)$$

Now, if $u < x + y + z < (u + \delta u)$, then

$$\iiint x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l + m + n + 1)}\left[(u + \delta u)^{l+m+n} - (u)^{l+m+n}\right]$$

$$= \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l + m + n + 1)} u^{l+m+n}\left[1 + \left(\frac{\delta u}{u}\right)^{l+m+n} - 1\right]$$

$$= \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l + m + n + 1)} u^{l+m+n}\left[1 + (l + m + n)\frac{\delta u}{u} + \ldots - 1\right]$$

$$= \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)} u^{l+m+n} (l+m+n) \frac{\delta u}{u}$$

$$= \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n)} u^{l+m+n-1}\, \delta u$$

Now, consider $\iiint f(x+y+z) x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$ under the condition $h_1 \le (x+y+z) \le h_2$. When $x+y+z$ lies between u and $u + \delta u$, the value of $f(x+y+z)$ can only differ from $f(u)$ by a small quantity of the same order as δu. Hence,

$$\iiint f(x+y+z) x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n)} \int f(u)\, u^{l+m+n-1}\, \delta u$$

where $x+y+z$ lies between u and $u + \delta u$.

Therefore, $\boxed{\iiint f(x+y+z) x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n)} \int_{h_1}^{h_2} f(u)\, u^{l+m+n-1}\, du.}$

ILLUSTRATIVE EXAMPLES

Example 1. *Evaluate* $\iiint \log(x+y+z)\, dx\, dy\, dz$, *the integral extending over all positive and zero values of* x, y, z *subject to* $x+y+z < 1$.

Sol. $0 \le x+y+z < 1$

$$\therefore \iiint \log(x+y+z)\, dx\, dy\, dz = \iiint x^{1-1} y^{1-1} z^{1-1} \log(x+y+z)\, dx\, dy\, dz$$

$$= \frac{\Gamma(1)\,\Gamma(1)\,\Gamma(1)}{\Gamma(3)} \int_0^1 t^{1+1+1-1} \log t\, dt \qquad | \text{ By Liouville's extension}$$

$$= \frac{1}{2}\int_0^1 t^2 \log t\, dt = \frac{1}{2}\left[\left(\frac{t^3}{3}\log t\right)_0^1 - \int_0^1 \frac{t^3}{3}\cdot\frac{1}{t}\, dt\right] = \frac{1}{2}\left[-\frac{1}{3}\left(\frac{t^3}{3}\right)_0^1\right] = -\frac{1}{18}.$$

Example 2. *Evaluate* $\iiint\cdots\int_n \frac{dx_1 dx_2 \ldots dx_n}{\sqrt{1 - x_1^2 - x_2^2 \ldots - x_n^2}}$, *integral being extended to all positive values of the variables for which the expression is real.*

Sol. The expression will be real, if

$$1 - x_1^2 - x_2^2 - \ldots - x_n^2 > 0$$

or

$$x_1^2 + x_2^2 + \ldots + x_n^2 < 1$$

Hence the given integral is extended for all positive value of the variables $x_1, x_2, \ldots, x_n$ such that $0 < x_1^2 + x_2^2 + \ldots x_n^2 < 1$.

Let us put $x_1^2 = u_1$ *i.e.*, $x_1 = \sqrt{u_1}$ so that, $dx_1 = \frac{1}{2\sqrt{u_1}} du_1$ etc.

Then the condition becomes, $0 < u_1 + u_2 + ... + u_n < 1$.

$\therefore$ Required integral $= \frac{1}{2^n} \iiint ... \int_n \frac{u_1^{-1/2} u_2^{-1/2} ... u_n^{-1/2}}{\sqrt{1 - u_1 - u_2 - ... - u_n}} du_1 du_2 ... du_n$

$$= \frac{1}{2^n} \iiint ... \int_n \frac{u_1^{1/2-1} u_2^{1/2-1} ... u_n^{1/2-1}}{\sqrt{1 - u_1 - u_2 - ... - u_n}} du_1 du_2 ... du_n$$

$$= \frac{1}{2^n} \frac{\Gamma(1/2)\Gamma(1/2)...\Gamma(1/2)}{\Gamma(1/2 + 1/2 + ... + 1/2)} \int_0^1 \frac{1}{\sqrt{1-u}} u^{(1/2+1/2+......+1/2)-1} du$$

| By Liouville's Extension

$$= \frac{1}{2^n} \cdot \frac{[\Gamma(1/2)]^n}{\Gamma(n/2)} \int_0^1 \frac{1}{\sqrt{1-u}} u^{\frac{n}{2}-1} du$$

$$= \frac{1}{2^n} \frac{(\sqrt{\pi})^n}{\Gamma(n/2)} \int_0^{\pi/2} \frac{1}{\sqrt{1-\sin^2\theta}} (\sin^2\theta)^{\frac{n}{2}-1} \cdot 2\sin\theta\cos\theta \, d\theta$$

(Putting $u = \sin^2\theta$)

$$= \frac{1}{2^{n-1}} \cdot \frac{(\sqrt{\pi})^n}{\Gamma(n/2)} \int_0^{\pi/2} \sin^{n-1}\theta \, d\theta = \frac{1}{2^{n-1}} \cdot \frac{(\pi)^{n/2}}{\Gamma(n/2)} \cdot \frac{\Gamma(n/2)\Gamma(1/2)}{2\Gamma\left(\frac{n+1}{2}\right)}$$

$$= \frac{1}{2^n} \frac{(\pi)^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}.$$

Example 3. *Show that* $\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3} = \frac{1}{2}\log 2 - \frac{5}{16}$, *the integral being taken throughout the volume bounded by planes* $x = 0$, $y = 0$, $z = 0$ *and* $x + y + z = 1$.

Sol. $0 \le x + y + z \le 1$

$$\therefore \quad \iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3} = \iiint \frac{x^{1-1} y^{1-1} z^{1-1}}{(x+y+z+1)^3} dx\,dy\,dz$$

$$= \frac{\Gamma(1)\Gamma(1)\Gamma(1)}{\Gamma(1+1+1)} \int_0^1 \frac{1}{(u+1)^3} u^{1+1+1-1} du$$

$$= \frac{1}{2} \int_0^1 \frac{u^2}{(u+1)^3} du$$

| By Liouville's extension

Put $u + 1 = t$ so that $du = dt$

$\therefore$ Required integral $= \frac{1}{2} \int_1^2 \frac{(t-1)^2}{t^3} dt = \frac{1}{2} \int_1^2 \left(\frac{t^2 - 2t + 1}{t^3}\right) dt$

$$= \frac{1}{2} \int_1^2 \left(\frac{1}{t} - \frac{2}{t^2} + \frac{1}{t^3}\right) dt = \frac{1}{2}\left[\log t + \frac{2}{t} - \frac{1}{2t^2}\right]_1^2$$

$$= \frac{1}{2}\left[\log 2 + 1 - \frac{1}{8} - 2 + \frac{1}{2}\right] = \frac{1}{2}\left[\log 2 - \frac{5}{8}\right] = \frac{1}{2}\log 2 - \frac{5}{16}.$$

Example 4. *Evaluate* $\iiint \sqrt{\dfrac{1-x^2-y^2-z^2}{1+x^2+y^2+z^2}}\, dx\, dy\, dz$, *integral being taken over all positive values of x, y, z such that* $x^2+y^2+z^2 \le 1$.

Sol. Putting $x^2 = u,\quad y^2 = v,\quad z^2 = w$ so that $u + v + w \le 1$

Also,
$$x = \sqrt{u} \quad \Rightarrow \quad dx = \frac{1}{2\sqrt{u}}\,du$$
$$y = \sqrt{v} \quad \Rightarrow \quad dy = \frac{1}{2\sqrt{v}}\,dv$$
$$z = \sqrt{w} \quad \Rightarrow \quad dz = \frac{1}{2\sqrt{w}}\,dw$$

$\therefore$ The given integral

$$= \iiint \sqrt{\frac{1-(u+v+w)}{1+(u+v+w)}}\;\frac{du\,dv\,dw}{8\sqrt{uvw}}$$

$$= \frac{1}{8}\iiint u^{1/2-1}\, v^{1/2-1}\, w^{1/2-1}\sqrt{\frac{1-(u+v+w)}{1+(u+v+w)}}\,du\,dv\,dw$$

$$= \frac{1}{8}\frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2}+\frac{1}{2})}\int_0^1 \sqrt{\frac{1-u}{1+u}}\;u^{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}-1}\,du \qquad \text{| Using Liouville's extension}$$

$$= \frac{1}{8}\frac{(\Gamma\frac{1}{2})^3}{\frac{1}{2}\Gamma(\frac{1}{2})}\int_0^1 \frac{(1-u)}{\sqrt{1-u^2}}\,u^{1/2}\,du = \frac{\pi}{4}\int_0^1 \frac{(1-\sqrt{t})}{\sqrt{1-t}}\,t^{1/4}\,\frac{dt}{2\sqrt{t}} \qquad \text{where } u^2 = t$$

$$= \frac{\pi}{8}\int_0^1 \frac{(1-\sqrt{t})\,t^{-1/4}}{\sqrt{1-t}}\,dt = \frac{\pi}{8}\left[\int_0^1 t^{\frac{3}{4}-1}(1-t)^{1/2-1}dt - \int_0^1 t^{5/4-1}(1-t)^{1/2-1}dt\right]$$

$$= \frac{\pi}{8}\left[\beta\left(\frac{3}{4},\frac{1}{2}\right) - \beta\left(\frac{5}{4},\frac{1}{2}\right)\right].$$

TEST YOUR KNOWLEDGE

1. Evaluate $\iiint e^{x+y+z}\,dx\,dy\,dz$, taken over positive octant such that $x + y + z \le 1$.
2. (*i*) Evaluate the integral $\iiint \dfrac{dx\,dy\,dz}{\sqrt{a^2-x^2-y^2-z^2}}$ the integral being extended to all positive values of the variables for which the expression is real. [*U.K.T.U. 2010*]

   (*ii*) Prove that $\iiint \dfrac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}} = \dfrac{\pi^2}{8}$, the integral being extended to all positive values of the variables for which the expression is real. (*A.K.T.U. 2016; U.P.T.U. 2014*)

3. Apply Dirichlet's integral to find the moment of inertia about z-axis of an octant of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

4. Show that $\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^2}=\frac{3}{4}-\log 2$, the integral being taken throughout the volume bounded by the planes $x=0, y=0, z=0$ and $x+y+z=1$.

5. Evaluate $\iiint (x+y+z)^9\,dx\,dy\,dz$ over the region defined by $x\ge 0, y\ge 0, z\ge 0$ and $x+y+z\le 1$.

6. Evaluate $\iiint \sqrt{\frac{1-x-y-z}{xyz}}\,dx\,dy\,dz$, extended to all positive values of the variables subject to the condition $x+y+z<1$.

7. Evaluate $\iint x^{l-1}y^{-l}e^{x+y}dx\,dy$, extended to all positive values subject to $x+y<h$.

8. Evaluate $\iiint xyz\sin(x+y+z)\,dx\,dy\,dz$, the integral being extended to all positive values of the variables subject to the condition $x+y+z\le\frac{\pi}{2}$. (U.K.T.U. 2011)

9. Evaluate $\iint \frac{\sqrt{1-(x^2/a^2)-(y^2/b^2)}}{\sqrt{1+(x^2/a^2)+(y^2/b^2)}}\,dx\,dy$, where $\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1$

10. Evaluate $\iiint \sqrt{a^2b^2c^2-b^2c^2x^2-c^2a^2y^2-a^2b^2z^2}\,dx\,dy\,dz$ taken throughout the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

11. Evaluate $\iiint_V e^{-(x+y+z)}\,dx\,dy\,dz$, where the region of integration is bounded by planes $x=0, y=0, z=0$ and $x+y+z=a, a>0$. [U.P.T.U. (SUM) 2008]

12. For all values of the variables for which $x^2+y^2+z^2+w^2$ is not less than a^2 and not greater than b^2, prove that:

$$\iiiint dx\,dy\,dz\,dw=\frac{\pi^2}{32}(b^4-a^4).$$

Answers

1. $\frac{e}{2}-1$
2. (i) $\frac{\pi^2a^2}{8}$
3. $\frac{\rho abc}{30}(a^2+b^2)\pi$
5. $\frac{1}{24}$
6. $\frac{1}{4}\pi^2$
7. $\frac{\pi}{\sin \pi l}(e^h-1)$
8. $\frac{1}{5!}\left[\frac{5}{16}\pi^4-15\pi^2+120\right]$
9. $\frac{\pi ab}{8}(\pi-2)$
10. $\frac{1}{4}\pi^2a^2b^2c^2$
11. $1-e^{-a}\left[1+a+\frac{a^2}{2}\right]$.

2.18 APPLICATION OF DEFINITE INTEGRALS TO VOLUMES AND SURFACE AREAS

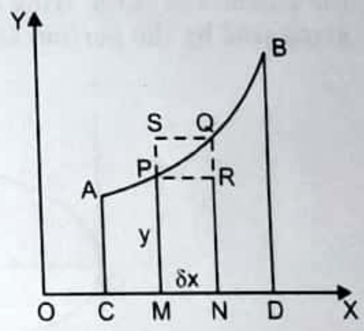
If a plane area is revolved about a fixed straight line lying in its own plane then the body so generated by the plane area is called the *solid of revolution* and the surface generated by the boundary of the plane area is called the *surface of revolution*. The fixed line about which the plane area rotates is called the axis of *revolution*.

For example, a right angled triangle when revolved about one of its sides (forming the right angle) generates a right circular cone.

Note: The section of a solid of revolution by a plane perpendicular to the axis of revolution is a circle having its centre on the axis of revolution.

2.19 VOLUME FORMULAE FOR CARTESIAN EQUATIONS

Prove that volume of the solid generated by revolution about x-axis, of the area bounded by the curve $y = f(x)$, the solid axis and ordinates $x = a, x = b$ is $\int_a^b \pi y^2 \, dx$, where $y = f(x)$ is a finite, continuous and single-valued function of x in the interval $a \le x \le b$.



Let AB be the curve $y = f(x)$ and CA, DB be the ordinates $x = a, x = b$ respectively. Let $P(x, y)$ be any point on the curve AB. Draw $PM \perp OX$.

$\therefore$ $$OM = x \text{ and } PM = y$$

Let V denote the volume of the solid generated by the revolution about X-axis of the area $ACMP$.

As x increases *i.e.*, MP moves towards the right, V also increases.

Thus the volume V depends on x and is therefore a function of x.

Let $Q(x + \delta x, y + \delta y)$ be a point on the curve in the neighbourhood of P. Then the volume of the solid generated by the revolution about x-axis of the area ACNQ will be $V + \delta V$, so that the volume of the solid generated by the revolution about x-axis of the area PMNQ is δV.

Complete the rectangle PRQS.

Then the volume of the solid generated by the revolution about the x-axis of the area $PMNQ$ lies between the right circular cylinders generated by the rectangles PMNR and SMNQ *i.e.*, δV lies between $\pi y^2 \, \delta x$ and $\pi (y + \delta y)^2 \, \delta x$

or $\dfrac{\delta V}{\delta x}$ lies between πy^2 and $\pi(y + \delta y)^2$

In the limiting position as $Q \to P$, $\delta x \to 0$ and $\therefore$ $\delta y \to 0$

$\therefore$ $\underset{\delta x \to 0}{\text{Lt}} \dfrac{\delta V}{\delta x}$ lies between πy^2 and $\underset{\delta y \to 0}{\text{Lt}} \pi(y + dy)^2$

or $\dfrac{dV}{dx}$ lies between πy^2 and a quantity which approaches πy^2.

$$\therefore \qquad \frac{dV}{dx} = \pi y^2$$

$$\int_c^b \pi y^2 \, dx = \int_a^b \frac{dV}{dx} \, dx = [V]_a^b$$

= (volume V when $x = b$) – (volume V when $x = a$)

= Volume generated by the area ACDB – 0

Hence the volume of the solid generated by the area ACDB about the x-axis is $\int_a^b \pi y^2\, dx$

Cor. Revolution about y-axis. *The volume of the solid generated by revolution about the y-axis of the area bounded by the curve $x = f(y)$, the y-axis and the abscissae $y = a$, $y = b$ is* $\int_a^b \pi x^2\, dy$.

The result follows immediately on interchanging x and y in the above proposition.

Note 1. If the generating curve is **symmetrical about the x-axis** and it is required to find the volume generated by the *revolution of the area about the x-axis,* in such a case we shall revolve only **one** of the symmetrical portions, because the part of the curve above the x-axis generates the same volume as the part of the curve below the x-axis when revolved about the x-axis. Thus the volume generated by the revolution of the area DCOAB about OX = volume generated by the area OABM or OCDM. (see Fig. 1)

Note 2. If the curve is *symmetrical about y-axis* and the same curve be made to revolve *about the x*-axis (the curve lying only on one side of the x-axis), then the volume generated = 2 × volume generated by the portion OAB lying to the right of y-axis. (see Fig. 2)

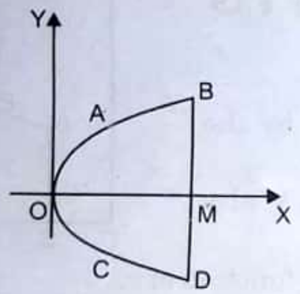


Fig. 1

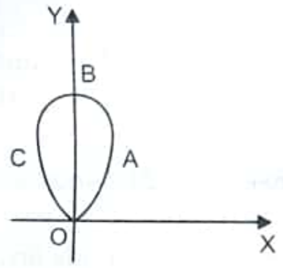


Fig. 2

ILLUSTRATIVE EXAMPLES

Example 1. (*a*) *Find the volume generated by rotating about the y-axis the area bounded by the co-ordinate axes and the graph of the curve $y = \cos x$ from $x = 0$ to $x = \frac{\pi}{2}$.*

(*b*) *The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ revolves about the axis of x. Show that the volume cut-off from one of the two solids thus obtained by a plane perpendicular to x-axis, and distant h from the vertex, is $\frac{\pi b^2 h^2 (3a + h)}{3a^2}$.*

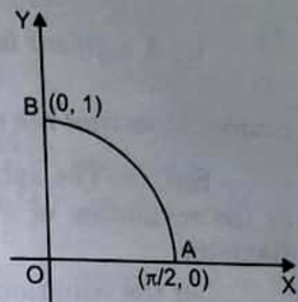
Sol. (*a*) The equation of the curve is $y = \cos x$ ...(1)

The graph of the curve from $x = 0$ to $x = \frac{\pi}{2}$ is as shown in the figure. Now the area OAB revolves about y-axis. For the area OAB, y varies from 1 to 0.

Volume of the surface generated by revolution about y-axis

$$= \int_{y=1}^{y=0} \pi x^2\, dy = \int_{x=0}^{x=\pi/2} \pi x^2 \frac{dy}{dx}\, dx$$

| $\because$ when $y = 1, x = 0$ and when $y = 0, x = \frac{\pi}{2}$



$$= \int_0^{\pi/2} \pi x^2 (-\sin x)\, dx \qquad \Big|\ \because \text{ From (1), } \frac{dy}{dx} = -\sin x$$

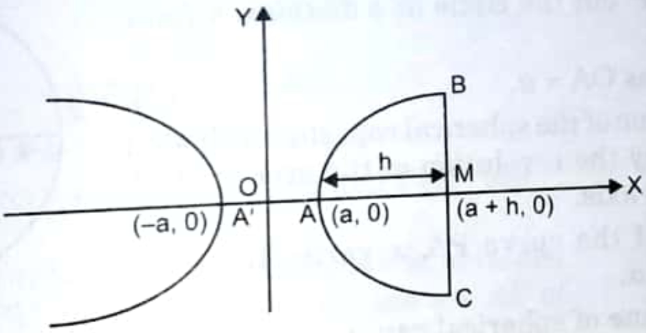
$$= -\pi \int_0^{\pi/2} x^2 \sin x\, dx$$ | Integrating by parts

$$= -\pi\left[\left|x^2 . (-\cos x)\right|_0^{\pi/2} - \int_0^{\pi/2} 2x(-\cos x)\, dx\right] = -2\pi \int_0^{\pi/2} x \cos x\, dx$$

| Integrating by parts

$$= \pi(\pi - 2)$$

(b) The given hyperbola is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ ...(1)



Volume of the solid cut-off by a plane ⊥ to x-axis and at a distance h from the vertex is the same as the volume obtained by rotating the portion of the curve from $x = a$ to $x = a + h$ about the x-axis.

$$\therefore \text{ Required volume} = \int_a^{a+h} \pi y^2\, dx = \pi \int_a^{a+h} \frac{b^2}{a^2}(x^2 - a^2)\, dx$$ | From (1)

$$= \pi \frac{b^2}{a^2}\left[\frac{x^3}{3} - a^2 x\right]_a^{a+h}$$

$$= \pi \frac{b^2}{a^2}\left[\frac{1}{3}(a^3 + 3a^2h + 3ah^2 + h^3 - a^3) - a^2h\right] = \pi \frac{b^2}{a^2}\left[\frac{3ah^2 + h^3}{3}\right] = \frac{\pi b^2 h^2 (3a + h)}{3a^2}.$$

Example 2. (a) *Find the volume of a sphere of radius a.*

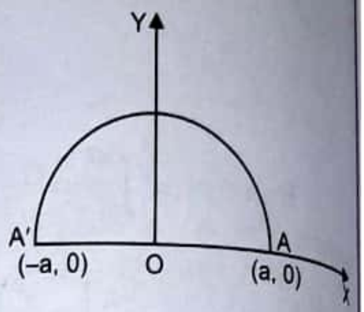
Or

The circle $x^2 + y^2 = a^2$ is revolved about x-axis. Find the volume of the sphere so formed.

(b) *Find the volume of a spherical cap of height h cut-off from a sphere of radius a.*

(c) *A segment is cut-off from a sphere of radius a by a plane at a distance* $\frac{a}{2}$ *from the centre. Show that the volume of the segment is* $\frac{5}{32}$ *of the volume of the sphere.*

Sol. (*a*) The sphere is the solid of revolution generated by the revolution of a semi-circular area about its bounding diameter.



Let the equation of circle of radius a be

$$x^2 + y^2 = a^2 \qquad \text{...(1)}$$

whose centre is O.

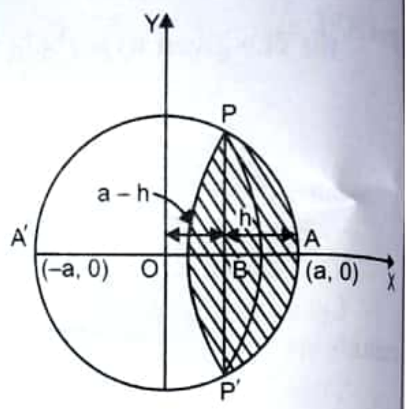
Now for the semi-circle above the x-axis, x varies from $-a$ to a.

∴ Required volume of sphere

$$= \int_{-a}^{a} \pi y^2\, dx = \pi \int_{-a}^{a} (a^2 - x^2)\,dx \qquad \Big| \text{ From (1)}$$

$$= \pi\left[a^2 x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{4}{3}\pi a^3.$$

(*b*) Let the equation of the circle be $x^2 + y^2 = a^2$ ...(1)



Let the plane PBP′ cut the circle at a distance h from A, so that BA = h and

∴ OB = $a - h$, as OA = a.

The required volume of the spherical cap (shown shaded in figure) is generated by the revolution of the area PBA of the circle (1) about the x-axis.

For the portion of the curve PA, x varies from OB to OA *i.e.*, from $a - h$ to a.

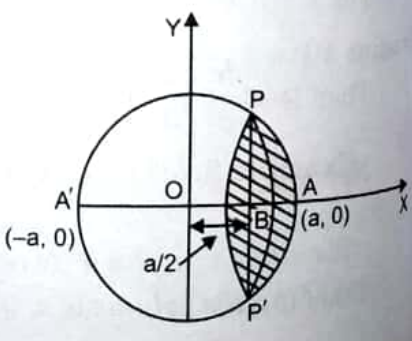
∴ Required volume of spherical cap

$$= \int_{a-h}^{a} \pi y^2\, dx = \int_{a-h}^{a} \pi(a^2 - x^2)\,dx \qquad \Big| \text{ From (1)}$$

$$= \pi\left[a^2 x - \frac{x^3}{3}\right]_{a-h}^{a} = \pi a^2[a - (a-h)] - \frac{\pi}{3}[a^3 - (a-h)^3]$$

$$= \pi a^2 h - \frac{\pi}{3}[3a^2 h - 3ah^2 + h^3] = \pi\left(ah^2 - \frac{h^3}{3}\right).$$

(*c*) Let the equation of the circle be $x^2 + y^2 = a^2$ ...(1)



Let the plane PBP′ cut the circle at a distance $\frac{a}{2}$ from the centre O of the circle. Then the required volume of the segment is generated by revolving the area PBA of the circle (1) about the x-axis.

Now for the arc PA, x varies from $\frac{a}{2}$ to a.

$\therefore$ Required volume

$$= \int_{a/2}^{a} \pi y^2\, dx = \pi \int_{a/2}^{a} (a^2 - x^2)\,dx \qquad \text{| Using (1)}$$

$$= \pi\left[a^2 x - \frac{x^3}{3}\right]_{a/2}^{a} = \pi\left[a^3 - \frac{a^3}{3} - \left(\frac{a^3}{2} - \frac{a^3}{24}\right)\right] = \frac{5\pi a^3}{24}$$

$$= \frac{5}{32}\left(\frac{4}{3}\pi a^3\right) = \frac{5}{32} \text{ (Volume of the sphere of radius } a)$$

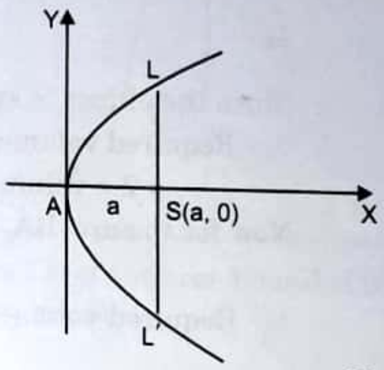
Example 3. *(a) The area of the parabola $y^2 = 4ax$ lying between the vertex and the latus rectum is revolved about the x-axis. Find the volume generated.*

(b) A paraboloid of revolution is generated by rotating the parabola $y^2 = 4ax$ about OX. Find the volume generated by that portion of the curve which lies between $x = 0$ and $x = h$. If R is the area of the cross-section at $x = h$, show that the volume is half that of a cylinder of base area R and length h.

Sol. (*a*) The equation of the parabola is $y^2 = 4ax$ ...(1)

Let A be its vertex and LSL' be the latus rectum.

The required volume is generated by the revolution of the area ASL about the x-axis and for the arc AL, x varies from 0 to a.



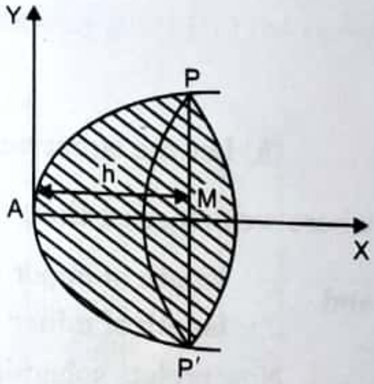
$$\therefore \text{ Required volume} = \int_0^a \pi y^2\, dx = \pi \int_0^a 4ax\, dx \quad \text{| From (1)}$$

$$= 4a\pi \left[\frac{x^2}{2}\right]_0^a = 2\pi a^3.$$

(*b*) The equation of the parabola is $y^2 = 4ax$ ...(1)

Let A(0, 0) be the vertex of the parabola and PMP′ the ordinate at $x = h$.

Then the required volume of the paraboloid is obtained by revolving the area APM about x-axis and for the arc AP of the curve x varies from 0 to h.



$\therefore$ Volume of the paraboloid of revolution is

$$V_1 = \int_0^h \pi y^2\, dx = \pi \int_0^h (4ax)\, dx \qquad \text{| Using (1)}$$

$$= 4a\pi \left[\frac{x^2}{2}\right]_0^h = 2\pi a h^2 \qquad \text{...(2)}$$

Now, $AM = h$ $\therefore$ $MP = \sqrt{4a\,.\,h} = 2\sqrt{ah}$ $\quad | \because\ y = \sqrt{4ax}$ | From (2)

The section of the paraboloid by a plane PMP' at a distance h from the vertex is a circle of radius MP = $2\sqrt{ah}$ and the centre M.

Then R is the area of this cross-section (circle)

$\therefore$
$$R = \pi\,.\,MP^2 = \pi.\,4ah = 4\pi ah \qquad \text{...(3)}$$

Now volume of cylinder of base area R and height h is

$$V_2 = \text{base area} \times \text{height} = Rh = 4\pi a h^2 \qquad \text{...(4) | Using (3)}$$

From (2) and (4), $\quad V_1 = \dfrac{1}{2} V_2.$

2.20 PROLATE AND OBLATE SPHEROIDS

(*i*) The solid formed by the revolution of the ellipse about the *major axis* is called a **prolate spheroid.**

(*ii*) The solid formed by revolving the ellipse about the *minor axis* is called an **oblate spheroid.**

Example 4. (*a*) *Find the volume of the solid generated by revolving the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *about the x-axis.*

(*b*) *Find the volume of the prolate spheroid generated by an ellipse whose major and minor axes are* $(24\pi)^{1/3}$ *and* $(3\pi)^{1/3}$.

Sol. (*a*) The equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$\Rightarrow \qquad y^2=\frac{b^2}{a^2}(a^2-x^2) \qquad \text{...(1)}$$

Since the ellipse is symmetrical about y-axis.

$\therefore$ Required volume generated by the ellipse by the revolution about x-axis.

= 2 × volume generated by the arc BA in first quadrant about the x-axis.

Now for the arc BA, x varies from 0 to a.

$$\therefore \quad \text{Required volume} \quad = 2\int_0^a \pi y^2\,dx = 2\pi\int_0^a \frac{b^2}{a^2}(a^2-x^2)\,dx \qquad |\text{ From (1)}$$

$$= 2\pi\frac{b^2}{a^2}\left[a^2x-\frac{x^3}{3}\right]_0^a = \frac{2\pi b^2}{a^2}\left(a^3-\frac{a^3}{3}\right)=\frac{4}{3}\pi ab^2.$$

(*b*) Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ...(1)

where we are given that

$$\left.\begin{aligned} &\text{length of major axis} = 2a = (24\pi)^{1/3}\\ \text{and} \quad &\text{length of minor axis} = 2b = (3\pi)^{1/3}\end{aligned}\right\} \qquad \text{...(2)}$$

Now prolate spheroid is the solid generated by revolving the ellipse (1) about the x-axis.

$$\therefore \quad \text{Required volume} = 2\int_0^a \pi y^2\,dx = 2\pi\int_0^a \frac{b^2}{a^2}(a^2-x^2)\,dx \qquad |\text{ From (1)}$$

$$= 2\pi\frac{b^2}{a^2}\left[a^2x-\frac{x^3}{3}\right]_0^a = 2\pi\frac{b^2}{a^2}\left(a^3-\frac{a^3}{3}\right)=\frac{4}{3}\pi ab^2$$

$$= \frac{1}{6}\pi\,.(2a)\,(2b)^2 = \frac{1}{6}\pi(24\pi)^{1/3}\,.(3\pi)^{2/3}$$

$$= \frac{1}{6}\pi\,.(24\pi\times 9\pi^2)^{1/3} = \frac{1}{6}\pi\,.\,6\pi = \pi^2.$$

Example 5. *A basin is formed by the revolution of the curve $x^3 = 64y$ $(y > 0)$ about the axis of y. If the depth of the basin is 8 cm, how many cubic cm of water will it hold?*

Sol. The equation of the generating curve is $x^3 = 64y$ $(y > 0)$ ...(1)

The curve is symmetrical in opposite quadrants.

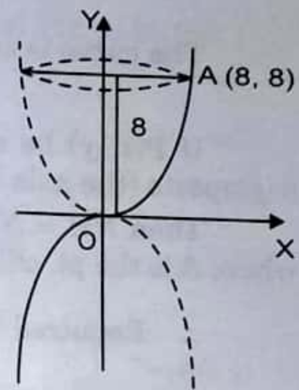
The shape of the curve is as shown in the figure.

Since the height of the basin is 8 cm, so when $y = 8$, from (1)

$$x^3 = 64 \times 8 \quad \therefore \quad x = 8.$$

Thus A(8, 8) is the point on the curve (1) at a height of 8 cm.

Thus the basin is formed by the revolution of the arc OA about y-axis, where A is the point (8, 8).



$$\therefore \text{ Required volume } = \int_0^8 \pi x^2\, dy = \pi \int_0^8 (64y)^{2/3}\, dy$$

$$= 16\pi . \left[\frac{y^{5/3}}{\frac{5}{3}} \right]_0^8 = \frac{48}{5} \pi[(8)^{5/3} - 0]$$

$$= \frac{1536}{5} \pi \text{ cubic cm.}$$

2.21 REVOLUTION ABOUT ANY AXIS

The volume of the solid generated by the revolution about any axis CD of the area bounded by the curve AB, the axis CD and the perpendiculars AC, BD on the axis, is

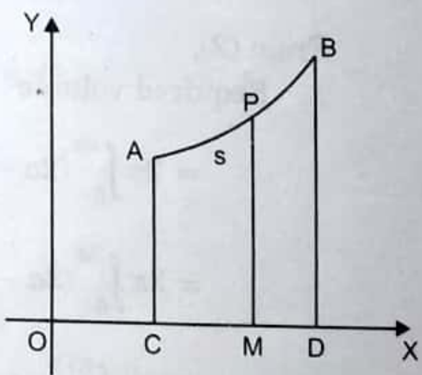
$$\int_{OC}^{OD} \pi (PM)^2 d(OM)$$

where O is a fixed point on the axis CD and PM is perpendicular from any point P of the curve AB on CD.

Take the fixed point O on CD as origin and CD, the axis of revolution as x-axis.

Let OY $\perp$ to CD be taken as y-axis.

Let P(x, y) be any point on the curve referred to OC and OY as axes.



Draw PM $\perp$ OC, so that OM = x and MP = y.

If OC = a and OD = b, then the required volume

$$= \int_0^b \pi y^2\, dx = \pi \int_a^b (\text{PM})^2 . d(\text{OM})$$

2.21.1 Working Rule

1. Take any point P(x, y) on the curve.
2. Draw PM $\perp$ on the line about which the curve is to be revolved and find the length of PM.
3. Find the distance OM of the foot of $\perp$ from a fixed point O (say) on the line and take its differential.
4. Then use the formula $\int \pi (\text{PM})^2 d(\text{OM})$ with proper limits for integration.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the volume of the solid formed by the revolution of the cissoid $y^2(2a - x) = x^3$ about its asymptote.*

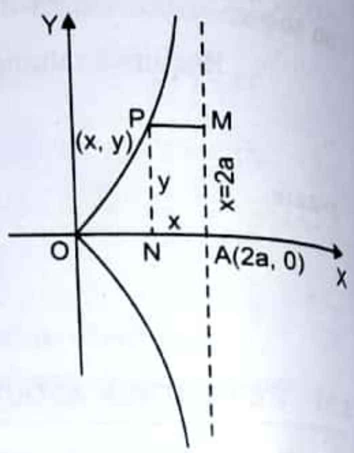
Sol. The equation of the curve is $y^2(2a - x) = x^3$ or $y^2 = \dfrac{x^3}{2a - x}$ ...(1)

The curve is symmetrical about the x-axis and the asymptote is the line

$$2a - x = 0 \quad \text{or} \quad x = 2a.$$

If P(x, y) be any point on the curve and PM $\perp$ on the asymptote (the axis of revolution), and $PN \perp OX$.

Then $PM = NA = OA - ON = 2a - x$ and $AM = NP = y$, where A is the pt. of intersection of the asymptote and the x-axis.



$\therefore$ Required volume $= 2\int \pi(\text{PM})^2 \, . \, d(\text{AM})$ ...(2)

Now $\quad \text{AM} = y = \dfrac{x^{3/2}}{\sqrt{2a - x}}$ | From (1)

$\therefore \quad d(\text{AM}) = dy$

$$= \frac{(2a - x)^{1/2} \, . \, \frac{3}{2} x^{1/2} - x^{3/2} \, . \, \frac{1}{2}(2a - x)^{-1/2} \, . (-1)}{2a - x} dx$$

$$= \frac{3x^{1/2}(2a-x) + x^{3/2}}{2(2a-x)^{3/2}} dx = \frac{\sqrt{x}(3a-x)}{(2a-x)^{3/2}} dx$$

From (2),

$\therefore$ Required volume

$$= 2\pi \int_0^{2a} (2a - x)^2 \, . \, \frac{\sqrt{x}(3a - x)}{(2a - x)^{3/2}} dx$$

$$= 2\pi \int_0^{2a} (3a - x)\sqrt{x}\sqrt{2a - x} \, dx$$

| Put $x = 2a \sin^2\theta \quad \therefore \quad dx = 4a \sin\theta \cos\theta \, d\theta$
when $x = 0, \theta = 0$, when $x = 2a, \theta = \pi/2$

$$= 2\pi \int_0^{\pi/2} (3a - 2a \sin^2\theta) \, . \, \sqrt{2a \sin^2\theta} \, . \, \sqrt{2a(1 - \sin^2\theta)} \, . \, 4a \sin\theta \cos\theta \, d\theta$$

$$= 16\pi a^3 \int_0^{\pi/2} (3 - 2\sin^2\theta) \, . \, \sin^2\theta \cos^2\theta \, d\theta$$

$$= 16\pi a^3 \int_0^{\pi/2} (3 \sin^2\theta \cos^2\theta - 2\sin^4\theta \cos^2\theta) \, d\theta$$

$$= 16\pi a^3 \left[3 \cdot \frac{1.1}{4.2} \, . \, \frac{\pi}{2} - 2 \, . \, \frac{3.1.1}{6.4.2} \, . \, \frac{\pi}{2}\right] = 16\pi a^3 \left[\frac{3\pi}{16} - \frac{\pi}{16}\right] = 2\pi^2 a^3.$$

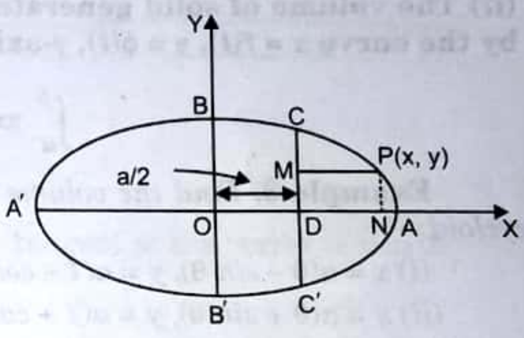
Example 2. *The ellipse $b^2x^2 + a^2y^2 = a^2b^2$ is divided into two parts by the line $x = \frac{a}{2}$, and the smaller part is rotated through four right angles about this line. Prove that the volume generated is $\pi a^2 b\left(\frac{3\sqrt{3}}{4} - \frac{\pi}{3}\right)$.*

Sol. The equation of the ellipse is $b^2x^2 + a^2y^2 = a^2b^2$ ...(1)

and CDC′ is the line $x = \frac{a}{2}$, about which the area CDC′AC is revolved. Let P(x, y) be any point on the arc CAC of the ellipse. From P draw PM ⊥ CC′ and PN ⊥ OX.

Then PM = DN = ON – OD = $x - \frac{a}{2}$ and PN = DM = y.

Since the arc CAC′ is symmetrical about the x-axis.



Required volume = $2\int \pi(\text{PM})^2 . d(\text{DM})$ ...(2)

From (1), $a^2y^2 = a^2b^2 - b^2x^2 = b^2(a^2 - x^2)$

$\therefore$ $\text{DM} = y = \frac{b}{a}\sqrt{a^2 - x^2}$

$\therefore$ $d(\text{DM}) = dy = \frac{-bx}{a\sqrt{a^2 - x^2}}dx$

$\therefore$ From (2), Required volume

$$= \int_{a/2}^{a}\left(x - \frac{a}{2}\right)^2 . \frac{-bx}{a\sqrt{a^2 - x^2}}dx$$

Put $x = a\sin\theta$ $\therefore$ $dx = a\cos\theta\, d\theta$
when $x = a/2, \theta = \pi/6$, when $x = a, \theta = \pi/2$

$$= 2\pi\int_{\pi/6}^{\pi/2}\left(a\sin\theta - \frac{a}{2}\right)^2 . \frac{-b . a\sin\theta}{a\sqrt{a^2(1 - \sin^2\theta)}} . a\cos\theta\, d\theta$$

$$= 2\pi\int_{\pi/6}^{\pi/2} -\frac{ba^2}{4}(2\sin\theta - 1)^2 . \sin\theta\, d\theta$$

$$= -\frac{\pi ba^2}{2}\int_{\pi/6}^{\pi/2}(4\sin^3\theta - 4\sin^2\theta + \sin\theta)\, d\theta$$

$$= -\frac{\pi ba^2}{2}\int_{\pi/6}^{\pi/2}\left[3\sin\theta - \sin 3\theta - 4\left(\frac{1-\cos 2\theta}{2}\right) + \sin\theta\right]d\theta$$

$$= -\frac{\pi ba^2}{2}\int_{\pi/6}^{\pi/2}(4\sin\theta - \sin 3\theta + 2\cos 2\theta - 2)\, d\theta$$

$$= -\frac{\pi ba^2}{2}\left[-4\cos\theta + \frac{\cos 3\theta}{3} + \sin 2\theta - 2\theta\right]_{\pi/6}^{\pi/2}$$

$$= -\frac{\pi ba^2}{2}\left[-2.\frac{\pi}{2}-\left(-4\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}-2.\frac{\pi}{6}\right)\right]$$

$$= -\frac{\pi ba^2}{2}\left[-\frac{2\pi}{3}+\frac{3\sqrt{3}}{2}\right] = \pi a^2 b\left(\frac{3\sqrt{3}}{4}-\frac{\pi}{3}\right). \qquad \text{(Numerically)}$$

2.22 VOLUME FORMULAE FOR PARAMETRIC EQUATIONS

(i) The volume of solid generated by revolution about x-axis, of the area bounded by the curve $x = f(t)$, $y = \phi(t)$, x-axis and ordinates, where $t = a$, $t = b$ is

$$\int_a^b \pi y^2 \frac{dx}{dt}\, dt.$$

(ii) The volume of solid generated by revolution about y-axis, of the area bounded by the curve $x = f(t)$, $y = \phi(t)$, y-axis and abscissae at the points, where $t = a$, $t = b$ is

$$\int_a^b \pi x^2 \frac{dy}{dt}\, dt.$$

Example 3. *Find the volume of the solid formed by the revolution of one arch of the cycloid:*

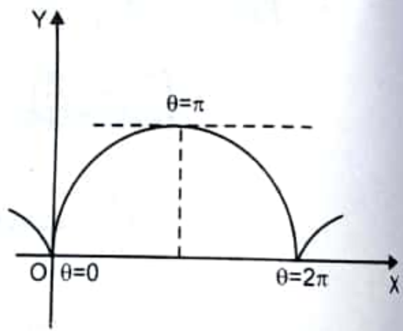
(i) $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ about its base.

(ii) $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ about its base.

(iii) $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ about the tangent at the vertex(i.e., about x-axis).

Sol. (*i*) The equations of the cycloid are

$$x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta)$$



The cycloid is symmetrical about the line through the point, where $\theta = \pi$, $\perp$ to x-axis.

For the first half of the cycloid in the first quadrant, θ varies from 0 to π. The base is x-axis.

$\therefore$ Required volume of the solid generated by revolution about its base (x-axis)

$$= 2\int_0^\pi \pi y^2.\frac{dx}{d\theta}.\,d\theta = 2\pi\int_0^\pi a^2(1-\cos\theta)^2.\,a(1-\cos\theta)\,d\theta \qquad \Big|\ \text{From (1)}$$

$$= 2\pi a^3\int_0^\pi\left(2\sin^2\frac{\theta}{2}\right)^2.\left(2\sin^2\frac{\theta}{2}\right)d\theta \qquad \Big|\ \begin{array}{l}\text{Put } \theta/2 = t \ \therefore\ d\theta = 2dt\\ \text{when } \theta = 0, t = 0\text{, when } \theta = \pi, t = \pi/2\end{array}$$

$$= 2\pi a^3\int_0^{\pi/2}(2\sin^2 t)^3.\,2dt = 32\pi a^3\int_0^{\pi/2}\sin^6 t\,dt = 32\pi a^3.\frac{5.3.1}{6.4.2}.\frac{\pi}{2} = 5\pi^2 a^3.$$

(*ii*) The equations of the cycloid are

$$x = a(\theta + \sin\theta),\ y = a(1 + \cos\theta)$$

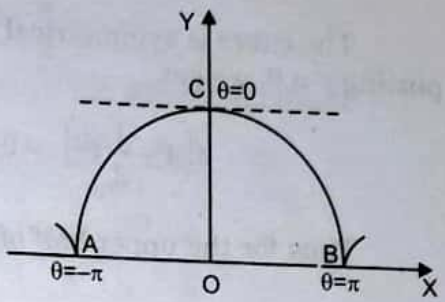
The curve is symmetrical about the y-axis and the base is AB, the x-axis.

For the half of the cycloid, θ varies from 0 to π.

∴ Required volume

$$= 2\int_0^\pi \pi y^2 \frac{dx}{d\theta}\, d\theta$$

$$= 2\pi \int_0^\pi a^2(1+\cos\theta)^2 \,.\, a(1+\cos\theta)\, d\theta$$



$$= 2\pi a^3 \int_0^\pi \left(2\cos^2\frac{\theta}{2}\right)^2 .2\cos^2\frac{\theta}{2} d\theta$$

$$= 2\pi a^3 \int_0^{\pi/2} 8\cos^6 t\,.2dt \qquad \text{where } \frac{\theta}{2} = t$$

$$= 32\pi a^3 \frac{5\cdot3\cdot1}{6\cdot4\cdot2}\cdot\frac{\pi}{2} = 5\pi^2 a^3.$$

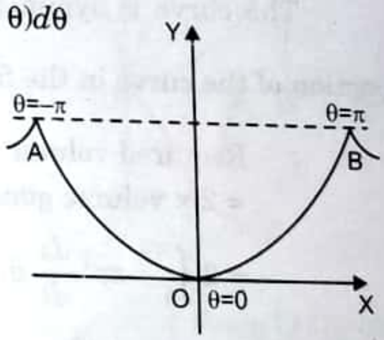
(*iii*) The equations of the cycloid are

$x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta)$

The cycloid is symmetrical about the y-axis and the tangent at the vertex is x-axis. For the half of the curve, θ varies from 0 to π.

∴ Required volume

$$= 2\int_0^\pi \pi y^2 \frac{dx}{d\theta}\, d\theta = 2\pi \int_0^\pi a^2\ (1-\cos\theta)^2\, a.(1+\cos\theta) d\theta$$



$$= 2\pi a^3 \int_0^\pi \left(2\sin^2\frac{\theta}{2}\right)^2 .\left(2\cos^2\frac{\theta}{2}\right) d\theta$$

$$= 2\pi a^3 \int_0^{\pi/2} (2\sin^2 t)^2.(2\cos^2 t)\,.2dt \text{ where } \frac{\theta}{2} = t$$

$$= 32\pi a^3 \int_0^{\pi/2} \sin^4 t\cos^2 t\, dt = 32\pi a^3 \,.\, \frac{3.1.1}{6.4.2}.\frac{\pi}{2} = \pi^2 a^3.$$

Example 4. (*a*) *Prove that the volume of the solid generated by the revolution about the x-axis of the loop of the curve* $x = t^2,\ y = t - \frac{1}{3}t^3$ *is* $\frac{3\pi}{4}$.

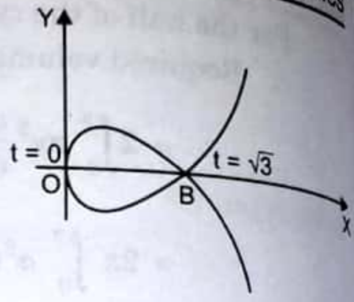
(*b*) *Find the volume of the spindle shaped solid generated by revolving the astroid* $x^{2/3} + y^{2/3} = a^{2/3}$ *about the x-axis.*

(*c*) *Find the volume of the solid generated by revolution of the tractrix*

$$x = a\cos t + \frac{a}{2}\log\tan^2\frac{t}{2},\ y = a\sin t \text{ about its asymptote.}$$

Sol. (*a*) The equations of the curve are

$$x = t^2, y = t - \frac{1}{3}t^2 \qquad ...(1)$$



The curve is symmetrical about the x-axis and for the loop, putting $y = 0$, we get

$$t\left(1 - \frac{1}{3}t^2\right) = 0 \quad \therefore \quad t = 0, \pm\sqrt{3}\,.$$

Thus for the upper half of the loop t-varies from 0 to $\sqrt{3}$.

$$\therefore \quad \text{Required volume} = \int_0^{\sqrt{3}} \pi y^2 \frac{dx}{dt}\,dt = \int_0^{\sqrt{3}} \pi\left(t - \frac{1}{3}t^3\right)^2 .2t\,dt$$

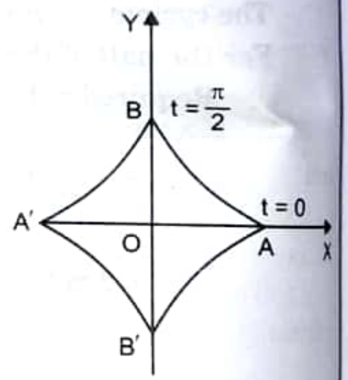
$$= 2\pi \int_0^{\sqrt{3}} t\left(t^2 + \frac{1}{9}t^6 - \frac{2}{3}t^4\right) dt$$

$$= 2\pi\left[\frac{t^4}{4} + \frac{1}{9}.\frac{t^8}{8} - \frac{2}{3}.\frac{t^6}{6}\right]_0^{\sqrt{3}} = 2\pi\left[\frac{9}{4} + \frac{1}{9}.\frac{81}{8} - \frac{2}{3}.\frac{27}{6}\right] = \frac{3\pi}{4}.$$

(*b*) The equation of the curve is $x^{2/3} + y^{2/3} = a^{2/3}$

The parametric equations of the curve are

$$x = a\cos^3 t,\ y = a\sin^3 t \qquad ...(1)$$



The curve is symmetrical about both the axes and for the portion of the curve in the first quadrant, t varies from 0 to $\frac{\pi}{2}$.

$\therefore$ Required volume

= 2 × volume generated by the arc in the first quadrant

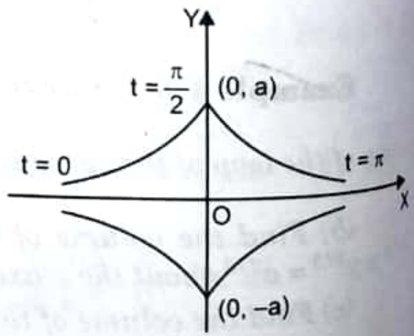
$$= 2\int_0^{\pi/2} \pi y^2 \frac{dx}{dt}\,dt = 2\pi \int_0^{\pi/2} a^2 \sin^6 t.(-\,3a\cos^2 t\sin t)\,dt$$

$$= -\,6\pi a^3 \int_0^{\pi/2} \sin^7 t.\cos^2 t\,dt = -\,6\pi a^3 .\frac{6.4.2.1}{9.7.5.3.1}$$

$$= \frac{-\,32}{105}\pi a^3 = \frac{32}{105}\pi a^3 \text{ (in magnitude).}$$

(*c*) The equations of the curve are

$$x = a\cos t + \frac{a}{2}\log\tan^2\frac{t}{2},\ y = a\sin t \qquad ...(1)$$



The curve is symmetrical about both the axes and the asymptote is $y = 0$ *i.e.*, x-axis.

$$\text{From (1), } \frac{dx}{dt} = -\,a\sin t + \frac{a}{2}.\frac{1}{\tan^2 t/2}.2\tan\frac{t}{2}.\sec^2\frac{t}{2}.\frac{1}{2}$$

$$= -\,a\sin t + \frac{a}{2\sin t/2\cos t/2}$$

$$= -a \sin t + \frac{a}{\sin t} = a\frac{(1-\sin^2 t)}{\sin t} = a\frac{\cos^2 t}{\sin t}$$

For the curve in the second quadrant, t varies from 0 to $\frac{\pi}{2}$

$\therefore$ Required volume = 2 × volume generated by the arc in the second quadrant

$$= 2\int_0^{\pi/2} \pi y^2 \frac{dx}{dt}\,dt = 2\pi\int_0^{\pi/2} a^2\sin^2 t.\frac{a\cos^2 t}{\sin t}\,dt$$

$$= 2\pi a^3\int_0^{\pi/2}\cos^2 t\sin t\,dt = -2\pi a^3\left[\frac{\cos^3 t}{3}\right]_0^{\pi/2} = \frac{2}{3}\pi a^3.$$

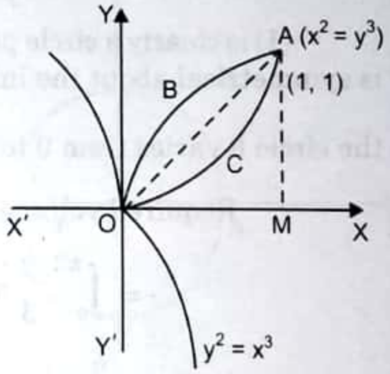
2.23 VOLUME BETWEEN TWO SOLIDS

The volume of the solid generated by the revolution about the x-axis, of the arc bounded by the curves $y = f(x)$, $y = \phi(x)$, and the ordinates $x = a$, $x = b$ is

$$\int_b^a \pi(y_1^2 - y_2^2)dx$$

where y_1 is the 'y' of the upper curve and y_2 that of the lower curve.

Example. *Find the volume of the solid of revolution abtained by rotating the area included between the curves $y^2 = x^3$ and $x^2 = y^3$ about the x-axis.*



Sol. The equations of the curves are

$$y^2 = x^3 \qquad ...(1)$$

and

$$x^2 = y^3 \qquad ...(2)$$

The curve (1) is symmetrical about x-axis and the curve (2) is symmetrical about the y-axis.

To find the points of intersection of (1) and (2), [eliminating y from (1) and (2)], we have

$$x^2 = y^3 = (y^2)^{3/2} = (x^3)^{3/2} = x^{9/2} \qquad | \text{ From (1)}$$

or $\quad x^2(1 - x^{5/2}) = 0 \quad \therefore \quad x = 0, 1$

When $x = 0$, then $y = 0$; when $x = 1$, $y = 1$ | From (1) or (2)

Hence the two curves meet in the points O(0, 0) and A(1, 1).

$$\therefore \text{ Required volume } = \pi\int_0^1 (y_1^2 - y_2^2)dx$$

where y_1 is the y of upper curve (2) and y_2 is the y of the lower curve (1)

$$= \pi\int_0^1 (x^{4/3} - x^3)dx \qquad \begin{array}{|l} \because \text{ From (2), } y_1^2 = y^2 = x^{4/3} \\ \text{and from (1), } y_2^2 = y^2 = x^3 \end{array}$$

$$= \pi\left[\frac{3}{7}x^{7/3} - \frac{x^4}{4}\right]_0^1 = \pi\left(\frac{3}{7} - \frac{1}{4}\right) = \frac{5}{28}\pi.$$

2.24 VOLUME FORMULAE FOR POLAR CURVES

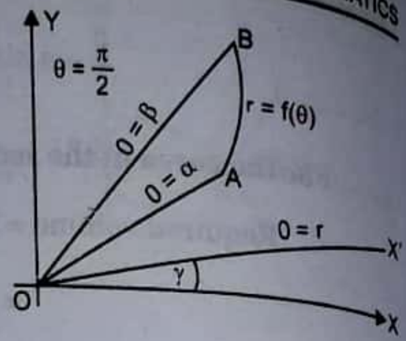
The volume of the solid generated by the revolution of the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \alpha$, $\theta = \beta$

(*i*) about the initial line OX ($\theta = 0$) is

$$\int_\alpha^\beta \frac{2}{3}\pi r^3 \sin\theta \, d\theta$$

(*ii*) about the line $OY\left(\theta = \frac{\pi}{2}\right)$ is $\int_\alpha^\beta \frac{2}{3}\pi r^3 \cos\theta \, d\theta$

(*iii*) about any line OX' ($\theta = \gamma$) is $\int_\alpha^\beta \frac{2}{3}\pi r^3 \sin(\theta - \gamma) d\theta$



Example 5. (*a*) *Find the volume of the solid generated by the revolution of $r = 2a\cos\theta$ about the initial line.*

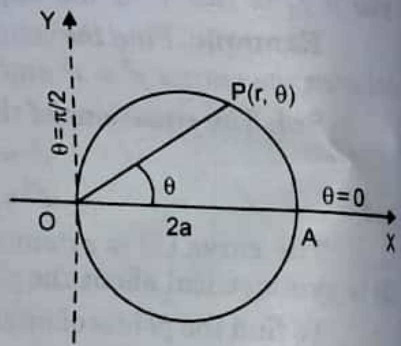
(*b*) *The cardioid $r = a(1 + \cos\theta)$, revolves about the initial line, find the volume of the solid generated.*

(*c*) *The arc of the cardioid $r = a(1 + \cos\theta)$, specified by $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ is rotated about the line $\theta = 0$, prove that the volume generated is $\frac{5}{2}\pi a^3$.*

Sol. (*a*) The equation of the curve is

$$r = 2a\cos\theta \quad \text{...(1)}$$

(1) is clearly a circle passing through the pole. The curve is symmetrical about the initial line and for the upper half of the circle θ varies from 0 to $\frac{\pi}{2}$.



$\therefore$ Required volume

$$= \int_0^{\pi/2} \frac{2}{3}\pi r^3 \sin\theta \, d\theta$$

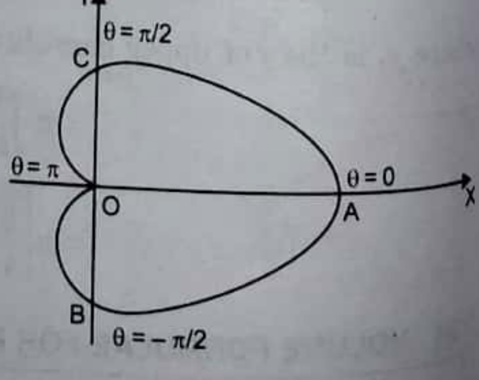
$$= \frac{2}{3}\pi \int_0^{\pi/2} (2a\cos\theta)^3 \sin\theta \, d\theta = \frac{16}{3}\pi a^3 \int_0^{\pi/2} \cos^3\theta \sin\theta d\theta$$

$$= -\frac{16}{3}\pi a^3 . \left[\frac{\cos^4\theta}{4}\right]_0^{\pi/2} = -\frac{4}{3}\pi a^3 (0-1) = \frac{4}{3}\pi a^3.$$

(*b*) The equation of the cardioid is $r = a(1 + \cos\theta)$.

The cardioid is symmetrical about the initial line and for the upper half of the curve, θ varies from 0 to π.

$\therefore$ Required volume



$$= \int_0^{\pi} \frac{2}{3}\pi r^3 \sin\theta \, d\theta$$

$$= \frac{2}{3}\pi \int_0^{\pi} a^3(1+\cos\theta)^3 . \sin\theta \, d\theta$$

$$= \frac{2}{3}\pi a^3 . \left|\frac{(1+\cos\theta)^4}{-4}\right|_0^{\pi} = \frac{8}{3}\pi a^3.$$

(c) The cardioid $r = a(1 + \cos\theta)$ is symmetrical about the initial line and the arc BAC of the curve revolves about the initial line.

For the upper half of the arc AC, θ varies from 0 to $\frac{\pi}{2}$

$\therefore$ Required volume

$$= \int_0^{\pi/2} \frac{2}{3}\pi r^3 \sin\theta\, d\theta = \frac{2}{3}\pi a^3 \int_0^{\pi/2} (1+\cos\theta)^3 \sin\theta\, d\theta$$

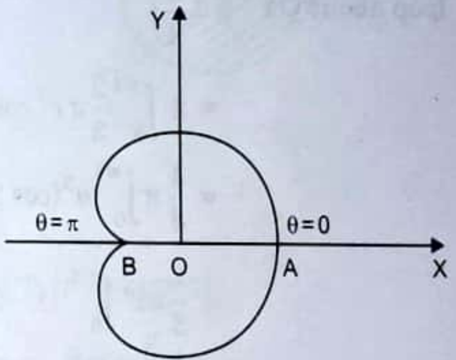
$$= -\frac{2}{3}\pi a^3 \cdot \left[\frac{(1+\cos\theta)^4}{4}\right]_0^{\pi/2} = -\frac{1}{6}\pi a^3(1-16) = \frac{15}{6}\pi a^3 = \frac{5}{2}\pi a^3.$$

Example 6. (a) *Show that the volume of the solid formed by the revolution of the curve $r = a + b\cos\theta$ $(a > b)$ about the initial line is $\frac{4}{3}\pi a(a^2 + b^2)$.*

(b) *Hence or otherwise show that the volume of solid generated by the revolution of the cardioid $r = a(1 + \cos\theta)$ about the initial line is $\frac{8}{3}\pi a^3$.*

Sol. (a) The equation of the curve is

$$r = a + b\cos\theta \;(a > b) \qquad \ldots(1)$$



The curve is symmetrical about the initial line and for the upper half of the curve θ varies from 0 to π.

$\therefore$ Required volume

$$= \int_0^{\pi} \frac{2}{3}\pi r^3 \sin\theta\, d\theta$$

$$= \frac{2}{3}\pi \int_0^{\pi} (a + b\cos\theta)^3 \sin\theta\, d\theta$$

$$= -\frac{2\pi}{3b} \int_0^{\pi} (a + b\cos\theta)^3 (-b\sin\theta\, d\theta)$$

$$= -\frac{2}{3}\frac{\pi}{b}\left[\frac{(a+b\cos\theta)^4}{4}\right]_0^{\pi} = -\frac{2\pi}{3b}\left[\frac{(a-b)^4}{4} - \frac{(a+b)^4}{4}\right] = \frac{\pi}{6b}[(a+b)^4 - (a-b)^4]$$

$$= \frac{4}{3}\pi a(a^2 + b^2) \qquad \ldots(2)$$

(b) Putting $b = a$, in (1), the curve becomes

$r = a\,(1 + \cos\theta)$ which is the given cardioid.

$\therefore$ Required volume by the revolution of the cardioid about the initial line [Putting $b = a$ in (2)] is

$$\frac{4}{3}\pi a(a^2 + a^2) = \frac{8}{3}\pi a^3.$$

Example 7. (*a*) *Find the volume of the solid generated by revolving one loop of the leminscate $r^2 = a^2 \cos 2\theta$ about the line $\theta = \frac{\pi}{2}$.*

(*b*) *The area of the inner loop of the curve $r = 1 + 2\cos\theta$ is rotated through two right angles about the initial line. Show that the volume of the solid so formed is $\frac{\pi}{12}$.*

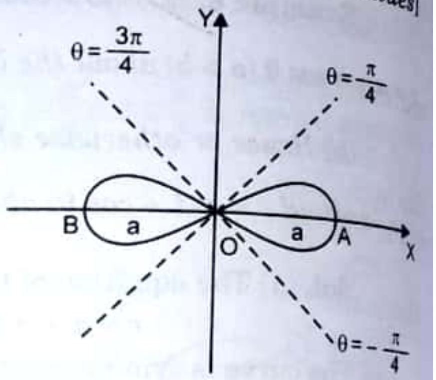
Sol. (*a*) The given curve is $r^2 = a^2 \cos 2\theta$ ...(1)

The curve is symmetrical about the initial line.

For a loop putting $r = 0$ in (1), we get $\cos 2\theta = 0$.

$\therefore \quad 2\theta = \pm\frac{\pi}{2} \quad \Rightarrow \quad \theta = \pm\frac{\pi}{4}$ [*Two consecutive values*]

Thus as θ varies from $-\frac{\pi}{4}$ to $\frac{\pi}{4}$, we get a loop.



For the upper half of this loop, θ varies from 0 to $\frac{\pi}{4}$.

$\therefore$ Required volume obtained by revolution of the loop about OY $\left(\theta = \frac{\pi}{2}\right)$

$$= 2\int_0^{\pi/4} \frac{2}{3}\pi r^3 \cos\theta \, d\theta$$

$$= \frac{4}{3}\pi \int_0^{\pi/4} a^3 (\cos 2\theta)^{3/2} \cos\theta \, d\theta$$

$$= \frac{4}{3}\pi a^3 \int_0^{\pi/4} (1 - 2\sin^2\theta)^{3/2} \cos\theta \, d\theta$$

Put $\sqrt{2}\sin\theta = \sin\phi$

$\therefore \quad \sqrt{2}\cos\theta \, d\theta = \cos\phi \, d\phi$.

$$= \frac{4}{3}\pi a^3 \int_0^{\pi/2} (1 - \sin^2\phi)^{3/2} \frac{1}{\sqrt{2}} \cos\phi \, d\phi$$

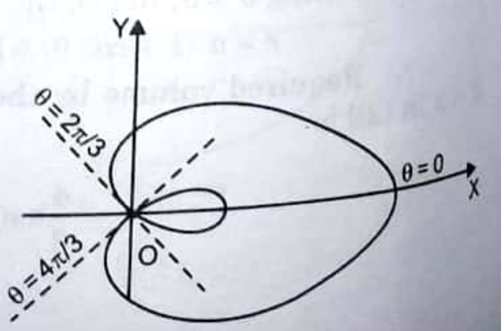
$$= \frac{4}{3\sqrt{2}}\pi a^3 \int_0^{\pi/2} \cos^4\phi \, d\phi = \frac{4}{3\sqrt{2}}\pi a^3 . \frac{3.1}{4.2}\frac{\pi}{2} = \frac{\pi^2 a^3}{4\sqrt{2}}.$$

(*b*) The equation of the curve is $r = 1 + 2\cos\theta$.

The curve is symmetrical about the initial line.

Also $\theta = 0$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$
$r = 3$	2	1	0	-1	0

Thus the inner loop is obtained as θ varies from $\frac{2\pi}{3}$ to $\frac{4\pi}{3}$. Since the loop (*or* curve) is symmetrical about the initial line, $\therefore$ For the lower half of the inner loop, θ varies from $\frac{2\pi}{3}$ to π.



$\therefore$ Required volume

$$= \int_{2\pi/3}^{\pi} \frac{2}{3}\pi r^3 \sin\theta \, d\theta$$

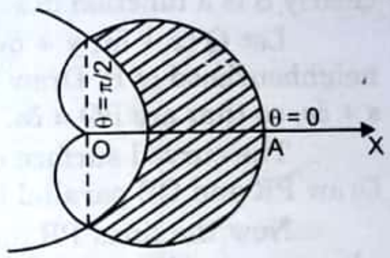
$$= \frac{2}{3}\pi \int_{2\pi/3}^{\pi} (1 + 2\cos\theta)^3 \sin\theta \, d\theta = \frac{2}{3}\pi . \left[\frac{(1+2\cos\theta)^4}{-8}\right]_{2\pi/3}^{\pi}$$

$$= -\frac{\pi}{12}[(-1)^4 - (1-1)^4] = -\frac{\pi}{12} = \frac{\pi}{12} \text{ (in magnitude)}$$

Example 8. *Show that if the area lying within the cardioid $r = 2a\,(1 + \cos\theta)$ and without the parabola $r\,(1 + \cos\theta) = 2a$, revolves about the initial line, the volume generated is $18\pi a^3$.*

Sol. The equation of the cardioid is $r = 2a\,(1 + \cos\theta)$ ...(1)

and that of the parabola is $r = \dfrac{2a}{1 + \cos\theta}$ ...(2)



Both the curves are symmetrical about the initial line. The upper half of the shaded area revolves about the initial line and it will generate the required volume.

The two curves intersect, where solving (1) and (2) [eliminating r],

$$2a(1 + \cos\theta) = \frac{2a}{1 + \cos\theta}$$

or $\quad (1 + \cos\theta)^2 = 1 \quad$ or $\quad 1 + \cos\theta = \pm 1$

Now either $\quad 1 + \cos\theta = 1 \quad$ *i.e.,* $\cos\theta = 0$ or $\theta = \pm\dfrac{\pi}{2}$

or $\quad 1 + \cos\theta = -1 \quad$ *i.e.,* $\quad \cos\theta = -2$, which is impossible.

Thus the two curves intersect at the points, where $\theta = \dfrac{\pi}{2}$. $\theta = -\dfrac{\pi}{2}$.

For the upper half, θ varies from 0 to $\dfrac{\pi}{2}$.

$\therefore$ Required volume

$$= \int_0^{\pi/2} \frac{2}{3}\pi[(r \text{ of outer curve})^3 - (r \text{ of inner curve})^3] \sin\theta \, d\theta$$

$$= \frac{2}{3}\pi \int_0^{\pi} \left[8a^3(1+\cos\theta)^3 - \frac{8a^3}{(1+\cos\theta)^3}\right] \sin\theta \, d\theta$$

$$= \frac{16}{3}\pi a^3 . \left[\frac{(1+\cos\theta)^4}{-4} - \frac{(1+\cos\theta)^{-2}}{(-2)(-1)}\right]_0^{\pi/2} = \frac{16}{3}\pi a^3\left[-\frac{1}{4}(1-16) - \frac{1}{2}\left(1 - \frac{1}{4}\right)\right]$$

$$= \frac{16}{3}\pi a^3\left[\frac{15}{4} - \frac{3}{8}\right] = 18\pi a^3.$$

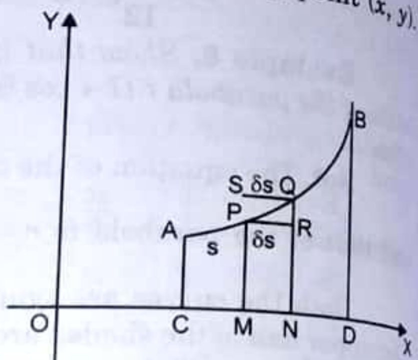
2.25 SURFACE OF THE SOLID OF REVOLUTION

Prove that curved surface of solid generated by revolution, about the x-axis, of the area bounded by the curve $y = f(x)$, x-axis and ordinates $x = a$, $x = b$ is

$$\int_{x=a}^{x=b} 2\pi\, y\, ds$$

where s is the length of arc of curve measured from a fixed point on it to any point (x, y).

Let AB be the curve $y = f(x)$ and A, B the points corresponding to $x = a$, $x = b$ respectively. Draw AC, BD $\perp$ s to x-axis.



Let P(x, y) be any point on the curve and let the arc AP be s. Draw PM $\perp$ OX.

If S denotes the curved surface of the solid generated by the revolution of the area ACMP about the x-axis, then clearly S is a function of s.

Let Q $(x + \delta x, y + \delta y)$ be a point on the curve in the neighbourhood of P. Draw QN $\perp$ OX, and let the arc AQ be $s + \delta s$, so that arc PQ $= \delta s$.

The curved surface of the solid of revolution of the area PMNQ about the x-axis is δS. Draw PR and QS parallel to x-axis and each equal in length to the arc PQ $= \delta s$.

Now the lines PR and QS generated cylinder, when the arc PQ revolves about x-axis, whose base radii are PM and NQ. The area of the curved surface generated by the arc PQ lies between the areas of the curved surfaces of the cylinders whose base radii are PM and NQ.

i.e., δS lies between $2\pi y\, \delta s$ and $2\pi(y + \delta y)\, \delta s$.

$\therefore$ $\dfrac{\delta S}{\delta s}$ lies between $2\pi y$ and $2\pi\,(y + \delta y)$.

Proceeding to limit as Q $\to$ P, *i.e.*, as $\delta x \to 0$ $\qquad (\because\ \delta y \to 0, \delta s \to 0)$

$\dfrac{\delta S}{\delta s}$ lies between $2\pi y$ and a quantity which approaches $2\pi y$.

$$\therefore \quad \frac{\delta S}{\delta s} = 2\pi y. \qquad \text{...(1)}$$

$$\therefore \int_{x=a}^{x=b} 2\pi\, y\, ds = \int_{x=a}^{x=b} \frac{dS}{ds}\, .ds = [S]_{x=a}^{x=b}$$

$=$ (value of S when $x = b$) $-$ (value of S when $x = a$)

$=$ area of the surface generated by the revolution of the area ACDB $-$ 0

$\therefore$ Surface area of the solid generated by the revolution of the area ACDB $= \displaystyle\int_{x=a}^{x=b} 2\pi\, y\, ds.$

2.25.1 Three Practical Forms of the Surface Formula

(i) **Surface formula for Cartesian equations.** *The curved surface of the solid generated by the revolution about the x-axis, of the bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$, $x = b$ is*

$$\int_{x=a}^{x=b} 2\pi\, y \frac{ds}{dx}\, dx, \text{ where } \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

(*ii*) **Surface formula for parametric equation.** *The curved surface of the solid generated by the revolution about the x-axis, of the area bounded by the curve $x = f(t)$, $y = \phi(t)$, the x-axis and the ordinates at the points, where $t = a$, $t = b$ is*

$$\int_{t=a}^{t=b} 2\pi\, y \frac{ds}{dt} \cdot dt, \text{ where } \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

(*iii*) **Surface formula for polar equations.** *The curved surface of the solid generated by the revolution, about the initial line, of the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \alpha$, $\theta = \beta$ is*

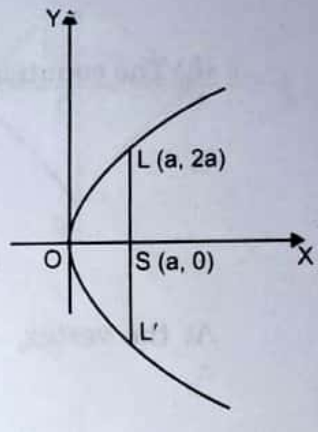
$$\int_{\theta=\alpha}^{\theta=\beta} 2\pi\, y \frac{ds}{d\theta} \cdot d\theta, \text{ where } \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

and $$y = r \sin \theta.$$

Example 9. *Find the area of the surface formed by the revolution of the parabola $y^2 = 4ax$ about the x-axis by the arc from the vertex to one end of the latus rectum.*

Sol. The equation of the parabola is $y^2 = 4ax$ ...(1)

Differentiating w.r.t. x, we get



$$2y\frac{dy}{dx} = 4a \quad \text{or} \quad \frac{dy}{dx} = \frac{2a}{y}$$

$$\therefore \quad \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{4a^2}{y^2}} = \sqrt{1 + \frac{4a^2}{4ax}} = \sqrt{\frac{x+a}{x}}$$

For the arc from the vertex O to L, the end of the latus rectum, x varies from 0 to a.

$$\therefore \text{ Required surface } = \int_{x=0}^{a} 2\pi y \frac{ds}{dx}\, dx$$

$$= \int_0^a 2\pi . \sqrt{4ax} . \sqrt{\frac{x+a}{x}}\, dx \qquad \Big| \because \ y = \sqrt{4ax} \quad \text{From (1)}$$

$$= 4\pi\sqrt{a} \int_0^a (x+a)^{1/2} dx = 4\pi\sqrt{a} . \frac{2}{3}\left[(x+a)^{3/2}\right]_0^a$$

$$= \frac{8\pi a^2}{3}(2\sqrt{2} - 1).$$

Example 10. (*a*) *Find the surface generated by the revolution of arc of the catenary $y = c \cosh \dfrac{x}{c}$ about the axis of x.*

(*b*) *For the catenary $y = a \cosh \left(\dfrac{x}{a}\right)$, prove that $aS = 2V = \pi a\,(ax + sy)$, where s is the length of the arc from the vertex, S and V are respectively the area of the curved surface and volume of the solid generated by the revolution of the arc about x-axis.*

Sol. (a) The equation of the curve is $y = c \cosh \dfrac{x}{c}$

$$\therefore \quad \frac{dy}{dx} = c \, . \, \sinh\left(\frac{x}{c}\right) . \frac{1}{c} = \sinh \frac{x}{c}$$

$$\therefore \quad \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\sinh^2 \frac{x}{c}} = \cosh \frac{x}{c}$$

If the arc be measured from the vertex (where $x = 0$) to any point (x, y) then

$$\text{The required surface} = \int_{x=0}^{x} 2\pi y \frac{ds}{dx}\, dx = \int_0^x 2\pi . c \cosh \frac{x}{c} . \cosh \frac{x}{x}\, dx = \int_0^x 2\pi c \cosh^2 \frac{x}{c}\, dx$$

$$= \pi c \int_0^x \left[1 + \cosh\left(\frac{2x}{c}\right)\right] dx \qquad \left|\ \because \cosh^2 \theta = \frac{1+\cosh 2\theta}{2}\right.$$

$$= \pi c \left[x + \frac{c}{2} \sinh\left(\frac{2x}{c}\right)\right]_0^x = \pi c \left[x + \frac{c}{2} \sinh\left(\frac{2x}{c}\right)\right].$$

(b) The equation of the catenary is $y = a \cosh \dfrac{x}{a}$

$$\therefore \quad \frac{dy}{dx} = \sinh \frac{x}{a}$$

$$\therefore \quad \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\sinh^2 \frac{x}{a}} = \cosh \frac{x}{a}$$

At the vertex, $x = a$.

$\therefore$ s = length of the arc from vertex to any point (x, y)

$$= \int_0^x \frac{ds}{dx} . dx = \int_0^x \cosh \frac{x}{a}\, dx = \left[a \sinh \frac{x}{a}\right]_0^x = a \sinh \frac{x}{a} \qquad \ldots(1)$$

S = Surface area generated by the arc

$$= \int_0^x 2\pi\, y \frac{ds}{dx} . \, dx = \int_0^x 2\pi . a \cosh \frac{x}{a} . \cosh \frac{x}{a} . \, dx$$

$$= 2\pi a \int_0^x \cosh^2 \frac{x}{a}\, dx = \pi a \int_0^x \left[1 + \cosh\left(\frac{2x}{a}\right)\right] dx$$

$$= \pi\, a \left[x + \frac{a}{2} \sinh\left(\frac{2x}{a}\right)\right] \qquad \ldots(2)$$

and V = the volume generated by the arc by the revolution about x-axis

$$= \int_0^x \pi\, y^2 dx = \pi \int_0^x a^2 \cosh^2 \frac{x}{a}\, dx$$

$$= \pi\, a^2 \int_0^x \frac{1}{2}\left[1 + \cosh\left(\frac{2x}{a}\right)\right] dx$$

$$= \frac{\pi a^2}{2}\left[x + \frac{a}{2} \sinh\left(\frac{2x}{a}\right)\right] \qquad \ldots(3)$$

From (2), $$aS = \pi a^2\left[x + \frac{a}{2}\sinh\left(\frac{2x}{a}\right)\right] \quad ...(4)$$

From (3), $$2V = \pi a^2\left[x + \frac{a}{2}\sinh\left(\frac{2x}{a}\right)\right] \quad ...(5)$$

From (1), $$\pi a(ax + sy) = \pi a\left[ax + a^2\sinh\frac{x}{a}\cosh\frac{x}{a}\right] = \pi a^2\left[x + \frac{a}{2}\sinh\left(\frac{2x}{a}\right)\right] \quad ...(6)$$

From (4), (5) and (6), it follows that $aS = 2V = a\pi\,(ax + sy)$.

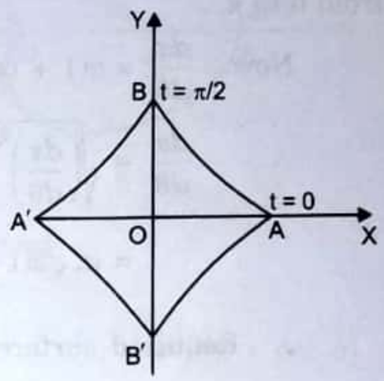
Example 11. *Find the surface of the solid generated by the revolution of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ (or $x = a\cos^3 t$, $y = a\sin^3 t$) about the x-axis.*

Sol. The parametric equations of the astroid are

$$x = a\cos^3 t,\ y = a\sin^3 t$$

The curve is symmetrical about both the axes and it lies in the square bounded by the lines $x = \pm a$, $y = \pm a$.

For the portion of the curve in the first quadrant, t varies from 0 to $\frac{\pi}{2}$.



Now $$\frac{dx}{dt} = -3a\cos^2 t\sin t,\quad \frac{dy}{dt} = 3a\sin^2 t\cos t$$

$$\frac{ds}{dt} = \sqrt{9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t}$$

$$= 3a\sin t\cos t$$

$\therefore$ $$\text{Required surface} = 2\int_{t=0}^{\pi/2} 2\pi y\frac{ds}{dt}\,dt$$

$$= 4\pi\int_0^{\pi/2} a\sin^3 t.\,3a\sin t\cos t\,dt = 12\pi a^2\int_0^{\pi/2}\sin^4 t\cos t\,dt$$

$$= 12\pi a^2\left[\frac{\sin^5 t}{5}\right]_0^{\pi/2} = \frac{12}{5}\pi a^2(1-0) = \frac{12}{5}\pi a^2.$$

Example 12. *Find the surface area of the solid generated by revolving the cycloid*

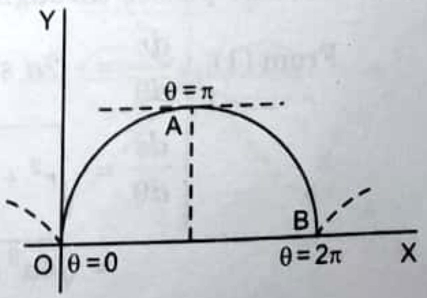
(i) $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ about the x-axis.

(ii) $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ about the tangent at the vertex.

Sol. (*i*) The equations of the cycloid are

$$x = a(\theta - \sin\theta),\quad y = a(1 - \cos\theta),$$

The cycloid is symmetrical about the line through the point $0 = \pi$, $\perp$ to the x-axis. For the first half of the cycloid, θ varies from 0 to π.



Now $$\frac{dx}{d\theta} = a(1 - \cos\theta),\quad \frac{dy}{d\theta} = a\sin\theta$$

$\therefore$ $$\frac{ds}{d\theta} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}$$

$$= a\sqrt{2(1-\cos\theta)} = 2a\sin\frac{\theta}{2}.$$

$$\therefore \quad \text{Required surface} = 2\int_0^\pi 2\pi y \frac{ds}{d\theta} d\theta = 4\pi \int_0^\pi a(1-\cos\theta) \,.\, 2a \sin\frac{\theta}{2}\, d\theta$$

$$= 4\pi \int_0^\pi a\left(2\sin^2\frac{\theta}{2}\right).\, 2a\sin\frac{\theta}{2}\, d\theta$$

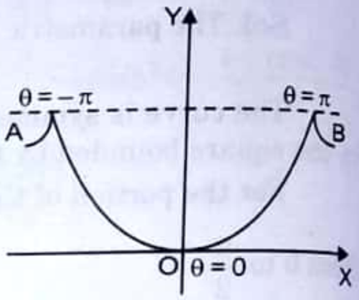
$$= 16\pi a^2 \int_0^\pi \sin^3\frac{\theta}{2}\, d\theta \qquad \Big| \text{ Put } \theta/2 = t \ \therefore\ d\theta = 2dt$$

$$= 16\pi a^2 \int_0^{\pi/2} \sin^3 t . 2\, dt = 32\pi a^2 \frac{2}{3.1} = \frac{64}{3}\pi a^2.$$

(*ii*) The equations of the cycloid are

$$x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta)$$

The cycloid is symmetrical about the y-axis and the tangent at the vertex is x-axis. For half of the curve, θ varies from 0 to π.



Now $\dfrac{dx}{d\theta} = a(1 + \cos\theta)$. $\quad \dfrac{dy}{d\theta} = a \sin\theta$

$$\therefore \quad \frac{ds}{d\theta} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}$$

$$= a.\sqrt{2(1+\cos\theta)} = 2a\cos\frac{\theta}{2}$$

$$\therefore \quad \text{Required surface} = 2\int_0^\pi 2\pi y\frac{ds}{d\theta}d\theta = 4\pi\int_0^\pi a(1-\cos\theta)\,.2\cos\frac{\theta}{2}\,d\theta$$

$$= 4\pi\int_0^\pi a\left(2\sin^2\frac{\theta}{2}\right).2a\cos\frac{\theta}{2}\,d\theta = 16\pi a^2\int_0^\pi \sin^2\frac{\theta}{2}\cos\frac{\theta}{2}d\theta$$

$$= 16\pi a^2 \left| \frac{\sin^3\theta/2}{3\,.\,\frac{1}{2}} \right|_0^\pi = \frac{32}{3}\pi a^2.$$

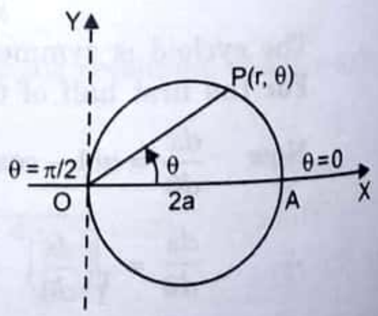
Example 13. (*a*) *Find the area of the surface of revolution formed by revolving the curve $r = 2a\cos\theta$ about the initial line.*

(*b*) *The curve $r = a(1 + \cos\theta)$ revolves about the initial line. Find the surface of the figure so formed.*

(*c*) *Find the surface of the solid generated by the revolution of the lemniscate $r^2 = a^2\cos 2\theta$ about the initial line.*

Sol. (*a*) The equation of the curve is $x = 2a\cos\theta$ ...(1)

Now (1) represents a circle passing through the pole and the initial line passes through the centre of the circle.



From (1), $\dfrac{dr}{d\theta} = -2a\sin\theta$

$$\therefore \quad \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

$$= \sqrt{4a^2\cos^2\theta + 4a^2\sin^2\theta} = 2a$$

$\therefore$ Required surface $= \int_0^{\pi/2} 2\pi y \frac{ds}{d\theta} d\theta$, where $y = r \sin \theta$

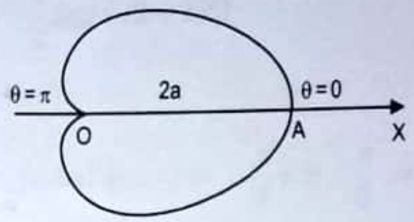
$$= 2\pi \int_0^{\pi/2} r \sin \theta . 2a \, d\theta = 4a\pi \int_0^{\pi/2} 2a \cos \theta \sin \theta \, d\theta$$

$$= 4\pi a^2 \int_0^{\pi/2} 2 \sin \theta \cos \theta \, d\theta = 4\pi a^2 . \left[\sin^2 \theta \right]_0^{\pi/2} = 4\pi a^2.$$

(*b*) The equation of the cardioid is $r = a(1 + \cos \theta)$

The cardioid is symmetrical about the initial line and for the upper half of the curve, θ varies from 0 to π.

Now from (1), $\frac{dr}{d\theta} = -a \sin \theta$



$$\therefore \quad \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

$$= \sqrt{a^2(1 + \cos \theta)^2 + a^2 \sin^2 \theta}$$

$$= a\sqrt{2(1 + \cos \theta)} = 2a \cos \frac{\theta}{2}$$

$\therefore$ Required volume $= \int 2\pi y \frac{ds}{d\theta} d\theta$, where $y = r \sin \theta$

$$= 2\pi \int_0^{\pi} a \sin \theta (1 + \cos \theta). 2a \cos \frac{\theta}{2} d\theta \qquad | \because r = a(1 + \cos \theta)$$

$$= 2\pi \int_0^{\pi} a \, 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \, . \, 2 \cos^2 \frac{\theta}{2} . 2a \cos \frac{\theta}{2} d\theta$$

$$= 16\pi a^2 \int_0^{\pi} \cos^4 \frac{\theta}{2} \sin \frac{\theta}{2} d\theta = 16\pi a^2 . \left[\frac{-\cos^5 \theta/2}{5.\frac{1}{2}} \right]_0^{\pi}$$

$$= -\frac{32}{5}\pi a^2 (0 - 1) = \frac{32}{5}\pi a^2.$$

(*c*) The equation of the lemniscate is $r = a^2 \cos 2\theta$ ...(1)

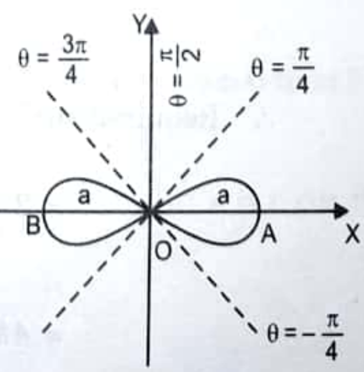
The curve is symmetrical about the initial line and the line $\theta = \frac{\pi}{2}$

For a loop, putting $r = 0$, we get

$$\cos 2\theta = 0$$

$$2\theta = \pm \frac{\pi}{2} \quad \text{or} \quad \theta = -\frac{\pi}{4}, \frac{\pi}{4}$$

[*two consecutive values*]



The curve consists of two equal loops and $\theta = \frac{-\pi}{4}, \theta = \frac{\pi}{4}$ are the tangents at pole. In the first quadrant, for half of the loop θ varies from 0 to $\frac{\pi}{4}$.

From (1), $2r\dfrac{dr}{d\theta} = -2a^2 \sin 2\theta$ or $\dfrac{dr}{d\theta} = \dfrac{-a^2 \sin 2\theta}{r}$

$$\therefore \quad \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2 \cos 2\theta + \frac{a^4 \sin^2 2\theta}{r^2}}$$

$$= \sqrt{a^2 \cos 2\theta + \frac{a^4 \sin^2 2\theta}{a^2 \cos 2\theta}} = \sqrt{\frac{a^2(\cos^2 2\theta + \sin^2 2\theta)}{\cos 2\theta}} = \frac{a}{\sqrt{\cos 2\theta}}$$

$$\therefore \text{ Required surface } = 2\int_0^{\pi/4} 2\pi y \frac{ds}{d\theta}.d\theta = 4\pi \int_0^{\pi/4} r \sin\theta.\frac{a}{\sqrt{\cos 2\theta}}.d\theta$$

$$= 4\pi \int_0^{\pi/4} a\sqrt{\cos 2\theta}.\sin\theta.\frac{a}{\sqrt{\cos 2\theta}} d\theta$$

$$= 4\pi a^2 \int_0^{\pi/4} \sin\theta d\theta = 4\pi a^2 \left(1 - \frac{1}{\sqrt{2}}\right).$$

2.26 REVOLUTION ABOUT Y-AXIS

The curved surface of solid generated by revolution about y-axis of the area bounded by the curve $x = f(y)$, y-axis and abscissae $y = a$, $y = b$ is

$$\int_{y=a}^{y=b} 2\pi x \, ds.$$

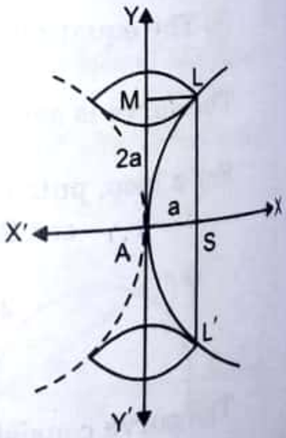
The result follows immediately on interchanging x and y in Art. 2.25.

Example 14. *The part of the parabola $y^2 = 4ax$ cut-off by the latus rectum revolves about the tangent at the vertex. Find the curved surface of the reel thus generated.*

Sol. The equation of the parabola is $y^2 = 4ax$ ...(1)

The required curved surface is generated by the revolution of the arc LAL′, cut-off by the latus rectum, about the tangent at the vertex *i.e.*, y-axis.

For half of the arc AL, x varies from 0 to a.



From (1), $y = 2\sqrt{a}\sqrt{x}$ $\therefore$ $\dfrac{dy}{dx} = \dfrac{\sqrt{a}}{\sqrt{x}}$

$$\therefore \quad \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{a}{x}} = \sqrt{\frac{x+a}{x}}.$$

$\therefore$ Required surface

$$= 2\int_0^a 2\pi x \frac{ds}{dx} dx = 4\pi \int_0^a x.\sqrt{\frac{x+a}{x}} dx$$

$$= 4\pi \int_0^a \sqrt{x^2 + ax}\, dx = 4\pi \int_0^a \sqrt{\left(x + \frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2} dx$$

$$= 4\pi\left[\frac{1}{2}\left(x+\frac{a}{2}\right)\sqrt{\left(x+\frac{a}{2}\right)^2-\left(\frac{a}{2}\right)^2}-\frac{1}{2}\cdot\frac{a^2}{4}\cosh^{-1}\left[\left(\frac{x+a/2}{a/2}\right)\right]\right]_0^a$$

$$= 4\pi\left[\frac{1}{2}\cdot\frac{3a}{2}\sqrt{\frac{9a^2}{4}-\frac{a^2}{4}}-\frac{a^2}{8}(\cosh^{-1}3-\cosh^{-1}1)\right]$$

$$= 4\pi\left[\frac{3a}{4}\cdot\sqrt{2}a-\frac{a^2}{8}\left\{\log(3+\sqrt{9-1})-\log(1+\sqrt{1-1})\right\}\right]$$

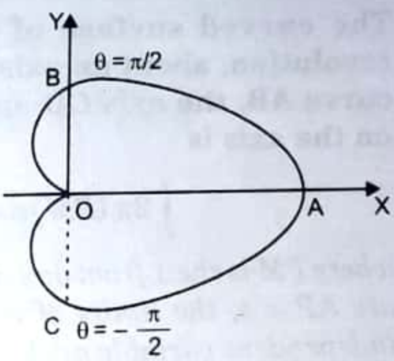
$$= 3\sqrt{2}\pi a^2-\frac{\pi a^2}{2}\log(3+2\sqrt{2})=\pi a^2\left[3\sqrt{2}-\frac{1}{2}\log(\sqrt{2}+1)^2\right]$$

$$= \pi a^2\left[3\sqrt{2}-\log(\sqrt{2}+1)\right].$$

Example 15. *The arc of the cardioid $r = a(1+\cos\theta)$ included between $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$ is rotated about the line $\theta=\frac{\pi}{2}$. Find the area of surface generated.*

Sol. The cardioid is $r = a(1+\cos\theta)$ ...(1)

The arc CAB $\left(\text{from } \theta=-\frac{\pi}{2} \text{ to } \theta=\frac{\pi}{2}\right)$ revolves about the line $\theta=\frac{\pi}{2}$, *i.e.*, the y-axis.



Also the curve is symmetrical about the initial line or x-axis.

From (1), $\quad \frac{dr}{d\theta} = -a\sin\theta$

$$\therefore \quad \frac{ds}{d\theta} = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2(1+\cos\theta)^2+a^2\sin^2\theta}$$

$$= a\sqrt{2(1+\cos\theta)} = 2a\cos\frac{\theta}{2}$$

$\therefore$ Required surface area

= 2 × surface generated by the revolution of arc AB

$$= 2\int_0^{\pi/2} 2\pi x\frac{ds}{d\theta}\,d\theta \quad \Big|\because \text{ for the arc AB, } \theta \text{ varies from } 0 \text{ to } \pi/2$$

$$= 4\pi\int_0^{\pi/2} r\cos\theta\,.\,2a\cos\frac{\theta}{2}\,d\theta \qquad \Big|\because x = r\cos\theta$$

$$= 8\pi a\int_0^{\pi/2} a(1+\cos\theta)\,.\cos\theta\,.\cos\frac{\theta}{2}\,d\theta$$

$$= 8\pi a^2 \int_0^{\pi/2} \left(2 - 2\sin^2\frac{\theta}{2}\right)\left(1 - 2\sin^2\frac{\theta}{2}\right)\cos\frac{\theta}{2}\, d\theta$$

$$= 16\pi a^2 \int_0^{\pi/2} \left(1 - 3\sin^2\frac{\theta}{2} + 2\sin^4\frac{\theta}{2}\right)\cos\frac{\theta}{2}\, d\theta$$

| Put $\sin\frac{\theta}{2} = t \quad \therefore \quad \frac{1}{2}\cos\frac{\theta}{2}\, d\theta = dt$
| when $\theta = 0, t = 0$; when $\theta = \pi/2, t = 1/\sqrt{2}$.

$$= 16\pi a^2 \int_0^{1/\sqrt{2}} (1 - 3t^2 + 2t^4) \,.\, 2dt$$

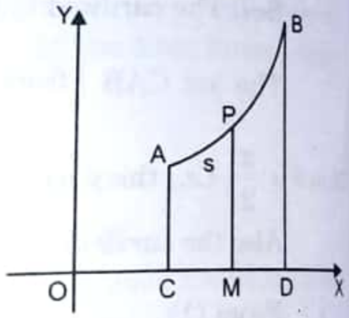
$$= 32\pi a^2 \left[t - t^3 + \frac{2t^5}{5}\right]_0^{1/\sqrt{2}} = \frac{96}{5\sqrt{2}}\pi a^2.$$

2.27 REVOLUTION ABOUT ANY AXIS

The curved surface of the solid generated by the revolution, about an axis CD of the area bounded by a curve AB, the axis CD and the perpendiculars AC, BD on the axis is

$$\int 2\pi (PM) ds$$

where PM is the $\perp$ from any point P of the curve on the axis, and arc AP = s, the limits of integration being the values of the independent variable at the ends of the revolving arc.



Example 16. *A quadrant of a circle of radius a revolves about its chord. Show that the surface of the spindle generated is $2\sqrt{2}\,\pi a^2\left(1 - \frac{\pi}{4}\right)$.*

Sol. The parametric equations of the circle are $x = a\cos\theta$, $y = a\sin\theta$.

$$\therefore \quad \frac{ds}{d\theta} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}$$

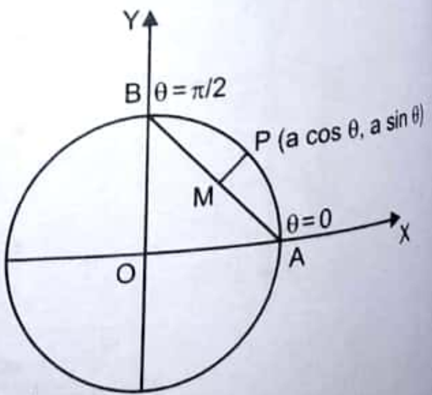
$$= \sqrt{a^2\sin^2\theta + a^2\cos^2\theta} = a.$$

Since OA = OB = a,

$\therefore$ Equation of the chord AB is

$$\frac{x}{a} + \frac{y}{a} = 1, \quad \text{or} \quad x + y = a \qquad ...(1)$$

Let P($a\cos\theta$, $a\sin\theta$) be any point on the arc AB. From P draw PM $\perp$ to chord AB, *i.e.*, (1).



$$\text{PM} = \frac{a\cos\theta + a\sin\theta - a}{\sqrt{2}}$$

$$\therefore \quad = \frac{a}{\sqrt{2}}(\cos\theta + \sin\theta - 1)$$

Also for the arc AB, θ varies from 0 to $\frac{\pi}{2}$.

$$\therefore \text{ Required surface } = \int_0^{\pi/2} 2\pi\,(\text{PM})\frac{ds}{d\theta}\,d\theta = 2\pi\int_0^{\pi/2}\frac{a}{\sqrt{2}}(\cos\theta + \sin\theta - 1).\,a\,d\theta$$

$$= \sqrt{2}\pi a^2[\sin\theta - \cos\theta - \theta]_0^{\pi/2} = \sqrt{2}\pi a^2\left[1 - \frac{\pi}{2} + 1\right]$$

$$= \sqrt{2}\pi a^2\left(2 - \frac{\pi}{2}\right) = 2\sqrt{2}\pi a^2\left(1 - \frac{\pi}{4}\right).$$

Example 17. *Find the area of the surface generated if an arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ revolves about the line $y = 2a$.*

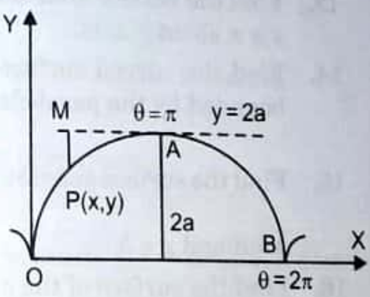
Sol. The equations of the cycloid are $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$

The cycloid is symmetrical about the line $\perp$ to x-axis, where $\theta = \pi$.

Now, $$\frac{dx}{d\theta} = a(1 - \cos\theta) \quad \text{and} \quad \frac{dy}{d\theta} = a\sin\theta$$

$$\therefore \quad \frac{ds}{d\theta} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}$$

$$= a\sqrt{2(1-\cos\theta)} = a\sqrt{4\sin^2\frac{\theta}{2}} = 2a\sin\frac{\theta}{2}$$



Now one arch OAB if the cycloid revolves about the line $y = 2a$, which is tangent to the cycloid at A, the vertex.

Let P(x, y) be any point on the arc OA.

Draw PM $\perp$ on MA, so that

$$\text{PM} = 2a - y = 2a - a(1 - \cos\theta) = a(1 + \cos\theta)$$

For the arc OA, θ varies from 0 to π.

$\therefore$ Required surface

= 2 × surface generated by the revolution of the arc OA about the line $y = 2a$

$$= 2\int_0^{\pi} 2\pi\,(\text{PM})\frac{ds}{d\theta}\,d\theta = 4\pi\int_0^{\pi} a(1 + \cos\theta).\,2a\sin\frac{\theta}{2}\,d\theta$$

$$= 4\pi\int_0^{\pi} a\left(2\cos^2\frac{\theta}{2}\right).\,2a\sin\frac{\theta}{2}\,d\theta$$

$$= 16\pi a^2\int_0^{\pi}\cos^2\frac{\theta}{2}\sin\frac{\theta}{2}\,d\theta = -32\pi a^2\left[\frac{\cos^3\theta/2}{3}\right]_0^{\pi} = \frac{32}{3}\pi a^2.$$

TEST YOUR KNOWLEDGE

1. Find the volume of a hemisphere.
[**Hint.** Hemisphere is generated by revolving a quadrant of the circle $x^2 + y^2 = a^2$ about x-axis, thus $0 \le x \le a$.]
2. The part of the curve $y^2 = x^2(1 - x^2)$ between $x = 0$ and $x = 1$ rotates about the x-axis. Obtain the volume of the solid thus generated.
3. Find the volume of the solid obtained by rotating the ellipse $x^2 + 9y^2 = 9$ about the x-axis.
4. The area of the parabola $y^2 = 4ax$ lying between the ordinates $x = 0$ and $x = a$ is revolved about the x-axis. Find the volume of the solid thus generated.
5. Find the volume of the solid formed by the revolution of the curve $yx^2 + y = 1$ about its asymptote.
6. The curve $y^2(a + x) = x^2(a - x)$ revolves about x-axis. Find the volume generated by the loop.
7. Find the volume of the solid formed by the revolution of the curve $xy^2 = 4(2 - x)$ about the y-axis.
8. The region enclosed by the curves $y = \sin x$, $y = \cos x$ and the x-axis, between $x = 0$ to $x = \frac{\pi}{2}$, is revolved about the x-axis. Find the volume of the solid thus obtained.

$$\left[\textbf{Hint. } V = \int_0^{\pi/4} \pi \sin^2 x\, dx + \int_{\pi/4}^{\pi/2} \pi \cos^2 x\, dx.\right]$$

9. Find the volume of the solid generated by the revolution of the cardioid $r = a(1 - \cos\theta)$ about the initial line.
10. Find the curved surface of a hemisphere of radius a.
11. The circle $x^2 + y^2 = a^2$ is revolved about x-axis. Find the area of the sphere generated.
12. Find the area of the surface generated by rotating about x-axis the arc of the curve $y = x^3$ between $x = 0$ and $x = 1$.
13. Find the surface area of the solid obtained by revolving the arc of the curve $y = \sin x$ from $x = 0$ to $x = \pi$ about x-axis.
14. Find the curved surface of the solid generated by the revolution about the x-axis of the area bounded by the parabola $y^2 = 4ax$, the x-axis and the ordinate (*i*) $x = h$ (*ii*) $x = 3a$.
15. Find the surface generated by the revolution of the curve $y = c\cosh\left(\frac{x}{c}\right)$ about the x-axis, between $x = a$ and $x = b$.
16. Find the surface of the solid generated by revolving the curve

$$x = e^t \sin t,\ y = e^t \cos t,\ 0 \le t \le \frac{\pi}{2},\ \text{about the axis of } x.$$

17. Find the surface of the solid formed by the revolution of the cardioid $r = a(1 - \cos\theta)$ about the initial line.
18. Find the area of the surface formed by the revolution of $y^2 = 4ax$ about y-axis by the arc from vertex to $x = \frac{a}{4}$.
19. Find the surface of the solid generated by the revolution of the ellipse $4x^2 + 5y^2 = 20$ about the minor axis.
20. Find the volume of the reel-shaped solid formed by the revolution about the y-axis of the part of the parabola $y^2 = 4ax$ cut-off by the latus rectum.

21. Prove that the volume of the solid generated by the revolution of an ellipse round its minor axis, is a mean proportional between those generated by the revolution of the ellipse and of auxiliary circle round the major axis.

22. Find the volume of the solid formed by the revolution of the loop about the x-axis, of the following curves:

(i) $y^2 (a - x) = x^2 (a + x)$ (ii) $y^2 (a + x) = x^2 (3a - x)$ (iii) $y^2 = x^2 (a - x)$.

23. Find the volume of the solid generated by the revolution of the curve $y(a^2 + x^2) = a^3$ about its asymptote.

24. Find the volume of the solid generated by the revolution of the area between the curve $xy^2 = 4a^2 (2a - x)$ and its asymptote about the asymptote.

25. The loop of the curve $2ay^2 = x(x - a)^2$ revolves about x-axis, find the volume of the solid so generated.

26. Find the volume of the solid obtained by revolving the loop of the curve $a^2y^2 = x^2(2a - x) (x - a)$ about x-axis.

27. Show that the volume of the solid generated by the revolution of the curve $(a - x) y^2 = a^2x$ about its asymptote is $\frac{1}{2} \pi^2 a^3$.

28. Find the volume of the torus obtained by rotating the area bounded by the circle $x^2 + y^2 = a^2$ around the line $x = b$ $(b > a)$.

29. Find the volume of the spindle formed by the revolution of a parabolic arc about the line joining the vertex to one extremity of the latus rectum.

30. A quadrant of a circle, of radius a, revolves about its chord. Show that the volume of the spindle generated is $\frac{\pi}{6\sqrt{2}} (10 - 3\pi)a^3$.

31. Find the volume of the solid generated by revolving about the y-axis the arca between the first arch of the cycloid $x = \theta + \sin \theta$, $y = 1 - \cos \theta$ and x-axis.

32. Prove that the volume of the solid generated by the revolution of the tractrix $x = a \cos t + \frac{a}{2} \log \tan^2 \frac{t}{2}$, $y = a \sin t$ about its asymptote equals half of a sphere of radius a.

33. Find the volume of the solid generated by the arc of the cissoid $x = 2a \sin^2 t$, $y = 2a \frac{\sin^3 t}{\cos t}$ about its asymptote.

34. The figure bounded by a parabola and the tangents at the extremities of its latus rectum revolves about the axis of the parabola. Find the volume of the solid thus generated.

35. Find the surface of a sphere of radius a.

36. Show that the surface of the spherical zone contained between two parallel planes is $2\pi ah$ where a is the radius of the sphere and h is the distance between planes.

37. Prove that the surface of the prolate spheroid formed by the revolution of the ellipse of eccentricity e about its major axis is equal to $2 \left(\sqrt{1 - e^2} + \frac{1}{e} \sin^{-1} e\right) \times$ area of ellipse.

38. Find the surface of the solid generated by the revolution of the ellipse $x^2 + 4y^2 = 16$ about its major axis.

39. Find the volume and surface of the right circular cone formed by the revolution of right angled triangle about a side which contains the right angle.

40. The portion between two consecutive cusps of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ is revolved about x-axis. Prove that the area of the surface so formed is equal to $\frac{64}{9}$ times the area of the cycloid.

41. Prove that the surface area of the solid generated by the revolution, about the x-axis of the loop of the curve $x = t^2$, $y = t - \frac{1}{3}t^3$ is 3π.

42. Prove that the surface of the solid generated by the revolution of the tractrix $x = a\cos t + \frac{1}{2}a\log\tan^2\frac{t}{2}$, $y = a\sin t$, about its asymptote is equal to the surface of a sphere of radius a.

43. Prove that the surface of the oblate spheroid formed by the revolution of the ellipse of semi-major axis a and eccentricity e is $2\pi a^2\left[1 + \frac{1-e^2}{1+e^2}\log\left(\frac{1+e}{1-e}\right)\right]$.

[**Hint.** Oblate spheroid is generated by the revolution of the ellipse about minor axis.]

44. A circular arc revolves about its chord. Find the area of the surface generated when 2α is the angle subtended by the arc at the centre.

Answers

1. $\frac{2}{3}\pi a^3$ **2.** $\frac{2\pi}{15}$ **3.** 4π **4.** $2\pi a^3$

5. $\frac{\pi^2}{2}$ **6.** $2\pi a^3\left(\log 2 - \frac{2}{3}\right)$ **7.** $4\pi^2$ **8.** $\frac{\pi(\pi-2)}{4}$

9. $\frac{8}{3}\pi a^3$ **10.** $2\pi a^2$ **11.** $4\pi a^2$ **12.** $\frac{\pi}{27}(10\sqrt{10} - 1)$

13. $2\pi\left[\sqrt{2} + \log(\sqrt{2}+1)\right]$ **14.** (*i*) $\frac{8}{3}\pi\sqrt{a}\,[(h+a)^{3/2} - a^{3/2}]$ (*ii*) $\frac{56}{3}\pi a^2$

15. $\pi c\left[(b-a) + \frac{c}{2}\left(\sinh\frac{2b}{c} - \sinh\frac{2a}{c}\right)\right]$ **16.** $\frac{2\sqrt{2}}{5}\pi(e^{\pi} - 2)$ **17.** $\frac{32}{5}\pi a^2$

18. $\frac{\pi a^2}{16}\left[3\sqrt{5} - 8\log\frac{\sqrt{5}+1}{2}\right]$ **19.** $10\pi\left[1 + \frac{2}{\sqrt{5}}\log\left(\frac{\sqrt{5}+1}{\sqrt{5}-1}\right)\right]$

20. $\frac{4}{5}\pi a^3$ **22.** (*i*) $2\pi a^3\left(\log 2 - \frac{2}{3}\right)$ (*ii*) $\pi a^3(8\log 2 - 3)$ (*iii*) $\frac{\pi}{12}a^4$

23. $\frac{\pi^2 a^3}{2}$ **24.** $4\pi^2 a^3$ **25.** $\frac{\pi a^3}{24}$ **26.** $\frac{23\pi a^3}{60}$

28. $\frac{\pi}{2}(6ab^2 + 4a^3 - 3\pi a^2 b)$ **29.** $\frac{2\pi a^3}{15\sqrt{5}}$ **31.** $\pi\left(\frac{3\pi^2}{2} - \frac{8}{3}\right)$

33. $2\pi^2 a^3$ **34.** $\frac{2}{3}\pi a^3$ **35.** $4\pi a^2$ **38.** $8\pi\left(1 + \frac{4}{3\sqrt{3}}\right)$

39. $\frac{1}{3}\pi r^2 h$, πrl **44.** $4\pi a^2(\sin\alpha - \alpha\cos\alpha)$.

ASSIGNMENT-II

1. Find the value of integral $\int_0^\infty e^{-ax}x^{n-1}dx$. (A.K.T.U. 2016)

2. Evaluate $\int_0^\infty \sqrt{x}\ e^{-x}\,dx$. (U.P.T.U. 2014)

3. Find the value of $\Gamma\left(-\frac{5}{2}\right)$. (U.P.T.U. 2014; M.T.U. 2012)

4. Evaluate $\Gamma(-3/2)$. (G.B.T.U. 2013)

5. Find the value of $\Gamma\left(-\frac{1}{2}\right)$. (G.B.T.U. 2011)

6. Evaluate : $\frac{\Gamma(8/3)}{\Gamma(2/3)}$

7. Show that $\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\,dt = \pi$.

8. Evaluate $\beta(2, 1) + \beta(1, 2)$.

9. Evaluate $\iint_D x^3y\,dx\,dy$ where D is the region enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ in the first quadrant.

10. Evaluate $\frac{\beta(m+1, n)}{\beta(m, n)}$. (G.B.T.U. 2011; U.K.T.U. 2012)

11. Evaluate $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$. (U.P.T.U. 2014)

12. Evaluate $\int_0^{\pi/2}\sin^7\theta\sqrt{\cos\theta}\,d\theta$ in terms of Beta function.

13. Evaluate:

    (i) $\Gamma(3.5)$ (ii) $\beta\left(\frac{1}{2},\frac{1}{2}\right)$

14. Evaluate $\int_0^\infty e^{-x^2}\,dx$. (A.K.T.U. 2011, 2018)

15. Evaluate $\int_0^\infty e^{-x^4}\,dx$.

16. Use Beta function to evaluate : $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx$. (M.T.U. 2013)

17. Find the total mass of the region in the case $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$ with density at any point being xyz.

18. Evaluate $\iint_D x^p y^q\,dx\,dy$, where D is the region bounded by $x = 0$, $y = 0$ and $\left(\frac{x}{a}\right)^m + \left(\frac{y}{b}\right)^n = 1$.

19. Find the mass of the region in xy-plane bounded by $x = 0, y = 0, x + y = 1$ with density $k\sqrt{xy}$.

20. Use Dirichlet's integral to evaluate $\iiint xyz \, dx\, dy\, dz$ throughout the volume bounded by $x = 0$, $y = 0, z = 0$ and $x + y + z = 1$. *(U.P.T.U. 2014)*

21. Evaluate (in terms of Beta and Gamma Functions)

(i) $\int_0^1 \sqrt[3]{x \log\left(\frac{1}{x}\right)}\, dx$ (ii) $\int_0^{\pi/2} \frac{\sqrt[3]{\sin^8 x}}{\sqrt{\cos x}}\, dx$

(iii) $\int_0^1 \left(\frac{x}{1-x^3}\right)^{1/2} dx$ *(U.P.T.U. 2014)* (iv) $\int_0^\infty \frac{x^2}{(1+x^4)^3}\, dx$

22. Evaluate: $\int_{-\infty}^{\infty} e^{-x^2}\, dx$

23. Evaluate: $\int_0^\infty \frac{x^4}{(x+1)^{10}}\, dx$

24. Evaluate: $\int_0^1 x^4 (1-x)^3\, dx$

25. Evaluate: $\int_0^1 \frac{dx}{\sqrt{-\ln x}}$.

26. Determine the area enclosed by the curve $\left(\frac{x}{a}\right)^{2m} + \left(\frac{y}{b}\right)^{2n} = 1$; m, n being positive integers.

27. Find the area enclosed by the curve $\left(\frac{x}{a}\right)^4 + \left(\frac{y}{b}\right)^{10} = 1$.

28. Compute the mass of a sphere of radius r if the density varies inversely as the square of the distance from the centre.

29. If V is the solid region in the first octant bounded by the unit sphere $x^2 + y^2 + z^2 = 1$, find the mass of the ellipsoid $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$ with density $\rho(x, y, z) = (x + y + z)^4$.

30. The parabolic arc $y = \sqrt{x}$, $1 \le x \le 2$ is revolved around x-axis. Find the volume of solid of revolution. *(A.K.T.U. 2017)*

Answers

1. $\frac{n!}{a^n}$ **2.** $\frac{1}{2}\sqrt{\pi}$ **3.** $\frac{-8}{15}\sqrt{\pi}$ **4.** $\frac{4}{3}\sqrt{\pi}$

5. $-2\sqrt{\pi}$ **6.** $\frac{10}{9}$ **8.** 1 **9.** $\frac{b^2 a^4}{24}$

10. $\dfrac{m}{m+n}$ 11. $\pi\sqrt{2}$ 12. $\dfrac{1}{2}\beta\left(4, \dfrac{3}{4}\right)$

13. (i) $\dfrac{15}{8}\sqrt{\pi}$ (ii) π 14. $\dfrac{1}{2}\sqrt{\pi}$ 15. $\dfrac{1}{4}\Gamma\left(\dfrac{1}{4}\right)$

16. 0 17. $\dfrac{1}{8}$ 18. $\dfrac{a^{p+1}b^{q+1}}{mn} \dfrac{\Gamma\left(\dfrac{p+1}{m}\right)\Gamma\left(\dfrac{q+1}{n}\right)}{\Gamma\left(\dfrac{p+1}{m}+\dfrac{q+1}{n}+1\right)}$

19. $\dfrac{k\pi}{24}$ 20. $\dfrac{1}{720}$

21. (i) $\left(\dfrac{3}{4}\right)^{4/3}\Gamma\left(\dfrac{4}{3}\right)$ (ii) $\dfrac{1}{2}\beta\left(\dfrac{11}{6}, \dfrac{1}{4}\right)$ (iii) $\dfrac{1}{3}\beta\left(\dfrac{5}{6}, \dfrac{1}{2}\right)$ (iv) $\dfrac{1}{4}\beta\left(\dfrac{3}{4}, \dfrac{9}{4}\right)$

22. $\sqrt{\pi}$ 23. $\dfrac{1}{630}$ 24. $\dfrac{1}{280}$ 25. $2\sqrt{\pi}$

26. $\dfrac{ab}{4mn} \cdot \dfrac{\Gamma\left(\dfrac{1}{2m}\right)\Gamma\left(\dfrac{1}{2n}\right)}{\Gamma\left(\dfrac{1}{2m}+\dfrac{1}{2n}+1\right)}$ 27. $\dfrac{ab}{40} \cdot \dfrac{\Gamma\left(\dfrac{1}{4}\right)\Gamma\left(\dfrac{1}{10}\right)}{\Gamma\left(\dfrac{27}{20}\right)}$ 28. $4k\pi r$

29. $\dfrac{4\pi\, abc\,(a^2+b^2+c^2)}{35}$ 30. $\dfrac{3\pi}{2}$

MODULE 3

Sequences and Series

3.1 SEQUENCE

A sequence is a function whose domain is the set N of all natural numbers whereas the range may be any set S. In other words, a sequence in a set S is a rule which assigns to each natural number a unique element of S.

3.2 REAL SEQUENCE

A real sequence is a function whose domain is the set N of all natural numbers and range a subset of the set R of real numbers.

Symbolically $f : N \to R$ (or $x : N \to R$ or $a : N \to R$) is a real sequence.

Note. If $x : N \to R$ be a sequence, the image of $n \in N$ instead of denoting it by $x(n)$, we shall generally denote it by x_n. Thus x_1, x_2, x_3 etc. are the real numbers associated to 1, 2, 3, etc. by this mapping. Also, the sequence $x : N \to R$ is denoted by $\{x_n\}$ or $< x_n >$.

$x_1, x_2, ...$ are called the first, second ...... terms of the sequence. The m^{th} and n^{th} terms x_m and x_n for $m \neq n$ are treated as distinct event if $x_m = x_n$ *i.e.*, the terms occurring at different positions are treated as distinct terms even if they have the same value.

3.3 RANGE OF A SEQUENCE

The set of all **distinct** terms of a sequence is called its range.

Note. In a sequence $\{x_n\}$, since $n \in N$ and N is an infinite set, **the number of terms of a sequence is always infinite.** The range of a sequence may be a finite set. *e.g.*, if $x_n = (-1)^n$ then $\{x_n\} = \{-1, 1, -1, 1, ...\}$

The range of sequence $\{x_n\} = \{-1, 1\}$ which is a finite set.

3.4 CONSTANT SEQUENCE

A sequence $\{x_n\}$ defined by $x_n = c \in R \quad \forall\ n \in N$ is called a constant sequence.

e.g., $\{x_n\} = \{c, c, c, ...\}$ is a constant sequence with range $= \{c\}$.

3.5 BOUNDED AND UNBOUNDED SEQUENCES

Bounded above sequence. A sequence $\{a_n\}$ is said to be bounded above if $\exists$ a real number K such that $a_n \leq K \quad \forall\ n \in N$.

Bounded below sequence. A sequence $\{a_n\}$ is said to be bounded below if $\exists$ a real number K such that $a_n \geq k \quad \forall\ n \in N$.

Bounded sequence. A sequence $\{a_n\}$ is said to be bounded when it is bounded both above and below.

$\Rightarrow$ A sequence $\{a_n\}$ is bounded if $\exists$ two real numbers k and K ($k \le$ K) such that

$$k \le a_n \le K \quad \forall\ n \in N.$$

Choosing M = maximum $\{|\ k\ |, |\ K\ |\}$, we can also define a sequence $\{a_n\}$ to be bounded if $|\ a_n\ | \le M \quad \forall \quad n \in N$.

Unbounded sequence. If $\exists$ no real number M such that $|\ a_n\ | \le M\ \forall\ n \in N$, then the sequence $\{a_n\}$ is said to be unbounded.

Examples (1). The sequence $\{a_n\}$ defined by $a_n = \dfrac{1}{n}$.

Here
$$\{a_n\} = \left\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\ldots\right\}$$

$$0 < a_n \le 1 \quad \forall \quad n \in N$$

$\therefore$ $\{a_n\}$ is bounded.

(2) The sequence $\{a_n\}$ defined by $a_n = 2^{n-1}$

Here
$$\{a_n\} = \{1, 2, 2^2, 2^3, \ldots\ldots\}.$$

Although $a_n \ge 1, \quad \forall \quad n \in N, \exists$ no real number K such that $a_n \le K$.

$\therefore$ The sequence is unbounded above.

3.6 CONVERGENT, DIVERGENT AND OSCILLATING SEQUENCES

Convergent sequence. A sequence $\{a_n\}$ is said to be convergent if $\underset{n \to \infty}{\text{Lt}}\ a_n$ is finite.

For example, consider the sequence $\dfrac{1}{2}, \dfrac{1}{2^2}, \dfrac{1}{2^3}, \ldots\ldots, \dfrac{1}{2^n}\ \ldots\ldots$

Here $a_n = \dfrac{1}{2^n}$, $\underset{n \to \infty}{\text{Lt}}\ a_n = \underset{n \to \infty}{\text{Lt}}\ \dfrac{1}{2^n} = 0$, which is finite.

$\Rightarrow$ The sequence $\{a_n\}$ is convergent.

Divergent sequence. A sequence $\{a_n\}$ is said to be divergent if $\underset{n \to \infty}{\text{Lt}}\ a_n$ is not finite, *i.e.*, if

$$\underset{n \to \infty}{\text{Lt}}\ a_n = +\infty \text{ or } -\infty.$$

For example:

(*i*) Consider the sequence $\{n^2\}$

Here $a_n = n^2$, $\underset{n \to \infty}{\text{Lt}}\ a_n = +\infty \quad \Rightarrow \quad$ The sequence $\{n^2\}$ is divergent.

(*ii*) Consider the sequence $\{-2^n\}$.

Here,
$$a_n = -2^n,\ \underset{n \to \infty}{\text{Lt}}\ a_n = \underset{n \to \infty}{\text{Lt}}\ (-2^n) = -\infty$$

$\Rightarrow$ The sequence $\{-2^n\}$ is divergent.

Oscillatory sequence. If a sequence $\{a_n\}$ neither converges to a finite number nor diverges to $+\infty$ or $-\infty$, it is called an oscillatory sequence. Oscillatory sequences are of two types :

(*i*) A bounded sequence which does not converge is said to oscillate finitely.

For example, consider the sequence $\{(-1)^n\}$.

Here
$$a_n = (-1)^n$$

It is a bounded sequence. $\underset{n \to \infty}{\text{Lt}}\ a_{2n} = \underset{n \to \infty}{\text{Lt}}\ (-1)^{2n} = 1$

$$\underset{n \to \infty}{\text{Lt}}\ a_{2n+1} = \underset{n \to \infty}{\text{Lt}}\ (-1)^{2n+1} = -1.$$

Thus $\underset{n \to \infty}{\text{Lt}}\ a_n$ does not exist $\Rightarrow$ the sequence does not converge.

Hence this sequence oscillates finitely.

(*ii*) An unbounded sequence which does not diverge is said to **oscillate infinitely.**

For example, consider the sequence $\{(-1)^n\, n\}$.

Here
$$a_n = (-1)^n\, n.$$

It is an unbounded sequence.

$$\underset{n \to \infty}{\text{Lt}}\ a_{2n} = \underset{n \to \infty}{\text{Lt}}\ (-1)^{2n} \,.\, 2n = \underset{n \to \infty}{\text{Lt}}\ 2n = +\infty$$

$$\underset{n \to \infty}{\text{Lt}}\ a_{2n+1} = \underset{n \to \infty}{\text{Lt}}\ (-1)^{2n+1}\,(2n+1) = \underset{n \to \infty}{\text{Lt}}\ -(2n+1) = -\infty.$$

Thus the sequence does not diverge.

Hence this sequence oscillates infinitely.

Note. When we say $\underset{n \to \infty}{\text{Lt}}\ a_n = l$, it means $\underset{n \to \infty}{\text{Lt}}\ a_{2n} = \underset{n \to \infty}{\text{Lt}}\ a_{2n+1} = l$

Similarly, $\underset{n \to \infty}{\text{Lt}}\ a_n = +\infty$ means $\underset{n \to \infty}{\text{Lt}}\ a_{2n} = \underset{n \to \infty}{\text{Lt}}\ a_{2n+1} = +\infty.$

3.7 MONOTONIC SEQUENCES

(*i*) A sequence $\{a_n\}$ is said to be **monotonically increasing** if $a_{n+1} \geq a_n \ \forall \ n \in \text{N}$.

i.e., if
$$a_1 \leq a_2 \leq a_3 \leq \ldots\ldots \leq a_n \leq a_{n+1} \leq \ldots\ldots$$

(*ii*) A sequence $\{a_n\}$ is said to be **monotonically decreasing** if $a_{n+1} \leq a_n \ \forall \ n \in \text{N}$.

i.e., if
$$a_1 \geq a_2 \geq a_3 \geq \ldots\ldots \geq a_n \geq a_{n+1} \geq \ldots\ldots$$

(*iii*) A sequence $\{a_n\}$ is said to be **monotonic** if it is either monotonically increasing or monotonically decreasing.

(*iv*) A sequence $\{a_n\}$ is said to be **strictly monotonically increasing** if
$$a_{n+1} > a_n \ \forall \ n \in \text{N}.$$

(*v*) A sequence $\{a_n\}$ is said to be **strictly monotonically decreasing** if
$$a_{n+1} < a_n \ \forall \ n \in \text{N}.$$

(*vi*) A sequence $\{a_n\}$ is said to be **strictly monotonic** if it is either strictly monotonically increasing or strictly monotonically decreasing.

3.8 LIMIT OF A SEQUENCE

A sequence $\{a_n\}$ is said to approach the limit l (say) when $n \to \infty$, if for each $\varepsilon > 0$, $\exists$ a + ve integer m (depending upon ε) such that $|\, a_n - l \,| < \varepsilon \ \forall \ n \geq m$.

In symbols, we write $\underset{n \to \infty}{\text{Lt}}\ a_n = l$.

Note. $|\, a_n - l \,| < \varepsilon \ \forall\, n \geq m \ \Rightarrow \ l - \varepsilon < a_n < l + \varepsilon$ for $n = m, m+1, m+2, \ldots\ldots$

3.9 EVERY CONVERGENT SEQUENCE IS BOUNDED

Let the sequence $\{a_n\}$ be convergent. Let it tend to the limit l.

Then given $\varepsilon > 0$, $\exists$ a + ve integer m, such that

$$|a_n - l| < \varepsilon \quad \forall \quad n \geq m$$

$$\Rightarrow \qquad l - \varepsilon < a_n < l + \varepsilon \quad \forall \quad n \geq m.$$

Let k and K be the least and the greatest of $a_1, a_2, a_3, \ldots\ldots, a_{m-1}, l - \varepsilon, l + \varepsilon$

Then $\qquad k \leq a_n \leq K \quad \forall \quad n \in N,$

$\Rightarrow$ the sequence $\{a_n\}$ is bounded.

The converse is not always true *i.e.*, a sequence may be bounded, yet it may not be convergent. *e.g.*, consider $a_n = (-1)^n$, then the sequence $\{a_n\}$ is bounded but not convergent since it does not have a unique limiting point.

3.10 CONVERGENCE OF MONOTONIC SEQUENCES

Theorem I. *The necessary and sufficient condition for the convergence of a monotonic sequence is that it is bounded.*

A monotonic increasing sequence which is bounded above converges.

A monotonic decreasing sequence which is bounded below converges.

Theorem II. *If a monotonic increasing sequence is not bounded above, it diverges to* $+\infty$.

Theorem III. *If a monotonic decreasing sequence is not bounded below, it diverges to* $-\infty$.

Theorem IV. *If* $\{a_n\}$ *and* $\{b_n\}$ *are two convergent sequences, then sequence* $\{a_n + b_n\}$ *is also convergent.*

Or

If $\quad Lt\, a_n = A \quad$ *and* $\quad Lt\, b_n = B, \quad$ *then* $\quad Lt\, (a_n + b_n) = A + B.$

Theorem V. *If* $\{a_n\}$ *and* $\{b_n\}$ *are two convergent sequences such that* $Lt\, a_n = A$ *and* $Lt\, b_n = B$, *then*

(*i*) sequence $\{a_n b_n\}$ is also convergent and converges to AB.

(*ii*) sequence $\left\{\dfrac{a_n}{b_n}\right\}$ is also convergent and converges to $\dfrac{A}{B}$, $(B \neq 0)$.

Theorem VI. *The sequence* $\{|\,a_n\,|\}$ *converges to zero if and only if the sequence* $\{a_n\}$ *converges to zero.*

Theorem VII. *If a sequence* $\{a_n\}$ *converges to a and* $a_n \geq 0 \quad \forall\, n$, *then* $a \geq 0$.

Theorem VIII. *If* $a_n \to a$, $b_n \to b$ *and* $a_n \leq b_n \quad \forall\, n$, *then* $a \leq b$.

Theorem IX. *If* $a_n \to l$, $b_n \to l$, *and* $a_n \leq c_n \leq b_n$, $\forall\, n$, *then* $c_n \to l$. $\qquad$ *(Squeeze Principle)*

ILLUSTRATIVE EXAMPLES

Example 1. *Give an example of a monotonic increasing sequence which is (i) convergent, (ii) divergent.*

Sol. (*i*) Consider the sequence $\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \ldots\ldots, \dfrac{n}{n+1}, \ldots\ldots$

Since $\dfrac{1}{2} < \dfrac{2}{3} < \dfrac{3}{4} < \ldots\ldots$ the sequence is monotonic increasing.

$$a_n = \frac{n}{n+1}, \quad \underset{n\to\infty}{\text{Lt}}\ a_n = \underset{n\to\infty}{\text{Lt}}\ \frac{n}{n+1} = \underset{n\to\infty}{\text{Lt}}\ \frac{1}{1+\frac{1}{n}} = 1$$

which is finite.

$\therefore$ The sequence is convergent.

(*ii*) Consider the sequence 1, 2, 3, ..., n, ...

Since $1 < 2 < 3 <$ ......, the sequence is monotonic increasing,

$$a_n = n, \quad \underset{n\to\infty}{\text{Lt}}\ a_n = \underset{n\to\infty}{\text{Lt}}\ n = \infty$$

$\therefore$ The sequence diverges to $+\infty$.

Example 2. *Give an example of a monotonic decreasing sequence which is*

(*i*) *convergent,* (*ii*) *divergent.*

Sol. (*i*) Consider the sequence $1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ...$

Since $1 > \frac{1}{2} > \frac{1}{3} >$ ......, the sequence is monotonic decreasing.

$$a_n = \frac{1}{n}, \quad \underset{n\to\infty}{\text{Lt}}\ a_n = \underset{n\to\infty}{\text{Lt}}\ \frac{1}{n} = 0.$$

$\therefore$ The sequence converges to 0.

(*ii*) Consider the sequence $-1, -2, -3, ..., -n, ...$

Since $-1 > -2 > -3 >$ ......, the sequence is monotonic decreasing.

$$a_n = -n, \quad \underset{n\to\infty}{\text{Lt}}\ a_n = \underset{n\to\infty}{\text{Lt}}\ (-n) = -\infty$$

$\therefore$ The sequence diverges to $-\infty$.

Example 3. *Discuss the convergence of the sequence* $\{a_n\}$ *where*

(*i*) $a_n = \dfrac{n+1}{n}$ (*ii*) $a_n = \dfrac{n}{n^2+1}$ (*iii*) $a_n = 1 + \dfrac{1}{3} + \dfrac{1}{3^2} + ... + \dfrac{1}{3^n}$.

Sol. (*i*)

$$a_n = \frac{n+1}{n}$$

$$a_{n+1} - a_n = \frac{n+2}{n+1} - \frac{n+1}{n} = \frac{-1}{n(n+1)} < 0 \ \forall\ n$$

$\Rightarrow$ $a_{n+1} < a_n \quad \forall \quad n$

$\Rightarrow$ $\{a_n\}$ is a decreasing sequence.

Also, $a_n = \dfrac{n+1}{n} = 1 + \dfrac{1}{n} > 1 \quad \forall \quad n$

$\Rightarrow$ $\{a_n\}$ is bounded below by 1,

$\because$ $\{a_n\}$ is decreasing and bounded below, it is convergent.

$$\underset{n\to\infty}{\text{Lt}}\ a_n = \underset{n\to\infty}{\text{Lt}}\ \left(1+\frac{1}{n}\right) = 1.$$

(*ii*)

$$a_n = \frac{n}{n^2+1}$$

$$a_{n+1} - a_n = \frac{n+1}{(n+1)^2+1} - \frac{n}{n^2+1} = \frac{(n+1)(n^2+1) - n(n^2+2n+2)}{(n^2+2n+2)(n^2+1)}$$

$$= \frac{-n^2 - n + 1}{(n^2 + 2n + 2)(n^2 + 1)} < 0 \; \forall n \;\; \Rightarrow \;\; a_{n+1} < a_n \forall n$$

$\Rightarrow$ $\{a_n\}$ is a decreasing sequence.

Also, $$a_n = \frac{n}{n^2 + 1} > 0 \;\; \forall \, n \;\; \Rightarrow \;\; \{a_n\} \text{ is bounded below by } 0.$$

$\because$ $\{a_n\}$ is decreasing and bounded below, it is convergent.

$$\underset{n \to \infty}{\text{Lt}} \, a_n = \underset{n \to \infty}{\text{Lt}} \frac{n}{n^2 + 1} = \underset{n \to \infty}{\text{Lt}} \frac{\frac{1}{n}}{1 + \frac{1}{n^2}} = 0.$$

(*iii*) $$a_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \ldots\ldots + \frac{1}{3^n} = \frac{1\left(1 - \frac{1}{3^{n+1}}\right)}{1 - \frac{1}{3}} = \frac{3}{2}\left(1 - \frac{1}{3^{n+1}}\right)$$

Now, $$a_{n+1} = 1 + \frac{1}{3} + \frac{1}{3^2} + \ldots\ldots + \frac{1}{3^n} + \frac{1}{3^{n+1}}$$

$\therefore$ $$a_{n+1} - a_n = \frac{1}{3^{n+1}} > 0 \;\; \forall \, n \;\; \Rightarrow \;\; a_{n+1} > a_n \;\; \forall \, n$$

$\Rightarrow$ $\{a_n\}$ is an increasing sequence.

Also, $$a_n = \frac{3}{2}\left(1 - \frac{1}{3^{n+1}}\right) < \frac{3}{2} \;\; \forall \, n \;\; \Rightarrow \;\; \{a_n\} \text{ is bounded above by } \frac{3}{2}.$$

$\because$ $\{a_n\}$ is increasing and bounded above, it is convergent.

$$\underset{n \to \infty}{\text{Lt}} \, a_n = \underset{n \to \infty}{\text{Lt}} \frac{3}{2}\left(1 - \frac{1}{3^{n+1}}\right) = \frac{3}{2}.$$

3.11 SERIES

An expression of the form $u_1 + u_2 + u_3 + \ldots + u_n + \ldots$ in which each term is obtained from its preceding term by some definite law is called a series. This definite law is known as law of formation of the series. $u_1, u_2, u_3, \ldots, u_n, \ldots$ are respectively known as first, second, third, ..., n^{th}, ..., term of the series.

3.12 FINITE SERIES

A series which contains a finite number of terms is called a finite series. Thus,

$$\sum_{r=1}^{n} u_r = u_1 + u_2 + u_3 + \ldots + u_r + \ldots + u_n \text{ is the finite series.}$$

3.13 INFINITE SERIES

A series which contains an infinite number of terms is called an infinite series. In other words, we can say that a series in which every term is followed by another is called an infinite series.

Thus,

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \dots + u_n + \dots \text{ is the infinite series.}$$

The above infinite series may simply be denoted by Σu_n.

The sum of the first n terms of the series is denoted by S_n.

Thus,

$$S_n = u_1 + u_2 + u_3 + \dots + \dots + u_n.$$

S_n is also called the n^{th} partial sum of $\sum_{n=1}^{\infty} u_n$.

Note. We shall consider the word series as infinite series onwards.

3.14 POSITIVE TERM SERIES

The series

$$\Sigma u_n = u_1 + u_2 + u_3 + \dots + u_n + \dots$$

is said to be a positive term series if every term of the series is positive. Similarly, we can define a negative term series.

3.15 ALTERNATING SERIES

If the terms of a series are alternately positive and negative beginning with the first term, then, the series is said to be an alternating series. Thus,

$$u_1 - u_2 + u_3 - u_4 + \dots + (-1)^{n+1} u_n + \dots$$

is an alternating series.

3.16 SOME IMPORTANT LIMITS

(a) $\lim\limits_{n \to \infty} \left(1 + \dfrac{x}{n}\right)^n = e^x$ (b) $\lim\limits_{n \to \infty} \dfrac{\log n}{n} = 0$

(c) $\lim\limits_{n \to \infty} n x^n = 0, \quad 0 < x < 1$

(d) $$\lim_{n \to \infty} \frac{A_1 n^p + A_2 n^{p-1} + \cdots + A_{p+1}}{B_1 n^q + B_2 n^{q-1} + \cdots + B_{q+1}}$$

$$= \begin{cases} 0 & \text{if } p < q \\ \dfrac{A_1}{B_1} & \text{if } p = q \\ \infty & \text{if } A_1 \text{ and } B_1 \text{ are of same signs and } p > q \\ -\infty & \text{if } A_1 \text{ and } B_1 \text{ are of opposite signs and } p > q \end{cases}$$

(e) $\lim\limits_{n \to \infty} x^n = 0, \quad (x < 1)$

3.17 CONVERGENCE AND DIVERGENCE OF A SERIES: DEFINITION

Let $\Sigma u_n = u_1 + u_2 + u_3 + \dots + u_n + \dots$ be an infinite series. Then,

$$S_n = u_1 + u_2 + u_3 + \dots + u_n$$

and we define the convergence and divergence of the given series as follows:

(*i*) If $\underset{n \to \infty}{\text{Lt}} S_n = S$, where S is a finite and unique quantity, then Σu_n is convergent. We say the series itself converges to S. S is known as sum of the series.

(*ii*) If $\underset{n \to \infty}{\text{Lt}} S_n = +\infty$ or $-\infty$, then Σu_n is divergent. We say that the series itself diverges to $+\infty$ or $-\infty$, as the case may be.

(*iii*) If $\underset{n \to \infty}{\text{Lt}} S_n$ is neither a finite unique quantity, nor $-\infty$ or $+\infty$, then the series Σu_n is said to be oscillatory or periodic convergent series. A series can oscillate finitely or infinitely.

(*a*) The series Σu_n is said to oscillate finitely if $\underset{n \to \infty}{\text{Lt}} S_n$ is a finite but not a unique quantity, *i.e.* $\underset{n \to \infty}{\text{Lt}} S_n$ fluctuates between finite limits.

(*b*) The series Σu_n is to oscillate infinitely if $\underset{n \to \infty}{\text{Lt}} S_n$ fluctuates between $-\infty$ and $+\infty$.

Examples. Consider the series $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots$

Here, $$S_n = \frac{1\left(1 - \frac{1}{2^n}\right)}{1 - \frac{1}{2}} = 2\left(1 - \frac{1}{2^n}\right)$$

Clearly, $\underset{n \to \infty}{\text{Lt}} S_n = 2$ which is a finite unique quantity, hence the series is convergent.

Again, consider the series $1 + 2 + 3 + 4 + 5 + \ldots$

Here, $$S_n = \frac{n(n+1)}{2}$$

Clearly, $$\underset{n \to \infty}{\text{Lt}} S_n = \infty$$

Hence, the series is divergent.

Convergence and Divergence of Infinite Series

Further, consisder the series $-1 - 2 - 3 - 4 - \ldots$

Here, $$S_n = -\frac{n(n+1)}{2}$$

Clearly, $$\underset{n \to \infty}{\text{Lt}} S_n = -\infty$$

Hence, the series is divergent.

Now, consider the series $1 - 1 + 1 - 1 + 1 - 1 + \ldots$

Here, $$S_n = \begin{Bmatrix} 0, \text{ if } n \text{ is even} \\ 1, \text{ if } n \text{ is odd} \end{Bmatrix}$$

Thus S_n oscillates (fluctuates) between 0 and 1 which are the finite limits. So, the series oscillates finitely.

In the last, consider the series $1 - 2 + 3 - 4 + 5 - 6 + \ldots$

Here, $S_1 = 1,\quad S_2 = -1$

$S_3 = 2,\quad S_4 = -2$

$S_5 = 3,\quad S_6 = -3$ and so on.

Thus, $$S_n = -\frac{n}{2} \text{ if } n \text{ is even}$$

and $$S_n = \left(\frac{n+1}{2}\right) \text{ if } n \text{ is odd}$$

Hence $$\underset{n \to \infty}{\text{Lt}} \; S_n = -\infty \text{ if } n \text{ is even}$$

and $$\underset{n \to \infty}{\text{Lt}} \; S_n = +\infty \text{ if } n \text{ is odd.}$$

Therefore, the series oscillates infinitely between $-\infty$ and $+\infty$.

Note 1 : If all the terms of a series are the same sign (either positive or negative), then this series can either be convergent or divergent and never an oscillatory one.

Note 2 : By "Lt" we shall mean $\underset{n \to \infty}{\text{Lt}}$.

3.18 GEOMETRIC SERIES

An infinite geometric series with common ratio r is

(*i*) convergent if $-1 < r < 1$ *i.e.* $|r| < 1$

(*ii*) divergent if $r > 1$

(*iii*) finitely oscillating if $r = -1$

(*iv*) infinitely oscillating if $r < -1$

Consider the infinite geometric series

$$1 + r + r^2 + r^3 + \cdots + r^{n-1} + r^n + \cdots$$

whose common ratio is r.

Case I. *When $-1 < r < 1$ i.e. $|r| < 1$.*

We know that, in this case as r is numerically less than 1,

$$\therefore \qquad S_n = \frac{1 - r^n}{1 - r}$$

Take limit as $n \to \infty$ on both sides, we have

$$\underset{n \to \infty}{\text{Lt}} \; S_n = \underset{n \to \infty}{\text{Lt}} \; \frac{1 - r^n}{1 - r}$$

$$= \frac{1}{1 - r} \qquad \left[\begin{array}{l}\text{as } r^n \to 0 \text{ when } n \to \infty, \\ |r| \text{ being less than } 1\end{array}\right]$$

which is a finite unique quantity, hence the series is convergent by definition of convergence of a series.

Case II. *When $r > 1$*

In this case, as $r > 1$

$$\therefore \quad S_n = \frac{r^n - 1}{r - 1}$$

$$\therefore \quad \underset{n \to \infty}{\text{Lt}} S_n = \underset{n \to \infty}{\text{Lt}} \frac{r^n - 1}{1 - r} = \infty \quad \left[\begin{array}{l} \because r^n \to \infty, \\ \text{as } n \to \infty, \\ r \text{ being greater than } 1 \end{array}\right]$$

Hence the series is divergent by definition of divergence of a series.

Also, when $r = 1$, the series becomes,

$$1 + 1 + 1 + 1 + 1 + 1 + \ldots$$

Thus, $S_n = n$ which tends to ∞ as $n \to \infty$, hence again the series is divergent.

Case III. *When $r = -1$*, then the series becomes $1 - 1 + 1 - 1 + 1 - 1 + \ldots$

As already proved, we see that

$$\underset{n \to \infty}{\text{Lt}} S_n = \begin{cases} 1, \text{ if } n \text{ is odd} \\ 0, \text{ if } n \text{ is even} \end{cases}$$

Here the limits are finite but not unique.

Hence the series is finitely oscillating series.

Case IV. *When $r < -1$.*

If $\quad r < -1$, then $-r > 1$

Let $\quad x = -r \quad$ then $x > 1$

and therefore $\quad x^n \to \infty$ as $n \to \infty$

Now, $$S_n = 1 + r + r^2 + \cdots + r^{n-1} = \frac{1 - r^n}{1 - r} = \frac{1 - (-x)^n}{1 + x}$$

$$= \frac{1 + x^n}{1 + x} \quad \text{or} \quad \frac{1 - x^n}{1 + x} \quad (\text{where } x > 1)$$

according as n is odd or even.

$$\therefore \quad \underset{n \to \infty}{\text{Lt}} S_n = \frac{1 + \infty}{1 + x} \quad \text{or} \quad \frac{1 - \infty}{1 + x} = +\infty \text{ or } -\infty$$

Hence the series is infinitely oscillating.

3.19 NECESSITY OF TESTS

It is clear that in order to decide whether a given series is convergent or divergent, we shall first determine S_n and then take its limit as $n \to \infty$. The value of limit will let us know about the convergency and divergency of the series. But there are many series in which the determination of S_n is not easy and rather impossible too. Therefore, some tests have been searched out to overcome this problem of deciding the convergence or divergence of a series without actually finding out the sum of its first n terms.

3.20 FUNDAMENTAL PROPERTIES

(*i*) *The nature of a series remains unaltered if the signs of all the terms are altogether changed.*

(ii) *The convergence or divergence of a series remains uneffected if a finite number of terms are added or neglected. (omitted or left)*

(iii) *If each term of a given series is multiplied or divided by some fixed quantity other than zero, then the new series so obtained will remain convergent or divergent according as it was orginally convergent or divergent.*

(iv) *If two infinite series are given, then the series formed by their sum will be :*

(a) *convergent if both the given series are convergent.*

(b) *divergent if any one of the given series is divergent.*

(v) *The series less than a convergent series is convergent, and greater than a divergent series is divergent.*

(vi) *If a series Σu_n converges to a fixed finite quantity S, then the series obtained by grouping the terms in brackets without affecting the order of the terms also converges to S.*

3.21 LIMIT u_n TEST

For a series Σu_n to be convergent, it is necessary but not sufficient that

$$\lim_{n \to \infty} u_n = 0.$$

3.22 A TEST FOR DIVERGENCE OF A SERIES

Statement. *If all the terms of an infinite series are positive and each term is greater than some fixed finite quantity however small, then the series will be divergent.*

Proof. Let $\Sigma u_n = u_1 + u_2 + u_3 + \ldots + u_n + \ldots$

be the given series. Let S_n denotes the sum of first n terms of the series, then,

$$S_n = u_1 + u_2 + u_3 + \ldots + u_n + \ldots,$$

let each term of the series be greater than some finite quantity say K.

Then we have, $S_n > K + K + K + \ldots + K$

$> nK$

Hence, $S_n \to \infty$ as $n \to \infty$

Hence, the series Σu_n is divergent.

3.23 LEIBNITZ'S TEST OR ALTERNATING SERIES TEST

Statement. An infinites series in which, the terms are alternately positive and negative is convergent if each term is numerically less than the preceding term and if $\underset{n \to \infty}{\text{Lt}} u_n = 0$.

Or

The infinite series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ is convergent if $u_n > 0$, $u_n < u_{n-1}$ and $\underset{n \to \infty}{\text{Lt}} u_n = 0$

Let the series be

$$u_1 - u_2 + u_3 - u_4 + u_5 - u_6 + \ldots$$

....

....

where $u_1 > u_2 > u_3 > u_4 > u_5 > u_6 > \ldots$

Let S_{2n} and S_{2n+1} represent the sum of first $2n$ and first $(2n + 1)$ terms of the series respectively.

Then, $S_{2n} = (u_1 - u_2) + (u_3 - u_4) + (u_5 - u_6) + \dots + (u_{2n-1} - u_{2n})$

$= \text{a positive quantity as } u_1 > u_2 > u_3 > \dots \text{ i.e. } > 0$ ...(1)

and $S_{2n+1} = u_1 - [(u_2 - u_3) + (u_4 - u_5) + \dots + (u_{2n} - u_{2n+1})]$

$= u_1 - \text{a positive quantity}$ $(\because u_2 > u_3 > u_4 \dots)$

$< u_1$ as S_{2n+1} cannot be negative ...(2)

Also, $S_{2n+1} = S_{2n} + u_{2n+1}$

$$\therefore \lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} u_{2n+1}$$

$$= \lim_{n\to\infty} S_{2n} + 0 \qquad \left(\because \lim_{n\to\infty} u_n = 0\right)$$

$$\therefore \lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n}$$

i.e. when $n \to \infty$, the sum of an odd number of terms of the series is same as sum of and even number of terms of the series.

Hence the series cannot be oscillating.

Now, from (1) and (2) it is clear that the above common limit must lie between 0 and u_1 and since u_1 is finite and definite, hence the series must be convergent.

Example 4. *Examine the series* $2 + \frac{3}{2} + \frac{4}{3} + \frac{5}{4} + \cdots$

Sol. We observe that in the given series, each term is positive and greater than 1 which is a finite quantity, hence the series is divergent.

Example 5. *Test the series* $\sqrt{\frac{1}{2}} + \sqrt{\frac{2}{3}} + \sqrt{\frac{3}{4}} + \sqrt{\frac{4}{5}} + \cdots$

Sol. If we leave the first term, the series becomes

$$\sqrt{\frac{2}{3}} + \sqrt{\frac{3}{4}} + \sqrt{\frac{4}{5}} + \cdots$$

It is to be noted that the nature of the series does not get affected by doing this.

Now in this series we observe that

(1) all the terms are positive.

(2) each term is greater than $\sqrt{\frac{1}{2}}$, which is a finite quantity.

Hence the series is divergent.

Example 6. *Test whether the series* $\frac{1}{1+2^{-1}} + \frac{2}{1+2^{-2}} + \frac{3}{1+2^{-3}} + \cdots$ *is convergent or divergent.*

Sol. The given series may be written as

$$\frac{1}{1+\frac{1}{2}}+\frac{2}{1+\frac{1}{2^2}}+\frac{3}{1+\frac{1}{2^3}}+\cdots$$

i.e. $$\frac{2}{3}+\frac{8}{5}+\frac{24}{9}+\cdots$$

We observe that

(*i*) all the terms are positive.

(*ii*) each term $> \frac{1}{2}$ which is a finite quantity.

Hence the given series is divergent.

Example 7. *Test the convergence of the series*

$$\sqrt{\frac{1}{2}}+\frac{2}{\sqrt{5}}+\frac{4}{\sqrt{17}}+\frac{8}{\sqrt{65}}+\cdots+\frac{2^n}{\sqrt{4^n+1}}+\cdots$$

Sol. Leaving the first term, we have

$$u_n=\frac{2^n}{\sqrt{4^n+1}}=\frac{2^n}{2^n\sqrt{1+4^{-n}}}=\frac{1}{\sqrt{1+4^{-n}}}$$

$$\therefore \qquad \operatorname*{Lt}_{n\to\infty} u_n=\operatorname*{Lt}_{n\to\infty}\frac{1}{\sqrt{1+4^{-n}}}=1\neq 0$$

Hence the given series is not convergent.

Example 8. *Prove that the series* $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$ *is convergent.*

Sol. We observe that

(*i*) the terms of the series are alternately positive and negative.

(*ii*) each term is numerically less than its preceding term as, $1>\frac{1}{2}>\frac{1}{3}>\frac{1}{4}>\ldots$

(*iii*) $u_n=\frac{1}{n}$

$\therefore \operatorname*{Lt}_{n\to\infty} u_n=0$

Thus all the three conditions of Leibnitz test are satisfied. Hence, the given series is convergent.

Example 9. *Test the convergence of the series*

$$2-\frac{3}{2}+\frac{4}{3}-\frac{5}{4}+\frac{6}{5}-\cdots$$

Sol. We observe that

(*i*) the terms are alternately positive and negative.

(*ii*) each term is numerically less than its preceding term as,

$$2 > \frac{3}{2} > \frac{4}{3} > \frac{5}{4} > \cdots$$

$$(iii)\ \lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{n+1}{n} = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right) = 1 \neq 0$$

Therefore, the third condition of Leibnitz test is not satisfied. Hence the series is not convergent. However, the given series can be written as

$$(1+1) - \left(1 + \frac{1}{2}\right) + \left(1 + \frac{1}{3}\right) - \left(1 + \frac{1}{4}\right) + \left(1 + \frac{1}{5}\right) - \cdots$$

i.e. $$(1 - 1 + 1 - 1 + 1 - \cdots) + \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots\right)$$

Here the series in first bracket is oscillatory whose sum of n terms is 1 or 0 according as n is odd or even. The series in second bracket is convergent as its sum is log 2. Therefore, the sum of n terms of given series as $n \to \infty$ is $1 + \log 2$ or $\log 2$ according as n is odd or even. Hence the given series is oscillatory.

TEST YOUR KNOWLEDGE

Test the convergence of the following infinite series (1 – 20):

1. $1 + 2 + 3 + 4 + \ldots$
2. $1 + 3 + 5 + 7 + \ldots$
3. $1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots$
4. $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$
5. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \ldots$
6. $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ldots$
7. $1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \ldots$
8. $\frac{1}{\sqrt{2}-1} - \frac{1}{\sqrt{3}-1} + \frac{1}{\sqrt{4}-1} - \frac{1}{\sqrt{5}-1} + \ldots$
9. $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$
10. $\frac{2}{1^3} - \frac{3}{2^3} + \frac{4}{3^3} - \frac{5}{4^3} + \ldots$
11. $\log\frac{1}{2} - \log\frac{2}{3} + \log\frac{3}{4} - \log\frac{4}{5} + \ldots$
12. $\log\frac{2}{1} - \log\frac{3}{2} + \log\frac{4}{3} - \log\frac{5}{4} + \ldots$
13. $\frac{1}{2} + \frac{1+a}{2+a} + \frac{1+2a}{2+2a} + \ldots + \frac{1+(n-1)a}{2+(n-1)a} + \ldots\ (a > 0)$
14. $\frac{a}{b} + \frac{a+x}{b+x} + \frac{a+2x}{b+2x} + \frac{a+3x}{b+3x} + \ldots$
    where a and b are finite and positive, $b > a$ and $x > 0$.

15. $\frac{1}{x} - \frac{1}{x+a} + \frac{1}{x+2a} - \frac{1}{x+3a} + \dots$; x and a being positive quantities.

16. $1 - \frac{1}{1+a} + \frac{1}{1+2a} - \frac{1}{1+3a} + \dots\ (a > 1)$

17. $\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - \frac{x^4}{1+x^4} + \dots\ (0 < x < 1)$

18. $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{-n}$

19. $1 - 2x + 3x^2 - 4x^3 + \dots\ (x < 1)$

20. $\frac{1}{xy} - \frac{1}{(x+1)(y+1)} + \frac{1}{(x+2)(y+2)} - \frac{1}{(x+3)(y+3)} + \dots$; $x, y > 0$

21. Test the series whose n^{th} terms are

(*i*) $\cos \frac{1}{n}$ (*ii*) $\frac{n^2-1}{2n^2-1}$ (*iii*) $\frac{n-1}{n}$

Answers

1. Divergent	**2.** Divergent	**3.** Convergent	**4.** Convergent
5. Convergent	**6.** Convergent	**7.** Convergent	**8.** Convergent
9. Convergent	**10.** Convergent	**11.** Convergent	**12.** Convergent
13. Divergent	**14.** Divergent.	**15.** Convergent	**16.** Convergent
17. Convergent	**18.** Divergent	**19.** Convergent	**20.** Convergent.
21. (*i*) Divergent	(*ii*) Divergent	(*iii*) Divergent	

3.24 COMPARISON TEST FOR POSITIVE TERM SERIES

Statement. *If the ratio of the corresponding terms of two series, in each of which all the terms are positive, be always a fixed finite non-zero quantity, then the two series are convergent or divergent simultaneously.*

Or

If Σu_n and Σv_n be two positive term series such that

$$\underset{n\to\infty}{\text{Lt}}\left(\frac{u_n}{v_n}\right)$$

is a fixed finite non-zero quantity, then both the series will converge or diverge simultaneously, i.e., the two series are either both convergent or both divergent.

Proof. Let $\Sigma u_n = u_1 + u_2 + u_3 + \dots$

and $\Sigma v_n = v_1 + v_2 + v_3 + \dots$

be two given positive term series. We are given that $\frac{u_1}{v_1}, \frac{u_2}{v_2}, \frac{u_3}{v_3}, \dots\dots\dots$ are all finite.

Let l_1 and l_2 be respectively the greatest and least values of these ratios then,

$$l_1 \geq \frac{u_1}{v_1} \geq l_2 \quad \text{or} \quad v_1 l_1 \geq u_1 \geq v_1 l_2$$

$$l_1 \geq \frac{u_2}{v_2} \geq l_2 \quad \text{or} \quad v_2 l_1 \geq u_2 \geq v_2 l_2$$

$$l_1 \geq \frac{u_3}{v_3} \geq l_2 \quad \text{or} \quad v_3 l_1 \geq u_3 \geq v_3 l_2$$

$$\begin{array}{ccc} \vdots & & \vdots \end{array}$$

and so on.

Adding the above results simultaneously, we get

$$v_1 l_1 + v_2 l_1 + v_3 l_1 + \ldots\ldots \geq u_1 + u_2 + u_3 + \ldots\ldots$$
$$\geq v_1 l_2 + v_2 l_2 + v_3 l_2 + \ldots\ldots$$

or, $(v_1 + v_2 + v_3 + \ldots)l_1 \geq u_1 + u_2 + u_3 + \ldots\ldots \geq (v_1 + v_2 + v_3 + \ldots)l_2$

or, $$l_1 \geq \frac{u_1 + u_2 + u_3 + \ldots\ldots}{v_1 + v_2 + v_3 + \ldots\ldots} \geq l_2$$

Since all the ratios are finite.

Therefore l_1 and l_2 are also finite. Hence $\dfrac{u_1 + u_2 + u_3 + \ldots}{v_1 + v_2 + v_3 + \ldots}$ which lies between l_1 and l_2 must be finite. Let it be l.

$$\therefore \quad \frac{u_1 + u_2 + u_3 + \ldots}{v_1 + v_2 + v_3 + \ldots} = l$$

or, $u_1 + u_2 + u_3 + \ldots = l(v_1 + v_2 + v_3 + \ldots)$

or, $\Sigma\, u_n = l\, \Sigma\, v_n$

Thus $\Sigma\, u_n$ is convergent or divergent, according as $\Sigma\, v_n$ is convergent or divergent.

3.25 WORKING RULE FOR COMPARISON TEST

Step 1. We find out the n^{th} term of the given series whose convergence or divergence is to be tested and denote it by u_n.

Step 2. Then for the sake of comparison, we construct a suitable test series whose convergence or divergence is already known to us. This series is called the Auxiliary Series. We find its n^{th} term and denote it by v_n.

Step 3. We determine $\underset{n \to \infty}{\text{Lt}} \dfrac{u_n}{v_n}$. If this limit comes out to be a fixed finite non-zero quality, then we say that the comparison test can be applied to test the convergence or divergence of the given series.

Step 4. If it is not so, then we cannot apply the comparison test.

Step 5. If the comparison test is applicable, then, we conclude that the given series will be convergent or divergent according as the auxiliary series Σv_n is convergent or divergent.

3.26 GENERAL METHOD FOR FINDING THE SUITABLE AUXILIARY SERIES

Let Σu_n be the given series. Write down u_n, the n^{th} term of the series. Retain the terms containing highest power of n in the numerator and denominator, separately and simultaneously and then neglect any numerical factors present therein. Thus what we get is n^{th} term of the auxiliary series *i.e.*, v_n. For example:

If we have u_n in the form

$$\frac{A_1 n^p + A_2 n^{p-1} + \ldots\ldots}{B_1 n^q + B_2 n^{q-1} + \ldots\ldots}$$

Then the n^{th} term of the auxiliary series will be

$$\frac{n^p}{n^q}$$

i.e.,
$$v_n = \frac{1}{n^{q-p}}$$

3.27 *p*-SERIES TEST OR HYPER-HARMONIC TEST

Statement. *The infinite series*

$$\Sigma \frac{1}{n^p} \text{ i.e., } \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots\ldots + \frac{1}{n^p} + \ldots$$

is convergent if $p > 1$ *and divergent if* $p \le 1$.

Proof.

Case I. *When* $p > 1$

Let the terms of the given series be grouped such that the first, second, third, ... groups conain 1 term, 2 terms, 2^2 terms, ..., respectively. Let S denote the sum of the series. Then, we have

$$S = \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \left(\frac{1}{8^p} + \frac{1}{9^p} + \ldots + \frac{1}{15^p}\right) + \ldots$$

Now, $\because$ $p > 1$

$\therefore$ $3^p > 2^p$

$\therefore$ $\frac{1}{3^p} < \frac{1}{2^p}$

$\therefore$ $\frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p}$

$$< \frac{2}{2^p}$$

$$< \frac{1}{2^{p-1}}$$

Similarly,

$$\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}$$

$$<\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}$$

i.e.,
$$<\frac{4}{4^p}$$

i.e.,
$$<\frac{1}{4^{p-1}}$$

i.e.,
$$<\frac{1}{2^{2(p-1)}}$$

and
$$\frac{1}{8^p}+\frac{1}{9^p}+\ldots+\frac{1}{15^p}$$

$$<\frac{1}{8^p}+\frac{1}{8^p}+\ldots+\frac{1}{8^p}$$

i.e.,
$$<\frac{8}{8^p}$$

i.e.,
$$<\frac{1}{8^{p-1}}$$

i.e.,
$$<\frac{1}{2^{3(p-1)}}$$

and so on.

Thus,
$$S<1+\frac{1}{2^{p-1}}+\frac{1}{2^{2(p-1)}}+\frac{1}{2^{3(p-1)}}+\ldots$$

Now the series on right-hand side is a geometric series whose common ratio is $\frac{1}{2^{p-1}}$

and as $p>1$, *i.e.*, $p-1>0$, $\frac{1}{2^{p-1}}<1$ and >0.

Hence the series on RHS is convergent.

Since S is less than a convergent series, therefore the given series is convergent when $p>1$.

Case II. *When $p = 1$, then the given series becomes*

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots$$

Let the terms of the given series be grouped such that first, second, third, fourth, fifth, ... groups contain 1 term, 2 terms, 2^2 terms, 2^3 terms, ... respectively. Then we have

$$S=1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{10}+\ldots\ldots+\frac{1}{16}\right)+\ldots\ldots$$

Now, since $3 < 4$

$$\therefore \quad \frac{1}{3} > \frac{1}{4}$$

$$\therefore \quad \frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4}$$

$$\text{or} \quad \frac{1}{3} + \frac{1}{4} > \frac{1}{2}$$

Similarly, $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$

$$> \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$$

$$> \frac{4}{8}$$

$$> \frac{1}{2}$$

and so on.

$$\therefore \quad S > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots$$

The series on RHS is divergent because each term of this series is +ve and greater than a fixed finite quantity $\frac{1}{4}$.

Thus, the sum of the given series S is greater than a divergent series.

Hence the given series is divergent when $p = 1$,

i.e., the series $\Sigma \frac{1}{n}$ is a divergent. The series $\Sigma \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$ is known as Harmonic series. Thus a harmonic series is divergent.

Case III. *When $p < 1$ or negative.*

In this case,

$$2^p < 2 \quad \therefore \quad \frac{1}{2^p} > \frac{1}{2}$$

$$3^p < 3 \quad \therefore \quad \frac{1}{3^p} > \frac{1}{3}$$

$$4^p < 4 \quad \therefore \quad \frac{1}{4^p} > \frac{1}{4}$$

and so on.

$$\therefore \quad S = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots$$

$$> 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

The series on RHS is divergent by virtue of case II. Hence sum of the given series S is greater than a divergent series.

Note : Two series are very commonly used as auxiliary series,

(1) Geometric series (2) $\Sigma \dfrac{1}{n^p}$ or p-series.

ILLUSTRATIVE EXAMPLES

Example 1. *Test the series:* $1 + \dfrac{1}{3} + \dfrac{1}{5} + \dfrac{1}{7} + \cdots$

Sol. Let $\Sigma u_n = 1 + \dfrac{1}{3} + \dfrac{1}{5} + \dfrac{1}{7} + \cdots$

Then, $u_n = \dfrac{1}{2n-1}$

$v_n = \dfrac{1}{n}$ where v_n is n^{th} term of the auxiliary series Σv_n.

$$\frac{u_n}{v_n} = \frac{n}{2n-1}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{n}{2n-1} = \underset{n\to\infty}{\text{Lt}} \frac{1}{2-\frac{1}{n}} = \frac{1}{2}$$

which is a fixed, finite, non-zero quantity.
Hence comparison test can be applied.

Now by p-test, we know that the series $\Sigma \dfrac{1}{n}$ is divergent as $p = 1$. Hence the given series Σu_n is also divergent by comparison test.

Example 2. *Test the series*

$$\frac{14}{1^3} + \frac{24}{2^3} + \frac{34}{3^3} + \cdots + \frac{10n+4}{n^3} + \cdots$$

Sol. Let $\Sigma u_n = \dfrac{14}{1^3} + \dfrac{24}{2^3} + \dfrac{34}{3^3} + \cdots + \dfrac{10n+4}{n^3} + \cdots$

Then, $u_n = \dfrac{10n+4}{n^3}$

$v_n = \dfrac{n}{n^3} = \dfrac{1}{n^2}$

where v_n is the n^{th} term of the auxiliary series Σv_n.

Now, $$\frac{u_n}{v_n} = \frac{10n+4}{n^3}\cdot n^2 = \frac{10n+4}{n} = 10 + \frac{4}{n}$$

$\therefore$ $$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = 10 \text{ which is a fixed, finite, non-zero quantity.}$$

Hence comparison test can be applied.

But by p-test, we know that the auxiliary series $\Sigma \frac{1}{n^2}$ is convergent as $p(=2) > 1$.

Hence by comparison test, the given series Σu_n is also convergent.

Example 3. *Test the convergence of the series*

$$\frac{\sqrt{1}}{1+\sqrt{1}} + \frac{\sqrt{2}}{2+\sqrt{2}} + \frac{\sqrt{3}}{3+\sqrt{3}} + \ldots$$

Sol. Let $\Sigma u_n = \frac{\sqrt{1}}{1+\sqrt{1}} + \frac{\sqrt{2}}{2+\sqrt{2}} + \frac{\sqrt{3}}{3+\sqrt{3}} + \ldots$

Then, $$u_n = \frac{\sqrt{n}}{n+\sqrt{n}}$$

$$v_n = \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$$

where v_n is the n^{th} term of auxiliary series Σv_n.

Now, $$\frac{u_n}{v_n} = \frac{\sqrt{n}}{n+\sqrt{n}}\cdot\sqrt{n} = \frac{n}{n+\sqrt{n}}$$

$\therefore$ $$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{n}{n+\sqrt{n}} = \underset{n\to\infty}{\text{Lt}} \frac{1}{1+\frac{1}{\sqrt{n}}} = 1$$

which is a fixed, finite and non-zero quantity.

Hence comparison test can be applied.

But by p-test, we know that the auxiliary series $\Sigma v_n = \Sigma \frac{1}{\sqrt{n}}$ is divergent as $p < 1$.

Hence by comparison test, the given series Σu_n is also divergent.

Example 4. *Test the series:*

$$1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \ldots$$

Sol. Leaving first term as it will not affect the nature of the series and let

$$\Sigma u_n = \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \ldots$$

Then, $$u_n = \frac{n^n}{(n+1)^{n+1}}$$

$$v_n = \frac{n^n}{n^{n+1}} = \frac{1}{n}$$

where v_n is the n^{th} term of auxiliary series Σv_n.

Now, $$\frac{u_n}{v_n} = \frac{n^n}{(n+1)^{n+1}} \cdot n = \frac{n^{n+1}}{(n+1)^{n+1}} = \frac{1}{\left(1+\frac{1}{n}\right)^{n+1}}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{1}{\left(1+\frac{1}{n}\right)^n} \cdot \frac{1}{\left(1+\frac{1}{n}\right)} = \frac{1}{e},$$

which is a fixed, finite and non-zero quantity.
Hence comparison test can be applied.

But, by p-test, we know that the auxiliary series $\Sigma v_n = \Sigma \frac{1}{n}$ is divergent as $p = 1$.

Hence, by comparison test, Σu_n is also divergent.

Example 5. *Test the series:*

$$\frac{2^p}{1^q} + \frac{3^p}{2^q} + \frac{4^p}{3^q} + \cdots$$

Sol. Let $\Sigma u_n = \frac{2^p}{1^q} + \frac{3^p}{2^q} + \frac{4^p}{3^q} + \cdots$

Then, $$u_n = \frac{(n+1)^p}{n^q}$$

$$v_n = \frac{n^p}{n^q}$$

where v_n is the n^{th} term of auxiliary series Σv_n.

Now, $$\frac{u_n}{v_n} = \frac{(n+1)^p}{n^q} \cdot \frac{n^q}{n^p} = \left(1+\frac{1}{n}\right)^p$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \left(1+\frac{1}{n}\right)^p = 1$$

which is a fixed, finite and non-zero quantity.
Hence comparison test can be applied.

But, by p-test, we know that the auxiliary series $\Sigma v_n = \Sigma \frac{1}{n^{q-p}}$ is convergent if $q - p > 1$ divergent if $q - p \leq 1$.

Hence by comparison test, the given series Σu_n is

convergent if $q - p > 1$ or $q > p + 1$

and divergent if $q - p \leq 1$ or $q \leq p + 1$.

Example 6. *Test the series*

$$\frac{1}{1\cdot 2\cdot 3} + \frac{3}{2\cdot 3\cdot 4} + \frac{5}{3\cdot 4\cdot 5} + \ldots$$

Sol. Let $\quad \Sigma u_n = \dfrac{1}{1\cdot 2\cdot 3} + \dfrac{3}{2\cdot 3\cdot 4} + \dfrac{5}{3\cdot 4\cdot 5} + \ldots$

Then, $\quad u_n = \dfrac{2n-1}{n(n+1)(n+2)}$

$$v_n = \frac{n}{n^3} = \frac{1}{n^2}$$

where v_n is the n^{th} term of auxiliary series Σv_n.

Now, $\quad \dfrac{u_n}{v_n} = \dfrac{2n-1}{n(n+1)(n+2)} \cdot n^2 = \dfrac{2 - \frac{1}{n}}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = 2$$

which is a fixed, finite and non-zero quantity.
Hence comparison test can be applied.

But by p-test, we know that the auxiliary series $\Sigma v_n = \Sigma \dfrac{1}{n^2}$ is convergent as $p(=2) > 1$.

Hence, by comparison test, the given series Σu_n is also convergent.

Example 7. *Test the series whose n^{th} term is* $\sqrt{n^3+1} - \sqrt{n^3}$.

Sol. Here, $\quad u_n = \sqrt{n^3+1} - \sqrt{n^3} = n^{3/2}\left[\left(1+\dfrac{1}{n^3}\right)^{\frac{1}{2}} - 1\right]$

$$= n^{3/2}\left[1 + \frac{1}{2}\cdot\frac{1}{n^3} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\cdot\frac{1}{n^6} + \ldots\ldots - 1\right] = \frac{1}{2n^{3/2}} - \frac{1}{8n^{9/2}} + \ldots$$

$$v_n = \frac{1}{n^{3/2}}$$

where v_n is the n^{th} term of auxiliary series Σv_n.

$$\frac{u_n}{v_n} = \frac{1}{2} - \frac{1}{8n^3} + \ldots$$

$$\underset{n \to \infty}{\text{Lt}} \frac{u_n}{v_n} = \frac{1}{2}$$

which is a fixed, finite and non-zero quantity.
Hence comparison test can be applied.

But by p-test, $\Sigma\, v_n = \Sigma \dfrac{1}{n^{3/2}}$ is convergent as $p > 1$.

Hence by comparison test, the given series $\Sigma\, u_n$ is also convergent.

TEST YOUR KNOWLEDGE

Test the convergence of the series (1–17) :

1. $1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \dfrac{1}{\sqrt{4}} + \ldots$

2. $\dfrac{1}{\sqrt{2}-1} + \dfrac{1}{\sqrt{3}-1} + \dfrac{1}{\sqrt{4}-1} + \ldots$

3. $\dfrac{2}{1} + \dfrac{3}{8} + \dfrac{4}{27} + \dfrac{5}{64} + \ldots$

4. $\dfrac{1}{2} + \dfrac{\sqrt{2}}{5} + \dfrac{\sqrt{3}}{10} + \ldots + \dfrac{\sqrt{n}}{n^2+1} + \ldots$

5. $\dfrac{1}{5} + \dfrac{\sqrt{2}}{7} + \dfrac{\sqrt{3}}{9} + \dfrac{\sqrt{4}}{11} + \ldots$

6. $\sqrt{\dfrac{1}{2^3}} + \sqrt{\dfrac{2}{3^3}} + \sqrt{\dfrac{3}{4^3}} + \ldots$

7. $\dfrac{1}{a \cdot 1^2 + b} + \dfrac{2}{a \cdot 2^2 + b} + \dfrac{3}{a \cdot 3^2 + b} + \ldots + \dfrac{n}{a \cdot n^2 + b} + \ldots;$

   a and b are finite and non-zero constants.

8. $\dfrac{2}{1} + \dfrac{3}{4} + \dfrac{4}{9} + \dfrac{5}{16} + \ldots$

9. $1 + \dfrac{1+2}{1+2^2} + \dfrac{1+3}{1+3^2} + \ldots$

10. $\dfrac{1}{1^p} + \dfrac{1}{3^p} + \dfrac{1}{5^p} + \dfrac{1}{7^p} + \ldots$

11. $\dfrac{2}{1^p} + \dfrac{3}{2^p} + \dfrac{4}{3^p} + \dfrac{5}{4^p} + \ldots$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(3n)^p}\,;\, p > 1$

13. $\dfrac{1}{a^2+1} + \dfrac{1}{2a^2+1} + \dfrac{1}{3a^2+1} + \ldots\ (a \neq 0)$

14. $\dfrac{1}{a+b} + \dfrac{1}{a+2b} + \dfrac{1}{a+3b} + \ldots + \dfrac{1}{a+nb} + \ldots$

15. $\dfrac{1}{a(a+b)} + \dfrac{1}{(a+2b)(a+3b)} + \dfrac{1}{(a+4b)(a+5b)} + \ldots;\ a, b \neq 0$

16. $\dfrac{\sqrt{3}}{1 \cdot 2} + \dfrac{\sqrt{5}}{3 \cdot 4} + \dfrac{\sqrt{7}}{5 \cdot 6} + \dfrac{\sqrt{9}}{7 \cdot 8} + \ldots$

17. $\dfrac{1}{(x-d)^2} + \dfrac{1}{(x-2d)^2} + \dfrac{1}{(x-3d)^2} + \ldots$

Test for convergence or divergence of the series whose n^{th} terms (general terms) are (18–40):

18. $\dfrac{1}{na+b}$

19. $\dfrac{2n^2+1}{3n^3+5n^2+6}$

20. $\dfrac{2n+1}{n(n+1)(n+2)}$

21. $\left(\dfrac{n^2+2}{2n^3+15}\right)^{1/3}$

22. $\dfrac{n^p}{(n+1)^{p+q}}$

23. $\sqrt{n+1}-\sqrt{n}$

24. $\dfrac{\sqrt{n+1}-\sqrt{n}}{n^p}$

25. $\sqrt{n^2+1}-n$

26. $\sqrt{n+1}-\sqrt{n-1}$

27. $\sqrt{n^2+1}-\sqrt{n^2-1}$

28. $\dfrac{\sqrt{n^2+1}-n}{n^p}$

29. $\sqrt{n^3+1}-\sqrt{n^3-1}$

30. $\sqrt{n^4+1}-\sqrt{n^4-1}$

31. $\sqrt{n^4+1}-n^2$

32. $(n^3+1)^{1/3}-n$

33. $\dfrac{\sqrt{n^2+n+1}-\sqrt{n^2-n+1}}{n}$

34. $\dfrac{1}{\sqrt{n}+\sqrt{n-1}}$

35. $\dfrac{n}{1+n\sqrt{n+1}}$

36. $\sqrt{\dfrac{n}{2+3n^3}}$

37. $\sin\dfrac{1}{n}$ D

38. $\cos\dfrac{1}{n}$ D

39. $\dfrac{1}{n}\sin\dfrac{1}{n}$ C

40. $\tan^{-1}\dfrac{1}{n}$ D

41. Test the series $\dfrac{1}{x}+\dfrac{1}{x-1}+\dfrac{1}{x+1}+\dfrac{1}{x-2}+\dfrac{1}{x+2}+\ldots$; x being a positive fraction.

Answers

1. Divergent
2. Divergent
3. Convergent
4. Convergent
5. Divergent
6. Divergent
7. Divergent
8. Divergent
9. Divergent
10. Convergent if $p>1$ and divergent if $p\le 1$
11. Convergent if $p>2$ and divergent if $p\le 2$
12. Convergent
13. Divergent
14. Divergent
15. Convergent
16. Convergent
17. Convergent
18. Divergent
19. Divergent
20. Convergent
21. Divergent
22. Convergent if $q>1$ and divergent if $q\le 1$
23. Divergent
24. Convergent if $p>\dfrac{1}{2}$ and divergent if $p\le\dfrac{1}{2}$
25. Divergent
26. Divergent
27. Divergent
28. Convergent if $p>0$ and divergent if $p\le 0$
29. Convergent
30. Convergent
31. Convergent
32. Convergent
33. Divergent
34. Divergent
35. Divergent
36. Convergent
37. Divergent
38. Divergent
39. Convergent
40. Divergent
41. Convergent

3.28 CAUCHY'S ROOT TEST OR RADICAL TEST

Statement. *An infinite series Σu_n of positive terms is convergent if $\underset{n\to\infty}{Lt} (u_n)^{1/n} < 1$, divergent if $\underset{n\to\infty}{Lt} (u_n)^{1/n} > 1$.*

The test fails if $\underset{n\to\infty}{Lt} (u_n)^{1/n} = 1$.

Proof. Let $\Sigma u_n = u_1 + u_2 + u_3 + \ldots + u_n + \ldots$ be the given infinite series of positive terms.

Consider an auxiliary series

$$\Sigma v_n = v_1 + v_2 + v_3 + \ldots v_n + \ldots$$
$$= r + r^2 + r^3 + \ldots + r^n + \ldots$$

which is an infinite geometric progression with common ratio r. We know that Σv_n will be convergent if $r < 1$ and divergent if $r \geq 1$.

Let $\dfrac{u_n}{v_n} = k$ where k is a fixed, finite, non-zero quantity.

then, $\dfrac{u_n}{r^n} = k$

or, $u_n = k\, r^n$

$$(u_n)^{1/n} = k^{1/n}\, r.$$

Then, $\underset{n\to\infty}{\text{Lt}} (u_n)^{1/n} = r$

Now, if $r < 1$ then $\underset{n\to\infty}{\text{Lt}} (u_n)^{1/n} < 1$ and Σv_n will be convergent.

Hence, by comparison test, Σu_n will also be convergent.

Again, if $r > 1$ then $\underset{n\to\infty}{\text{Lt}} (u_n)^{1/n} > 1$ and Σv_n is divergent, hence, by comparison test, Σu_n will also be divergent.

If $r = 1$ then $\underset{n\to\infty}{\text{Lt}} (u_n)^{1/n} = 1$ *i.e.*, r may be infinitesimally less or greater than 1 so nothing can be said about the nature of the series in this case.

Hence this test fails if $r = 1$ *i.e.*, when $\underset{n\to\infty}{\text{Lt}} (u_n)^{1/n} = 1$.

ILLUSTRATIVE EXAMPLES

Example 1. *Test the series:* $\Sigma\left(1 - \dfrac{1}{n}\right)^{n^2}$.

Sol. Here, $u_n = \left(1 - \dfrac{1}{n}\right)^{n^2}$

$$(u_n)^{1/n} = \left(1 - \frac{1}{n}\right)^n$$

$$\underset{n\to\infty}{\text{Lt}}\ (u_n)^{1/n} = \underset{n\to\infty}{\text{Lt}}\left[1 + \frac{(-1)}{n}\right]^n = \frac{1}{e} < 1$$

Hence by Cauchy's root test, $\Sigma\, u_n$ is convergent.

Example 2. *Test the series :*

$$\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \ldots$$

Sol. Let the given series be denoted by $\Sigma\, u_n$. Then,

$$u_n = \left[\frac{(n+1)^{n+1}}{n^{n+1}} - \frac{n+1}{n}\right]^{-n}$$

$$(u_n)^{1/n} = \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]^{-1} = \left(\frac{n+1}{n}\right)^{-1}\left[\left(\frac{n+1}{n}\right)^n - 1\right]^{-1}$$

$$= \left(1 + \frac{1}{n}\right)^{-1}\left[\left(1 + \frac{1}{n}\right)^n - 1\right]^{-1}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\ (u_n)^{1/n} = \frac{1}{e-1} < 1 \qquad (\because\ e = 2.718)$$

Hence, by Cauchy's root test, Σu_n is convergent.

Example 3. *Test the series :*

$$\frac{1}{2} + \frac{2}{3}x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3 + \ldots$$

Sol. Neglecting the first term, as it will not affect the nature of the series.

Let the given series be denoted by Σu_n. Then,

$$u_n = \left(\frac{n+1}{n+2}\right)^n x^n$$

$$\therefore \quad (u_n)^{1/n} = \left(\frac{n+1}{n+2}\right)x$$

$$\underset{n\to\infty}{\text{Lt}}\ (u_n)^{1/n} = \underset{n\to\infty}{\text{Lt}}\ \frac{1 + \frac{1}{n}}{1 + \frac{2}{n}} \cdot x = x$$

Hence, by Cauchy's root test,

$\Sigma\, u_n$ is convergent if $x < 1$ and divergent if $x > 1$.

When $x = 1$, $\quad u_n = \left(\dfrac{n+1}{n+2}\right)^n$

$$\underset{n\to\infty}{\text{Lt}}\ u_n = \underset{n\to\infty}{\text{Lt}} \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n} = \frac{1}{e} \neq 0$$

Hence by limit u_n test, Σu_n is divergent.

Therefore, Σu_n is convergent if $x < 1$ and divergent if $x \geq 1$.

TEST YOUR KNOWLEDGE

Test the convergence of the series (1 – 8) whose n^{th} term is:

1. $\left(1+\frac{1}{n}\right)^{n^2}$
2. $\left(1+\frac{1}{n}\right)^{-n^2}$
3. $\left[\log_e\left(1+\frac{1}{n}\right)\right]^n$
4. $(n^{1/n}-1)^n$
5. $\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$
6. $\frac{n^{n^2}}{(n+1)^{n^2}}$
7. $\frac{n^{n^2}}{\left(n+\frac{1}{4}\right)^{n^2}}$
8. $\left(\frac{1+nx}{n}\right)^n$

Test the series :

9. $\sum_{n=2}^{\infty}\left(\frac{1}{\log n}\right)^n$
10. $1+\frac{x}{2}+\frac{x^2}{3^2}+\frac{x^3}{4^3}+\ldots$
11. $\sum_{n=1}^{\infty}\frac{(n+1)^n\, x^n}{n^{n+1}}$
12. (i) $a+b+a^2+b^2+a^3+b^3+\ldots$; a and b are positive numbers.

    (ii) $\frac{1}{3}+\frac{1}{7}+\cdots+\frac{1}{S_n}+\ldots$

    where, $S_n = 1+2+2^2+\ldots+2^{n-1}$.

Answers

1. Divergent
2. Convergent
3. Convergent
4. Convergent
5. Convergent.
6. Convergent
7. Convergent
8. Convergent if $x < 1$ and divergent if $x \geq 1$
9. Convergent
10. Convergent
11. Convergent if $x < 1$ and divergent if $x \geq 1$
12. (i) Convergent when $0 < a < 1$ and $0 < b < 1$ and divergent otherwise

    (ii) Convergent

3.29 D' ALEMBERT'S TEST OR RATIO TEST

Statement. *An infinite series Σu_n of positive terms is convergent if from and after some fixed terms $\frac{u_{n+1}}{u_n} < k < 1$,*

where k is fixed number and is divergent if $\frac{u_{n+1}}{u_n} > 1$.

Proof. After leaving a finite number of terms, if necessary, let the given series be denoted by

$$\Sigma u_n = u_1 + u_2 + u_3 + \ldots$$

and let $\dfrac{u_2}{u_1} < k$

$$\frac{u_3}{u_2} < k$$

$$\frac{u_4}{u_3} < k$$

.
.
.

and so on. (where $k < 1$)

Now, $u_1 + u_2 + u_3 + u_4 + \ldots$

$$= u_1\left[1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \frac{u_4}{u_3}\cdot\frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \ldots\right]$$

$$< u_1(1 + k + k^2 + k^3 + \ldots)$$

$$< \frac{u_1}{1-k} \quad (\text{as } k < 1)$$

Hence the given series is convergent.

Again, let $\dfrac{u_2}{u_1} > 1,\ \dfrac{u_3}{u_2} > 1,\ \dfrac{u_4}{u_3} > 1, \ldots$ and so on.

then, $u_2 > u_1$

$u_3 > u_2$

$u_4 > u_3$ ... and so on.

i.e., $S_n = u_1 + u_2 + u_3 + \ldots + u_n > nu_1$

$\therefore$ $\underset{n\to\infty}{\text{Lt}}\ S_n > \underset{n\to\infty}{\text{Lt}}\ nu_1$

i.e., $\underset{n\to\infty}{\text{Lt}}\ S_n \to \infty$

Hence the series Σu_n is divergent.

When $\underset{n\to\infty}{\text{Lt}}\ \dfrac{u_{n+1}}{u_n} = 1$, it means that $\dfrac{u_{n+1}}{u_n}$ may be infinitesimally less or greater than 1 so nothing can be said about the nature of the series in this case hence this test fails when $\underset{n\to\infty}{\text{Lt}}\ \dfrac{u_{n+1}}{u_n} = 1$ and we determine the convergence or divergence of the given series by some other suitable test in this case.

Remark. In practice, ratio test is applied in its following form:
The series Σu_n of positive terms is,

convergent if $\displaystyle \operatorname*{Lt}_{n \to \infty} \frac{u_n}{u_{n+1}} > 1,$

divergent if $\displaystyle \operatorname*{Lt}_{n \to \infty} \frac{u_n}{u_{n+1}} < 1$

and test fails if $\displaystyle \operatorname*{Lt}_{n \to \infty} \frac{u_n}{u_{n+1}} = 1.$

ILLUSTRATIVE EXAMPLES

Example 1. *Test the series:*

$$\frac{2}{1^2+1} + \frac{2^2}{2^2+1} + \frac{2^3}{3^2+1} + \cdots$$

Sol. Let the given series be denoted by Σu_n. Then,

$$u_n = \frac{2^n}{n^2+1}$$

$$\therefore \qquad u_{n+1} = \frac{2^{n+1}}{(n+1)^2+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{2^n}{n^2+1} \cdot \frac{(n+1)^2+1}{2^{n+1}} = \frac{\left(1+\frac{1}{n}\right)^2 + \frac{1}{n^2}}{1+\frac{1}{n^2}} \cdot \frac{1}{2}$$

$$\therefore \qquad \operatorname*{Lt}_{n \to \infty} \frac{u_n}{u_{n+1}} = \frac{1}{2} < 1$$

Hence by ratio test, Σu_n is divergent.

Example 2. *Test the series*

$$1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \cdots + \frac{x^n}{n^2+1} + \cdots$$

Sol. Leaving first term as it will not affect the nature of the series and let the given series be denoted by Σu_n then,

$$u_n = \frac{x^n}{n^2+1}$$

$$u_{n+1} = \frac{x^{n+1}}{(n+1)^2+1}$$

$$\therefore \qquad \frac{u_n}{u_{n+1}} = \frac{x^n}{n^2+1} \cdot \frac{(n+1)^2+1}{x^{n+1}} = \frac{\left(1+\frac{1}{n}\right)^2 + \frac{1}{n^2}}{1+\frac{1}{n^2}} \cdot \frac{1}{x}$$

$$\operatorname*{Lt}_{n \to \infty} \frac{u_n}{u_{n+1}} = \frac{1}{x}$$

Hence by ratio test, $\Sigma\, u_n$ is

convergent if $\quad \frac{1}{x} > 1 \quad$ or $\quad x < 1$

divergent if $\quad \frac{1}{x} < 1 \quad$ or $\quad x > 1$

and the test fails if $\frac{1}{x} = 1$ or $x = 1$

When $x = 1$, $\qquad u_n = \frac{1}{n^2+1}$

$$v_n = \frac{1}{n^2}$$

where v_n is n^{th} term of auxiliary series $\Sigma\, v_n$.

$$\frac{u_n}{v_n} = \frac{n^2}{n^2+1} = \frac{1}{1+\frac{1}{n^2}}$$

$$\operatorname*{Lt}_{n \to \infty} \frac{u_n}{u_{n+1}} = 1$$

which is a fixed, finite, non-zero quantity. Hence comparison test can be applied.

But, by p-test, $\Sigma \frac{1}{n^2}$ is convergent as $p = 2\ (> 1)$.

Hence by comparison test, $\Sigma\, u_n$ is also convergent.

Finally, $\Sigma\, u_n$ is convergent if $x \le 1$ and divergent if $x > 1$.

Example 3. *Test the series:*

$$\frac{1}{1 \cdot 2 \cdot 3} + \frac{x}{4 \cdot 5 \cdot 6} + \frac{x^2}{7 \cdot 8 \cdot 9} + \ldots$$

Sol. Let the given series be denoted by $\Sigma\, u_n$. Then,

$$u_n = \frac{x^{n-1}}{(3n-2)(3n-1)3n}$$

$$u_{n+1} = \frac{x^n}{(3n+1)(3n+2)(3n+3)}$$

$$\frac{u_n}{u_{n+1}} = \frac{(3n+1)(3n+2)(3n+3)}{(3n-2)(3n-1)3n} \cdot \frac{1}{x}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{\left(3+\frac{1}{n}\right)\left(3+\frac{2}{n}\right)\left(1+\frac{1}{n}\right)}{\left(3-\frac{2}{n}\right)\left(3-\frac{1}{n}\right)} \cdot \frac{1}{x} = \frac{1}{x}$$

Hence by ratio test, Σu_n is

convergent if $\frac{1}{x} > 1$ *i.e.* $x < 1$

divergent if $\frac{1}{x} < 1$ *i.e.* $x > 1$

and the test fails if $\frac{1}{x} = 1$ *i.e.* $x = 1$

When $x = 1$, $u_n = \frac{1}{(3n-2)(3n-1)3n}$

$$v_n = \frac{1}{n^3}$$

where v_n is n^{th} term of auxiliary series Σv_n.

$$\frac{u_n}{v_n} = \frac{n^3}{(3n-2)(3n-1)3n}$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{1}{3\left(3-\frac{1}{n}\right)\left(3-\frac{2}{n}\right)} = \frac{1}{27}$$

which is a fixed, finite and non-zero quantity.
Hence comparison test can be applied.

But, $\Sigma \frac{1}{n^3}$ is convergent as $p(=3) > 1$ by p-test.

Hence the given series Σu_n is also convergent.
Finally, Σu_n is convergent if $x \le 1$ and divergent if $x > 1$.

TEST YOUR KNOWLEDGE

Test the series (1 – 28):

1. $\frac{1}{1+2} + \frac{2}{1+2^2} + \frac{3}{1+2^3} + \ldots$

2. $\frac{1!}{5} + \frac{2!}{5^2} + \frac{3!}{5^3} + \ldots$

3. $\frac{2!}{3} + \frac{3!}{3^2} + \frac{4!}{3^3} + \ldots$

4. $1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \ldots$

5. $1+\frac{2^p}{2!}+\frac{3^p}{3!}+\frac{4^p}{4!}+\ldots$

6. $1+2x+3x^2+4x^3+\ldots$

7. $1+3x+5x^2+7x^3+\ldots$

8. $\Sigma\frac{n+1}{n^2}x^n$

9. $x+\frac{3}{5}x^2+\frac{8}{10}x^3+\frac{15}{17}x^4+\ldots+\frac{n^2-1}{n^2+1}x^n+\ldots$

10. $1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots$

11. $\frac{2}{1!}+\frac{3}{2!}+\frac{4}{3!}+\frac{5}{4!}+\ldots$

12. $1+\frac{3}{2!}+\frac{5}{3!}+\frac{7}{4!}+\ldots$

13. $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$

14. $1^2+2^2x+3^2x^2+4^2x^3+\ldots$

15. $2x+\frac{3x^2}{8}+\frac{4}{27}x^3+\ldots+\frac{n+1}{n^3}x^n+\ldots$

16. $1+\frac{x^2}{2}+\frac{x^4}{4}+\frac{x^6}{6}+\ldots\ (x>0)$

17. $\frac{x}{1\cdot 2}+\frac{x^2}{3\cdot 4}+\frac{x^3}{5\cdot 6}+\frac{x^4}{7\cdot 8}+\ldots$

18. $\frac{1}{2\sqrt{1}}+\frac{x^2}{3\sqrt{2}}+\frac{x^4}{4\sqrt{3}}+\frac{x^6}{5\sqrt{4}}+\ldots$

19. $\Sigma\left(\sqrt{n^2+1}-n\right)x^{2n}$

20. $\Sigma\left(\frac{n^3+a}{2^n+a}\right)$

21. $\frac{1}{x-1}+\frac{x}{x^2-1}+\frac{x^2}{x^3-1}+\ldots\ (x>1)$

22. $\Sigma\frac{3n-1}{2^n}$

23. $\Sigma\frac{n!}{n^n}$

24. $\Sigma(3n-1)2^n$

25. $\Sigma\frac{(2x)^n}{n^2},\ 0<x<1$

26. $\Sigma\frac{x^n}{n(n+1)}$

27. $\Sigma\frac{x^n}{a+\sqrt{n}}$

28. $\Sigma\frac{1}{x^n+x^{-n}}$

29. Show that the series

$$1+\frac{x\log a}{1!}+\frac{x^2(\log a)^2}{2!}+\ldots$$ is convergent for all values of x.

30. Show that the series

$$1+\frac{\alpha+1}{\beta+1}+\frac{(\alpha+1)(2\alpha+1)}{(\beta+1)(2\beta+1)}+\frac{(\alpha+1)(2\alpha+1)(3\alpha+1)}{(\beta+1)(2\beta+1)(3\beta+1)}+\ldots$$

converges if $\beta>\alpha>0$ and diverges if $\alpha\geq\beta>0$. Given that, $\alpha>0,\beta>0$.

Answers

1. Convergent 2. Divergent 3. Divergent 4. Convergent
5. Convergent 6. Convergent if $x < 1$ and divergent if $x \geq 1$
7. Convergent if $x < 1$ and divergent if $x \geq 1$ 8. Convergent if $x < 1$ and divergent if $x \geq 1$
9. Convergent if $x < 1$ and divergent if $x \geq 1$ 10. Convergent 11. Convergent
12. Convergent 13. Convergent for all values of x
14. Convergent if $x < 1$ and divergent if $x \geq 1$ 15. Convergent if $x \leq 1$ and divergent if $x > 1$
16. Convergent if $x^2 < 1$ and divergent if $x^2 \geq 1$ 17. Convergent if $x \leq 1$ and divergent if $x > 1$
18. Convergent if $x^2 \leq 1$ and divergent if $x^2 > 1$ 19. Convergent if $x^2 < 1$ and divergent if $x^2 \geq 1$
20. Convergent 21. Divergent 22. Convergent 23. Convergent
24. Divergent 25. Convergent if $x \leq \frac{1}{2}$ and divergent if $x > \frac{1}{2}$
26. Convergent if $x \leq 1$ and divergent if $x > 1$
27. Convergent if $x < 1$ and divergent if $x \geq 1$
28. Convergent if $x > 1$ or $x < 1$ and divergent if $x = 1$.

3.30 AN IMPORTANT COMPARISON TEST

Statement. If Σu_n and Σv_n are two series of positive terms and if from and after some fixed terms,

(*i*) $\dfrac{u_{n+1}}{u_n} < \dfrac{v_{n+1}}{v_n}$ for all values of n then Σu_n will be convergent if Σv_n is convergent and if,

(*ii*) $\dfrac{u_{n+1}}{u_n} > \dfrac{v_{n+1}}{v_n}$ for all values of n then Σu_n will be divergent if Σv_n is divergent.

Note. For the sake of convenience, the above results are used in the following form:

(*a*) If $\dfrac{u_n}{u_{n+1}} > \dfrac{v_n}{v_{n+1}}$ and Σv_n is convergent then Σu_n is also convergent.

(*b*) If $\dfrac{u_n}{u_{n+1}} > \dfrac{v_n}{v_{n+1}}$ and Σv_n is divergent then Σu_n is also divergent.

3.31 RAABE'S TEST

Statement. *The series Σu_n of positive terms is*

convergent if $\quad \underset{n \to \infty}{Lt}\; n\left(\dfrac{u_n}{u_{n+1}} - 1\right) > 1,$

divergent if $\quad \underset{n \to \infty}{Lt}\; n\left(\dfrac{u_n}{u_{n+1}} - 1\right) < 1$

This test fails if $\quad \underset{n \to \infty}{Lt}\; n\left(\dfrac{u_n}{u_{n+1}} - 1\right) = 1.$

Proof. The given series is $\Sigma u_n = u_1 + u_2 + u_3 + \dots$

Compare this series with auxiliary series $\Sigma v_n = \Sigma \frac{1}{n^p}$ which by p-test, is convergent or divergent according as $p > 1$ or $p < 1$.

So, $$v_n = \frac{1}{n^p}$$

$\therefore$ $$v_{n+1} = \frac{1}{(n+1)^p}$$

$$\frac{v_n}{v_{n+1}} = \left(\frac{n+1}{n}\right)^p = \left(1+\frac{1}{n}\right)^p$$

$$= 1 + \frac{p}{n} + \frac{p(p-1)}{2!} \cdot \frac{1}{n^2} + \dots \qquad | \text{ By Binomial theorem}$$

$$n\left(\frac{v_n}{v_{n+1}} - 1\right) = p + \frac{p(p-1)}{2!} \cdot \frac{1}{n} + \dots$$

$$\underset{n \to \infty}{\text{Lt}}\ n\left(\frac{v_n}{v_{n+1}} - 1\right) = p \qquad \dots(1)$$

Now, let $\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$ then from eqn. (1),

$$\underset{n \to \infty}{\text{Lt}}\ n\left(\frac{u_n}{u_{n+1}} - 1\right) > p$$

Again, if Σv_n is convergent then $p > 1$

Hence, Σu_n is convergent if $\underset{n \to \infty}{\text{Lt}}\ n\left(\frac{u_n}{u_{n+1}} - 1\right) > 1$

Further, let $\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}}$ then from eqn. (1),

$$\underset{n \to \infty}{\text{Lt}}\ n\left(\frac{u_n}{u_{n+1}} - 1\right) < p$$

Again, if Σv_n is divergent then $p < 1$

Hence, Σu_n is convergent if $\underset{n \to \infty}{\text{Lt}}\ n\left(\frac{u_n}{u_{n+1}} - 1\right) < 1$.

Note. Raabe's test is applied only when ratio test fails, *i.e.*, when $\underset{n \to \infty}{\text{Lt}}\ \frac{u_n}{u_{n+1}} = 1$.

3.32 LOGARITHMIC TEST

Statement. *The series Σu_n of positive terms is*

convergent if $\quad \underset{n \to \infty}{Lt} \; n \log \dfrac{u_n}{u_{n+1}} > 1,$

divergent if $\quad \underset{n \to \infty}{Lt} \; n \log \dfrac{u_n}{u_{n+1}} < 1$

This test fails if $\quad \underset{n \to \infty}{Lt} \; n \log \dfrac{u_n}{u_{n+1}} = 1.$

Proof. The given series is $\Sigma u_n = u_1 + u_2 + u_3 + \ldots$

Compare this series with auxiliary series $\Sigma v_n = \Sigma \dfrac{1}{n^p}$ which by p-test, is convergent or divergent according as $p > 1$ or $p < 1$.

So, $$v_n = \frac{1}{n^p}$$

$\therefore$ $$v_{n+1} = \frac{1}{(n+1)^p}$$

$$\frac{v_n}{v_{n+1}} = \left(\frac{n+1}{n}\right)^p = \left(1 + \frac{1}{n}\right)^p$$

$$\log \frac{v_n}{v_{n+1}} = p \log\left(1 + \frac{1}{n}\right) = p\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots\right]$$

or $$n \log \frac{v_n}{v_{n+1}} = p\left(1 - \frac{1}{2n} + \frac{1}{3n^2} - \ldots\right)$$

$$\underset{n \to \infty}{\text{Lt}} \; n \log \frac{v_n}{v_{n+1}} = p \qquad \ldots(1)$$

Now, let $\dfrac{u_n}{u_{n+1}} > \dfrac{v_n}{v_{n+1}}$ then from eqn. (1),

$$\underset{n \to \infty}{\text{Lt}} \; n \log \frac{u_n}{u_{n+1}} > p$$

Again, if Σv_n is convergent then $p > 1$.

Hence Σu_n is convergent if $\underset{n \to \infty}{\text{Lt}} \; n \log \dfrac{u_n}{u_{n+1}} > 1$

Further, let $\dfrac{u_n}{u_{n+1}} < \dfrac{v_n}{v_{n+1}}$, then from eqn. (1)

$$\underset{n \to \infty}{\text{Lt}}\ n \log \frac{u_n}{u_{n+1}} < p$$

Again, if Σv_n is divergent then $p < 1$

Hence, Σu_n is divergent $\underset{n \to \infty}{\text{Lt}}\ n \log \dfrac{u_n}{u_{n+1}} < 1$.

Note. Logarithmic test is applied when ratio test fails and is preferred when e or powers are involved.

ILLUSTRATIVE EXAMPLES

Example 1. *Test the convergence of the series:*

$$1 + \frac{2}{3} \cdot \frac{1}{4} + \frac{2 \cdot 4}{3 \cdot 5} \cdot \frac{1}{6} + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7} \cdot \frac{1}{8} + \ldots$$

Sol. Leaving the first term, let the given series be denoted by $\Sigma\, u_n$. Then,

$$u_n = \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{3 \cdot 5 \cdot 7 \cdots (2n+1)} \cdot \frac{1}{2n+2}$$

$$\therefore \qquad u_{n+1} = \frac{2 \cdot 4 \cdot 6 \cdots (2n)(2n+2)}{3 \cdot 5 \cdot 7 \cdots (2n+1)(2n+3)} \cdot \frac{1}{2n+4}$$

$$\frac{u_n}{u_{n+1}} = \frac{2n+3}{2n+2} \cdot \frac{2n+4}{2n+2} = \frac{\left(2 + \frac{3}{n}\right)\left(2 + \frac{4}{n}\right)}{\left(2 + \frac{2}{n}\right)\left(2 + \frac{2}{n}\right)}$$

$$\therefore \qquad \underset{n \to \infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = 1.$$

Hence Ratio test fails.

Now, $$\frac{u_n}{u_{n+1}} - 1 = \frac{(2n+3)(2n+4)}{(2n+2)(2n+2)} - 1 = \frac{6n+8}{(2n+2)^2}$$

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{6n^2 + 8n}{(2n+2)^2}$$

$$\underset{n \to \infty}{\text{Lt}}\ n\left(\frac{u_n}{u_{n+1}} - 1\right) = \underset{n \to \infty}{\text{Lt}} \frac{6 + \frac{8}{n}}{\left(2 + \frac{2}{n}\right)^2} = \frac{3}{2}\ (> 1)$$

Hence, by Raabe's test, Σu_n is convergent.

Example 2. *Test for convergence of the series*

$$1 + \frac{x}{2} + \frac{1 \cdot 3}{2 \cdot 4} x^2 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} x^3 + \ldots,\ x > 0$$

Sol. Neglecting the first term, let the given series be denoted by Σu_n. Then,

$$u_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} x^n$$

$$\therefore \qquad u_{n+1} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)}{2 \cdot 4 \cdot 6 \cdots (2n)(2n+2)} x^{n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{2n+2}{2n+1} \cdot \frac{1}{x}$$

$$\operatorname*{Lt}_{n \to \infty} \frac{u_n}{u_{n+1}} = \operatorname*{Lt}_{n \to \infty} \frac{2 + \frac{2}{n}}{2 + \frac{1}{n}} \cdot \frac{1}{x} = \frac{1}{x}$$

Hence by ratio test, Σu_n is

convergent if $\qquad \frac{1}{x} > 1 \qquad$ *i.e.,* $\qquad x < 1$

divergent if $\qquad \frac{1}{x} < 1 \qquad$ *i.e.,* $\qquad x > 1$

and the test fails if $\qquad \frac{1}{x} = 1 \qquad$ *i.e.,* $\qquad x = 1.$

when $x = 1$,

$$\frac{u_n}{u_{n+1}} - 1 = \frac{2n+2}{2n+1} - 1 = \frac{1}{2n+1}$$

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{n}{2n+1} = \frac{1}{2 + \frac{1}{n}}$$

$$\therefore \qquad \operatorname*{Lt}_{n \to \infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{1}{2} < 1$$

Hence the given series is divergent when $x = 1$ by Raabe's test.

Finally, Σu_n is convergent if $x < 1$ and divergent if $x \geq 1$.

Example 3. *Test the series:*

$$x + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \frac{4^4 x^4}{4!} + \frac{5^5 x^5}{5!} + \ldots, x > 0.$$

Sol. Let the given series be denoted by Σu_n then,

$$u_n = \frac{n^n x^n}{n!}$$

$$\therefore \qquad u_{n+1} = \frac{(n+1)^{n+1} x^{n+1}}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}} = \frac{n^n}{n!} \cdot \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{1}{x}$$

$$= \frac{n^n}{(n+1)^n} \cdot \frac{1}{x} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \cdot \frac{1}{x}$$

$$\underset{n \to \infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{1}{ex}$$

Hence by ratio test, $\Sigma\, u_n$ is

convergent if $\frac{1}{ex} > 1$ *i.e.,* $x < \frac{1}{e}$

divergent if $\frac{1}{ex} < 1$ *i.e.,* $x > \frac{1}{e}$

and the test fails if $\frac{1}{ex} = 1$ *i.e.,* $x = \frac{1}{e}$.

when $x = \frac{1}{e}$,

$$\frac{u_n}{u_{n+1}} = \frac{e}{\left(1+\frac{1}{n}\right)^n}$$

Applying logarithmic test, we have

$$\log \frac{u_n}{u_{n+1}} = \log\left\{\frac{e}{\left(1+\frac{1}{n}\right)^n}\right\} = \log e - n \log\left(1+\frac{1}{n}\right)$$

$$= 1 - n\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots\right] = \frac{1}{2n} - \frac{1}{3n^2} + \ldots$$

$$n \log \frac{u_n}{u_{n+1}} = \frac{1}{2} - \frac{1}{3n} + \ldots$$

$$\therefore \quad \underset{n \to \infty}{\text{Lt}}\, n \log \frac{u_n}{u_{n+1}} = \frac{1}{2} < 1.$$

Hence by logarithmic test, Σu_n is divergent.

TEST YOUR KNOWLEDGE

Find whether the following series are convergent or divergent:

1. $\frac{1^2}{4^2}+\frac{1^2\cdot 5^2}{4^2\cdot 8^2}+\frac{1^2\cdot 5^2\cdot 9^2}{4^2\cdot 8^2\cdot 12^2}+\frac{1^2\cdot 5^2\cdot 9^2\cdot 13^2}{4^2\cdot 8^2\cdot 12^2\cdot 16^2}+\ldots$

2. $1+\frac{2^2}{3\cdot 4}+\frac{2^2\cdot 4^2}{3\cdot 4\cdot 5\cdot 6}+\frac{2^2\cdot 4^2\cdot 6^2}{3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}+\ldots$

3. $\frac{x}{1}+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7}+\ldots$

4. $1+\frac{3}{7}x+\frac{3\cdot 6}{7\cdot 10}x^2+\frac{3\cdot 6\cdot 9}{7\cdot 10\cdot 13}x^3+\ldots, x>0$

5. $2+\frac{3}{2}x+\frac{4}{3}x^2+\frac{5}{4}x^3+\ldots, x>0$

6. $1+\frac{1}{2}\cdot\frac{x^2}{4}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^4}{8}+\frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8\cdot 10}\cdot\frac{x^6}{12}+\ldots$

7. $x^2+\frac{2^2}{3\cdot 4}x^4+\frac{2^2\cdot 4^2}{3\cdot 4\cdot 5\cdot 6}x^6+\frac{2^2\cdot 4^2\cdot 6^2}{3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}x^8+\ldots$

8. $\Sigma\frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots 2n}\cdot\frac{x^{2n}}{2n}$

9. $\Sigma\frac{1\cdot 2\cdot 3\ldots n}{4\cdot 7\cdot 10\ldots(3n+1)}x^n$

10. $x\log x+x^2\log 2x+x^3\log 3x+\ldots+x^n\log nx+\ldots$

11. $x^2(\log 2)^q+x^3(\log 3)^q+x^4(\log 4)^q+\ldots$

12. $1+a+\frac{a(a+1)}{1\cdot 2}+\frac{a(a+1)(a+2)}{1\cdot 2\cdot 3}+\ldots$

13. $\Sigma\frac{a^n}{a^n+x^n}\ (a\neq 0)$

14. $x\log x+x^2\log(2x)+x^3\log(3x)+\ldots+x^n\log(nx)+\ldots$

15. $1+\frac{x}{2}+\frac{2!}{3^2}x^2+\frac{3!}{4^3}x^3+\ldots$

16. $1+\frac{2x}{2!}+\frac{3^2x^2}{3!}+\frac{4^3x^3}{4!}+\ldots, x>0$

17. $x^2(\log 2)^q+x^3(\log 3)^q+x^4(\log 4)^q+\ldots$

18. (*i*) $\Sigma\left(\frac{nx}{n+1}\right)^n$ (*ii*) $\frac{a+x}{1!}+\frac{(a+2x)^2}{2!}+\frac{(a+3x)^3}{3!}+\ldots$

Answers

1. Convergent
2. Convergent
3. Convergent if $x^2\le 1$ and divergent if $x^2>1$
4. Convergent if $x\le 1$ and divergent if $x>1$
5. Convergent if $x<1$ and divergent if $x\ge 1$
6. Convergent if $x^2\le 1$ and divergent if $x^2>1$
7. Convergent if $x^2\le 1$ and divergent if $x^2>1$
8. Convergent if $x^2\le 1$ and divergent if $x^2>1$
9. Convergent if $x<3$ and divergent if $x\ge 3$
10. Convergent if $x<1$ and divergent if $x\ge 1$
11. Convergent if $x<1$ and divergent if $x\ge 1$

12. Convergent if $a \le 0$ and divergent if $a > 0$
13. Convergent if $x > a$ and divergent if $x \le a$
14. Convergent if $x < 1$ and divergent if $x \ge 1$
15. Convergent if $x < e$ and divergent if $x \ge e$
16. Convergent if $x \le \frac{1}{e}$ and divergent if $x > \frac{1}{e}$
17. Convergent if $x < 1$ and divergent if $x \ge 1$
18. (*i*) Convergent if $x < 1$ and divergent if $x \ge 1$ (*ii*) Convergent if $x < \frac{1}{e}$ and divergent if $x \ge \frac{1}{e}$.

3.33 PERIODIC FUNCTIONS

A function $f(x)$ which satisfies the relation $f(x + T) = f(x)$ for all real x and some fixed T is called a periodic function. The *smallest positive number* T, for which this relation holds, is called the **period** of $f(x)$.

If T is the period of $f(x)$, then $f(x) = f(x + T) = f(x + 2T) = \ldots\ldots = f(x + nT) = \ldots\ldots$

Also, $f(x) = f(x - T) = f(x - 2T) = \ldots\ldots = f(x - nT) = \ldots\ldots$

$\therefore$ $f(x) = f(x \pm nT)$, where n is a positive integer.

Thus, $f(x)$ repeats itself after periods of T.

For example, sin x, cos x, sec x and cosec x are periodic functions with period 2π.

Since,
$$\tan(\theta + \pi) = \frac{\sin(\pi + \theta)}{\cos(\pi + \theta)} = \frac{-\sin\theta}{-\cos\theta} = \tan\theta$$

and
$$\cot(\theta + \pi) = \frac{\cos(\pi + \theta)}{\sin(\pi + \theta)} = \frac{-\cos\theta}{-\sin\theta} = \cot\theta.$$

Therefore tan θ and cot θ are periodic functions with period π.

The functions sin nx and cos nx are periodic with period $\frac{2\pi}{n}$.

The sum of a number of periodic functions is also periodic. If T_1 and T_2 are the periods of $f(x)$ and $g(x)$, then the period of $a\,f(x) + b\,g(x)$ is the least common multiple of T_1 and T_2.

For example, cos x, cos $2x$, cos $3x$ are periodic functions with periods 2π, π and $\frac{2\pi}{3}$ respectively.

$\therefore$ $f(x) = \cos x + \frac{1}{2}\cos 2x + \frac{1}{3}\cos 3x$ is also periodic with period 2π, the LCM of 2π, π and $\frac{2\pi}{3}$.

3.34 FOURIER SERIES

Expansion of a function $f(x)$ in a series of sines and cosines of multiples of x was developed by French Mathematician and physicist Jacques Fourier.

We have seen how a function can be expanded inpowers of x by Maclaurin's theorem but that expansion was possible only when the function and its derivatives are continuous. A need arises to expand functions which have discontinuities in their values or derivatives.

By Fourier series, we can expand both type of functions under certain conditions as an infinite series of sines and cosines of x and its integral multiples.

Fourier series for the function $f(x)$ in the interval $c < x < c + 2\pi$ is given by

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx}$$

where $$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx$$

Above formulae are also called **Euler's formulae.** Constants a_0, a_n and b_n are called **Fourier coefficients** of $f(x)$.

Remark. To write $\frac{a_0}{2}$ instead of a_0 is a conventional device to be able to get more symmetric formulae for the coefficients.

Note. To determine a_0, a_n and b_n, we shall need the following results : (m and n are integers)

(i) $\int_c^{c+2\pi} \sin nx\,dx = \left(\frac{-\cos nx}{n}\right)_c^{c+2\pi} = 0, n \neq 0$; $\int_c^{c+2\pi} \cos nx\,dx = \left(\frac{\sin nx}{n}\right)_c^{c+2\pi} = 0, n \neq 0$

(ii) $\int_c^{c+2\pi} \sin mx \cos nx\,dx = 0, m \neq n$ (iii) $\int_c^{c+2\pi} \cos mx \cos nx\,dx = 0, m \neq n$

(iv) $\int_c^{c+2\pi} \sin mx \sin nx\,dx = 0, m \neq n$

(v) $\int_c^{c+2\pi} \cos^2 nx\,dx = \pi, n \neq 0$; $\int_c^{c+2\pi} \sin^2 nx\,dx = \pi, n \neq 0$

(vi) $\int_c^{c+2\pi} \sin nx \cos nx\,dx = \frac{1}{2}\int_c^{c+2\pi} \sin 2nx\,dx = 0, n \neq 0$

(vii) $\int e^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx) + c$

(viii) $\int e^{ax} \cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx) + c$

(ix) To integrate the product of two functions, one of which is a power of x, we apply the *generalised rule of integration by parts.* If dashes denote differentiation and suffixes denote integration w.r.t. x, the rule can be stated as follows:

$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\ldots$, where u and v are functions of x.

i.e., Integral of the product of two functions

= I function × integral of II – go on differentiating I, integrating II, signs alternately +ve and –ve.

[Simplification should be done only when the integration is over.]

e.g.,

$$\int x^2 \cos nx\,dx = x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)$$

$$= \frac{x^2}{n}\sin nx + \frac{2x}{n^2}\cos nx - \frac{2}{n^3}\sin nx.$$

(x) $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$

$$\sin\left(n+\frac{1}{2}\right)\pi = (-1)^n \text{ and } \cos\left(n+\frac{1}{2}\right)\pi = 0,$$ where n is an integer.

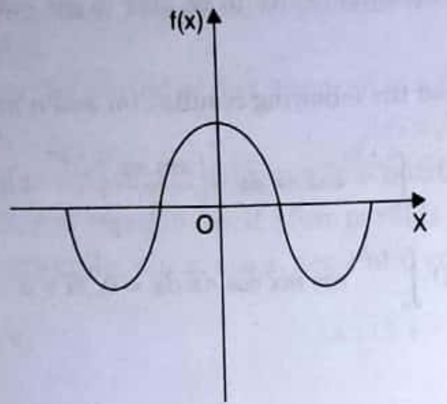
(xi) Even and Odd Functions

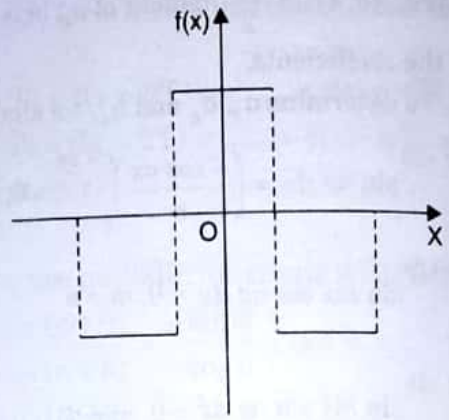
A function $f(x)$ is said to be *even* if $f(-x) = f(x)$ *e.g.*, x^2, $\cos x$, $\sin^2 x$ are even functions.

The graph of an even function is symmetrical about the y-axis.

Here y-axis is a mirror for the reflection of the curve.

$$\int_{-\pi}^{\pi} f(x)\,dx = 2\int_0^{\pi} f(x)\,dx$$



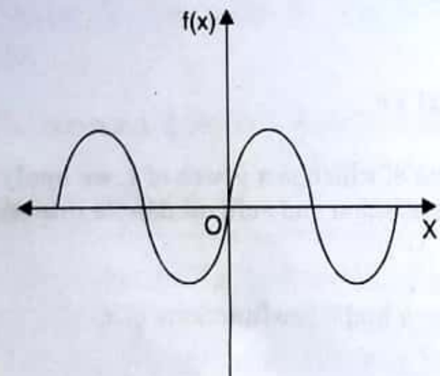


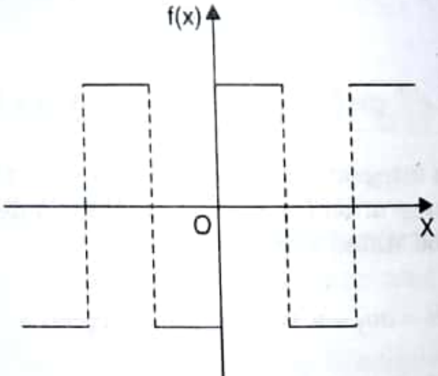
Graphs of even functions

A function $f(x)$ is said to be *odd* if $f(-x) = -f(x)$ *e.g.*, x^3, $\sin x$, $\tan^3 x$ are odd functions.

The graph of an odd function is symmetrical about the origin.

$$\int_{-\pi}^{\pi} f(x)\,dx = 0$$





Graphs of odd functions

The product of two even functions or two odd functions is an even function while the product of an even function and an odd function is an odd function.

3.35 EULER'S FORMULAE

The Fourier series for the function $f(x)$ in the interval $c < x < c + 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{...(1)}$$

In finding the coefficients a_0, a_n and b_n, we assume that the series on the right hand side of eqn. (1) is uniformly convergent for $c < x < c + 2\pi$ and it can be integrated term by term in the given interval.

To find a_0, integrate both sides of (1) w.r.t. x between the limits c to $c + 2\pi$.

$$\int_c^{c+2\pi} f(x)\,dx = \frac{a_0}{2}\int_c^{c+2\pi} dx + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)dx + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)dx$$

$$= \frac{a_0}{2}(c + 2\pi - c) + 0 + 0 \qquad \text{[By formula } (i) \text{ above]}$$

$$= a_0\pi$$

$$\therefore \quad \boxed{a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx}$$

To find a_n, multiply both sides of (1) by $\cos nx$ and integrate w.r.t. x between the limits c to $c + 2\pi$.

$$\int_c^{c+2\pi} f(x)\cos nx\,dx = \frac{a_0}{2}\int_c^{c+2\pi} \cos nx\,dx + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)\cos nx\,dx$$

$$+ \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)\cos nx\,dx$$

$$= 0 + a_n\pi + 0 \qquad \text{[By formulae } (i), (v) \text{ and } (vi)]$$

$$= a_n\pi$$

$$\therefore \quad \boxed{a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx}$$

To find b_n, multiply both sides of (1) by $\sin nx$ and integrate w.r.t. x between the limits c to $c + 2\pi$.

$$\int_c^{c+2\pi} f(x)\sin nx\,dx = \frac{a_0}{2}\int_c^{c+2\pi} \sin nx\,dx + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)\sin nx\,dx$$

$$+ \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)\sin nx\,dx$$

$$= 0 + 0 + b_n\pi \qquad \text{[By formulae } (i), (vi) \text{ and } (v)]$$

$$= b_n\pi$$

$$\therefore \qquad \boxed{b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx}$$

These values of a_0, a_n and b_n are called **Euler's formulae.**

Cor. 1. *If c = 0, the interval becomes $0 < x < 2\pi$, and the formulae reduce to*

$$\boxed{a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx\,; \quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx\,; \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx}$$

Cor. 2. *If c = $-\pi$, the interval becomes $-\pi < x < \pi$, and the formulae reduce to*

$$\boxed{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx\,; a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx\,; b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx}$$

Cor. 3. When $f(x)$ is an odd function $\quad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$

Since cos nx is an even function, therefore, $f(x)$ cos nx is an odd function.

$$\therefore \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0$$

Since sin nx is an odd function, therefore, $f(x)$ sin nx is an even function.

$$\therefore \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Hence, if a periodic function $f(x)$ is odd, its Fourier expansion contains only sine terms.

i.e.,
$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad \text{where} \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx}$$

Cor. 4. When $f(x)$ is an even function $\quad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

Since cos nx is an even function, therefore, $f(x)$ cos nx is an even function.

$$\therefore \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

Since sin nx is an odd function, therefore, $f(x)$ sin nx is an odd function.

$$\therefore \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0$$

Hence, if a periodic function $f(x)$ is even, its Fourier expansion contains only cosine terms.

i.e.,
$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \text{ where } a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \text{ and } a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx}$$

3.36 DIRICHLET'S CONDITIONS

The sufficient conditions for the uniform convergence of a Fourier series are called Dirichlet's conditions (after Dirichlet, a German mathematician). All the functions that normally arise in engineering problems satisfy these conditions and hence they can be expressed as a Fourier series.

Any function f(x) can be expressed as a Fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

where a_0, a_n, b_n are constants, provided

(i) f(x) is periodic, single valued and finite

(ii) f(x) has a finite number of finite discontinuities in any one period.

(iii) f(x) has a finite number of maxima and minima.

When these conditions are satisfied, the Fourier series converges to $f(x)$ at every point of continuity. At a point of discontinuity, the sum of the series is equal to the mean of the limits on the right and left

i.e.,

$$\tfrac{1}{2}[f(x+0)+f(x-0)]$$

where $f(x+0)$ and $f(x-0)$ denote the limit on the right and the limit on the left respectively.

ILLUSTRATIVE EXAMPLES

Example 1. *Obtain the Fourier series to represent* $f(x) = \frac{1}{4}(\pi - x)^2$ *in the interval* $0 \le x \le 2\pi$. *Hence obtain the following relations:* [A.K.T.U. 2017]

(i) $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots=\frac{\pi^2}{6}$ (ii) $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots=\frac{\pi^2}{12}$

(iii) $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots=\frac{\pi^2}{8}$.

Sol. Let $$f(x) = \frac{1}{4}(\pi - x)^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \ldots(1)$$

By Euler's formulae, we have

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{4}(\pi-x)^2\,dx = \frac{1}{4\pi}\left[\frac{(\pi-x)^3}{-3}\right]_0^{2\pi} = -\frac{1}{12\pi}[-\pi^3-\pi^3] = \frac{\pi^2}{6}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{4}(\pi-x)^2\cos nx\,dx$$

$$= \frac{1}{4\pi}\left[\left\{(\pi-x)^2\frac{\sin nx}{n}\right\}_0^{2\pi} + \int_0^{2\pi} 2(\pi-x)\frac{\sin nx}{n}\,dx\right]$$

$$= \frac{1}{4\pi}\cdot\frac{2}{n}\left[\left\{(\pi - x)\left(\frac{-\cos nx}{n}\right)\right\}_0^{2\pi} - \int_0^{2\pi}(-1)\left(\frac{-\cos nx}{n}\right)dx\right]$$

$$= \frac{-1}{2\pi n^2}(-\pi-\pi) = \frac{1}{n^2} \qquad \mid \because \ \cos 2n\pi = 1 \text{ and } \sin 2n\pi = 0,\ n \in \mathrm{I}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{4}(\pi - x)^2 \sin nx\,dx$$

$$= \frac{1}{4\pi}\left[\left\{(\pi - x)^2\left(\frac{-\cos nx}{n}\right)\right\}_0^{2\pi} - \int_0^{2\pi} 2(\pi - x)\frac{\cos nx}{n}\,dx\right]$$

$$= \frac{1}{4\pi}\left[\left(-\frac{\pi^2}{n}+\frac{\pi^2}{n}\right) - \frac{2}{n}\int_0^{2\pi}(\pi - x)\cos nx\,dx\right]$$

$$= -\frac{1}{2\pi n}\left[\left\{(\pi - x)\frac{\sin nx}{n}\right\}_0^{2\pi} - \int_0^{2\pi}(-1)\frac{\sin nx}{n}\,dx\right]$$

$\frac{1}{4}(\pi-x)^2$

$$= -\frac{1}{2\pi n^2}\left(\frac{-\cos nx}{n}\right)_0^{2\pi} = 0 \qquad \mid \because \cos 2n\pi = 1,\ n \in \mathrm{I} \text{ and } \cos 0 = 1$$

$$\therefore \quad f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{\cos nx}{n^2} = \frac{\pi^2}{12} + \frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \ldots\ldots \qquad \ldots(2)$$

(*i*) Putting $x = 0$ in eqn. (2), we get

$$\frac{\pi^2}{4} = \frac{\pi^2}{12} + \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\right)$$

$$\Rightarrow \quad \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots \qquad \ldots(3)$$

(*ii*) Putting $x = \pi$ in eqn. (2), we get

$$0 = \frac{\pi^2}{12} + \left[\left(\frac{-1}{1^2}\right) + \frac{1}{2^2} + \left(\frac{-1}{3^2}\right) + \frac{1}{4^2} + \ldots\right]$$

$$\Rightarrow \quad \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots \qquad \ldots(4)$$

(*iii*) Adding equations (3) and (4), we get

$$\frac{\pi^2}{6} + \frac{\pi^2}{12} = 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right)$$

$$\Rightarrow \quad \frac{\pi^2}{4} = 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right)$$

$$\Rightarrow \quad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

Example 2. *Obtain the Fourier series for $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$.*

Sol. Let $f(x) = e^{-x} = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

Here, $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx = \frac{1}{\pi}\Big[-e^{-x}\Big]_0^{2\pi} = \frac{1-e^{-2\pi}}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}(-\cos nx + n\sin nx)\right]_0^{2\pi} = \frac{1-e^{-2\pi}}{\pi(1+n^2)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}(-\sin nx - n\cos nx)\right]_0^{2\pi} = \frac{1-e^{-2\pi}}{\pi}\cdot\frac{n}{1+n^2}$$

$$\therefore \quad e^{-x} = \frac{1-e^{-2\pi}}{2\pi} + \frac{1-e^{-2\pi}}{\pi}\sum_{n=1}^{\infty}\frac{\cos nx}{1+n^2} + \frac{1-e^{-2\pi}}{\pi}\sum_{n=1}^{\infty}\frac{n\sin nx}{1+n^2}.$$

Example 3. *Expand $f(x) = x \sin x$, $0 < x < 2\pi$ as a Fourier series.* (U.P.T.U. 2015)

Sol. Let $f(x) = x\sin x = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

By Euler's formulae, we have

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x\,dx$$

$$= \frac{1}{\pi}\Big[x(-\cos x) - 1.(-\sin x)\Big]_0^{2\pi} = \frac{1}{\pi}[-2\pi] = -2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos nx\,dx = \frac{1}{2\pi}\int_0^{2\pi} x(2\cos nx\sin x)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x[\sin(n+1)x - \sin(n-1)x]\,dx$$

$$= \frac{1}{2\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - 1.\left\{-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left\{-\frac{\cos 2(n+1)\pi}{n+1} + \frac{\cos 2(n-1)\pi}{n-1}\right\}\right]$$

$$= -\frac{1}{n+1} + \frac{1}{n-1} = \frac{2}{n^2-1},\ n \neq 1$$

When $n = 1$, we have

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos x \, dx = \frac{1}{2\pi}\int_0^{2\pi} x \sin 2x \, dx$$

$$= \frac{1}{2\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - 1.\left(-\frac{\sin 2x}{4}\right)\right]_0^{2\pi} = \frac{1}{2\pi}[-\pi] = -\frac{1}{2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx \, dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin nx \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(2 \sin nx \sin x) \, dx = \frac{1}{2\pi}\int_0^{2\pi} x\,[\cos (n-1)x - \cos (n+1)x] \, dx.$$

$$= \frac{1}{2\pi}\left[x\left\{\frac{\sin (n-1)x}{n-1} - \frac{\sin (n+1)x}{n+1}\right\} - 1.\left\{-\frac{\cos (n-1)x}{(n-1)^2} + \frac{\cos (n+1)x}{(n+1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\cos 2(n-1)\pi}{(n-1)^2} - \frac{\cos 2(n+1)\pi}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right] = 0, \; n \neq 1$$

When $n = 1$, we have

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin x \, dx = \frac{1}{2\pi}\int_0^{2\pi} x(1 - \cos 2x) \, dx$$

$$= \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - 1.\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi(2\pi) - \frac{4\pi^2}{2} - \frac{1}{4} + \frac{1}{4}\right] = \frac{1}{2\pi}(2\pi^2) = \pi$$

$$\therefore \; f(x) = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + \sum_{n=2}^{\infty} a_n \cos nx + \sum_{n=2}^{\infty} b_n \sin nx$$

$$= -1 - \frac{1}{2}\cos x + \pi \sin x + \sum_{n=2}^{\infty} \frac{2}{n^2 - 1} \cos nx.$$

Example 4. *Find the Fourier series to represent e^{ax} in the interval $-\pi < x < \pi$.*

Sol. Let $f(x) = e^{ax} = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

Here, $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax} \, dx = \frac{1}{\pi}\left[\frac{e^{ax}}{a}\right]_{-\pi}^{\pi} = \frac{1}{\pi a}(e^{a\pi} - e^{-a\pi}) = \frac{2 \sinh a\pi}{\pi a}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax} \cos nx \, dx$$

$$= \frac{1}{\pi}\left[\frac{e^{ax}}{a^2 + n^2}(a \cos nx + n \sin nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(a^2+n^2)}[ae^{a\pi}\cos n\pi - ae^{-a\pi}\cos n\pi]$$

$$= \frac{a\cos n\pi\,(e^{a\pi}-e^{-a\pi})}{\pi(a^2+n^2)} = \frac{2a(-1)^n\sinh a\pi}{\pi(a^2+n^2)}$$

Similarly, $$b_n = \frac{2n(-1)^n\sinh a\pi}{\pi(a^2+n^2)}$$

$$\therefore \quad e^{ax} = \frac{\sinh a\pi}{a\pi} + \sum_{n=1}^{\infty}\frac{2a(-1)^n\sinh a\pi}{\pi(a^2+n^2)}\cos nx + \sum_{n=1}^{\infty}\frac{2n(-1)^n\sinh a\pi}{\pi(a^2+n^2)}\sin nx.$$

Example 5. *Find a Fourier series to represent $x - x^2$ from $x = -\pi$ to $x = \pi$. Hence show that*

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{12}.$$

Sol. Let $$x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

By Euler's formulae, we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{\pi}x\,dx - \int_{-\pi}^{\pi}x^2\,dx\right]$$

$$= -\frac{2}{\pi}\int_0^{\pi}x^2\,dx = -\frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi} = -\frac{2}{3}\pi^2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{\pi}x\cos nx\,dx - \int_{-\pi}^{\pi}x^2\cos nx\,dx\right]$$

$$= -\frac{2}{\pi}\int_0^{\pi}x^2\cos nx\,dx = -\frac{2}{\pi}\left[\left(x^2\frac{\sin nx}{n}\right)_0^{\pi} - \int_0^{\pi}2x\frac{\sin nx}{n}\,dx\right]$$

$$= \frac{4}{\pi n}\int_0^{\pi}x\sin nx\,dx = \frac{4}{\pi n}\left[\left\{x\left(\frac{-\cos nx}{n}\right)\right\}_0^{\pi} - \int_0^{\pi}\left(\frac{-\cos nx}{n}\right)dx\right]$$

$$= -\frac{4}{\pi n^2}(\pi\cos n\pi) = -\frac{4}{n^2}(-1)^n \qquad \mid \because \quad \cos n\pi = (-1)^n,\ n \in I$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{\pi}x\sin nx\,dx - \int_{-\pi}^{\pi}x^2\sin nx\,dx\right]$$

$$= \frac{2}{\pi}\int_0^{\pi}x\sin nx\,dx = \frac{2}{\pi}\left[\left\{x\left(\frac{-\cos nx}{n}\right)\right\}_0^{\pi} - \int_0^{\pi}1.\left(\frac{-\cos nx}{n}\right)dx\right]$$

$$= -\frac{2}{n}\cos n\pi = -\frac{2}{n}(-1)^n \qquad \mid \because \quad \sin n\pi = 0,\ n \in I$$

$$\therefore \quad x - x^2 = -\frac{\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

$$= -\frac{\pi^2}{3} - 4\left[-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \ldots\ldots\right] - 2\left[\frac{-\sin x}{1} + \frac{\sin 2x}{2} - \frac{\sin 3x}{3} + \ldots\ldots\right]$$

$$= -\frac{\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \ldots\ldots\right] + 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\ldots\right]$$

Putting $x = 0$, we get

$$0 = -\frac{\pi^2}{3} + 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots\right)$$

$$\Rightarrow \quad \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{12}.$$

Example 6. *Find the Fourier series for the function* $f(x) = x + x^2, -\pi < x < \pi$.

[G.B.T.U. (AG) SUM 2010]

Hence show that:

(*i*) $\dfrac{\pi^2}{6} = 1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \ldots$ (U.K.T.U. 2011) (*ii*) $\dfrac{\pi^2}{12} = \dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \ldots$

Sol. Let the Fourier series be

$$x + x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots(1)$$

Here, $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\,dx + \int_{-\pi}^{\pi} x^2\,dx\right] = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{3}\pi^2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\cos nx\,dx + \int_{-\pi}^{\pi} x^2\cos nx\,dx\right] = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\left(x^2\,\frac{\sin nx}{n}\right)_0^{\pi} - \int_0^{\pi} 2x\,.\,\frac{\sin nx}{n}\,dx\right] = -\frac{4}{n\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= -\frac{4}{n\pi}\left[\left\{x\,.\left(\frac{-\cos nx}{n}\right)\right\}_0^{\pi} - \int_0^{\pi} 1.\left(\frac{-\cos nx}{n}\right)dx\right]$$

$$= -\frac{4}{n\pi}\left(\frac{-\pi}{n}\cos n\pi\right) = \frac{4}{n^2}\cos n\pi = \frac{4}{n^2}(-1)^n$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx \qquad \Big|\; \because \int_{-\pi}^{\pi} x^2\sin nx\,dx = 0$$

$$= \frac{2}{\pi}\left(\frac{-\pi}{n}\cos n\pi\right) = -\frac{2}{n}(-1)^n \qquad | \text{ as above}$$

$\therefore$ From (1), $x + x^2 = \dfrac{\pi^2}{3} + 4\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$

$$\Rightarrow \quad f(x) = \frac{\pi^2}{3} + 4\left[\frac{-1}{1^2}\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \ldots\right]$$

$$- 2\left[\frac{-1}{1}\sin x + \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + \ldots\right] \qquad \ldots(2)$$

We observe that the series on the R.H.S. given by eqn. (2) always represents $x + x^2$ for all values of x except the end points $-\pi$ or π.

At the point of discontinuity,

$$f(-\pi) = \frac{1}{2}\,(\text{LHL} + \text{RHL}) = \frac{1}{2}\,[f(-\pi - 0) + f(-\pi + 0)]$$

$$= \frac{1}{2}\,[f(\pi - 0) + f(-\pi + 0)] \qquad | \because f(x) \text{ is periodic with period } 2\pi$$

$$= \frac{1}{2}\,[\pi + \pi^2 + (-\pi) + (-\pi)^2] = \pi^2$$

Putting $x = -\pi$ in eqn. (2), we get

$$\pi^2 = \frac{\pi^2}{3} + 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\right]$$

$$\Rightarrow \quad \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots \qquad \ldots(3)$$

Again, putting $x = 0$ in eqn. (2), we get

$$0 = \frac{\pi^2}{3} + 4\left[\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \ldots\right]$$

$$\Rightarrow \quad \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots \qquad \ldots(4)$$

Example 7. *Express $f(x) = |\,x\,|, -\pi < x < \pi$, as Fourier series. Hence show that*

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}.$$

Sol. Since $f(-x) = |-x| = |\,x\,| = f(x)$

$\therefore$ $f(x)$ is an even function and hence $b_n = 0$

Let $\quad f(x) = |\,x\,| = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos nx$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} |x|\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} |x|\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1.\left(-\frac{\cos nx}{n^2}\right)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi n^2}[(-1)^n - 1] = \begin{cases} 0 & , \text{ if } n \text{ is even} \\ \frac{-4}{\pi n^2}, & \text{ if } n \text{ is odd} \end{cases}$$

$$\therefore \quad |x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right)$$

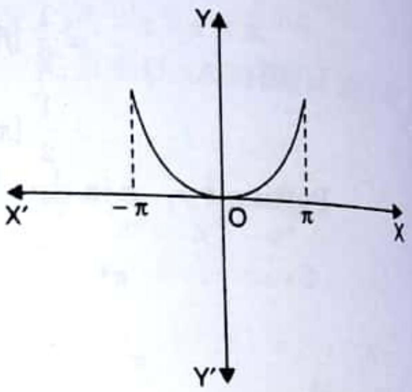
Putting $x = 0$ in the above result, we get $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$.

Example 8. *Obtain the Fourier series for the function $f(x) = x^2, -\pi \le x \le \pi$. Sketch the graph of $f(x)$. Hence show that:*

(i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots.. = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

(ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots.. = \frac{\pi^2}{12}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.



Graph of $f(x) = x^2$

Sol. Since $f(-x) = (-x)^2 = x^2 = f(x)$.

$\therefore$ $f(x)$ is an even function and hence $b_n = 0$

Let $$f(x) = x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where $$a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx = \frac{2}{\pi}\int_0^\pi x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^\pi = \frac{2}{3}\pi^2$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^\pi x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\left[\left(x^2\,\frac{\sin nx}{n}\right)_0^\pi - \int_0^\pi 2x\,.\,\frac{\sin nx}{n}\,dx\right]$$

$$= -\frac{4}{\pi n}\int_0^\pi x\sin nx\,dx = -\frac{4}{\pi n}\left[\left\{x\left(\frac{-\cos nx}{n}\right)\right\}_0^\pi - \int_0^\pi 1.\left(\frac{-\cos nx}{n}\right)dx\right]$$

$$= \frac{4}{\pi n^2}(\pi \cos n\pi) = \frac{4}{n^2}(-1)^n$$

$$\therefore \quad x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx$$

$$= \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \ldots\ldots\right) \quad \ldots(1)$$

Putting $x = \pi$ in (1), we get

$$\pi^2 = \frac{\pi^2}{3} - 4\left(-\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots\right)$$

$$\Rightarrow \qquad \frac{2\pi^2}{3} = 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots\right)$$

$$\therefore \qquad \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{6} \qquad \ldots(2)$$

Putting $x = 0$ in (1), we get

$$0 = \frac{\pi^2}{3} - 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots\right)$$

$$\therefore \qquad \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{12} \qquad \ldots(3)$$

Adding (2) and (3), we get

$$2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\right) = \frac{\pi^2}{4}$$

$$\therefore \qquad \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}.$$

Example 9. *Expand the function $f(x) = x \sin x$ as a Fourier series in the interval $-\pi \le x \le \pi$. Deduce that* $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} + \ldots\ldots = \frac{\pi - 2}{4}$.

Sol. Since $x \sin x$ is an even function of x, $b_n = 0$

Let $\qquad f(x) = x \sin x = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos nx$

where $\qquad a_0 = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} x \sin x\, dx = \dfrac{2}{\pi}\Big[x(-\cos x) - 1.(-\sin x)\Big]_0^{\pi} = \dfrac{2}{\pi}(-\pi \cos \pi) = 2$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos nx\, dx = \frac{1}{\pi}\int_0^{\pi} x(2\cos nx \sin x)\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x[\sin(n+1)x - \sin(n-1)x]\, dx$$

$$= \frac{1}{\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - 1.\left\{-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi\left\{-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right\}\right] = \frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1}, n \ne 1$$

When n is odd, $n \neq 1$, $n-1$ and $n+1$ are even

$$\therefore \qquad a_n = \frac{1}{n-1} - \frac{1}{n+1} = \frac{2}{n^2-1}$$

When n is even, $n-1$ and $n+1$ are odd

$$\therefore \qquad a_n = \frac{-1}{n-1} + \frac{1}{n+1} = \frac{-2}{n^2-1}$$

When $n = 1$, we have

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x \, dx = \frac{1}{\pi}\int_0^{\pi} x \sin 2x \, dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - 1.\left(-\frac{\sin 2x}{4}\right)\right]_0^{\pi} = \frac{1}{\pi}\left[-\frac{\pi \cos 2\pi}{2}\right] = -\frac{1}{2}$$

$$\therefore \qquad x \sin x = 1 - \frac{1}{2}\cos x - 2\left(\frac{\cos 2x}{2^2-1} - \frac{\cos 3x}{3^2-1} + \frac{\cos 4x}{4^2-1} - \frac{\cos 5x}{5^2-1} + \ldots\ldots\right)$$

Putting $\quad x = \frac{\pi}{2}$, we get

$$\frac{\pi}{2} = 1 - 2\left(\frac{-1}{2^2-1} + \frac{1}{4^2-1} - \frac{1}{6^2-1} + \ldots\ldots\right)$$

$$\Rightarrow \qquad \frac{\pi}{2} - 1 = 2\left(\frac{1}{3} - \frac{1}{15} + \frac{1}{35} - \ldots\ldots\right)$$

$$\Rightarrow \qquad \frac{\pi-2}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \ldots\ldots$$

Example 10. *Show that for* $-\pi < x < \pi$,

$$\sin ax = \frac{2 \sin a\pi}{\pi}\left(\frac{\sin x}{1^2-a^2} - \frac{2\sin 2x}{2^2-a^2} + \frac{3\sin 3x}{3^2-a^2} - \ldots\ldots\right).$$

Sol. Since $\sin ax$ is an odd function of x, $\therefore \; a_0 = 0$ and $a_n = 0$

Let $\sin ax = \sum_{n=1}^{\infty} b_n \sin nx$

where $\qquad b_n = \frac{2}{\pi}\int_0^{\pi} \sin ax \sin nx \, dx = \frac{1}{\pi}\int_0^{\pi} [\cos(n-a)x - \cos(n+a)x] \, dx$

$$= \frac{1}{\pi}\left[\frac{\sin(n-a)x}{n-a} - \frac{\sin(n+a)x}{n+a}\right]_0^{\pi} = \frac{1}{\pi}\left[\frac{\sin(n-a)\pi}{n-a} - \frac{\sin(n+a)\pi}{n+a}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n(-\sin a\pi)}{n-a} - \frac{(-1)^n \sin a\pi}{n+a}\right] = -\frac{(-1)^n \sin a\pi}{\pi}\left[\frac{1}{n-a} + \frac{1}{n+a}\right]$$

$$= (-1)^{n+1} \cdot \frac{2n \sin a\pi}{\pi(n^2 - a^2)}$$

$$\therefore \quad \sin ax = \frac{2 \sin a\pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2 - a^2} \sin nx$$

$$= \frac{2 \sin a\pi}{\pi} \left(\frac{\sin x}{1^2 - a^2} - \frac{2 \sin 2x}{2^2 - a^2} + \frac{3 \sin 3x}{3^2 - a^2} - \ldots\ldots \right).$$

TEST YOUR KNOWLEDGE

1. Expand in a Fourier series the function $f(x) = x$ in the interval $0 < x < 2\pi$.
2. Express $f(x) = \frac{1}{2}(\pi - x)$ in a Fourier series in the interval $0 < x < 2\pi$. Deduce that
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$
3. Prove that for all values of x between $-\pi$ and π, $\frac{1}{2}x = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \ldots\ldots$
4. Obtain the Fourier series to represent e^x in the interval $0 < x < 2\pi$.
5. Find the Fourier series to represent e^x in the interval $-\pi < x < \pi$.
6. Find the Fourier series to represent the function $f(x) = |\sin x|, -\pi < x < \pi$.
7. Expand $f(x) = |\cos x|$ as a Fourier series in the interval $-\pi < x < \pi$. *[M.T.U. (SUM) 2011]*
8. Prove that in the interval $-\pi < x < \pi$,
$$x \cos x = -\frac{1}{2}\sin x + 2\sum_{n=2}^{\infty} \frac{n(-1)^n}{n^2 - 1} \sin nx.$$
9. Prove that for $-\pi < x < \pi$, $\frac{x(\pi^2 - x^2)}{12} = \frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \frac{\sin 4x}{4^3} + \ldots\ldots$
10. Obtain a Fourier series to represent $f(x) = x^2$ in the interval $(0, 2\pi)$ and hence deduce that
$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}$$
11. Obtain a Fourier series to represent e^{-ax} from $x = -\pi$ to $x = \pi$. Hence derive series for $\frac{\pi}{\sinh \pi}$.
12. Prove that in the range $-\pi < x < \pi$, $\cosh ax = \frac{2a}{\pi}\sinh a\pi \left[\frac{1}{2a^2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + a^2} \cos nx \right]$. *(A.K.T.U. 2016)*
13. Find Fourier series of $f(x) = x^3$ in $(-\pi, \pi)$.
14. Expand the function $f(x) = \frac{\pi^2}{12} - \frac{x^2}{4}$ in Fourier series in the interval $(-\pi, \pi)$.

15. Find the Fourier series expansion for $f(x) = x + \frac{x^2}{4}$, $-\pi \le x \le \pi$

16. Express $f(x) = \cos wx$, $-\pi < x < \pi$, where w is a fraction, as a Fourier series. Hence prove that

$$\cot\theta = \frac{1}{\theta} + \frac{2\theta}{\theta^2 - \pi^2} + \frac{2\theta}{\theta^2 - 4\pi^2} + \ldots\ldots$$

17. Find the Fourier series of $f(x) = \frac{3x^2 - 6\pi x + 2\pi^2}{12}$ in the interval $(0, 2\pi)$. Hence, deduce that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

(M.T.U. 2012)

Answers

1. $f(x) = \pi - 2\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ **2.** $f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n}$

4. $e^x = \frac{e^{2\pi} - 1}{2\pi} + \frac{e^{2\pi} - 1}{\pi} \sum_{n=1}^{\infty} \left(\frac{\cos nx}{1+n^2} - \frac{n}{1+n^2}\sin nx\right)$

5. $e^x = \frac{2\sinh\pi}{\pi}\left[\frac{1}{2} - \left(\frac{1}{2}\cos x - \frac{1}{5}\cos 2x + \frac{1}{10}\cos 3x - \ldots\ldots\right) - \left(\frac{1}{2}\sin x - \frac{2}{5}\sin 2x + \frac{3}{10}\sin 3x - \ldots\ldots\right)\right]$

6. $|\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \ldots\ldots + \frac{\cos 2nx}{4n^2 - 1} + \ldots\ldots\right)$

7. $|\cos x| = \frac{2}{\pi} + \frac{4}{\pi}\left(\frac{\cos 2x}{3} - \frac{\cos 4x}{15} + \ldots\ldots\right)$ **10.** $x^2 = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos nx}{n^2} - 4\pi \sum_{n=1}^{\infty} \frac{\sin nx}{n}$

11. $e^{-ax} = \frac{2\sinh a\pi}{\pi}\left[\left(\frac{1}{2a} - \frac{a\cos x}{1^2 + a^2} + \frac{a\cos 2x}{2^2 + a^2} - \ldots\ldots\right) - \left(\frac{\sin x}{1^2 + a^2} - \frac{2\sin 2x}{2^2 + a^2} + \frac{3\sin 3x}{3^2 + a^2} - \ldots\ldots\right)\right]$

$$\frac{\pi}{\sinh\pi} = 2\left[\frac{1}{2^2+1} - \frac{1}{3^2+1} + \frac{1}{4^2+1} - \ldots\ldots\right]$$

13. $f(x) = 2\sum_{n=1}^{\infty}\left(\frac{6}{n^3} - \frac{\pi^2}{n}\right)(-1)^n \sin nx$ **14.** $f(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\cos nx}{n^2}$

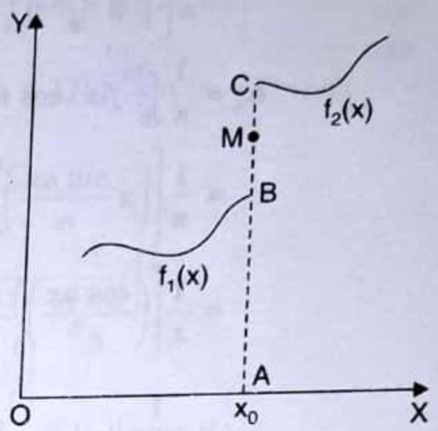
15. $f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\sin nx$

16. $\cos wx = \frac{2w\sin w\pi}{\pi}\left(\frac{1}{2w^2} + \frac{\cos x}{1^2 - w^2} - \frac{\cos 2x}{2^2 - w^2} + \frac{\cos 3x}{3^2 - w^2} - \ldots\ldots\right)$

17. $f(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$.

3.37 FOURIER SERIES FOR DISCONTINUOUS FUNCTIONS

In art. 3.35, we derived Euler's formulae for a_0, a_n, b_n on the assumption that $f(x)$ is continuous in $(c, c + 2\pi)$. However, if $f(x)$ has finitely many points of finite discontinuity, even then it can be expressed as a Fourier series. The integrals for a_0, a_n, b_n are to be evaluated by breaking up the range of integration.



Let $f(x)$ be defined by $f(x) = \begin{cases} f_1(x), & c < x < x_0 \\ f_2(x), & x_0 < x < c + 2\pi \end{cases}$

where x_0 is the point of finite discontinuity in the interval $(c, c + 2\pi)$.

The values of a_0, a_n, b_n are given by

$$a_0 = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)\, dx + \int_{x_0}^{c+2\pi} f_2(x)\, dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x) \cos nx\, dx + \int_{x_0}^{c+2\pi} f_2(x) \cos nx\, dx\right]$$

$$b_n = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x) \sin nx\, dx + \int_{x_0}^{c+2\pi} f_2(x) \sin nx\, dx\right]$$

At $x = x_0$, there is a finite jump in the graph of the function. Both the limits $f(x_0 - 0)$ and $f(x_0 + 0)$ exist but are unequal. The sum of the Fourier series $= \frac{1}{2}[f(x_0 - 0) + f(x_0 + 0)]$ $= \frac{1}{2}$ (AB + AC) = AM, where M is the mid-point of BC.

When $f(x)$ is discontinuous, the graph of the partial sum $S_n(x)$ of the Fourier series representation of the function exhibits over and undershoots close to the discontinuities. This is called the **Gibbs phenomenon** and it persists for all values of n. This behaviour reflects the way the continuous function $S_n(x)$ obtained from the Fourier series approximates the behaviour of $f(x)$ at a point of discontinuity. Increasing n simply moves the under and overshoots closer to the discontinuity while leaving their size approximately the same.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the Fourier series to represent the function f(x) given by*

$$f(x) = \begin{cases} x & \text{for } 0 \le x \le \pi \\ 2\pi - x & \text{for } \pi \le x \le 2\pi \end{cases}$$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$. [G.B.T.U. (SUM) 2010]

Sol. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$, where

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx = \frac{1}{\pi}\left[\int_0^{\pi} x\, dx + \int_{\pi}^{2\pi} (2\pi - x)\, dx\right]$$

$$= \frac{1}{\pi}\left[\left|\frac{x^2}{2}\right|_0^\pi + \left|2\pi x - \frac{x^2}{2}\right|_\pi^{2\pi}\right] = \frac{1}{\pi}\left[\frac{\pi^2}{2} + (4\pi^2 - 2\pi^2) - \left(2\pi^2 - \frac{\pi^2}{2}\right)\right] = \pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\left[\int_0^\pi x\cos nx\, dx + \int_\pi^{2\pi}(2\pi - x)\cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\left(x\frac{\sin nx}{n}\right)_0^\pi - \int_0^\pi 1\cdot\frac{\sin nx}{n}dx + \left\{(2\pi - x)\frac{\sin nx}{n}\right\}_\pi^{2\pi} + \int_\pi^{2\pi}\frac{\sin nx}{n}dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\cos nx}{n^2}\right)_0^\pi - \left(\frac{\cos nx}{n^2}\right)_\pi^{2\pi}\right]$$

$$= \frac{1}{\pi n^2}[(-1)^n - 1 - 1 + (-1)^n] = \frac{2}{\pi n^2}[(-1)^n - 1] = \begin{cases} -\dfrac{4}{\pi n^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even}\end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\left[\int_0^\pi x\sin nx\, dx + \int_\pi^{2\pi}(2\pi - x)\sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\left\{x\left(\frac{-\cos nx}{n}\right)\right\}_0^\pi + \int_0^\pi \frac{\cos nx}{n}dx\right.$$

$$\left. + \left\{(2\pi - x)\left(\frac{-\cos nx}{n}\right)\right\}_\pi^{2\pi} - \int_\pi^{2\pi}(-1)\left(\frac{-\cos nx}{n}\right)dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\cos n\pi + \frac{\pi}{n}\cos n\pi\right] = 0$$

$$\therefore \quad f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right)$$

Putting $x = 0$, we get $0 = \dfrac{\pi}{2} - \dfrac{4}{\pi}\left(\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots\ldots\right)$

$$\Rightarrow \quad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$$

Example 2. *If* $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ \sin x, & 0 \le x \le \pi \end{cases}$

Prove that: $$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2 - 1}.$$ [G.B.T.U. (AG) 2012]

Hence show that:

(i) $\dfrac{1}{1\cdot 3} + \dfrac{1}{3\cdot 5} + \dfrac{1}{5\cdot 7} + \ldots\ldots = \dfrac{1}{2}$ (ii) $\dfrac{1}{1\cdot 3} - \dfrac{1}{3\cdot 5} + \dfrac{1}{5\cdot 7} - \ldots\ldots = \dfrac{\pi - 2}{4}$.

Sol. Let $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Then $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\, dx + \int_0^\pi \sin x\, dx\right] = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_0^{\pi}\sin x\cos nx\,dx\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi} 2\cos nx\sin x\,dx = \frac{1}{2\pi}\int_0^{\pi}[\sin(n+1)x - \sin(n-1)x]\,dx$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi},\ n \neq 1$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{2\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \begin{cases} \frac{1}{2\pi}\left(-\frac{1}{n+1} + \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right), & \text{when } n \text{ is odd} \\ \frac{1}{2\pi}\left(\frac{1}{n+1} - \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right), & \text{when } n \text{ is even} \end{cases}$$

$$= \begin{cases} 0, \text{ when } n \text{ is odd, } i.e., n = 3, 5, 7, \ldots.. \\ -\frac{2}{\pi(n^2-1)}, \text{ when } n \text{ is even} \end{cases}$$

When $n = 1$, we have

$$a_1 = \frac{1}{\pi}\int_0^{\pi}\sin x\cos x\,dx = \frac{1}{2\pi}\int_0^{\pi}\sin 2x\,dx = \frac{1}{2\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_0^{\pi}\sin x\sin nx\,dx\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi} 2\sin nx\sin x\,dx = \frac{1}{2\pi}\int_0^{\pi}[\cos(n-1)x - \cos(n+1)x]\,dx$$

$$= \frac{1}{2\pi}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_0^{\pi} = 0,\ n \neq 1$$

When $n = 1$, we have

$$b_1 = \frac{1}{\pi}\int_0^{\pi}\sin x\sin x\,dx = \frac{1}{2\pi}\int_0^{\pi}(1-\cos 2x)\,dx = \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{1}{2}$$

$\therefore$

$$f(x) = \frac{1}{\pi} - \frac{2}{\pi}\left[\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \ldots\ldots\right] + \frac{1}{2}\sin x$$

$$= \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{(2n)^2-1} \quad \ldots(1)$$

Putting $x = 0$ in (1), we have

$$0 = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$

$\Rightarrow$

$$\frac{1}{2} = \sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)} = \frac{1}{1\cdot3} + \frac{1}{3\cdot5} + \frac{1}{5\cdot7} + \ldots\ldots$$

Putting $x = \frac{\pi}{2}$ in (1), we have

$$1 = \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos n\pi}{4n^2 - 1}$$

$$\Rightarrow \quad \frac{1}{2} - \frac{1}{\pi} = -\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1}$$

$$\Rightarrow \quad \frac{\pi - 2}{4} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)(2n+1)} = -\left(-\frac{1}{1\cdot3} + \frac{1}{3\cdot5} - \frac{1}{5\cdot7} + \ldots\ldots\right)$$

$$\Rightarrow \quad \frac{1}{1\cdot3} - \frac{1}{3\cdot5} + \frac{1}{5\cdot7} - \ldots\ldots = \frac{\pi-2}{4}.$$

Example 3. *Obtain Fourier series for the function*

$$f(x) = \begin{cases} x, & -\pi < x < 0 \\ -x, & 0 < x < \pi \end{cases}$$

and hence show that:

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}.$$ (A.K.T.U. 2015, 17, 18)

Sol. Let $\quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots(1)$

Then, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} x\, dx + \int_0^{\pi} -x\, dx\right]$

$$= \frac{1}{\pi}\left[\left(\frac{x^2}{2}\right)_{-\pi}^{0} - \left(\frac{x^2}{2}\right)_0^{\pi}\right] = \frac{1}{\pi}\left(0 - \frac{\pi^2}{2} - \frac{\pi^2}{2}\right) = -\pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} x\cos nx\, dx + \int_0^{\pi} -x\cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\left(x\frac{\sin nx}{n}\right)_{-\pi}^{0} - \int_{-\pi}^{0} 1\cdot\frac{\sin nx}{n}\, dx + \left(-x\cdot\frac{\sin nx}{n}\right)_0^{\pi} - \int_0^{\pi}(-1)\frac{\sin nx}{n}\, dx\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2}\left(\cos nx\right)_{-\pi}^{0} - \frac{1}{n^2}\left(\cos nx\right)_0^{\pi}\right] = \frac{1}{\pi}\left[\left\{\frac{1-(-1)^n}{n^2}\right\} - \left\{\frac{(-1)^n - 1}{n^2}\right\}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\{1-(-1)^n\}}{n^2}\right] = \frac{2}{\pi n^2}\{1-(-1)^n\} = \left.\begin{matrix} 0, & \text{if } n \text{ is even} \\ \frac{4}{\pi n^2}, & \text{if } n \text{ is odd} \end{matrix}\right\}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} x\sin nx\, dx + \int_0^{\pi}(-x)\sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\left\{x\cdot\left(-\frac{\cos nx}{n}\right)\right\}_{-\pi}^{0} - \int_{-\pi}^{0} 1.\left(-\frac{\cos nx}{n}\right)dx\right.$$

$$\left. + \left\{(-x)\left(-\frac{\cos nx}{n}\right)\right\}_0^{\pi} - \int_0^{\pi}(-1)\cdot\left(-\frac{\cos nx}{n}\right)dx\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi}{n}(-1)^n + \frac{1}{n}.\pi(-1)^n\right] = 0$$

$$\therefore \quad \text{From (1),} \quad f(x) = -\frac{\pi}{2} + \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right] \qquad \ldots(2)$$

At the point of discontinuity,

$$f(0) = \frac{1}{2}\,[f(0-0) + f(0+0)] = \frac{1}{2}\,(0-0) = 0$$

Putting $x = 0$ is above expression, we get

$$0 = -\frac{\pi}{2} + \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\right)$$

$$\Rightarrow \quad \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}.$$

Example 4. *Obtain Fourier series for the function f(x) given by*

$$f(x) = \begin{cases} 1 + \dfrac{2x}{\pi}, & -\pi \le x \le 0 \\ 1 - \dfrac{2x}{\pi}, & 0 \le x \le \pi \end{cases} \qquad \text{(U.P.T.U. 2014; U.K.T.U. 2011)}$$

Hence deduce that $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots\ldots = \dfrac{\pi^2}{8}$.

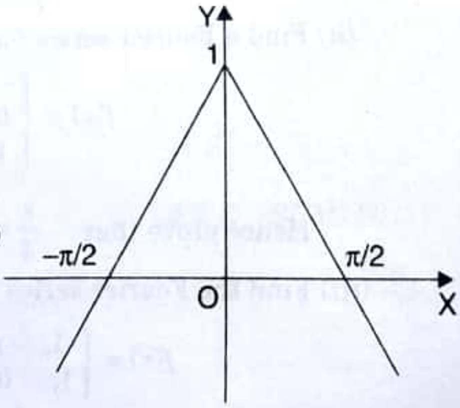
Sol. When $\quad -\pi \le x \le 0, \qquad 0 \le -x \le \pi$

$$\therefore \quad f(-x) = 1 - \frac{2(-x)}{\pi} = 1 + \frac{2x}{\pi} = f(x)$$

When $\quad 0 \le x \le \pi, \qquad -\pi \le -x \le 0$

$$\therefore \quad f(-x) = 1 + \frac{2(-x)}{\pi} = 1 - \frac{2x}{\pi} = f(x)$$

$\Rightarrow$ $f(x)$ is an even function of x in $[-\pi, \pi]$. This is also clear from its graph which is symmetrical about the y-axis.



$$\therefore \quad b_n = 0$$

Let $\quad f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos nx$

Here, $\quad a_0 = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x)\,dx = \dfrac{2}{\pi}\displaystyle\int_0^{\pi}\left(1 - \dfrac{2x}{\pi}\right)dx = \dfrac{2}{x}\left[x - \dfrac{x^2}{\pi}\right]_0^{\pi} = 0$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\left(1-\frac{2x}{\pi}\right)\frac{\sin nx}{n} - \left(-\frac{2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[-\frac{2\cos n\pi}{\pi n^2} + \frac{2}{\pi n^2}\right] = \frac{4}{\pi^2 n^2}[1-(-1)^n]$$

$$\therefore \quad f(x) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}[1-(-1)^n]\frac{\cos nx}{n^2}$$

$$= \frac{4}{\pi^2}\left(\frac{2\cos x}{1^2} + \frac{2\cos 3x}{3^2} + \frac{2\cos 5x}{5^2} + \ldots\ldots\right) = \frac{8}{\pi^2}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right)$$

Putting $x = 0$, we get $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$. $\quad | \because f(0) = 1.$

TEST YOUR KNOWLEDGE

1. Find the Fourier series to represent the function

$$f(x) = \begin{cases} -k, & \text{when } -\pi < x < 0 \\ k, & \text{where } 0 < x < \pi \end{cases}$$

Also deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots$ *(A.K.T.U. 2016)*

2. (*i*) Find the Fourier series for the function

$$f(x) = \begin{cases} -1, & -\pi < x < -\pi/2 \\ 0, & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 1, & \frac{\pi}{2} < x < \pi \end{cases}$$ *(G.B.T.U. 2012)*

Hence deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots$

(*ii*) Find a Fourier series for the function defined by:

$$f(x) = \begin{cases} -1, & \text{for } -\pi < x < 0 \\ 0, & \text{for } x = 0 \\ 1, & \text{for } 0 < x < \pi \end{cases}$$

Hence prove that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots\ldots$

(*iii*) Find the Fourier series expansion of the following function of period 2π defined as

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 \le x < \pi \end{cases}$$

Hence evaluate $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ *(G.B.T.U. 2013)*

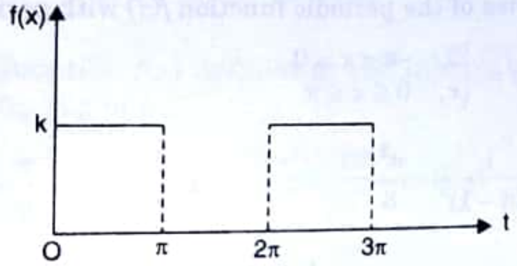
3. Find the Fourier series for the function

$$f(x) = \begin{cases} -\frac{\pi}{4}, & \text{for } -\pi < x < 0 \\ \frac{\pi}{4}, & \text{for } 0 < x < \pi \end{cases}$$

and $f(-\pi) = f(0) = f(\pi) = 0$, $f(x) = f(x + 2\pi)$ for all x.

Deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots$

4. (i) Obtain the Fourier series for the square waveform upto 4 terms as shown in the following figure: [G.B.T.U. (C.O.) 2010]



(ii) Obtain Fourier series of the function:

$$f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

and deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$. [U.K.T.U. 2012 ; G.B.T.U. (AG) 2011]

5. Find the Fourier series of the following function :

$$f(x) = \begin{cases} x^2, & -\pi < x < 0 \\ -x^2, & 0 < x < \pi \end{cases}.$$

6. An alternating current after passing through a rectifier has the form

$$i = \begin{cases} I_0 \sin x & \text{for} \quad 0 \le x \le \pi \\ 0 & \text{for} \quad \pi \le x \le 2\pi. \end{cases}$$

where I_0 is the maximum current and the period is 2π. Express i as a Fourier series.

7. Find the Fourier series of the function defined as

$$f(x) = \begin{cases} x + \pi, & \text{for } 0 \le x \le \pi \\ -x - \pi, & \text{for } -\pi \le x < 0 \end{cases} \quad \text{and} \quad f(x + 2\pi) = f(x).$$

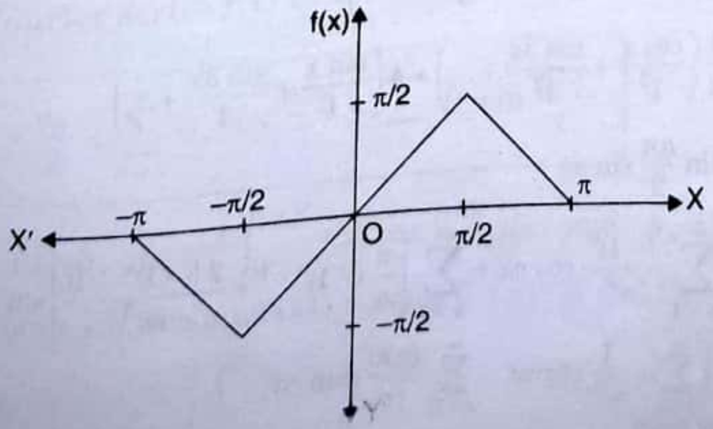
8. (i) Find the Fourier series for $f(x)$ in the interval $(-\pi, \pi)$ when

$$f(x) = \begin{cases} \pi + x, & -\pi < x < 0 \\ \pi - x, & 0 < x < \pi \end{cases}$$

(ii) Find the Fourier series for following periodic function:

$$f(x) = \begin{cases} \pi + x, & -\pi < x < 0 \\ -\pi - x, & 0 < x < \pi \end{cases}$$ [M.T.U. (SUM) 2011]

9. Find the Fourier series expansion of $f(x)$ as shown in figure:



10. (*i*) Find the Fourier series of

$$f(x) = \begin{cases} 0, & \text{when } -\pi \le x \le 0 \\ x^2, & \text{when } 0 \le x \le \pi \end{cases}$$

which is assumed to be periodic with period 2π. (*M.T.U.* 2011)

(*ii*) Find the Fourier series of the periodic function $f(x)$ with period 2π defined as follows:

$$f(x) = \begin{cases} 0, & -\pi < x \le 0 \\ x, & 0 \le x \le \pi \end{cases}$$

Hence prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$. (*U.P.T.U.* 2013, 2014)

Answers

1. $f(x) = \dfrac{4k}{\pi}\left(\sin x + \dfrac{\sin 3x}{3} + \dfrac{\sin 5x}{5} + \ldots\ldots\right)$

2. (*i*) $f(x) = \dfrac{2}{\pi}\left[\sin x - \sin 2x + \dfrac{\sin 3x}{3} - \ldots\right]$ (*ii*) $f(x) = \dfrac{4}{\pi}\left(\sin x + \dfrac{\sin 3x}{3} + \dfrac{\sin 5x}{5} + \ldots\ldots\right)$

(*iii*) $f(x) = \dfrac{4}{\pi}\left(\sin x + \dfrac{\sin 3x}{3} + \dfrac{\sin 5x}{5} + \ldots\ldots\right); \dfrac{\pi}{4}$

3. $f(x) = \dfrac{\sin x}{1} + \dfrac{\sin 3x}{3} + \dfrac{\sin 5x}{5} + \dfrac{\sin 7x}{7} + \ldots$

4. (*i*) $f(t) = \dfrac{k}{2} + \dfrac{2k}{\pi}\left(\sin t + \dfrac{1}{3}\sin 3t + \dfrac{1}{5}\sin 5t + \ldots\right)$

(*ii*) $f(x) = -\dfrac{\pi}{4} - \dfrac{2}{\pi}\left[\dfrac{1}{1^2}\cos x + \dfrac{1}{3^2}\cos 3x + \dfrac{1}{5^2}\cos 5x + \ldots\right] + \displaystyle\sum_{n=1}^{\infty}\left(\frac{1 - 2\cos n\pi}{n}\right)\sin nx$

5. $f(x) = \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\left[\frac{\pi^2}{n}\cos n\pi + \frac{2}{n^3}(1 - \cos n\pi)\right]\sin nx$

6. $i = \dfrac{I_0}{\pi} + \dfrac{I_0}{2}\sin x - \dfrac{2I_0}{\pi}\left(\dfrac{\cos 2x}{2^2 - 1} + \dfrac{\cos 4x}{4^2 - 1} + \dfrac{\cos 6x}{6^2 - 1} + \ldots\ldots\right)$

7. $f(x) = \dfrac{\pi}{2} - \dfrac{4}{\pi}\left[\dfrac{\cos x}{1^2} + \dfrac{\cos 3x}{3^2} + \ldots\right] + 4\left[\dfrac{\sin x}{1} + \dfrac{\sin 3x}{3} + \ldots\right]$

8. (*i*) $f(x) = \dfrac{\pi}{2} + \dfrac{4}{\pi}\left(\dfrac{\cos x}{1^2} + \dfrac{\cos 3x}{3^2} + \dfrac{\cos 5x}{5^2} + \ldots\right)$

(*ii*) $f(x) = -\dfrac{\pi}{2} + \dfrac{4}{\pi}\left(\dfrac{\cos x}{1^2} + \dfrac{\cos 3x}{3^2} + \ldots\right) + 4\left(\dfrac{\sin x}{1} + \dfrac{\sin 3x}{3} + \ldots\right)$

9. $f(x) = \dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin nx$

10. (*i*) $f(x) = \dfrac{\pi^2}{6} + 2\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx + \sum_{n=1}^{\infty}\left[\frac{\pi}{n}(-1)^{n+1} + \frac{2\{(-1)^n - 1\}}{\pi n^3}\right]\sin nx$

(*ii*) $f(x) = \dfrac{\pi}{4} - \dfrac{2}{\pi}\displaystyle\sum_{\substack{n=1 \\ (n \text{ is odd})}}^{\infty}\frac{1}{n^2}\cos nx - \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$

3.38 CHANGE OF INTERVAL

In many engineering problems, it is desired to expand a function in a Fourier series over an interval of length $2l$ and not 2π. In order to apply foregoing theory, this interval must be transformed into an interval of length 2π. This can be achieved by a transformation of the variable.

Consider a periodic function $f(x)$ defined in the interval $c < x < c + 2l$. To change the interval into one of length 2π, we put

$$\frac{x}{l} = \frac{z}{\pi} \quad \text{or} \quad z = \frac{\pi x}{l} \quad \text{so that}$$

when $\quad x = c, \quad z = \frac{\pi c}{l} = d \text{ (say)}$

and $\quad$ when $\quad x = c + 2l, \quad z = \frac{\pi(c + 2l)}{l} = \frac{\pi c}{l} + 2\pi = d + 2\pi.$

Thus the function $f(x)$ of period $2l$ in $(c, c + 2l)$ is transformed to the function $f\left(\frac{lz}{\pi}\right) = F(z)$, say, of period 2π in $(d, d + 2\pi)$ and the latter function can be expressed as the Fourier series

$$F(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nz + \sum_{n=1}^{\infty} b_n \sin nz \qquad \text{...(1)}$$

where $\quad a_0 = \frac{1}{\pi}\int_d^{d+2\pi} F(z)\, dz \,; \quad a_n = \frac{1}{\pi}\int_d^{d+2\pi} F(z) \cos nz\, dz \,; \quad$ and $\quad b_n = \frac{1}{\pi}\int_d^{d+2\pi} F(z) \sin nz\, dz$

Now making the inverse substitution $z = \frac{\pi x}{l}$, $dz = \frac{\pi}{l}\, dx$

When $\quad z = d, \quad x = c \quad$ and when $\quad z = d + 2\pi, x = c + 2l.$

The expression (1) becomes $F(z) = F\left(\frac{\pi x}{l}\right) = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$

and the coefficients a_0, a_n, b_n from (2) reduce to

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\, dx \,; \quad a_n = \frac{1}{l}\int_c^{c+2l} f(x) \cos \frac{n\pi x}{l}\, dx \,; \quad \text{and} \quad b_n = \frac{1}{l}\int_c^{c+2l} f(x) \sin \frac{n\pi x}{l}\, dx$$

Hence the Fourier series $f(x)$ in the interval $c < x < c + 2l$ is given by

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}}$$

where $\quad a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\, dx, \quad a_n = \frac{1}{l}\int_c^{c+2l} f(x) \cos \frac{n\pi x}{l}\, dx \quad$ and $\quad b_n = \frac{1}{l}\int_c^{c+2l} f(x) \sin \frac{n\pi x}{l}\, dx.$

Cor. 1. If we put $c = 0$, the interval becomes $0 < x < 2l$ and the above results reduce to

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\, dx \,; \quad a_n = \frac{1}{l}\int_0^{2l} f(x) \cos \frac{n\pi x}{l}\, dx \,; \quad \text{and} \quad b_n = \frac{1}{l}\int_0^{2l} f(x) \sin \frac{n\pi x}{l}\, dx.$$

Cor. 2. If we put $c = -l$, the interval becomes $-l < x < l$ and the above results reduce to

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx, \quad a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx, \quad \text{and} \quad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx.$$

Cor. 3. If $f(x)$ is an even function, we have

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx, \quad a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx \quad \text{and} \quad b_n = 0$$

Cor. 4. If $f(x)$ is an odd function, we have $a_0 = 0$, $a_n = 0$ and $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx.$

ILLUSTRATIVE EXAMPLES

Example 1. *Obtain the Fourier series expansion of*

$$f(x) = \left(\frac{\pi - x}{2}\right) \text{ for } 0 < x < 2.$$

[M.T.U. 2011]

Sol. Let $f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\dfrac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\dfrac{n\pi x}{l}$

Here, $\quad l = 1$

$$\therefore \quad \frac{\pi - x}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x \quad \text{...(1)}$$

Here,

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = \int_0^2 \left(\frac{\pi - x}{2}\right) dx = \frac{1}{2}\left(\pi x - \frac{x^2}{2}\right)_0^2 = \frac{1}{2}(2\pi - 2) = \pi - 1$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos n\pi x\,dx = \int_0^2 \left(\frac{\pi - x}{2}\right)\cos n\pi x\,dx$$

$$= \frac{1}{2}\left[\left\{(\pi - x)\frac{\sin n\pi x}{n\pi}\right\}_0^2 - \int_0^2 (-1)\frac{\sin n\pi x}{n\pi}\,dx\right] = \frac{1}{2n\pi}\left(\frac{-\cos n\pi x}{n\pi}\right)_0^2 = 0$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin n\pi x\,dx = \int_0^2 \left(\frac{\pi - x}{2}\right)\sin n\pi x\,dx$$

$$= \frac{1}{2}\left[\left\{(\pi - x)\left(\frac{-\cos n\pi x}{n\pi}\right)\right\}_0^2 - \int_0^2 (-1)\left(\frac{-\cos n\pi x}{n\pi}\right) dx\right]$$

$$= -\frac{1}{2n\pi}[(\pi - 2) - \pi] = \frac{1}{n\pi}$$

Hence, from (1),

$$\frac{\pi - x}{2} = \frac{(\pi - 1)}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\sin n\pi x$$

Example 2. *Find Fourier expansion for the function $f(x) = x - x^2$, $-1 < x < 1$.*

Sol. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$

Then $a_0 = \int_{-1}^{1} (x - x^2)\, dx = \int_{-1}^{1} x\, dx - \int_{-1}^{1} x^2\, dx = 0 - 2\int_0^1 x^2\, dx = -2\left[\frac{x^3}{3}\right]_0^1 = -\frac{2}{3}$

$$a_n = \int_{-1}^{1} (x - x^2) \cos n\pi x\, dx = \int_{-1}^{1} x \cos n\pi x\, dx - \int_{-1}^{1} x^2 \cos n\pi x\, dx$$

$$= 0 - 2\int_0^1 x^2 \cos n\pi x\, dx = -2\left[\left(x^2 \frac{\sin n\pi x}{n\pi}\right)_0^1 - \int_0^1 2x \cdot \frac{\sin n\pi x}{n\pi}\, dx\right]$$

$$= \frac{4}{n\pi}\int_0^1 x \sin n\pi x\, dx = \frac{4}{n\pi}\left[\left\{x \cdot \left(\frac{-\cos n\pi x}{n\pi}\right)\right\}_0^1 - \int_0^1 \left(\frac{-\cos n\pi x}{n\pi}\right) dx\right]$$

$$= -\frac{4}{n^2\pi^2}\cos n\pi = \frac{-4(-1)^n}{n^2\pi^2}$$

$$b_n = \int_{-1}^{1} (x - x^2) \sin n\pi x\, dx = \int_{-1}^{1} x \sin n\pi x\, dx - \int_{-1}^{1} x^2 \sin n\pi x\, dx$$

$$= 2\int_0^1 x \sin n\pi x\, dx - 0 = 2\left[x\left(-\frac{\cos n\pi x}{n\pi}\right) - 1 \cdot \left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$= 2\left[-\frac{\cos n\pi}{n\pi}\right] = \frac{-2(-1)^n}{n\pi}$$

$$\therefore\ x - x^2 = -\frac{1}{3} + \frac{4}{\pi^2}\left(\frac{\cos \pi x}{1^2} - \frac{\cos 2\pi x}{2^2} + \frac{\cos 3\pi x}{3^2} - \ldots\ldots\right)$$

$$+ \frac{2}{\pi}\left(\frac{\sin \pi x}{1} - \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} - \ldots\ldots\right)$$

Example 3. *Find the Fourier series to represent $f(x) = x^2 - 2$, when $-2 \le x \le 2$.*

Sol. Since $f(x)$ is an even function, $b_n = 0$.

Let
$$f(x) = x^2 - 2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2}$$

Then,
$$a_0 = \frac{2}{2}\int_0^2 (x^2 - 2)\, dx = \left[\frac{x^3}{3} - 2x\right]_0^2 = \frac{8}{3} - 4 = -\frac{4}{3}$$

$$a_n = \frac{2}{2}\int_0^2 (x^2 - 2) \cos \frac{n\pi x}{2}\, dx$$

$$= \left\{(x^2-2)\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right\}_0^2 - \int_0^2 (2x)\frac{\sin\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)}\,dx$$

$$= -\frac{4}{n\pi}\int_0^2 x\sin\frac{n\pi x}{2}\,dx$$

$$= -\frac{4}{n\pi}\left[\left\{x\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right)\right\}_0^2 - \int_0^2\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right)dx\right]$$

$$= \frac{8}{n^2\pi^2}(2\cos n\pi) = \frac{16\cos n\pi}{n^2\pi^2} = \frac{16(-1)^n}{n^2\pi^2}$$

$$\therefore \quad x^2-2 = -\frac{2}{3} - \frac{16}{\pi^2}\left(\cos\frac{\pi x}{2} - \frac{1}{4}\cos\pi x + \frac{1}{9}\cos\frac{3\pi x}{2} - \ldots\ldots\right).$$

Example 4. *Expand $f(x) = e^{-x}$ as a Fourier series in the interval $(-l, l)$.*

Sol. Let $f(x) = e^{-x} = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\dfrac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\dfrac{n\pi x}{l}$

Then $$a_0 = \frac{1}{l}\int_{-l}^{l} e^{-x}\,dx = \frac{1}{l}\left[-e^{-x}\right]_{-l}^{l} = \frac{1}{l}(e^l - e^{-l}) = \frac{2\sinh l}{l}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} e^{-x}\cos\frac{n\pi x}{l}\,dx = \frac{1}{l}\left[\frac{e^{-x}}{1+\left(\frac{n\pi}{l}\right)^2}\left(-\cos\frac{n\pi x}{l} + \frac{n\pi}{l}\sin\frac{n\pi x}{l}\right)\right]_{-l}^{l}$$

$$= \frac{l}{l^2+(n\pi)^2}\left[-e^{-l}\cos n\pi + e^l\cos n\pi\right] = \frac{2l\cos n\pi}{l^2+(n\pi)^2}\left(\frac{e^l - e^{-l}}{2}\right)$$

$$= \frac{2l(-1)^n\sinh l}{l^2+n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} e^{-x}\sin\frac{n\pi x}{l}\,dx = \frac{1}{l}\left[\frac{e^{-x}}{1+\left(\frac{n\pi}{l}\right)^2}\left(-\sin\frac{n\pi x}{l} - \frac{n\pi}{l}\cos\frac{n\pi x}{l}\right)\right]_{-l}^{l}$$

$$= -\frac{l}{l^2+(n\pi)^2}\left[\frac{n\pi}{l}(e^{-l} - e^l)\cos n\pi\right] = \frac{2n\pi\cos n\pi}{l^2+(n\pi)^2}\left(\frac{e^l - e^{-l}}{2}\right) = \frac{2n\pi(-1)^n\sinh l}{l^2+n^2\pi^2}$$

$$\therefore \quad e^{-x} = \frac{\sinh l}{l} + 2l\sinh l\sum_{n=1}^{\infty}\frac{(-1)^n}{l^2+n^2\pi^2}\cos\frac{n\pi x}{l} + 2\pi\sinh l\sum_{n=1}^{\infty}\frac{n(-1)^n}{l^2+n^2\pi^2}\sin\frac{n\pi x}{l}.$$

Example 5. *Obtain Fourier series for* $f(x) = \begin{cases} \pi x, & 0 \le x \le 1 \\ \pi(2-x), & 1 \le x \le 2 \end{cases}$.

Sol. Let $f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$

Then $a_0 = \int_0^2 f(x)\,dx = \int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx = \pi\left[\dfrac{x^2}{2}\right]_0^1 + \pi\left[2x - \dfrac{x^2}{2}\right]_1^2$

$$= \pi\left(\frac{1}{2}\right) + \pi\left[(4-2) - \left(2 - \frac{1}{2}\right)\right] = \pi$$

$$a_n = \int_0^2 f(x)\cos n\pi x\,dx = \int_0^1 \pi x \cos n\pi x\,dx + \int_1^2 \pi(2-x)\cos n\pi x\,dx$$

$$= \left[\pi x \cdot \frac{\sin n\pi x}{n\pi} - \pi\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_0^1 + \left[\pi(2-x)\cdot\frac{\sin n\pi x}{n\pi} - (-\pi)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$= \left[\frac{\cos n\pi}{n^2\pi} - \frac{1}{n^2\pi}\right] + \left[-\frac{\cos 2n\pi}{n^2\pi} + \frac{\cos n\pi}{n^2\pi}\right] = \frac{2}{n^2\pi}(\cos n\pi - 1) = \frac{2}{n^2\pi}[(-1)^n - 1]$$

$= 0$ or $-\dfrac{4}{n^2\pi}$ according as n is even or odd.

$$b_n = \int_0^2 f(x)\sin n\pi x\,dx = \int_0^1 \pi x \sin n\pi x\,dx + \int_1^2 \pi(2-x)\sin n\pi x\,dx$$

$$= \left[\pi x\left(-\frac{\cos n\pi x}{n\pi}\right) - \pi\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_0^1 + \left[\pi(2-x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-\pi)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$= \left[-\frac{\cos n\pi}{n}\right] + \left[\frac{\cos n\pi}{n}\right] = 0$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \ldots\ldots\right).$$

TEST YOUR KNOWLEDGE

1. Find a Fourier series for $f(t) = 1 - t^2$ when $-1 \le t \le 1$.
2. (*i*) Expand $f(x)$ in Fourier series in the interval $(-2, 2)$ when $f(x) = \begin{cases} 0, & -2 < x < 0 \\ 1, & 0 < x < 2. \end{cases}$

   (*ii*) Find the Fourier series of $f(x) = \begin{cases} 0 &, \text{ if } -2 \le x \le -1 \\ 1+x, & \text{if } -1 \le x \le 0 \\ 1-x, & \text{if } 0 \le x \le 1 \\ 0 &, \text{ if } 1 \le x \le 2 \end{cases}$ **[Hint.** $f(x)$ is an even function**]**
3. Develop $f(x)$ in a Fourier series in the interval $(0, 2)$ if $f(x) = \begin{cases} x, & 0 < x < 1 \\ 0, & 1 < x < 2. \end{cases}$
4. (*i*) Find the Fourier expansion for $f(x) = \pi x$ from $x = -c$ to $x = c$.

   (*ii*) Find the Fourier series expansion of $f(x) = 1 + |x|$ defined in $-3 < x < 3$.

   [*M.T.U. (SUM) 2011*]

5. Find the Fourier expansion for the function $f(x) = x - x^3$ in the interval $-1 < x < 1$.

6. Find the Fourier series for the function given by $f(t) = \begin{cases} t, & 0 < t < 1 \\ 1-t, & 1 < t < 2. \end{cases}$ (U.P.T.U. 2015)

Hence deduce that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$. (G.B.T.U. 2011)

7. Find the Fourier series for the function $f(x) = \begin{cases} 0, & \text{when } -2 < x < -1 \\ k, & \text{when } -1 < x < 1 \\ 0, & \text{when } 1 < x < 2 \end{cases}$.

8. A sinusoidal voltage E sin ωt is passed through a half-wave rectifier which clips the negative portion of the wave. Expand the resulting periodic function $u(t) = \begin{cases} 0, & \text{when } -\frac{T}{2} < t < 0 \\ E \sin \omega t, & \text{when } 0 < t < \frac{T}{2} \end{cases}$ and $T = \frac{2\pi}{\omega}$, in a Fourier series.

9. Obtain Fourier series of the function $F(x) = \begin{cases} 1 + \frac{2x}{l}, & -l < x < 0 \\ 1 - \frac{2x}{l}, & 0 < x < l \end{cases}$

10. Find the Fourier series expansion of the periodic function whose definition in one period is $f(x) = 4 - x^2$, $-2 \le x \le 2$. Also, prove that:

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{12}.$$

Answers

1. $1 - t^2 = \frac{2}{3} + \frac{4}{\pi^2}\left(\cos \pi t - \frac{\cos 2\pi t}{2^2} + \frac{\cos 3\pi t}{3^2} - \ldots\ldots\right)$

2. (i) $f(x) = \frac{1}{2} + \frac{2}{\pi^2}\left(\sin \frac{\pi x}{2} + \frac{1}{3}\sin \frac{3\pi x}{2} + \frac{1}{5}\sin \frac{5\pi x}{2} + \ldots\ldots\right)$

(ii) $f(x) = \frac{1}{4} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{\left(1 - \cos \frac{n\pi}{2}\right)}{n^2} \cos \frac{n\pi x}{2}$

3. $f(x) = \frac{1}{4} - \frac{2}{\pi^2}\left(\cos \pi x + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \ldots\ldots\right) + \frac{1}{\pi}\left(\sin \pi x - \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} + \ldots\ldots\right)$

4. (i) $f(x) = 2c\left[\sin\left(\frac{\pi x}{c}\right) - \frac{1}{2}\sin\left(\frac{2\pi x}{c}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{c}\right) - \ldots\ldots\right]$

(ii) $f(x) = \frac{5}{2} - \frac{12}{\pi^2}\left[\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right]$

5. $f(x) = \frac{12}{\pi^3}\left(\sin \pi x - \frac{\sin 2\pi x}{2^3} + \frac{\sin 3\pi x}{3^3} - \ldots\ldots\right)$

6. $f(t) = -\frac{4}{\pi^2}\left(\cos \pi t + \frac{\cos 3\pi t}{3^2} + \frac{\cos 5\pi t}{5^2} + \ldots\ldots\right) + \frac{2}{\pi}\left(\sin \pi t + \frac{\sin 3\pi t}{3} + \ldots\ldots\right)$

7. $f(x) = \frac{k}{2} + \frac{2k}{\pi}\left(\cos\frac{\pi x}{2} - \frac{1}{3}\cos\frac{3\pi x}{2} + \frac{1}{5}\cos\frac{5\pi x}{2} - \ldots\ldots\right)$

8. $u(t) = \frac{E}{\pi} + \frac{E}{2}\sin\omega t - \frac{2E}{\pi}\left(\frac{\cos 2\omega t}{1\cdot 3} + \frac{\cos 4\omega t}{3\cdot 5} + \frac{\cos 6\omega t}{5\cdot 7} + \ldots\ldots\right)$

9. $F(x) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\{1-(-1)^n\}\frac{\cos n\frac{\pi x}{l}}{n^2}$

10. $f(x) = \frac{8}{3} - \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{2}$

3.39 HALF RANGE SERIES

Sometimes it is required to expand a function $f(x)$ in the range $(0, \pi)$ in a Fourier series of period 2π or more generally in the range $(0, l)$ in a Fourier series of period $2l$.

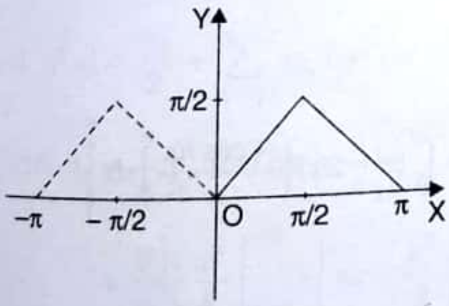
If it is required to expand $f(x)$ in the interval $(0, l)$, then it is immaterial what the function may be outside the range $0 < x < l$. We are free to choose it arbitrarily in the interval $(-l, 0)$.

If we extend the function $f(x)$ by reflecting it in the y-axis so that $f(-x) = f(x)$, then the extended function is even for which $b_n = 0$. The Fourier expansion of $f(x)$ will contain only cosine terms.

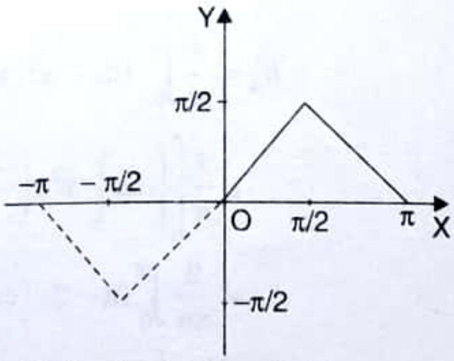
If we extend the function $f(x)$ by reflecting it in the origin so that $f(-x) = -f(x)$, then the extended function is odd for which $a_0 = a_n = 0$. The Fourier expansion of $f(x)$ will contain only sine terms.

For example, consider the function

$$f(x) = \begin{cases} x, & 0 < x < \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} < x < \pi \end{cases}$$



(Reflection in the y-axis)



(Reflection in the origin)

Hence a function $f(x)$ defined over the interval $0 < x < l$ is capable of two distinct half-range series.

The half-range cosine series is

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}}$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx\,;\quad a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx.$

The half-range sine series is

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}} \quad \text{where} \quad b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx.$$

Cor. If the range is $0 < x < \pi$, then

(*i*) The half-range cosine series is

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx}$$

where $a_0 = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x)\, dx$; $a_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x) \cos nx\, dx.$

(*ii*) The half-range sine series is

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin nx} \quad \text{where} \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Expand $\pi x - x^2$ in a half-range sine series in the interval $(0, \pi)$ up to the first three terms.*

Sol. Let $\pi x - x^2 = \sum_{n=1}^{\infty} b_n \sin nx$, where

$$b_n = \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2) \sin nx\, dx$$

$$= \frac{2}{\pi}\left[\left\{(\pi x - x^2)\left(\frac{-\cos nx}{n}\right)\right\}_0^{\pi} - \int_0^{\pi} (\pi - 2x)\left(\frac{-\cos nx}{n}\right) dx\right]$$

$$= \frac{2}{\pi n}\int_0^{\pi} (\pi - 2x) \cos nx\, dx$$

$$= \frac{2}{\pi n}\left[\left\{(\pi - 2x)\frac{\sin nx}{n}\right\}_0^{\pi} - \int_0^{\pi} (-2)\frac{\sin nx}{n}\, dx\right]$$

$$= \frac{4}{\pi n^2}\left(\frac{-\cos nx}{n}\right)_0^{\pi} = \frac{4}{\pi n^3}\left\{1 - (-1)^n\right\}$$

$$= 0 \quad \text{or} \quad \frac{8}{\pi n^3} \text{ according as } n \text{ is even or odd.}$$

$$\therefore \qquad \pi x - x^2 = \frac{8}{\pi}\left(\sin x + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \ldots\ldots\right).$$

$C + 2\pi$
$\cancel{C} + 2\pi =$
$2\pi = 2$

Example 2. *If* $f(x) = \begin{cases} x, & 0 < x < \dfrac{\pi}{2} \\ \pi - x, & \dfrac{\pi}{2} < x < \pi \end{cases}$

show that (i) $f(x) = \dfrac{4}{\pi}\left[\sin x - \dfrac{\sin 3x}{3^2} + \dfrac{\sin 5x}{5^2} - \dots\dots\right]$ (U.P.T.U. 2014)

(ii) $f(x) = \dfrac{\pi}{4} - \dfrac{2}{\pi}\left[\dfrac{\cos 2x}{1^2} + \dfrac{\cos 6x}{3^2} + \dfrac{\cos 10x}{5^2} + \dots\dots\right].$

Sol. (*i*) **For the half-range sine series,**

Let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

where, $b_n = \dfrac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx = \dfrac{2}{\pi}\left[\int_0^{\pi/2} x \sin nx\, dx + \int_{\pi/2}^{\pi} (\pi - x)\sin nx\, dx\right]$

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - 1.\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi/2} + \frac{2}{\pi}\left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right] + \frac{2}{\pi}\left[\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{n^2}\sin\frac{n\pi}{2}\right] = \frac{4}{\pi n^2}\sin\frac{n\pi}{2}$$

When n is even, $b_n = 0$.

$$\therefore \quad f(x) = \frac{4}{\pi}\left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \dots\dots\right]$$

(*ii*) **For the half-range cosine series,**

Let $f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

Then $a_0 = \dfrac{2}{\pi}\int_0^{\pi} f(x)\, dx = \dfrac{2}{\pi}\left[\int_0^{\pi/2} x\, dx + \int_{\pi/2}^{\pi} (\pi - x)\, dx\right]$

$$= \frac{2}{\pi}\left[\left|\frac{x^2}{2}\right|_0^{\pi/2} + \left|\pi x - \frac{x^2}{2}\right|_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{8} + \left(\pi^2 - \frac{\pi^2}{2}\right) - \left(\frac{\pi^2}{2} - \frac{\pi^2}{8}\right)\right] = \frac{2}{\pi}\left[\frac{\pi^2}{4}\right] = \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx = \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos nx\, dx + \int_{\pi/2}^{\pi} (\pi - x)\cos nx\, dx\right]$$

$$= \frac{2}{\pi}\left[x.\frac{\sin nx}{n} - 1\cdot\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi/2} + \frac{2}{\pi}\left[(\pi - x).\frac{\sin nx}{n} - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2} - \frac{1}{n^2}\right] + \frac{2}{\pi}\left[-\frac{\cos n\pi}{n^2} - \frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2}\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{n^2}\cos\frac{n\pi}{2} - \frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi n^2}\left[2\cos\frac{n\pi}{2} - \cos n\pi - 1\right]$$

$$\therefore \quad a_1 = 0, a_2 = \frac{2}{\pi . 2^2}(2\cos\pi - \cos 2\pi - 1) = \frac{-2}{\pi . 1^2},$$

$$a_3 = 0, a_4 = 0, a_5 = 0, a_6 = \frac{2}{\pi \cdot 10^2}(2\cos 3\pi - \cos 6\pi - 1) = \frac{-2}{\pi . 5^2},$$

$$a_7 = 0, a_8 = 0, a_9 = 0, a_{10} = \frac{2}{\pi . 10^2}(2\cos 5\pi - \cos 10\pi - 1) = \frac{-2}{\pi . 5^2}, \ldots\ldots$$

Hence $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \ldots\ldots\right].$

Example 3. *Find a series of cosines of multiples of x which will represent x sin x in the interval (0, π) and show that* $\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \ldots\ldots = \frac{\pi - 2}{4}.$ (A.K.T.U. 2017)

Sol. Let $x \sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

Here, $a_0 = \frac{2}{\pi}\int_0^{\pi} x \sin x \, dx = \frac{2}{\pi}\Big[x(-\cos x) - 1.(-\sin x)\Big]_0^{\pi} = \frac{2}{\pi}[-\pi\cos\pi] = 2$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos nx \, dx = \frac{1}{\pi}\int_0^{\pi} x\,(2\cos nx \sin x)\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\,[\sin(n+1)x - \sin(n-1)x]\, dx$$

$$= \frac{1}{\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - 1.\left\{-\frac{\sin(n+1)x}{(n+1)^2} - \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\pi\cos(n+1)\pi}{n+1} + \frac{\pi\cos(n-1)\pi}{n-1}\right] \quad \text{when } n \neq 1$$

$$= -\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} = (-1)^{n-1}\left[\frac{1}{n-1} - \frac{1}{n+1}\right] = \frac{2(-1)^{n-1}}{(n-1)(n+1)}$$

When $n = 1$, we have

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x \, dx = \frac{1}{\pi}\int_0^{\pi} x \sin 2x \, dx = \frac{1}{\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - 1.\left(-\frac{\sin 2x}{2^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\pi\cos 2\pi}{2}\right] = -\frac{1}{2}$$

$$\therefore \quad x\sin x = 1 - \frac{1}{2}\cos x - 2\left(\frac{\cos 2x}{1\cdot 3} - \frac{\cos 3x}{2\cdot 4} + \frac{\cos 4x}{3\cdot 5} - \ldots\ldots\right).$$

Putting $x = \frac{\pi}{2}$, we get

$$\frac{\pi}{2} = 1 - 2\left(\frac{-1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{-1}{5\cdot 7} - \ldots\ldots\right)$$

$$\Rightarrow \quad 1 + \frac{2}{1\cdot 3} - \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} - \ldots\ldots = \frac{\pi}{2}$$

$$\Rightarrow \quad \frac{2}{1\cdot 3} - \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} - \ldots\ldots = \frac{\pi}{2} - 1$$

Hence,
$$\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \ldots\ldots = \frac{\pi - 2}{4}.$$

Example 4. *Obtain the half-range sine series for e^x in $0 < x < 1$.*

Sol. Let
$$e^x = \sum_{n=1}^{\infty} b_n \sin n\pi x, \text{ (since } l = 1)$$

Here,
$$b_n = 2\int_0^1 e^x \sin n\pi x \, dx = 2\left[\frac{e^x}{1+(n\pi)^2}(\sin n\pi x - n\pi \cos n\pi x)\right]_0^1$$

$$= 2\left[\frac{e}{1+(n\pi)^2}(-n\pi\cos n\pi) - \frac{1}{1+(n\pi)^2}(-n\pi)\right]$$

$$= \frac{2}{1+n^2\pi^2}\,[-en\pi(-1)^n + n\pi] = \frac{2n\pi}{1+n^2\pi^2}\,[1 - e(-1)^n]$$

Hence,
$$e^x = 2\pi \sum_{n=1}^{\infty} \frac{n[1-e(-1)^n]}{1+n^2\pi^2} \sin n\pi x.$$

Example 5. *Develop $f(x) = \sin\left(\frac{\pi x}{l}\right)$ in half-range cosine series in the range $0 < x < l$. Graph the corresponding periodic continuation of $f(x)$.*

Sol. Let
$$\sin\left(\frac{\pi x}{l}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

where,
$$a_0 = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l}\,dx = \frac{2}{l}\left[-\frac{\cos\frac{\pi x}{l}}{\frac{\pi}{l}}\right]_0^l = -\frac{2}{\pi}\,[\cos\pi - 1] = \frac{4}{\pi}$$

$$a_n = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l}\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\int_0^l \left[\sin(n+1)\frac{\pi x}{l} - \sin(n-1)\frac{\pi x}{l}\right] dx$$

$$= \frac{1}{l}\left[-\frac{\cos(n+1)\frac{\pi x}{l}}{(n+1)\frac{\pi}{l}} + \frac{\cos(n-1)\frac{\pi x}{l}}{(n-1)\frac{\pi}{l}}\right]_0^l$$

$$= \frac{1}{\pi}\left[\left\{-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right\} - \left\{-\frac{1}{n+1} + \frac{1}{n-1}\right\}\right]$$

$$= \frac{1}{\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

When n is odd, $$a_n = \frac{1}{\pi}\left[-\frac{1}{n+1}+\frac{1}{n-1}+\frac{1}{n+1}-\frac{1}{n-1}\right] = 0\ ;\ n \neq 1$$

When n is even, $$a_n = \frac{1}{\pi}\left[\frac{1}{n+1}-\frac{1}{n-1}+\frac{1}{n+1}-\frac{1}{n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{n+1}-\frac{1}{n-1}\right] = -\frac{4}{\pi(n+1)(n-1)}$$

When $n = 1$, $$a_1 = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l}\cos\frac{\pi x}{l}\,dx$$

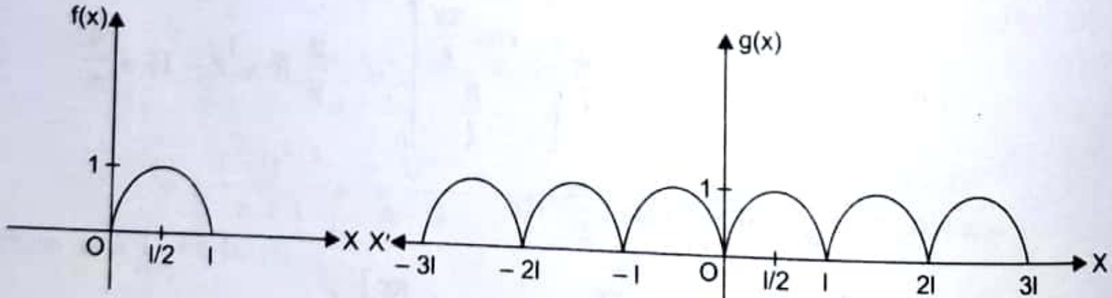
$$= \frac{1}{l}\int_0^l \sin\frac{2\pi x}{l}\,dx = \frac{1}{l}\left(\frac{-\cos\frac{2\pi x}{l}}{\frac{2\pi}{l}}\right)_0^l = \frac{1}{2\pi}(1-\cos 2\pi) = 0$$

$$\therefore \qquad \sin\left(\frac{\pi x}{l}\right) = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos\frac{2\pi x}{l}}{1.3} + \frac{\cos\frac{4\pi x}{l}}{3.5} + \frac{\cos\frac{6\pi x}{l}}{5.7} + \ldots\ldots\right].$$

In $0 < x < l$, $f(x)$ is neither periodic nor odd nor even. So, we construct

$$g(x) = \begin{cases} \sin\dfrac{\pi x}{l} & \text{in } 0 < x < l \\ -\sin\dfrac{\pi x}{l} & \text{in } -l < x < 0 \end{cases}$$

which is even, periodic with period $2l$.



Example 6. *Obtain a half-range cosine series for*

$$f(x) = \begin{cases} kx, & \textit{for} \quad 0 < x < \dfrac{l}{2} \\ k(l-x), & \textit{for} \quad \dfrac{l}{2} \le x \le l \end{cases}.$$

Deduce the sum of the series $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots\ldots$

Sol. Let $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

where, $$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\left[\int_0^{l/2} kx\,dx + \int_{l/2}^{l} k(l-x)\,dx\right]$$

$$= \frac{2}{l}\left[\left|\frac{kx^2}{2}\right|_0^{l/2} + \left|k\left(lx - \frac{x^2}{2}\right)\right|_{l/2}^{l}\right]$$

$$= \frac{2}{l}\left[\frac{kl^2}{8} + k\left(l^2 - \frac{l^2}{2}\right) - k\left(\frac{l^2}{2} - \frac{l^2}{8}\right)\right] = \frac{2}{l}\left(\frac{kl^2}{4}\right) = \frac{kl}{2}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\left[\int_0^{l/2} kx\,.\cos\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} k(l-x)\,.\cos\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2}{l}\left[\left|kx\,.\frac{l}{n\pi}\sin\frac{n\pi x}{l} + k\,.\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right|_0^{l/2} + \left|k(l-x)\,.\frac{l}{n\pi}\sin\frac{n\pi x}{l} - k\,.\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right|_{l/2}^{l}\right]$$

$$= \frac{2}{l}\left[\left|\frac{kl^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{kl^2}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)\right| + \left|\frac{-kl^2}{n^2\pi^2}\cos n\pi - \frac{kl^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{kl^2}{n^2\pi^2}\cos\frac{n\pi}{2}\right|\right]$$

$$= \frac{2}{l}\left[\frac{2kl^2}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{kl^2}{n^2\pi^2} - \frac{kl^2}{n^2\pi^2}\cos n\pi\right] = \frac{2kl}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - 1 - \cos n\pi\right]$$

When n is odd, $\cos\frac{n\pi}{2} = 0$ and $\cos n\pi = -1$ $\quad\therefore\quad a_n = 0 \Rightarrow a_1 = a_3 = a_5 = \ldots\ldots = 0$

When n is even, $$a_2 = \frac{2kl}{2^2\pi^2}[2\cos\pi - 1 - \cos 2\pi] = -\frac{8kl}{2^2\pi^2};$$

$$a_4 = \frac{2kl}{4^2\pi^2}[2\cos 2\pi - 1 - \cos 4\pi] = 0$$

$$a_6 = \frac{2kl}{6^2\pi^2}[2\cos 3\pi - 1 - \cos 6\pi]$$

$$= \frac{2kl}{6^2\pi^2}(-2-1-1) = -\frac{8kl}{6^2\pi^2} \text{ and so on.}$$

$$\therefore \quad f(x) = \frac{kl}{4} - \frac{8kl}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{6^2}\cos\frac{6\pi x}{l} + \ldots\ldots\right) \quad \ldots(1)$$

Putting $x = l$, $f(l) = 0$

$\therefore$ From (1), we have $0 = \frac{kl}{4} - \frac{8kl}{\pi^2}\left(\frac{1}{2^2} + \frac{1}{6^2} + \ldots\ldots\right)$

$$\Rightarrow \qquad \frac{1}{2^2}+\frac{1}{6^2}+\ldots\ldots=\frac{\pi^2}{32}$$

$$\Rightarrow \qquad \frac{1}{2^2}\left[\frac{1}{1^2}+\frac{1}{3^2}+\ldots\ldots\right]=\frac{\pi^2}{32}$$

Hence $$\frac{1}{1^2}+\frac{1}{3^2}+\ldots\ldots=\frac{\pi^2}{8}.$$

Example 7. *Let* $f(x)=\begin{cases}\omega x, & when \quad 0\le x\le \frac{l}{2}.\\ \omega(l-x), & when \quad \frac{l}{2}\le x\le l\end{cases}$

Show that $$f(x)=\frac{4\omega l}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sin\frac{(2n+1)\pi x}{l}.$$

Hence obtain the sum of the series

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots$$

(G.B.T.U. 2011)

Sol. For half-range sine series,

Let $f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ ...(1)

where, $$b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$=\frac{2}{l}\left[\int_0^{l/2}\omega x\sin\frac{n\pi x}{l}\,dx+\int_{l/2}^{l}\omega(l-x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$=\frac{2}{l}\left[\omega\cdot\left\{x\cdot\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right\}_0^{l/2}-\int_0^{l/2}\omega\cdot\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)dx\right.$$

$$\left.+\,\omega\left\{(l-x)\cdot\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right\}_{l/2}^{l}-\int_{l/2}^{l}(-\omega)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)dx\right]$$

$$=\frac{2}{l}\left[\omega\cdot\frac{l}{2}\cdot\frac{l}{n\pi}\left(-\cos\frac{n\pi}{2}\right)+\frac{\omega l}{n\pi}\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)_0^{l/2}\right.$$

$$\left.+\,\omega\cdot\frac{l}{2}\cdot\frac{l}{n\pi}\cos\frac{n\pi}{2}-\frac{\omega l}{n\pi}\left\{\sin\left(\frac{n\pi x}{l}\right)\Big/\frac{n\pi}{l}\right\}_{l/2}^{l}\right]$$

$$=\frac{2}{l}\left[\frac{\omega l^2}{n^2\pi^2}\sin\frac{n\pi}{2}-\frac{\omega l^2}{n^2\pi^2}\left(0-\sin\frac{n\pi}{2}\right)\right]=\frac{4\omega l}{n^2\pi^2}\sin\frac{n\pi}{2} \qquad ...(2)$$

$\therefore$ From (1), $$f(x) = \frac{4\omega l}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{l}$$

$$= \frac{4\omega l}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \sin (2n+1) \frac{\pi}{2} \sin \frac{(2n+1)\pi x}{l}$$

$$= \frac{4\omega l}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \sin \frac{(2n+1)\pi x}{l} \qquad ...(3)$$

Put $x = \frac{l}{2}$ in eqn.(3), we get

$$\frac{\omega l}{2} = \frac{4\omega l}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \sin (2n+1) \frac{\pi}{2} = \frac{4\omega l}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} (-1)^n$$

$$\Rightarrow \quad \frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$$

or $$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8}.$$

Example 8. *Expand $f(x) = x$ as a half range* (U.K.T.U. 2011)

(*i*) *sine series in* $0 < x < 2$

(*ii*) *cosine series in* $0 < x < 2$.

Sol. (*i*) Let $$x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2} \qquad ...(1)$$

where, $$b_n = \int_0^2 x \sin \frac{n\pi x}{2}\, dx = \left\{ x \left(\frac{-\cos \frac{n\pi x}{2}}{\frac{n\pi}{2}} \right) \right\}_0^2 - \int_0^2 \left(\frac{-\cos \frac{n\pi x}{2}}{\frac{n\pi}{2}} \right) dx$$

$$= -\frac{4}{n\pi} \cos n\pi = -\frac{4}{n\pi} (-1)^n$$

Hence from (1),

$$x = -\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin \frac{n\pi x}{2}$$

(*ii*) Let $$x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2} \qquad ...(2)$$

where, $$a_0 = \int_0^2 x\, dx = \left(\frac{x^2}{2} \right)_0^2 = 2$$

and $$a_n = \int_0^2 x \cos \frac{n\pi x}{2}\, dx = \left(x \frac{\sin \frac{n\pi x}{2}}{\frac{n\pi}{2}} \right)_0^2 - \int_0^2 \frac{\sin \frac{n\pi x}{2}}{\frac{n\pi}{2}}\, dx$$

$$= -\frac{2}{n\pi}\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right)_0^2 = \frac{4}{n^2\pi^2}(\cos n\pi - 1)$$

Hence, $$x = 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(\cos n\pi - 1)}{n^2}\cos\frac{n\pi x}{2}.$$

TEST YOUR KNOWLEDGE

1. Obtain cosine and sine series for $f(x) = x$ in the interval $0 \le x \le \pi$. Hence show that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$.
2. Find Fourier half-range even expansion of the function $f(x) = \left(-\frac{x}{l}\right) + 1$; $0 \le x \le l$.
3. Find the half-range cosine series for the function $f(x) = (x-1)^2$ in the interval $0 < x < 1$. Hence show that

   (i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{6}$ (ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{12}$

   (iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\ldots = \frac{\pi^2}{8}$.
4. Express $\sin x$ as a cosine series in $0 < x < \pi$.
5. Given the function $f(x) = x$, $0 < x < 1$, find:

   (i) Fourier cosine series for $f(x)$ (ii) Fourier sine series for $f(x)$
6. (i) If

$$f(x) = \begin{cases} \frac{\pi}{3}, & 0 \le x \le \frac{\pi}{3} \\ 0, & \frac{\pi}{3} \le x \le \frac{2\pi}{3} \\ -\frac{\pi}{3}, & \frac{2\pi}{3} \le x \le \pi \end{cases}$$

   then show that $f(x) = \frac{2}{\sqrt{3}}\left[\cos x - \frac{\cos 5x}{5} + \frac{\cos 7x}{7} - \ldots\ldots\right]$

   (ii) If

$$f(x) = \begin{cases} mx, & 0 \le x \le \frac{\pi}{2} \\ m(\pi - x), & \frac{\pi}{2} \le x \le \pi \end{cases}$$

   then show that $f(x) = \frac{4m}{\pi}\left[\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \ldots\ldots\right]$.
7. Prove that for all values of x lying between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, $x = \frac{4}{\pi}\left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \ldots\ldots\right]$.

8. (i) Obtain the half-range sine series for the function $f(x) = x^2$ is the interval $0 < x < 3$. *(U.P.T.U. 2015)*

(ii) Expand $f(x) = k$ for $0 < x < 2$ in a half-range sine series.

(iii) Find the half-range sine series of $f(x) = lx - x^2$ in the interval $(0, l)$. Hence, deduce that

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots$$ *(M.T.U. 2012)*

9. Show that the series $\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2n\pi x}{l}$ represents $\frac{l}{2} - x$ when $0 < x < l$.

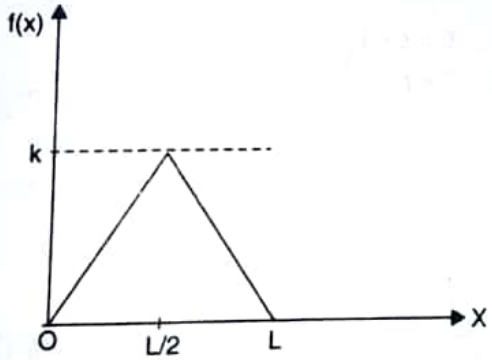
10. (i) Find the half-range sine series for *(U.K.T.U. 2012)*

$$f(x) = \begin{cases} \frac{1}{4} - x, & 0 < x < \frac{1}{2} \\ x - \frac{3}{4}, & \frac{1}{2} < x < 1. \end{cases}$$

(ii) Represent the following function by Fourier sine series

$$f(x) = \begin{cases} 1, \text{ when } & 0 < x < \frac{l}{2} \\ 0, \text{ when } & \frac{l}{2} < x < l. \end{cases}$$

11. (i) Find the half-range expansions of the function whose graph is given in followng figure.

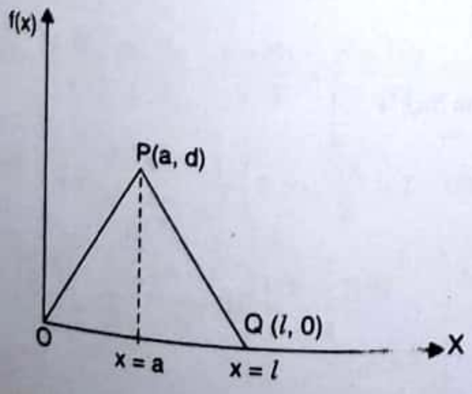


Or

(ii) Find the two half-range expansions of

$$f(x) = \begin{cases} \frac{2kx}{L}, & \text{if } 0 < x < \frac{L}{2} \\ \frac{2k(L-x)}{L}, & \text{if } \frac{L}{2} < x < L \end{cases}$$

12. Find the half period sine series for $f(x)$ given in the range $(0, l)$ by the graph OPQ as shown in figure:



$$\left[\textbf{Hint: } f(x) = \begin{cases} \frac{dx}{a}, & 0 < x < a \\ \frac{d(l-x)}{l-a}, & a < x < l \end{cases}\right]$$

13. (i) Show that in the range $0 < x < \pi$, $c = \frac{4}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\right] c$; where c is the constant function.

(ii) Show that in the interval (0, 1), $\cos \pi x = \frac{8}{\pi}\sum_{n=1}^{\infty} \frac{n}{4n^2 - 1} \sin 2n\pi x$.

14. (i) Find the half-range cosine series for the function $f(x) = x(\pi - x)$; $0 < x < \pi$ (U.P.T.U. 2015)

(ii) Find the half-range cosine series expansion of $f(x) = x - x^2$ in $0 < x < 1$. (G.B.T.U. 2012, 2013)

(iii) Find the half-range sine series for the function $f(t) = t - t^2$ in the interval $0 < t < 1$. (M.T.U. 2011)

(iv) Find the half-range sine series for the function $f(x) = 2x - 1$; $0 < x < 1$.

(v) Find the half-range sine series for $f(x) = x + x^2$, $0 < x < 1$. (U.P.T.U. 2014)

(vi) Expand $f(x) = 2x - 1$ as a cosine series in $0 < x < 2$. (A.K.T.U. 2017)

15. (i) Find the half range Fourier sine series of $f(x) = \begin{cases} \sin x, & 0 \le x \le \frac{\pi}{4} \\ \cos x, & \frac{\pi}{4} \le x \le \frac{\pi}{2} \end{cases}$. [M.T.U. (SUM) 2011]

(ii) Find the Fourier half-range cosine series of the function

$$F(t) = \begin{Bmatrix} 2t \quad , & 0 < t < 1 \\ 2(2 - t), & 1 < t < 2 \end{Bmatrix}$$ [A.K.T.U. 2018]

16. Find the half-range sine expansion of $f(t) = \begin{cases} t, & 0 < t < 2 \\ 4 - t, & 2 < t < 4 \end{cases}$ (U.P.T.U. 2015)

Answers

1. $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right)$; $f(x) = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots\right)$

2. $f(x) = \frac{1}{2} + \frac{4}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \ldots\right]$

3. $f(x) = \frac{1}{3} + \frac{4}{\pi^2}\left(\cos \pi x + \frac{\cos 2\pi x}{2^2} + \frac{\cos 3\pi x}{3^2} + \ldots\right)$

4. $\sin x = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos 2x}{1 \cdot 3} + \frac{\cos 4x}{3 \cdot 5} + \frac{\cos 6x}{5 \cdot 7} + \ldots\right]$

5. (i) $f(x) = \frac{1}{2} - \frac{4}{\pi^2}\left[\frac{1}{1^2}\cos \pi x + \frac{1}{3^2}\cos 3\pi x + \frac{1}{5^2}\cos 5\pi x + \ldots\right]$

(ii) $f(x) = -\frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin n\pi x$

8. (i) $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{3}$, where $b_n = \frac{18}{n\pi}(-1)^{n+1} + \frac{36}{n^3\pi^3}[(-1)^n - 1]$

(ii) $f(x) = \frac{4k}{\pi}\left[\sin \frac{\pi x}{2} + \frac{1}{3}\sin \frac{3\pi x}{2} + \frac{1}{5}\sin \frac{5\pi x}{2} + \ldots\right]$

(iii) $f(x) = \frac{8l^2}{\pi^3} \sum_{\substack{n=1 \\ (n \text{ is odd})}}^{\infty} \frac{1}{n^3} \sin \frac{n\pi x}{l}$

10. (i) $f(x) = \left(\frac{1}{\pi} - \frac{4}{\pi^2}\right)\sin \pi x + \left(\frac{1}{3\pi} - \frac{4}{3^2\pi^2}\right)\sin 3\pi x + \left(\frac{1}{5\pi} - \frac{4}{5^2\pi^2}\right)\sin 5\pi x + \ldots$

(ii) $f(x) = \frac{2}{\pi}\left[\sin \frac{\pi x}{l} + \sin \frac{2\pi x}{l} + \frac{1}{3}\sin \frac{3\pi x}{l} + \frac{1}{5}\sin \frac{5\pi x}{l} + \ldots\ldots\right]$

11. (i) $f(x) = \frac{k}{2} + \frac{4k}{\pi^2} \sum_{n=1}^{\infty} \left(\frac{2\cos \frac{n\pi}{2} - \cos n\pi - 1}{n^2}\right) \cos \frac{n\pi x}{L}$

(ii) $f(x) = \frac{8k}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{L}$

12. $f(x) = \sum_{n=1}^{\infty} \frac{2dl^2}{a(l-a)\, n^2\pi^2} \sin \frac{n\pi a}{l} \sin \frac{n\pi x}{l}$

14. (i) $f(x) = \frac{\pi^2}{6} - 4\left(\frac{\cos 2x}{2^2} + \frac{\cos 4x}{4^2} + \ldots\ldots\right)$ (ii) $f(x) = \frac{1}{6} - \frac{4}{\pi^2} \sum_{\substack{n=2 \\ (n \text{ is even})}}^{\infty} \frac{1}{n^2} \cos n\pi x$

(iii) $f(t) = \frac{8}{\pi^3}\left[\sin \pi t + \frac{1}{3^3}\sin 3\pi t + \frac{1}{5^3}\sin 5\pi t + \ldots\ldots\right]$

(iv) $f(x) = -\frac{2}{\pi}\left(\sin 2\pi x + \frac{1}{2}\sin 4\pi x + \frac{1}{3}\sin 6\pi x + \ldots\ldots\right)$

(v) $f(x) = 4 \sum_{n=1}^{\infty} \left(\frac{1}{n\pi} - \frac{1}{n^3\pi^3}\right)(1 - \cos n\pi) \sin n\pi x$

(vi) $f(x) = 1 + \frac{8}{\pi^2} \sum_{n=1}^{\infty} \left\{\frac{(-1)^n - 1}{n^2}\right\} \cos \frac{n\pi x}{2}$

15. (i) $f(x) = \frac{4\sqrt{2}}{\pi}\left[\frac{\sin 2x}{1 \cdot 3} - \frac{\sin 6x}{5 \cdot 7} + \frac{\sin 10x}{9 \cdot 11} - \ldots\ldots\right]$

(ii) $F(t) = 1 + \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}\left(2\cos \frac{n\pi}{2} - 1 - \cos n\pi\right) \cos \frac{n\pi t}{2}$

16. $f(t) = \frac{16}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \sin \frac{(2n+1)\pi t}{4}$

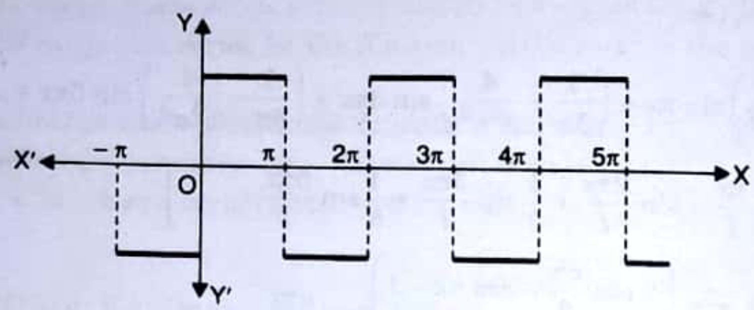
3.40 FOURIER SERIES OF DIFFERENT WAVEFORMS

Some typical waveforms usually met with in communication engineering are:

(1) **Square waveform.** This waveform is the extension of the function

$$f(x) = \begin{cases} -k, & -\pi < x < 0 \\ k, & 0 < x < \pi \end{cases}$$

where $f(x + 2\pi) = f(x)$ and k is a constant. It has the graph as shown in the diagram:



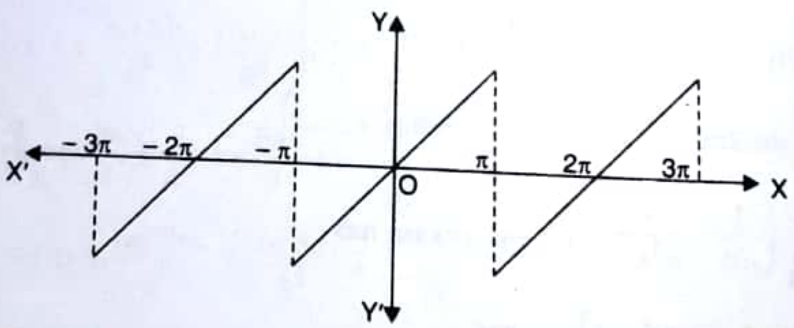
Fourier series is

$$f(x) = \frac{4k}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + ...\right)$$

(2) **Saw-toothed waveform.** This waveform is the extension of the function

$$f(x) = x, \quad -\pi < x < \pi; \quad f(x + 2\pi) = f(x)$$

This function represents the discontinuous function called **saw-toothed waveform.** It has the graph as shown in the diagram:



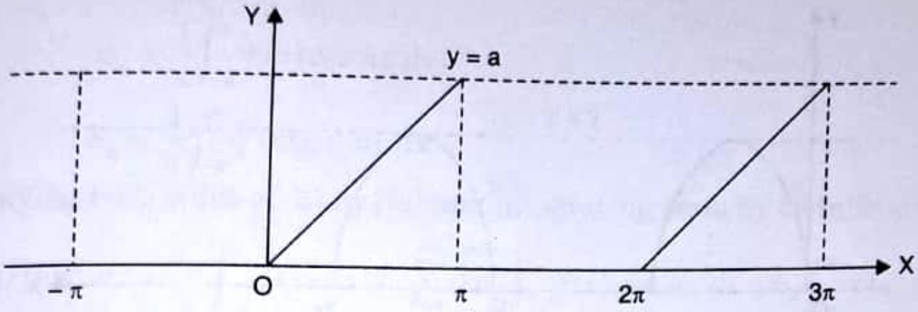
Fourier series is

$$f(x) = 2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + ...\right]$$

(3) **Modified saw-toothed waveform.** This wave is the extension of the function

$$f(x) = \begin{Bmatrix} 0, & -\pi < x \le 0 \\ x, & 0 \le x < \pi \end{Bmatrix}$$

It has the graph as shown in the diagram:



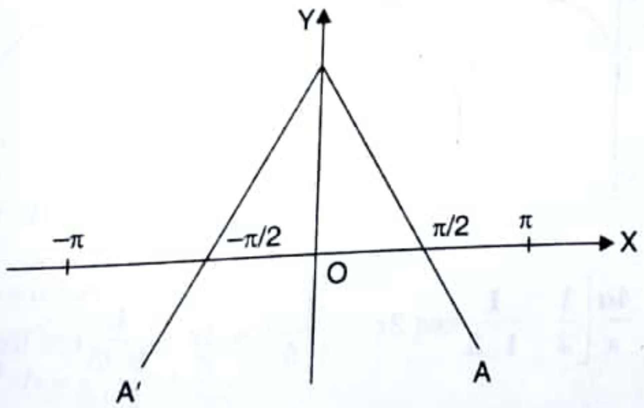
Fourier series is

$$F(x) = \frac{a}{4} - \frac{2a}{\pi^2}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right) + \frac{a}{\pi}\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\ldots\right)$$

(4) **Triangular waveform.** This waveform is the extension of the function

$$f(x) = \begin{cases} 1 + \left(\frac{2x}{\pi}\right), & -\pi \le x \le 0 \\ 1 - \left(\frac{2x}{\pi}\right), & 0 \le x \le \pi \end{cases}$$

It has the graph as shown in the diagram:



Fourier series is

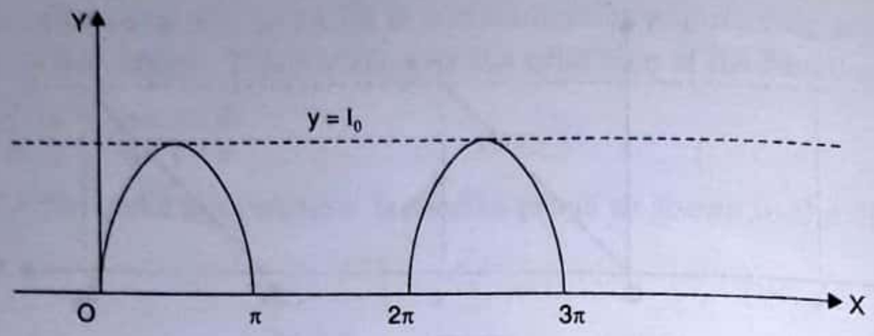
$$f(x) = \frac{8}{\pi^2}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\ldots\right]$$

(5) **Half-wave rectifier.** This wave rectifier is the extension of the function

$$f(x) = \begin{cases} I_0 \sin x, & 0 \le x \le \pi \\ 0, & \pi \le x \le 2\pi \end{cases}$$

where I_0 is the maximum current and the period is 2π.

It has the graph as shown in the diagram :



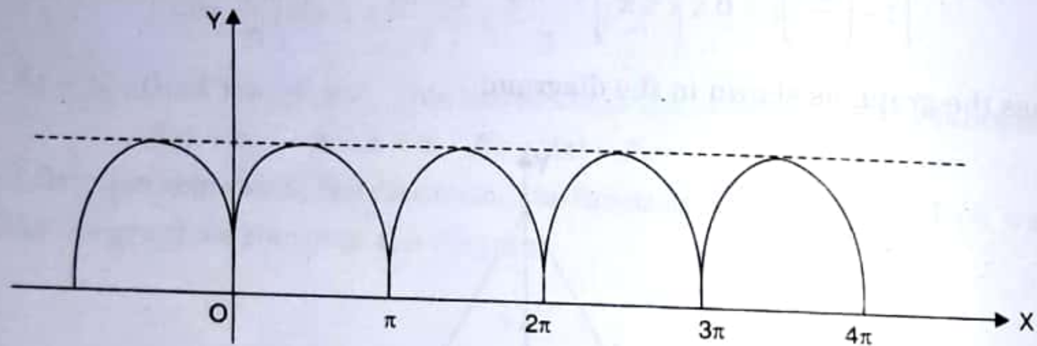
Fourier series is

$$f(x) = \frac{I_0}{\pi} + \frac{I_0}{2} \sin x - \frac{2I_0}{\pi}\left[\frac{\cos 2x}{2^2 - 1} + \frac{\cos 4x}{4^2 - 1} + \frac{\cos 6x}{6^2 - 1} + \dots\dots\right]$$

(6) **Full-wave rectifier**

This waveform is an extension of the function $f(x) = a \sin x,\ 0 < x \le \pi$.

It has the graph as shown in the diagram:



Fourier series is

$$f(x) = \frac{4a}{\pi}\left[\frac{1}{2} - \frac{1}{1 \cdot 3} \cos 2x - \frac{1}{3 \cdot 5} \cos 4x - \frac{1}{5 \cdot 7} \cos 6x - \dots\dots\right].$$

3.41 PARSEVAL'S IDENTITY

If Fourier series for f(x) converges uniformly in $(-\pi, \pi)$, *then*

$$\boxed{\int_{-\pi}^{\pi} [f(x)]^2\, dx = \pi\left[\frac{a_0{}^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)\right]}$$

Proof. The Fourier series for $f(x)$ in $(-\pi, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad \dots(1)$$

where
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \quad ...(2)$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad ...(3)$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \quad ...(4)$$

Multiplying both sides of (1) by $f(x)$ and integrating term by term from $-\pi$ to π, we get

$$\int_{-\pi}^{\pi} [f(x)]^2\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi} f(x)\,dx + \sum_{n=1}^{\infty}\left[a_n\int_{-\pi}^{\pi} f(x)\cos nx\,dx + b_n\int_{-\pi}^{\pi} f(x)\sin nx\,dx\right]$$
$$= \frac{a_0}{2}(\pi a_0) + \sum_{n=1}^{\infty}[a_n(\pi a_n) + b_n(\pi b_n)] \quad | \text{ Using (2), (3) and (4)}$$
$$\Rightarrow \int_{-\pi}^{\pi} [f(x)]^2\,dx = \pi\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right].$$

Note. Similarly, we can obtain the following Parseval's formulae:

(1) If Fourier series for $f(x)$ converges uniformly in $(-l, l)$ then
$$\int_{-l}^{l} [f(x)]^2\,dx = l\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right].$$

(2) If Fourier series for $f(x)$ converges uniformly in $(0, 2l)$ then
$$\int_{0}^{2l} [f(x)]^2\,dx = l\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right].$$

(3) For half-range sine series in $(0, l)$, we have
$$\int_{0}^{l} [f(x)]^2\,dx = \frac{l}{2}\sum_{n=1}^{\infty} b_n^2$$

(4) For half-range cosine series in $(0, l)$, we have
$$\int_{0}^{l} [f(x)]^2\,dx = \frac{l}{2}\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2\right]$$

In all the above formulae, the left hand side terms are related to root mean square (rms) value.

3.42 ROOT MEAN SQUARE (RMS) VALUE

The root mean square value of the function $y = f(x)$ over an interval (a, b) is defined as
$$\bar{y} = \sqrt{\left\{\frac{\int_a^b [f(x)]^2\,dx}{b-a}\right\}}$$

The use of root mean square value of a periodic function is frequently made in the theory of mechanical vibrations and in electric circuit theory.

Example. *Obtain the Fourier series for $y = x^2$ in $-\pi < x < \pi$ and hence show that*

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\ldots\ldots=\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}.$$

Sol. Since $y = x^2$ is an even function $\therefore\ b_n = 0$

Let
$$y=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx \qquad \ldots(1)$$

where,
$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\,dx=\frac{2}{\pi}\int_0^{\pi}x^2\,dx=\frac{2}{3}\pi^2$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\cos nx\,dx=\frac{2}{\pi}\int_0^{\pi}x^2\cos nx\,dx$$

$$=\frac{2}{\pi}\left[\left(x^2\frac{\sin nx}{n}\right)_0^{\pi}-\int_0^{\pi}2x.\frac{\sin nx}{n}\,dx\right]=-\frac{4}{\pi n}\int_0^{\pi}x\sin nx\,dx$$

$$=-\frac{4}{\pi n}\left[\left\{x\left(\frac{-\cos nx}{n}\right)\right\}_0^{\pi}-\int_0^{\pi}1.\left(\frac{-\cos nx}{n}\right)dx\right]$$

$$\Rightarrow\qquad a_n=\frac{4}{\pi n^2}(\pi\cos n\pi)=\frac{4}{n^2}(-1)^n \qquad \ldots(2)$$

$\therefore$ From (1),
$$x^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

Applying Parseval's identity, we get

$$\int_{-\pi}^{\pi}(x^2)^2\,dx=\pi\left[\frac{1}{2}\left(\frac{2\pi^2}{3}\right)^2+\sum_{n=1}^{\infty}\frac{16}{n^4}(-1)^{2n}\right]$$

$$\Rightarrow\qquad 2\int_0^{\pi}x^4\,dx=\pi\left[\frac{2\pi^4}{9}+16\sum_{n=1}^{\infty}\frac{1}{n^4}\right]$$

$$\Rightarrow\qquad \frac{2}{5}\pi^4=\frac{2}{9}\pi^4+16\sum_{n=1}^{\infty}\frac{1}{n^4}$$

or
$$\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}$$

TEST YOUR KNOWLEDGE

1. Using the Fourier series expansion of $f(x) = |x|$ in $(-\pi, \pi)$, show that

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$ (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.

2. Using the Fourier series for $f(x) = x$ is $0 < x < 1$, show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$.

3. Expand $f(x) = x - \dfrac{x^2}{2}$ in $(0, 2)$ as Fourier sine series and hence evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^6}$.

4. By using the sine series for $f(x) = 1$ in $0 < x < \pi$, show that $\dfrac{\pi^2}{8} = 1 + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \dfrac{1}{7^2} + \ldots\ldots$.

5. If $f(x) = \begin{cases} \pi x, & 0 < x < 1 \\ \pi(2-x), & 1 < x < 2 \end{cases}$, then using half range cosine series expansion, show that $\dfrac{1}{1^4} + \dfrac{1}{3^4} + \dfrac{1}{5^4} + \ldots. = \dfrac{\pi^4}{96}$.

Answer

3. $f(x) = \dfrac{16}{\pi^2} \displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin \frac{(2n-1)\pi x}{2}$; $\dfrac{\pi^4}{960}$.

3.43 PRACTICAL HARMONIC ANALYSIS

Sometimes the function is not given by a formula but by a graph or by a table of corresponding values. The process of finding Fourier series for a function given by such values of the function and independent variable is called **Harmonic Analysis.**

Fourier constants are evaluated by the following formulae:

(1) $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = 2.\left[\frac{1}{2\pi - 0}\int_0^{2\pi} f(x)\,dx\right]$$

$$= 2\,[\text{ mean value of } f(x) \text{ in } (0, 2\pi)] \qquad \left|\ \because\ \text{Mean} = \frac{1}{b-a}\int_a^b f(x)\,dx\right.$$

(2) $$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= 2.\left[\frac{1}{2\pi - 0}\int_0^{2\pi} f(x)\cos nx\,dx\right]$$

$$= 2\,[\text{mean value of } f(x)\cos nx \text{ in } (0, 2\pi)].$$

(3) $$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= 2.\left[\frac{1}{2\pi - 0}\int_0^{2\pi} f(x)\sin nx\,dx\right] = 2\,[\text{mean value of } f(x)\sin nx \text{ in } (0, 2\pi)]$$

Here mean value is also referred as 'average'.

The term ($a_1 \cos x + b_1 \sin x$) in Fourier series is called the **fundamental** or **first harmonic.** The term ($a_2 \cos 2x + b_2 \sin 2x$) in Fourier series is called the **second harmonic** and so on. Similarly, we can obtain the harmonics for Fourier series in $(0, 2l)$, half-range sine and cosine series.

ILLUSTRATIVE EXAMPLES

Example 1. *Obtain the first three coefficients in the Fourier cosine series for y, where y is given in the following table:*

x	*0*	*1*	*2*	*3*	*4*	*5*
y	*4*	*8*	*15*	*7*	*6*	*2*

Sol. Taking the interval as $\pi/3$, Fourier cosine series to represent y in $(0, 5)$ *i.e.*, in $(0, 2\pi)$ is

$$y = \frac{a_0}{2} + a_1 \cos \frac{\pi x}{3} + a_2 \cos \frac{2\pi x}{3} + \ldots\ldots$$

where $a_0 = 2$ [mean value of y in $(0, 5)$]

$$a_1 = 2\left[\text{mean value of } y \cos \frac{\pi x}{3} \text{ in } (0,5)\right]$$

$$a_2 = 2\left[\text{mean value of } y \cos \frac{2\pi x}{3} \text{ in } (0,5)\right]$$

Let us form the following table:

x	$\frac{\pi x}{3}$	$\cos \frac{\pi x}{3}$	$\cos \frac{2\pi x}{3}$	y	$y \cos \frac{\pi x}{3}$	$y \cos \frac{2\pi x}{3}$
0	0	1	1	4	4	4
1	$\frac{\pi}{3}$	$\frac{1}{2}$	$-\frac{1}{2}$	8	4	-4
2	$\frac{2\pi}{3}$	$-\frac{1}{2}$	$-\frac{1}{2}$	15	-7.5	-7.5
3	π	-1	1	7	-7	7
4	$\frac{4\pi}{3}$	$-\frac{1}{2}$	$-\frac{1}{2}$	6	-3	-3
5	$\frac{5\pi}{3}$	$\frac{1}{2}$	$-\frac{1}{2}$	2	1	-1
		Total		$\Sigma y = 42$	$\Sigma y \cos \frac{\pi x}{3} = -8.5$	$\Sigma y \cos \frac{2\pi x}{3} = -4.5$

Hence, $a_0 = 2\left[\frac{42}{6}\right] = 14$, $a_1 = 2\left[\frac{-8.5}{6}\right] = -2.83$, $a_2 = 2\left[\frac{-4.5}{6}\right] = -1.5$

Hence the required series is

$$y = 7 - 2.83 \cos \frac{\pi x}{3} - 1.5 \cos \frac{2\pi x}{3}.$$

Example 2. *Find the Fourier series as far as the second harmonic to represent the function given by the following table:*

x	0	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
$f(x)$	2.34	3.01	3.69	4.15	3.69	2.20	0.83	0.51	0.88	1.09	1.19	1.64

Sol.

x	$\sin x$	$\sin 2x$	$\cos x$	$\cos 2x$	$f(x)$	$f(x) \sin x$	$f(x) \sin 2x$	$f(x) \cos x$	$f(x) \cos 2x$
0	0	0	1	1	2.34	0	0	2.340	2.340
30°	0.50	0.87	.87	.5	3.01	1.505	2.619	2.619	1.505
60°	0.87	0.87	.50	– .5	3.69	3.210	3.210	1.845	– 1.845
90°	1	0	0	– 1	4.15	4.150	0	0	– 4.150
120°	0.87	– 0.87	– .50	– .5	3.69	3.210	– 3.210	– 1.845	– 1.845
150°	0.50	– 0.87	– .87	.5	2.20	1.100	– 1.914	– 1.914	1.100
180°	0	0	– 1	1	.83	0	0	– .830	.830
210°	– 0.50	0.87	– .87	.5	.51	– .255	.444	– .444	.255
240°	– 0.87	0.87	– .50	– .5	.88	– .766	.766	– .440	– .440
270°	– 1	0	0	– 1	1.09	– 1.090	0	0	– 1.090
300°	– .87	– 0.87	.50	– .5	1.19	– 1.035	– 1.035	.595	– .595
330°	– .50	– 0.87	.87	.5	1.64	– .820	– 1.427	1.427	.820
	Total	(Σ)	=		25.22	9.209	– .547	3.353	– 3.115

$$a_0 = 2 \text{ [Mean of } f(x)] = 2\left(\frac{25.22}{12}\right) = 4.203$$

$$a_1 = 2 \text{ [Mean of } f(x) \cos x] = 2\left(\frac{3.353}{12}\right) = .559$$

$$a_2 = 2 \text{ [Mean of } f(x) \cos 2x] = 2\left(\frac{-3.115}{12}\right) = -.519$$

$$b_1 = 2 \text{ [Mean of } f(x) \sin x] = 2\left(\frac{9.209}{12}\right) = 1.535$$

$$b_2 = 2 \text{ [Mean of } f(x) \sin 2x] = 2\left(\frac{-.547}{12}\right) = -.091$$

$\therefore$ Fourier series is

$$f(x) = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \ldots.$$

$$= 2.1015 + (.559 \cos x + 1.535 \sin x) + (-.519 \cos 2x - .091 \sin 2x) + \ldots\ldots$$

TEST YOUR KNOWLEDGE

1. A machine completes its cycle of operations every time as certain pulley completes a revolution. The displacement $f(x)$ of a point on a certain portion of the machine is given in the table given below for 12 positions of the pulley, x being the angle in degree turned through by the pulley. Find the Fourier series to represent $f(x)$ for all values of x

x	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$f(x)$	7.976	8.026	7.204	5.676	3.674	1.764	0.552	0.262	0.904	2.492	4.736	6.824

2. Obtain the constant term and the coefficient of the first sine and cosine terms in the Fourier series of $f(x)$ as given in the following table: (M.T.U. 2012)

x	0	1	2	3	4	5
$f(x)$	9	18	24	28	26	20

3. The displacement y of a part of a mechanism is tabulated with corresponding angular movement $x°$ of the crank. Express y as a Fourier series neglecting the harmonics above the third:

$x°$	0	30	60	90	120	150	180	210	240	270	300	330
y	1.80	1.10	.30	.16	1.50	1.30	2.16	1.25	1.30	1.52	1.76	2.00

4. Following table gives the variations of periodic current over a period:

t (sec.)	0	T/6	T/3	T/2	$\frac{2T}{3}$	$\frac{5T}{6}$	T
A(amp.)	1.98	1.30	1.05	1.30	– 0.88	– 0.25	1.98

Show that there is a direct current part of 0.75 amp. in the variable current and obtain the amplitude of the first harmonic.

[**Hint:** Here length of interval is T $\therefore$ $c = T/2$

Hence $A = \frac{a_0}{2} + \left(a_1 \cos \frac{2\pi t}{T} + b_1 \sin \frac{2\pi t}{T}\right) + \ldots\ldots$].

5. In a machine, the displacement y of a given point for a certain angle θ is as follows:

$\theta°$	0	30	60	90	120	150	180	210	240	270	300	330
y	7.8	8.1	7.4	5.7	3.6	2.1	1.1	0.4	0.9	1.9	3.8	6.2

Express y as a Fourier series neglecting the harmonics on and above the third.

6. The turning moment T on the crank-shaft of a steam engine for the crank angle θ degrees is given as follows:

θ	0	30	60	90	120	150	180
T	0	6.1	8.2	7.9	5.5	2.5	0

Find the coefficient of $\sin 2\theta$ if T is expanded in a series of sines.

7. The turning moment T is given for a series of values of the crank angle θ degrees. Expand T in a half-range cosine series.

θ	0	30	60	90	120	150	180
T	0	52	81	79	55	27	0

8. Determine the first two harmonics of the Fourier series for the following values:

α	30	60	90	120	150	180	210	240	270	300	330	360
y	3.11	3.23	4	4.25	3.71	2.69	1.23	.87	.98	1.24	1.44	1.86

9. Analyse the following data harmonically and express y interms of a Fourier series : [G.B.T.U. (A.G.) 2010]

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
y	1.0	1.4	1.9	1.7	1.5	1.2	1.0

10. In a machine, the displacement y of a given point is given for a certain angle θ as follows :

θ	0	30	60	90	120	150	180	210	240	270	300	330
y	7.9	8	7.2	5.6	3.6	1.7	0.5	0.2	0.9	2.5	4.7	6.8

Find the coefficient of sin 2θ in the Fourier series representing the above variation.

Answers

1. $f(x) = 4.17 + (2.45 \cos x + 3.16 \sin x) + (0.12 \cos 2x + 0.03 \sin 2x) + (0.08 \cos 3x + 0.01 \sin 3x) + \ldots$
2. $f(x) = 20.83 - \left(8.33 \cos \frac{\pi x}{3} + 1.156 \sin \frac{\pi x}{3}\right) + \ldots$
3. $y = 1.26 + (0.04 \cos x - 0.63 \sin x) + (0.53 \cos 2x - 0.23 \sin 2x) + (-0.1 \cos 3x + 0.085 \sin 3x)$.
4. 1.072.
5. $4.085 + (3.39 \cos \theta + 1.85 \sin \theta) + (-0.04 \cos 2\theta + 0.16 \sin 2\theta)$.
6. 1.56
7. $42 + 9.93 \cos x - 30.71 \cos 2x + \ldots$
8. $(-0.25 \cos x + 1.63 \sin x)$; $(-0.08 \cos 2x - 0.32 \sin 2x)$.
9. $y = 1.45 + (-.37 \cos x + .17 \sin x) + (-.1 \cos 2x - .06 \sin 2x) + .03 \cos 3x + \ldots$
10. -0.0731.

ASSIGNMENT-III

(2 Marks Questions for Section–A)

1. Find the constant term if the function $f(x) = x + x^2$ is expanded in Fourier series defined in $(-1, 1)$. (G.B.T.U. 2012)
2. Find the constant term if $f(x) = x^2$ is expanded in Fourier series in $(-\pi, \pi)$. (U.P.T.U. 2015)
3. If $f(x) = |x|$ is expanded in Fourier series defined in $(-1, 1)$ then find the constant term. (U.P.T.U. 2013)
4. Find the constant term when $f(x) = |x|$ is expanded in Fourier series in the interval $(-2, 2)$. (A.K.T.U. 2017).
5. Define periodic functions and find the period of $f(x) = \cos x + \frac{1}{2} \cos 2x + \frac{1}{3} \cos 3x$. [G.B.T.U. (AG) 2012]

6. For an even function defined in the interval (0, 2p), write down the Fourier series. [G.B.T.U. (AG) 2012]
7. State Dirichlet's conditions for the expansion of $f(x)$ in Fourier series. (U.P.T.U. 2015, 17)
8. If $f(x) = 1$ is expanded in a Fourier sine series in $(0, \pi)$, then find the value of b_n. (G.B.T.U. 2013)
9. If $f(x) = 1$, $0 < x < \pi$ is expanded in half range cosine series then find the value of a_0. (U.P.T.U. 2014)
10. Find the value of the Fourier coefficient a_0 for the function

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$ (A.K.T.U. 2016)

11. If $F(x) = \begin{cases} -x, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$, then find F(0). (U.P.T.U. 2014)
12. Find the period of $\sin nx$.
13. Find Fourier half-range sine series for the function $f(x) = x$, $0 < x < 2$. [G.B.T.U. (SUM) 2010]
14. What does the Fourier coefficient a_0 in Fourier series expansion of a function represent? (G.B.T.U. 2010)
15. If $F(x) = x \sin x$ in $(-\pi, \pi)$ then find b_n.
16. What is the smallest period of the function $f(x) = \sin\left(\dfrac{2n\pi x}{k}\right)$.
17. If $F(x) = x^2$ in $-2 < x < 2$ and $F(x + 4) = F(x)$, then find the value of a_n.
18. What is the product of two odd functions. (G.B.T.U. 2011)
19. If $F(x) = x$ is expanded in a Fourier sine series in $(0, \pi)$ then find b_n. [M.T.U. (SUM) 2011]
20. If $F(x) = x \cos x$ is expanded in a Fourier series in $(-\pi, \pi)$ then find a_0. [G.B.T.U. (AG) 2011]
21. Find the Fourier coefficient for the function $f(x) = x^2$, $0 < x < 2\pi$. (A.K.T.U. 2017)

Answers

1. $\dfrac{2}{3}$
2. $\dfrac{2}{3}\pi^2$
3. 1
4. $\dfrac{a_0}{2} = 1$
5. 2π
6. $F(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$
8. $\dfrac{2}{n\pi}[1 - (-1)^n]$
9. 2
10. $\dfrac{\pi}{2}$
11. 0
12. $\dfrac{2\pi}{n}$
13. $x = -\dfrac{4}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n}\sin\dfrac{n\pi x}{2}$
14. Mean value of the function
15. 0
16. $\dfrac{k}{n}$
17. $\int_0^2 x^2 \cos\dfrac{n\pi x}{2}\,dx$
18. Even function
19. $-\dfrac{2}{n}(-1)^n$
20. 0
21. $a_0 = \dfrac{8}{3}\pi^2$, $a_n = \dfrac{4}{n^2}$, $b_n = -\dfrac{4\pi}{n}$.

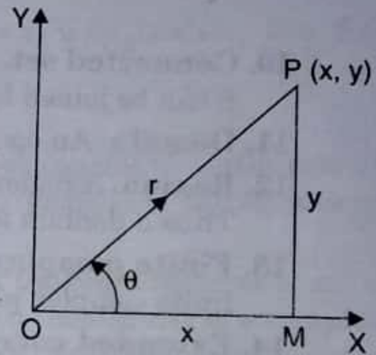
MODULE 4

Complex Variable – Differentiation

4.1 INTRODUCTION

A complex number z is an ordered pair (x, y) of real numbers and is written as

$$z = x + iy, \quad \text{where } i = \sqrt{-1}.$$



The real numbers x and y are called the real and imaginary parts of z. In the Argand's diagram, the complex number z is represented by the point $P(x, y)$. If (r, θ) are the polar coordinates of P, then $r = \sqrt{x^2 + y^2}$ is called the modulus of z and is denoted by $|\ z\ |$. Also $\theta = \tan^{-1} \dfrac{y}{x}$ is called the argument of z and is denoted by arg. z. Every non-zero complex number z can be expressed as

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

If $z = x + iy$, then the complex number $x - iy$ is called the conjugate of the complex number z and is denoted by $\bar{z}$.

Clearly,
$$|\bar{z}| = |z|, |z|^2 = z\bar{z},$$

$$\text{Re}(z) = \frac{z + \bar{z}}{2}, \quad \text{Im}(z) = \frac{z - \bar{z}}{2i}.$$

4.2 DEFINITIONS

Let S be a non-empty set of complex numbers and δ be a positive real number.

1. **Circle.** $|z - a| = r$ represents a circle C with centre at the point a and radius r.
2. **Open disk.** The set of points which satisfies the equation $|z - z_0| < \delta$ defines an open disk of radius d with centre at $z_0 = (x_0, y_0)$. This set consists of all points which lie inside circle C.
3. **Closed disk.** The set of points which satisfies the equation $|z - z_0| \leq \delta$ defines a closed disk of radius d with centre at $z_0 = (x_0, y_0)$. This set consists of all points which lie inside and on the boundary of circle C.
4. **Annulus.** The set of points which lie between two concentric circles $C_1 : |z - a| = r_1$ and $C_2 : |z - a| = r_2$ defines an open annulus *i.e.*, the set of points which satisfies the inequality $r_1 < |z - a| < r_2$.

The set of points which satisfies the inequality $r_1 \leq |z - a| \leq r_2$ defines a closed annulus. It is to be noted that $r_1 \leq |z - a| <$ ... ither open nor closed.

5. **Neighbourhood.** δ-Neighbourhood of a point z_0 is the set of all points z for which $|z - z_0| < \delta$ where d is a positive constant. If we exclude the point z_0 from the open disk $|z - z_0| < d$ then it is called the deleted neighbourhood of the point z_0 and is written as $0 < |z - z_0| < \delta$.
6. **Interior and exterior points.** A point z is an interior point of S if all the points in some δ-neighbourhood of z are in S and an exterior point of S if they are outside S.
7. **Boundary point.** A point z is a boundary point of S if every δ-neighbourhood of z contains at least one point of S and at least one point not in S. For example, the points on the circle $|z - z_0| = r$ are the boundary points for the disk $|z - z_0| \le r$.
8. **Open and closed sets.** A set S is open if every point of S is an interior point while a set S is closed if every boundary point of S belongs to S. *e.g.* $S = \{z : |z - z_0| < r\}$ is open set while $S = \{z : |z - z_0| \le r\}$ is closed set.
9. **Bounded set.** An open set S is bounded if ∃ a positive real number M such that $|z| \le M$ for all $z \in S$ otherwise unbounded.

   For example: the set $S = \{z : |z - z_0| < r\}$ is a bounded set while the set
   $$S = \{z : |z - z_0| > r\} \text{ is an unbounded set.}$$
10. **Connected set.** An open set S is connected if any two points z_1 and z_2 belonging to S can be joined by a polygonal line which is totally contained in S.
11. **Domain.** An open connected set is called a domain denoted by D.
12. **Region.** A region is a domain together with all, some or none of its boundary points. Thus a domain is always a region but a region may or may not be a domain.
13. **Finite complex plane.** The complex plane without the point $z = \infty$ is called the finite complex plane.
14. **Extended complex plane.** The complex plane to which the point $z = \infty$ has been added is called the extended complex plane.

4.3 FUNCTION OF A COMPLEX VARIABLE

If x and y are real variables, then $z = x + iy$ is called a **complex variable.** If corresponding to each value of a complex variable $z(= x + iy)$ in a given region R, there correspond one or more values of another complex variable $w\ (= u + iv)$, then w is called a **function of the complex variable** z and is denoted by

$$w = f(z) = u + iv$$

For example, if $\quad w = z^2 \quad$ where $z = x + iy$ and $w = f(z) = u + iv$

then $\quad u + iv = (x + iy)^2 = (x^2 - y^2) + i(2xy)$

$\Rightarrow \quad u = x^2 - y^2 \quad \text{and} \quad v = 2xy$

Thus u and v, the real and imaginary parts of w, are functions of the real variables x and y.

$\therefore \quad w = f(z) = u(x, y) + iv(x, y)$

If to each value of z, there corresponds one and only one value of w, then w is called a *single-valued function* of z. If to each value of z, there correspond more than one values of w, then w is called a *multi-valued function of* z. For example, $w = \sqrt{z}$ is a multi-valued function.

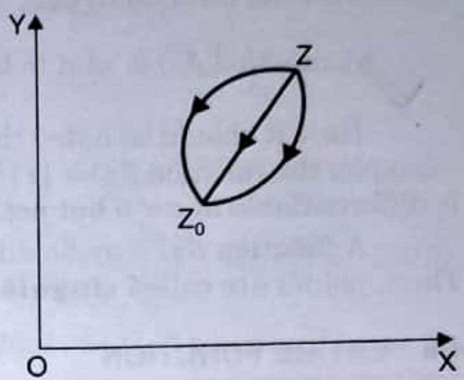
To represent $w = f(z)$ graphically, we take two Argand diagrams: one to represent the point z and the other to represent w. The former diagram is called the XOY-plane or the z-plane and the latter UOV-plane or the w-plane.

4.4 LIMIT OF $f(z)$

A function $f(z)$ tends to the limit l as z tends to z_0 *along any path*, if to each positive arbitrary number ε, however small, there corresponds a positive number δ, such that

$$| f(z) - l | < \varepsilon \quad \text{whenever} \quad 0 < | z - z_0 | < \delta$$

and we write $\quad \operatorname*{Lt}_{z \to z_0} f(z) = l$, where l is finite



Note. In real variables, $x \to x_0$ implies that x approaches x_0 along the number line, either from left or from right. In complex variables, $z \to z_0$ implies that z approaches z_0 along any path, straight or curved, since the two points representing z and z_0 in a complex plane can be joined by an infinite number of curves.

4.5 CONTINUITY OF $f(z)$

A single-valued function $f(z)$ is said to be continuous at a point $z = z_0$ if $f(z_0)$ exists, $\lim_{z \to z_0} f(z)$ exists and $\operatorname*{Lt}_{z \to z_0} f(z) = f(z_0)$.

A function $f(z)$ is said to be continuous in a region R of the z-plane if it is continuous at every point of the region. A function $f(z)$ which is not continuous at z_0 is said to be discontinuous at z_0.

If the function $f(z) = u + iv$ is continuous at $z_0 = x_0 + iy_0$ then the real functions u and v are also continuous at the point (x_0, y_0). Therefore, we can discuss the continuity of a complex valued function by studying the continuity of its real and imaginary parts. If $f(z)$ and $g(z)$ are continuous at a point z_0 then the functions $f(z) \pm g(z)$, $f(z)\,g(z)$ and $\dfrac{f(z)}{g(z)}$, where $g(z_0) \neq 0$ are also continuous at z_0.

If $f(z)$ is continuous in a closed region S then it is bounded in S *i.e.*, $|f(z)| \leq M \quad \forall\, z \in S$.

Also, the function $f(z)$ is continuous at $z = \infty$ if the function $f\left(\dfrac{1}{\xi}\right)$ is continuous at $\xi = 0$.

4.6 DERIVATIVE OF $f(z)$

Let $w = f(z)$ be a single-valued function of the variable $z(= x + iy)$, then the derivative or differential co-efficient of $w = f(z)$ is defined as

$$\boxed{\frac{dw}{dz} = f'(z) = \operatorname*{Lt}_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z}}$$

provided the limit exists, independent of the manner in which $\delta z \to 0$.

4.7 ANALYTIC FUNCTION

[A.K.T.U 2012]

A function $f(z)$ is said to be **analytic** at a point z_0 if it is one-valued and differentiable not only at z_0 but at every point of some neighbourhood of z_0. For example: $e^x(\cos y + i \sin y)$. A function $f(z)$ is said to be analytic in a certain domain D if it is analytic at every point of D.

The terms 'regular', 'holomorphic' and 'monogenic' are also sometimes used as synonymous with the term analytic.

A function $f(z)$ is said to be analytic at $z = \infty$ if the function $f\left(\frac{1}{z}\right)$ is analytic at $z = 0$.

Here it should be noted that analyticity implies differentiability but not vice versa. For example, the function $f(z) = |z|^2$ is differentiable only at $z = 0$ and nowhere else therefore $f(z)$ is differentiable at $z = 0$ but not analytic anywhere.

A function $f(z)$ may be differentiable in a domain except for a finite number of points. These points are called **singular points** or **singularities** of $f(z)$ in that domain.

4.8 ENTIRE FUNCTION

A function $f(z)$ which is analytic at every point of the finite complex plane is called an entire function. Since the derivative of a polynomial exists at every point, a polynomial of any degree is an entire function. Rational functions with non-zero denominators are also entire functions.

4.9 NECESSARY AND SUFFICIENT CONDITIONS FOR f(z) TO BE ANALYTIC

[M.T.U. 2012]

The necessary and sufficient conditions for the function

$$w = f(z) = u(x, y) + iv(x, y)$$

to be analytic in a region R, are

(*i*) $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ *are continuous functions of x and y in the region R.*

(*ii*) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$.

The conditions in (ii) are known as Cauchy-Riemann equations or briefly C-R equations.

Proof. (*a*) **Necessary Condition.** Let $w = f(z) = u(x, y) + iv(x, y)$ be analytic in a region R, then $\dfrac{dw}{dz} = f'(z)$ exists uniquely at every point of that region.

Let δx and δy be the increments in x and y respectively. Let δu, δv and δz be the corresponding increments in u, v and z respectively. Then,

$$f'(z) = \underset{\delta z \to 0}{\text{Lt}} \frac{f(z + \delta z) - f(z)}{\delta z} = \underset{\delta z \to 0}{\text{Lt}} \frac{(u + \delta u) + i(v + \delta v) - (u + iv)}{\delta z}$$

$$= \underset{\delta z \to 0}{\text{Lt}} \left(\frac{\delta u}{\delta z} + i\frac{\delta v}{\delta z}\right) \quad \ldots(1)$$

Since the function $w = f(z)$ is analytic in the region R, the limit (1) must exist independent of the manner in which $\delta z \to 0$, *i.e.*, along whichever path δx and $\delta y \to 0$.

First, let $\delta z \to 0$ along a line parallel to x-axis so that $\delta y = 0$ and $\delta z = \delta x$.

[since $z = x + iy$, $z + \delta z = (x + \delta x) + i(y + \delta y)$ and $\delta z = \delta x + i\delta y$]

$\therefore$ From (1), $$f'(z) = \underset{\delta x \to 0}{\text{Lt}} \left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \ldots(2)$$

Now, let $\delta z \to 0$ along a line parallel to y-axis so that $\delta x = 0$ and $\delta z = i\delta y$.

$\therefore$ From (1), $\quad f'(z) = \underset{\delta y \to 0}{\text{Lt}} \left(\dfrac{\delta u}{i\,\delta y} + i\,\dfrac{\delta v}{i\,\delta y} \right) = \dfrac{1}{i}\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial y}$

$$= \frac{\partial v}{\partial y} - i\,\frac{\partial u}{\partial y} \qquad \text{...(3)} \quad \Big|\; \because\; \frac{1}{i} = -i$$

From (2) and (3), we have $\dfrac{\partial u}{\partial x} + i\,\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial y} - i\,\dfrac{\partial u}{\partial y}$

Equating the real and imaginary parts, $\boxed{\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}}$ and $\boxed{\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}}$ *(U.P.T.U. 2015)*

Hence the necessary condition for f(z) to be analytic is that the C-R equations must be satisfied.

(b) **Sufficient Condition.** Let $f(z) = u + iv$ be a single-valued function possessing partial derivatives $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ at each point of a region R and satisfying C-R equations.

i.e.,
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

We shall show that $f(z)$ is analytic, *i.e.,* $f'(z)$ exists at every point of the region R.

By Taylor's theorem for functions of two variables, we have, on omitting second and higher degree terms of δx and δy,

$$f(z + \delta z) = u(x + \delta x, y + \delta y) + iv(x + \delta x, y + \delta y)$$

$$= \left[u(x, y) + \left(\frac{\partial u}{\partial x}\,\delta x + \frac{\partial u}{\partial y}\,\delta y \right) \right] + i \left[v(x, y) + \left(\frac{\partial v}{\partial x}\,\delta x + \frac{\partial v}{\partial y}\,\delta y \right) \right]$$

$$= [u(x, y) + iv(x, y)] + \left(\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \right) \delta x + \left(\frac{\partial u}{\partial y} + i\,\frac{\partial v}{\partial y} \right) \delta y$$

$$= f(z) + \left(\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \right) \delta x + \left(\frac{\partial u}{\partial y} + i\,\frac{\partial v}{\partial y} \right) \delta y$$

or
$$f(z + \delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \right) \delta x + \left(\frac{\partial u}{\partial y} + i\,\frac{\partial v}{\partial y} \right) \delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \right) \delta x + \left(-\frac{\partial v}{\partial x} + i\,\frac{\partial u}{\partial x} \right) \delta y \qquad \Big|\; \text{Using C-R equations}$$

$$= \left(\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \right) \delta x + \left(\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \right) i\delta y \qquad \Big|\; \because\; -1 = i^2$$

$$= \left(\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \right) (\delta x + i\delta y) = \left(\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \right) \delta z \qquad \Big|\; \because\; \delta x + i\delta y = \delta z$$

$$\Rightarrow \quad \frac{f(z + \delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}$$

$\therefore$
$$f'(z) = \underset{\delta z \to 0}{\text{Lt}} \frac{f(z + \delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}$$

Thus $f'(z)$ exists, because $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}$ exist.

Hence $f(z)$ is analytic.

Note 1. The real and imaginary parts of an analytic function are called **conjugate functions.** Thus, if $f(z) = u(x, y) + iv(x, y)$ is an analytic function, then $u(x, y)$ and $v(x, y)$ are conjugate functions. The relation between two conjugate functions is given by C-R equations.

Note 2. When a function $f(z)$ is known to be analytic, it can be differentiated in the ordinary way as if z is a real variable.

Thus,
$$f(z) = z^2 \quad \Rightarrow \quad f'(z) = 2z$$
$$f(z) = \sin z \quad \Rightarrow \quad f'(z) = \cos z \text{ etc.}$$

4.10 CAUCHY-RIEMANN EQUATIONS IN POLAR COORDINATES (A.K.T.U. 2016)

Let (r, θ) be the polar coordinates of the point whose cartesian coordinates are (x, y), then

$$x = r\cos\theta,\ y = r\sin\theta,$$
$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$
$$\therefore \quad u + iv = f(z) = f(re^{i\theta}) \qquad \ldots(1)$$

Differentiating (1) partially w.r.t. r, we have

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta})\,.\,e^{i\theta} \qquad \ldots(2)$$

Differentiating (1) partially w.r.t. θ, we have

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta})\,.\,ire^{i\theta} = ir\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) \qquad |\text{ Using (2)}$$
$$= -r\frac{\partial v}{\partial r} + ir\frac{\partial u}{\partial r}$$

Equating real and imaginary parts, we get

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \quad \text{and} \quad \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$$

or $\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}}$ and $\boxed{\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}}$, which is the polar form of C-R equations.

4.11 DERIVATIVE OF *w*, i.e., *f′(z)* IN POLAR COORDINATES

$$w = f(z)$$
$$\therefore \quad \frac{dw}{dz} = f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(u + iv) = \frac{\partial w}{\partial x}$$
$$= \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial \theta}\frac{\partial \theta}{\partial x}$$
$$= \cos\theta\,\frac{\partial w}{\partial r} - \left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right)\frac{\sin\theta}{r}$$
$$= \cos\theta\,\frac{\partial w}{\partial r} - \left(-r\frac{\partial v}{\partial r} + ir\frac{\partial u}{\partial r}\right)\frac{\sin\theta}{r}$$

$\because\ r^2 = x^2 + y^2$

$\therefore\ \partial r/\partial x = \cos\theta$ as $x = r\cos\theta$

and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\therefore\ \frac{\partial \theta}{\partial x} = \frac{-\sin\theta}{r}$ as $y = r\sin\theta$

$$= \cos\theta \frac{\partial w}{\partial r} - i\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)\sin\theta = \cos\theta\frac{\partial w}{\partial r} - i\sin\theta\frac{\partial w}{\partial r}$$

$$\Rightarrow \quad \boxed{\frac{dw}{dz} = (\cos\theta - i\sin\theta)\frac{\partial w}{\partial r}} \qquad ...(1)$$

which is the result in terms of $\frac{\partial w}{\partial r}$.

Again,
$$\frac{dw}{dz} = \frac{\partial w}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial w}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} = \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)\cos\theta - \frac{\partial w}{\partial \theta}\cdot\frac{\sin\theta}{r}$$

$$= \left(\frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{i}{r}\frac{\partial u}{\partial \theta}\right)\cos\theta - \frac{\sin\theta}{r}\frac{\partial w}{\partial \theta} = -\frac{i}{r}\left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right)\cos\theta - \frac{\sin\theta}{r}\frac{\partial w}{\partial \theta}$$

$$= -\frac{i}{r}\frac{\partial w}{\partial \theta}\cos\theta - \frac{\sin\theta}{r}\frac{\partial w}{\partial \theta}$$

$$\Rightarrow \quad \boxed{\frac{dw}{dz} = -\frac{i}{r}(\cos\theta - i\sin\theta)\frac{\partial w}{\partial \theta}}$$

which is the result in terms of $\frac{\partial w}{\partial \theta}$.

4.12 HARMONIC FUNCTION

[*M.T.U. 2014, G.B.T.U. 2012*]

A function of x, y which possesses continuous partial derivatives of the first and second orders and satisfies Laplace's equation is called a Harmonic function.

4.13 THEOREM

If $f(z) = u + iv$ is an analytic function then u and v are both harmonic functions.

Proof. Let $f(z) = u + iv$ be analytic in some region of the z-plane, then u and v satisfy C-R equations.

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad ...(1)$$

and
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad ...(2)$$

Differentiating eqn. (1) partially w.r.t. x and eqn. (2) w.r.t. y, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \qquad ...(3)$$

and
$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \qquad ...(4)$$

Assuming $\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$ and adding equations (3) and (4), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad ...(5)$$

Now, differentiating eqn. (1) partially w.r.t. y and eqn. (2) w.r.t. x, we get

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2} \quad ...(6)$$

and

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2} \quad ...(7)$$

Assuming $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$ and subtracting eqn. (7) from eqn. (6), we get

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad ...(8)$$

Equations (5) and (8) show that the real and imaginary parts u and v of an analytic function satisfy the Laplace's equation. Hence u and v are harmonic functions.

Note. Here u and v are called conjugate harmonic functions.

4.14 ORTHOGONAL SYSTEM

[M.T.U. 2012]

Every analytic function $f(z) = u + iv$ defines two families of curves $u(x, y) = c_1$ and $v(x, y) = c_2$, which form an orthogonal system.

Consider the two families of curves

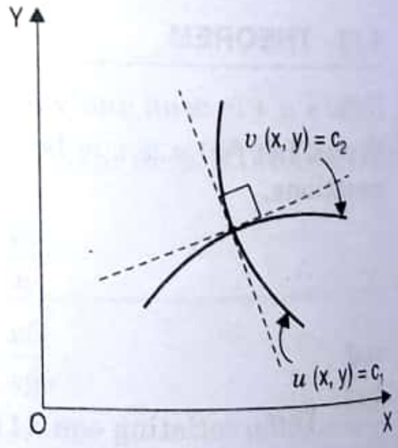
$$u(x, y) = c_1 \quad ...(1)$$

and

$$v(x, y) = c_2 \quad ...(2)$$

Differentiating eqn. (1) w.r.t. x, we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = m_1 \text{ (say)}$$



Similarly, from eqn. (2), we get $\frac{dy}{dx} = -\frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} = m_2$ (say)

$$\therefore \qquad m_1 m_2 = \frac{\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x}}{\frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y}} \quad ...(3)$$

Since $f(z)$ is analytic, u and v satisfy C-R equations

i.e.,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\therefore \text{ From (3),} \quad m_1 m_2 = \frac{\frac{\partial v}{\partial y} \cdot \frac{\partial v}{\partial x}}{-\frac{\partial v}{\partial x} \cdot \frac{\partial v}{\partial y}} = -1$$

Thus, the product of the slopes of the curves (1) and (2) is -1. Hence the curves intersect at right angles, *i.e.*, they form an orthogonal system.

4.15 THEOREM

An analytic function with constant modulus is constant.

Proof. Let $f(z) = u + iv$ be an analytic function with constant modulus. Then,

$$| f(z) | = | u + iv | = \text{constant}$$

$$\Rightarrow \quad \sqrt{u^2 + v^2} = \text{constant} = c \text{ (say)}$$

Squaring both sides, we get

$$u^2 + v^2 = c^2 \qquad ...(1)$$

Differentiating eqn. (1) partially w.r.t. x, we get

$$2u \frac{\partial u}{\partial x} + 2v \frac{\partial v}{\partial x} = 0$$

$$\Rightarrow \quad u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial x} = 0 \qquad ...(2)$$

Again, differentiating eqn. (1) partially w.r.t. y, we get

$$2u \frac{\partial u}{\partial y} + 2v \frac{\partial v}{\partial y} = 0$$

$$\Rightarrow \quad u \frac{\partial u}{\partial y} + v \frac{\partial v}{\partial y} = 0$$

$$\Rightarrow \quad u\left(-\frac{\partial v}{\partial x}\right) + v\left(\frac{\partial u}{\partial x}\right) = 0 \qquad ...(3) \quad \left| \because \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ and } \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \right.$$

Squaring and adding eqns. (2) and (3), we get

$$(u^2 + v^2)\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right\} = 0$$

$$\Rightarrow \quad \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = 0 \qquad | \because u^2 + v^2 = c^2 \neq 0$$

$$\Rightarrow \quad | f'(z) |^2 = 0 \qquad \left| \because f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \right.$$

$$\Rightarrow \quad | f'(z) | = 0$$

$$\Rightarrow \quad f(z) \text{ is constant.}$$

4.16 APPLICATION OF ANALYTIC FUNCTIONS TO FLOW PROBLEMS

Since the real and imaginary parts of an analytic function satisfy the Laplace's equation in two variables, these conjugate functions provide solutions to a number of field and flow problems.

For example, consider the two dimensional irrotational motion of an incompressible fluid, in planes parallel to xy-plane.

Let $\vec{V}$ be the velocity of a fluid particle, then it can be expressed as

$$\vec{V} = v_x \hat{i} + v_y \hat{j} \qquad ...(1)$$

Since the motion is irrotational, there exists a scalar function $\phi(x, y)$, such that

$$\vec{V} = \nabla\phi(x, y) = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} \quad \ldots(2)$$

From (1) and (2), we have $\quad v_x = \dfrac{\partial \phi}{\partial x} \quad$ and $\quad v_y = \dfrac{\partial \phi}{\partial y} \quad \ldots(3)$

The scalar function $\phi(x, y)$, which gives the velocity components, is called the **velocity potential function** or simply the **velocity potential.**

Also the fluid being incompressible, div $\vec{V} = 0$

$$\Rightarrow \quad \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}\right)(v_x\hat{i} + v_y\hat{j}) = 0$$

$$\Rightarrow \quad \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \quad \ldots(4)$$

Substituting the values of v_x and v_y from (3) in (4), we get

$$\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) = 0 \quad \text{or} \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Thus, the function ϕ is harmonic and can be treated as real part of an analytic function

$$w = f(z) = \phi(x, y) + i\,\psi(x, y)$$

For interpretation of conjugate function $\psi(x, y)$, the slope at any point of the curve $\psi(x, y) = c'$ is given by

$$\frac{dy}{dx} = -\frac{\dfrac{\partial \psi}{\partial x}}{\dfrac{\partial \psi}{\partial y}} = \frac{\dfrac{\partial \phi}{\partial y}}{\dfrac{\partial \phi}{\partial x}} \quad | \text{ By C-R equations}$$

$$= \frac{v_y}{v_x} \quad | \text{ By (3)}$$

This shows that the resultant velocity $\sqrt{{v_x}^2 + {v_y}^2}$ of the fluid particle is along the tangent to the curve $\psi(x, y) = c'$ *i.e.*, the fluid particles move along this curve. Such curves are known as **stream lines** and $\psi(x, y)$ is called the **stream function**. The curves represented by $\phi(x, y) = c$ are called **equipotential lines**.

Since $\phi(x, y)$ and $\psi(x, y)$ are conjugate functions of analytic function $w = f(z)$, the equipotential lines $\phi(x, y) = c$ and the stream lines $\psi(x, y) = c'$, intersect each other orthogonally.

Now, $$\frac{dw}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y} \quad | \text{ By C-R equations}$$

$$= v_x - iv_y \quad | \text{ By (3)}$$

$\therefore$ The magnitude of resultant velocity $= \left|\dfrac{dw}{dz}\right| = \sqrt{{v_x}^2 + {v_y}^2}$

The function $w = f(z)$ which fully represents the flow pattern is called the **complex potential**.

In the study of electrostatics and gravitational fields, the curves $\phi(x, y) = c$ and $\psi(x, y) = c'$ are called **equipotential lines** and **lines of force** respectively. In heat flow problems, the curves $\phi(x, y) = c$ and $\psi(x, y) = c'$ are known as **isothermals** and **heat flow lines** respectively.

4.17 DETERMINATION OF THE CONJUGATE FUNCTION

If $f(z) = u + iv$ is an analytic function where both $u(x, y)$ and $v(x, y)$ are conjugate functions, then we determine the other function v when one of these say u is given as follows:

$\because$ $$v = v(x, y)$$

$\therefore$ $$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$\Rightarrow$ $$dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \qquad \ldots(1) \qquad | \text{ By C-R eqns.}$$

$$M = -\frac{\partial u}{\partial y}, \quad N = \frac{\partial u}{\partial x}$$

$\therefore$ $$\frac{\partial M}{\partial y} = -\frac{\partial^2 u}{\partial y^2} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x^2}$$

Now, $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ gives}$$

$$-\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2}$$

or $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

which is true as u being a harmonic function satisfies Laplace's equation.

$\therefore$ dv is exact.

$\therefore$ dv can be integrated to get v.

However, if we are to construct $f(z) = u + iv$ when only u is given, we first of all find v by above procedure and then write $f(z) = u + iv$.

Similarly, if we are to determine u and only v is given then we use $du = \frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy$ and integrate it to find u. Consequently $f(z) = u + iv$ can also be determined.

4.18 MILNE'S THOMSON METHOD

With the help of this method, we can directly construct $f(z)$ in terms of z without first finding out v when u is given or u when v is given.

$$z = x + iy$$

$$\bar{z} = x - iy$$

$\Rightarrow$ $$x = \frac{1}{2}(z + \bar{z}) \text{ and } y = \frac{1}{2i}(z - \bar{z})$$

$\therefore$ $$f(z) = u(x, y) + iv(x, y)$$

$$= u\left\{\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right\} + iv\left\{\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right\} \qquad \ldots(1)$$

Relation (1) is an identity in z and $\bar{z}$. Putting $\bar{z} = z$, we get

$$f(z) = u(z, 0) + iv(z, 0) \quad ...(2)$$

Now,

$$f(z) = u + iv$$

$$\Rightarrow \quad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \quad | \text{ By C-R eqns.}$$

$$= \phi_1(x, y) - i\,\phi_2(x, y)$$

| where $\phi_1(x, y) = \frac{\partial u}{\partial x}$ and $\phi_2(x, y) = \frac{\partial u}{\partial y}$

| Replacing x by z and y by 0

Now,

$$f'(z) = \phi_1(z, 0) - i\,\phi_2(z, 0)$$

Integrating, we get

$$\boxed{f(z) = \int \{\phi_1(z, 0) - i\,\phi_2(z, 0)\}\,dz + c} \quad | \ c \text{ is an arbitrary constant.}$$

Hence the function is constructed directly in terms of z.

Similarly if $v(x, y)$ is given, then

$$\boxed{f(z) = \int [\psi_1(z, 0) + i\psi_2(z, 0)]\,dz + c} \quad \Bigg| \ \psi_1(x, y) = \frac{\partial v}{\partial y} \text{ and } \psi_2(x, y) = \frac{\partial v}{\partial x}$$

Milne's Thomson method can easily be grasped by going through the steps involved in following various cases.

Case I. *When only real part* $u(x, y)$ *is given.*

To construct analytic function $f(z)$ directly in terms of z when only real part u is given, we use the following steps:

1. Find $\frac{\partial u}{\partial x}$

2. Write it as equal to $\phi_1(x, y)$

3. Find $\frac{\partial u}{\partial y}$

4. Write it as equal to $\phi_2(x, y)$

5. Find $\phi_1(z, 0)$ by replacing x by z and y by 0 in $\phi_1(x, y)$.

6. Find $\phi_2(z, 0)$ by replacing x by z and y by 0 in $\phi_2(x, y)$.

7. $f(z)$ is obtained by the formula

$$f(z) = \int \{\phi_1(z, 0) - i\phi_2(z, 0)\}\,dz + c \text{ directly in terms of } z.$$

Case II. *When only imaginary part* $v(x, y)$ *is given.*

To construct analytic function $f(z)$ directly in terms of z when only imaginary part v is given, we use the following steps :

1. Find $\frac{\partial v}{\partial y}$

2. Write it as equal to $\psi_1(x, y)$

3. Find $\frac{\partial v}{\partial x}$

4. Write it as equal to $\psi_2(x, y)$
5. Find $\psi_1(z, 0)$ by replacing x by z and y by 0 in $\psi_1(x, y)$
6. Find $\psi_2(z, 0)$ by replacing x by z and y by 0 in $\psi_2(x, y)$
7. $f(z)$ is obtained by the formula

$$f(z) = \int \{\psi_1(z, 0) + i\psi_2(z, 0)\}\, dz + c \text{ directly in terms of } z.$$

Case III. *When* $u - v$ *is given.*

To construct analytic function $f(z)$ directly in terms of z when $u - v$ is given, we follow the following steps:

1. $f(z) = u + iv$ ...(1)
2. $i\,f(z) = iu - v$ ...(2)
3. Add (1) and (2) to get

$$(1 + i)\, f(z) = (u - v) + i(u + v)$$

or,
$$F(z) = U + iV$$

where
$$F(z) = (1 + i)\, f(z),\ U = u - v \text{ and } V = u + v$$

4. Since $u - v$ is given hence $U(x, y)$ is given
5. Find $\frac{\partial U}{\partial x}$
6. Write it as equal to $\phi_1(x, y)$
7. Find $\frac{\partial U}{\partial y}$
8. Write it as equal to $\phi_2(x, y)$
9. Find $\phi_1(z, 0)$
10. Find $\phi_2(z, 0)$
11. $F(z)$ is obtained by the formula

$$F(z) = \int \{\phi_1(z, 0) - i\phi_2(z, 0)\}\, dz + c$$

12. $f(z)$ is determined by $\quad f(z) = \dfrac{F(z)}{1+i}$ directly in terms of z.

Case IV. *When* $u + v$ *is given.*

To construct analytic function $f(z)$ directly in terms of z when $u + v$ is given, we follow the following steps:

1. $f(z) = u + iv$ ...(1)
2. $i\,f(z) = iu - v$ ...(2)
3. Add (1) and (2) to get

$$(1 + i)\, f(z) = (u - v) + i(u + v)$$

$$\Rightarrow \qquad F(z) = U + iV$$

where,
$$F(z) = (1 + i)\, f(z),\ U = u - v \text{ and } V = u + v$$

4. Since $u + v$ is given hence $V(x, y)$ is given
5. Find $\frac{\partial V}{\partial y}$
6. Write it as equal to $\psi_1(x, y)$

7. Find $\dfrac{\partial V}{\partial x}$

8. Write it as equal to $\psi_2(x, y)$

9. Find $\psi_1(z, 0)$

10. Find $\psi_2(z, 0)$

11. F(z) is obtained by the formula

$$F(z) = \int \{\psi_1(z, 0) + i\psi_2(z, 0)\}\, dz + c$$

12. $f(z)$ is determined by $f(z) = \dfrac{F(z)}{1+i}$ directly in terms of z.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the values of c_1 and c_2 such that the function*

$$f(z) = x^2 + c_1y^2 - 2xy + i\,(c_2x^2 - y^2 + 2xy)$$

is analytic. Also find $f'(z)$. (A.K.T.U. 2017)

Sol. Here, $$f(z) = (x^2 + c_1y^2 - 2xy) + i\,(c_2x^2 - y^2 + 2xy) \quad \text{...(1)}$$

Comparing (1) with $f(z) = u(x, y) + iv(x, y)$, we get

$$u(x, y) = x^2 + c_1y^2 - 2xy \quad \text{...(2)}$$

and $$v(x, y) = c_2x^2 - y^2 + 2xy \quad \text{...(3)}$$

For the function $f(z)$ to be analytic, it should satisfy Cauchy-Riemann equations.

Now from (2), $\dfrac{\partial u}{\partial x} = 2x - 2y$ and $\dfrac{\partial u}{\partial y} = 2c_1y - 2x$

Also, from (3), $\dfrac{\partial v}{\partial x} = 2c_2x + 2y$ and $\dfrac{\partial v}{\partial y} = -2y + 2x$

Cauchy-Riemann eqns. are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$\Rightarrow$ $2x - 2y = -2y + 2x$ which is true.

and $$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\Rightarrow$ $$2c_1y - 2x = -2c_2x - 2y \quad \text{...(4)}$$

Comparing the coefficients of x and y in eqn. (4), we get

$$2c_1 = -2 \quad \Rightarrow \quad c_1 = -1$$

and $$-2 = -2c_2 \quad \Rightarrow \quad c_2 = 1$$

Hence $c_1 = -1$ and $c_2 = 1$

Now, $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2x - 2y + i(2c_2x + 2y)$$

$$= 2x - 2y + i(2x + 2y) \quad | \because c_2 = 1$$

$$= 2(x + iy) + 2i(x + iy)$$

$$= 2z + 2iz = 2(1 + i)z.$$

Example 2. *Find p such that the function f(z) expressed in polar coordinates as $f(z) = r^2 \cos 2\theta + ir^2 \sin p\theta$ is analytic.*

Sol. Let $f(z) = u + iv$, then $u = r^2 \cos 2\theta$, $v = r^2 \sin p\theta$

$$\frac{\partial u}{\partial r} = 2r \cos 2\theta, \quad \frac{\partial v}{\partial r} = 2r \sin p\theta$$

$$\frac{\partial u}{\partial \theta} = -2r^2 \sin 2\theta, \quad \frac{\partial v}{\partial \theta} = pr^2 \cos p\theta$$

For $f(z)$ to be analytic, $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

$\therefore$ $\quad 2r \cos 2\theta = pr \cos p\theta$ and $2r \sin p\theta = 2r \sin 2\theta$

Both these equations are satisfied if $p = 2$.

Example 3. (*i*) *Prove that the function sinh z is analytic and find its derivative.*

(A.K.T.U. 2017)

(*ii*) *Show that f(z) = log z is analytic everywhere in the complex plane except at the origin and that its derivative is* $\left(\frac{1}{z}\right)$.

Sol. (*i*) Here $\quad f(z) = u + iv = \sinh z = \sinh (x + iy) = \sinh x \cos y + i \cosh x \sin y$

$\therefore$ $\quad u = \sinh x \cos y$ and $v = \cosh x \sin y$

$$\frac{\partial u}{\partial x} = \cosh x \cos y, \quad \frac{\partial u}{\partial y} = -\sinh x \sin y$$

$$\frac{\partial v}{\partial x} = \sinh x \sin y, \quad \frac{\partial v}{\partial y} = \cosh x \cos y$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Thus C-R equations are satisfied.

Since $\sinh x$, $\cosh x$, $\sin y$ and $\cos y$ are continuous functions, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are also continuous functions satisfying C-R equations.

Hence $f(z)$ is analytic everywhere.

Now, $\quad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \cosh x \cos y + i \sinh x \sin y = \cosh (x + iy) = \cosh z.$

(*ii*) Here, $\quad f(z) = u + iv = \log z = \log (x + iy)$

Let $\quad x = r \cos \theta$ and $y = r \sin \theta$ so that

$$x + iy = r(\cos \theta + i \sin \theta) = re^{i\theta}$$

$$\log (x + iy) = \log (r e^{i\theta}) = \log r + i\theta = \frac{1}{2} \log (x^2 + y^2) + i \tan^{-1}\left(\frac{y}{x}\right)$$

Separating real and imaginary parts, we get

$$u = \frac{1}{2} \log (x^2 + y^2) \quad \text{and} \quad v = \tan^{-1}\left(\frac{y}{x}\right)$$

Now, $$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}, \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$$

and $$\frac{\partial v}{\partial x} = \frac{-y}{x^2+y^2}, \frac{\partial v}{\partial y} = \frac{x}{x^2+y^2}$$

We observe that the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are satisfied except when $x^2 + y^2 = 0$ *i.e.*, when $x = 0, y = 0$

Also derivatives are continuous except at origin.

Hence the function $f(z) = \log z$ is analytic everywhere in the complex plane except at the origin.

Also, $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x-iy}{x^2+y^2} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{1}{x+iy} = \frac{1}{z}$$

Example 4. *Show that the function e^x (cos y + i sin y) is holomorphic and find its derivative.*

Sol. $f(z) = e^x \cos y + i\, e^x \sin y = u + iv$

Here, $u = e^x \cos y, \quad v = e^x \sin y$

$$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial v}{\partial x} = e^x \sin y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y \qquad \frac{\partial v}{\partial y} = e^x \cos y$$

Since, $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

hence, C-R equations are satisfied. Also first order partial derivatives of u and v are continuous everywhere. Therefore $f(z)$ is analytic.

Now, $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + i\, e^x \sin y$$

$$= e^x (\cos y + i \sin y) = e^x \cdot e^{iy} = e^{x+iy} = e^z$$

Example 5. *If n is real, show that r^n (cos nθ + i sin nθ) is analytic except possibly when r = 0 and that its derivative is*

$$nr^{n-1} [\cos (n-1)\theta + i \sin (n-1)\theta].$$

Sol. Let $w = f(z) = u + iv = r^n (\cos n\theta + i \sin n\theta)$

Here, $u = r^n \cos n\theta, \quad v = r^n \sin n\theta$

then, $$\frac{\partial u}{\partial r} = nr^{n-1} \cos n\theta \qquad \frac{\partial v}{\partial r} = nr^{n-1} \sin n\theta$$

$$\frac{\partial u}{\partial \theta} = -nr^n \sin n\theta \qquad \frac{\partial v}{\partial \theta} = nr^n \cos n\theta$$

Thus, we see that, $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

$\therefore$ Cauchy-Riemann equations are satisfied. Also first order partial derivatives of u and v are continuous everywhere.

Hence $f(z)$ is analytic if $f'(z)$ or $\frac{dw}{dz}$ exists for all finite values of z.

We have,
$$\frac{dw}{dz} = (\cos\theta - i\sin\theta)\frac{\partial w}{\partial r}$$
$$= (\cos\theta - i\sin\theta)\cdot nr^{n-1}(\cos n\theta + i\sin n\theta)$$
$$= nr^{n-1}[\cos(n-1)\theta + i\sin(n-1)\theta]$$

This exists for all finite values of r including zero, except when $r = 0$ and $n \le 1$.

Example 6. *Show that if $f(z)$ is analytic and Re $f(z)$ = constant then $f(z)$ is a constant.*

Sol. Since the function $f(z) = u(x, y) + iv(x, y)$ is analytic, it satisfies the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Also, Re $f(z)$ = constant, therefore $u(x, y) = c_1$

$$\therefore \quad \frac{\partial u}{\partial x} = 0 = \frac{\partial u}{\partial y}.$$

Using C-R equations, $\frac{\partial v}{\partial x} = 0 = \frac{\partial v}{\partial y}$

Hence $v(x, y) = c_2$ = a real constant

Therefore $f(z) = u(x, y) + iv(x, y) = c_1 + ic_2$ = a complex constant.

Example 7. *Given that $u(x, y) = x^2 - y^2$ and $v(x, y) = -\left(\frac{y}{x^2+y^2}\right)$.*

Prove that both u and v are harmonic functions but $u + iv$ is not an analytic function of z.

Sol.
$$u = x^2 - y^2$$
$$\frac{\partial u}{\partial x} = 2x \quad \Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = 2$$
$$\frac{\partial u}{\partial y} = -2y \quad \Rightarrow \quad \frac{\partial^2 u}{\partial y^2} = -2$$

Since $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ Hence $u(x, y)$ is harmonic.

Also,
$$v = \frac{-y}{x^2+y^2}$$
$$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2} \quad \Rightarrow \quad \frac{\partial^2 v}{\partial x^2} = \frac{2y^3 - 6x^2y}{(x^2+y^2)^3}$$

$$\frac{\partial v}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \Rightarrow \frac{\partial^2 v}{\partial y^2} = \frac{6x^2 y - 2y^3}{(x^2 + y^2)^3}$$

Since, $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$. Hence $v(x, y)$ is also harmonic.

But, $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} \neq -\frac{\partial u}{\partial y}$

Therefore $u + iv$ is not an analytic function of z.

Example 8. *If ϕ and ψ are functions of x and y satisfying Laplace's equation, show that $s + it$ is analytic, where*

$$s = \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x} \text{ and } t = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}.$$

[U.K.T.U. 2010, G.B.T.U. (C.O.) 2011]

Sol. Since ϕ and ψ are functions of x and y satisfying Laplace's equations,

$$\therefore \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad ...(1)$$

and $$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0. \quad ...(2)$$

For the function $s + it$ to be analytic,

$$\frac{\partial s}{\partial x} = \frac{\partial t}{\partial y} \quad ...(3)$$

and $$\frac{\partial s}{\partial y} = -\frac{\partial t}{\partial x} \quad ...(4)$$

must satisfy.

Now, $$\frac{\partial s}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x}\right) = \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x^2} \quad ...(5)$$

$$\frac{\partial t}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}\right) = \frac{\partial^2 \phi}{\partial y \partial x} + \frac{\partial^2 \psi}{\partial y^2} \quad ...(6)$$

$$\frac{\partial s}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x}\right) = \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \psi}{\partial y \partial x} \quad ...(7)$$

and $$\frac{\partial t}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}\right) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x \partial y}. \quad ...(8)$$

From (3), (5) and (6), we have

$$\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 \phi}{\partial y \partial x} + \frac{\partial^2 \psi}{\partial y^2} \Rightarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

which is true by (2).

Again from (4), (7) and (8), we have

$$\frac{\partial^2\phi}{\partial y^2}-\frac{\partial^2\psi}{\partial y\partial x}=-\frac{\partial^2\phi}{\partial x^2}-\frac{\partial^2\psi}{\partial x\partial y} \Rightarrow \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$$

which is also true by (1).

Hence the function $s + it$ is analytic.

Example 9. *Verify if* $f(z)=\dfrac{xy^2(x+iy)}{x^2+y^4}$, $z \neq 0$; $f(0) = 0$ *is analytic or not?*

Sol.
$$u+iv=\frac{xy^2(x+iy)}{x^2+y^4}\ ;\ z\neq 0$$

$$\therefore \qquad u=\frac{x^2y^2}{x^2+y^4},\ v=\frac{xy^3}{x^2+y^4}$$

At the origin,
$$\frac{\partial u}{\partial x}=\lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x}=\lim_{x\to 0}\frac{0-0}{x}=0$$

$$\frac{\partial u}{\partial y}=\lim_{y\to 0}\frac{u(0,y)-u(0,0)}{y}=\lim_{y\to 0}\frac{0-0}{y}=0$$

$$\frac{\partial v}{\partial x}=\lim_{x\to 0}\frac{v(x,0)-v(0,0)}{x}=\lim_{x\to 0}\frac{0-0}{x}=0$$

$$\frac{\partial v}{\partial y}=\lim_{y\to 0}\frac{v(0,y)-v(0,0)}{y}=\lim_{y\to 0}\frac{0-0}{y}=0$$

Since,
$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad\text{and}\quad \frac{\partial u}{\partial y}=\frac{-\partial v}{\partial x}$$

Hence Cauchy-Riemann equations are satisfied at the origin.

But
$$f'(0)=\lim_{z\to 0}\frac{f(z)-f(0)}{z}=\lim_{\substack{x\to 0\\ y\to 0}}\left[\frac{xy^2(x+iy)}{x^2+y^4}-0\right]\cdot\frac{1}{x+iy}=\lim_{\substack{x\to 0\\ y\to 0}}\frac{xy^2}{x^2+y^4}$$

Let $z \to 0$ along the real axis $y = 0$, then

$$f'(0)=0$$

Again let $z \to 0$ along the curve $x = y^2$ then

$$f'(0)=\lim_{x\to 0}\frac{x^2}{x^2+x^2}=\frac{1}{2}$$

which shows that $f'(0)$ does not exist since the limit is not unique along two different paths. Hence $f(z)$ is not analytic at origin although Cauchy-Riemann equations are satisfied there.

Example 10. *Show that the function defined by $f(z) = \sqrt{|xy|}$ is not regular at the origin, although Cauchy-Riemann equations are satisfied there.* (A.K.T.U. 2017)

Sol. Let $$f(z) = u(x, y) + iv(x, y) = \sqrt{|xy|}$$

then $$u(x, y) = \sqrt{|xy|}, \; v(x, y) = 0$$

At the origin (0, 0), we have

$$\frac{\partial u}{\partial x} = \underset{x \to 0}{\text{Lt}} \frac{u(x, 0) - u(0, 0)}{x} = \underset{x \to 0}{\text{Lt}} \frac{0 - 0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \underset{y \to 0}{\text{Lt}} \frac{u(0, y) - u(0, 0)}{y} = \underset{y \to 0}{\text{Lt}} \frac{0 - 0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \underset{x \to 0}{\text{Lt}} \frac{v(x, 0) - v(0, 0)}{x} = \underset{x \to 0}{\text{Lt}} \frac{0 - 0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \underset{y \to 0}{\text{Lt}} \frac{v(0, y) - v(0, 0)}{y} = \underset{y \to 0}{\text{Lt}} \frac{0 - 0}{y} = 0$$

Clearly, $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence C-R equations are satisfied at the origin.

Now $$f'(0) = \underset{z \to 0}{\text{Lt}} \frac{f(z) - f(0)}{z} = \underset{z \to 0}{\text{Lt}} \frac{\sqrt{|xy|} - 0}{x + iy}$$

If $z \to 0$ along the line $y = mx$, we get

$$f'(0) = \underset{x \to 0}{\text{Lt}} \frac{\sqrt{|mx^2|}}{x(1 + im)} = \underset{x \to 0}{\text{Lt}} \frac{\sqrt{|m|}}{1 + im}$$

Now this limit is not unique since it depends on m. Therefore, $f'(0)$ does not exist.

Hence the function $f(z)$ is not regular at the origin.

Example 11. *Prove that the function $f(z)$ defined by*

$$f(z) = \frac{x^3(1 + i) - y^3(1 - i)}{x^2 + y^2}, \; z \neq 0 \text{ and } f(0) = 0$$

is continuous and the Cauchy-Riemann equations are satisfied at the origin, yet $f'(0)$ does not exist. (A.K.T.U. 2015, 2017)

Sol. Here, $$f(z) = \frac{(x^3 - y^3) + i(x^3 + y^3)}{x^2 + y^2}, \; z \neq 0$$

Let $$f(z) = u + iv = \frac{x^3 - y^3}{x^2 + y^2} + i\,\frac{x^3 + y^3}{x^2 + y^2},$$

then $$u = \frac{x^3 - y^3}{x^2 + y^2}, \; v = \frac{x^3 + y^3}{x^2 + y^2}$$

Since $$z \neq 0 \;\Rightarrow\; x \neq 0, y \neq 0$$

$\therefore$ u and v are rational functions of x and y with non-zero denominators. Thus, u, v and hence $f(z)$ *are continuous functions when $z \neq 0$*. To test them for continuity at $z = 0$, on changing u, v to polar co-ordinates by putting $x = r \cos \theta$, $y = r \sin \theta$, we get

$$u = r(\cos^3\theta - \sin^3\theta) \text{ and } v = r(\cos^3\theta + \sin^3\theta)$$

When $z \to 0, r \to 0$

$$\therefore \quad \underset{z \to 0}{\text{Lt}}\ u = \underset{r \to 0}{\text{Lt}}\ r(\cos^3\theta - \sin^3\theta) = 0$$

Similarly, $\quad \underset{z \to 0}{\text{Lt}}\ v = 0$

$$\therefore \quad \underset{z \to 0}{\text{Lt}}\ f(z) = 0 = f(0)$$

$\Rightarrow$ **$f(z)$ is continuous at $z = 0$.**

Hence $f(z)$ is continuous for all values of z.

At the origin $(0, 0)$, we have

$$\frac{\partial u}{\partial x} = \underset{x \to 0}{\text{Lt}}\ \frac{u(x, 0) - u(0, 0)}{x} = \underset{x \to 0}{\text{Lt}}\ \frac{x - 0}{x} = 1$$

$$\frac{\partial u}{\partial y} = \underset{y \to 0}{\text{Lt}}\ \frac{u(0, y) - u(0, 0)}{y} = \underset{y \to 0}{\text{Lt}}\ \frac{-y - 0}{y} = -1$$

$$\frac{\partial v}{\partial x} = \underset{x \to 0}{\text{Lt}}\ \frac{v(x, 0) - v(0, 0)}{x} = \underset{x \to 0}{\text{Lt}}\ \frac{x - 0}{x} = 1$$

$$\frac{\partial v}{\partial y} = \underset{y \to 0}{\text{Lt}}\ \frac{v(0, y) - v(0, 0)}{y} = \underset{y \to 0}{\text{Lt}}\ \frac{y - 0}{y} = 1$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence C-R equations are satisfied at the origin.

Now, $$f'(0) = \underset{z \to 0}{\text{Lt}}\ \frac{f(z) - f(0)}{z} = \underset{z \to 0}{\text{Lt}}\ \frac{(x^3 - y^3) + i(x^3 + y^3) - 0}{(x^2 + y^2)(x + iy)}$$

Let $z \to 0$ along the line $y = x$, then

$$f'(0) = \underset{x \to 0}{\text{Lt}}\ \frac{0 + 2ix^3}{2x^3(1+i)} = \frac{i}{1+i} = \frac{i(1-i)}{2} = \frac{1+i}{2} \qquad ...(1)$$

Also, let $z \to 0$ along the x-axis (*i.e.* $y = 0$), then

$$f'(0) = \underset{x \to 0}{\text{Lt}}\ \frac{x^3 + ix^3}{x^3} = 1 + i \qquad ...(2)$$

Since the limits (1) and (2) are different, $f'(0)$ does not exist.

Example 12. *Show that the function $f(z) = e^{-z^{-4}}$, $z \neq 0$ and $f(0) = 0$ is not analytic at $z = 0$, although Cauchy-Riemann equations are satisfied at this point.*

Sol. Here, $$f(z) = e^{-z^{-4}} = e^{-(x+iy)^{-4}}$$

$$= e^{-\frac{1}{(x+iy)^4} \cdot \frac{(x-iy)^4}{(x-iy)^4}} = e^{-\left\{\frac{(x-iy)^4}{(x^2+y^2)^4}\right\}}$$

$$= e^{-\frac{1}{(x^2+y^2)^4}[(x^4+y^4-6x^2y^2)-4ixy(x^2-y^2)]}$$

$$\Rightarrow \qquad u + iv = e^{-\left[\frac{x^4+y^4-6x^2y^2}{(x^2+y^2)^4}\right]}\left[\cos\frac{4xy(x^2-y^2)}{(x^2+y^2)^4} + i\sin\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}\right]$$

$$\therefore \qquad u = e^{-\left[\frac{x^4+y^4-6x^2y^2}{(x^2+y^2)^4}\right]}\cos\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}$$

and
$$v = e^{-\left[\frac{x^4+y^4-6x^2y^2}{(x^2+y^2)^4}\right]}\sin\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}$$

At $z = 0$,
$$\frac{\partial u}{\partial x} = \lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x}$$

$$= \lim_{x\to 0}\frac{e^{-x^{-4}}-0}{x} = \lim_{x\to 0}\frac{1}{xe^{x^{-4}}}$$

$$= \lim_{x\to 0}\frac{1}{x\left[1+\frac{1}{x^4}+\frac{1}{2x^8}+\ldots\ldots\right]} = \lim_{x\to 0}\frac{1}{x+\frac{1}{x^3}+\frac{1}{2x^7}+\ldots\ldots} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0}\frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0}\frac{-e^{-y^{-4}}}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0}\frac{v(x,0)-v(0,0)}{x} = \lim_{x\to 0}\frac{0}{x} = 0$$

and
$$\frac{\partial v}{\partial y} = \lim_{y\to 0}\frac{v(0,y)-v(0,0)}{y} = \lim_{y\to 0}\frac{0}{y} = 0.$$

Hence Cauchy-Riemann Conditions are satisfied at $z = 0$.

But
$$f'(0) = \lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{z\to 0}\frac{e^{-z^{-4}}}{z}$$

$$= \lim_{r\to 0}\frac{e^{-(re^{i\pi/4})^{-4}}}{re^{i\pi/4}}\ ; \text{ if } z \to 0 \text{ along } z = re^{i\pi/4}$$

$$= \lim_{r\to 0}\frac{e^{r^{-4}}}{re^{i\pi/4}} = \infty$$

which shows that $f'(z)$ does not exist at $z = 0$. Hence $f(z)$ is not analytic at $z = 0$.

Example 13. *(i) Examine the nature of the function*

$$f(z) = \begin{cases}\dfrac{x^2y^5(x+iy)}{x^4+y^{10}}, & z \neq 0\\ 0, & z = 0\end{cases}$$ *in the region including the origin.* [A.K.T.U. 2016]

(ii) If $f(z) = \begin{cases}\dfrac{x^3y(y-ix)}{x^6+y^2}, & z \neq 0\\ 0, & z = 0\end{cases}$*, prove that* $\dfrac{f(z)-f(0)}{z} \to 0$ *as* $z \to 0$ *along any radius vector but not as* $z \to 0$ *in any manner and also that* $f(z)$ *is not analytic at* $z = 0$.

[G.B.T.U. 2013, U.K.T.U. 2010]

Sol. (*i*) Here, $$u + iv = \frac{x^2y^5(x+iy)}{x^4+y^{10}}\ ; z \neq 0$$

$\therefore$ $$u = \frac{x^3y^5}{x^4+y^{10}},\ v = \frac{x^2y^6}{x^4+y^{10}}$$

At the origin, $$\frac{\partial u}{\partial x} = \lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0}\frac{0-0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0}\frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0}\frac{0-0}{y} = 0$$

Similarly, $$\frac{\partial v}{\partial x} = 0 = \frac{\partial v}{\partial y}$$

Since $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence Cauchy-Riemann equations are satisfied at the origin

But $$f'(0) = \lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{\substack{x\to 0\\ y\to 0}}\left[\frac{x^2y^5(x+iy)}{x^4+y^{10}} - 0\right].\frac{1}{x+iy}$$

$$= \lim_{\substack{x\to 0\\ y\to 0}}\frac{x^2y^5}{x^4+y^{10}}$$

Let $z \to 0$ along the radius vector $y = mx$, then

$$f'(0) = \lim_{x\to 0}\frac{m^5x^7}{x^4+m^{10}x^{10}} = \lim_{x\to 0}\frac{m^5x^3}{1+m^{10}x^6} = 0$$

Again let $z \to 0$ along the curve $y^5 = x^2$

$$f'(0) = \lim_{x\to 0}\frac{x^4}{x^4+x^4} = \frac{1}{2}$$

which shows that $f'(0)$ does not exist. Hence $f(z)$ is not analytic at origin although Cauchy-Riemann equations are satisfied there.

(*ii*) $$\frac{f(z)-f(0)}{z} = \left[\frac{x^3y(y-ix)}{x^6+y^2} - 0\right].\frac{1}{x+iy} = \frac{-ix^3y(x+iy)}{(x^6+y^2)}.\frac{1}{x+iy} = -i\,\frac{x^3y}{x^6+y^2}$$

Let $z \to 0$ along radius vector $y = mx$ then,

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{x\to 0}\frac{-ix^3(mx)}{x^6+m^2x^2} = \lim_{x\to 0}\frac{-imx^2}{x^4+m^2} = 0$$

Hence $\dfrac{f(z)-f(0)}{z} \to 0$ as $z \to 0$ along any radius vector.

Now let $z \to 0$ along a curve $y = x^3$ then,

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{x\to 0}\frac{-ix^3.x^3}{x^6+x^6} = \frac{-i}{2}$$

Hence $\dfrac{f(z)-f(0)}{z}$ does not tend to zero as $z \to 0$ along the curve $y = x^3$.

We observe that $f'(0)$ does not exist hence $f(z)$ is not analytic at $z = 0$.

Example 14. *Show that the following functions are harmonic and find their harmonic conjugate functions.*

(*i*) $u = \frac{1}{2} \log(x^2 + y^2)$ (U.P.T.U. 2015) (*ii*) $v = \sinh x \cos y$.

Sol. (*i*)
$$u = \frac{1}{2} \log (x^2 + y^2) \quad ...(1)$$

$$\frac{\partial u}{\partial x} = \frac{1}{2} \cdot \frac{1}{x^2 + y^2} \cdot 2x = \frac{x}{x^2 + y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2 + y^2) \cdot 1 - x \cdot 2x}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \quad ...(2)$$

Also,
$$\frac{\partial u}{\partial y} = \frac{1}{2} \cdot \frac{1}{x^2 + y^2} \cdot 2y = \frac{y}{x^2 + y^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2 + y^2) \cdot 1 - y \cdot 2y}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2} \quad ...(3)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \quad \text{[From (2) and (3)]}$$

Since u satisfies Laplace's equation hence u is a harmonic function.

Let
$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$= \left(-\frac{\partial u}{\partial y}\right) dx + \left(\frac{\partial u}{\partial x}\right) dy \quad \text{[Using C-R equations]}$$

$$= \left(\frac{-y}{x^2 + y^2}\right) dx + \left(\frac{x}{x^2 + y^2}\right) dy$$

$$= \frac{x\,dy - y\,dx}{(x^2 + y^2)} = d\left[\tan^{-1}\left(\frac{y}{x}\right)\right]$$

Integration yields,
$$v = \tan^{-1}\left(\frac{y}{x}\right) + c \quad | \; c \text{ is a constant}$$

which is the required harmonic conjugate function of u.

(*ii*)
$$v = \sinh x \cos y \quad ...(1)$$

$$\frac{\partial v}{\partial x} = \cosh x \cos y \Rightarrow \frac{\partial^2 v}{\partial x^2} = \sinh x \cos y \quad ...(2)$$

$$\frac{\partial v}{\partial y} = -\sinh x \sin y \Rightarrow \frac{\partial^2 v}{\partial y^2} = -\sinh x \cos y \quad ...(3)$$

Since,
$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Hence v is harmonic.

Now, $$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = \frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy$$
$$= -\sinh x \sin y \, dx - \cosh x \cos y \, dy$$
$$= -[\sinh x \sin y \, dx + \cosh x \cos y \, dy]$$
$$= -d(\cosh x \sin y).$$

Integration yields, $u = -\cosh x \sin y + c$ | c is a constant

Example 15. (*i*) *Show that the function* $u(x, y) = x^4 - 6x^2y^2 + y^4$ *is harmonic. Also find the analytic function* $f(z) = u(x, y) + iv(x, y)$.

(*ii*) *Show that the function* $u = x^3 - 3xy^2$ *is harmonic and find the corresponding analytic function.*

(*iii*) *Show that* $e^x \cos y$ *is a harmonic function, find the analytic function of which it is real part.*

Sol. (*i*) $$u = x^4 - 6x^2y^2 + y^4$$

$\therefore$ $$\frac{\partial u}{\partial x} = 4x^3 - 12xy^2 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 12x^2 - 12y^2$$

$$\frac{\partial u}{\partial y} = -12x^2y + 4y^3 \Rightarrow \frac{\partial^2 u}{\partial y^2} = -12x^2 + 12y^2$$

Since, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \quad \therefore \quad u(x, y)$ is a harmonic function.

Now, let $$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = \left(-\frac{\partial u}{\partial y}\right) dx + \frac{\partial u}{\partial x} dy$$ | By C-R eqns.
$$= (12x^2y - 4y^3)\, dx + (4x^3 - 12xy^2)\, dy$$
$$= (12x^2y \, dx + 4x^3 dy) - (4y^3 \, dx + 12xy^2 \, dy)$$
$$= d(4x^3y) - d(4xy^3)$$

Integration yields, $v = 4x^3y - 4xy^3 + c$

Hence $$f(z) = u + iv = x^4 - 6x^2y^2 + y^4 + i(4x^3y - 4xy^3 + c)$$
$$= (x + iy)^4 + c_1 = z^4 + c_1$$ | where $c_1 = ic$

(*ii*) $$u = x^3 - 3xy^2$$

$\therefore$ $$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 6x$$

$$\frac{\partial u}{\partial y} = -6xy \Rightarrow \frac{\partial^2 u}{\partial y^2} = -6x$$

Since, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \quad \therefore \quad u$ is a harmonic function.

Now, $$dv = \frac{dv}{dx} dx + \frac{\partial v}{\partial y} dy = \left(-\frac{\partial u}{\partial y}\right) dx + \frac{\partial u}{\partial x} dy$$ | By C-R eqns.

$$= 6xy\,dx + (3x^2 - 3y^2)\,dy = (6xy\,dx + 3x^2\,dy) - 3y^2\,dy$$
$$= d(3x^2y) - d(y^3)$$

Integration yields,

$$v = 3x^2y - y^3 + c$$

$\therefore$ $$f(z) = u + iv = x^3 - 3xy^2 + i(3x^2y - y^3 + c)$$
$$= (x + iy)^3 + ic = z^3 + c_1 \qquad \text{(where } c_1 = ic)$$

(*iii*) Let $$u = e^x \cos y$$

$\therefore$ $$\frac{\partial u}{\partial x} = e^x \cos y \quad \Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = e^x \cos y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y \quad \Rightarrow \quad \frac{\partial^2 u}{\partial y^2} = -e^x \cos y$$

Since $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \quad \therefore \quad u$ is a harmonic function.

Let $$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = \left(-\frac{\partial u}{\partial y}\right) dx + \left(\frac{\partial u}{\partial x}\right) dy \qquad | \text{ By C-R eqns.}$$
$$= e^x \sin y\,dx + e^x \cos y\,dy = d\,(e^x \sin y)$$

Integration yields,

$$v = e^x \sin y + c$$

Hence $$f(z) = u + iv = e^x \cos y + i\,(e^x \sin y + c)$$
$$= e^x(\cos y + i \sin y) + c_1 \qquad | \text{ where } c_1 = ic$$
$$= e^{x+iy} + c_1 = e^z + c_1.$$

Example 16. *(i) In a two-dimensional fluid flow, the stream function is* $\psi = -\dfrac{y}{x^2 + y^2}$, *find the velocity potential* ϕ. [M.T.U. 2014]

(ii) An electrostatic field in the xy-plane is given by the potential function $\phi = 3x^2y - y^3$, *find the stream function and hence find complex potential.* (G.B.T.U. 2011, 2013)

Sol. (*i*) $$\psi = -\frac{y}{x^2 + y^2} \qquad \ldots(1)$$

$$\frac{\partial \psi}{\partial x} = \frac{2xy}{(x^2 + y^2)^2}, \qquad \frac{\partial \psi}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

We know that, $$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = \frac{\partial \psi}{\partial y} dx - \frac{\partial \psi}{\partial x} dy$$

$$= \frac{(y^2 - x^2)}{(x^2 + y^2)^2} dx - \frac{2xy}{(x^2 + y^2)^2} dy$$

$$= \frac{(x^2 + y^2)dx - 2x^2\,dx - 2xy\,dy}{(x^2 + y^2)^2}$$

$$= \frac{(x^2 + y^2)\, d(x) - x(2x\, dx + 2y\, dy)}{(x^2 + y^2)^2}$$

$$= \frac{(x^2 + y^2)\, d(x) - x d(x^2 + y^2)}{(x^2 + y^2)^2} = d\left(\frac{x}{x^2 + y^2}\right).$$

Integration yields, $\phi = \dfrac{x}{x^2 + y^2} + c$, where c is a constant.

(*ii*) Let $\psi(x, y)$ be the stream function.

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = \left(-\frac{\partial \phi}{\partial y}\right) dx + \left(\frac{\partial \phi}{\partial x}\right) dy$$

$$= \{-(3x^2 - 3y^2)\}\, dx + 6xy\, dy$$

$$= -3x^2\, dx + (3y^2\, dx + 6xy\, dy)$$

$$= -d(x^3) + 3d(xy^2)$$

Integrating, we get $\psi = -x^3 + 3xy^2 + c$ | c is a constant

Complex potential is given by

$$w = \phi + i\psi = 3x^2y - y^3 + i(-x^3 + 3xy^2 + c)$$

or $$w = -i[x^3 - iy^3 + 3ix^2y - 3xy^2 - c]$$

or $$w = -i\,[(x + iy)^3 - c]$$

$\Rightarrow$ $$w = -iz^3 + c_1$$ | where $c_1 = ic$

Example 17. (*i*) *If $u = e^x(x \cos y - y \sin y)$ is a harmonic function, find an analytic function $f(z) = u + iv$ such that $f(1) = e$.*

(*ii*) *Determine an analytic function $f(z)$ in terms of z whose real part is $e^{-x}(x \sin y - y \cos y)$.*

[M.T.U. 2012, G.B.T.U. 2011, U.P.T.U. 2014]

Sol. (*i*) We have, $u = e^x(x \cos y - y \sin y)$

$$\frac{\partial u}{\partial x} = e^x(x \cos y - y \sin y) + e^x \cos y = \phi_1(x, y)$$ | say

$$\frac{\partial u}{\partial y} = e^x\,[-x \sin y - y \cos y - \sin y] = \phi_2(x, y)$$ | say

$\therefore$ $$\phi_1(z, 0) = e^z z + e^z = (z + 1)\, e^z$$

$$\phi_2(z, 0) = 0$$

By Milne's Thomson method,

$$f(z) = \int \{\phi_1(z, 0) - i\, \phi_2(z, 0)\}\, dz + c$$ | c is a constant

$$= \int (z + 1)\, e^z\, dz + c = (z - 1)\, e^z + e^z + c = ze^z + c \quad ...(1)$$

| From (1)

$$f(1) = e + c$$ | $f(1) = e$ (given)

$$e = e + c$$

$\Rightarrow$ $$c = 0$$

$\therefore$ From (1), $f(z) = ze^z$.

(*ii*) $$u = e^{-x}(x \sin y - y \cos y)$$

$$\frac{\partial u}{\partial x} = e^{-x} \sin y - e^{-x}\,(x \sin y - y \cos y) = \phi_1(x, y)$$ | say

$$\frac{\partial u}{\partial y} = e^{-x}(x \cos y - \cos y + y \sin y) = \phi_2(x, y) \qquad | \text{ say}$$

$\therefore \qquad \phi_1(z, 0) = 0 \quad \text{and} \quad \phi_2(z, 0) = e^{-z}(z - 1)$

By Milne's Thomson method,

$$f(z) = \int \{\phi_1(z, 0) - i\phi_2(z, 0)\} \, dz + c$$

$$= -i \int e^{-z}(z-1) \, dz + c$$

$$= -i\left[(z-1)(-e^{-z}) - \int (-e^{-z}) \, dz\right] + c$$

$$= -i\, [(1-z)\, e^{-z} - e^{-z}] + c$$

$\Rightarrow \qquad f(z) = ize^{-z} + c \qquad$ | where c is a constant

Example 18. (*i*) *Determine the analytic function whose real part is $e^{2x}(x \cos 2y - y \sin 2y)$.*
(*ii*) *Find an analytic function whose imaginary part is $e^{-x}(x \cos y + y \sin y)$.*

Sol. (*i*) Let $f(z) = u + iv$ be the required analytic function.

Here, $\qquad u = e^{2x}(x \cos 2y - y \sin 2y)$

$$\therefore \qquad \frac{\partial u}{\partial x} = e^{2x}(2x \cos 2y - 2y \sin 2y + \cos 2y) = \phi_1(x, y) \qquad | \text{ say}$$

and
$$\frac{\partial u}{\partial y} = -e^{2x}(2x \sin 2y + \sin 2y + 2y \cos 2y) = \phi_2(x, y) \qquad | \text{ say}$$

Now, $\qquad \phi_1(z, 0) = e^{2z}(2z + 1)$

$\qquad \phi_2(z, 0) = -e^{2z}(0) = 0$

By Milne's Thomson method,

$$f(z) = \int \{\phi_1(z, 0) - i\,\phi_2(z, 0)\} \, dz + c = \int e^{2z}(2z+1) \, dz + c$$

$$= (2z+1)\, \frac{e^{2z}}{2} - \int 2 \cdot \frac{e^{2z}}{2} \, dz + c \qquad | \text{ Integrating by parts}$$

$$= (2z+1)\, \frac{e^{2z}}{2} - \frac{1}{2}\, e^{2z} + c$$

$f(z) = ze^{2z} + c$ where c is an arbitrary constant.

(*ii*) Let $f(z) = u + iv$ be the required analytic function.

Here $\qquad v = e^{-x}(x \cos y + y \sin y)$

$$\frac{\partial v}{\partial y} = e^{-x}(-x \sin y + y \cos y + \sin y) = \psi_1(x, y) \qquad | \text{ say}$$

$$\frac{\partial v}{\partial x} = e^{-x} \cos y - e^{-x}(x \cos y + y \sin y) = \psi_2(x, y) \qquad | \text{ say}$$

$\therefore \qquad \psi_1(z, 0) = 0$

$\qquad \psi_2(z, 0) = e^{-z} - e^{-z} z = (1 - z)\, e^{-z}$

By Milne's Thomson method,

$$f(z) = \int [\psi_1(z, 0) + i\, \psi_2(z, 0)] \, dz + c = i \int (1-z)\, e^{-z} \, dz + c$$

$$= i\left[(1-z)(-e^{-z}) - \int (-1)(-e^{-z})\,dz\right] + c$$

$$= i\,[(z-1)\,e^{-z} + e^{-z}] + c$$

$$\Rightarrow \quad f(z) = ize^{-z} + c$$

Example 19. (*i*) *Let $f(z) = u(r, \theta) + iv(r, \theta)$ be an analytic function. If $u = -r^3 \sin 3\theta$, then construct the corresponding analytic function $f(z)$ in terms of z.*

(*ii*) *Find the analytic function $f(z) = u + iv$, given that $v = \left(r - \dfrac{1}{r}\right) \sin \theta \,;\, r \neq 0$.*

Sol. (*i*) $$u = -r^3 \sin 3\theta$$

$$\Rightarrow \quad \frac{\partial u}{\partial r} = -3r^2 \sin 3\theta, \quad \frac{\partial u}{\partial \theta} = -3r^3 \cos 3\theta$$

We know that, $$dv = \frac{\partial v}{\partial r}\,dr + \frac{\partial v}{\partial \theta}\,d\theta = \left(-\frac{1}{r}\frac{\partial u}{\partial \theta}\right) dr + \left(r\frac{\partial u}{\partial r}\right) d\theta$$

$$= (3r^2 \cos 3\theta)\,dr - (3r^3 \sin 3\theta)\,d\theta$$

$$\Rightarrow \quad dv = d\,(r^3 \cos 3\theta)$$

Integration yields,

$$v = r^3 \cos 3\theta + c$$

$$\therefore \quad f(z) = u + iv = -r^3 \sin 3\theta + ir^3 \cos 3\theta + ic$$

$$= ir^3 (\cos 3\theta + i \sin 3\theta) + c_1 \qquad | \; c_1 = ic$$

$$= i(re^{i\theta})^3 + c_1$$

$$\Rightarrow \quad f(z) = iz^3 + c_1 \qquad | \because \; z = re^{i\theta}$$

(*ii*) $$v = \left(r - \frac{1}{r}\right) \sin \theta$$

$$\frac{\partial v}{\partial r} = \left(1 + \frac{1}{r^2}\right) \sin \theta, \quad \frac{\partial v}{\partial \theta} = \left(r - \frac{1}{r}\right) \cos \theta$$

We know that, $$du = \frac{\partial u}{\partial r}\,dr + \frac{\partial u}{\partial \theta}\,d\theta = \left(\frac{1}{r}\frac{\partial v}{\partial \theta}\right) dr + \left(-r\frac{\partial v}{\partial r}\right) d\theta$$

$$= \left(1 - \frac{1}{r^2}\right) \cos \theta\,dr - \left(r + \frac{1}{r}\right) \sin \theta\,d\theta$$

$$\Rightarrow \quad du = d\,(r \cos \theta) + d\left(\frac{1}{r} \cos \theta\right)$$

Integration yields, $$u = \left(r + \frac{1}{r}\right) \cos \theta + c$$

$$\therefore \quad f(z) = u + iv = \left(r + \frac{1}{r}\right) \cos \theta + c + i\left(r - \frac{1}{r}\right) \sin \theta$$

$$= re^{i\theta} + \frac{1}{r} e^{-i\theta} + c$$

$$\Rightarrow \qquad f(z) = z + \frac{1}{z} + c.$$

Example 20. *If* $u - v = (x - y)(x^2 + 4xy + y^2)$ *and* $f(z) = u + iv$ *is an analytic function of* $z = x + iy$, *find* $f(z)$ *in terms of* z.

Sol. Here, $f(z) = u + iv$

$\therefore \qquad if(z) = iu - v$

Adding $\qquad (1 + i) f(z) = (u - v) + i(u + v)$

Let $\qquad (1 + i) f(z) = F(z)$, $u - v = U$, $u + v = V$, then

$$F(z) = U + iV$$

Now, $\qquad U = u - v = (x - y)(x^2 + 4xy + y^2)$

$$\Rightarrow \qquad \frac{\partial U}{\partial x} = x^2 + 4xy + y^2 + (x - y)(2x + 4y) = 3x^2 + 6xy - 3y^2 = \phi_1(x, y) \quad | \text{ say}$$

and

$$\frac{\partial U}{\partial y} = -(x^2 + 4xy + y^2) + (x - y)(4x + 2y) = 3x^2 - 6xy - 3y^2 = \phi_2(x, y) \quad | \text{ say}$$

Now, $\qquad \phi_1(z, 0) = 3z^2, \quad \phi_2(z, 0) = 3z^2$

By Milne's Thomson method,

$$F(z) = \int [\phi_1(z, 0) - i\phi_2(z, 0)]\, dz + c = \int [3z^2 - i(3z^2)]\, dz + c$$

$$F(z) = (1 - i) z^3 + c$$

$$\Rightarrow \qquad (1 + i) f(z) = (1 - i) z^3 + c$$

or,

$$f(z) = \left(\frac{1-i}{1+i}\right) z^3 + \frac{c}{1+i} = \left(\frac{-2i}{2}\right) z^3 + c_1 \qquad \left(\text{where } c_1 = \frac{c}{1+i}\right)$$

or,

$$f(z) = -iz^3 + c_1.$$

Example 21. *If* $u + v = \dfrac{2 \sin 2x}{e^{2y} + e^{-2y} - 2 \cos 2x}$ *and* $f(z) = u + iv$ *is an analytic function of* $z = x + iy$, *find* $f(z)$ *in terms of* z.

Sol. Let $\qquad f(z) = u + iv \qquad \ldots(1)$

Multiplying both sides by i

$$i f(z) = iu - v \qquad \ldots(2)$$

Adding (1) and (2), we get

$$(1 + i) f(z) = (u - v) + i(u + v) \qquad \ldots(3)$$

$$\Rightarrow \qquad F(z) = U + iV \qquad \ldots(4)$$

where $\qquad F(z) = (1 + i) f(z) \qquad \ldots(5)$

$$U = u - v \quad \text{and} \quad V = u + v \qquad \ldots(6)$$

It means that we have been given

$$V = \frac{\sin 2x}{\cosh 2y - \cos 2x} \qquad \text{...(7)} \quad \Big| \because \quad e^{2y} + e^{-2y} = 2\cosh 2y$$

Now,
$$\frac{\partial V}{\partial y} = \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2} = \psi_1(x, y) \qquad \text{| say}$$

and
$$\frac{\partial V}{\partial x} = \frac{2\cos 2x(\cosh 2y - \cos 2x) - 2\sin^2 2x}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2\cos 2x\cosh 2y - 2}{(\cosh 2y - \cos 2x)^2} = \psi_2(x, y) \qquad \text{| say}$$

$\therefore$
$$\psi_1(z, 0) = 0$$

$$\psi_2(z, 0) = \frac{2(\cos 2z - 1)}{(1 - \cos 2z)^2} = \frac{-2}{1 - \cos 2z} = \frac{-2}{1 - 1 + 2\sin^2 z} = -\operatorname{cosec}^2 z$$

By Milne's Thomson method, we have

$$F(z) = \int \{\psi_1(z, 0) + i\,\psi_2(z, 0)\}\, dz + c$$

$$= \int -i \operatorname{cosec}^2 z\, dz + c = i \cot z + c$$

Replacing $F(z)$ by $(1 + i) f(z)$, from eqn. (5), we get

$$(1 + i) f(z) = i \cot z + c$$

$$\Rightarrow \qquad f(z) = \frac{i}{1+i} \cot z + \frac{c}{1+i}$$

$$\therefore \qquad f(z) = \frac{1}{2}(1 + i)\cot z + c_1, \qquad \text{where } c_1 = \frac{c}{1+i}.$$

Example 22. *If $f(z) = u + iv$ is an analytic function of z and $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - 2\cosh y}$, prove that $f(z) = \dfrac{1}{2}\left[1 - \cot \dfrac{z}{2}\right]$ when $f\left(\dfrac{\pi}{2}\right) = 0$.*

Sol. Let
$$f(z) = u + iv \qquad \text{...(1)}$$

$\therefore$
$$i f(z) = iu - v$$

Add,
$$(1 + i) f(z) = (u - v) + i(u + v) \qquad \text{...(2)}$$

$\Rightarrow$
$$F(z) = U + iV \qquad \text{...(3)}$$

where
$$u - v = U,\; u + v = V \quad \text{and} \quad (1 + i) f(z) = F(z).$$

We have,
$$u - v = \frac{\cos x + \sin x - e^{-y}}{2\cos x - 2\cosh y}$$

or,
$$U = \frac{\cos x + \sin x - \cosh y + \sinh y}{2\cos x - 2\cosh y} \qquad [\because \quad e^{-y} = \cosh y - \sinh y]$$

$$= \frac{1}{2} + \frac{\sin x + \sinh y}{2(\cos x - \cosh y)} \qquad \text{...(4)}$$

Diff. (4) w.r.t. x partially, we get

$$\frac{\partial U}{\partial x} = \frac{1}{2}\left[\frac{(\cos x - \cosh y)\cos x - (\sin x + \sinh y)(-\sin x)}{(\cos x - \cosh y)^2}\right]$$

$$\phi_1(x, y) = \frac{1}{2}\left[\frac{1 - \cosh y \cos x + \sinh y \sin x}{(\cos x - \cosh y)^2}\right]$$

$$\phi_1(z, 0) = \frac{1}{2}\left[\frac{1 - \cos z}{(\cos z - 1)^2}\right] = \frac{1}{2(1 - \cos z)}. \quad \text{...(5)}$$

Diff. (4) partially w.r.t. y, we get

$$\frac{\partial U}{\partial y} = \frac{1}{2}\left[\frac{(\cos x - \cosh y).\cosh y - (\sin x + \sinh y)(-\sinh y)}{(\cos x - \cosh y)^2}\right]$$

$$\phi_2(x, y) = \frac{1}{2}\left[\frac{\cos x \cosh y + \sin x \sinh y - 1}{(\cos x - \cosh y)^2}\right]$$

$$\therefore \quad \phi_2(z, 0) = \frac{1}{2}\left[\frac{\cos z - 1}{(\cos z - 1)^2}\right] = \frac{1}{2}.\left(\frac{-1}{1 - \cos z}\right). \quad \text{...(6)}$$

By Milne's Thomson Method,

$$F(z) = \int [\phi_1(z, 0) - i\,\phi_2(z, 0)]\, dz + c$$

$$= \int \left[\frac{1}{2}.\frac{1}{(1 - \cos z)} + \frac{i}{2}.\frac{1}{1 - \cos z}\right] dz + c$$

$$= \frac{1+i}{2}\int \frac{1}{2\sin^2 z/2}\, dz + c = \frac{1+i}{4}\int \operatorname{cosec}^2 (z/2)\, dz + c$$

$$= \left(\frac{1+i}{4}\right).\frac{(-\cot z/2)}{\left(\frac{1}{2}\right)} + c = -\left(\frac{1+i}{2}\right)\cot\frac{z}{2} + c$$

or, $$(1 + i)\, f(z) = -\left(\frac{1+i}{2}\right)\cot\frac{z}{2} + c$$

$$\Rightarrow \quad f(z) = -\frac{1}{2}\cot\frac{z}{2} + \frac{c}{1+i} \quad \text{...(7)}$$

$$f\left(\frac{\pi}{2}\right) = -\frac{1}{2}\cot\frac{\pi}{4} + \frac{c}{1+i} \quad \text{[From (7)]}$$

$$0 = -\frac{1}{2} + \frac{c}{1+i} \;\Rightarrow\; \frac{c}{1+i} = \frac{1}{2} \quad \text{...(8)}$$

$$\therefore \quad \text{From (7), } f(z) = -\frac{1}{2}\cot\frac{z}{2} + \frac{1}{2} = \frac{1}{2}\left(1 - \cot\frac{z}{2}\right). \quad \text{[Using (8)]}$$

Example 23. *(i) If f(z) is a regular function of z, prove that*

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\,|f'(z)|^2. \qquad \text{(U.P.T.U. 2015)}$$

(ii) If f(z) is a harmonic function of z, show that

$$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2+\left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = |f'(z)|^2$$

Sol. (i) Let $f(z) = u + iv$ so that $|f(z)| = \sqrt{u^2+v^2}$

or
$$|f(z)|^2 = u^2 + v^2 = \phi(x, y) \text{ (say)}$$

$\therefore$
$$\frac{\partial \phi}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 \phi}{\partial x^2} = 2\left[u\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial u}{\partial x}\right)^2+v\frac{\partial^2 v}{\partial x^2}+\left(\frac{\partial v}{\partial x}\right)^2\right]$$

Similarly,
$$\frac{\partial^2 \phi}{\partial y^2} = 2\left[u\frac{\partial^2 u}{\partial y^2}+\left(\frac{\partial u}{\partial y}\right)^2+v\frac{\partial^2 v}{\partial y^2}+\left(\frac{\partial v}{\partial y}\right)^2\right]$$

Adding, we get

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2} = 2\left[u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+v\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right] \qquad ...(1)$$

Since $f(z) = u + iv$ is a regular function of z, u and v satisfy C-R equations and Laplace's equation.

$\therefore$
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0 = \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}$$

$\therefore$ From (1), we get

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2} = 2\left[0+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+0+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial u}{\partial x}\right)^2\right]$$

$$= 4\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right] \qquad ...(2)$$

Now,
$$f(z) = u + iv$$

$\therefore$
$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{and} \quad |f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2$$

From (2), we get

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi = 4\,|f'(z)|^2 \quad \text{or} \quad \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\,|f'(z)|^2.$$

(*ii*) We have, $f(z) = u + iv$ ...(1)

$\therefore$ $|f(z)| = \sqrt{u^2 + v^2}$ ...(2)

Partially differentiating eqn. (2) w.r.t. x and y, we get

$$\frac{\partial}{\partial x}|f(z)| = \frac{1}{2}(u^2 + v^2)^{-1/2}\left(2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}\right) = \frac{u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}}{|f(z)|} \quad ...(3)$$

Similarly, $$\frac{\partial}{\partial y}|f(z)| = \frac{u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y}}{|f(z)|} \quad ...(4)$$

Squaring and adding (3) and (4), we get

$$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = \frac{\left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right)^2 + \left(u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y}\right)^2}{|f(z)|^2}$$

$$= \frac{\left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right)^2 + \left(-u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}\right)^2}{|f(z)|^2} \quad | \text{ Using C-R eqns.}$$

$$= \frac{(u^2 + v^2)\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]}{|f(z)|^2}$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 \quad \left| \because \ |f(z)|^2 = u^2 + v^2\right.$$

$$= |f'(z)|^2 \quad \left| \because \ f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right.$$

TEST YOUR KNOWLEDGE

1. (*i*) Determine a, b, c, d so that the function $f(z) = (x^2 + axy + by^2) + i(cx^2 + dxy + y^2)$ is analytic.

   (*ii*) Find the constants a, b, c such that the function $f(z)$ where
   $f(z) = -x^2 + xy + y^2 + i(ax^2 + bxy + cy^2)$ is analytic. Express $f(z)$ in terms of z.
   (*M.T.U. 2013*)

   (*iii*) Find the value of the constants a and b such that the following function $f(z)$ is analytic.
   $f(z) = \cos x(\cosh y + a \sinh y) + i \sin x(\cosh y + b \sinh y)$

   (*iv*) Determine p such that the function $f(z) = \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}\frac{px}{y}$ is an analytic function.
   Also find $f'(z)$. (*M.T.U. 2012*)

2. Show that

   (*a*) $f(z) = xy + iy$ is everywhere continuous but is not analytic.

   (*b*) $f(z) = z + 2\bar{z}$ is not analytic anywhere in the complex plane.

   (*c*) $f(z) = z\,|\,z\,|$ is not analytic anywhere.
   (*U.K.T.U. 2010*)

3. Discuss the analyticity of the following functions:

(i) $\sin z$ (ii) $\cosh z$ (iii) $\frac{1}{z}$ (iv) z^3.

4. (i) Define analytic function. Discuss the analyticity and differentiability of $f(z) = |z|^4$ at $z = 0$. (G.B.T.U. 2012)

(ii) Define analytic function. Discuss the analyticity of $f(z) = \text{Re}\,(z^3)$ in the complex plane. (U.P.T.U. 2014)

5. Show that the polar form of Cauchy-Riemann equations are $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$. Deduce that $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$.

6. (i) Show that an analytic function $f(z)$, whose derivative is identically zero, is constant.

(ii) It is given that a function $f(z)$ and its conjugate $\overline{f(z)}$ are both analytic. Determine the function $f(z)$.

(iii) Show that if $f(z)$ is analytic and Im $f(z)$ = constant then $f(z)$ is a constant.

(iv) Show that if $f(z)$ is differentiable at a point z, then $|f'(z)|^2 = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

7. (i) Show that the function $f(z)$ defined by $f(z) = \frac{x^3y^5(x+iy)}{x^6+y^{10}}$, $z \neq 0$, $f(0) = 0$, is not analytic at the origin even though it satisfies Cauchy-Riemann equations at the origin. (G.B.T.U. 2011)

(ii) Show that for the function

$$f(z) = \begin{cases} \frac{(\bar{z})^2}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

the Cauchy-Riemann equations are satisfied at the origin. Does $f'(0)$ exist?

(iii) Show that for the function

$$f(z) = \begin{cases} \frac{2xy\,(x+iy)}{x^2+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

the C-R equations are satisfied at origin but derivative of $f(z)$ does not exist at origin. (A.K.T.U. 2016)

(iv) Give an example of a function in which Cauchy-Riemann equations are satisfied yet the function is not analytic at the origin. Justify your answer. (A.K.T.U. 2018)

8. (i) If u is a harmonic function then show that $w = u^2$ is not a harmonic function unless u is a constant.

(ii) If $f(z)$ is an analytic function, show that $|f(z)|$ is not a harmonic function.

9. (i) Show that the function $u(x, y) = 2x + y^3 - 3x^2y$ is harmonic. Find its conjugate harmonic function $v(x, y)$ and the corresponding analytic function $f(z)$.

(ii) Show that the function $v(x, y) = e^x \sin y$ is harmonic. Find its conjugate harmonic function $u(x, y)$ and the corresponding analytic function $f(z)$.

(iii) Define a harmonic function and conjugate harmonic function. Find the harmonic conjugate of the function $u(x, y) = 2x\,(1 - y)$.

(iv) Show that the function $u = e^{-2xy} \sin (x^2 - y^2)$ is harmonic. (U.K.T.U. 2011)

(v) Show that $u(x, y) = x^3 - 4xy - 3xy^2$ is harmonic. Find its harmonic conjugate $v(x, y)$ and the corresponding analytic function $f(z) = u + iv$. (G.B.T.U. 2013)

10. (*i*) Show that the function $u(r, \theta) = r^2 \cos 2\theta$ is harmonic. Find its conjugate harmonic function and the corresponding analytic function $f(z)$.

(*ii*) Determine constant '*b*' such that $u = e^{bx} \cos 5y$ is harmonic.

(*iii*) Define Harmonic function. Show that the function $v = \log(x^2 + y^2) + x - 2y$ is harmonic. Also find the analytic function $f(z) = u + iv$. (G.B.T.U. 2012)

(*iv*) Show that $v(x, y) = e^{-x}(x \cos y + y \sin y)$ is harmonic. Find its harmonic conjugate. (U.P.T.U. 2014)

(*v*) Verify that the given function $u(x, y)$ is harmonic and find its conjugate harmonic function. Express $u + iv$ as an analytic function $f(z)$.

$$u(x, y) = x^2 - y^2 - y.$$ (A.K.T.U. 2016)

11. Determine the analytic function $f(z)$ in terms of z whose real part is

(*i*) $\frac{1}{2}\log(x^2 + y^2)$ (U.K.T.U. 2011) (*ii*) $\cos x \cosh y$

(*iii*) $e^{-x}(x \cos y + y \sin y)$; $f(0) = 1$ (*iv*) $(x - y)(x^2 + 4xy + y^2)$. (G.B.T.U. 2012)

(*v*) $\dfrac{\sin 2x}{\cosh 2y - \cos 2x}$ (*vi*) $\dfrac{\sin 2x}{\cosh 2y + \cos 2x}$.

12. Find the regular function $f(z)$ in terms of z whose imaginary part is

(*i*) $\dfrac{x - y}{x^2 + y^2}$ (*ii*) $\cos x \cosh y$ (*iii*) $\sinh x \cos y$

(*iv*) $6xy - 5x + 3$ (*v*) $\dfrac{x}{x^2 + y^2} + \cosh x \cos y.$ (*vi*) $e^x(x \sin y + y \cos y)$

(U.P.T.U. 2015)

13. Prove that $u = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic. Find a function v such that $f(z) = u + iv$ is analytic. Also express $f(z)$ in terms of z.

14. (*i*) An electrostatic field in the xy-plane is given by the potential function $\phi = x^2 - y^2$, find the stream function.

(*ii*) If the potential function is $\log(x^2 + y^2)$, find the flux function and the complex potential function.

15. (*i*) In a two dimensional fluid flow, the stream function is $\psi = \tan^{-1}\left(\dfrac{y}{x}\right)$, find the velocity potential ϕ.

(*ii*) If $w = \phi + i\psi$ represents the complex potential for an electric field and $\psi = x^2 - y^2 + \dfrac{x}{x^2 + y^2}$, determine the function ϕ.

(*iii*) If $u = (x - 1)^3 - 3xy^2 + 3y^2$, determine v so that $u + iv$ is a regular function of $x + iy$. (U.K.T.U. 2010)

16. If $f(z)$ is an analytic function of z, prove that $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right) |\text{Re } f(z)|^2 = 2\, |f'(z)|^2$.

(G.B.T.U. 2012)

17. Find an analytic function $f(z) = u(r, \theta) + iv(r, \theta)$ such that $v(r, \theta) = r^2 \cos 2\theta - r \cos \theta + 2$.

18. If $f(z) = u + iv$ is an analytic function, find $f(z)$ in terms of z if

(*i*) $u - v = e^x(\cos y - \sin y)$ (A.K.T.U. 2016) (*ii*) $u + v = \dfrac{x}{x^2 + y^2}$, when $f(1) = 1$

(*iii*) $u - v = \dfrac{e^y - \cos x + \sin x}{\cosh y - \cos x}$ when $f\left(\dfrac{\pi}{2}\right) = \dfrac{3 - i}{2}$.

19. (i) If $f(z) = u + iv$ is an analytic function of $z = x + iy$ and $u + v = (x + y)(2 - 4xy + x^2 + y^2)$, then construct $f(z)$ in terms of z.

(ii) If $f(z) = u + iv$ is an analytic function of $z = x + iy$ and $u - v = e^{-x}[(x - y)\sin y - (x + y)\cos y]$, then construct $f(z)$ in terms of z.

20. If $f = u + iv$ is analytic show that $g = -v + iu$ is also analytic. Also show that u and $-v$ are conjugate harmonic.

21. Show that the function

(i) $f(z) = \dfrac{z}{z+1}$ is analytic at $z = \infty$. (ii) $f(z) = z$ is not analytic at $z = \infty$.

22. If $f(z) = u(x, y) + iv(x, y)$ where $x = \dfrac{z + \bar{z}}{2}$, $y = \dfrac{z - \bar{z}}{2i}$ is continuous as a function of two variables z and $\bar{z}$, then show that $\dfrac{\partial f}{\partial \bar{z}} = 0$ is equivalent to the Cauchy-Riemann equations.

$$\left[\textbf{Hint. } \frac{\partial f}{\partial \bar{z}} = \left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \bar{z}}\right) + i\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \bar{z}}\right)\right]$$

23. (i) Show that a harmonic function satisfies the formal differential equation $\dfrac{\partial^2 u}{\partial z \partial \bar{z}} = 0$.

(ii) If $w = f(z)$ is a regular function of z, prove that $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 0$. Further, if $|f'(z)|$ is the product of a function of x and function of y, show that $f'(z) = \exp.(\alpha z^2 + \beta z + \gamma)$ where α is real and β, γ are complex constants.

24. If $f(z) = u + iv$ is an analytic function of $z = x + iy$, find $f(z)$ in terms of z if

(i) $3u + v = 3\sin x\cosh y + \cos x\sinh y$ (ii) $u - 2v = \cos x\cosh y + 2\sin x\sinh y$

(iii) $2u - v = e^x(2\cos y - \sin y)$

25. (i) If $f'(z) = f(z)$ for all z, then show that $f(z) = ke^z$, where k is an arbitrary constant.

(ii) Find an analytic function $f(z)$ such that $\text{Re}[f'(z)] = 3x^2 - 4y - 3y^2$ and $f(1 + i) = 0$

(iii) Let $f(z) = u + iv$ and $g(z) = v + iu$ be analytic functions for all z. Let $f(0) = 1$ and $g(0) = i$. Obtain the value of $h(z)$ at $z = 1 + i$ where $h(z) = f'(z) + g'(z) + 2f(z)\,g(z)$.

(iv) If $f(z) = u + iv$ is an analytic function of z and ϕ is a function of u and v, then show that

$$\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 = \left[\left(\frac{\partial \phi}{\partial u}\right)^2 + \left(\frac{\partial \phi}{\partial v}\right)^2\right]|f'(z)|^2$$

Answers

1. (i) $a = 2, b = -1, c = -1, d = 2$ (ii) $a = -\dfrac{1}{2}, b = -2, c = \dfrac{1}{2}$; $f(z) = -\dfrac{1}{2}(2 + i)z^2$

(iii) $a = -1, b = -1$ (iv) $p = -1, f'(z) = \dfrac{1}{z}$

6. (ii) constant function

7. (ii) No (iv) $f(z) = \dfrac{xy^2(x + iy)}{x^2 + y^4}, z \neq 0; f(0) = 0$

9. (i) $v = 2y - 3xy^2 + x^3 + c$; $f(z) = 2z + iz^3 + ic$ (ii) $u = e^x\cos y + c$; $f(z) = e^z + c$

(iii) $v(x, y) = x^2 - y^2 + 2y + c$ (v) $v(x, y) = 2x^2 - 2y^2 + 3x^2y - y^3 + c, f(z) = z^3 + 2iz^2 + c$

10. (*i*) $v = r^2 \sin 2\theta + c$; $f(z) = z^2 + ic$ (*ii*) $b = \pm 5$

(*iii*) $f(z) = -2z + i(2 \log z + z) + c$ (*iv*) $u(x, y) = e^{-x} (x \sin y - y \cos y) + c$

(*v*) $v(x, y) = 2xy + x$; $f(z) = z^2 + iz$

11. (*i*) $\log z + c$ (*ii*) $\cos z + c$ (*iii*) $1 + ze^{-z}$

(*iv*) $(1 - i) z^3 + c$ (*v*) $\cot z + c$ (*vi*) $\tan z + c$

12. (*i*) $\dfrac{1+i}{z} + c$ (*ii*) $i \cos z + c$ (*iii*) $i \sinh z + c$

(*iv*) $3z^2 - 5iz + c$ (*v*) $\dfrac{i}{z} + i \cosh z + c$ (*vi*) $z e^z + c$

13. $v = x^2 - y^2 + 2xy - 2y - 3x + c$, $f(z) = (1 + i)z^2 - (2 + 3i)z + ic$

14. (*i*) $\psi = 2xy + c$ (*ii*) $2 \tan^{-1}\left(\dfrac{y}{x}\right) + c$, $2 \log z + ic$

15. (*i*) $\dfrac{1}{2} \log (x^2 + y^2) + c$ (*ii*) $-2xy + \dfrac{y}{x^2 + y^2} + c$ (*iii*) $v = 3y (1 + x^2) - y^3$

17. $i(z^2 - z + 2) + c$

18. (*i*) $e^z + c$ (*ii*) $\dfrac{1}{1+i}\left(\dfrac{i}{z} + 1\right)$ (*iii*) $\cot \dfrac{z}{2} + \dfrac{1}{2} (1 - i)$

19. (*i*) $2z + iz^3 + c$ (*ii*) $ize^{-z} + c$

24. (*i*) $f(z) = \sin z + c$ (*ii*) $f(z) = \cos z + c$ (*iii*) $f(z) = e^z + c$

25. (*ii*) $f(z) = z^3 + 2iz^2 + 6 - 2i$ (*iii*) $2i$

4.19 TRANSFORMATION OR MAPPING

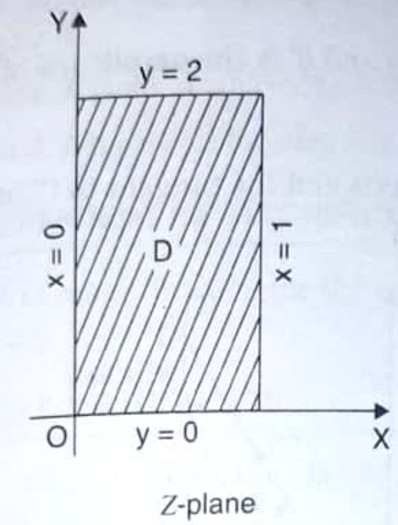
We know that the real function $y = f(x)$ can be represented graphically by a curve in the xy-plane. Also, the real function $z = f(x, y)$ can be represented by a surface in three dimensional space. However, this method of graphical representation fails in the case of complex functions because a complex function $w = f(z)$ *i.e.*, $u + iv = f(x + iy)$ involves four real variables, two independent variables x, y and two dependent variables u, v. Thus a four dimensional region is required to represent it graphically in the cartesian fashion. As it is not possible, we choose, two complex planes and call them z-plane and w-plane. In the z-plane, we plot the point $z = x + iy$ and in the w-plane, we plot the corresponding point $w = u + iv$. Thus the function $w = f(z)$ defines a correspondence between points of these two planes. If the point z describes some curve C in the z-plane, the point w will move along a corresponding curve C′ in the w-plane, since to each (x, y) there corresponds a point (u, v). *The function* $w = f(z)$ *thus defines a* **mapping or transformation** *of the z-plane into the w-plane.*

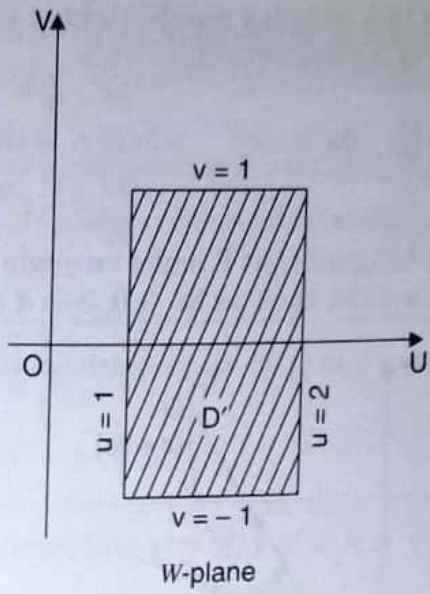
For example, consider the transformation $w = z + (1 - i)$. Let us determine the region D′ of the w-plane corresponding to the rectangular region D in the z-plane bounded by $x = 0$, $y = 0$, $x = 1$ and $y = 2$.

Since, $$w = z + (1 - i), \text{ we have}$$
$$u + iv = (x + iy) + (1 - i) = (x + 1) + i(y - 1)$$

Thus, $$u = x + 1 \quad \text{and} \quad v = y - 1$$

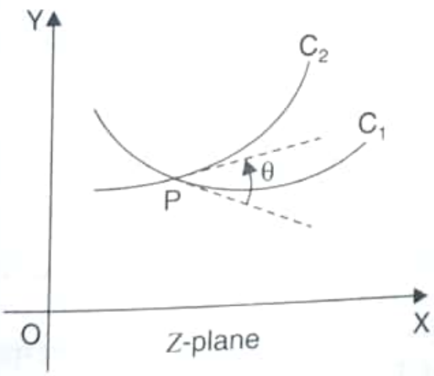
Hence the lines $x = 0$, $y = 0$, $x = 1$ and $y = 2$ in the z-plane are mapped onto the lines $u = 1$, $v = -1$, $u = 2$ and $v = 1$ in the w-plane. The regions D and D′ are shown shaded in the figure given below.

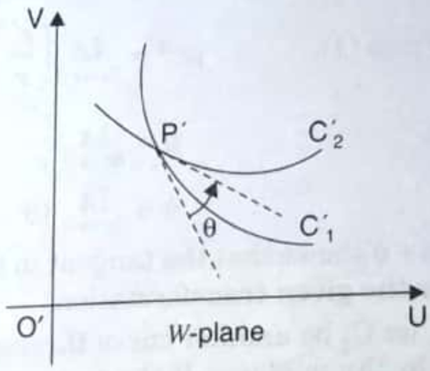




4.20 CONFORMAL TRANSFORMATION

Suppose two curves C_1, C_2 in the z-plane intersect at the point P and the corresponding curves C_1', C_2' in the w-plane intersect at P′ under the transformation $w = f(z)$. *If the angle of intersection of the curves at P is the same as the angle of intersection of the curves at P′, both in magnitude and sense, then the transformation is said to be conformal at P.*





Definition. A transformation which preserves angles both in magnitude and sense between every pair of curves through a point is said to be conformal at the point.

The conditions under which the transformation $w = f(z)$ is conformal are given by the following theorem.

4.21 THEOREM

If $f(z)$ is analytic and $f'(z) \neq 0$ in a region R of the z-plane, then the mapping $w = f(z)$ is conformal at all points of R.

Proof. Let P(z) be a point in the region R of the z-plane and P′(w) the corresponding point in the region R′ of the w-plane. Suppose P moves on a curve C and P′ moves on the corresponding curve C′. Let Q($z + \delta z$) be a neighbouring point on C and Q′ ($w + \delta w$) the corresponding point on C′ so that $\overrightarrow{PQ} = \delta z$ and $\overrightarrow{P'Q'} = \delta w$.

Hence, two curves cut orthogonally.

However, since $\qquad \dfrac{\partial u}{\partial x} = 4x, \; \dfrac{\partial u}{\partial y} = 2y$

$$\frac{\partial v}{\partial x} = -\frac{y^2}{x^2}, \; \frac{\partial v}{\partial y} = \frac{2y}{x}$$

The Cauchy-Riemann equations are not satisfied by u and v.

Hence the function $u + iv$ is not analytic. So the transformation is not conformal.

Example 2. *For the conformal transformation $w = z^2$, show that*

(*a*) *The coefficient of magnification at $z = 2 + i$ is $2\sqrt{5}$.*

(*b*) *The angle of rotation at $z = 2 + i$ is $\tan^{-1} 0.5$.*

Sol. Let $\qquad w = f(z) = z^2$

$\therefore \qquad f'(z) = 2z$

$$f'(2 + i) = 2(2 + i) = 4 + 2i.$$

(*a*) Coefficient of magnification at $z = 2 + i$ is $= | f'(2 + i) |$

$$= | 4 + 2i | = \sqrt{16 + 4} = 2\sqrt{5}.$$

(*b*) Angle of rotation at $z = 2 + i$ is = amp. $[f'(2 + i)] = \tan^{-1}\left(\dfrac{2}{4}\right) = \tan^{-1}(0.5)$.

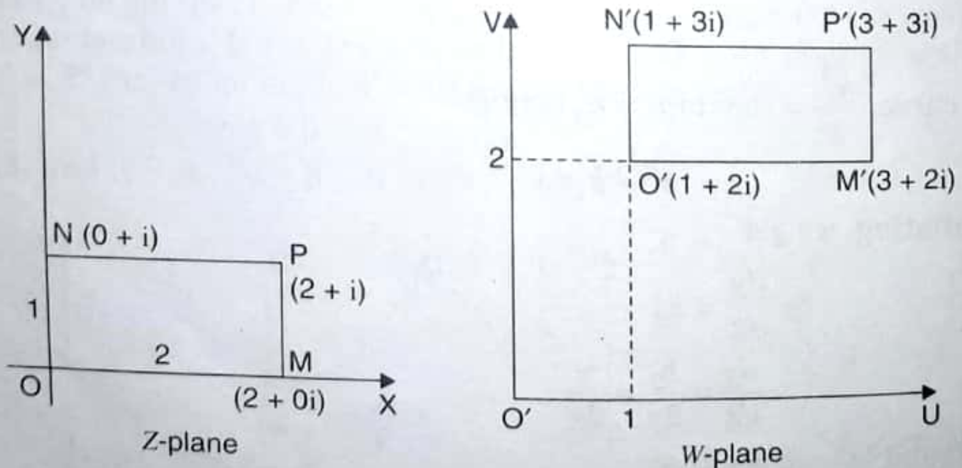
4.24 SOME STANDARD TRANSFORMATIONS

4.24.1 Translation: $w = z + c$, where c is a complex constant

Let $\qquad z = x + iy, \quad c = a + ib$ and $w = u + iv$

then the transformation becomes $u + iv = (x + iy) + (a + ib) = (x + a) + i(y + b)$

so that $\qquad u = x + a$ and $v = y + b$



Thus the transformation is a mere translation of the axes and preserves the shape and size.

For example, the rectangle OMPN in z-plane is transformed to rectangle O′M′P′N′ in the w-plane under the transformation $w = z + (1 + 2i)$.

Example 3. *Let a rectangular domain R be bounded by $x = 0$, $y = 0$, $x = 2$, $y = 1$. Determine the region R′ of w-plane into which R is mapped under the transformation $w = z + (1 - 2i)$.*

Sol.

$$w = z + (1 - 2i)$$

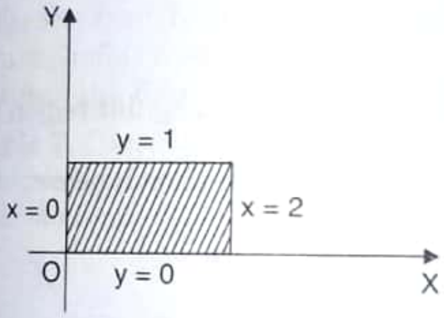
$$u + iv = x + iy + 1 - 2i = (x + 1) + i(y - 2)$$

$$\therefore \quad u = x + 1, \; v = y - 2$$

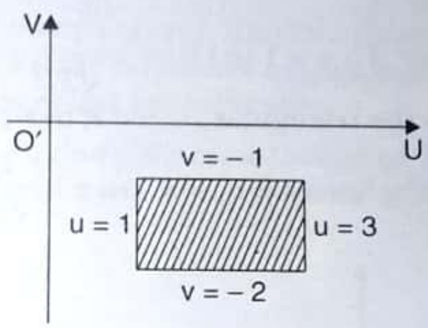
By the map $u = x + 1$, the lines $x = 0$, $x = 2$ are mapped respectively on the lines $u = 1$, $u = 3$.

By the map $v = y - 2$, the lines $y = 0$, $y = 1$ are mapped on $v = -2$, $v = -1$ respectively.

The required image is rectangle R′ bounded by $u = 1$, $u = 3$, $v = -2$, $v = -1$ in w-plane.



Z-plane



W-plane

Similarly, we can show that each point of R is mapped into one and only one point of R′ and conversely. Hence the transformation performs a translation of rectangle.

Here the rectangle R is translated in the direction of vector $\alpha = 1 - 2i$.

Example 4. *Find the image of $2x + y - 3 = 0$ under the transformation $w = z + 2i$.*

Sol.

$$w = z + 2i$$

$$\Rightarrow \quad u + iv = x + iy + 2i$$

$$\Rightarrow \quad u + iv = x + i(y + 2)$$

Equating real and imaginary parts, we get

$$u = x$$

$$\Rightarrow \quad v = y + 2$$

$$x = u$$

$$y = v - 2$$

$$\Rightarrow \quad 2x + y - 3 = 0 \quad | \text{ given}$$

$$\Rightarrow \quad 2u + v - 2 = 3$$

$$2u + v = 5$$

It is another straight line.

4.24.2 Rotation

By the transformation $w = ze^{i\theta_0}$, figures in z-plane are rotated through an angle θ_0. If $\theta_0 > 0$, the rotation is anticlockwise and if $\theta_0 < 0$, the rotation is clockwise.

Example 5. *Consider the transformation $w = z\,e^{i\pi/4}$ and determine the region R′ in w-plane corresponding to the triangular region R bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$ in z-plane.*

Sol.

$$w = z\,e^{i\pi/4}$$

$$\Rightarrow \quad u + iv = (x + iy)\left(\frac{1+i}{\sqrt{2}}\right)$$

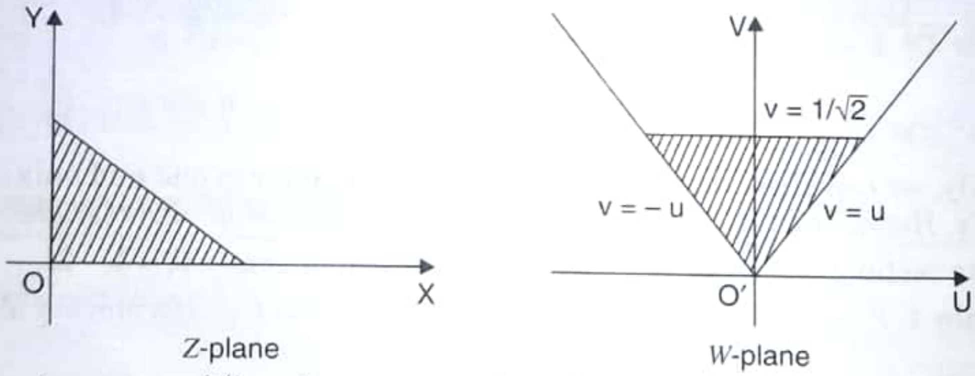
$$\Rightarrow \quad u = \frac{1}{\sqrt{2}}(x - y), \qquad v = \frac{1}{\sqrt{2}}(x + y) \qquad \ldots(1)$$

Put $x = 0$, $\quad u = -\frac{1}{\sqrt{2}}\,y, \qquad v = \frac{1}{\sqrt{2}}\,y$ or $v = -u$

Put $y = 0$, $\quad u = \frac{1}{\sqrt{2}}\,x, \qquad v = \frac{1}{\sqrt{2}}\,x$ or $v = u$

Putting $x + y = 1$ in (1), $v = \frac{1}{\sqrt{2}}$

Hence the triangular region R in z-plane is mapped on a triangular region R′ of w-plane bounded by the lines $v = u$, $v = -u$, $v = \frac{1}{\sqrt{2}}$. The two regions are shown below:



The mapping $w = z\,e^{i\pi/4}$ performs a rotation of R through an angle $\pi/4$.

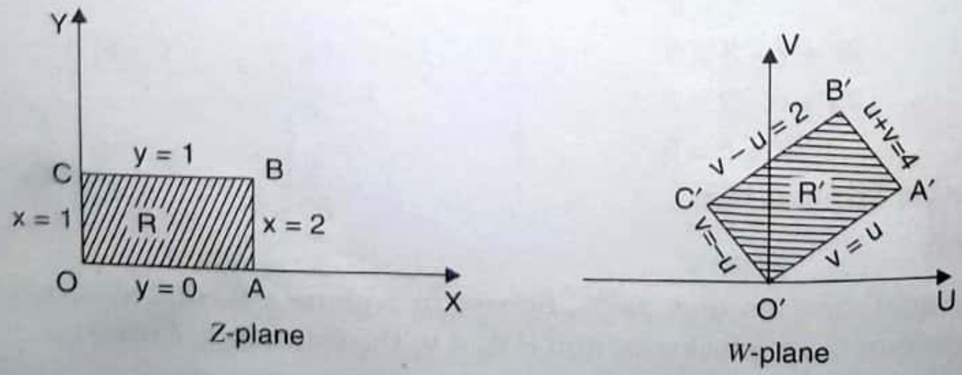
Example 6. *Determine the region in the w-plane in which the rectangle bounded by the lines $x = 0$, $y = 0$, $x = 2$ and $y = 1$ is mapped under the transformation $w = \sqrt{2}\,e^{i\pi/4}\,z$.*

Sol.

$$w = \sqrt{2}\,e^{i\pi/4}\,z$$

$$u + iv = (x + iy)\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$= (x + iy)(1 + i) = (x - y) + i(y + x)$$



$$u = x - y$$

$\Rightarrow$ $$v = x + y.$$

and

Put $x = 0$, $u = -y$, $v = y$ so that $v = -u$

Put $y = 0$, $u = x$, $v = x$ so that $v = u$

Put $x = 2$, $u = 2 - y$, $v = 2 + y$ so that $u + v = 4$

Put $y = 1$, $u = x - 1$, $v = x + 1$ so that $v - u = 2$.

Hence the rectangular region R in z-plane is mapped on a rectangular region R′ of w-plane bounded by the lines $v = -u$, $v = u$, $u + v = 4$ and $v - u = 2$. These two regions are shown in the figure.

4.24.3 Magnification (*Stretching*)

Consider the transformation (map) $\boldsymbol{w = az}$, *where a is real*. The two figures in z-plane and w-plane are similar and similarly situated about their respective origins but the figure in w-plane is a times the figure in z-plane. Such map is called *magnification*.

Example 7. *Consider the transformation $w = 2z$ and determine the region R′ of w-plane into which the triangular region R enclosed by the lines $x = 0$, $y = 0$, $x + y = 1$ in the z-plane is mapped under the map.*

Sol. $$w = 2z$$

$\Rightarrow$ $$u + iv = 2(x + iy)$$

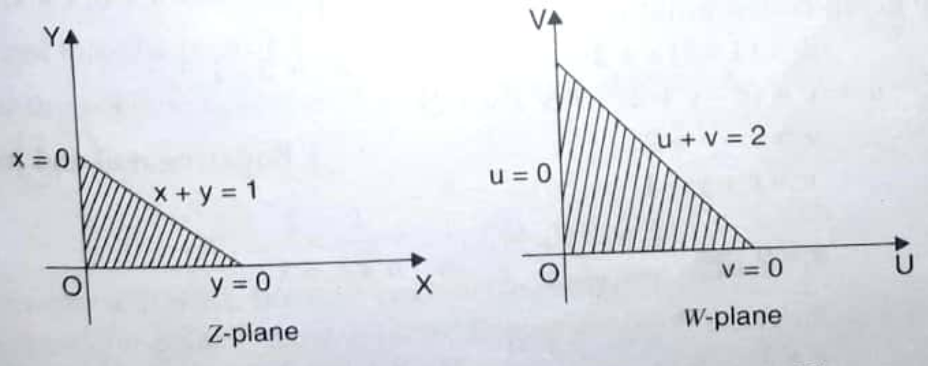
$\Rightarrow$ $$u = 2x,\ v = 2y$$

$$x = 0 \Rightarrow u = 0$$

$$y = 0 \Rightarrow v = 0$$

$$x + y = 1,\ u = 2x,\ v = 2y \Rightarrow u + v = 2$$

Hence the region R′ is triangular bounded by the lines $u = 0$, $v = 0$, $u + v = 2$.



This transformation $w = 2z$ performs a magnification of R into R′.

Example 8. *Draw the image of the square whose vertices are at (0, 0), (1, 0), (1, 1), (0, 1) in the z-plane under the transformation $w = (1 + i)z$. What has this transformation done to the original square?*

Sol. $$w = (1 + i)z = (1 + i)(x + iy)$$

$\Rightarrow$ $$u + iv = (x - y) + i(x + y)$$

$\therefore$ $u = x - y$ | Equating real and imaginary parts

$v = x + y$

The given square is bounded by the lines

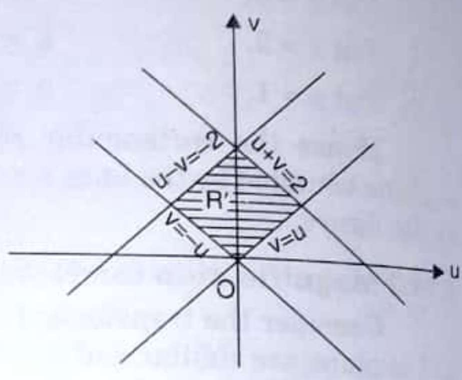
$x = 0, x = 1, y = 0$ and $y = 1$.

Now, $x = 0 \Rightarrow \left.\begin{array}{l} u = -y \\ v = y \end{array}\right\} \Rightarrow u + v = 0$

$x = 1 \Rightarrow \left.\begin{array}{l} u = 1 - y \\ v = 1 + y \end{array}\right\} \Rightarrow u + v = 2$

$y = 0 \Rightarrow \left.\begin{array}{l} u = x \\ v = x \end{array}\right\} \Rightarrow u = v$

$y = 1 \Rightarrow \left.\begin{array}{l} u = x - 1 \\ v = x + 1 \end{array}\right\} \Rightarrow u - v = -2$



(Image in w-plane)

This transformation has magnified and rotated the original square anticlockwise at an angle of 45°.

Example 9. *Find the image of the region $y > 1$ under the transformation $w = (1 - i) z$.*

Sol. $w = (1 - i) z = (1 - i)(x + iy)$

$\Rightarrow$ $u + iv = x + y + i (y - x)$

$\therefore$ $u = x + y$

$v = y - x$

$\Rightarrow$ $u + v = 2y$

The image of the region $y > 1$ will be $u + v > 2$.

Example 10. *Find the image of the region bounded by (0, 0), (1, 0), (1, 2), (0, 2) by the transformation $w = (1 + i) z + 2 - i$. Sketch the image.*

Sol. The given region is a rectangle R bounded by the lines $x = 0, x = 1, y = 0$ and $y = 2$ in z-plane. The given transformation is

$$w = (1 + i) z + 2 - i = (1 + i)(x + iy) + 2 - i$$

$\Rightarrow$ $u + iv = (x - y + 2) + i (x + y - 1)$

$\therefore$ $u = x - y + 2$ | Equating real and imaginary parts

and $v = x + y - 1$

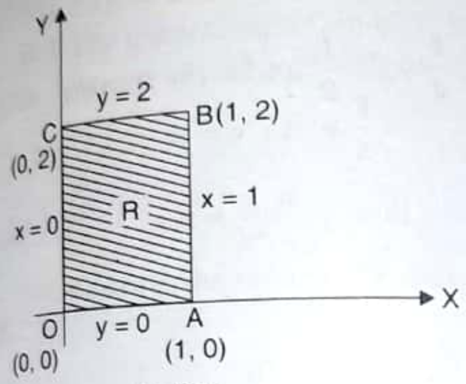
Now, $x = 0 \Rightarrow \left.\begin{array}{l} u = -y + 2 \\ v = y - 1 \end{array}\right\} \Rightarrow u + v = 1$

$x = 1 \Rightarrow \left.\begin{array}{l} u = -y + 3 \\ v = y \end{array}\right\} \Rightarrow u + v = 3$

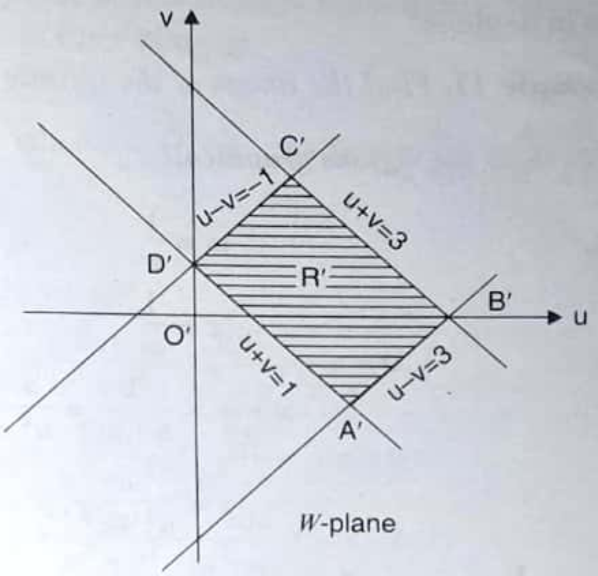
$y = 0 \Rightarrow \left.\begin{array}{l} u = x + 2 \\ v = x - 1 \end{array}\right\} \Rightarrow u - v = 3$

$y = 2 \Rightarrow \left.\begin{array}{l} u = x \\ v = x + 1 \end{array}\right\} \Rightarrow u - v = -1$

The required image R′ in w-plane is the region bounded by the lines $u + v = 1$, $u + v = 3$, $u - v = 3$ and $u - v = -1$.



Z-plane



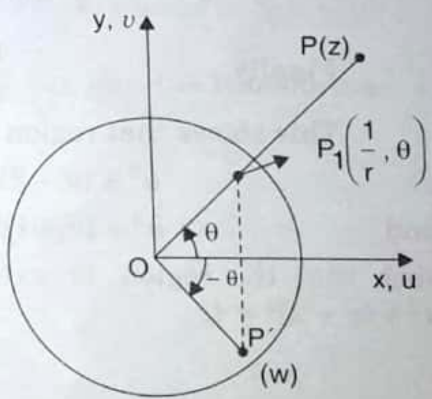
W-plane

4.24.4 Inversion: $w = \frac{1}{z}$

Let $\quad z = re^{i\theta}$ and $w = Re^{i\phi}$

then the transformation becomes $Re^{i\phi} = \frac{1}{r}e^{-i\theta}$ so that $R = \frac{1}{r}$ and $\phi = -\theta$.

Thus under the transformation $w = \frac{1}{z}$, a point $P(r, \theta)$ in z-plane is mapped into the point $P\left(\frac{1}{r}, -\theta\right)$.



Consider the w-plane superposed on the z-plane. If P is (r, θ) and P_1 is $\left(\frac{1}{r}, \theta\right)$, then

$$OP_1 = \frac{1}{r} = \frac{1}{OP} \quad i.e., \ OP \cdot OP_1 = 1$$

so that P_1 is inverse of P w.r.t. the unit circle with centre O.

[*The inverse of a point P w.r.t. a circle having centre O and radius k is defined as the point Q on OP such that $OP \cdot OQ = k^2$*]

The reflection P′ of P_1 in the real axis represents $w = \frac{1}{z}$. Thus the transformation $w = \frac{1}{z}$ is an inversion of z w.r.t. the unit circle $| z | = 1$ followed by reflection of the inverse into the real axis.

Obviously, the transformation $w = \frac{1}{z}$ maps the interior of the unit circle $| z | = 1$ into the exterior of the unit circle $| w | = 1$ and the exterior of $| z | = 1$ into the interior of $| w | = 1$.

However, the origin $z = 0$ is mapped to the point $w = \infty$, called the *point at infinity*.

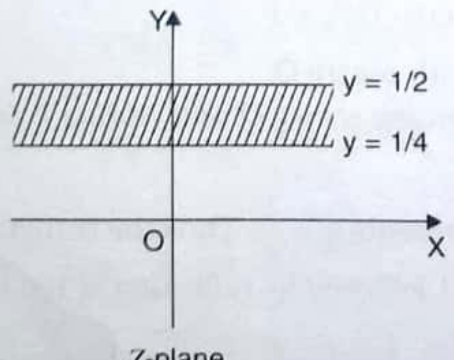
By means of this transformation $w = \frac{1}{z}$, figures in z-plane are mapped upon the reciprocal figures in w-plane.

Example 11. *Find the image of the infinite strip $\frac{1}{4} \le y \le \frac{1}{2}$ under the transformation $w = \frac{1}{z}$. Also show the regions graphically.*

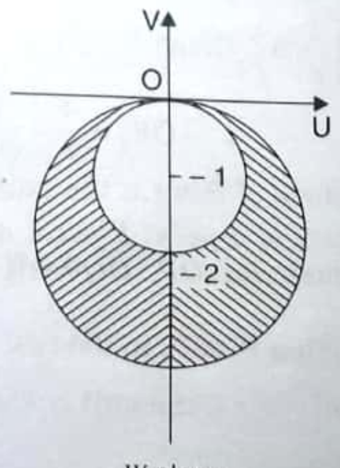
Sol.
$$w = \frac{1}{z}$$
$$\Rightarrow \quad z = \frac{1}{w} \quad \text{...(1)}$$
$$\Rightarrow \quad x + iy = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$$
$$\Rightarrow \quad x = \frac{u}{u^2+v^2},\ y = \frac{-v}{u^2+v^2}$$
$$y < \frac{1}{2} \Rightarrow \frac{-v}{u^2+v^2} < \frac{1}{2} \Rightarrow -2v < u^2+v^2 \Rightarrow u^2+(v+1)^2 > 1 \quad \text{...(2)}$$
$$y > \frac{1}{4} \Rightarrow \frac{-v}{u^2+v^2} > \frac{1}{4} \Rightarrow -4v > u^2+v^2 \Rightarrow u^2+(v+2)^2 < 4 \quad \text{...(3)}$$

Finally,
$$\frac{1}{4} < y < \frac{1}{2} \Rightarrow u^2+(v+2)^2 < 4 \text{ and } u^2+(v+1)^2 > 1$$

This shows that region R′ in w-plane is bounded by two circles
$$u^2 + (v+2)^2 = 4$$
and
$$u^2 + (v+1)^2 = 1$$
such that the region is exterior to the circle $u^2 + (v+1)^2 = 1$ and interior to the circle $u^2 + (v+2)^2 = 4$.



Z-plane



W-plane

Example 12. *Show that the transformation* $w = \dfrac{1}{z}$ *maps a circle in z-plane to a circle in w-plane or to a straight line if the circle in z-plane passes through the origin.*

Sol. The general equation of any circle in the z-plane is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...(1)}$$

Let $\quad w = u + iv = \dfrac{1}{z}$, then $z = \dfrac{1}{w}$ or $x + iy = \dfrac{1}{u+iv} = \dfrac{u-iv}{u^2+v^2}$

so that $\quad x = \dfrac{u}{u^2+v^2}$ and $y = \dfrac{-v}{u^2+v^2}$

Substituting the values of x and y in (1), we get

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} + \frac{2gu}{u^2+v^2} - \frac{2fv}{u^2+v^2} + c = 0$$

or
$$\frac{u^2+v^2}{(u^2+v^2)^2} + \frac{2gu}{u^2+v^2} - \frac{2fv}{u^2+v^2} + c = 0$$

or
$$\frac{1}{u^2+v^2} + \frac{2gu}{u^2+v^2} - \frac{2fv}{u^2+v^2} + c = 0$$

or
$$c(u^2+v^2) + 2gu - 2fv + 1 = 0 \quad \text{...(2)}$$

If $c \neq 0$, the circle (1) does not pass through the origin and equation (2) represents a circle in the w-plane.

If $c = 0$, the circle (1) passes through the origin and equation (2) reduces to $2gu - 2fv + 1 = 0$ which is a straight line in the w-plane.

Regarding a straight line as a circle of infinite radius, we can say that the transformation $w = \dfrac{1}{z}$ maps circles into circles.

Example 13. *Find the image of* $|z - 3i| = 3$ *under the mapping* $w = \dfrac{1}{z}$.

Sol.
$$z = \frac{1}{w} \Rightarrow x + iy = \frac{u-iv}{u^2+v^2}$$

$\therefore$
$$x = \frac{u}{u^2+v^2},\ y = \frac{-v}{u^2+v^2}$$

The given curve is
$$|z - 3i| = 3$$

or
$$|x + iy - 3i| = 3$$
$$x^2 + (y-3)^2 = 9$$

$\Rightarrow$
$$\frac{u^2}{(u^2+v^2)^2} + \left(\frac{-v}{u^2+v^2} - 3\right)^2 = 9$$

$\Rightarrow$
$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} + 9 + \frac{6v}{u^2+v^2} = 9$$

$\Rightarrow$
$$(u^2+v^2)(6v+1) = 0$$

$6v + 1 = 0$ is the reqd. image.

Aliter.
$$z - 3i = 3\,e^{i\theta}$$
$$z = 3i + 3\,e^{i\theta}$$
$$w = \frac{1}{z} = \frac{1}{3i + 3e^{i\theta}}$$
$$\Rightarrow \quad 3(u + iv) = \frac{1}{i + \cos\theta + i\sin\theta} = \frac{\cos\theta - i(1+\sin\theta)}{\cos^2\theta + (1+\sin\theta)^2}$$
$$3v = \frac{-(1+\sin\theta)}{2(1+\sin\theta)}$$
$$\Rightarrow \quad 6v + 1 = 0.$$

Example 14. *Show that under transformation $w = \frac{1}{z}$, the image of hyperbola $x^2 - y^2 = 1$ is the lemniscate $\rho^2 = \cos 2\phi$.*

Sol.
$$x^2 - y^2 = 1$$
$$\Rightarrow \quad r^2 \cos 2\theta = 1 \qquad \ldots(1) \quad | \text{ Polar form}$$
and
$$w = \frac{1}{z}$$
or
$$z = \frac{1}{w}$$
or
$$re^{i\theta} = \frac{1}{\rho e^{i\phi}} = \frac{e^{-i\phi}}{\rho} \qquad | \text{ Putting } z = re^{i\theta} \text{ and } w = \rho e^{i\phi}$$
$$\therefore \quad r = \frac{1}{\rho} \quad \text{and} \quad \theta = -\phi$$
$\therefore$ From (1),
$$\frac{1}{\rho^2}\cos 2(-\phi) = 1 \quad \Rightarrow \quad \rho^2 = \cos 2\phi.$$

Example 15. *Find the image of the half plane $x > c$ when $c > 0$ under the transformation $w = \frac{1}{z}$.*

Sol.
$$w = \frac{1}{z}$$
$$\Rightarrow \quad z = \frac{1}{w}$$
$$\therefore \quad x + iy = \frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2}$$
Separating real and imaginary parts,
$$\Rightarrow \quad x = \frac{u}{u^2 + v^2},\ y = \frac{-v}{u^2 + v^2}$$
Now,
$$x > c$$
$$\Rightarrow \quad \frac{u}{u^2 + v^2} > c$$
$$\Rightarrow \quad c\,(u^2 + v^2) - u < 0$$

The image is the circle in w-plane with centre at $\left(\frac{1}{2c}, 0\right)$ and radius $\frac{1}{2c}$ with $c > 0$.

Example 16. *Find the image of the circle $|z - 1| = 1$ in the complex plane under the mapping $w = \frac{1}{z}$.*

Sol.
$$z = \frac{1}{w}$$

$\Rightarrow$
$$x + iy = \frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2}$$

Separating real and imaginary parts, we get
$$x = \frac{u}{u^2 + v^2},\ y = \frac{-v}{u^2 + v^2}$$

The given circle is
$$|z - 1| = 1$$
$$|x - 1 + iy| = 1$$
$\Rightarrow$
$$(x - 1)^2 + y^2 = 1$$
$\Rightarrow$
$$\left(\frac{u}{u^2 + v^2} - 1\right)^2 + \left(\frac{-v}{u^2 + v^2}\right)^2 = 1$$
$\Rightarrow$
$$\frac{u^2}{(u^2 + v^2)^2} + 1 - \frac{2u}{u^2 + v^2} + \frac{v^2}{(u^2 + v^2)^2} = 1$$
$\Rightarrow$
$$1 - 2u = 0$$
$\Rightarrow$
$$u = \frac{1}{2}$$

which is a line in w-plane.

TEST YOUR KNOWLEDGE

1. (*i*) Find the image of $x = 2$ under the transformation $w = \frac{1}{z}$.

   (*ii*) What is the image of the line $x = k$ under the transformation $w = \frac{1}{z}$.

   (*iii*) Find the image of $|z - 2i| = 2$ under the mapping $w = \frac{1}{z}$.

2. Determine the image of $1 < x < 2$ under the mapping $w = \frac{1}{z}$.
3. (*i*) Find the image of the circle $|z| = \lambda$ under the transformation $w = 5z$
   (*ii*) Find the image of the circle $|z| = 2$ under the transformation $w = 3z$.
   (*iii*) Find the image of the circle $|z| = 3$ under the transformation $w = 3z$.
4. Find the image of $0 \le x \le 2$ under the transformation $w = iz$.
5. What is the region of the w-plane into which the rectangular region in the z-plane bounded by the lines $x = 0, y = 0, x = 1$ and $y = 2$ is mapped under the transformation $w = z + (2 - i)$?

6. Find the image of the circle $|z| = 2$ under the transformation $w = z + 3 + 2i$.
7. Find the image of the triangle with vertices at i, $1 + i$, $1 - i$ in the z-plane under the transformation $w = 3z + 4 - 2i$.
8. Find the image of the semi-infinite strip $x > 0$, $0 < y < 2$, under the transformation $w = iz + 1$. Show the region graphically.
9. Find the image of the following curves under the mapping $w = \dfrac{1}{z}$
   (i) the line $y - x + 1 = 0$ (ii) the circle $|z - 3| = 5$.
10. Determine the region in the w-plane in which rectangle bounded by the lines $x = 0$, $y = 0$, $x = 2$ and $y = 3$ is mapped under the transformation $w = \sqrt{2}\, e^{i\pi/4} \,.\, z$.

Answers

1. (i) $\left(u - \dfrac{1}{4}\right)^2 + v^2 = \left(\dfrac{1}{4}\right)^2$ (ii) $\left(u - \dfrac{1}{2k}\right)^2 + v^2 = \left(\dfrac{1}{2k}\right)^2$ (iii) $4v + 1 = 0$
2. Region between the circles $\left(u - \dfrac{1}{2}\right)^2 + v^2 = \left(\dfrac{1}{2}\right)^2$ and $\left(u - \dfrac{1}{4}\right)^2 + v^2 = \left(\dfrac{1}{4}\right)^2$
3. (i) $u^2 + v^2 = (5\lambda)^2$ (ii) $u^2 + v^2 = 36$ (iii) $u^2 + v^2 = 36$.
4. $0 \le v \le 2$.
5. Rectangular region bounded by the lines $u = 2$, $v = -1$, $u = 3$, $v = 1$
6. $(u - 3)^2 + (v - 2)^2 = 4$
7. A triangle with vertices $(4, 1)$, $(7, 1)$ and $(7, -5)$.
8. $-1 < u < 1$, $v > 0$
9. (i) $u^2 + v^2 - u - v = 0$ (ii) $\left|w + \dfrac{3}{16}\right| = \dfrac{5}{16}$
10. Rectangular region bounded by the lines $v = -u$, $v = u$, $u + v = 4$, $v - u = 6$.

4.25 BILINEAR TRANSFORMATION

A transformation of the form $w = \dfrac{az + b}{cz + d}$ ...(1)

where a, b, c, d are complex constants and $ad - bc \neq 0$ is called a **bilinear transformation.**

Such type of transformation was first studied by Mobius and hence it is sometimes called **Mobius transformation.**

The transformation given by (1) is conformal, since

$$\frac{dw}{dz} = \frac{ad - bc}{(cz + d)^2} \neq 0$$

The inverse mapping of (1) is $z = \dfrac{-dw + b}{cw - a}$, which is also a bilinear transformation.

The transformation (1) can be written as

$$cwz + wd - az - b = 0$$

which is linear both in w and z and hence the name bilinear transformation. The expression $(ad - bc)$ is called the determinant of bilinear transformation.

From (1), we observe that each point in the z-plane except the point $z = -\frac{d}{c}$ maps into a unique point in the w-plane. Similarly, from (2), we observe that each point in the w-plane except the point $w = \frac{a}{c}$ maps into a unique point in the z-plane. Considering the two exceptional points as points at infinity in respective planes, we can say that there is one to one correspondence between all points in the two planes.

Every bilinear transformation $w = \frac{az+b}{cz+d}$, $ad - bc \neq 0$ is the combination of basic transformations

(*i*) translation : $w = z + c$ (*ii*) rotation : $w = ze^{i\theta_0}$

(*iii*) magnification : $w = cz$ (*iv*) inversion : $w = \frac{1}{z}$

By actual division, we have $w = \frac{a}{c} + \frac{bc - ad}{c^2} \cdot \frac{1}{z + \frac{d}{c}}$

Taking $w_1 = z + \frac{d}{c}$, $w_2 = \frac{1}{w_1}$, $w_3 = \frac{bc - ad}{c^2} w_2$, we get $w = \frac{a}{c} + w_3$

Thus, by these transformations, we successively pass from z-plane to w_1-plane, from w_1-plane to w_2-plane, from w_2-plane to w_3-plane and finally from w_3-plane to w-plane.

Since each of these auxiliary transformations maps circles into circles, hence a *bilinear transformation also maps circles into circles.*

4.26 THEOREM

The bilinear transformation $w = \frac{az+b}{cz+d}$ *transforms the circle arg* $\frac{z - z_1}{z - z_2} = \lambda$ *into similar circle arg* $\frac{w - w_1}{w - w_2}$ *= constant where* w_1, w_2 *correspond to* z_1, z_2 *respectively.*

Proof.

$$w_1 = \frac{az_1 + b}{cz_1 + d},\ w_2 = \frac{az_2 + b}{cz_2 + d} \text{ so that,}$$

$$\frac{w - w_1}{w - w_2} = \frac{\dfrac{az+b}{cz+d} - \dfrac{az_1+b}{cz_1+d}}{\dfrac{az+b}{cz+d} - \dfrac{az_2+b}{cz_2+d}} = \frac{cz_2 + d}{cz_1 + d} \cdot \frac{z - z_1}{z - z_2}$$

$$= \mu \cdot \frac{z - z_1}{z - z_2} \qquad \text{where } \mu \text{ is complex}$$

$\therefore$

$$\arg.\left(\frac{w - w_1}{w - w_2}\right) = \arg \mu + \arg\left(\frac{z - z_1}{z - z_2}\right) = \arg \mu + \lambda$$

Thus, $\arg\left(\dfrac{w-w_1}{w-w_2}\right) = k$, where k is real, which is a circle in w-plane passing through two fixed points w_1, w_2 which are images of z_1, z_2.

4.27 CROSS-RATIO

If there are four points z_1, z_2, z_3, z_4 taken in order, then the ratio $\dfrac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$ is called the cross-ratio of z_1, z_2, z_3, z_4.

4.28 THEOREM

A bilinear transformation preserves cross-ratio of four points.

Proof. Let bilinear transformation be $w = \dfrac{az+b}{cz+d}$.

Since w_1, w_2, w_3, w_4 are images of z_1, z_2, z_3, z_4 respectively, thus

$$w_1 = \frac{az_1+b}{cz_1+d},\ w_2 = \frac{az_2+b}{cz_2+d}$$

$\therefore$
$$w_1 - w_2 = \frac{(ad-bc)}{(cz_1+d)(cz_2+d)}(z_1-z_2)$$

Similarly,
$$w_2 - w_3 = \frac{ad-bc}{(cz_2+d)(cz_3+d)}(z_2-z_3)$$

$$w_3 - w_4 = \frac{ad-bc}{(cz_3+d)(cz_4+d)}(z_3-z_4)$$

$$w_4 - w_1 = \frac{ad-bc}{(cz_4+d)(cz_1+d)}(z_4-z_1)$$

From above results,

$$\frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$$

$$\Rightarrow \quad (w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4).$$

Note. In the bilinear transformation $w = \dfrac{az+b}{cz+d}$, $ad - bc \neq 0$ dividing the numerator and denominator of the right hand side by one of the four constants, we observe that there are only three independent constants. Hence *three independent conditions are required to determine a bilinear transformation.*

4.29 INVARIANT OR FIXED POINTS

The points which coincide with their transformations are called invariant points of the transformation. Fixed points of a transformation $w = f(z)$ are obtained by the equation $z = f(z)$.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the bilinear transformation which maps the points $z = 1, -i, -1$ to the points $w = i, 0, -i$ respectively. Show also that transformation maps the region outside the circle $|z| = 1$ into the half-plane $R(w) \geq 0$.*

Sol. The bilinear transformation mapping $z = 1, -i, -1$ into $w = i, 0, -i$ respectively is given by,

$$\frac{(w-i)(0+i)}{(w+i)(-i)} = \frac{(z-1)(1-i)}{(z+1)(-i-1)}$$

$$\frac{i-w}{i+w} = \frac{(z-1)(i-1)}{(z+1)(i+1)}$$

$$\frac{i-w}{i+w} = \frac{(i-1)\,z+1-i}{(i+1)\,z+1+i}$$

$$\Rightarrow \quad \frac{2i}{-2w} = \frac{2iz+2}{-2z-2i} \quad \text{(Applying Componendo and Dividendo)}$$

$$\Rightarrow \quad \frac{i}{-w} = \frac{iz+1}{-(z+i)}$$

$$\Rightarrow \quad w = \frac{i(z+i)}{iz+1} = \frac{iz-1}{iz+1} \quad \text{...(1)}$$

which is the required transformation

Eqn. (1) may also be written as,

$$z = i\left(\frac{w+1}{w-1}\right)$$

$|z| \geq 1$ is transformed into

$$\left|\frac{w+1}{w-1}\right| \; |i| \geq 1$$

or $$|w+1|^2 \geq |w-1|^2$$

or $$|u+iv+1|^2 \geq |u+iv-1|^2$$

or $$(u+1)^2 + v^2 \geq (u-1)^2 + v^2$$

or $$u \geq 0$$

or $$R(w) \geq 0.$$

Thus exterior of the circle $|z| = 1$ is transformed into the half-plane $R(w) \geq 0$.

Example 2. *Find the bilinear transformation which maps the points $z = 0, -1, i$ onto $w = i, 0, \infty$. Also, find the image of the unit circle $|z| = 1$.*

Sol. The bilinear transformation mapping $z = 0, -1, i$ into $w = i, 0, \infty$ respectively is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\Rightarrow \quad \frac{(w-w_1)\left(\dfrac{w_2}{w_3}-1\right)}{\left(\dfrac{w}{w_3}-1\right)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

| Since w_3 is ∞ $\therefore$ formula cannot be applied directly

$$\Rightarrow \quad \frac{(w-i)(-1)}{(-1)(0-i)} = \frac{(z-0)(-1-i)}{(z-i)(-1-0)}$$

$$\Rightarrow \quad \left(\frac{w-i}{-i}\right) = \frac{z(1+i)}{z-i}$$

$$\Rightarrow \quad w - i = \frac{(-i+1)z}{z-i}$$

$$\Rightarrow \quad w = \frac{(1-i)z}{z-i} + i$$

$$\Rightarrow \quad w = \frac{z+1}{z-i} \qquad \ldots(1)$$

which is the required bilinear transformation.

Eqn. (1), can be written as

$$z = \frac{iw+1}{w-1} \qquad \ldots(2)$$

Now,
$$|z| = 1$$

$$\Rightarrow \quad \left|\frac{iw+1}{w-1}\right| = 1$$

$$\Rightarrow \quad |1+iw| = |w-1|$$

$$\Rightarrow \quad |1+i(u+iv)| = |u+iv-1|$$

$$\Rightarrow \quad |1-v+iu| = |u-1+iv|$$

$$\Rightarrow \quad (1-v)^2 + u^2 = (u-1)^2 + v^2$$

$$\Rightarrow \quad 1 + v^2 - 2v + u^2 = u^2 + 1 - 2u + v^2$$

$$\Rightarrow \quad u - v = 0 \quad \text{or} \quad v = u$$

Hence the image of unit circle $|z| = 1$ in z-plane is a straight line making an angle $\pi/4$ to u-axis and passing through origin in w-plane.

Example 3. *Determine the bilinear transformation which maps $z_1 = 0, z_2 = 1, z_3 = \infty$ onto $w_1 = i, w_2 = -1, w_3 = -i$ respectively.*

Sol. We know that,

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\Rightarrow \quad \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)\left(\dfrac{z_2}{z_3}-1\right)}{\left(\dfrac{z}{z_3}-1\right)(z_2-z_1)} \qquad | \because z_3 = \infty$$

$$\Rightarrow \quad \frac{(w-i)(i-1)}{(w+i)(-1-i)} = \frac{(z-0)(-1)}{(-1)(1)}$$

$$\Rightarrow \quad \frac{w-i}{w+i} = \left(\frac{1+i}{1-i}\right) z \quad \Rightarrow \quad \frac{w-i}{w+i} = iz$$

$$\Rightarrow \quad \frac{2w}{-2i} = \frac{iz+1}{iz-1} \qquad \text{| Applying C and D formula}$$

$$\Rightarrow \quad w = \frac{z-i}{iz-1}$$

Example 4. *Find the bilinear transformation which maps the points $z = 1, i, -1$ into the points $w = i, 0, -i$. Hence find the image of $|z| < 1$.*

Sol. We have

$$\frac{(w-i)(i)}{(w+i)(-i)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\Rightarrow \quad \frac{w-i}{w+i} = i\left(\frac{z-1}{z+1}\right)$$

$$\Rightarrow \quad \frac{2w}{-2i} = \frac{iz-i+z+1}{iz-i-z-1} = \frac{(i+1)z-(i-1)}{(i-1)z-(i+1)} = -i\left(\frac{z-i}{z+i}\right) \qquad \text{| Applying C and D formula}$$

$$\Rightarrow \quad w = \frac{i-z}{i+z} \qquad \ldots(1)$$

which is the required bilinear transformation.

(1) can be rewritten as

$$z = i\left(\frac{1-w}{1+w}\right)$$

$\therefore$ $|z| < 1$ is mapped into the region

$$\left| i\left(\frac{1-w}{1+w}\right) \right| < 1$$

$$\Rightarrow \quad \frac{|i||1-w|}{|1+w|} < 1$$

$$\Rightarrow \quad |1-w| < |1+w| \qquad | \because |i| = 1$$

$$\Rightarrow \quad |1-u-iv| < |1+u+iv|$$

$$\Rightarrow \quad (1-u)^2 + v^2 < (1+u)^2 + v^2$$

$$\Rightarrow \quad 1 + u^2 + v^2 - 2u < 1 + u^2 + v^2 + 2u$$

$$\Rightarrow \quad u > 0$$

Hence the interior of the circle $|z| = 1$ in z-plane is mapped into the entire half of the w-plane to the right of the imaginary axis.

Example 5. *Find a bilinear transformation which maps the points $i, -i, 1$ of the z-plane into $0, 1, \infty$ of the w-plane respectively.*

Sol. We know that the bilinear transformation mapping $z = z_1, z_2, z_3$ into $w = w_1, w_2, w_3$ respectively is,

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \qquad \text{...(1)}$$

Here $w_1 = 0, w_2 = 1, w_3 = \infty$ and $z_1 = i, z_2 = -i, z_3 = 1$

Eqn. (1) may be rewritten as,

or
$$\frac{(w-w_1)\left(\dfrac{w_2}{w_3}-1\right)}{\left(\dfrac{w}{w_3}-1\right)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

or
$$\frac{(w-0)\left(\dfrac{1}{\infty}-1\right)}{\left(\dfrac{w}{\infty}-1\right)(1-0)} = \frac{(z-i)(-i-1)}{(z-1)(-i-i)}$$

$\Rightarrow$
$$w = \frac{(z-i)(i+1)}{(z-1)(2i)} = \frac{(i-1)z+(i+1)}{-2z+2}.$$

Example 6. *Prove that, in general, in the bilinear transformation* $w = \dfrac{az+b}{cz+d}$, *there are two values of z (invariant points) for which w = z but there is only one if* $(a-d)^2 + 4bc = 0$.

Show that if there are distinct invariant points p and q, the transformation may be put in the form

$$\frac{w-p}{w-q} = k\,\frac{z-p}{z-q}$$

and that if there is only one invariant point p, the transformation may be put in the form

$$\frac{1}{w-p} = \frac{1}{z-p} + k.$$

Sol. The bilinear transformation is

$$w = \frac{az+b}{cz+d}$$

If $w = z$ then it gives, $z = \dfrac{az+b}{cz+d}$

i.e., $\quad cz^2 + (d-a)\,z - b = 0$

Let p, q be the roots, then

$$p = \frac{a-d+\sqrt{(a-d)^2+4bc}}{2c}$$

and
$$q = \frac{a-d-\sqrt{(a-d)^2+4bc}}{2c}$$

Thus, p and q as expressed above are the two invariant points of the transformation. They reduce to one if

$$(a-d)^2 + 4bc = 0$$

and then
$$p = q = \frac{a-d}{2c} \qquad \text{...(1)}$$

When p and q are distinct points, we have

$$cp^2 + (d - a)\, p - b = 0 \quad \Rightarrow \quad b - pd = cp^2 - ap$$

$$cp^2 + (d - a)\, q - b = 0 \quad \Rightarrow \quad b - qd = cq^2 - aq \qquad \text{...(2)}$$

Now,

$$\frac{w - p}{w - q} = \frac{\dfrac{az + b}{cz + d} - p}{\dfrac{az + b}{cz + d} - q}$$

$$= \frac{(a - cp)\, z + (b - pd)}{(a - cq)\, z + (b - qd)} = \frac{(a - cp)\, z + cp^2 - ap}{(a - cq)\, z + cq^2 - aq}$$

$$= \frac{(a - cp)(z - p)}{(a - cq)(z - q)}$$

$$\frac{w - p}{w - q} = k\, \frac{z - p}{z - q} \quad \text{where } k = \frac{a - cp}{a - cq}$$

When there is only one invariant point p:

$$\frac{1}{w - p} = \frac{1}{\dfrac{az + b}{cz + d} - p} = \frac{cz + d}{(a - cp)\, z + b - pd}$$

$$= \frac{cz + d}{(a - cp)\, z + cp^2 - ap}$$

$$= \frac{cz + d}{(a - cp)(z - p)} = \frac{cz + a - 2cp}{(a - cp)(z - p)} \qquad | \text{ Using (1)}$$

$$= \frac{(a - cp) + c(z - p)}{(a - cp)(z - p)} = \frac{1}{z - p} + \frac{c}{a - cp}$$

$$= \frac{1}{z - p} + k \qquad \left|\ \text{where } k = \frac{c}{a - cp}\right.$$

or

$$\frac{1}{w - p} = \frac{1}{z - p} + k.$$

Example 7. *What is the form of a bilinear transformation which has one fixed point α and the other fixed point ∞?*

Sol. Consider the transformation

$$w = \frac{az + b}{cz + d} \qquad \text{...(1)}$$

In this case, $c = 0$, $a - d \neq 0$ | Refer Example 6

$\therefore$

$$w = \frac{a}{d}\, z + \frac{b}{d} \qquad \text{...(2)}$$

Let finite fixed point be α, then

$$\alpha = \frac{a}{d}\, \alpha + \frac{b}{d} \qquad \text{...(3)}$$

(2) – (3) gives, $w - \alpha = \dfrac{a}{d}(z - \alpha)$ $\mid \lambda = a/d$

$\Rightarrow$ $w - \alpha = \lambda(z - \alpha)$.

Example 8. *Find two bilinear transformations whose fixed points are 1 and 2.*

Sol. Let the bilinear transformation be

$$w = \frac{az + b}{cz + d}$$

Fixed points are given by

$$z = \frac{az + b}{cz + d}$$

$\Rightarrow$ $cz^2 - (a - d)z - b = 0$

If the fixed points are 1 and 2 this equation is same as

$$(z - 1)(z - 2) = 0$$

$\Rightarrow$ $z^2 - 3z + 2 = 0$

$\therefore$ $\dfrac{a - d}{c} = 3$ and $-\dfrac{b}{c} = 2$

$\Rightarrow$ $b = -2c$ and $d = a - 3c$

Hence the transformation, $w = \dfrac{az - 2c}{cz + a - 3c}$ has its fixed points at $z = 1$ and $z = 2$.

Taking $a = 1, c = -1$ and $a = 2, c = -1$, we get two particular cases.

$$w = \frac{z + 2}{4 - z} \quad \text{and} \quad w = \frac{2(z + 1)}{5 - z}$$

Note. Some values of a and c *i.e.*, $a = 1, c = 1$ will not give a bilinear transformation.

Example 9. *Show that the transformation* $w = \dfrac{2z + 3}{z - 4}$ *maps the circle* $x^2 + y^2 - 4x = 0$ *onto the straight line* $4u + 3 = 0$. *Explain why the curve obtained is not a circle?*

Sol. The given transformation is $w = \dfrac{2z + 3}{z - 4}$

The inverse transformation is $z = \dfrac{4w + 3}{w - 2}$ ...(1)

Now the equation $x^2 + y^2 - 4x = 0$ can be written as $z\bar{z} - 2(z + \bar{z}) = 0$

Substituting for z and $\bar{z}$ from (1), we get

$$\frac{4w + 3}{w - 2} \cdot \frac{4\bar{w} + 3}{\bar{w} - 2} - 2\left(\frac{4w + 3}{w - 2} + \frac{4\bar{w} + 3}{\bar{w} - 2}\right) = 0$$

or $16w\bar{w} + 12w + 12\bar{w} + 9 - 2(4w\bar{w} + 3\bar{w} - 8w - 6 + 4w\bar{w} + 3w - 8\bar{w} - 6) = 0$

or $22(w + \bar{w}) + 33 = 0$ or $22(2u) + 33 = 0$ or $4u + 3 = 0$.

Explanation. This is possible under bilinear transformation since we regard a straight line as a particular case of circle.

For, $aw\bar{w} + \bar{b}w + b\bar{w} + c = 0$ where c is real, is an equation of a circle and if $a = 0$, then the same equation represents a straight line.

Example 10. *Discuss the application of the transformation* $w = \dfrac{iz+1}{z+i}$ *to the areas in the z-plane which are respectively inside and outside the unit circle with its centre at the origin.*

Sol.

$$w = \frac{iz+1}{z+i} = \frac{ix-y+1}{x+iy+i} = \frac{(1-y)+ix}{x+i(y+1)}$$

$$\Rightarrow \quad u + iv = \frac{-i(1-y^2)+ix^2+x(1-y)+x(1+y)}{x^2+(1+y)^2}$$

$$\Rightarrow \quad v = \frac{x^2+y^2-1}{x^2+(1+y)^2} = \frac{|z|^2-1}{x^2+(1+y)^2} \qquad ...(1)$$

The denominator is essentially positive.

From (1), it is clear that

(*i*) $v = 0$ (real axis of w-plane) corresponds to

$$|z|^2 - 1 = 0 \quad \text{or} \quad |z| = 1$$

(*i.e.*, boundary of the circle $|z| = 1$)

(*ii*) $v > 0$ (upper half of w-plane) corresponds to

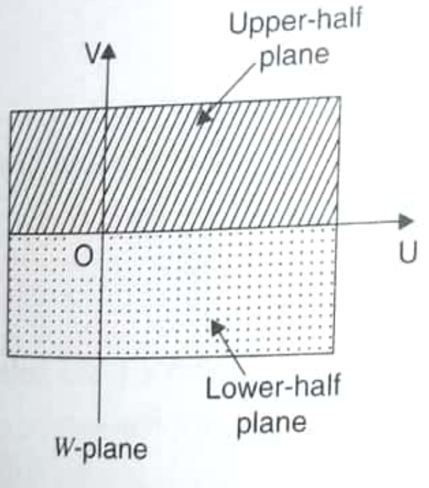
$$|z|^2 - 1 > 0 \quad \text{or} \quad |z| > 1$$

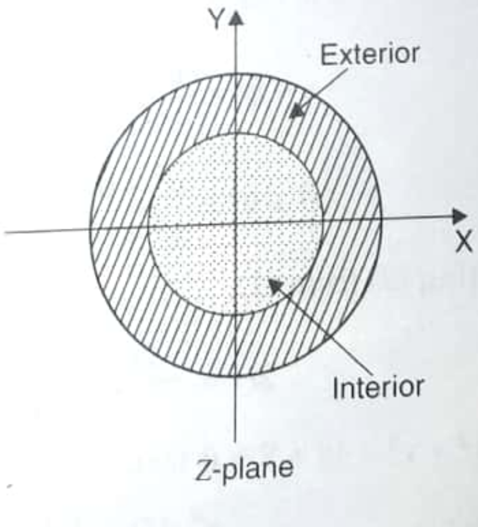
(*i.e.*, exterior of circle $|z| = 1$)

(*iii*) $v < 0$ (lower half of w-plane) corresponds to $|z| < 1$

(*i.e.*, interior of circle $|z| = 1$)

This concludes the problem.





Example 11. *Show that the transformation* $w = \dfrac{5-4z}{4z-2}$ *transform the circle* $|z| = 1$ *into a circle of radius unity in w-plane and find the centre of the circle.*

Sol. We have, $$w = \frac{5-4z}{4z-2}$$

$$\Rightarrow \quad z = \frac{2w+5}{4w+4}$$

$$|z| = 1 \quad \Rightarrow \quad z\bar{z} = 1$$

$$\Rightarrow \quad \frac{2w+5}{4w+4} \cdot \frac{2\bar{w}+5}{4\bar{w}+4} = 1$$

$$\Rightarrow \quad 4w\bar{w} + 25 + 10(w + \bar{w}) = 16\,(w\bar{w} + 1 + w + \bar{w}) \qquad ...(1)$$

$$\Rightarrow \quad 4(u^2 + v^2) + 25 + 20u = 16\,[u^2 + v^2 + 1 + 2u]$$

$$\Rightarrow \quad u^2 + v^2 + u - \frac{3}{4} = 0 \qquad ...(2)$$

Comparing this with $$u^2 + v^2 + 2gu + 2fv + c = 0$$

$$\Rightarrow \quad g = 1/2, f = 0$$

Centre = $(-g, -f) = (-1/2, 0)$

Radius = $\sqrt{g^2 + f^2 - c} = \sqrt{\frac{1}{4} + 0 + \frac{3}{4}} = 1$

Hence (2) represents circle with its centre at $(-1/2, 0)$ and of radius unity in w-plane.

Example 12. *Find the image of $x^2 + y^2 - 4y + 2 = 0$ under the mapping $w = \frac{z-i}{iz-1}$.*

Sol. $$w\,(iz - 1) = z - i$$

$$\Rightarrow \quad z = \frac{w-i}{iw-1}$$

$$x + iy = \frac{u + i(v-1)}{iu - (v+1)} \qquad ...(1)$$

$$\therefore \quad x - iy = \frac{u - i(v-1)}{-iu - (v+1)} \qquad ...(2)$$

$$\therefore \quad x^2 + y^2 = \frac{u^2 + (v-1)^2}{u^2 + (v+1)^2}$$

Subtracting (2) from (1),

$$2iy = \frac{-2iu^2 - 2i(v^2 - 1)}{u^2 + (v+1)^2}$$

Hence, $x^2 + y^2 - 4y + 2 = 0$ transforms into

$$\frac{u^2 + (v-1)^2}{u^2 + (v+1)^2} + 4 \cdot \frac{u^2 + (v^2 - 1)}{u^2 + (v+1)^2} + 2 = 0$$

$$\Rightarrow \quad u^2 + (v-1)^2 + 4\,[u^2 + (v^2 - 1)] + 2\,[u^2 + (v+1)^2] = 0$$

$$\Rightarrow \quad 7\,(u^2 + v^2) + 2v - 1 = 0$$

Example 13. *Show that under the transformation* $w = \dfrac{z-i}{z+i}$, *the real axis in z-plane is mapped into the circle* $|w| = 1$. *What portion of the z-plane corresponds to the interior of the circle?*

Sol.
$$w = \frac{z-i}{z+i}$$

So the real axis in z-plane $(y = 0)$ *i.e.*, $z = x$ transforms into

$$w = \frac{x-i}{x+i} = \frac{(x-i)^2}{x^2+1} = \frac{x^2-1-2ix}{x^2+1}$$

$$|w| = \sqrt{\left(\frac{x^2-1}{x^2+1}\right)^2 + \left(\frac{2x}{x^2+1}\right)^2} = 1$$

Hence the real axis in z-plane transforms into the circle $|w| = 1$.

Put $z = i$, we get $w = 0$. Hence the half z-plane above the real axis corresponds to the interior of the circle $|w| = 1$.

Example 14. *If a is any real positive number, show that the transformation* $w = \dfrac{z-a}{z+a}$ *transforms conformally the plane* $x > 0$ *to the unit circle* $|w| < 1$. *What are the transforms of* $|w|$ = *constant and arg* w = *constant in z-plane?*

Sol.
$$w = \frac{z-a}{z+a}$$

$$\rho e^{i\phi} = \frac{(x-a)+iy}{(x+a)+iy} \qquad \text{...(1)}$$

Then,
$$\rho e^{-i\phi} = \frac{(x-a)-iy}{(x+a)-iy}$$

$\therefore$
$$\rho^2 = \frac{(x-a)^2+y^2}{(x+a)^2+y^2}$$

Interior of unit circle *i.e.*, $|w| < 1$ *i.e.*, $\rho < 1$ is

$$(x-a)^2 + y^2 < (x+a)^2 + y^2$$

$\Rightarrow$
$$-2ax < 2ax$$

or
$$x > 0 \ (a \text{ positive})$$

The circles $|w| = k$ transform into

$$(x-a)^2 + y^2 = k^2\,[(x+a)^2 + y^2]$$

$\Rightarrow$
$$x^2 + y^2 + a^2 - 2ax\left(\frac{1+k^2}{1-k^2}\right) = 0 \qquad \text{...(2)}$$

There are a set of coaxal circles is z-plane.

Take logarithm of both sides of (1),

$$\log \rho + i\phi = \log\{(x-a)+iy\} - \log\{(x+a)+iy\}$$

$$\therefore \qquad \phi = \tan^{-1}\left(\frac{y}{x-a}\right) - \tan^{-1}\left(\frac{y}{x+a}\right)$$

or
$$\tan\phi = \frac{\frac{y}{x-a} - \frac{y}{x+a}}{1 + \frac{y^2}{x^2-a^2}} = \frac{y(x+a) - y(x-a)}{x^2 - a^2 + y^2}$$

Hence the lines $\phi = \alpha$ transform into

$$\tan\alpha = \frac{2ay}{x^2 + y^2 - a^2}$$

$\Rightarrow \quad x^2 + y^2 - a^2 - 2ay \cot\alpha = 0$ which are coaxal circles orthogonal to (2).

Example 15. *Show that* $w = \dfrac{i-z}{i+z}$ *maps the real axis of the z-plane into the circle* $|\,w\,| = 1$ *and the half-plane* $y > 0$ *into the interior of the unit circle* $|\,w\,| = 1$ *in the w-plane.*

Sol. $$w = \frac{i-z}{i+z} \qquad \text{...(1)}$$

(*i*) $$|\,w\,| = 1 \;\Rightarrow\; \left|\frac{i-z}{i+z}\right| = 1$$

$$\Rightarrow \quad |\,i - z\,| = |\,i + z\,|$$

$$\Rightarrow \quad |\,i - x - iy\,| = |\,i + x + iy\,|$$

$$\Rightarrow \quad x^2 + (1-y)^2 = x^2 + (1+y)^2$$

$$\Rightarrow \quad 1 + y^2 - 2y = 1 + y^2 + 2y$$

$$\Rightarrow \quad 4y = 0 \;\Rightarrow\; y = 0 \qquad \text{(real axis of } z\text{-plane)}$$

Hence the given transformation maps the real axis of z-plane into the unit circle $|\,w\,| = 1$

(*ii*) $$|\,w\,| < 1 \;\Rightarrow\; \left|\frac{i-z}{i+z}\right| < 1$$

$$\Rightarrow \quad |\,i - x - iy\,| < |\,i + x + iy\,|$$

$$\Rightarrow \quad x^2 + (1-y)^2 < x^2 + (1+y)^2$$

$$\Rightarrow \quad x^2 + 1 + y^2 - 2y < x^2 + 1 + y^2 + 2y$$

$$\Rightarrow \quad 4y > 0 \;\Rightarrow\; y > 0$$

Hence the given transformation maps the half-plane $y > 0$ of the z-plane into the interior of the unit circle $|\,w\,| = 1$ in the w-plane.

Example 16. *Show that the transformation* $w = i\left(\dfrac{1-z}{1+z}\right)$ *transforms the circle* $|\,z\,| = 1$ *onto the real axis of the w-plane and the interior of the circle into the upper half of the w-plane.*

Sol. $$w = i\left(\frac{1-z}{1+z}\right)$$

$$\frac{w}{i} = \frac{1-z}{1+z}$$

$\Rightarrow$ $$\frac{w+i}{w-i} = \frac{2}{-2z}$$

$\Rightarrow$ $$z = \frac{i-w}{i+w}$$

$\Rightarrow$

(i) $$|z| = 1$$

$\Rightarrow$ $$\left|\frac{i-w}{i+w}\right| = 1$$

$\Rightarrow$ $$|i-w| = |i+w|$$

$\Rightarrow$ $$|i-u-iv| = |i+u+iv|$$

$\Rightarrow$ $$u^2 + (1-v)^2 = u^2 + (1+v)^2$$

$\Rightarrow$ $$1 + v^2 - 2v = 1 + v^2 + 2v$$

$\Rightarrow$ $$4v = 0 \Rightarrow v = 0 \qquad \text{(real axis of } w\text{-plane)}$$

Hence, the given transformation transforms the circle $|z| = 1$ onto the real axis of the w-plane.

(ii) $$|z| < 1$$

$\Rightarrow$ $$\left|\frac{i-w}{i+w}\right| < 1$$

$\Rightarrow$ $$|i-w| < |i+w|$$

$\Rightarrow$ $$|i-u-iv| < |i+u+iv|$$

$\Rightarrow$ $$u^2 + (1-v)^2 < u^2 + (1+v)^2$$

$\Rightarrow$ $$1 + v^2 - 2v < 1 + v^2 + 2v$$

$\Rightarrow$ $$4v > 0 \Rightarrow v > 0$$

Hence, the given transformation transforms the interior of the circle $|z| = 1$ into the upper half of w-plane.

Example 17. *Find the condition that the transformation $w = \dfrac{az+b}{cz+d}$ transforms the unit circle in the w-plane into straight line in the z-plane.*

Sol. The given transformation is

$$w = \frac{az+b}{cz+d} = \frac{a}{c} \cdot \frac{z+\frac{b}{a}}{z+\frac{d}{c}}$$

Clearly, under this transformation, the unit circle $|w| = 1$ in w-plane transforms into

$$1 = \left|\frac{a}{c}\right| \left|\frac{z+\frac{b}{a}}{z+\frac{d}{c}}\right|$$

$$\Rightarrow \qquad \left|\frac{z+\frac{b}{a}}{z+\frac{d}{c}}\right| = \left|\frac{c}{a}\right| \quad \text{in } z\text{-plane}$$

This represents a line if $\left|\frac{c}{a}\right| = 1$ or $|\, a \,| = |\, c \,|$

The corresponding line is $\left|\frac{z+\frac{b}{a}}{z+\frac{d}{c}}\right| = 1.$

Note. The equation $\left|\frac{z-p}{z-q}\right| = k$ represents a line or a circle according as k is or is not equal to 1.

Example 18. *Prove that* $w = \frac{z}{1-z}$ *maps the upper half of the z-plane onto the upper half of the w-plane. What is the image of the circle* $|\, z \,| = 1$ *under this transformatio̧n?*

Sol.
$$w = \frac{z}{1-z} \qquad \text{...(1)}$$

$$\Rightarrow \qquad u + iv = \frac{x+iy}{1-x-iy} \cdot \frac{1-x+iy}{1-x+iy} = \frac{(x+iy)(1-x+iy)}{(1-x)^2+y^2}$$

$$= \left[\frac{x(1-x)-y^2}{(1-x)^2+y^2}\right] + i\left[\frac{y(1-x)+xy}{(1-x)^2+y^2}\right]$$

$$\Rightarrow \qquad u = \frac{x(1-x)-y^2}{(1-x)^2+y^2}, \; v = \frac{y}{(1-x)^2+y^2}$$

Now, upper half of w-plane is

$$v > 0$$

$$\Rightarrow \qquad \frac{y}{(1-x)^2+y^2} > 0 \qquad \Rightarrow \quad y > 0$$

which is the upper half of z-plane.

Hence the given transformation maps the upper half of w-plane onto the upper half of z-plane.

From (1),
$$z = \frac{w}{1+w} \qquad \text{...(2)}$$

Now,
$$|\, z \,| = 1 \qquad \Rightarrow \qquad \left|\frac{w}{1+w}\right| = 1$$

$$\Rightarrow \qquad |\, w \,| = |\, 1 + w \,|$$

$$\Rightarrow \qquad |\, u + iv \,| = |\, 1 + u + iv \,|$$

$$\Rightarrow \qquad u^2 + v^2 = (1 + u)^2 + v^2$$

$$\Rightarrow \qquad u^2 + v^2 = 1 + u^2 + 2u + v^2$$

$$2u + 1 = 0$$

$\Rightarrow$ which is a straight line.

Hence, the image of the circle $|z| = 1$ is the straight line $2u + 1 = 0$.

Example 19. *Show that the map of the real axis of the z-plane on the w-plane by the transformation $w = \dfrac{1}{z+i}$ is a circle and find its centre and radius.*

Sol.
$$w = \frac{1}{z+i} \qquad \text{...(1)}$$

$\Rightarrow$
$$z = \frac{1-iw}{w} \qquad \text{...(2)}$$

$$= \left\{\frac{1-i(u+iv)}{u+iv}\right\} \cdot \frac{u-iv}{u-iv} = \left(\frac{1+v-iu}{u^2+v^2}\right)(u-iv)$$

$\Rightarrow$
$$x + iy = \left[\frac{u}{u^2+v^2}\right] + i\left[\frac{-u^2 - v(1+v)}{u^2+v^2}\right]$$

$\Rightarrow$
$$x = \frac{u}{u^2+v^2},\ y = -\left(\frac{u^2+v^2+v}{u^2+v^2}\right)$$

Real axis of the z-plane is
$$y = 0$$

$\Rightarrow$
$$-\left(\frac{u^2+v^2+v}{u^2+v^2}\right) = 0 \quad \Rightarrow \quad u^2 + v^2 + v = 0 \qquad \text{...(3)}$$

Comparing eqn. (3) with

$u^2 + v^2 + 2gu + 2fv + c = 0$, we get
$$g = 0, f = \frac{1}{2}, c = 0$$

$\therefore$
$$\text{centre} = (-g, -f) = \left(0, -\frac{1}{2}\right)$$

and
$$\text{radius} = \sqrt{g^2 + f^2 - c} = \sqrt{0 + \frac{1}{4} - 0} = \frac{1}{2}$$

Hence eqn. (3) represents circle with its centre at $\left(0, -\dfrac{1}{2}\right)$ and radius $\dfrac{1}{2}$ in w-plane.

Example 20. *Find the invariant points of the transformation $w = -\left(\dfrac{2z+4i}{iz+1}\right)$. Prove also that these two points together with any point z and its image w, form a set of four points having a constant cross ratio.*

Sol. The invariant points of the transformation are given by

$$z = -\left(\frac{2z+4i}{iz+1}\right) \quad \Rightarrow \quad iz^2 + 3z + 4i = 0$$

$\Rightarrow$
$$z^2 - 3iz + 4 = 0 \quad \Rightarrow \quad (z-4i)(z+i) = 0$$

$\therefore$ The invariant points are $4i$ and $-i$.

Taking $z_1 = z,\ z_2 = w = -\left(\dfrac{2z+4i}{iz+1}\right),\ z_3 = 4i$ and $z_4 = -i$; the cross-ratio of the four points z_1, z_2, z_3 and z_4 is given by

$$(z_1, z_2, z_3, z_4) = \frac{\left(z+\dfrac{2z+4i}{iz+1}\right)(4i+i)}{(z+i)\left(4i+\dfrac{2z+4i}{iz+1}\right)} = \frac{5i(iz^2+3z+4i)}{(z+i)(-2z+8i)} = \frac{5i(iz^2+3z+4i)}{2i(iz^2+3z+4i)}$$

$$= \frac{5}{2} = \text{a constant (independent of } z).$$

TEST YOUR KNOWLEDGE

1. Find the critical points of the transformation $w^2 = (z-\alpha)(z-\beta)$ and express them in terms of w.
2. (*i*) Obtain the invariant points of the transformation $w = 2 - \dfrac{2}{z}$.

   (*ii*) Find the fixed points under the transformation $w = \dfrac{2z-5}{z+4}$.

   (*iii*) Find the invariant points of the transformation $w = \dfrac{1+z}{1-z}$.

   (*iv*) Find the fixed points of $w = \dfrac{3z-4}{z-1}$.
3. Find the invariant points of the transformations:

   (*i*) $w = \dfrac{2z+6}{z+7}$ (*ii*) $w = iz^2$ (*iii*) $w = \dfrac{3z-5i}{iz-1}$

   (*iv*) $w = z^3$ (*v*) $w = \dfrac{z-1-i}{z+2}$ (*vi*) $w = \dfrac{6z-9}{z}$.
4. (*i*) Find the bilinear transformation which transforms the points $z = \infty, i, 0$ into $w = 0, i, \infty$ respectively.

   (*ii*) Find a bilinear transformation which takes the points $(1, i, \infty)$ to $(\infty, i, 1)$.
5. Find the bilinear transformation which maps $1, i, -1$ to $2, i, -2$ respectively. Find the fixed and critical points of the transformation.
6. Find the bilinear transformation which maps the points $z = 1, -i, -1$ into the points $w = i, 0, -i$.
7. Find the bilinear transformation which maps the points $z = -i, 0, i$ into points $w = -1, i, 1$. Into what curve, the y-axis is transformed to this transformation ?
8. Find the bilinear transformation which maps the points $z = 0, -i, -1$ into the points $w = i, 1, 0$. Find the image of the line $y = mx$ under this transformation.
9. Prove that if $w = \dfrac{az+b}{cz+d}$ and $ad - bc = 1$; the linear magnification is $(cz+d)^{-2}$.
10. Find the bilinear transformation which maps the points $z = 0, -i, 2i$ into the points $w = 5i, \infty, \dfrac{-i}{3}$ respectively.

11. Determine the bilinear transformation that maps the points $-1, 0, 1$ in the z-plane, onto the points $0, i, 3i$ in the w-plane.
12. Find the bilinear transformation which maps the points $z = 0, 1, \infty$ into the points $w = -5, -1, 3$ respectively.
13. Find the bilinear transformation which maps the points $-2, 0, 2$ into the points $w = 0, i, -i$ respectively
14. Obtain the bilinear transformation which map the points $z = 1, i, -1$ into the points $w = 0, 1, \infty$. Show that the transformation maps the interior of the unit circle of the z-plane onto the upper half of the w-plane.
15. Find the bilinear transformation which maps $z = 0$ onto $w = -i$ and has -1 and 1 as the invariant points. Also show that under this transformation the upper half of the z-plane maps onto the interior of the unit circle in the w-plane.
16. Find the bilinear mapping which maps $-1, 0, 1$ of the z-plane onto $-1, -i, 1$ of the w-plane. Show that under this mapping, the upper half of the z-plane maps onto the interior of the unit circle $|w| = 1$.
17. Find the bilinear transformation which maps the points $z = 0$, $z = 1$ and $z = \infty$ into the points $w = i$, $w = 1$ and $w = -i$.
18. (*i*) Determine the bilinear transformation that maps the points $-1, 0, 1$ in the z-plane onto the points $0, i, 3i$ in the w-plane.

    (*ii*) Find the bilinear transformation which maps the points $-2, 0, 2$ into the points $w = 0, i, -i$ respectively.
19. Show that the relation $w = \dfrac{iz + 2}{4z + i}$ transforms the real axis in the z-plane into a circle in the w-plane. Find the centre and the radius of the circle. Also find the point in z-plane which is mapped on the centre of the circle.
20. Express the relation $w = \dfrac{13iz + 75}{3z - 5i}$ in the form $\dfrac{w - a}{w - b} = k\left(\dfrac{z - a}{z - b}\right)$ where a, b, k are constants.

Answers

1. $\dfrac{\alpha+\beta}{2}, \alpha, \beta$

2. (*i*) $1 \pm i$ (*ii*) $-1 \pm 2i$ (*iii*) $i, -i$ (*iv*) 2

3. (*i*) $1, -6$ (*ii*) $0, -i$ (*iii*) $i, -5i$ (*iv*) $0, \pm 1$

   (*v*) $-i, -1 + i$ (*vi*) 3

4. (*i*) $w = -\dfrac{1}{z}$ (*ii*) $w = \dfrac{z - (1 + 2i)}{z - 1}$ **5.** $i, 2i$ **6.** $w = \dfrac{z + i}{z - i}$

7. $w = \dfrac{i(1 - z)}{1 + z}$; $u^2 + v^2 = 1$ **8.** $w = \dfrac{i(1 + z)}{1 - z}$; $m(u^2 + v^2) + 2u - m = 0$

10. $w = \dfrac{3z - 5i}{iz - 1}$ **11.** $w = -3i\left(\dfrac{1 + z}{z - 3}\right)$ **12.** $w = 4\left(\dfrac{z - 1}{z + 1}\right) - 1$

13. $w = -\dfrac{i}{2}\left(\dfrac{-z + 6}{3z - 2} + 1\right)$ **14.** $w = -i\left(\dfrac{z - 1}{z + 1}\right)$

15. $w = \frac{1}{2}\left[1 - i + \frac{3z - 1 + iz - i}{1 - iz}\right]$ 16. $w = i\left(\frac{z - i}{z + i}\right)$ 17. $w = \frac{z + i}{1 + zi}$

18. (i) $w = -3i\left[\frac{1 + z}{z - 3}\right]$ (ii) $w = -\frac{i}{2}\left[\frac{-z + 6}{3z - 2} + 1\right]$

19. centre $= \left(0, -\frac{7}{8}\right)$; radius $= \frac{9}{8}$; point $= \frac{i}{4}$

20. $a = 3i - 4$, $b = 3i + 4$ (if $a \neq b$) ; $k = -\frac{1}{5}(4 + 3i)$

4.30 NATURE OF BILINEAR TRANSFORMATION

(1) A linear fractional transformation with one fixed point z_0 is called parabolic and is expressible as

$$\frac{1}{w - z_0} = \frac{1}{z - z_0} + h \qquad \text{if } z_0 \neq \infty$$

or

$$w = z + h \qquad \text{if } z_0 = \infty$$

(2) A linear fractional transformation with two different fixed points z_1 and z_2 is given as,

$$\frac{w - z_1}{w - z_2} = k\left(\frac{z - z_1}{z - z_2}\right) \text{ if } z_1, z_2 \neq \infty$$

If $z_2 = \infty$, then it becomes $w - z_1 = k\,(z - z_1)$.

(3) A transformation with two different fixed points is called hyperbolic if $k > 0$ and elliptic if $k = e^{i\alpha}$; $\alpha \neq 0$ or $|\,k\,| = 1$ and Loxodromic if $k = e^{i\alpha}\,.\,a$ where $a \neq 0$, $\alpha \neq 0$, a, α are real, $a > 0$.

Example. *Find the fixed points and the normal of the following bilinear transformations:*

(i) $w = \frac{z}{z - 2}$ (ii) $\frac{3iz + 1}{z + i} = w$

Is any of these transformation, hyperbolic, elliptic or parabolic?

Sol. (i) Fixed points are given by,

$$z = \frac{z}{z - 2}$$

$$z^2 - 3z = 0 \quad \Rightarrow \quad z = 0, 3$$

Hence fixed points are 0, 3.

Normal Form

$$\frac{w - 0}{w - 3} = \frac{z}{z - 2} \cdot \frac{1}{\left(\frac{z}{z - 2} - 3\right)} = \frac{z}{z - 2} \cdot \frac{z - 2}{6 - 2z} = -\frac{1}{2}\left(\frac{z - 0}{z - 3}\right)$$

$$\therefore \qquad \frac{w - 0}{w - 3} = k\left(\frac{z - 0}{z - 3}\right) \text{ with } k = -\frac{1}{2}$$

This is normal form.

Here, $$k = -\frac{1}{2} = \frac{1}{2}e^{i\pi}.$$

Hence map is loxodromic.

(*ii*) Fixed points are given by

$$z = \frac{3iz+1}{z+i}$$

or $$(z-i)^2 = 0 \quad \text{or} \quad z = i, i$$

Hence there is only one distinct fixed point i.

$$w - i = \frac{3iz+1}{z+i} - i$$

$$w - i = \frac{2iz+2}{z+i}$$

or $$\frac{1}{w-i} = \frac{z+i}{2(iz+1)} = \frac{z+i}{2i(z-i)} = \frac{1}{2i}\left(1+\frac{2i}{z-i}\right) = \frac{1}{z-i} + \frac{1}{2i}$$

or $$\frac{1}{w-i} = \frac{1}{z-i} - \frac{i}{2}$$ which is normal form.

Hence, map is parabolic.

TEST YOUR KNOWLEDGE

1. Find the fixed points and the normal of the following linear fractional transformations :

(*i*) $w = \dfrac{z-1}{z+1}$ (*ii*) $w = \dfrac{z}{2-z}$

Also find the nature of these transformations.

Answer

1. (*i*) Elliptic (*ii*) Hyperbolic.

ASSIGNMENT-IV

1. Define analytic function and state the necessary and sufficient condition for function to be analytic. *(M.T.U. 2012)*
2. If $f(z) = u + iv$ is analytic, then show that the family of curves $u(x, y) = c_1$ and $v(x, y) = c_2$ are mutually orthogonal. *(M.T.U. 2012)*
3. Using the Cauchy-Riemann equations, show that $f(z) = |z|^2$ is not analytic at any point. *(M.T.U. 2013)*

4. Find the constants a, b and c such that the function $f(z) = -x^2 + xy + y^2 + i(ax^2 + bxy + y^2)$ is analytic. (M.T.U. 2013)
5. Define analytic function with an example. (A.K.T.U. 2018)
6. Find the values of a and b for which the function $f(z) = \cos x\,(\cosh y + a \sinh y) + i \sin x\,(\cosh y + b \sinh y)$ is analytic.
7. If $f(z) = u + iv$ is an analytic function and $u = x^2 - y^2 - y$, then find its conjugate harmonic function $v(x, y)$. (A.K.T.U. 2016)
8. If $f(z) = u + iv$ is an analytic function and $v = y^2 - x^2$, then find its conjugate harmonic function $u(x, y)$.
9. If $u = \dfrac{x^2 - y^2}{(x^2 + y^2)^2}$ is the real part of analytic function $f(z) = u + iv$, then find $f(z)$ in terms of z.
10. Let $u(x, y) = 2x(1 - y)$ for all real x and y. Find a function $v(x, y)$ so that $f(z) = u + iv$ is analytic.
11. If $u(x, y) = x^3y - xy^3$ is the real part of analytic function $f(z) = u(x, y) + (x, y)$, then find its conjugate harmonic function $v(x, y)$. (M.T.U. 2014)
12. Define Harmonic function.
13. If $u(x, y) = x^2 - y^2$, prove that u satisfies Laplace equation. (A.K.T.U. 2016)
14. Let $u(x, y) = x^3 + ax^2y + bxy^2 + 2y^3$ be a harmonic function and $v(x, y)$ its harmonic conjugate. If $u(0, 0) = 1$, then find $|a + b + v(1, 1)|$.
15. Prove that $\sinh z$ is analytic. (A.K.T.U. 2017)
16. Find the image of the straight line $2x - y + 3 = 0$ in the w-plane under the transformation $w = z - 2$.
17. Find the points of invariant of the transformation $w = \dfrac{2z + 3}{z + 2}$.
18. Show that an analytic function with constant real part is constant.
19. If $u + iv$ is analytic, show that $v - iu$ and $-v + iu$ are also analytic.
20. Write the Cauchy's Reimann conditions in polar coordinates system. (A.K.T.U. 2016)

Answers

4. $a = -\dfrac{1}{2}, b = -2, c = \dfrac{1}{2}$
6. $a = -1, b = -1$
7. $2xy + x + c$
8. $2xy + c$
9. $\dfrac{1}{z^2} + c$
10. $x^2 - (y - 1)^2$
11. $x^4 + y^4 - 6x^2y^2 + c$
14. 10
16. $2u - v + 7 = 0$
17. $z = \pm\sqrt{3}$
20. $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$ and $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$

MODULE 5

Complex Variable – Integration

5.1 COMPLEX INTEGRALS

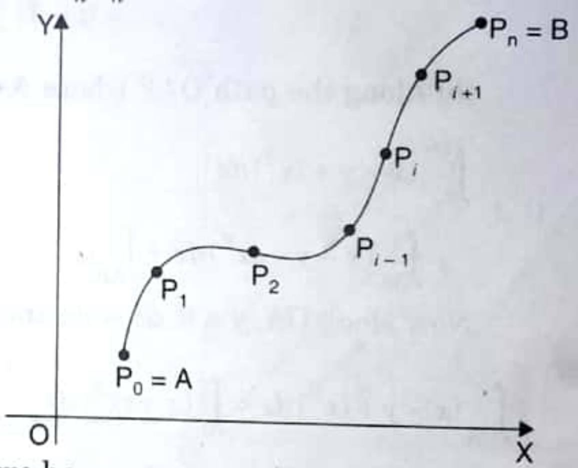
In case of real variable, the path of integration of $\int_a^b f(x)\,dx$ is always along the x-axis from $x = a$ to $x = b$. But in case of a complex function $f(z)$, the path of the definite integral $\int_a^b f(z)\,dz$ can be along any curve from $z = a$ to $z = b$.

Let $f(z)$ be a continuous function of the complex variable $z = x + iy$ defined at all points of a curve C having end points A and B. Divide the curve C into n parts at the points

$$A = P_0(z_0),\ P_1(z_1),\ \ldots\ldots,\ P_i(z_i),\ \ldots\ldots,\ P_n(z_n) = B.$$

Let $\delta z_i = z_i - z_{i-1}$ and ξ_i be any point on the arc $P_{i-1}P_i$. Then the limit of the sum

$\sum_{i=1}^{n} f(\xi_i)\,\delta z_i$ as $n \to \infty$ and each $\delta z_i \to 0$, if it exists, is called the **line integral of f(z)** along the curve C. It is denoted by $\int_C f(z)\,dz$.



In case the points P_0 and P_n coincide so that C is a closed curve, then this integral is called **contour integral** and is denoted by $\oint_C f(z)\,dz$.

If $f(z) = u(x, y) + iv(x, y)$, then since $dz = dx + i\,dy$, we have

$$\int_C f(z)\,dz = \int_C (u + iv)(dx + i\,dy)$$

$$= \int_C (udx - vdy) + i\int_C (vdx + udy)$$

which shows that the evaluation of the line integral of a complex function can be reduced to the evaluation of two line integrals of real functions.

Moreover, the value of the integral depends on the path of integration unless the integrand is analytic.

When the same path of integration is used in each integral, then

$$\int_a^b f(z)\,dz = -\int_b^a f(z)\,dz$$

If c is a point on the arc joining a and b, then

$$\int_a^b f(z)\,dz = \int_a^c f(z)\,dz + \int_c^b f(z)\,dz.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Evaluate* $\int_0^{1+i} (x - y + ix^2)\,dz$.

(a) along the straight line from $z = 0$ to $z = 1 + i$

(b) along the real axis from $z = 0$ to $z = 1$ and then along a line parallel to imaginary axis from $z = 1$ to $z = 1 + i$.

(c) along the imaginary axis from $z = 0$ to $z = i$ and then along a line parallel to real axis from $z = i$ to $z = 1 + i$.

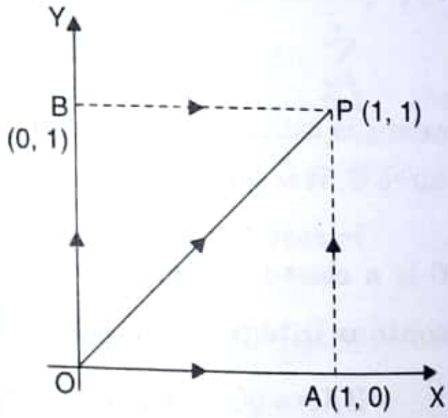
Sol. (*a*) Along the straight line OP joining O($z = 0$) and P($z = 1 + i$), $y = x$, $dy = dx$ and x varies from 0 to 1.

$$\therefore \quad \int_0^{1+i} (x - y + ix^2)\,dz = \int_0^{1+i} (x - y + ix^2)(dx + i\,dy)$$

$$= \int_0^1 (x - x + ix^2)(dx + idx) = \int_0^1 ix^2(1+i)\,dx$$

$$= (i-1)\left(\frac{x^3}{3}\right)_0^1 = -\frac{1}{3} + \frac{1}{3}i.$$

(*b*) Along the path OAP where A is $z = 1$



$$\int_0^{1+i} (x - y + ix^2)dz$$

$$= \int_{OA} (x - y + ix^2)dz + \int_{AP} (x - y + ix^2)\,dz \qquad ...(1)$$

Now along OA, $y = 0$, $dz = dx$ and x varies from 0 to 1.

$$\therefore \quad \int_{OA} (x - y + ix^2)dz = \int_0^1 (x + ix^2)dx = \left[\frac{x^2}{2} + i\frac{x^3}{3}\right]_0^1 = \frac{1}{2} + \frac{1}{3}i$$

Also along AP, $x = 1$, $dz = idy$ and y varies from 0 to 1

$$\therefore \quad \int_{AP} (x - y + ix^2)\,dz = \int_0^1 (1 - y + i)\,idy = \left[(-1+i)y - i\frac{y^2}{2}\right]_0^1 = -1 + i - \frac{1}{2}i = -1 + \frac{1}{2}i$$

Hence from (1), $\displaystyle\int_0^{1+i} (x - y + ix^2)\,dz = \left(\frac{1}{2} + \frac{1}{3}i\right) + \left(-1 + \frac{1}{2}i\right) = -\frac{1}{2} + \frac{5}{6}i.$

(*c*) Along the path OBP where B is $z = i$

$$\int_0^{1+i} (x - y + ix^2)\,dz = \int_{OB} (x - y + ix^2)dz + \int_{BP} (x - y + ix^2)\,dz \qquad ...(2)$$

Now along OB, $x = 0$, $dz = idy$ and y varies from 0 to 1

$$\therefore \quad \int_{OB} (x - y + ix^2)dz = \int_0^1 (-y)\,idy = -i\left[\frac{y^2}{2}\right]_0^1 = -\frac{1}{2}i$$

Also, along BP, $y = 1$, $dz = dx$ and x varies from 0 to 1

$$\therefore \quad \int_{BP} (x - y + ix^2)\,dz = \int_0^1 (x - 1 + ix^2)\,dx = \left[\frac{x^2}{2} - x + i\frac{x^3}{3}\right]_0^1 = -\frac{1}{2} + \frac{1}{3}i$$

Hence from (2), $\displaystyle\int_0^{1+i} (x - y + ix^2)\,dz = -\frac{1}{2}i + \left(-\frac{1}{2} + \frac{1}{3}i\right) = -\frac{1}{2} - \frac{1}{6}i.$

Note. The values of the integral are different along the three different paths.

Example 2. *Evaluate* $\displaystyle\int_0^{1+i} (x^2 - iy)\,dz$ *along the paths*

(a) $y = x$ *(b)* $y = x^2$. (G.B.T.U. 2010)

Sol. (*a*) Along the line $y = x$,

$$dy = dx \text{ so that } dz = dx + idx = (1 + i)\,dx$$

$$\int_0^{1+i} (x^2 - iy)\,dz = \int_0^1 (x^2 - ix)(1 + i)\,dx$$

$$= (1 + i)\left[\frac{x^3}{3} - i\frac{x^2}{2}\right]_0^1 = (1 + i)\left(\frac{1}{3} - \frac{1}{2}i\right) = \frac{5}{6} - \frac{1}{6}i.$$

(*b*) Along the parabola $y = x^2$, $dy = 2x\,dx$ so that

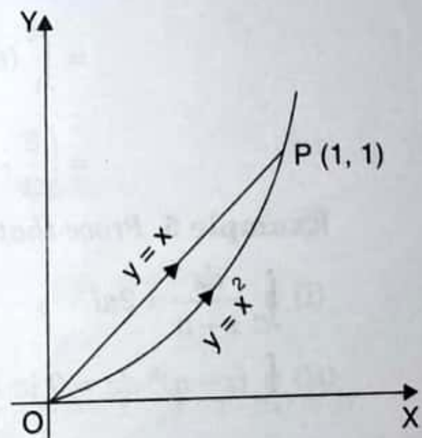
$$dz = dx + 2ix\,dx = (1 + 2ix)\,dx$$

and x varies from 0 to 1.

$$\therefore \quad \int_0^{1+i} (x^2 - iy)\,dz = \int_0^1 (x^2 - ix^2)(1 + 2ix)\,dx$$

$$= (1 - i)\left[\frac{x^3}{3} + i\frac{x^4}{2}\right]_0^1$$

$$= (1 - i)\left(\frac{1}{3} + \frac{1}{2}i\right) = \frac{5}{6} + \frac{1}{6}i.$$



Example 3. *Evaluate* $\displaystyle\int_0^{2+i} (\bar{z})^2\,dz$, *along*

(a) the real axis from $z = 0$ to $z = 2$ and then along a line parallel to y-axis from $z = 2$ to $z = 2 + i$. (U.K.T.U. 2011)

(b) along the line $2y = x$.

Sol. $(\bar{z})^2 = (x - iy)^2 = (x^2 - y^2) - 2ixy$

(*a*) Along the path OAP where A is (2, 0) and P is (2, 1).

$$\int_0^{2+i} (\bar{z})^2\,dz = \int_{OA} (x^2 - y^2 - 2ixy)\,dz + \int_{AP} (x^2 - y^2 - 2ixy)\,dz \qquad \ldots(1)$$

Now, along OA, $y = 0$, $dz = dx$ and x varies from 0 to 2

$$\therefore \quad \int_{OA} (x^2 - y^2 - 2ixy)\,dz = \int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$$

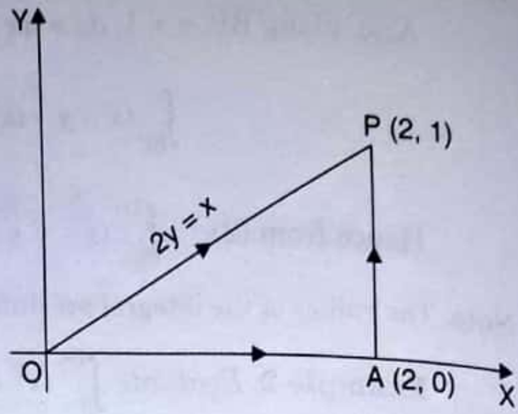
Also, along AP, $x = 2$, $dz = idy$ and y varies from 0 to 1

$\therefore \int_{AP} (x^2 - y^2 - 2ixy)\, dz = \int_0^1 (4 - y^2 - 4iy)\, idy$

$= \left[4iy - i\dfrac{y^3}{3} + 2y^2\right]_0^1 = 4i - \dfrac{1}{3}i + 2 = 2 + \dfrac{11}{3}i$

Hence from (1), we have

$$\int_0^{2+i} (\bar{z})^2\, dz = \frac{8}{3} + 2 + \frac{11}{3}i = \frac{14}{3} + \frac{11}{3}i.$$



(*b*) Along the line OP, $2y = x$, $dx = 2dy$

so that

$$dz = 2dy + i\, dy = (2 + i)\, dy$$

and y varies from 0 to 1.

$$\therefore \int_0^{2+i} (\bar{z})^2\, dz = \int_0^{2+i} (x^2 - y^2 - 2ixy)\, dz = \int_0^1 (4y^2 - y^2 - 4iy^2)(2 + i)\, dy$$

$$= (2 + i)(3 - 4i) \int_0^1 y^2\, dy = (10 - 5i)\left[\frac{y^3}{3}\right]_0^1 = \frac{10}{3} - \frac{5}{3}i.$$

Example 4. *Integrate $f(z) = x^2 + ixy$ from A(1, 1) to B(2, 4) along the curve $x = t$, $y = t^2$.*

Sol. Equations of the path of integration are $x = t$, $y = t^2$

$$\therefore \quad dx = dt, \quad dy = 2t\, dt$$

At A(1, 1), $t = 1$ and at B(2, 4), $t = 2$

$$\therefore \int_A^B f(z)\, dz = \int_A^B (x^2 + ixy)(dx + idy) = \int_1^2 (t^2 + it^3)(dt + 2it\, dt)$$

$$= \int_1^2 (t^2 - 2t^4)\, dt + i\int_1^2 3t^3\, dt = \left[\frac{t^3}{3} - \frac{2t^5}{5}\right]_1^2 + i\left[\frac{3t^4}{4}\right]_1^2$$

$$= \left(\frac{8}{3} - \frac{64}{5}\right) - \left(\frac{1}{3} - \frac{2}{5}\right) + i\left(12 - \frac{3}{4}\right) = -\frac{151}{5} + \frac{45}{4}i.$$

Example 5. *Prove that*

(*i*) $\oint_C \dfrac{dz}{z - a} = 2\pi i$

(*ii*) $\oint_C (z - a)^n\, dz = 0$ *[n is an integer $\neq -1$] where C is the circle $|\, z - a\,| = r$.*

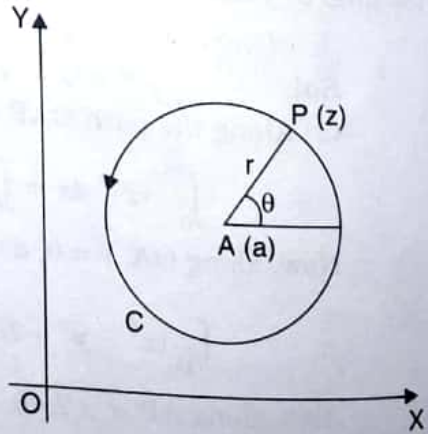
(G.B.T.U. 2011)

Sol. The equation of the circle C is

$$|\, z - a\,| = r \quad \text{or} \quad z - a = re^{i\theta}$$

where θ varies from 0 to 2π as z describes C once in the anti-clockwise direction.

Also $\qquad dz = ire^{i\theta}\, d\theta.$



(*i*) $\displaystyle \oint_C \frac{dz}{z - a} = \int_0^{2\pi} \frac{ire^{i\theta}\, d\theta}{re^{i\theta}} = i\int_0^{2\pi} d\theta = 2\pi i$

(*ii*) $\displaystyle \oint_C (z - a)^n\, dz = \int_0^{2\pi} r^n\, e^{ni\theta} \,.\, ire^{i\theta}\, d\theta$

$$= ir^{n+1} \int_0^{2\pi} e^{i(n+1)\theta}\, d\theta$$

$$= ir^{n+1} \left[\frac{e^{i(n+1)\theta}}{i(n+1)} \right]_0^{2\pi} \qquad [\because \; n \neq -1]$$

$$= \frac{r^{n+1}}{n+1} \, [e^{i2(n+1)\pi} - 1]$$

$$= 0. \qquad [\because \; e^{i2(n+1)\pi} = \cos 2(n+1)\pi + i \sin 2(n+1)\pi = 1]$$

Example 6. *Evaluate the integral* $\int_c |z|\, dz$, *where c is the contour*

(i) The straight line from $z = -i$ *to* $z = i$

(ii) The left half of the unit circle $|z| = 1$ *from* $z = -i$ *to* $z = i$.

Sol. (*i*) The straight line from $z = -i$ to $z = i$ is $x = 0$

i.e., $z = iy$ so that $dz = idy$

$$\therefore \quad \int_c |z|\, dz = \int_{-1}^{1} |iy|\, i\, dy = i \int_{-1}^{0} (-y)\, dy + i \int_0^1 y\, dy$$

$$= -i \left(\frac{y^2}{2} \right)_{-1}^{0} + i \left(\frac{y^2}{2} \right)_0^1 = -i \left(-\frac{1}{2} \right) + i \left(\frac{1}{2} \right) = i.$$

(*ii*) For a point on the unit circle $|z| = 1$,

$$z = e^{i\theta}$$

$$\therefore \quad dz = ie^{i\theta}\, d\theta.$$

The points $z = -i$ and i correspond to $\theta = \frac{3\pi}{2}$ and $\theta = \frac{\pi}{2}$ respectively.

$$\therefore \quad \int_c |z|\, dz = \int_{3\pi/2}^{\pi/2} 1 . e^{i\theta}\, id\theta = \left(e^{i\theta} \right)_{3\pi/2}^{\pi/2} = e^{i\pi/2} - e^{3i\pi/2}$$

$$= \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} - \cos \frac{3\pi}{2} - i \sin \frac{3\pi}{2} = 0 + i - 0 - i(-1) = 2i.$$

Example 7. *Evaluate the integral* $\int_c \log z\, dz$, *where c is the unit circle* $|z| = 1$.

Sol. Here, $\qquad c \equiv |z| = 1 \qquad \ldots(1)$

$$\int_c \log z dz = \int_c \log (x + iy)\, dz$$

$$= \int_c \left[\frac{1}{2} \log (x^2 + y^2) + i \tan^{-1} \frac{y}{x} \right] dz$$

$$= i \int_c \tan^{-1} \left(\frac{y}{x} \right) dz. \qquad \ldots(2) \quad (\because \; x^2 + y^2 = 1)$$

On the unit circle, $\quad z = e^{i\theta}$

$$\therefore \quad dz = ie^{i\theta}\, d\theta.$$

Now (2) becomes,

$$\int_c \log z\, dz = i\int_0^{2\pi} \tan^{-1}(\tan\theta)\, ie^{i\theta}\, d\theta = -\int_0^{2\pi} \theta e^{i\theta}\, d\theta$$

$$= -\left[\left(\theta\frac{e^{i\theta}}{i}\right)_0^{2\pi} - \int_0^{2\pi} 1.\frac{e^{i\theta}}{i}\, d\theta\right] = -\left[\frac{2\pi}{i}e^{2\pi i} - \frac{1}{i}\left(\frac{e^{i\theta}}{i}\right)_0^{2\pi}\right]$$

$$= -\left[\frac{2\pi}{i}e^{2\pi i} + e^{2\pi i} - 1\right] = 2\pi i e^{2\pi i} + 1 - e^{2\pi i} = 2\pi i \quad \Big| \because \; e^{2\pi i} = 1$$

TEST YOUR KNOWLEDGE

1. Evaluate $\int_0^{3+i} z^2\, dz$, along

   (a) the line $y = \frac{x}{3}$ (b) the real axis to 3 and then vertically to $3 + i$

   (c) the parabola $x = 3y^2$.

2. Find the value of the integral $\int_0^{1+i} (x - y - ix^2)\, dz$, along real axis from $z = 0$ to $z = 1$ and then along a line parallel to imaginary axis from $z = 1$ to $z = 1 + i$. [G.B.T.U. (C.O.) 2011]

3. Evaluate $\int_0^{4+2i} \bar{z}\, dz$ along the curve given by $z = t^2 + it$.

4. (i) Evaluate $\oint_C |z|^2\, dz$ around the square with vertices at (0, 0), (1, 0), (1, 1) and (0, 1).

   (ii) Show that $\oint_C (z + 1)\, dz = 0$ where C is the boundary of the square whose vertices are at the points $z = 0, z = 1, z = 1 + i$ and $z = i$.

5. (a) Evaluate $\int_C \left[(x + y)\, dx + x^2 y\, dy\right]$

   (i) along $y = x^2$ having (0, 0), (3, 9) as end points.

   (ii) along $y = 3x$ between the same points.

   (b) Evaluate $\int_{(0,0)}^{(1,1)} (3x^2 + 4xy + 3y^2)\, dx + 2(x^2 + 3xy + 4y^2) dy$

   (i) along $y = x^2$ (ii) along $y = x$

   Does the value of the integral depend upon the path?

6. (i) Evaluate $\oint_C (y - x - 3x^2 i)\, dz$ where C is the straight line from $z = 0$ to $z = 1 + i$.

   (ii) Evaluate $\int_{1-i}^{2+3i} (z^2 + z)\, dz$ along the line joining the points (1, – 1) and (2, 3).

7. (i) Evaluate $\oint_C (z - z^2)\, dz$ where C is the upper half of the circle $|z| = 1$. What is the value of this integral if C is lower half of the given circle?

(ii) Evaluate the integral $\int_C (z - z^2)\, dz$ where C is the upper half of the circle $|z - 2| = 3$. What is the value of the integral if C is the lower half of the circle? *(M.T.U. 2013)*
[Hint: $z = 2 + 3e^{i\theta}$]

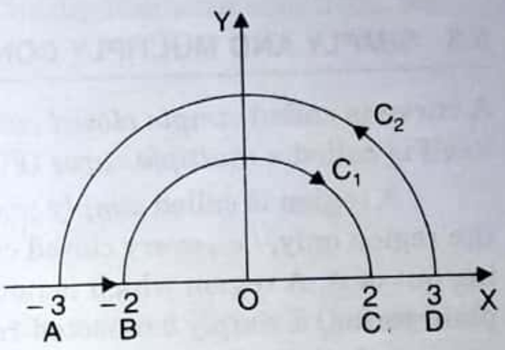
8. Prove that $\int_C \frac{1}{z}\, dz = -\pi i$ or πi according as C is the semi-circular arc $|z| = 1$ from $z = -1$ to $z = 1$ above or below the real axis.

9. Evaluate $\int_{1-i}^{2+i} (2x + iy + 1)\, dz$ along

(a) the straight line joining $(1 - i)$ to $(2 + i)$ (b) the curve $x = t + 1$, $y = 2t^2 - 1$.

10. Evaluate the line integral $\int_C (3y^2 dx + 2y\, dy)$ where C is the circle $x^2 + y^2 = 1$ counter clockwise from (1, 0) to (0, 1).

11. Evaluate the integral $I = \int_C \left(\frac{z}{\bar{z}}\right) dz$ where C is the boundary of the half annulus as given in adjoining figure.



12. Evaluate the integral $\int_C z^2 dz$ where C is the arc of the circle $|z| = 2$ from $\theta = 0$ to $\theta = \frac{\pi}{3}$.

13. Evaluate $\int_C \frac{2z + 3}{z}\, dz$ where C is

(i) upper half of the circle $|z| = 2$ in clockwise direction.
(ii) lower half of the circle $|z| = 2$ in anticlockwise direction.
(iii) the circle $|z| = 2$ in anticlockwise direction.

14. Evaluate the integral $\int_C \text{Re}\,(z^2)\, dz$ from 0 to $2 + 4i$ along the line segment joining the points (0, 0) and (2, 4).

15. Evaluate $\int_0^{3+i} (\bar{z})^2\, dz$ along the real axis from $z = 0$ to $z = 3$ and then along a line parallel to imaginary axis from $z = 3$ to $z = 3 + i$. *(G.B.T.U. 2013)*

16. Integrate $f(z) = \text{Re}\,(z)$ from $z = 0$ to $z = 1 + 2i$.

(i) along straight line joining $z = 0$ to $z = 1 + 2i$.
(ii) along the real axis from $z = 0$ to $z = 1$ and then along a line parallel to imaginary axis from $z = 1$ to $z = 1 + 2i$. *(U.P.T.U. 2014)*

Answers

1. (a) $6 + \frac{26}{3}i$ (b) $6 + \frac{26}{3}i$ (c) $6 + \frac{26}{3}i$ 2. $\frac{3}{2} + \frac{i}{6}$

3. $10 - \frac{8}{3}i$ 4. (i) $-1 + i$

5. (*a*) (*i*) 256.5 (*ii*) 200.25 (*b*) (*i*) 26/3 (*ii*) 26/3 ; No

6. (*i*) $1 - i$ (*ii*) $\frac{1}{6}(64i - 103)$ **7.** (*i*) $\frac{2}{3}; -\frac{2}{3}$ (*ii*) 66, – 66

9. (*a*) $4 + 8i$ (*b*) $4 + \frac{25}{3}i$ **10.** – 1 **11.** $\frac{4}{3}$

12. $\frac{-16}{3}$ **13.** (*i*) $8 - 3\pi i$ (*ii*) $8 + 3\pi i$ (*iii*) $6\pi i$

14. $-8(1 + 2i)$ **15.** $12 + \frac{26}{3}i$ **16.** (*i*) $\frac{1 + 2i}{2}$ (*ii*) $\frac{1}{2} + 2i$.

5.2 SIMPLY AND MULTIPLY CONNECTED DOMAINS

A domain in which every closed curve can be shrunk to a point without passing out of the region is called a *simply connected domain.* If a domain is not simply connected, then it is called *multiply connected domain.*

5.3 SIMPLY AND MULTIPLY CONNECTED REGIONS

A curve is called *simple closed curve* if it does not cross itself (Fig. *a*). A curve which crosses itself is called a *multiple curve* (Fig. *b*).

A region is called *simply connected* if every closed curve in the region encloses points of the region only, *i.e.*, every closed curve lying in it can be contracted indefinitely without passing out of it. A region which is not simply connected is called a *multiply connected* region. In plain terms, a simply connected region is one which has no holes. Figure *c* shows a multiply connected region R enclosed between two separate curves C_1 and C_2. (There can be more than two separate curves). We can convert a multiply connected region into a simply connected one, by giving it one or more cuts (*e.g.* along the dotted line AB).

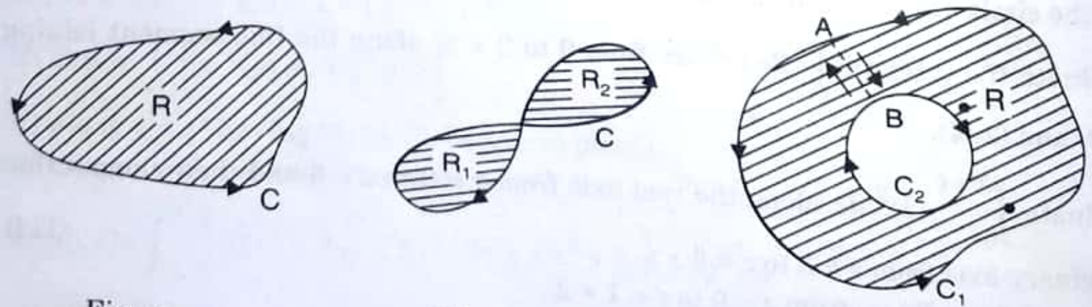


Fig. *a* Fig. *b* Fig. *c*

Remark 1. Jordan arc is a continuous arc without multiple points.

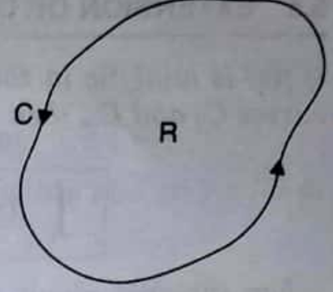
Remark 2. Contour is a Jordan curve consisting of continuous chain of a finite number of regular arcs.

5.4 CAUCHY'S INTEGRAL THEOREM

(*A.K.T.U. 2016*)

Statement. *If $f(z)$ is an analytic function and $f'(z)$ is continuous at each point within and on a simple closed curve C, then*

$$\boxed{\oint_C f(z)\,dz = 0.}$$



Proof. Let R be the region bounded by the curve C.

Let $\quad f(z) = u(x, y) + iv(x, y)$, then

$$\oint_C f(z)\,dz = \oint_C (u + iv)(dx + idy)$$

$$= \oint_C (udx - vdy) + i\oint_C (vdx + udy) \qquad \text{...(1)}$$

Since $f'(z)$ is continuous, the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous in R. Hence by Green's Theorem, we have

$$\oint_C f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy \qquad \text{...(2)}$$

Now $f(z)$ being analytic at each point of the region R, by Cauchy-Riemann equations, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Thus, the two double integrals in (2) vanish.

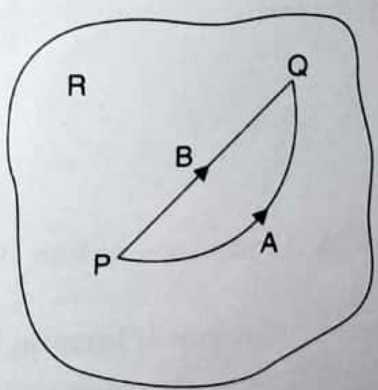
Hence $\qquad \oint_C f(z)\,dz = 0.$

However Cauchy with the help of Goursat developed the revised form of Cauchy's fundamental theorem which states that

"If $f(z)$ is analytic and one valued within and on a simple closed contour C then $\int_C f(z)\,dz = 0.$"

Goursat showed that for the truth of the original theorem, the assumption of continuity of $f'(z)$ is unnecessary and Cauchy's theorem holds if $f(z)$ is analytic within and on C.

Corollary. *If $f(z)$ is analytic in a region R and P, Q are two points in R, then $\int_P^Q f(z)\,dz$ is independent of the path joining P and Q and lying entirely in R.*



Let PAQ and PBQ be any two paths joining P and Q.

By Cauchy's theorem,

$$\int_{PAQBP} f(z)\,dz = 0$$

$$\Rightarrow \quad \int_{PAQ} f(z)\,dz + \int_{QBP} f(z)\,dz = 0$$

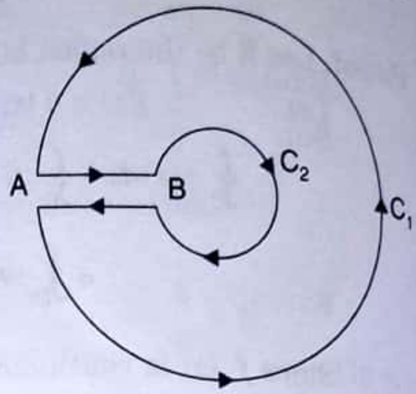
$$\Rightarrow \quad \int_{PAQ} f(z)\,dz - \int_{PBQ} f(z)\,dz = 0$$

Hence $$\int_{PAQ} f(z)\,dz = \int_{PBQ} f(z)\,dz.$$

5.5 EXTENSION OF CAUCHY'S THEOREM TO MULTIPLY CONNECTED REGION

If $f(z)$ is analytic in the region R between two simple closed curves C_1 and C_2, then

$$\boxed{\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz}$$



when integral along each curve is taken in anti-clockwise direction.

Proof. $\int f(z)dz = 0$

where the path of integration is along AB and curve C_2 in clockwise direction and along BA and along C_1 in anti-clockwise direction.

$$\int_{AB} f(z)dz + \int_{C_2} f(z)dz + \int_{BA} f(z)dz + \int_{C_1} f(z)dz = 0$$

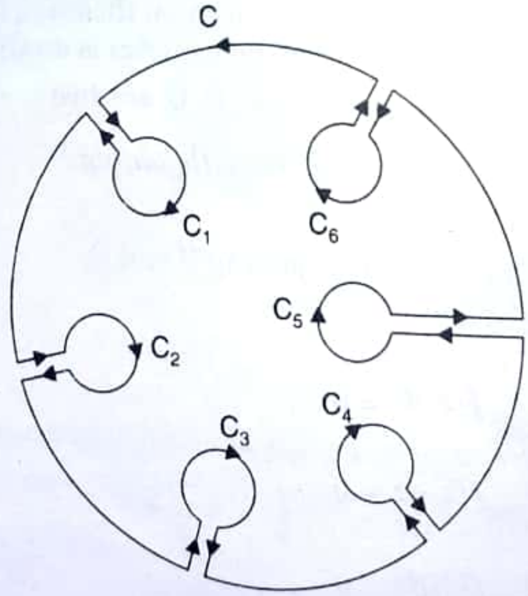
or $$\int_{C_2} f(z)dz + \int_{C_1} f(z)dz = 0 \qquad \left|\because \int_{AB} f(z)dz = -\int_{BA} f(z)dz\right.$$

Reversing the direction of the integral around C_2, we get

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

However, if a closed curve C contains non-intersecting closed curves $C_1, C_2, \ldots, C_n$, then by introducing cross-cuts, it can be shown that

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \cdots\cdots + \oint_{C_n} f(z)dz.$$



ILLUSTRATIVE EXAMPLES

Example 1. *Evaluate* $\oint_C (x^2 - y^2 + 2ixy)\,dz$, *where C is the contour* $|z| = 1$.

Sol. $f(z) = x^2 - y^2 + 2ixy = (x + iy)^2 = z^2$ is analytic everywhere within and on $|z| = 1$.

$\therefore$ By Cauchy's integral theorem, $\oint_C f(z)\,dz = 0$.

Example 2. *Evaluate* $\oint_C (3z^2 + 4z + 1)\,dz$ *where C is the arc of the cycloid* $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ *between* $(0, 0)$ *and* $(2\pi a, 0)$.

Sol. Here, $f(z) = 3z^2 + 4z + 1$ is analytic everywhere so that the integral is independent of the path of integration and depends only on the end points $z_1 = 0 + i0$ and $z_2 = 2\pi a + i0$.

$$\therefore \int_C (3z^2 + 4z + 1)\,dz = \int_0^{2\pi a} (3z^2 + 4z + 1)\,dz = \left[z^3 + 2z^2 + z\right]_0^{2\pi a} = 2\pi a\,(4\pi^2 a^2 + 4\pi a + 1).$$

Example 3. *Evaluate:* $\oint_C \dfrac{2z^2 + 5}{(z + 2)^3 (z^2 + 4)}\,dz$, *where C is the square with vertices at* $1 + i, 2 + i, 2 + 2i, 1 + 2i$.

Sol. Here, $$f(z) = \frac{2z^2 + 5}{(z + 2)^3 (z^2 + 4)}$$

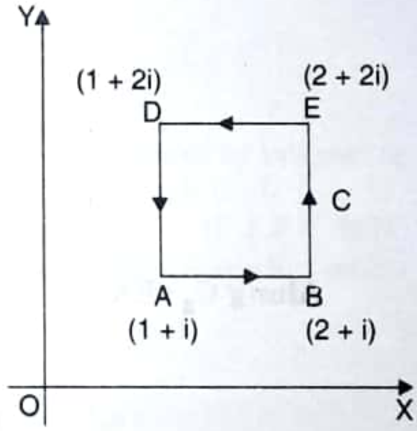
Singularities are given by

$(z + 2)^3 (z^2 + 4) = 0$

$z = -2$ (order 3), $\pm 2i$ (simple poles)

Since the singularities donot lie inside the contour C hence by Cauchy's integral theorem,

$$\oint_C \frac{2z^2 + 5}{(z + 2)^3 (z^2 + 4)}\,dz = 0.$$



Example 4. *Evaluate* $\oint_C (5z^4 - z^3 + 2)\,dz$ *around*

(i) *unit circle* $|z| = 1$

(ii) *square with vertices* $(0, 0), (1, 0), (1, 1), (1, 0)$

(iii) *curve consisting of the parabola* $y = x^2$ *from* $(0, 0)$ *to* $(1, 1)$ *and* $y^2 = x$ *from* $(1, 1)$ *to* $(0, 0)$.

Sol. $f(z) = 5z^4 - z^3 + 2$ is analytic everywhere. So, by Cauchy integral theorem,

$$\oint_C f(z)\,dz = 0$$

$\therefore$ For all given curves, $\oint_C (5z^4 - z^3 + 2)\,dz = 0$.

Example 5. *Verify Cauchy theorem by integrating* e^{iz} *along the boundary of the triangle with the vertices at the points* $1 + i$, $-1 + i$ *and* $-1 - i$. (G.B.T.U. 2012, 2013; M.T.U. 2012)

6. Evaluate the following integrals:

(i) $\oint_C \frac{z^3+z+1}{z^2-3z+2}\,dz$, where C is the ellipse $4x^2+9y^2=1$

(ii) $\oint_C \frac{z+4}{z^2+2z+5}\,dz$, where C is the circle $|z+1|=1$

(iii) $\oint_C \frac{z^2-z+1}{z-1}\,dz$, where C is the circle $|z|=\frac{1}{2}$.

7. Evaluate $I=\oint_C \frac{dz}{z-2}$ around a triangle with vertices at (0, 0), (1, 0) and (0, 1).

8. State and prove Cauchy's integral theorem. Hence evaluate $\int_C \frac{z^2+5z+6}{z-2}dz$ where C: $|z|=\frac{3}{2}$. [*M.T.U. 2014, G.B.T.U. (C.O.) 2011*]

9. Evaluate $\oint_C \frac{e^{3iz}}{(z+\pi)^3}\,dz$, where C is the circle $|z-\pi|=3.2$.

10. (i) Verify Cauchy's theorem for the function $f(z)=3z^2+iz-4$ along the perimeter of square with vertices $1\pm i, -1\pm i$. (*G.B.T.U. 2011*)

(ii) Verify Cauchy's theorem for the function $f(z)=4z^2+iz-3$ along the positively oriented square with vertices (1, 0), (– 1, 0), (0, 1) and (0, – 1). (*M.T.U. 2012*)

(iii) Verify Cauchy's theorem for $f(z)=z^2+3z+2$ where c is the perimeter of square with vertices $1\pm i, -1\pm i$. (*G.B.T.U. 2012*)

Answers

2. 0 in all cases	**3.** (i) 0	(ii) 0
5. (i) 0	(ii) 0	(iii) 0
6. (i) 0	(ii) 0	(iii) 0
7. 0	**8.** 0	**9.** 0.

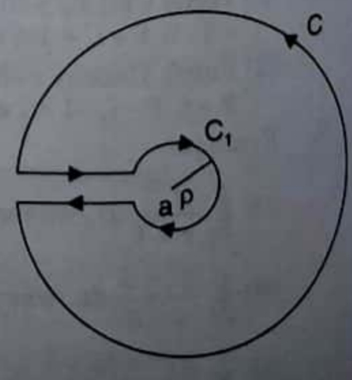
5.6 CAUCHY'S INTEGRAL FORMULA

(*M.T.U. 2012, U.P.T.U. 2014; G.B.T.U. 2011, 2013*)

Statement. *If f(z) is analytic within and on a closed curve C and a is any point within C, then*

$$\boxed{f(a)=\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz.}$$

Proof. Consider the function $\frac{f(z)}{z-a}$, which is analytic at every point within C except at $z=a$. Draw a circle C_1 with a as centre and radius ρ such that C_1 lies entirely inside C. Thus $\frac{f(z)}{z-a}$ is analytic in the region between C and C_1.



∴ By Cauchy's theorem, we have

$$\oint_C \frac{f(z)}{z-a}\,dz=\oint_{C_1} \frac{f(z)}{z-a}\,dz \quad ...(1)$$

Now, the equation of circle C_1 is

$$|z-a| = \rho \quad \text{or} \quad z - a = \rho e^{i\theta}$$

so that
$$dz = i\rho e^{i\theta}\, d\theta$$

$$\therefore \quad \oint_{C_1} \frac{f(z)}{z-a}\, dz = \int_0^{2\pi} \frac{f(a+\rho e^{i\theta})}{\rho e^{i\theta}} \cdot i\rho e^{i\theta}\, d\theta = i\int_0^{2\pi} f(a+\rho e^{i\theta})\, d\theta$$

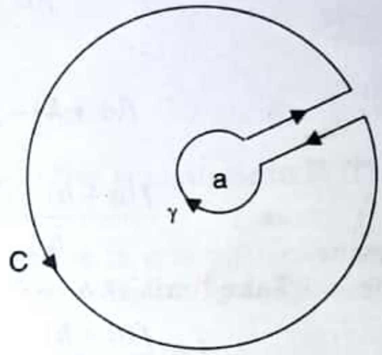
Hence by (1), we have $\oint_C \frac{f(z)}{z-a}\, dz = i\int_0^{2\pi} f(a+\rho e^{i\theta})\, d\theta$ ...(2)

In the limiting form, as the circle C_1 shrinks to the point a, *i.e.*, $\rho \to 0$, then from (2),

$$\oint_C \frac{f(z)}{z-a}\, dz = i\int_0^{2\pi} f(a)\, d\theta = if(a)\int_0^{2\pi} d\theta = 2\pi i f(a)$$

Hence
$$\boxed{f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\, dz}$$

Aliter: About the point $z = a$, describe a small circle γ of radius r lying entirely within C. Consider the function $\frac{f(z)}{z-a}$.



This function is analytic in the region between C and γ. Hence by Cauchy's theorem for multiply connected region, we have

$$\int_C \frac{f(z)}{z-a}\, dz = \int_\gamma \frac{f(z)}{z-a}\, dz$$

$$\Rightarrow \quad \int_C \frac{f(z)}{z-a}\, dz - \int_\gamma \frac{f(a)}{z-a}\, dz = \int_\gamma \frac{f(z)-f(a)}{z-a}\, dz$$

$$\Rightarrow \quad \int_C \frac{f(z)}{z-a}\, dz - f(a)\int_\gamma \frac{dz}{z-a} = \int_\gamma \frac{f(z)-f(a)}{z-a}\, dz$$

$$\Rightarrow \quad \int_C \frac{f(z)}{z-a}\, dz - 2\pi i f(a) = \int_\gamma \frac{f(z)-f(a)}{z-a}\, dz$$

$\because \int_\gamma \frac{dz}{z-a} = 2\pi i$ since $|z-a| = r$ on γ

$$\Rightarrow \quad \left|\int_C \frac{f(z)}{z-a}\, dz - 2\pi i f(a)\right| = \left|\int_\gamma \frac{f(z)-f(a)}{z-a}\, dz\right|$$

$$\leq \int_\gamma \frac{|f(z)-f(a)|}{|z-a|}\,|dz|$$

$$\leq \frac{\varepsilon}{r}\int_\gamma |dz|$$

$\because$ $f(z)$ is continuous at $z = a$
$\therefore$ $|f(z)-f(a)| < \varepsilon$
and $|z-a| = r$ for z on γ

$$\leq \frac{\varepsilon}{r} \cdot 2\pi r$$

$$\leq 2\pi\varepsilon \to 0 \text{ as } \varepsilon \to 0$$

$$\Rightarrow \quad \int_C \frac{f(z)}{z-a}\, dz - 2\pi i f(a) = 0$$

$$\Rightarrow \quad \boxed{f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\, dz}$$

5.7 CAUCHY'S INTEGRAL FORMULA FOR THE DERIVATIVE OF AN ANALYTIC FUNCTION

[U.P.T.U. (C.O.) 2010]

If a function f(z) is analytic in a region D, then its derivative at any point z = a of D is also analytic in D and is given by

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz$$

where C is any closed contour in D surrounding the point z = a.

Proof. Let $a + h$ be a point in the neighbourhood of the point a. Then by Cauchy's Integral Formula

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$

$$f(a+h) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a-h}\,dz$$

$$\therefore \quad f(a+h) - f(a) = \frac{1}{2\pi i}\int_C \left\{\frac{1}{z-a-h} - \frac{1}{z-a}\right\} f(z)\,dz = \frac{h}{2\pi i}\int_C \frac{f(z)dz}{(z-a-h)(z-a)}$$

$$\Rightarrow \quad \frac{f(a+h)-f(a)}{h} = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-a-h)(z-a)}$$

Take limit as $h \to 0$

$$\underset{h\to 0}{\text{Lt}} \frac{f(a+h)-f(a)}{h} = \underset{h\to 0}{\text{Lt}} \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-a-h)(z-a)}$$

$$\Rightarrow \quad \boxed{f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz} \qquad \text{...(1)}$$

Since a is any point of the region D, so by (1) it is clear that $f'(a)$ is analytic in D. Thus, *the derivative of an analytic function is also analytic.*

5.8 THEOREM

(A.K.T.U. 2016)

If a function f(z) is analytic in a domain D, then at any point z = a of D, f(z) has derivatives of all orders, all of which are again analytic functions in D, their values are given by

$$\boxed{f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz}$$

where C is any closed contour in D surrounding the point z = a.

Proof. We shall prove this theorem by Mathematical Induction.

Let the theorem be true for $n = m$. Then

$$f^m(a) = \frac{m!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{m+1}}\,dz \text{ is true.}$$

$$\Rightarrow \quad \frac{f^m(a+h)-f^m(a)}{h} = \frac{m!}{2\pi i}\,\frac{1}{h}\left[\int_C \frac{f(z)dz}{(z-a-h)^{m+1}} - \int_C \frac{f(z)dz}{(z-a)^{m+1}}\right\}$$

$$= \frac{m!}{2\pi i}\cdot\frac{1}{h}\cdot\int_C \frac{1}{(z-a)^{m+1}}\left\{\left(1-\frac{h}{z-a}\right)^{-(m+1)} - 1\right\} f(z)\,dz$$

$$= \frac{m!}{2\pi i}\cdot\frac{1}{h}\cdot\int_C \frac{1}{(z-a)^{m+1}}\left\{(m+1)\frac{h}{z-a} + \frac{(m+1)(m+2)}{2!}\frac{h^2}{(z-a)^2} + \cdots\right\} f(z)\,dz$$

Take limit as $h \to 0$

$$\lim_{h\to 0}\frac{f^m(a+h) - f^m(a)}{h} = \frac{(m+1)!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{m+2}}\,dz$$

$$\Rightarrow \qquad f^{m+1}(a) = \frac{(m+1)!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{m+2}}\,dz$$

Hence the theorem is true for $n = m + 1$ if the theorem is true for $n = m$. But we know by Cauchy's Integral formula for the derivative of a function that the theorem is true for $n = 1$. Hence the theorem must be true for $n = 2, 3, 4,$ ...... and so on *i.e.*, for all +ve integral values of n. Thus,

$$\boxed{f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz} \qquad ...(1)$$

Since a is any point of the region D, so by (1) it is clear that $f^n(a)$ is analytic in D. Thus the derivatives of $f(z)$ of all orders are analytic if $f(z)$ is analytic.

Thus, *if a function of a complex variable has a first derivative in a simply connected region, all its higher derivatives exist in that region. This property is not exhibited by the functions of real variables.*

5.9 CAUCHY'S INEQUALITY

If $f(z)$ is analytic within a circle C given by $|z-a| = R$ and if $|f(z)| \le M$ on C, then

$$|f^n(a)| \le \frac{Mn!}{R^n}.$$

Proof.

$$f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^{n+1}}$$

$$\Rightarrow \qquad |f^n(a)| = \left|\frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^{n+1}}\right|$$

$$\le \frac{n!}{|2\pi i|}\int_C \frac{|f(z)||dz|}{|(z-a)^{n+1}|}$$

$$\le \frac{n!}{2\pi}\frac{M}{R^{n+1}}\int_0^{2\pi} R\,d\theta$$

$$\le \frac{n!}{2\pi}\frac{M}{R^{n+1}}\,2\pi R \le \frac{Mn!}{R^n}.$$

$\because \quad z - a = R\,e^{i\theta}$

$\therefore \quad dz = i\,R\,e^{i\theta}\,d\theta$

$\therefore \quad |dz| = |i\,R\,e^{i\theta}\,d\theta| = R\,d\theta$

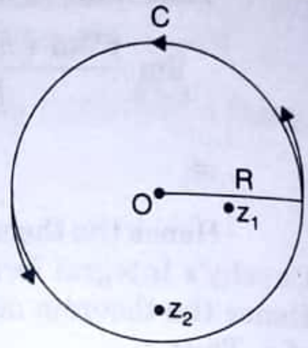
5.10 LIOUVILLE'S THEOREM

If a function $f(z)$ is analytic for all finite values of z and is bounded, then it is a constant.

Proof. Since $f(z)$ is bounded, so $|f(z)| \le M$ where M is a positive constant.

Let z_1, z_2 be any two points of the z-plane. Take the contour C to be a large circle with its centre at the origin and radius R enclosing the points z_1 and z_2 so that $R > |z_1|$ and $R > |z_2|$.

By Cauchy's Integral formula, we have



$$f(z_1) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_1} dz$$

$$f(z_2) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_2} dz$$

$$\Rightarrow \quad f(z_1) - f(z_2) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_1} - \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_2} dz$$

$$= \frac{1}{2\pi i}\int_C \frac{(z_1 - z_2) f(z)}{(z - z_1)(z - z_2)} dz$$

$$\Rightarrow \quad |f(z_1) - f(z_2)| = \left| \frac{1}{2\pi i}\int_C \frac{(z_1 - z_2) f(z) dz}{(z - z_1)(z - z_2)} \right|$$

$$\le \left| \frac{1}{2\pi i} \right| \int_C \frac{|z_1 - z_2| |f(z)| |dz|}{|z - z_1| |z - z_2|}$$

$$\le \frac{1}{2\pi} |z_1 - z_2| M \int_C \frac{|dz|}{\{|z| - |z_1|\}\{|z| - |z_2|\}} \qquad \Big| \because \ |f(z)| \le M$$

$$\le \frac{1}{2\pi} |z_1 - z_2| M \int_C \frac{|dz|}{\{R - |z_1|\}\{R - |z_2|\}} \qquad \Big| \because \ |z| = R$$

$$\le \frac{|z_1 - z_2| M}{2\pi \{R - |z_1|\}\{R - |z_2|\}} \int_0^{2\pi} R d\theta \qquad \because \ z = Re^{i\theta}$$

$$\therefore \ dz = R ie^{i\theta} d\theta$$

$$\therefore \ |dz| = R\, d\theta$$

$$\le \frac{|z_1 - z_2| M\, 2\pi R}{2\pi \{R - |z_1|\}\{R - |z_2|\}}$$

$$\to 0 \text{ as } R \to \infty$$

Hence $\qquad f(z_1) = f(z_2)$

Since this holds for all values of z_1 and z_2, therefore $f(z)$ is constant.

Note. A function which is analytic in the whole of the z-plane is called an Integral (Entire) function.

5.11 ROUCHE'S THEOREM

If $f(z)$ and $g(z)$ are analytic within and on a closed curve C and $|g(z)| < |f(z)|$ on C then $f(z)$ and $f(z) + g(z)$ have same number of zeros inside C.

5.12 FUNDAMENTAL THEOREM OF ALGEBRA

Every polynomial of degree n has exactly n zeros.

Proof. Consider the polynomial

$$a_0 + a_1z + a_2z^2 + \ldots\ldots + a_nz^n \ (a_n \neq 0)$$

Let $\qquad f(z) = a_nz^n$

and $\qquad g(z) = a_0 + a_1z + a_2z^2 + \ldots\ldots + a_{n-1}z^{n-1}$.

Let C be a circle with centre at the origin and radius R > 1 then on C, we have

$$\frac{|g(z)|}{|f(z)|} = \frac{|a_0 + a_1z + a_2z^2 + \ldots\ldots + a_{n-1}z^{n-1}|}{|a_nz^n|}$$

$$\leq \frac{|a_0|\mathrm{R}^{n-1} + |a_1|\mathrm{R}^{n-1} + |a_2|\mathrm{R}^{n-1} + \ldots\ldots + |a_{n-1}|\mathrm{R}^{n-1}}{|a_n|\mathrm{R}^n}$$

$$= \frac{|a_0| + |a_1| + |a_2| + \ldots\ldots + |a_{n-1}|}{|a_n|\mathrm{R}}$$

Since R is arbitrary, choose R such that

$$\mathrm{R} > \frac{|a_0| + |a_1| + |a_2| + \ldots\ldots + |a_{n-1}|}{|a_n|}$$

which makes $\dfrac{|g(z)|}{|f(z)|} < 1$ on C.

Hence by Rouche's theorem, the given polynomial $g(z) + f(z)$ has the same number of zeros as $f(z)$. But $f(z)$ *i.e.*, a_nz^n has clearly n zeros located at $z = 0$ therefore $g(z) + f(z)$ also has n zeros.

ILLUSTRATIVE EXAMPLES

Example 1. *Evaluate* $\oint_C \frac{e^{-z}}{z+1}dz$, *where C is the circle* $|z| = 2$.

Sol. $f(z) = e^{-z}$ is an analytic function.

The point $a = -1$ lies inside the circle $|z| = 2$.

∴ By Cauchy's integral formula,

$$\oint_C \frac{e^{-z}}{z+1}dz = 2\pi i(e^{-z})_{z=-1} = 2\pi ie.$$

Example 2. *Evaluate the following integral:*

$$\int_C \frac{1}{z}\cos z\,dz$$

where C is the ellipse $9x^2 + 4y^2 = 1$.

Sol. Pole is given by $z = 0$. The given ellipse encloses the simple pole.

$\therefore$ By Cauchy's integral formula,

$$\int_C \frac{\cos z}{z} dz = 2\pi i (\cos z)_{z=0} = 2\pi i.$$

Example 3. (*i*) *Use Cauchy Integral formula to evaluate*

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$$

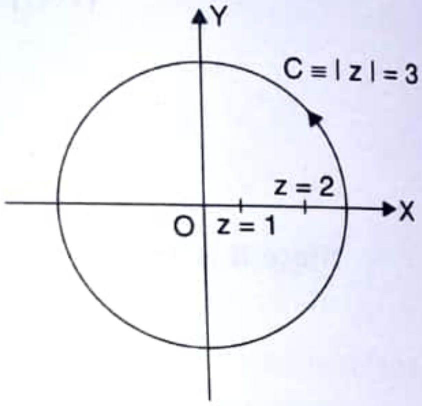
where C is the circle $|z| = 3$. [G.B.T.U. 2010; G.B.T.U. (C.O.) 2011]

(*ii*) *Evaluate:* $\int_C \frac{\sin \pi z + \cos \pi z}{(z-1)(z-2)} dz$, *where C is the circle* $|z| = 4$.

Sol. (*i*) The integrand has singularities given by

$$(z-1)(z-2) = 0 \quad \Rightarrow \quad z = 1, 2$$

The given circle $|z| = 3$ with centre at $z = 0$ and radius 3 encloses both the singularities.



$$\therefore \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = \int_{C_1} \frac{\left(\dfrac{\sin \pi z^2 + \cos \pi z^2}{z-2}\right)}{z-1} dz + \int_{C_2} \frac{\left(\dfrac{\sin \pi z^2 + \cos \pi z^2}{z-1}\right)}{z-2} dz$$

$$= 2\pi i \left[\frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\right]_{z=1} + 2\pi i \left[\frac{\sin \pi z^2 + \cos \pi z^2}{z-1}\right]_{z=2}$$

$$= 2\pi i \left(\frac{0-1}{-1}\right) + 2\pi i \left(\frac{0+1}{1}\right) = 2\pi i + 2\pi i = 4\pi i.$$

(*ii*) Singularities are given by

$$(z-1)(z-2) = 0 \quad \Rightarrow \quad z = 1, 2$$

The given circle $|z| = 4$ with centre at $z = 0$ and radius 4 encloses both the singularities.

$$\therefore \int_C \frac{\sin \pi z + \cos \pi z}{(z-1)(z-2)} dz = \int_{C_1} \frac{\left(\dfrac{\sin \pi z + \cos \pi z}{z-2}\right)}{z-1} dz + \int_{C_2} \frac{\left(\dfrac{\sin \pi z + \cos \pi z}{z-1}\right)}{z-2} dz$$

$$= 2\pi i \left[\frac{\sin \pi z + \cos \pi z}{z-2}\right]_{z=1} + 2\pi i \left[\frac{\sin \pi z + \cos \pi z}{z-1}\right]_{z=2}$$

$$= 2\pi i \left[\frac{-1}{-1}\right] + 2\pi i \left[\frac{1}{1}\right] = 2\pi i + 2\pi i = 4\pi i.$$

Example 4. (*i*) *Evaluate the following integral using Cauchy Integral formula*

$$\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz, \quad \textit{where C is the circle } |z| = 3/2.$$

(U.P.T.U. 2015)

(*ii*) *Use Cauchy-integral formula to evaluate*

$$\int_C \frac{z}{z^2-3z+2}\,dz, \quad \textit{where C is the circle } |z-2| = \frac{1}{2}.$$

Sol. (*i*) Poles of the integrand are $z = 0, 1, 2$. These are simple poles.

Given circle $|z| = \frac{3}{2}$ with centre at $z = 0$ and radius $\frac{3}{2}$ encloses two poles $z = 0$ and $z = 1$.

$$\therefore \quad \int_C \frac{4-3z}{z(z-1)(z-2)}\,dz = \int_{C_1} \frac{\frac{4-3z}{(z-1)(z-2)}}{z}\,dz + \int_{C_2} \frac{\frac{4-3z}{z(z-2)}}{(z-1)}\,dz$$

$$= 2\pi i \left[\frac{4-3z}{(z-1)(z-2)}\right]_{z=0} + 2\pi i \left[\frac{4-3z}{z(z-2)}\right]_{z=1} = 2\pi i.$$

(*ii*) Poles of the integrand are given by

$$z^2 - 3z + 2 = 0 \quad \Rightarrow \quad z = 1, 2$$

Both are simple poles. The given circle $|z-2| = \frac{1}{2}$ with centre at $z = 2$ and radius $\frac{1}{2}$ encloses only one of the poles at $z = 2$.

$\therefore$ By Cauchy's integral formula,

$$\int_C \frac{z}{z^2-3z+2}\,dz = \int_C \frac{\left(\frac{z}{z-1}\right)}{z-2}\,dz = 2\pi i \left[\frac{z}{z-1}\right]_{z=2} = 2\pi i \left(\frac{2}{1}\right) = 4\pi i.$$

Example 5. *Evaluate by Cauchy's integral formula*

$$\int_C \frac{dz}{z(z+\pi i)}, \quad \textit{where C is } |z+3i| = 1.$$

Sol. Poles of the integrand are $z = 0, -\pi i$ (simple poles)

The given curve C is a circle with centre at $z = -3i$, *i.e.*, at $(0, -3)$ and radius 1.

Clearly, only the pole $z = -\pi i$ lies inside the circle.

$$\therefore \quad \int_C \frac{dz}{z(z+\pi i)} = \int_C \frac{\left(\frac{1}{z}\right)}{z+\pi i}\,dz$$

$$= 2\pi i \left(\frac{1}{z}\right)_{z=-\pi i} \qquad | \text{ By Cauchy's Integral formula}$$

$$= \frac{2\pi i}{-\pi i} = -2$$

Example 6. *Evaluate* $\oint_C \frac{z^2+1}{z^2-1}\, dz$ *where C is circle,*

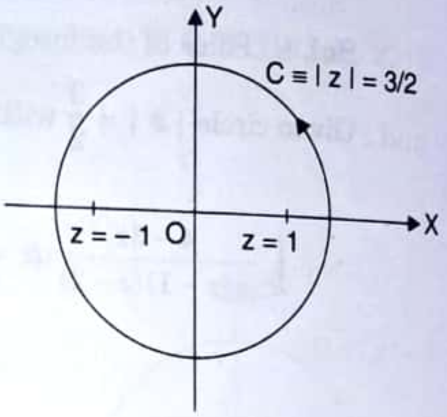
(*i*) $|z| = 3/2$ (A.K.T.U. 2016) (*ii*) $|z-1| = 1$ (U.P.T.U. 2014) (*iii*) $|z| = 1/2$. (U.P.T.U. 2014)

Sol. The integrand has singularities given by

$$z^2 - 1 = 0 \Rightarrow z = \pm 1.$$

(*i*) The given curve C is a circle with centre at origin (0, 0) and radius 3/2.

Both the singularities $z = 1$ and $z = -1$ lie inside the circle $|z| = 3/2$.



$$\therefore \quad \oint_C \frac{z^2+1}{z^2-1}\, dz = \oint_{C_1} \frac{\left(\frac{z^2+1}{z+1}\right)}{z-1}\, dz + \oint_{C_2} \frac{\left(\frac{z^2+1}{z-1}\right)}{z+1}\, dz$$

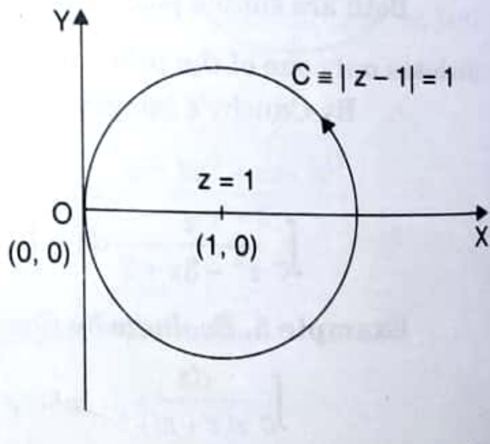
$$= 2\pi i \left(\frac{z^2+1}{z+1}\right)_{z=1} + 2\pi i \left(\frac{z^2+1}{z-1}\right)_{z=-1}$$

| By Cauchy's Integral formula

$$= 2\pi i\,(1) + 2\pi i\,(-1) = 0$$

(*ii*) The given curve C is a circle with centre at (1, 0) and radius 1.

Only the singularity $z = 1$ lie inside the given circle $|z-1| = 1$.



$$\therefore \quad \oint_C \frac{z^2+1}{z^2-1}\, dz = \oint_C \frac{\left(\frac{z^2+1}{z+1}\right)}{z-1}\, dz$$

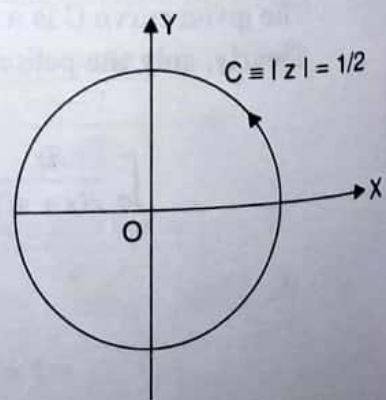
$$= 2\pi i \left(\frac{z^2+1}{z+1}\right)_{z=1} = 2\pi i$$

| By Cauchy's Integral formula

(*iii*) The given curve C is a circle with centre at origin (0, 0) and radius $\frac{1}{2}$. Clearly both the singularities $z = 1$ and $z = -1$ lie outside the given circle $|z| = \frac{1}{2}$.

Hence, by Cauchy's Integral theorem

$$\oint_C \frac{z^2+1}{z^2-1}\, dz = 0.$$



Example 7. (*i*) *Use Cauchy's integral formula to show that*

$$\int_C \frac{e^{zt}}{z^2+1}\,dz = 2\pi i \sin t \text{ if } t > 0 \text{ and C is the circle } |z| = 3.$$

(*ii*) *Evaluate the following complex integration using Cauchy's integral formula*

$$\int_C \frac{3z^2+z+1}{(z^2-1)(z+3)}\,dz \quad \text{where C is the circle } |z| = 2.$$

Sol. (*i*) Singularities of the integrand are given by

$$z^2 + 1 = 0 \quad \Rightarrow \quad z = \pm i \quad \text{(order 1)}$$

The circle $|z| = 3$ has centre at $z = 0$ and radius 3. It encloses both the singularities $z = i$ and $z = -i$.

Now, $$\int_C \frac{e^{zt}}{z^2+1}\,dz = \int_C \frac{e^{zt}}{(z-i)(z+i)}\,dz = \int_{C_1} \frac{\left(\frac{e^{zt}}{z+i}\right)}{z-i}\,dz + \int_{C_2} \frac{\left(\frac{e^{zt}}{z-i}\right)}{z+i}\,dz$$

$$= 2\pi i \left(\frac{e^{zt}}{z+i}\right)_{z=i} + 2\pi i \left(\frac{e^{zt}}{z-i}\right)_{z=-i} = \pi\,(e^{it} - e^{-it}) = 2\pi i \sin t$$

(*ii*) Poles of the integrand are given by

$$(z^2-1)(z+3) = 0 \quad \Rightarrow \quad z = 1, -1, -3 \text{ (simple poles)}$$

The circle $|z| = 2$ has centre at $z = 0$ and radius 2 clearly the poles $z = 1$ and $z = -1$ lie inside the given circle while the pole $z = -3$ lie outside it.

$$\therefore \quad \int_C \frac{3z^2+z+1}{(z^2-1)(z+3)}\,dz = \int_{C_1} \frac{\left\{\frac{3z^2+z+1}{(z+1)(z+3)}\right\}}{z-1}\,dz + \int_{C_2} \frac{\left\{\frac{3z^2+z+1}{(z-1)(z+3)}\right\}}{z+1}\,dz$$

$$= 2\pi i \left[\frac{3z^2+z+1}{(z+1)(z+3)}\right]_{z=1} + 2\pi i \left[\frac{3z^2+z+1}{(z-1)(z+3)}\right]_{z=-1}$$

| Using Cauchy's Integral formula

$$= 2\pi i \left(\frac{5}{8}\right) + 2\pi i \left(-\frac{3}{4}\right) = 2\pi i \left(\frac{-1}{8}\right) = -\frac{\pi i}{4}$$

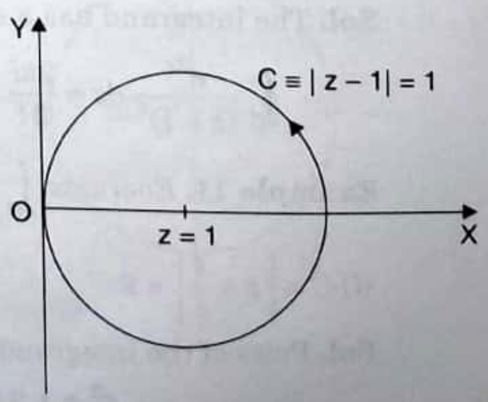
Example 8. *Integrate $(z^3 - 1)^{-2}$ the counterclockwise sense around the circle $|z - 1| = 1$.*

Sol. Singularities of integrand are given by

$$(z^3 - 1)^2 = 0$$

$$\Rightarrow \quad (z-1)^2 (z^2 + z + 1)^2 = 0$$

$$\Rightarrow \quad z = 1, \frac{-1 \pm i\sqrt{3}}{2}$$



Singularities are of second order.

The circle $|z - 1| = 1$ has centre at $z = 1$ and radius 1. Clearly, only $z = 1$ lies inside the circle $|z - 1| = 1$

Now, $$\int_C \frac{dz}{(z^3-1)^2} = \int_C \frac{\left\{\dfrac{1}{(z^2+z+1)^2}\right\}}{(z-1)^2}\,dz$$

$$= \frac{2\pi i}{1!}\left[\frac{d}{dz}\left\{\frac{1}{(z^2+z+1)^2}\right\}\right]_{z=1}$$ | Using Cauchy's Integral formula for derivatives

$$= 2\pi i\left[\frac{-2(2z+1)}{(z^2+z+1)^3}\right]_{z=1} = -4\pi i\left(\frac{3}{27}\right) = -\frac{4\pi i}{9}$$

Example 9. *Evaluate:* $\int_C \frac{e^z}{(z^2+\pi^2)^2}\,dz$, *where C is* $|z| = 4$.

Sol. Singularities of the integrand are given by

$$(z^2+\pi^2)^2 = 0 \Rightarrow z = \pm \pi i \qquad \text{(order 2)}$$

The given curve C is a circle with centre at origin and radius 4. The circle encloses both the singularities.

$$\therefore \quad \oint_C \frac{e^z}{(z^2+\pi^2)^2}\,dz = \oint_{C_1} \frac{\left\{\dfrac{e^z}{(z+\pi i)^2}\right\}}{(z-\pi i)^2}\,dz + \int_{C_2} \frac{\left\{\dfrac{e^z}{(z-\pi i)^2}\right\}}{(z+\pi i)^2}\,dz$$

$$= 2\pi i\left[\frac{d}{dz}\left\{\frac{e^z}{(z+\pi i)^2}\right\}\right]_{z=\pi i} + 2\pi i\left[\frac{d}{dz}\left\{\frac{e^z}{(z-\pi i)^2}\right\}\right]_{z=-\pi i}$$

| By C–I formula for derivatives

$$= 2\pi i\left[\frac{e^z(z+\pi i-2)}{(z+\pi i)^3}\right]_{z=\pi i} + 2\pi i\left[\frac{e^z(z-\pi i-2)}{(z-\pi i)^3}\right]_{z=-\pi i}$$

$$= \left(\frac{\pi i-1}{2\pi^2}\right) + \left(\frac{\pi i+1}{2\pi^2}\right) = \frac{i}{\pi}.$$

Example 10. *Use Cauchy's integral formula to evaluate*

$$\oint_C \frac{e^{2z}}{(z+1)^4}\,dz \text{ where C is the circle } |z| = 3.$$ (A.K.T.U. 2017)

Sol. The integrand has a singularity at $z = -1$ which lies within the circle $|z| = 3$.

$$\oint_C \frac{e^{2z}}{(z+1)^4}\,dz = \frac{2\pi i}{3!}\left\{\frac{d^3}{dz^3}(e^{2z})\right\}_{z=-1} = \frac{\pi i}{3}(8e^{2z})_{z=-1} = \frac{8\pi i}{3e^2}.$$

Example 11. *Evaluate* $\int_C \frac{z}{z^2+1}\,dz$, *where*

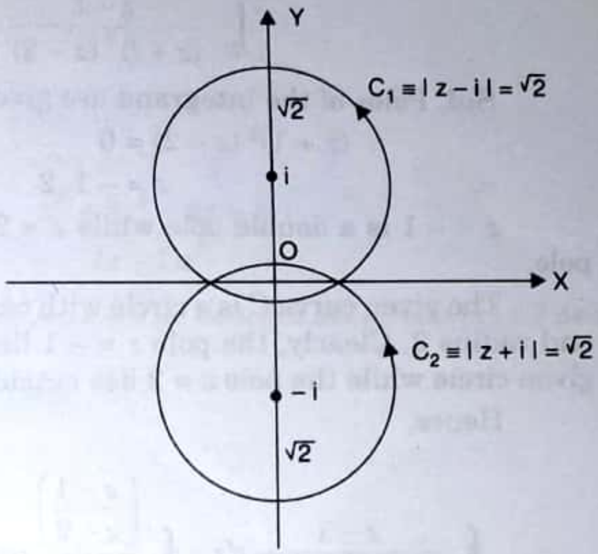
(*i*) $C \equiv \left|z+\frac{1}{z}\right| = 2$ (*ii*) $C \equiv |z+i| = 1$.

Sol. Poles of the integrand are given by

$$z^2+1 = 0 \Rightarrow z = \pm i$$

Integrand has two simple poles $z = i$ and $z = -i$

(*i*) The given curve is



$$\left| z + \frac{1}{z} \right| = 2$$

$$\Rightarrow \quad \left| x + iy + \frac{1}{x + iy} \right| = 2$$

$$\Rightarrow \quad \left| \frac{x^2 - y^2 + 2ixy + 1}{x + iy} \right| = 2$$

$$\Rightarrow \quad (x^2 - y^2 + 1)^2 + 4x^2y^2 = 4x^2 + 4y^2$$

$$\Rightarrow \quad x^4 + y^4 - 2x^2y^2 + 1 + 2x^2 - 2y^2 + 4x^2y^2 = 4x^2 + 4y^2$$

$$\Rightarrow \quad (x^2 + y^2)^2 - 2(x^2 + y^2) + 1 = 4y^2$$

$$\Rightarrow \quad x^2 + y^2 - 1 = \pm 2y$$

$$\Rightarrow \quad x^2 + (y \pm 1)^2 = 2$$

Above eqn. represents two circles with centres (0, 1), (0, – 1) and radius $\sqrt{2}$.

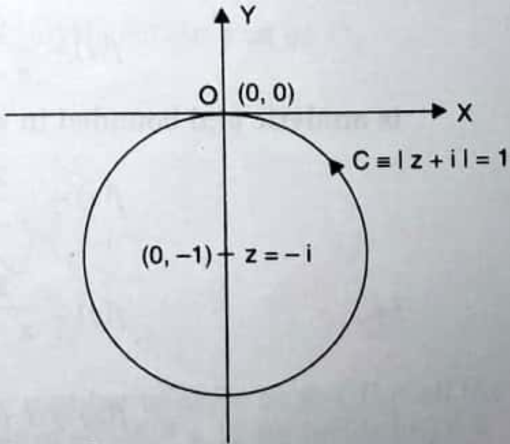
$$\int_C \frac{z}{z^2 + 1} dz = \int_{C_1} \frac{z}{z^2 + 1} dz + \int_{C_2} \frac{z}{z^2 + 1} dz$$

$$= \int_{C_1} \frac{\left(\frac{z}{z + i} \right)}{z - i} dz + \int_{C_2} \frac{\left(\frac{z}{z - i} \right)}{z + i} dz$$

$$= 2\pi i \left(\frac{z}{z + i} \right)_{z = i} + 2\pi i \left(\frac{z}{z + i} \right)_{z = -i}$$

$$= 2\pi i \left(\frac{1}{2} \right) + 2\pi i \left(\frac{1}{2} \right) = 2\pi i.$$

(*ii*) The given curve $|z + i| = 1$ is a circle with centre at $z = -i$ and radius 1. Clearly only the pole $z = -i$ lies inside the circle $|z + i| = 1$



$$\int_C \frac{z}{z^2 + 1} dz = \int_C \frac{\left(\frac{z}{z - i} \right)}{z + i} dz$$

$$= 2\pi i \left(\frac{z}{z - i} \right)_{z = -i}$$

$= \pi i$ | By Cauchy Integral formula

Example 12. *Evaluate by using Cauchy Integral formula*

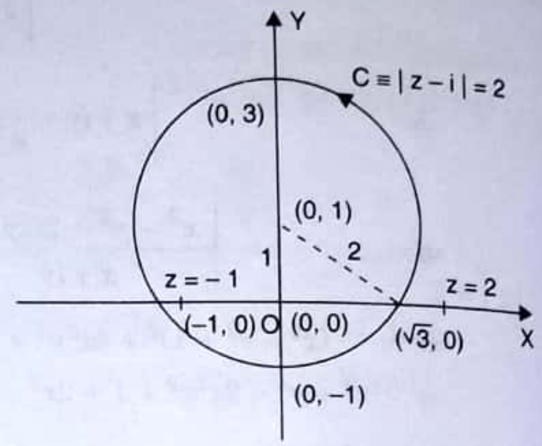
$$\int_C \frac{z-1}{(z+1)^2(z-2)}\,dz, \quad \text{where } C \text{ is } |z-i| = 2.$$

Sol. Poles of the integrand are given by

$$(z+1)^2(z-2) = 0$$

$$\Rightarrow \qquad z = -1, 2$$

$z = -1$ is a double pole while $z = 2$ is a simple pole.



The given curve C is a circle with centre at (0, 1) and radius 2. Clearly, the pole $z = -1$ lies inside the given circle while the pole $z = 2$ lies outside it.

Hence,

$$\oint_C \frac{z-1}{(z+1)^2(z-2)}\,dz = \oint_C \frac{\left(\frac{z-1}{z-2}\right)}{(z+1)^2}\,dz$$

$$= \frac{2\pi i}{1!}\left\{\frac{d}{dz}\left(\frac{z-1}{z-2}\right)\right\}_{z=-1}$$

$$= 2\pi i\left\{\frac{-1}{(z-2)^2}\right\}_{z=-1} = -\frac{2\pi i}{9}.$$

Example 13. *The function f(z) has a double pole at z = 0 with residue 2, a simple pole at z = 1 with residue 2 is analytic at all other finite points of the plane and is bounded as* $|z| \to \infty$. *If f(2) = 5 and f(−1) = 2, find f(z).*

Sol. Given that $f(z)$ has a simple pole at $z = 1$ and residue at it is 2. Also $f(z)$ has a double pole at $z = 0$ and residue at it is 2. Hence the principal part of $f(z)$ is $\frac{2}{z-1} + \left(\frac{2}{z} + \frac{b}{z^2}\right)$.

Since $f(z)$ is analytic at all other points except $z = 1$ and $z = 0$ and $|f(z)|$ is bounded therefore,

$$f(z) - \frac{2}{z-1} - \frac{2}{z} - \frac{b}{z^2}.$$

is analytic and bounded in whole z-plane. Hence by Liouville's theorem,

$$f(z) - \frac{2}{z-1} - \frac{2}{z} - \frac{b}{z^2} \text{ is constant}$$

i.e.,

$$f(z) - \frac{2}{z-1} - \frac{2}{z} - \frac{b}{z^2} = a \quad (a \text{ is constant})$$

$$\therefore \qquad f(z) = a + \frac{2}{z-1} + \frac{2}{z} + \frac{b}{z^2}$$

According to the problem,

$$f(2) = 5 = a + 2 + 1 + \frac{b}{4}$$

and $$f(-1) = 2 = a - 1 - 2 + b$$

On solving, we get, $a = 1, b = 4$

$$\therefore \quad f(z) = 1 + \frac{2}{z-1} + \frac{2}{z} + \frac{4}{z^2} = \frac{z^3 + 3z^2 + 2z - 4}{(z-1)z^2}.$$

Example 14. *Prove that all the roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.*

Sol. Let C_1 be the circle $|z| = 1$ and $C_2 = |z| = 2$.

The given polynomial is

$$z^7 - 5z^3 + 12 = 0$$

Consider the circle $|z| = 1$

Let $f(z) = 12$ and $g(z) = z^7 - 5z^3$

We observe that both $f(z)$ and $g(z)$ are analytic within and on C_1.

On C_1, we have

$$\frac{|g(z)|}{|f(z)|} = \frac{|z^7 - 5z^3|}{12} \leq \frac{|z|^7 + 5|z|^3}{12}$$

$$= \frac{1+5}{12} \qquad | \because \quad |z| = 1 \text{ on } C_1$$

$$= \frac{1}{2}$$

Thus, $$\frac{|g(z)|}{|f(z)|} < 1 \text{ on } C_1.$$

Hence by Rouche's theorem, $f(z) + g(z)$ has the same number of zeros inside C_1 as $f(z)$. But $f(z) = 12$ has no zeros hence $f(z) + g(z) \equiv z^7 - 5z^3 + 12$ has no zero inside C_1.

Now consider the circle $C_2 \equiv |z| = 2$.

Let $f(z) = z^7$ and $g(z) = 12 - 5z^3$.

Since both $f(z)$ and $g(z)$ are polynomials hence are analytic within and on C_2.

On C_2, we have $$\frac{|g(z)|}{|f(z)|} = \frac{|12 - 5z^3|}{|z^7|}$$

$$< \frac{|12| + 5|z|^3}{|z|^7}$$

$$= \frac{12 + 5\,.\,(2)^3}{(2)^7} = \frac{52}{128} < 1.$$

Hence by Rouche's theorem, $f(z) + g(z)$ has the same number of zeros as $f(z)$. But all the seven zeros of z^7 lie inside $|z| = 2$ hence all the seven zeros of $z^7 - 5z^3 + 12$ lie inside $|z| = 2$.

Hence all the roots of eqn. $z^7 - 5z^3 + 12 = 0$ lie inside $|z| = 2$ but outside $|z| = 1$ *i.e.*, all the roots of the given equation lie between $|z| = 1$ and $|z| = 2$.

Example 15. *Use Rouche theorem to show that the equation $z^5 + 15z + 1 = 0$ has one root in the disc $|z| < \frac{3}{2}$ and four roots in annulus $\frac{3}{2} < |z| < 2$.*

Sol. Let $\quad C_1 \equiv |z| = 2,\ C_2 \equiv |z| = \frac{3}{2}$

Consider $\quad C_1 \equiv |z| = 2$

Let $\quad f(z) = z^5 \quad$ and $\quad g(z) = 15z + 1$

We see that $f(z)$ and $g(z)$ being polynomials are analytic with and on C_2.

$$\text{Also,}\quad \frac{|g(z)|}{|f(z)|} = \frac{|15z+1|}{|z^5|} < \frac{15|z|+1}{|z|^5}$$

$$= \frac{15.2+1}{(2)^5} \text{ on circle } |z| = 2$$

$$= \frac{31}{32} < 1$$

showing that $\frac{|g(z)|}{|f(z)|} < 1$ on $|z| = 2$. Hence by Rouche's theorem, $f(z) + g(z)$ has all five zeros inside $|z| = 2$ as $f(z) = z^5$.

Now, consider $\quad C_2 \equiv |z| = \frac{3}{2}$

Let, $\quad f(z) = 15z \quad$ and $\quad g(z) = z^5 + 1$

We see that both $f(z)$ and $g(z)$ are analytic within and on C_2.

$$\text{Also,}\quad \frac{|g(z)|}{|f(z)|} = \frac{|z^5+1|}{|15z|}$$

$$< \frac{|z|^5+1}{15|z|} = \frac{\left(\frac{243}{32}\right)+1}{15\left(\frac{3}{2}\right)} = \frac{55}{144} < 1$$

showing that, $\quad |g(z)| < |f(z)|$

$\therefore$ By Rouche's theorem, $f(z) + g(z)$ has one zero inside $|z| = \frac{3}{2}$ as $f(z) = 15z$ has.

Consequently, remaining four of its zeros must lie in the annulus $\frac{3}{2} < |z| < 2$.

TEST YOUR KNOWLEDGE

1. Evaluate $\oint_C \frac{z^2+5}{z-3}\,dz$, where C is the circle $|z| = 4$.
2. Evaluate $\int_C \frac{e^z}{z^2+1}\,dz$ over the circular path $|z| = 2$.
3. Evaluate $\oint_C \frac{3z^2+7z+1}{z+1}\,dz$, where C is the circle $|z| = 1.5$.

4. Evaluate $\oint_C \frac{\cos z}{z-\pi}\, dz$, where C is the circle $|z-1| = 3$.

5. Evaluate the complex integration

(i) $\int_C \left\{\frac{\cos \pi z^2 + \sin \pi z^2}{(z+1)(z+2)}\right\} dz$ where C is the circle $|z| = 3$.

(ii) $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-3)}\, dz$ where C: $|z| = 2$ **(M.T.U. 2013)**

6. (i) Evaluate $\oint_C \frac{z\, dz}{(z-1)(z-3)}$, where C is the circle

(a) $|z| = 3$ (b) $|z| = 3/2$.

(ii) Evaluate $\oint_C \frac{e^z}{(z-1)(z-4)}\, dz$, where C is the circle $|z| = 2$.

(iii) Evaluate using Cauchy's integral formula:

$$\int_C \frac{e^{2z}}{(z-1)(z-2)}\, dz \quad \text{where C is the circle } |z| = 3.$$

(iv) State Cauchy's integral formula. Hence evaluate: **(G.B.T.U. 2011, 2012)**

$$\int_C \frac{\exp(i\pi z)}{(2z^2 - 5z + 2)}\, dz$$

where C is the unit circle with centre at origin and having positive orientation.

7. (i) Evaluate $\oint_C \frac{e^z}{z(z+1)}\, dz$, where C is the circle $|z| = \frac{1}{4}$.

(ii) Using Cauchy Integral formula, evaluate $\int_C \frac{dz}{z^2 - 1}$ where $C \equiv |z| = 2$.

(iii) Evaluate $\int_C \frac{2z+1}{z^2+z}\, dz$ where C is $|z| = \frac{1}{2}$.

8. Evaluate $\oint_C \frac{\cos \pi z}{z^2 - 1}\, dz$ around a rectangle with vertices

(a) $2 \pm i, -2 \pm i$ (b) $-i, 2-i, 2+i, i$.

9. Integrate $\frac{e^z}{z^2+1}$ around the contour C, where C is

(i) $|z - i| = 1$ (ii) $|z + i| = 1$

10. Show that $\oint_C \frac{e^z}{z}\, dz = 2\pi i$, $C \equiv |z| = 1$. Hence show that

$$\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta)\, d\theta = 2\pi \quad \text{and} \quad \int_0^{2\pi} e^{\cos\theta} \sin(\sin\theta)\, d\theta = 0$$

11. Evaluate $\oint_C \frac{dz}{z^2+9}$, where C is

(i) $|z - 3i| = 4$ (ii) $|z + 3i| = 2$ (iii) $|z| = 5$

12. Evaluate:

(i) $\int_C \frac{z+4}{z^2+2z+5}\, dz$; $C \equiv | z+1-i | = 2$ (ii) $\int_C \frac{z^3-6}{2z-i}\, dz$; $C \equiv | z | = 1$

(iii) $\int_C \frac{\tan z}{z^2-1}\, dz$; $C \equiv | z | = 3/2$ (iv) $\int_C \frac{2z^2+z}{z^2-1}\, dz$; $C \equiv | z-1 | = 1$.

13. Evaluate by Cauchy-Integral formula: $\oint_C \frac{z^2+1}{z^2-1}\, dz$, where C is

(i) $| z-1 | = 1$ (ii) $| z+1 | = 1$ (iii) $| z-i | = 1$.

14. Evaluate the following integrals:

(i) $\oint_C \frac{\cos 2\pi z}{(2z-1)(z-3)}\, dz$; $C \equiv | z | = 1$ (ii) $\oint_C \frac{z^4-3z^2+6}{(z+i)^3}\, dz$; $C \equiv | z | = 2$

(iii) $\oint_C \frac{\cosh z}{z^4}\, dz$; $C \equiv | z | = 1/2$

15. Evaluate $\oint_C \frac{\sin^2 z}{\left(z-\frac{\pi}{6}\right)^3}\, dz$, where C is the circle $| z | = 1$.

16. (i) Evaluate $\oint_C \frac{e^{-2z}\, dz}{(z+1)^3}$, where C is the circle $| z | = 2$.

(ii) Evaluate the integral $\int \frac{e^{2z}}{(z+1)^5}\, dz$ around the boundary of the circle $|z| = 2$.

(U.P.T.U. 2015)

17. (i) Evaluate: $\oint_C \frac{e^{3z}}{(z-\log 2)^4}\, dz$, where C is the square with vertices at $\pm 1 \pm i$.

(ii) Evaluate: $\oint_C \frac{dz}{z^2(z^2-4)e^z}$, where $C \equiv | z | = 1$. (G.B.T.U. 2013)

18. Evaluate $\oint_C \frac{e^z}{z(1-z)^3}\, dz$, where C is

(i) $| z | = \frac{1}{2}$ (ii) $| z-1 | = \frac{1}{2}$ (iii) $| z | = 2$

19. Integrate $\frac{\sin 2z}{(z+3)(z+1)^2}$ around the contour C, where C is a rectangle with vertices at $3 \pm i, -2 \pm i$.

20. Evaluate: $\oint_C \frac{z^3-z}{(z-2)^3}\, dz$, where C is

(i) $| z | = 3$ (ii) $| z-2 | = 1$ (iii) $| z | = 1$

21. Using Cauchy-integral formula, evaluate:

(i) $\oint_C \frac{\cos z}{(z-\pi i)^2}\, dz$; $C \equiv | z | = 5$ (ii) $\oint_C \frac{e^z}{z^3}\, dz$; $C \equiv | z | = 1$

(iii) $\oint_C \frac{e^z}{(z-1)^2(z^2+4)}\, dz$; $C \equiv | z-1 | = \frac{1}{2}$

(iv) $\oint_C \frac{e^{zt}}{(z^2+1)^2}\, dz$; $C \equiv | z | = 3$, $t > 0$.

22. Evaluate $\int_C \frac{\sin z}{z^2 - iz + 2}\, dz$, where C is

(i) $|z+2| = 2$

(ii) A rectangle with vertices at $(1, 0), (1, 3), (-1, 3)$ and $(-1, 0)$

(iii) A rectangle with vertices at $(2, 0), (2, 3), (-2, 3)$ and $(-2, -3)$.

23. Evaluate the integrals

(i) $\oint_C \frac{e^z + \sin \pi z}{(z-1)(z-3)^2(z+4)}\, dz, \quad C \equiv |z| = 2$ (ii) $\oint_C \frac{z+1}{z^2 - 9}\, dz; \quad C \equiv |z+3| = 1$

24. Show that

(i) $\oint_C \frac{dz}{(z^2+4)^2} = \frac{\pi}{16}; \; C \equiv |z - i| = 2$ (A.K.T.U. 2018)

(ii) $\oint_C \frac{e^z}{z^2(z+1)^3}\, dz = \left(\frac{11}{e} - 4\right)\pi i; \; C \equiv |z| = 2$

(iii) $\oint_C \frac{dz}{(z^2+4)^3} = 0; \; C \equiv |z-1| = 4$

25. Evaluate by Cauchy integral formula

$$\oint_C \frac{z^2 - 2z}{(z+1)^2 (z^2+4)}\, dz$$

where C is the circle $|z| = 3$. (A.K.T.U. 2016)

26. Let $P(z) = a + bz + cz^2$ and $\oint \frac{P(z)}{z}\, dz = \oint_C \frac{P(z)}{z^2}\, dz = \oint_C \frac{P(z)}{z^3}\, dz = 2\pi i$ where C is the circle $|z| = 1$.

Evaluate $P(z)$.

27. If $f(\xi) = \int_C \frac{3z^2 + 7z + 1}{z - \xi}\, dz$, where C is the circle $x^2 + y^2 = 4$, find the values of $f(3)$, $f'(1-i)$ and $f''(1-i)$.

28. (i) Evaluate: $\int_C \frac{(1+z)\sin z}{(2z-3)^2}\, dz$, where $C \equiv |z - i| = 2$ counter-clockwise. (U.P.T.U. 2014)

(ii) Evaluate $\oint_C \frac{z}{(z^2 - 6z + 25)^2}\, dz$ by Cauchy integral formula, where C is $|z - 3 - 4i| = 4$.

29. The only singularities of a single valued function $f(z)$ are poles of order 1 and 2 at $z = -1$ and $z = -2$ with residues at these poles 1 and 2 respectively. If $f(0) = \frac{7}{4}$ and $f(1) = \frac{5}{2}$, determine the function.

30. The only singularities of a single valued function $f(z)$ are poles of order 2 and 1 at $z = 1$ and $z = 2$ with residues of these poles 1 and 3 respectively. If $f(0) = \frac{3}{2}$, $f(-1) = 1$, determine the function.

31. If $a > e$, use Rouche's theorem to prove that the equation $e^z = az^n$ has n roots inside the circle $|z| = 1$.

Answers

1. $28\pi i$ **2.** $2\pi i \sin 1$ **3.** $-6\pi i$ **4.** $-2\pi i$

5. (i) $-4\pi i$ (ii) πi

6. (i) (a) $2\pi i$ (b) $-\pi i$ (ii) $-\frac{2}{3}\pi i e$ (iii) $2\pi i\,(e^4 - e^2)$ (iv) $\frac{2\pi}{3}$

7. (i) $2\pi i$ (ii) 0 (iii) $2\pi i$

8. (a) 0 (b) $-\pi i$

9. (i) $\pi(\cos 1 + i \sin 1)$ (ii) $-\pi(\cos 1 - i \sin 1)$

11. (i) $\frac{\pi}{3}$ (ii) $-\frac{\pi}{3}$ (iii) 0

12. (i) $\frac{\pi}{2}(3 + 2i)$ (ii) $\frac{\pi}{8} - 6\pi i$ (iii) $2\pi i \tan 1$ (iv) $3\pi i$

13. (i) $2\pi i$ (ii) $-2\pi i$ (iii) 0

14. (i) $\frac{2\pi i}{5}$ (ii) $-18\pi i$ (iii) 0 **15.** πi

16. (i) $4\pi i e^2$ (ii) $\frac{4\pi i}{3e^2}$

17. (i) $72\pi i$ (ii) $-\frac{\pi i}{2}$

18. (i) $2\pi i$ (ii) $-\pi i e$ (iii) $\pi i\,(2 - e)$

19. $\frac{\pi i}{2}(4 \cos 2 + \sin 2)$ **20.** (i) $12\pi i$ (ii) $12\pi i$ (iii) 0

21. (i) $2\pi \sinh \pi$ (ii) πi (iii) $\frac{6\pi e}{25}$ (iv) $\pi i\,(\sin t - t \cos t)$

22. (i) 0 (ii) $\frac{2\pi i}{3} \sinh 2$ (iii) $\frac{2\pi i}{3}(\sinh 2 + \sinh 1)$

23. (i) $\frac{\pi i e}{10}$ (ii) $\frac{2\pi i}{3}$ **25.** 0

26. $P(z) = 1 + z + z^2$ **27.** $f(3) = 0, f'(1 - i) = 2\pi(6 + 13i), f''(1 - i) = 12\pi i$

28. (i) $\frac{\pi i}{2}\left(\frac{5}{2} \cos \frac{3}{2} + \sin \frac{3}{2}\right)$ (ii) $\frac{3\pi}{128}$.

29. $1 + \frac{1}{z+1} + \frac{2}{z-2} + \frac{3}{(z-2)^2}$ **30.** $\frac{2z^3 - 4z^2 + 3z - 3}{(z-2)(z-1)^2}$

5.13 REPRESENTATION OF A FUNCTION BY POWER SERIES

A series of the form $\sum_{n=0}^{\infty} a_n z^n$ or $\sum_{n=0}^{\infty} a_n (z - a)^n$ whose terms are variable is called a power series, where z is a complex variable and a_n, a are complex constants. The second form can be reduced to first form merely by substitution $z = \zeta + a$ or by changing the origin.

Every complex function $f(z)$ which is analytic in a domain D can be represented by a power series valid in some circular region R about a point z_0. Both the circular region R and the point z_0 lie inside D. Such a power series is **Taylor's series**. If $f(z)$ is not analytic at a point z_0, we can still expand $f(z)$ in an infinite series having both positive and negative powers of $z - z_0$. This series is called the **Laurent's series**.

5.14 TAYLOR'S SERIES

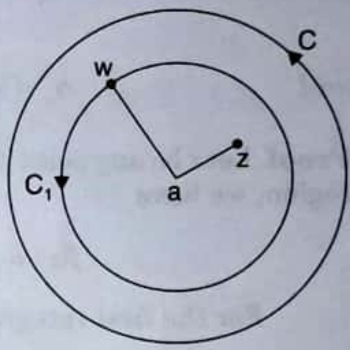
If $f(z)$ is analytic inside a circle C with centre at a, then for all z inside C,

$$f(z) = f(a) + (z-a)\, f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots + \frac{(z-a)^n}{n!} f^n(a) + \cdots$$

Or

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n, \quad \text{where} \quad a_n = \frac{f^{(n)}(a)}{n!}.$$

Proof. Let z be any point inside the circle C. Draw a circle C_1 with centre at a and radius smaller than that of C such that z is an interior point of C_1. Let w be any point on C_1, then



$$|z-a| < |w-a| \quad i.e., \left|\frac{z-a}{w-a}\right| < 1$$

Now, $$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{w-a}\left[1 - \frac{z-a}{w-a}\right]^{-1}$$

Expanding RHS by binomial theorem as $\left|\frac{z-a}{w-a}\right| < 1$, we get

$$\frac{1}{w-z} = \frac{1}{w-a}\left[1 + \frac{z-a}{w-a} + \left(\frac{z-a}{w-a}\right)^2 + \cdots + \left(\frac{z-a}{w-a}\right)^n + \cdots\right] \qquad ...(1)$$

This series converges uniformly since $\left|\frac{z-a}{w-a}\right| < 1$. Multiplying both sides of eqn. (1) by $\frac{1}{2\pi i} f(w)$ and integrating term by term w.r.t. w, over C_1, we get

$$\frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{w-z}\, dw = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{w-a}\, dw + \frac{z-a}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^2}\, dw + \frac{(z-a)^2}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^3}\, dw$$
$$+ \cdots + \frac{(z-a)^n}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^{n+1}}\, dw + \cdots \qquad ...(2)$$

$$\Rightarrow \quad \boxed{f(z) = f(a) + (z-a)\, f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots + \frac{(z-a)^n}{n!} f^n(a) + \cdots} \qquad ...(3)$$

which is the required *Taylor's series* for $f(z)$ about $z = a$.

Cor. 1. Putting $z = a + h$ in (3), we get

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^n}{n!} f^n(a) + \cdots$$

Cor. 2. If $a = 0$, the series (3) becomes

$$f(z) = f(0) + zf'(0) + \frac{z^2}{2!} f''(0) + \cdots + \frac{z^n}{n!} f^n(0) + \cdots$$

This series is called **Maclaurin's series.**

5.15 LAURENT'S SERIES

If $f(z)$ is analytic inside and on the boundary of the annular (ring shaped) region R bounded by two concentric circles C_1 and C_2 of radii r_1 and r_2 ($r_1 > r_2$) respectively having centre at a, then for all z in R,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

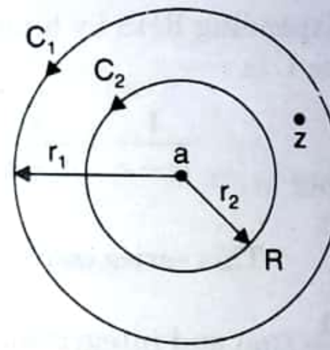
where, $$a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{(w-a)^{n+1}} dw \; ; n = 0, 1, 2, \ldots$$

and $$b_n = \frac{1}{2\pi i} \oint_{C_2} \frac{f(w)}{(w-a)^{-n+1}} dw \; ; n = 1, 2, 3, \ldots$$

Proof. Let z be any point in the region R, then by Cauchy's integral formula for double connected region, we have

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{w-z} dw - \frac{1}{2\pi i} \oint_{C_2} \frac{f(w)}{w-z} dw \qquad \ldots(1)$$

For the first integral in (1), w lies on C_1



$$\therefore \quad |z-a| < |w-a| \quad \text{i.e.,} \quad \left| \frac{z-a}{w-a} \right| < 1$$

Now, $$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{w-a}\left(1 - \frac{z-a}{w-a}\right)^{-1}$$

$$= \frac{1}{w-a}\left[1 + \frac{z-a}{w-a} + \left(\frac{z-a}{w-a}\right)^2 + \cdots\right]$$

Multiplying both sides by $\frac{1}{2\pi i} f(w)$ and integrating term by term w.r.t. w, along the circle C_1, we get

$$\frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{w-z} dw = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{w-a} dw + \frac{z-a}{2\pi i} \oint_{C_1} \frac{f(w)}{(w-a)^2} dw + \frac{(z-a)^2}{2\pi i} \oint_{C_1} \frac{f(w)}{(w-a)^3} dw + \cdots$$

$$= a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n \quad \ldots(2) \qquad \left[\because \; a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{(w-a)^{n+1}} dw, n = 0, 1, 2, \cdots \right]$$

For the second integral in (1), w lies on C_2

$$\therefore \quad |w-a| < |z-a| \quad \text{i.e.,} \quad \left| \frac{w-a}{z-a} \right| < 1$$

Now, $$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = -\frac{1}{z-a}\left(1 - \frac{w-a}{z-a}\right)^{-1}$$

$$= -\frac{1}{z-a}\left[1 + \frac{w-a}{z-a} + \left(\frac{w-a}{z-a}\right)^2 + \cdots\right]$$

Multiplying both sides by $-\dfrac{1}{2\pi i} f(w)$ and integrating term by term w.r.t. w, along the circle C_2, we get

$$-\frac{1}{2\pi i}\oint_{C_2} \frac{f(w)}{w-z}\,dw = \frac{1}{z-a}\cdot\frac{1}{2\pi i}\oint_{C_2} f(w)\,dw + \frac{1}{(z-a)^2}\cdot\frac{1}{2\pi i}\oint_{C_2}(w-a)\,f(w)\,dw$$

$$+ \frac{1}{(z-a)^3}\cdot\frac{1}{2\pi i}\oint_{C_2}(w-a)^2 f(w)\,dw + \cdots$$

$$= b_1(z-a)^{-1} + b_2(z-a)^{-2} + b_3(z-a)^{-3} + \cdots$$

$$= \sum_{n=1}^{\infty} b_n (z-a)^{-n} \quad \ldots(3) \qquad \left[\because\ b_n = \frac{1}{2\pi i}\oint_{C_2}\frac{f(w)}{(w-a)^{-n+1}}\,dw,\ n = 1, 2, 3, \cdots\right]$$

Substituting from (2) and (3) in (1), we get

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}}$$

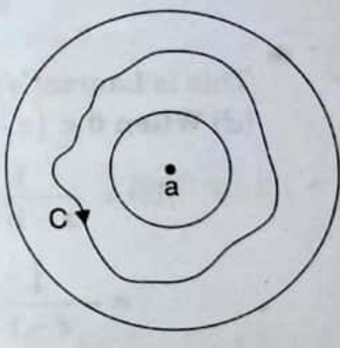
Note 1. In case $f(z)$ is analytic inside C_1, then $b_n = 0$ and $a_n = \dfrac{1}{2\pi i}\displaystyle\oint_{C_1}\frac{f(w)}{(w-a)^{n+1}}\,dw = \frac{f^n(a)}{n!}$

and Laurent's series reduces to Taylor's series.

Note 2. If C is any simple closed curve which lies in the ring-shaped region R and encloses the circle C_1, then

$$\oint_{C_1}\frac{f(w)}{(w-a)^{n+1}}\,dw = \oint_{C}\frac{f(w)}{(w-a)^{n+1}}\,dw$$

and
$$\oint_{C_2}\frac{f(w)}{(w-a)^{-n+1}}\,dw = \oint_{C}\frac{f(w)}{(w-a)^{-n+1}}\,dw$$



$\therefore$ Laurent's series can be written as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n, \quad \text{where } a_n = \frac{1}{2\pi i}\oint_{C}\frac{f(w)}{(w-a)^{n+1}}\,dw.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Expand* $\dfrac{1}{z^2 - 3z + 2}$ *in the region*

(*a*) $|z| < 1$ (*b*) $1 < |z| < 2$ (U.P.T.U. 2015)

(*c*) $|z| > 2$ (*d*) $0 < |z - 1| < 1$. (G.B.T.U. 2010)

Sol. Here $f(z) = \dfrac{1}{z^2 - 3z + 2} = \dfrac{1}{(z-1)(z-2)} = \dfrac{1}{z-2} - \dfrac{1}{z-1}$ | Partial Fractions

(*a*) **When $|z| < 1$**

$\therefore$
$$f(z) = \frac{1}{-2\left(1 - \frac{z}{2}\right)} + \frac{1}{1-z}$$

[Arranged suitably to make the binomial expansions valid]

$$= -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} + (1-z)^{-1} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n + \sum_{n=0}^{\infty} z^n$$

This is a series in positive powers of z, so it is an expansion of $f(z)$ in Taylor's series within the circle $|z| = 1$.

(*b*) **When $1 < |z| < 2$**

$$\therefore \quad f(z) = \frac{1}{-2\left(1-\frac{z}{2}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)} = -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

This is a series in positive and negative powers of z, so it is an expansion of $f(z)$ in Laurent's series within the annulus $1 < |z| < 2$.

(*c*) **When $|z| > 2$**

$$\therefore \quad f(z) = \frac{1}{z\left(1-\frac{2}{z}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)} = \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

This is Laurent's series within the annulus $2 < |z| < R$, where R is large.

(*d*) **When $0 < |z-1| < 1$**

$$f(z) = \frac{1}{(z-1)-1} - \frac{1}{z-1} = -\frac{1}{1-(z-1)} - \frac{1}{z-1} = -(z-1)^{-1} - [1-(z-1)]^{-1}$$

$$= -\frac{1}{z-1} - \sum_{n=0}^{\infty}(z-1)^n.$$

This is also Laurent's series within the annulus $0 < |z-1| < 1$.

Example 2. *Show that when* $|z+1| < 1$, $z^{-2} = 1 + \sum_{n=1}^{\infty}(n+1)(z+1)^n$.

Sol. $f(z) = z^{-2} = \dfrac{1}{z^2} = \dfrac{1}{[(z+1)-1]^2} = \dfrac{1}{[1-(z+1)]^2} = [1-(z+1)]^{-2}$

$$= 1 + 2(z+1) + 3(z+1)^2 + 4(z+1)^3 + \ldots$$

[By Binomial theorem, since $|z+1| < 1$]

$$= 1 + \sum_{n=1}^{\infty}(n+1)(z+1)^n.$$

Example 3. *Expand* $\cos z$ *in a Taylor's series about* $z = \dfrac{\pi}{4}$.

Sol. Here, $f(z) = \cos z,\ f'(z) = -\sin z,\ f''(z) = -\cos z,\ f'''(z) = \sin z, \ldots$

$$\therefore \quad f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}},\quad f'\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}},\quad f''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}},\quad f'''\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}, \ldots$$

Hence, $\cos z = f(z)$

$$= f\left(\frac{\pi}{4}\right) + \left(z - \frac{\pi}{4}\right) f'\left(\frac{\pi}{4}\right) + \frac{\left(z - \frac{\pi}{4}\right)^2}{2!} f''\left(\frac{\pi}{4}\right) + \frac{\left(z - \frac{\pi}{4}\right)^3}{3!} f'''\left(\frac{\pi}{4}\right) + \ldots$$

$$= \frac{1}{\sqrt{2}}\left[1 - \left(z - \frac{\pi}{4}\right) - \frac{1}{2!}\left(z - \frac{\pi}{4}\right)^2 + \frac{1}{3!}\left(z - \frac{\pi}{4}\right)^3 + \cdots\right]$$

Example 4. *Expand the function* $\dfrac{\sin z}{z - \pi}$ *about* $z = \pi$.

Sol. Putting $z - \pi = t$, we have

$$\frac{\sin z}{z - \pi} = \frac{\sin(\pi + t)}{t} = \frac{-\sin t}{t} = -\frac{1}{t}\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} \cdots\right)$$

$$= -1 + \frac{t^2}{3!} - \frac{t^4}{5!} + \cdots = -1 + \frac{(z - \pi)^2}{3!} - \frac{(z - \pi)^4}{5!} + \cdots$$

Example 5. *Expand* $f(z) = \dfrac{z}{(z + 1)(z + 2)}$ *about* $z = -2$.

Sol. To expand $f(z)$ about $z = -2$, *i.e.*, in powers of $z + 2$, we put $z + 2 = t$.

$$\therefore \quad f(z) = \frac{z}{(z + 1)(z + 2)} = \frac{t - 2}{(t - 1)t} = \frac{2 - t}{t(1 - t)} = \frac{2 - t}{t}(1 - t)^{-1}$$

$$= \frac{2 - t}{t}(1 + t + t^2 + t^3 + \cdots) \qquad \text{for } 0 < |t| < 1$$

$$= \frac{1}{t}(2 + t + t^2 + t^3 + \cdots) = \frac{2}{t} + 1 + t + t^2 + \cdots$$

$$= \frac{2}{z + 2} + 1 + (z + 2) + (z + 2)^2 + \cdots \qquad \text{for } 0 < |z + 2| < 1$$

which is Laurent's series.

Example 6. *Expand the following function in a Laurent's series:*

(*i*) $f(z) = \dfrac{e^z}{(z - 1)^2}$ *about* $z = 1$. (*ii*) $f(z) = \dfrac{1}{z(z - 1)(z - 2)}$ *for* $|z - 1| < 1$

Sol. (*i*) $f(z) = \dfrac{e^z}{(z - 1)^2}$

Put $z - 1 = t$ then $z = 1 + t$

$$\therefore \quad f(z) = \frac{e^{1+t}}{t^2} = \frac{e}{t^2}\left[1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right] = e\left[\frac{1}{t^2} + \frac{1}{t} + \frac{1}{2!} + \frac{t}{3!} + \cdots\right]$$

$$= e\left[\frac{1}{(z - 1)^2} + \frac{1}{z - 1} + \frac{1}{2!} + \frac{z - 1}{3!} + \cdots\right].$$

(*ii*) $$f(z) = \frac{1}{z(z-1)(z-2)} = \frac{1}{2z} - \frac{1}{z-1} + \frac{1}{2(z-2)}$$ | Partial fractions

$$= \frac{1}{2(z-1+1)} - \frac{1}{z-1} + \frac{1}{2(z-1-1)}$$

$$= \frac{1}{2}\{1+(z-1)\}^{-1} - \frac{1}{z-1} - \frac{1}{2}\{1-(z-1)\}^{-1}$$ $| \because |z-1| < 1$

$$= \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n (z-1)^n - \frac{1}{z-1} - \frac{1}{2}\sum_{n=0}^{\infty} (z-1)^n$$

This is a series in positive and negative powers of $(z-1)$ hence it is an expansion of $f(z)$ in a Laurent's series for $|z-1| < 1$.

Example 7. *Expand the following function in a Laurent's series about the point $z = 0$:*

$$f(z) = \frac{1-\cos z}{z^3}.$$

Sol. $$f(z) = \frac{1-\cos z}{z^3} = \frac{1}{z^3}\left[1 - \left\{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots\right\}\right]$$

$$= \frac{1}{z^3}\left(\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \ldots\right) = \frac{1}{2!z} - \frac{1}{4!}z + \frac{1}{6!}z^3 - \ldots$$

Example 8. *Find the terms in the Laurent's expansion of $\frac{1}{z(e^z-1)}$ for the region $0 < |z| < 2\pi$.*

Sol. $$f(z) = \frac{1}{z(e^z-1)} = \frac{1}{z\left[1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots - 1\right]}$$

$$= z^{-1}\left(z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots\right)^{-1} = z^{-2}\left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \ldots\right)^{-1}$$

$$= z^{-2}\left[1 - \left(\frac{z}{2} + \frac{z^2}{6} + \frac{z^3}{24} + \frac{z^4}{120} + \ldots\right) + \frac{1}{4}z^2\left(1 + \frac{z}{3} + \frac{z^2}{12} + \ldots\right)^2 \right.$$
$$\left. - \frac{1}{8}z^3\left(1 + \frac{z}{3} + \ldots\right)^3 + \frac{z^4}{16}(1+\ldots)^4 + \ldots\right]$$

$$= z^{-2}\left[1 - \frac{z}{2} + z^2\left(\frac{1}{4} - \frac{1}{6}\right) - z^3\left(\frac{1}{8} - \frac{1}{6} + \frac{1}{24}\right)\right.$$
$$\left. + z^4\left(\frac{1}{16} - \frac{1}{8} + \frac{1}{24} + \frac{1}{36} - \frac{1}{120}\right) + \ldots\right]$$

$$= z^{-2}\left[1 - \frac{z}{2} + \frac{z^2}{12} - \frac{z^4}{720} + \ldots\right] = z^{-2} - \frac{1}{2}z^{-1} + \frac{1}{12} - \frac{z^2}{720} + \ldots$$

The singularities of $\frac{1}{z(e^z - 1)}$ are given by $z = 0$, $e^z = 1$ *i.e.*, $z = 0, \pm 2\pi i, \pm 4\pi i, \ldots\ldots$.

Hence the above expansion is valid for the region $0 < |z| < 2\pi$.

Example 9. *Using Taylor's theorem, show that:*

$$\log z = (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \ldots \text{ where } |z-1| < 1.$$

Sol. $f(z) = \log z$, $\quad f(1) = 0 \quad | \because \; a = 1$ and $\log 1 = 0$

Now, $f'(z) = \frac{1}{z}$, $\quad f'(1) = 1$

$f''(z) = -\frac{1}{z^2}$, $\quad f''(1) = -1$

$f'''(z) = \frac{2}{z^3}$, $\quad f'''(1) = 2$

$f^{(iv)}(z) = \frac{-6}{z^4}$, $\quad f^{(iv)}(1) = -6$ and so on.

We know that,

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \frac{(z-a)^3}{3!}f'''(a) + \cdots$$

$$= f(1) + (z-1)f'(1) + \frac{(z-1)^2}{2!}f''(1) + \frac{(z-1)^3}{3!}f'''(1) + \cdots$$

$$= 0 + (z-1)(1) + \frac{(z-1)^2}{2!}(-1) + \frac{(z-1)^3}{3!}(2) + \cdots$$

$$= (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \ldots$$

Example 10. *Find the Taylor's or Laurent's series which represent the function* $\frac{1}{(1+z^2)(z+2)}$ *when*

(*i*) $|z| < 1$ (*ii*) $1 < |z| < 2$ (*iii*) $|z| > 2$.

Sol. Let $f(z) = \frac{1}{(1+z^2)(z+2)} = \frac{1}{5}\left\{\frac{1}{z+2} - \frac{z-2}{1+z^2}\right\}$

(*i*) $|z| < 1$

$$f(z) = \frac{1}{5}\cdot\frac{1}{2}\left(1 + \frac{1}{2}z\right)^{-1} + \frac{2-z}{5}(1+z^2)^{-1}$$

| Binomial expansion of $(1+z)^{-1}$ is valid only when $|z| < 1$

$$= \frac{1}{10} \sum_{0}^{\infty} (-1)^n \left(\frac{z}{2}\right)^n + \frac{2-z}{5} \sum_{0}^{\infty} (-1)^n z^{2n}$$

This is a series in positive powers of z, so it is an expansion of $f(z)$ in a Taylor's series within the circle $|z| = 1$.

Remark. If $|z| < 1$, $(1+z)^{-1} = \sum_{n=0}^{\infty} (-1)^n z^n$; $(1-z)^{-1} = \sum_{n=0}^{\infty} z^n$.

(*ii*) **1 < | z | < 2**

$$f(z) = \frac{1}{5} \cdot \frac{1}{2}\left(1 + \frac{1}{2}z\right)^{-1} + \frac{2-z}{5} \cdot \frac{1}{z^2}\left(1 + \frac{1}{z^2}\right)^{-1}$$

$$= \frac{1}{10} \sum_{0}^{\infty} (-1)^n \left(\frac{z}{2}\right)^n + \frac{2-z}{5z^2} \sum_{0}^{\infty} (-1)^n \left(\frac{1}{z^2}\right)^n$$

This is a series in positive and negative powers of z, so it is an expansion of $f(z)$ in a Laurent's series within the annulus $1 < |z| < 2$.

(*iii*) **| z | > 2**

$$f(z) = \frac{1}{5} \cdot \frac{1}{z}\left(1 + \frac{2}{z}\right)^{-1} - \frac{1}{5}(z-2)\frac{1}{z^2}\left(1 + \frac{1}{z^2}\right)^{-1}$$

$$= \frac{1}{5z} \sum_{0}^{\infty} (-1)^n \left(\frac{2}{z}\right)^n - \frac{1}{5}\left(\frac{1}{z} - \frac{2}{z^2}\right) \sum_{0}^{\infty} (-1)^n \left(\frac{1}{z^2}\right)^n$$

This is Laurent's series within the annulus $2 < |z| < R$, where R is large.

Example 11. *Find the Taylor's and Laurent's series which represent the function*

$$\frac{z^2 - 1}{(z+2)(z+3)} \text{ when} \qquad \text{(U.K.T.U. 2011)}$$

(*i*) $|z| < 2$ (*ii*) $2 < |z| < 3$ (*iii*) $|z| > 3$.

Sol. Let $f(z) = \dfrac{z^2 - 1}{(z+2)(z+3)} = 1 + \dfrac{3}{z+2} - \dfrac{8}{z+3}$.

(*i*) **| z | < 2**

$$f(z) = 1 + \frac{3}{2}\left(1 + \frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1 + \frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{2} \sum_{0}^{\infty} (-1)^n \left(\frac{z}{2}\right)^n - \frac{8}{3} \sum_{0}^{\infty} (-1)^n \left(\frac{z}{3}\right)^n$$

It is a Taylor's series within a circle $|z| = 2$.

(*ii*) **2 < | z | < 3**

$$f(z) = 1 + \frac{3}{z}\left(1 + \frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1 + \frac{z}{3}\right)^{-1}$$

| Arranging suitably to make the binomial expansion valid for $2 < |z| < 3$

$$= 1 + \frac{3}{z}\sum_0^\infty (-1)^n \left(\frac{2}{z}\right)^n - \frac{8}{3}\sum_0^\infty (-1)^n \left(\frac{z}{3}\right)^n$$

It is a Laurent's series within the annulus $2 < |z| < 3$.

(*iii*) $|z| > 3$

$$f(z) = 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= 1 + \frac{3}{z}\sum_0^\infty (-1)^n \left(\frac{2}{z}\right)^n - \frac{8}{z}\sum_0^\infty (-1)^n \left(\frac{3}{z}\right)^n$$

It is a Laurent's series within the annulus $3 < |z| < R$, where R is large.

Example 12. *Expand* $\dfrac{1}{(z+1)(z+3)}$ *in the regions*

(*i*) $|z| < 1$ (A.K.T.U. 2016) (*ii*) $1 < |z| < 3$

(*iii*) $|z| > 3$ (*iv*) $1 < |z+1| < 2$.

Sol. $$f(z) = \frac{1}{(z+1)(z+3)} = \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z+3}\right]$$

(*i*) $|z| < 1$

$$f(z) = \frac{1}{2}\left[(1+z)^{-1} - \frac{1}{3}\left(1+\frac{z}{3}\right)^{-1}\right] = \frac{1}{2}\left[\sum_0^\infty (-1)^n z^n - \frac{1}{3}\sum_0^\infty (-1)^n \left(\frac{z}{3}\right)^n\right]$$

It is a Taylor's series within a circle $|z| = 1$.

(*ii*) $1 < |z| < 3$

$$f(z) = \frac{1}{2}\cdot\left[\frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{3}\left(1+\frac{z}{3}\right)^{-1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z}\sum_0^\infty (-1)^n \left(\frac{1}{z}\right)^n - \frac{1}{3}\sum_0^\infty (-1)^n \left(\frac{z}{3}\right)^n\right]$$

It is a Laurent's series within the annulus $1 < |z| < 3$.

(*iii*) $|z| > 3$

$$f(z) = \frac{1}{2}\left[\frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{z}\left(1+\frac{3}{z}\right)^{-1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z}\sum_0^\infty (-1)^n \left(\frac{1}{z}\right)^n - \frac{1}{z}\sum_0^\infty (-1)^n \left(\frac{3}{z}\right)^n\right]$$

It is a Laurent's series within the annulus $3 < |z| < R$ where R is large.

(*iv*) $$1 < |z+1| < 2$$

$\Rightarrow$ $$1 < |u| < 2 \quad \text{where } z + 1 = u$$

$$f(z) = \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z+3}\right] = \frac{1}{2}\left[\frac{1}{u} - \frac{1}{u+2}\right]$$

$$= \frac{1}{2}\cdot\frac{2}{u(u+2)} = \frac{1}{u(u+2)} = \frac{1}{2u}\left(1+\frac{u}{2}\right)^{-1} = \frac{1}{2u}\sum_0^\infty (-1)^n \left(\frac{u}{2}\right)^n$$

$$= \frac{1}{2(z+1)} \sum_0^{\infty} (-1)^n \left(\frac{z+1}{2}\right)^n$$

It is Laurent's series in the annulus $1 < |z+1| < 2$.

Example 13. *Find the Laurent's expansion for:*

$$f(z) = \frac{7z-2}{z^3 - z^2 - 2z} \qquad \text{(U.K.T.U. 2010)}$$

in the regions given by:

(i) $0 < |z+1| < 1$ *(ii)* $1 < |z+1| < 3$ *(iii)* $|z+1| > 3$.

Sol. We have

$$f(z) = \frac{7z-2}{z^3 - z^2 - 2z} = \frac{1}{z} - \frac{3}{z+1} + \frac{2}{z-2} = \frac{1}{(z+1)-1} - \frac{3}{z+1} + \frac{2}{(z+1)-3}$$

(i) **$0 < |z+1| < 1$**

$$f(z) = -\{1-(z+1)\}^{-1} - \frac{3}{z+1} - \frac{2}{3}\left\{1 - \left(\frac{z+1}{3}\right)\right\}^{-1}$$

$$= -\frac{3}{z+1} - \sum_{n=0}^{\infty} (z+1)^n - \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{z+1}{3}\right)^n$$

This is a series in negative and positive powers of $(z+1)$ hence it is an expansion of $f(z)$ in Laurent's series within the annulus $0 < |z+1| < 1$.

(ii) **$1 < |z+1| < 3$**

$$f(z) = \frac{1}{z+1}\left(1 - \frac{1}{z+1}\right)^{-1} - \frac{3}{z+1} - \frac{2}{3}\left\{1 - \left(\frac{z+1}{3}\right)\right\}^{-1}$$

$$= \frac{1}{z+1}\sum_{n=0}^{\infty}\left(\frac{1}{z+1}\right)^n - \frac{3}{z+1} - \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{z+1}{3}\right)^n$$

This is also a series in negative and positive powers of $(z+1)$ hence it is an expansion of $f(z)$ in Laurent's series within the annulus $1 < |z+1| < 3$.

(iii) **$|z+1| > 3$**

$$f(z) = \frac{1}{z+1}\left(1 - \frac{1}{z+1}\right)^{-1} - \frac{3}{z+1} + \frac{2}{z+1}\left(1 - \frac{3}{z+1}\right)^{-1}$$

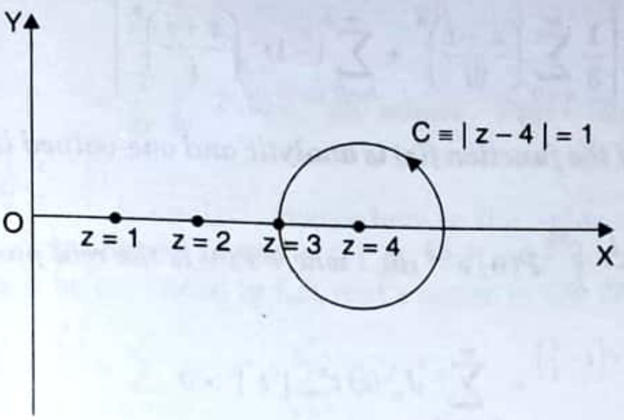
$$= \frac{1}{z+1}\sum_{n=0}^{\infty}\left(\frac{1}{z+1}\right)^n - \frac{3}{z+1} + \frac{2}{z+1}\sum_{n=0}^{\infty}\left(\frac{3}{z+1}\right)^n$$

This is a series in negative powers of $(z+1)$ hence it is an expansion of $f(z)$ in Laurent's series within the annulus $3 < |z+1| < R$ where R is large.

Example 14. *(i) Obtain the Taylor's series expansion of* $f(z) = \dfrac{1}{z^2 - 4z + 3}$ *about the point* $z = 4$. *Find its region of convergence.* uptu 2013-14

(ii) Obtain Taylor's series expansion of $f(z) = \dfrac{1}{z^2+4}$ *about the point* $z = -i$. *Find the region of convergence.*

Sol. (*i*) If the centre of the circle is at $z = 4$, then the distances of the singularities $z = 1$ and $z = 3$ from centre are 3 and 1. Hence if a circle is drawn with centre at $z = 4$ and radius 1 then within a circle $|z - 4| = 1$, the given function $f(z)$ is analytic hence it can be expanded in Taylor's series within the circle $|z - 4| = 1$ which is therefore the circle of convergence.



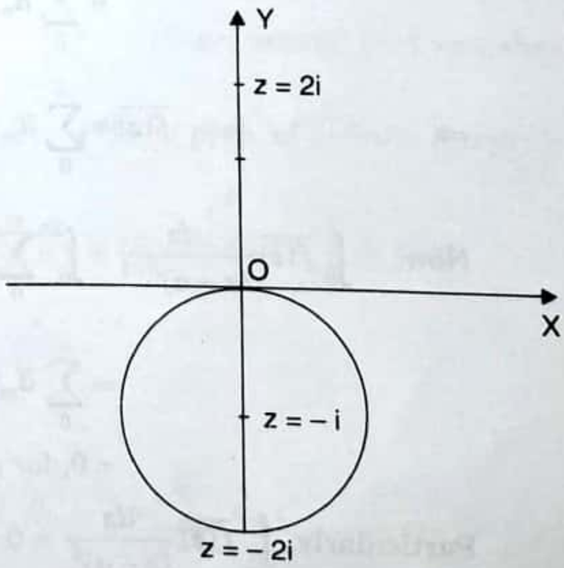
$$f(z) = \frac{1}{(z-1)(z-3)} = \frac{1}{2}\left[\frac{1}{z-3} - \frac{1}{z-1}\right] = \frac{1}{2}\left[\frac{1}{z-4+1} - \frac{1}{z-4+3}\right]$$

$$= \frac{1}{2}\left[\{1+(z-4)\}^{-1} - \frac{1}{3}\left\{1+\left(\frac{z-4}{3}\right)\right\}^{-1}\right]$$

$$\Rightarrow \qquad f(z) = \frac{1}{2}\left[\sum_{n=0}^{\infty}(-1)^n (z-4)^n - \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-4}{3}\right)^n\right]$$

(*ii*) If the centre of the circle is at $z = -i$, then the distances of the singularities $z = 2i$ and $z = -2i$ from centre are 3 and 1 respectively.

Hence if a circle is drawn with centre at $z = -i$ and radius 1 then within a circle $|z + i| = 1$, the given function $f(z)$ is analytic hence it can be expanded in Taylor's series within the circle $|z + i| = 1$ which is therefore the circle of convergence.



$$f(z) = \frac{1}{z^2+4} = \frac{1}{(z-2i)(z+2i)}$$

$$= \frac{1}{4i}\left(\frac{1}{z-2i} - \frac{1}{z+2i}\right)$$

$$= \frac{1}{4i}\left[\frac{1}{(z+i)-3i} - \frac{1}{(z+i)+i}\right]$$

$$= \frac{1}{4i}\left[\frac{-1}{3i}\left\{1-\left(\frac{z+i}{3i}\right)\right\}^{-1} - \frac{1}{i}\left\{1+\left(\frac{z+i}{i}\right)\right\}^{-1}\right]$$

$$= \frac{1}{4i}\left[\frac{i}{3}\sum_{n=0}^{\infty}\left(\frac{z+i}{3i}\right)^n + i\sum_{n=0}^{\infty}(-1)^n\left(\frac{z+i}{i}\right)^n\right]$$

$$= \frac{1}{4}\left[\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z+i}{3i}\right)^n + \sum_{n=0}^{\infty}(-1)^n\left(\frac{z+i}{i}\right)^n\right]$$

Example 15. *(i) If the function f(z) is analytic and one-valued in* $|z-a| < R$, *prove that for* $0 < r < R$,

$$f'(a) = \frac{1}{\pi r}\int_0^{2\pi} P(\theta)\, e^{-i\theta}\, d\theta \quad \text{where } P(\theta) \text{ is the real part of } (a + re^{i\theta}).$$

(ii) Prove that: $e^{\frac{1}{2}z\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(z)\, t^n,\ |t| > 0$

where $\quad J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - z\sin\theta)\, d\theta$

Sol. (*i*) $\because$ $f(z)$ is regular in $|z-a| < R$

$\therefore$ $f(z)$ is regular in $|z-a| = r$ $\qquad | \because r < R$

$\therefore$ $f(z)$ can be expanded in a Taylor's series within the circle $|z-a| = r$. Thus,

$$f(z) = \sum_0^{\infty} a_m (z-a)^m \text{ where } z - a = re^{i\theta}$$

$$= \sum_0^{\infty} a_m\, r^m\, e^{mi\theta} \qquad \text{...(1)}$$

$$\Rightarrow \qquad \overline{f(z)} = \sum_0^{\infty} \bar{a}_m\, r^m\, e^{-mi\theta} \qquad \text{...(2)}$$

Now, $\displaystyle\int_C \overline{f(z)}\frac{dz}{(z-a)^{n+1}} = \int_0^{2\pi}\sum_0^{\infty}\bar{a}_m\, r^m\, e^{-mi\theta}\frac{re^{i\theta}\, id\theta}{r^{n+1}\, e^{i(n+1)\theta}}$

$$= \sum_0^{\infty}\bar{a}_m\, r^{m-n}\, i\int_0^{2\pi} e^{-i(m+n)\theta}\, d\theta$$

$$= 0, \text{ for all values of } n \qquad \text{...(3)}$$

Particularly, $\displaystyle\int_C \overline{f(z)}\frac{dz}{(z-a)^2} = 0 \qquad \text{...(4)}$

We know that $\quad \displaystyle f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^2} = \frac{1}{2\pi i}\int_C \frac{f(z)+\overline{f(\bar{z})}}{(z-a)^2}\, dz \qquad |$ Using (4)

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a+re^{i\theta})+\overline{f(a+re^{i\theta})}}{r^2 e^{2i\theta}}\, re^{i\theta}\, id\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{2\,\mathrm{Re}(a+re^{i\theta})}{re^{i\theta}}\, d\theta \qquad \Big|\ \because\ z+\bar{z} = 2\,\mathrm{Re}(z)$$

$$= \frac{1}{\pi r}\int_0^{2\pi} \mathrm{P}(\theta)\, e^{-i\theta}\, d\theta \text{ where } \mathrm{P}(\theta) = \mathrm{Re}\,(a+re^{i\theta}).$$

(*ii*) The function $e^{\frac{1}{2}z\left(t-\frac{1}{t}\right)}$ is analytic everywhere in the t-plane except at $t = 0$ and $t = \infty$ *i.e.,* it is analytic in the ring shaped region $r \le |\, t\,| \le \mathrm{R}$ where r is small and R is large. Therefore this function can be expanded in Laurent's series in the form

$$e^{\frac{1}{2}z\left(t-\frac{1}{t}\right)} = \sum_{n=0}^{\infty} a_n t^n + \sum_{n=1}^{\infty} b_n t^{-n}$$

where $$a_n = \frac{1}{2\pi i}\int_{\mathrm{C}} e^{\frac{1}{2}z\left(t-\frac{1}{t}\right)} \frac{dt}{t^{n+1}} \text{ and } b_n = \frac{1}{2\pi i}\int_{\mathrm{C}} e^{\frac{1}{2}z\left(t-\frac{1}{t}\right)} \frac{dt}{t^{-n+1}}$$

where C is any circle with centre as origin.

Taking $\mathrm{C} \equiv |\, t\,| = 1$ so that $t = e^{i\theta}$ and $dt = ie^{i\theta}\, d\theta$, we get

$$a_n = \frac{1}{2\pi i}\int_0^{2\pi} e^{\frac{z}{2}(e^{i\theta}-e^{-i\theta})} \frac{ie^{i\theta}\, d\theta}{e^{(n+1)i\theta}}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{iz\sin\theta}\, e^{-ni\theta}\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{-i(n\theta - z\sin\theta)}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cos\,(n\theta - z\sin\theta)\, d\theta \qquad |\text{ Since second part vanishes}$$

$$\Rightarrow \qquad a_n = \frac{1}{\pi}\int_0^{\pi} \cos\,(n\theta - z\sin\theta)\, d\theta \qquad |\text{ Using prop. of definite integrals}$$

Clearly, the function $e^{\frac{1}{2}z\left(t-\frac{1}{t}\right)}$ remains unaltered if t is replaced by $-\frac{1}{t}$ so that $b_n = (-1)^n a_n$. Therefore,

$$e^{\frac{1}{2}z\left(t-\frac{1}{t}\right)} = \sum_{n=0}^{\infty} a_n t^n + \sum_{n=1}^{\infty} b_n t^{-n}$$

$$= \sum_{n=0}^{\infty} a_n t^n + \sum_{n=1}^{\infty} (-1)^n a_n t^{-n} = \sum_{n=-\infty}^{\infty} a_n t^n$$

Here, a_n is $\mathrm{J}_n(z)$ hence, $e^{\frac{1}{2}z\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} \mathrm{J}_n(z)\, t^n$

where,

$$\mathrm{J}_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos\,(n\theta - z\sin\theta)\, d\theta.$$

Example 16. *Prove that* $\cosh\left(z+\frac{1}{z}\right) = a_0 + \sum_{1}^{\infty} a_n\left(z^n + \frac{1}{z^n}\right)$

where $a_n = \frac{1}{2\pi}\int_0^{2\pi} \cosh\theta \cosh(2\cos\theta)\, d\theta.$ (M.T.U. 2013)

Sol. The function $f(z) = \cosh\left(z+\frac{1}{z}\right)$ is analytic everywhere in the finite part of the plane except at $z = 0$ *i.e.*, it is analytic in the annulus $r \le |z| \le R$ where r is small and R is large. Hence $f(z)$ can be expanded in Laurent's series in the annulus $r < |z| < R$. Thus,

$$\cosh\left(z+\frac{1}{z}\right) = \sum_{0}^{\infty} a_n z^n + \sum_{1}^{\infty} b_n z^{-n}$$

where $$a_n = \frac{1}{2\pi i}\int_C \frac{\cosh\left(z+\frac{1}{z}\right)}{z^{n+1}}\, dz \quad \text{and} \quad b_n = \frac{1}{2\pi i}\int_C \frac{\cosh\left(z+\frac{1}{z}\right)}{z^{-n+1}}\, dz$$

where C is any circle lying in the annulus with origin as centre.

$$a_n = \frac{1}{2\pi i}\int_C \frac{\cosh\left(z+\frac{1}{z}\right)}{z^{n+1}}\, dz$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{\cosh(2\cos\theta)\, ie^{i\theta}\, d\theta}{e^{i(n+1)\theta}}$$

| Take C as a circle $|z| = 1$ on which $z = e^{i\theta}$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\, e^{-in\theta}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\cos n\theta\, d\theta \qquad \Big|\ \because \int_0^{2\pi} \cosh(2\cos\theta)\sin n\theta\, d\theta = 0$$

$$b_n = a_{-n}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\cos(-n\theta)\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\cos n\theta\, d\theta = a_n$$

Hence, $$\cosh\left(z+\frac{1}{z}\right) = \sum_{0}^{\infty} a_n z^n + \sum_{1}^{\infty} b_n z^{-n} = \sum_{0}^{\infty} a_n z^n + \sum_{1}^{\infty} a_n z^{-n} \qquad \big|\ \because a_n = b_n$$

$$= a_0 + \sum_{1}^{\infty} a_n z^n + \sum_{1}^{\infty} a_n z^{-n} = a_0 + \sum_{1}^{\infty} a_n (z^n + z^{-n}).$$

Example 17. *If C is a closed contour around origin, prove that* $\left(\dfrac{a^n}{n!}\right)^2 = \dfrac{1}{2\pi i}\displaystyle\int_C \dfrac{a^n e^{az}}{n!\, z^{n+1}}\, dz.$

Hence deduce $\displaystyle\sum_0^\infty \left(\frac{a^n}{n!}\right)^2 = \frac{1}{2\pi}\int_0^{2\pi} e^{2a\cos\theta}\, d\theta.$

Sol. Let $f(z) = e^{az}$

$\therefore \quad f^n(z) = a^n e^{az}$

$\therefore \quad f^n(0) = a^n$

$$\Rightarrow \quad a^n = \frac{1}{2\pi i}\, n! \int_C \frac{f(z)\, dz}{z^{n+1}}$$

$$\Rightarrow \quad \left(\frac{a^n}{n!}\right)^2 = \frac{1}{2\pi i}\,\frac{1}{n!}\int_C \frac{a^n e^{az}}{z^{n+1}}\, dz$$

$$\Rightarrow \quad \sum_0^\infty \left(\frac{a^n}{n!}\right)^2 = \sum_0^\infty \frac{1}{2\pi i\, n!}\int_C \frac{a^n e^{az}}{z^{n+1}}\, dz = \frac{1}{2\pi i}\int_C \sum_0^\infty \frac{a^n e^{az}}{n!\, z^{n+1}}\, dz$$

$$= \frac{1}{2\pi i}\int_C e^{az} \sum_0^\infty \frac{a^n}{n!}\,\frac{1}{z^{n+1}}\, dz = \frac{1}{2\pi i}\int_C e^{az}\left\{\sum_0^\infty \left(\frac{a}{z}\right)^n \frac{1}{n!}\right\}\frac{dz}{z}$$

$$= \frac{1}{2\pi i}\int_C e^{az}\, e^{(a/z)}\,\frac{dz}{z} = \frac{1}{2\pi i}\int_C e^{a\left(z+\frac{1}{z}\right)}\,\frac{dz}{z}$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} e^{2a\cos\theta}\,\frac{i e^{i\theta}\, d\theta}{e^{i\theta}}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{2a\cos\theta}\, d\theta$$

where the circle C is taken as $|z| = 1$ so that $z = e^{i\theta}$ on C
$\therefore \quad dz = i e^{i\theta}\, d\theta$

Example 18. *Prove that for real k,* $k^2 < 1$; $\displaystyle\sum_{n=0}^\infty k^n \sin(n+1)\theta = \frac{\sin\theta}{1 - 2k\cos\theta + k^2}$

and

$$\sum_{n=0}^\infty k^n \cos(n+1)\theta = \frac{\cos\theta - k}{1 - 2k\cos\theta + k^2}.$$

Sol. $$\frac{1}{z-k} = \frac{1}{z}\left(1 - \frac{k}{z}\right)^{-1} = \frac{1}{z}\sum_{n=0}^\infty \left(\frac{k}{z}\right)^n = \sum_{n=0}^\infty \frac{k^n}{z^{n+1}}\ ;\ |z| > k \qquad ...(1)$$

Again, put $z = e^{i\theta}$ in (1),

$$\frac{1}{z-k} = \frac{1}{e^{i\theta} - k} = \sum_{n=0}^\infty k^n e^{-(n+1)i\theta}$$

$$\Rightarrow \quad \frac{1}{\cos\theta + i\sin\theta - k} = \sum_{n=0}^{\infty} k^n \left[\cos(n+1)\theta - i\sin(n+1)\theta\right]$$

$$\Rightarrow \quad \frac{(\cos\theta - k) - i\sin\theta}{1 - 2k\cos\theta + k^2} = \sum_{n=0}^{\infty} k^n \left[\cos(n+1)\theta - i\sin(n+1)\theta\right] \quad \text{...(2)}$$

Comparing real and imaginary parts of (2), we get the required results.

Example 19. (*i*) *Show that* $\operatorname{cosec} z = \frac{1}{z} + \frac{1}{3!}z + \frac{7}{360}z^3 + \ldots\ldots;\ 0 < |z| < \pi.$

(*ii*) *Find the Taylor's series expansion of* $f(z) = \frac{a}{bz+c}$ *about the point* z_0.

Sol. (*i*) $\operatorname{cosec} z = \frac{1}{\sin z}$ has singular points at $z = 0, \pm n\pi$.

We expand the series in $0 < |z| < \pi$.

$$\operatorname{cosec} z = \frac{1}{\sin z} = \frac{1}{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots} = \frac{1}{z\left[1 - \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^6}{7!} + \ldots\ldots\right)\right]}$$

$$= \frac{1}{z}\left[1 + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^6}{7!} + \ldots\ldots\right) + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^6}{7!} + \ldots\ldots\right)^2 + \ldots\ldots\right]$$

$$= \frac{1}{z}\left[1 + \frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^4}{(3!)^2} + \ldots\ldots\right] = \frac{1}{z} + \frac{z}{3!} + \left\{\frac{1}{(3!)^2} - \frac{1}{5!}\right\} z^4 + \ldots\ldots$$

(*ii*)
$$f(z) = \frac{a}{bz+c} = \frac{a}{b(z-z_0) + bz_0 + c} = \frac{1}{bz_0 + c}\left[\frac{a}{1 + \frac{b(z-z_0)}{bz_0+c}}\right]$$

$$= \frac{a}{d}\left[\frac{1}{1 + e(z-z_0)}\right] \quad \Big| \text{ where } bz_0 + c = d, \frac{b}{d} = e$$

$$= \frac{a}{d}\sum_{n=0}^{\infty} (-1)^n e^n (z-z_0)^n \text{ if } |e(z-z_0)| < 1$$

$$= \frac{a}{bz_0 + c}\sum_{n=0}^{\infty} (-1)^n \left(\frac{b}{bz_0+c}\right)^n (z-z_0)^n \text{ if } |z - z_0| < \frac{1}{e}.$$

Example 20. *Find Taylor's series expansion of* $\frac{4z-1}{z^4-1}$ *about the point* $z = 0$.

(M.T.U. 2012)

Sol.
$$f(z) = \frac{4z-1}{z^4-1} = \frac{4z-1}{(z-1)(z+1)(z^2+1)}$$

$$= \frac{3}{4}\left(\frac{1}{z-1}\right) + \frac{5}{4(z+1)} + \frac{\left(-2z + \frac{1}{2}\right)}{z^2+1}$$

Expanding about the point $z = 0$, we get

$$f(z) = -\frac{3}{4}(1-z)^{-1} + \frac{5}{4}(1+z)^{-1} + \left(-2z+\frac{1}{2}\right)\left(1+z^2\right)^{-1}$$

$$= -\frac{3}{4}\sum_{n=0}^{\infty} z^n + \frac{5}{4}\sum_{n=0}^{\infty}(-1)^n z^n + \left(-2z+\frac{1}{2}\right)\sum_{n=0}^{\infty}(-1)^n z^{2n}.$$

TEST YOUR KNOWLEDGE

Expand the following functions as a Taylor's series (1–3):

1. (*i*) $\log(1+z)$ about $z = 0$
   (*ii*) $\tan^{-1} z$ in powers of z
   (*iii*) $\sin^{-1} z$ in powers of z
2. (*i*) $\sin z$ about $z = \frac{\pi}{4}$ (*ii*) $\tan^{-1} z$ about $z = \frac{\pi}{4}$ (*U.P.T.U. 2015*)
3. $\dfrac{z}{(z+1)(z+2)}$ about $z = 2$.

Expand the following functions in Laurent's series (4–6):

4. $\dfrac{1}{z-2}$, for $|z| > 2$
5. $\dfrac{1}{z^2-4z+3}$, for $1 < |z| < 3$
6. $\dfrac{1}{z(z-1)(z-2)}$, for $|z| > 2$
7. (*i*) Find Taylor's expansion of $\dfrac{2z^3+1}{z(z+1)}$ about the point $z = 1$.
   (*ii*) Define the Laurent series expansion of a function. Expand $f(z) = e^{\frac{z}{(z-2)}}$ in a Laurent series about the point $z = 2$.
8. Expand $f(z) = \dfrac{(z-2)(z+2)}{(z+1)(z+4)}$ in the region:
   (*a*) $|z| < 1$ (*b*) $1 < |z| < 4$ (*c*) $|z| > 4$.
9. Expand the function $f(z) = \dfrac{1}{z^2-z-6}$ about (*i*) $z = -1$ (*ii*) $z = 1$
10. (*i*) Find the Laurent's series expansion of the function $f(z) = \dfrac{z^2-6z-1}{(z-1)(z-3)(z+2)}$ in the region $3 < |z+2| < 5$.
    (*ii*) Find the Laurent's series expansion of $f(z) = \dfrac{7z-2}{z(z+1)(z+2)}$ in the region $1 < |z+1| < 3$.
    (*A.K.T.U. 2017*)
11. (*i*) Obtain the Taylor series expansion of $f(z) = \dfrac{1}{z^2+(1+2i)z+2i}$ about $z = 0$.

(ii) Expand $f(z) = \dfrac{z}{(z-1)(2-z)}$ is Laurent series valid for

(a) $|z-1| > 1$ and (b) $0 < |z-2| < 1$ (G.B.T.U. 2011, 2013)

(iii) Expand $f(z) = \dfrac{z}{(z-1)(z-2)}$ in Laurent's series valid for region:

(a) $|z-1| > 1$ (b) $0 < |z-2| < 1$ (M.T.U. 2014)

12. Find Laurent's series of $f(z) = \dfrac{1}{z^2+1}$ about its singular points. Determine the region of convergence.

13. Find all possible Taylor's and Laurent's series expansions of the function $f(z) = \dfrac{1}{(z+1)(z+2)^2}$ about the point $z = 1$. Consider the regions

(i) $|z-1| < 2$ (ii) $2 < |z-1| < 3$ (iii) $|z-1| > 3$

14. The series expansions of the functions $\dfrac{1}{1-z}$ and $\dfrac{1}{z-1}$ are

$$\frac{1}{1-z} = 1 + z + z^2 + \ldots\ldots \text{ and } \frac{1}{z-1} = \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \ldots\ldots\right)$$

Adding, we get $(1 + z + z^2 + \ldots\ldots) + \dfrac{1}{z}\left(1 + \dfrac{1}{z} + \dfrac{1}{z^2} + \ldots\ldots\right) = 0$

Is this result true? If not, give the reason.

15. Expand $f(z) = \dfrac{7z^2 + 9z - 18}{z^3 - 9z}$ in Laurent series valid for the regions:

(i) $0 < |z| < 3$ (ii) $|z| > 3$ (A.K.T.U. 2016)

16. If $f(z) = \dfrac{z+4}{(z+3)(z-1)^2}$, find Laurent's series expansion in (i) $0 < |z-1| < 4$ and (ii) $|z-1| > 4$. (M.T.U. 2013)

17. Expand $f(z) = \dfrac{z}{(z^2-1)(z^2+4)}$ in Laurent series in $1 < |z| < 2$. (G.B.T.U. 2011, 2012)

18. Find all Taylor and Laurent series expansion of the following function about $z = 0$.

$$f(z) = \frac{-2z+3}{z^2 - 3z + 2}.$$ (U.P.T.U. 2014)

Answers

1. (i) $z - \dfrac{z^2}{2} + \dfrac{z^3}{3} - \dfrac{z^4}{4} + \ldots\ldots$ (ii) $z - \dfrac{z^3}{3} + \dfrac{z^5}{5} - \ldots\ldots$ (iii) $z + \dfrac{z^3}{6} + \dfrac{3}{40}z^5 + \ldots$

2. (i) $\dfrac{1}{\sqrt{2}}\left[1 + \left(z - \dfrac{\pi}{4}\right) - \dfrac{1}{2!}\left(z - \dfrac{\pi}{4}\right)^2 - \dfrac{1}{3!}\left(z - \dfrac{\pi}{4}\right)^3 + \ldots\ldots\right]$

(ii) $\tan^{-1} z = \tan^{-1}\left(\dfrac{\pi}{4}\right) + \left(z - \dfrac{\pi}{4}\right) \cdot \dfrac{16}{\pi^2 + 16} - 64\pi\,\dfrac{(z - \pi/4)^2}{(\pi^2+16)^2} + \ldots$

3. $\left(\dfrac{1}{2} - \dfrac{1}{3}\right) - \left(\dfrac{1}{2^3} - \dfrac{1}{3^2}\right)(z-2) + \left(\dfrac{1}{2^5} - \dfrac{1}{3^3}\right)(z-2)^2 - \ldots\ldots$

4. $f(z) = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n$ 5. $f(z) = -\frac{1}{6}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n - \frac{1}{2z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$

6. $f(z) = \frac{1}{2z} - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n + \frac{1}{2z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n$

7. (i) $f(z) = 2z - 2 + \sum_{n=0}^{\infty}(-1)^n (z-1)^n + \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-1}{2}\right)^n$ (ii) $f(z) = e\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{2}{z-2}\right)^n$

8. (a) $f(z) = 1 - \sum_{n=0}^{\infty}(-1)^n z^n - \sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{4}\right)^n$

(b) $f(z) = 1 - \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z}\right)^n - \sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{4}\right)^n$

(c) $f(z) = 1 - \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z}\right)^n - \frac{4}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{4}{z}\right)^n$

9. (i) $f(z) = -\frac{1}{20}\sum_{n=0}^{\infty}\left(\frac{z+1}{4}\right)^n - \frac{1}{5}\sum_{n=0}^{\infty}(-1)^n (z+1)^n$

(ii) $f(z) = -\frac{1}{10}\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n - \frac{1}{15}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-1}{3}\right)^n$

10. (i) $f(z) = \frac{1}{z+2}\sum_{n=0}^{\infty}\left(\frac{3}{z+2}\right)^n + \frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{z+2}{5}\right)^n + \frac{1}{z+2}$.

(ii) $f(z) = \frac{9}{z+1} - \frac{1}{z+1}\sum_{n=0}^{\infty}\frac{1}{(z+1)^n} - \frac{8}{z+1}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(z+1)^n}$

11. (i) $f(z) = \frac{1}{1-2i}\left[\frac{1}{2i}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2i}\right)^n - \sum_{n=0}^{\infty}(-1)^n z^n\right]$

(ii) (a) $f(z) = \frac{1}{z-1} - \frac{2}{z-1}\sum_{n=0}^{\infty}\frac{1}{(z-1)^n}$ (b) $f(z) = \sum_{n=0}^{\infty}(-1)^n (z-2)^n - \frac{2}{z-2}$

(iii) (a) $f(z) = \frac{-1}{z-1} + \frac{2}{z-1}\sum_{n=0}^{\infty}\left(\frac{1}{z-1}\right)^n$ (b) $f(z) = \frac{2}{z-2} - \sum_{n=0}^{\infty}(-1)^n (z-2)^n$

12. (i) $f(z) = \dfrac{1}{2i(z-i)} \displaystyle\sum_{n=0}^{\infty} (-1)^n \left(\dfrac{z-i}{2i}\right)^n ; |z-i| < 2$

(ii) $f(z) = \dfrac{-1}{2i(z+i)} \displaystyle\sum_{n=0}^{\infty} \left(\dfrac{z+i}{2i}\right)^n ; \quad |z+i| < 2$

13. (i) $\dfrac{1}{2}\displaystyle\sum_{n=0}^{\infty} (-1)^n \left(\dfrac{z-1}{2}\right)^n - \dfrac{1}{3}\sum_{n=0}^{\infty} (-1)^n \left(\dfrac{z-1}{3}\right)^n - \dfrac{1}{9}\sum_{n=0}^{\infty} (-1)^n (n+1)\left(\dfrac{z-1}{3}\right)^n$

(ii) $\dfrac{1}{z-1}\displaystyle\sum_{n=0}^{\infty} (-1)^n \left(\dfrac{2}{z-1}\right)^n - \dfrac{1}{3}\sum_{n=0}^{\infty} (-1)^n \left(\dfrac{z-1}{3}\right)^n - \dfrac{1}{9}\sum_{n=0}^{\infty} (-1)^n (n+1)\left(\dfrac{z-1}{3}\right)^n$

(iii) $\dfrac{1}{z-1}\displaystyle\sum_{n=0}^{\infty} (-1)^n \left(\dfrac{2}{z-1}\right)^n - \dfrac{1}{z-1}\sum_{n=0}^{\infty} (-1)^n \left(\dfrac{3}{z-1}\right)^n - \dfrac{1}{(z-1)^2}\sum_{n=0}^{\infty} (-1)^n (n+1)\left(\dfrac{3}{z-1}\right)^n$

14. *No*. The first series is valid for $|z| < 1$ and the second series is valid for $|z| > 1$. There is no common point where both the series are valid.

15. (i) $f(z) = \dfrac{2}{z} + \dfrac{1}{3}\displaystyle\sum_{n=0}^{\infty} (-1)^n \left(\dfrac{z}{3}\right)^n - \dfrac{4}{3}\sum_{n=0}^{\infty} \left(\dfrac{z}{3}\right)^n$ (ii) $f(z) = \dfrac{2}{z} + \dfrac{1}{z}\displaystyle\sum_{n=0}^{\infty} (-1)^n \left(\dfrac{3}{z}\right)^n + \dfrac{4}{z}\sum_{n=0}^{\infty} \left(\dfrac{3}{z}\right)^n$

16. (i) $f(z) = \dfrac{1}{64}\displaystyle\sum_{n=0}^{\infty} (-1)^n \left(\dfrac{z-1}{4}\right)^n - \dfrac{1}{16(z-1)} + \dfrac{5}{4(z-1)^2}$

(ii) $f(z) = \dfrac{1}{16(z-1)}\displaystyle\sum_{n=0}^{\infty} (-1)^n \left(\dfrac{4}{z-1}\right)^n - \dfrac{1}{16(z-1)} + \dfrac{5}{4(z-1)^2}$

17. $f(z) = \dfrac{1}{10}\left[\dfrac{1}{z}\displaystyle\sum_{n=0}^{\infty} \left(\dfrac{1}{z}\right)^n + \dfrac{1}{z}\sum_{n=0}^{\infty} (-1)^n \left(\dfrac{1}{z}\right)^n + \dfrac{1}{2i}\sum_{n=0}^{\infty} \left(\dfrac{z}{2i}\right)^n - \dfrac{1}{2i}\sum_{n=0}^{\infty} (-1)^n \left(\dfrac{z}{2i}\right)^n\right]$.

18. (i) $f(z) = \displaystyle\sum_{n=0}^{\infty} z^n + \dfrac{1}{2}\sum_{n=0}^{\infty} \left(\dfrac{z}{2}\right)^n ; |z| < 1$

(ii) $f(z) = -\dfrac{1}{z}\displaystyle\sum_{n=0}^{\infty} \left(\dfrac{1}{z}\right)^n + \dfrac{1}{2}\sum_{n=0}^{\infty} \left(\dfrac{z}{2}\right)^n ; 1 < |z| < 2$

(iii) $f(z) = -\dfrac{1}{z}\displaystyle\sum_{n=0}^{\infty} \left(\dfrac{1}{z}\right)^n - \dfrac{1}{z}\sum_{n=0}^{\infty} \left(\dfrac{2}{z}\right)^n ; |z| > 2$

5.16 ZERO OF AN ANALYTIC FUNCTION

A zero of an analytic function $f(z)$ is a value of z such that $f(z) = 0$.

If $f(z)$ is analytic in the neighbourhood of $z = a$, then by Taylor's theorem

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + a_3(z-a)^3 + \ldots\ldots + a_n(z-a)^n + \ldots\ldots\infty$$

If $a_0 = a_1 = a_2 = \ldots\ldots = a_{n-1} = 0$ but $a_n \neq 0$, then $f(z)$ is said to have a zero of order n at $z = a$. The zero is said to be simple if $n = 1$.

$$\because \quad a_n = \frac{f^n(a)}{n!}$$

$\therefore$ for a zero of order m at $z = a$,

$$f(a) = f'(a) = f''(a) = \ldots\ldots = f^{n-1}(a) = 0 \text{ but } f^n(a) \neq 0.$$

Thus in the neighbourhood of the zero at $z = a$ of order n,

$$f(z) = a_n(z-a)^n + a_{n+1}(z-a)^{n+1} + \ldots\ldots = (z-a)^n\,[a_n + a_{n+1}(z-a) + \ldots\ldots] = (z-a)^n\,\phi(z)$$

where $\phi(z) = a_n + a_{n+1}(z-a) + \ldots\ldots$ is analytic and non-zero at and in the neighbourhood of $z = a$.

5.17 SINGULARITY

[*M.T.U.* 2013]

A singularity of a function $f(z)$ is a point at which the function ceases to be analytic.

5.18 ISOLATED AND NON-ISOLATED SINGULARITY

[*M.T.U.* 2012]

If $z = a$ is a singularity of $f(z)$ and if there is no other singularity within a small circle surrounding the point $z = a$, then $z = a$ is said to be an isolated singularity of the function $f(z)$, otherwise it is called non-isolated.

Example. Consider the function $f(z) = \dfrac{z+1}{z(z-2)}$.

It is analytic everywhere except at $z = 0$ and $z = 2$. Thus $z = 0$ and $z = 2$ are the only singularities of this function. There are no other singularities of $f(z)$ in the neighbourhood of $z = 0, z = 2$. Hence $z = 0$ and $z = 2$ are the isolated singularities of this function.

Again, consider the function

$$f(z) = \frac{1}{\tan\left(\frac{\pi}{z}\right)} = \cot\left(\frac{\pi}{z}\right)$$

It is not analytic at the points where $\tan\left(\frac{\pi}{z}\right) = 0 = \tan n\pi$ *i.e.*, at the points where $\frac{\pi}{z} = n\pi$

$$\Rightarrow \quad z = \frac{1}{n} \quad (n = 1, 2, 3, \ldots\ldots)$$

Thus $z = 1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots, z = 0$ are the singularities of the function all of which are isolated except $z = 0$ because in the neighbourhood of $z = 0$, there are infinite number of other singularities $z = \frac{1}{n}$ where n is large. Therefore, $z = 0$ is the non-isolated singularity of the given function.

5.19 TYPES OF SINGULARITY

Let $f(z)$ be analytic within a domain D except at $z = a$ which is an isolated singularity. Draw a circle C with its centre $z = a$ and radius as small as we wish and another large concentric circle C of any radius R lying wholly within the domain D. Now in the annulus between these two circles, $f(z)$ is analytic. If z is any point of the annulus, then by Laurent's theorem,

$$f(z) = \sum_0^\infty a_n (z-a)^n + \sum_1^\infty b_n (z-a)^{-n} \quad \text{where } 0 < |z-a| < R.$$

The second term $\sum_1^\infty b_n (z-a)^{-n}$ on the RHS is called the **Principal Part** of $f(z)$ at the isolated singularity $z = a$. Now there arise three possibilities :

(*i*) All b_n's are zero $\Rightarrow$ no term in P.P. **(Removable singularity)** (*M.T.U. 2012*)

(*ii*) Infinite number of terms in P.P. **(Essential singularity)** (*M.T.U. 2012*)

(*iii*) Finite number of terms in P.P. **(Pole)**

(*i*) **Removable Singularity.** Here $f(z) = \sum_0^\infty a_n (z-a)^n$ which is analytic for $|z-a| < R$ except at $z = a$. Let $\varphi(z)$ be the sum function of the power series $\sum_0^\infty a_n (z-a)^n$. Now $\varphi(z)$ differs from $f(z)$ only at $z = a$, where there is singularity. To avoid this singularity, we can suitably define $f(z)$ at $z = a$, so that we have

$$\phi(z) = \begin{cases} f(z) & \text{for } 0 < |z-a| < R \\ a_0 & \text{for } z = a \end{cases}$$

This type of singularity which can be made to disappear by defining the function suitably is called *removable singularity*.

Example. The function $\dfrac{\sin(z-a)}{z-a}$ has removable singularity at $z = a$ because

$$\frac{\sin(z-a)}{z-a} = \frac{1}{z-a}\left\{(z-a) - \frac{(z-a)^3}{3!} + \frac{(z-a)^5}{5!} - \ldots\ldots\right\}$$

$$= 1 - \frac{(z-a)^2}{3!} + \frac{(z-a)^4}{5!} - \ldots\ldots$$

has no terms containing negative powers of $z - a$. However this singularity can be removed and the function made analytic by defining

$$\frac{\sin(z-a)}{z-a} = 1 \text{ at } z = a.$$

(*ii*) **Essential Singularity.** Here the series $\sum_1^\infty (z-a)^{-n}$ does not terminate.

Example. $f(z) = \sin\left(\dfrac{1}{z-a}\right)$ has essential singularity at $z = a$, because

$$\sin\left(\frac{1}{z-a}\right) = \frac{1}{z-a} - \frac{1}{3!}\frac{1}{(z-a)^3} + \frac{1}{5!}\frac{1}{(z-a)^5} - \ldots\ldots$$

has infinite number of terms in the negative powers of $z - a$.

(*iii*) **Pole.** Here the series $\sum_{1}^{\infty}(z-a)^n$ consists of finite number of terms. Then $z = a$ is said to be a pole of order m of the function $f(z)$. When $m = 1$, the pole is said to be simple.

Example. $f(z) = \dfrac{\sin(z-a)}{(z-a)^4}$ has a pole at $z = a$ because

$$\frac{\sin(z-a)}{(z-a)^4} = \frac{1}{(z-a)^4}\left[(z-a) - \frac{(z-a)^3}{3!} + \frac{(z-a)^5}{5!} - \frac{(z-a)^7}{7!} + \ldots\ldots\right]$$

$$= \frac{1}{(z-a)^3} - \frac{1}{3!}\frac{1}{(z-a)} + \frac{1}{5!}(z-a) - \frac{1}{7!}(z-a)^3 + \ldots\ldots$$

has finite number of terms (here first two terms only) in negative powers of $z - a$.

Thus if $z = a$ is a pole of order m of the function $f(z)$, then

$$f(z) = \sum_{0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \ldots\ldots + \frac{b_m}{(z-a)^m}$$

$$= \frac{1}{(z-a)^m}\left[\left(\sum_{0}^{\infty} a_n (z-a)^{n+m}\right) + \{b_m + b_{m-1}(z-a) + \ldots\ldots + b_1(z-a)^m\}\right]$$

$$= \frac{1}{(z-a)^m}\,\varphi(z)$$

Clearly, $\varphi(z) \to b_m$ as $z \to a$. Hence $\varphi(z)$ is analytic in the neighbourhood of the pole $z = a$.

5.20 THEOREMS

(1) The limit point of the zeros of a function f(z) is an isolated essential singularity.

Proof. Let $z_1, z_2, z_3, \ldots\ldots$ be an infinite set of zeros of $f(z)$. Let z_0 be their limit point.

(*i*) If z_0 is a point of the set, then z_0 will be a zero of $f(z)$ and will have in its neighbourhood, a cluster of zeros. But zeros are isolated, so, z_0 cannot be a zero of $f(z)$ unless $f(z)$ is identically zero in D.

(*ii*) If $f(z)$ is not identically zero in D, then z_0 is not a zero of $f(z)$. But z_0 is surrounded by many zeros. So z_0 is a singularity. Also z_0 is not a pole since $f(z)$ does not tend to infinity in the neighbourhood of z_0. Therefore z_0 is an essential singularity. But the singularity is isolated since in the neighbourhood of z_0, $f(z)$ is analytic. Hence z_0 is an isolated essential singularity.

(2) The limit point of the poles of a function f(z) is a non-isolated essential singularity.

Proof. Let $p_1, p_2, p_3, \ldots\ldots$ be an infinite set of poles of $f(z)$. Let p_0 be their limit point.

(*i*) If p_0 is a point of the set, then p_0 will be a pole of $f(z)$ and will have in its neighbourhood a cluster of poles. But poles are isolated, so, p_0 cannot be a pole of $f(z)$.

(*ii*) p_0 cannot be a zero of $f(z)$ since the function is not analytic (has poles) in the neighbourhood of p_0. So, p_0 is an essential singularity. This singularity is not isolated, since these are poles around p_0. Hence p_0 is a non-isolated essential singularity.

5.21 DETECTION OF SINGULARITY

(1) Removable Singularity: $\lim_{z \to a} f(z)$ exists and is finite.

Example. $$f(z) = \frac{z^2 - a^2}{z - a}$$
$$\lim_{z \to a} f(z) = 2a$$

So, $f(z)$ has a removable singularity at $z = a$.

(2) Pole: $\lim_{z \to a} f(z) = \infty$.

Example. $$f(z) = \frac{z^2 + a^2}{z - a}$$
$$\lim_{z \to a} f(z) = \infty.$$

So, $f(z)$ has a pole at $z = a$.

Moreover, the pole is said to be of order n, if there are n terms in the principal part.

Example. $$\frac{e^{z-a}}{(z-a)^2} = \frac{1}{(z-a)^2}\left[1 + (z-a) + \frac{(z-a)^2}{2!} + \ldots\ldots\right]$$
$$= \frac{1}{(z-a)^2} + \frac{1}{(z-a)} + \frac{1}{2!} + \ldots\ldots$$

Since there are only two terms in the negative powers of $z - a$ *i.e.*, there are only 2 (a finite number) terms in the principal part of the function. Hence the function has a pole of order 2.

(3) Essential singularity: $\lim_{z \to a} f(z)$ does not exist.

Example. $\lim_{z \to a} e^{\frac{1}{z-a}}$ does not exist, so $f(z)$ has an essential singularity at $z = a$.

ILLUSTRATIVE EXAMPLES

Example 1. *Find out the zero and discuss the nature of the singularity of* $f(z) = \frac{z-2}{z^2} \sin \frac{1}{z-1}$.

Sol. Zeros of $f(z)$ are given by $f(z) = 0$

$$\Rightarrow \quad z - 2 = 0, \sin \frac{1}{z-1} = 0$$

$$\Rightarrow \quad z = 2, \frac{1}{z-1} = n\pi \ (n = 0, \pm 1, \pm 2, \ldots\ldots)$$

$$\Rightarrow \quad z = 2, 1 + \frac{1}{n\pi} \ (n = 0, \pm 1, \pm 2, \ldots\ldots)$$

Clearly, $z = 1$ is an isolated essential singularity.

Poles of $f(z)$ are given by

$$z^2 = 0$$
$$\Rightarrow \quad z = 0$$

Hence $z = 0$ is a pole of order 2.

Example 2. *Show that the function* e^z *has an isolated essential singularity at* $z = \infty$.

Sol. Put $z = \dfrac{1}{\rho}$ $\qquad e^{\frac{1}{\rho}} = 1 + \dfrac{1}{\rho} + \dfrac{1}{2!} \cdot \dfrac{1}{\rho^2} + \ldots\ldots\infty$

We have an infinite number of terms in the negative powers of ρ. So the function $e^{\frac{1}{\rho}}$ has an isolated essential singularity at $\rho = 0$. This implies that e^z has an isolated essential singularity at $z = \infty$.

Example 3. *Discuss singularity of* $\dfrac{1}{1-e^z}$ *at* $z = 2\pi i$.

Sol.
$$f(z) = \frac{1}{1-e^z}$$

For poles
$$1 - e^z = 0$$
$$\Rightarrow \quad e^z = 1 = e^{2n\pi i}$$
$$\Rightarrow \quad z = 2n\pi i \ (n = 0, \pm 1, \pm 2, \ldots\ldots)$$

Clearly, $z = 2\pi i$ is a simple pole.

Example 4. *Discuss singularity of* $\dfrac{\cot \pi z}{(z-a)^2}$ *at* $z = a$ *and* $z = \infty$. [U.P.T.U. (C.O.) 2008]

Sol.
$$f(z) = \frac{\cot \pi z}{(z-a)^2} = \frac{\cos \pi z}{\sin \pi z\,(z-a)^2}$$

For poles $\quad \sin \pi z (z-a)^2 = 0$
$$\Rightarrow \quad z = a, \ \pi z = n\pi \ (n \in \mathrm{I})$$
$$\Rightarrow \quad z = a, n$$

Clearly, $z = \infty$ is the limit point of these poles. Hence $z = \infty$ is a non-isolated essential singularity. Also $z = a$, being repeated twice, gives a double pole.

Example 5. *Discuss the nature of singularity of* $f(z) = \dfrac{z - \sin z}{z^3}$ *at* $z = 0$.

Sol.
$$f(z) = \frac{1}{z^3}(z - \sin z) = \frac{1}{z^3}\left[z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots\ldots\right)\right]$$
$$= \frac{1}{z^3}\left(\frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} - \ldots\ldots\right) = \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - \ldots\ldots$$

Since, there is no term in the principal part of given function hence $z = 0$ is a **removable singularity.**

TEST YOUR KNOWLEDGE

1. Discuss singularity of $\dfrac{1}{\sin z - \cos z}$ at $z = \dfrac{\pi}{4}$.
2. Discuss the nature of singularity of the function $f(z) = z \operatorname{cosec} z$ at $z = \infty$.
3. What is the nature of the singularity at $z = \infty$ of the function $f(z) = \cos z - \sin z$?
4. Discuss the singularity of the function $f(z) = \dfrac{1}{\cos \frac{1}{z}}$ at $z = 0$.
5. Discuss the nature of singularity of $f(z) = \sin \dfrac{1}{z}$ at $z = 0$.
6. Find the singularity of the function $g(z) = \dfrac{e^{1/z}}{z^2}$.
7. Prove that the singularity of $\cot z$ at $z = \infty$ is a non-isolated essential singularity.
8. Find the nature of singularities of the following functions:

   (*i*) $\dfrac{1-e^z}{1+e^z}$ at $z = \infty$ (*ii*) $\operatorname{cosec} \dfrac{1}{z}$ at $z = 0$.
9. Discuss singularity of $f(z) = \sin \dfrac{1}{1-z}$ at $z = 1$.

Answers

1. Simple pole **2.** Non-isolated essential **3.** Isolated essential
4. Non-isolated essential **5.** Isolated essential
6. Isolated essential singularity ($z = 0$)
8. (*i*) Non-isolated essential, (*ii*) Non-isolated essential **9.** Isolated essential.

5.22 DEFINITION OF THE RESIDUE AT A POLE

Let $z = a$ be a pole of order m of a one valued function $f(z)$ and γ any circle of radius r with centre at $z = a$ which does not contain any other singularities except at $z = a$, then $f(z)$ is analytic within the annulus $r < | z - a | < R$ hence it can be expanded within the annulus in a Laurent's series in the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n} \quad ...(1)$$

where
$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{(z-a)^{n+1}} \quad ...(2)$$

and
$$b_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{-n+1}}\, dz \quad ...(3)$$

$| z - a | = r$ being the circle γ.

Particularly, $$\boxed{b_1 = \frac{1}{2\pi i}\int_\gamma f(z)\,dz}$$

The coefficient b_1 is called residue of $f(z)$ at the pole $z = a$. It is denoted by symbol Res. $(z = a) = b_1$.

5.23 RESIDUE AT INFINITY

Residue of $f(z)$ at $z = \infty$ is defined as $-\dfrac{1}{2\pi i}\displaystyle\int_C f(z)\,dz$ where the integration is taken round C in anti-clockwise direction.

5.24 CAUCHY'S RESIDUE THEOREM OR THE THEOREM OF RESIDUES

[M.T.U. 2013, G.B.T.U. (C.O.) 2011]

Let $f(z)$ be one valued and analytic within and on a closed contour C except at a finite number of poles $z_1, z_2, z_3, \ldots\ldots, z_n$ and let $R_1, R_2, R_3, \ldots\ldots, R_n$ be respectively the residues of $f(z)$ at these poles, then

$$\boxed{\begin{aligned}\oint_C f(z)\,dz &= 2\pi i\,(R_1 + R_2 + R_3 + \cdots\cdots + R_n)\\ &= 2\pi i\,(\textit{Sum of the residues at the poles within C}).\end{aligned}}$$

Proof. Let $\gamma_1, \gamma_2, \gamma_3, \ldots\ldots, \gamma_n$ be the circles with centres at $z_1, z_2, z_3, \ldots\ldots, z_n$ respectively and radii so small that they lie entirely within the closed curve C and do not overlap. Then $f(z)$ is analytic within the region enclosed by the curve C and these circles. Hence by Cauchy's theorem for multi-connected regions, we have

$$\int_C f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz + \cdots\cdots + \int_{\gamma_n} f(z)\,dz$$

But by definition of residue,

$$R_1 = \frac{1}{2\pi i}\int_{\gamma_1} f(z)\,dz$$

$$\Rightarrow \qquad \int_{\gamma_1} f(z)\,dz = 2\pi i\,R_1$$

Similarly, $$\int_{\gamma_2} f(z)\,dz = 2\pi i\,R_2$$

$$\int_{\gamma_3} f(z)\,dz = 2\pi i\,R_3$$

$$\vdots \qquad \vdots$$

$$\int_{\gamma_n} f(z)\,dz = 2\pi i\,R_n$$

Hence, $$\begin{aligned}\int_C f(z)\,dz &= 2\pi i\,R_1 + 2\pi i R_2 + 2\pi i\,R_3 + \cdots\cdots + 2\pi i\,R_n\\ &= 2\pi i(R_1 + R_2 + R_3 + \cdots\cdots + R_n).\end{aligned}$$

5.25 METHODS OF FINDING OUT RESIDUES

(1) *If $f(z)$ has a simple pole (i.e., pole of order 1) at $z = a$, then*

$$\boxed{\textbf{Res}\ \{f(z)\} = \underset{z \to a}{\textbf{Lt}}\ (z - a)\, f(z).}$$

Since $z = a$ is a pole of order 1, the Laurent's series becomes

$$f(z) = a_0 + a_1(z - a) + a_2(z - a)^2 + \ldots\ldots + b_1(z - a)^{-1}.$$

Multiplying both sides by $(z - a)$, we get

$$(z - a)\, f(z) = a_0(z - a) + a_1(z - a)^2 + a_2(z - a)^3 + \ldots\ldots + b_1$$

$$\therefore \quad \underset{z \to a}{\text{Lt}}\ (z - a)\, f(z) = b_1 = \text{Res}\ \{f(z)\}$$

(2) *If $f(z)$ has a pole of order m at $z = a$, then*

$$\textbf{Res}\ \{f(z)\} = \frac{1}{(m-1)!}\ \underset{z \to a}{\textbf{Lt}}\ \frac{d^{m-1}}{dz^{m-1}}\ [(z - a)^m f(z)]$$

Since $z = a$ is a pole of order m, the Laurent's series becomes

$$f(z) = a_0 + a_1(z - a) + a_2(z - a)^2 + \ldots\ldots + b_1(z - a)^{-1} + b_2(z - a)^{-2} + \ldots\ldots + b_m(z - a)^{-m}$$

Multiplying both sides by $(z - a)^m$, we get

$$(z - a)^m f(z) = a_0(z - a)^m + a_1(z - a)^{m+1} + a_2(z - a)^{m+2} + \ldots\ldots$$
$$+ b_1(z - a)^{m-1} + b_2(z - a)^{m-2} + \ldots\ldots + b_m$$

Differentiating both sides $(m - 1)$ times w.r.t. z and taking the limit as $z \to a$, we get

$$\underset{z \to a}{\text{Lt}}\ \frac{d^{m-1}}{dz^{m-1}}\ [(z - a)^m f(z)] = b_1(m - 1)\,!$$

or
$$\frac{1}{(m-1)!}\ \underset{z \to a}{\text{Lt}}\ \frac{d^{m-1}}{dz^{m-1}}\ [(z - a)^m f(z)] = b_1 = \text{Res}\ \{f(z)\}.$$

Or

$$\boxed{\text{Res. } \{f(z)\} = \frac{1}{(m-1)!}\left[\frac{d^{m-1}}{dz^{m-1}}\{(z - a)^m\ f(z)\}\right]_{z=a}}$$

(3) If $f(z)$ is of the form given by

$$f(z) = \frac{\phi(z)}{\psi(z)}\ ;\ \psi(a) = 0,\ \phi(a) \neq 0$$

where $z = a$ is the simple pole of $f(z)$, then Residue of $f(z)$ at $z = a$ is $= \dfrac{\phi(a)}{\psi'(a)}$.

(4) Residue of $f(z)$ at $z = a$ pole (simple or of order m)

$= \text{Coefficient of } \dfrac{1}{t}$ in $f(a + t)$ expanded in powers of t, where t, is sufficiently small.

(5) Residue of $f(z)$ at $z = \infty$

$$= \lim_{z \to \infty} \{-zf(z)\}$$

Or

$$= -\left[\text{coefficient of } \frac{1}{z} \text{ in the expansion of } f(z) \text{ for values of } z \text{ in the neighbourhood of } z = \infty\right].$$

ILLUSTRATIVE EXAMPLES

Example 1. *Determine the poles of the following functions and residue at each pole:*

(i) $\dfrac{z^2}{(z-1)(z-2)^2}$ (ii) $\dfrac{1}{z^4+1}$ (iii) $\dfrac{1-e^{2z}}{z^4}$

Sol. (i) $$f(z) = \frac{z^2}{(z-1)(z-2)^2}.$$

Poles are given by

$$(z-1)(z-2)^2 = 0 \Rightarrow z = 1, 2.$$

$z = 1$ is a simple pole while $z = 2$ is a double pole.

Residue of $f(z)$ at simple pole ($z = 1$) is

$$R_1 = \lim_{z \to 1} (z-1) \cdot \frac{z^2}{(z-1)(z-2)^2} = \lim_{z \to 1} \frac{z^2}{(z-2)^2} = \frac{(1)^2}{(1-2)^2} = 1.$$

Residue of $f(z)$ at double pole ($z = 2$) is

$$R_2 = \frac{1}{(2-1)!}\left[\frac{d}{dz}\left\{(z-2)^2 \cdot \frac{z^2}{(z-1)(z-2)^2}\right\}\right]_{z=2}$$

$$= \left[\frac{d}{dz}\left(\frac{z^2}{z-1}\right)\right]_{z=2} = \left[\frac{(z-1) \cdot 2z - z^2}{(z-1)^2}\right]_{z=2} = \left[\frac{z^2-2z}{(z-1)^2}\right]_{z=2} = 0$$

(ii) $$f(z) = \frac{1}{z^4+1}$$

Poles of $f(z)$ are given by

$$z^4 + 1 = 0$$

$$\Rightarrow \quad z = (-1)^{1/4} = \{e^{(2n+1)\pi i}\}^{1/4}$$

$\therefore$ Poles are, $z = e^{(2n+1)\pi i/4}$ where, $n = 0, 1, 2, 3, \ldots$

These are all of order 1 since the four factors occur linearly in $z^4 + 1$.

Since the roots repeat themselves, we can write them more conveniently as $e^{(2n+1)\pi i/4}$ where, $n = -2, -1, 0, 1$

i.e., $e^{m\pi i/4}$ where, $m = \pm 1, \pm 3$. Let denote it by z_m.

Residue at ($z = z_m$) is

$$R = \lim_{z \to z_m} (z - z_m) \cdot \frac{1}{z^4+1} = \lim_{z \to z_m} \frac{1}{4z^3} \qquad \text{| By L' Hospital's Rule}$$

$$= \frac{1}{4z_m^3} = \frac{z_m}{4z_m^4} = -\frac{z_m}{4} = -\frac{1}{4} e^{m\pi i/4} \text{ } h \text{ where, } m = \pm 1, \pm 3.$$

(*iii*) Pole of $\frac{1-e^{2z}}{z^4}$ is evidently $z = 0$. But this is not of the fourth order since

$$\frac{1-e^{2z}}{z^4} = \frac{1}{z^4}\left[1-\left\{1+2z+\frac{4z^2}{2}+\frac{8z^3}{6}+\frac{16z^4}{24}+\ldots\ldots\right\}\right]$$

$$= -\frac{\left(2+2z+\frac{4}{3}z^2+\frac{2}{3}z^3+\ldots\ldots\right)}{z^3} \qquad \ldots(1)$$

Therefore, the pole is of order 3.

Residue at this pole is

$$\text{R} = \lim_{z\to 0}\frac{1}{2!}\frac{d^2}{dz^2}\left\{\frac{(1-e^{2z})z^3}{z^4}\right\}$$

$$= \lim_{z\to 0}\frac{1}{2!}\left[\frac{d^2}{dz^2}\left\{\frac{1}{z}\left(1-1-2z-\frac{4z^2}{2!}-\frac{8z^3}{3!}-\ldots\ldots\right)\right\}\right]$$

$$= \lim_{z\to 0}\frac{1}{2}\left[\frac{d^2}{dz^2}\left(-2-2z-\frac{4}{3}z^2-\ldots\ldots\right)\right]$$

$$= \lim_{z\to 0}\frac{1}{2}\left[-\frac{8}{3}-\frac{2}{3}.6z-\ldots\ldots\right] = -\frac{4}{3}.$$

Example 2. *Find the residue at $z = 0$ of the following functions:*

(*i*) $\frac{1+e^z}{\sin z + z\cos z}$ (*ii*) $z\cos\frac{1}{z}$.

Sol. (*i*) $z = 0$ is a pole of order 1.

$$\text{Residue} = \lim_{z\to 0}\frac{z(1+e^z)}{\sin z + z\cos z} = \lim_{z\to 0}\frac{1+e^z}{\left(\frac{\sin z}{z}\right)+\cos z} = \frac{1+1}{1+1} = 1.$$

(*ii*) Expanding the function in powers of z, we have

$$z\cos\frac{1}{z} = z\left[1-\frac{1}{2z^2}+\frac{1}{4!\,z^4}-\ldots\ldots\right] = z-\frac{1}{2z}+\frac{1}{24z^3}-\ldots\ldots$$

This is the Laurent 's expansion about $z = 0$.

The coefficient of $\frac{1}{z}$ in it is $-\frac{1}{2}$. So the residue of $z\cos\frac{1}{z}$ at $z = 0$ is $-\frac{1}{2}$.

Example 3. (*a*) *Give an example of a function having residue at infinity yet analytic there.*

(*b*) *Find the residue of $f(z) = \frac{z^3}{z^2-1}$ at $z = \infty$.*

Sol. (*a*) $$f(z) = \frac{z^2}{(z-\alpha)(z-\beta)(z-\gamma)}$$

Residue of $f(z)$ at $z = \infty$ is

$$= \lim_{z\to\infty}\left\{-z\cdot\frac{z^2}{(z-\alpha)(z-\beta)(z-\gamma)}\right\}$$

$$= \lim_{z\to\infty}\frac{-1}{\left(1-\frac{\alpha}{z}\right)\left(1-\frac{\beta}{z}\right)\left(1-\frac{\gamma}{z}\right)} = -1$$

Now, $$f\left(\frac{1}{\lambda}\right) = \frac{\frac{1}{\lambda^2}}{\frac{(1-\alpha\lambda)}{\lambda}\cdot\frac{(1-\beta\alpha)}{\lambda}\cdot\frac{(1-\gamma\lambda)}{\lambda}} = \frac{\lambda}{(1-\alpha\lambda)(1-\beta\lambda)(1-\gamma\lambda)}$$

At $\lambda = 0$, $$f\left(\frac{1}{\lambda}\right) = 0\ (\neq\infty)$$

$\therefore$ $f\left(\frac{1}{\lambda}\right)$ is analytic at $\lambda = 0$

$\Rightarrow$ $f(z)$ is analytic at $z = \infty$.

(*b*) Required residue $= \underset{z\to\infty}{\text{Lt}}\left(-z\cdot\frac{z^3}{z^2-1}\right)$ which does not exist.

Hence, $$f(z) = \frac{z^3}{z^2\left(1-\frac{1}{z^2}\right)} = z\left(1-\frac{1}{z^2}\right)^{-1} = z\left(1+\frac{1}{z^2}+\frac{1}{z^4}+\ldots\ldots\right) = z+\frac{1}{z}+\frac{1}{z^3}+\ldots\ldots$$

Required residue $= -\left(\text{coefficient of } \frac{1}{z}\right) = -1.$

Example 4. *Evaluate* $\oint_C \frac{e^z}{(z+1)^2}\,dz$, *where C is the circle* $|z-1| = 3$.

Sol. Here $f(z) = \frac{e^z}{(z+1)^2}$ has only one singular point $z = -1$ which is a pole of order 2 and it lies inside the circle $|z-1| = 3$.

Residue of $f(z)$ at $z = -1$ is $\underset{z\to-1}{\text{Lt}}\frac{d}{dz}[(z+1)^2 f(z)] = \underset{z\to-1}{\text{Lt}}\frac{d}{dz}(e^z) = \underset{z\to-1}{\text{Lt}}\ e^z = e^{-1}$

$\therefore$ By Residue theorem, we have $\oint_C \frac{e^z}{(z+1)^2}\,dz = 2\pi i(e^{-1}) = \frac{2\pi i}{e}$.

Example 5. *Determine the poles of the function* $f(z) = \dfrac{z^2}{(z-1)^2(z+2)}$ *and the residue at each pole. Hence evaluate* $\int_C \dfrac{z^2}{(z-1)^2(z+2)}\,dz$ *where* $C \equiv |z| = 3.$ (U.P.T.U. 2015)

Sol. The function $f(z)$ has a pole of order 2 at $z = 1$ and a simple pole at $z = -2$.

Residue of $f(z)$ at $z = 1$ is

$$R_1 = \underset{z\to 1}{\text{Lt}} \frac{d}{dz}[(z-1)^2 f(z)] = \underset{z\to 1}{\text{Lt}} \frac{d}{dz}\left(\frac{z^2}{z+2}\right)$$

or
$$R_1 = \underset{z\to 1}{\text{Lt}} \frac{(z+2).2z - z^2.1}{(z+2)^2} = \underset{z\to 1}{\text{Lt}} \frac{z^2+4z}{(z+2)^2} = \frac{5}{9}$$

Residue of $f(z)$ at $z = -2$ is

$$R_2 = \underset{z\to -2}{\text{Lt}} [(z+2) f(z)] = \underset{z\to -2}{\text{Lt}} \frac{z^2}{(z-1)^2} = \frac{4}{9}.$$

Since both the poles lie inside the given curve $C \equiv |z| = 3$,

$$\therefore \quad \int_C \frac{z^2}{(z-1)^2(z+2)}\,dz = 2\pi i(R_1 + R_2) = 2\pi i\left[\frac{5}{9}+\frac{4}{9}\right] = 2\pi i.$$

| By Cauchy's Residue theorem

Example 6. *Determine the poles of the following function and residues at each pole:*

$f(z) = \dfrac{z-1}{(z+1)^2(z-2)}$ *and hence evaluate* $\oint_C f(z)\,dz$*, where C is the circle* $|z-i| = 2.$

(U.K.T.U. 2011)

Sol. Poles of $f(z)$ are given by

$(z+1)^2(z-2) = 0 \quad \Rightarrow \quad z = -1$ (double pole), 2(simple pole)

Residue of $f(z)$ at $z = -1$ is

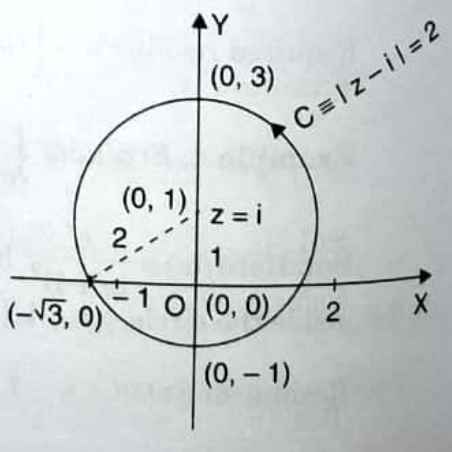
$$R_1 = \frac{1}{(2-1)!}\left[\frac{d}{dz}\left\{(z+1)^2 . \frac{z-1}{(z+1)^2(z-2)}\right\}\right]_{z=-1}$$

$$= \left[\frac{d}{dz}\left(\frac{z-1}{z-2}\right)\right]_{z=-1} = \left[\frac{-1}{(z-2)^2}\right]_{z=-1} = \frac{-1}{9}$$

Residue of $f(z)$ at $z = 2$ is

$$R_2 = \underset{z\to 2}{\text{Lt}} (z-2)\frac{z-1}{(z+1)^2(z-2)}$$

$$= \underset{z\to 2}{\text{Lt}} \frac{z-1}{(z+1)^2} = \frac{1}{9}$$



The given curve $C \equiv |z-i| = 2$ is a circle whose centre is at $z = i$ [*i.e.*, at (0, 1)] and radius is 2. Clearly, only the pole $z = -1$ lies inside the curve C.

Hence, by Cauchy's residue theorem

$$\oint_C f(z)\,dz = 2\pi i\,(R_1) = 2\pi i\left(\frac{-1}{9}\right) = -\frac{2\pi i}{9}.$$

Example 7. *Evaluate* $\int_C \frac{z^2-2z}{(z+1)^2(z^2+4)}\,dz$, *where C is the circle* $|z| = 10$.

Sol. Singularities are given by

$$(z+1)^2(z^2+4) = 0 \;\Rightarrow\; z = -1 \text{ (double pole)}, \pm 2i \text{ (simple poles)}$$

All the poles lie inside the given circle $c \equiv |z| = 10$.

$\therefore$ Residue (at $z = -1$) is

$$R_1 = \frac{1}{2-1!}\left[\frac{d}{dz}\left\{(z+1)^2\frac{z^2-2z}{(z+1)^2(z^2+4)}\right\}\right]_{z=-1}$$

$$= \left[\frac{d}{dz}\left(\frac{z^2-2z}{z^2+4}\right)\right]_{z=-1} = \left[\frac{2z^2+8z-8}{(z^2+4)^2}\right]_{z=-1} = -\frac{14}{25}$$

Residue (at $z = 2i$) is

$$R_2 = \lim_{z\to 2i}(z-2i)\frac{z^2-2z}{(z+1)^2(z-2i)(z+2i)}$$

$$= \frac{-4-4i}{(2i+1)^2(4i)} = \frac{1+i}{3i+4} = \frac{7+i}{25}$$

Similarly, Residue (at $z = -2i$) is

$$R_3 = \frac{7-i}{25}$$

By Cauchy's Residue theorem,

$$\int_C \frac{z^2-2z}{(z+1)^2(z^2+4)}\,dz = 2\pi i\,(R_1+R_2+R_3) = 2\pi i\left[-\frac{14}{25}+\frac{7+i}{25}+\frac{7-i}{25}\right] = 0.$$

Example 8. *Evaluate* $\int_C \frac{12z-7}{(z-1)^2(2z+3)}\,dz$, *where C is the circle*

(*i*) $|z| = 2$ (G.B.T.U. 2011) (*ii*) $|z+i| = \sqrt{3}$.

Sol.
$$f(z) = \frac{12z-7}{(z-1)^2(2z+3)}.$$

Poles are given by

$$z = 1 \text{ (double pole)} \quad\text{and}\quad z = -\frac{3}{2} \text{ (simple pole)}$$

Residue at ($z = 1$) is

$$R_1 = \frac{1}{(2-1)!}\left[\frac{d}{dz}\left\{(z-1)^2\cdot\frac{12z-7}{(z-1)^2(2z+3)}\right\}\right]_{z=1}$$

$$= \left[\frac{d}{dz}\left(\frac{12z-7}{2z+3}\right)\right]_{z=1} = \left[\frac{(2z+3).12-(12z-7).2}{(2z+3)^2}\right]_{z=1}$$

$$= \frac{60-10}{25} = 2.$$

Residue at simple pole $\left(z = -\frac{3}{2}\right)$ is

$$R_2 = \lim_{z \to -3/2} \left(z + \frac{3}{2}\right) . \frac{12z-7}{(z-1)^2 (2z+3)}$$

$$= \lim_{z \to -3/2} \frac{1}{2} . \frac{(12z-7)}{(z-1)^2} = -2.$$

(*i*) The contour $|z| = 2$ encloses both the poles 1 and $-\frac{3}{2}$.

$\therefore$ The given integral $= 2\pi i (R_1 + R_2) = 2\pi i (2-2) = 0$.

(*ii*) The contour $|z+i| = \sqrt{3}$ is a circle of radius $\sqrt{3}$ and centre at $z = -i$. The distances of the centre from $z = 1$ and $-\frac{3}{2}$ are respectively $\sqrt{2}$ and $\sqrt{\frac{13}{4}}$. The first of these is $< \sqrt{3}$ and the second is $> \sqrt{3}$.

$\therefore$ The second contour includes only the first singularity $z = 1$.

Hence, the given integral $= 2\pi i (R_1) = 2\pi i (2) = 4\pi i$.

Example 9. *Evaluate* $\oint_C \frac{z-3}{z^2+2z+5} dz$, *where C is the circle*

(*i*) $|z| = 1$ (*ii*) $|z+1-i| = 2$ (*iii*) $|z+1+i| = 2$.

(G.B.T.U. 2013)

Sol. The poles of $f(z) = \frac{z-3}{z^2+2z+5}$ are given by

$$z^2 + 2z + 5 = 0 \Rightarrow z = -1 \pm 2i$$

(*i*) Both the poles lie outside the circle $|z| = 1$.

$\therefore$ By Cauchy's integral theorem, we have $\oint_C \frac{z-3}{z^2+2z+5} dz = 0$

(*ii*) Only the pole $z = -1+2i$ lies inside the circle $|z+1-i| = 2$

Residue of $f(z)$ at $z = -1+2i$ is

$$\operatorname*{Lt}_{z \to -1+2i} (z+1-2i) f(z) = \operatorname*{Lt}_{z \to \alpha} \frac{(z-\alpha)(z-3)}{z^2+2z+5}, \text{ where } \alpha = -1-2i \qquad \left| \text{ Form } \frac{0}{0} \right.$$

$$= \operatorname*{Lt}_{z \to \alpha} \frac{(z-\alpha)+(z-3)}{2z+2} \qquad | \text{ By L' Hospital's Rule}$$

$$= \frac{\alpha-3}{2\alpha+2} = \frac{-1+2i-3}{-2+4i+2} = \frac{i-2}{2i}$$

$\therefore$ By Cauchy's residue theorem, $\oint_C \frac{z-3}{z^2+2z+5} dz = 2\pi i \left(\frac{i-2}{2i}\right) = \pi(i-2)$.

(*iii*) Only the pole $z = -1 - 2i$ lies inside the circle $|z + 1 + i| = 2$.

Residue of $f(z)$ at $z = -1 - 2i$ is

$$\underset{z \to -1-2i}{\text{Lt}} (z + 1 + 2i) f(z) = \underset{z \to \beta}{\text{Lt}} \frac{(z-\beta)(z-3)}{z^2+2z+5}, \text{ where } \beta = -1 - 2i \qquad \Big| \text{ Form } \frac{0}{0}$$

$$= \underset{z \to \beta}{\text{Lt}} \frac{(z-\beta)+(z-3)}{2z+2} \qquad | \text{ By L' Hospital's Rule}$$

$$= \frac{\beta - 3}{2\beta + 2} = \frac{-4-2i}{-4i} = \frac{i+2}{2i}$$

$\therefore$ By Cauchy's residue theorem, $\oint_C \frac{z-3}{z^2+2z+5} dz = 2\pi i \left(\frac{i+2}{2i}\right) = \pi(i+2)$.

Example 10. *Find the residue of* $f(z) = \dfrac{z^3}{(z-1)^4 (z-2)(z-3)}$ *at its pole and hence evaluate* $\int_C f(z)\, dz$, *where C is the circle* $|z| = 5/2$.

Sol. Poles of $f(z)$ are given by $(z-1)^4(z-2)(z-3) = 0 \quad \Rightarrow \quad z = 1, 2, 3$

$z = 1$ is a pole of order 4 while $z = 2$ and $z = 3$ are simple poles.

Residue of $f(z)$ at $z = 2$ is

$$R_1 = \underset{z \to 2}{\text{Lt}} (z-2) \cdot \frac{z^3}{(z-1)^4 (z-2)(z-3)} = \underset{z \to 2}{\text{Lt}} \frac{z^3}{(z-1)^4 (z-3)} = \frac{8}{(-1)} = -8$$

Residue of $f(z)$ at $z = 3$ is

$$R_2 = \underset{z \to 3}{\text{Lt}} (z-3) \cdot \frac{z^3}{(z-1)^4 (z-2)(z-3)} = \underset{z \to 3}{\text{Lt}} \frac{z^3}{(z-1)^4 (z-2)} = \frac{27}{16}$$

Residue of $f(z)$ at $z = 1$ is

$$R_3 = \frac{1}{(4-1)!} \left[\frac{d^3}{dz^3} \left\{(z-1)^4 \cdot \frac{z^3}{(z-1)^4 (z-2)(z-3)}\right\}\right]_{z=1}$$

$$= \frac{1}{6}\left[\frac{d^3}{dz^3}\left\{\frac{z^3}{(z-2)(z-3)}\right\}\right]_{z=1} = \frac{1}{6}\left[\frac{d^3}{dz^3}\left\{z + 5 + \frac{19z-30}{z^2-5z+6}\right\}\right]_{z=1}$$

$$= \frac{1}{6}\left[\frac{d^3}{dz^3}\left\{z+5+\frac{27}{z-3} - \frac{8}{z-2}\right\}\right]_{z=1} = \frac{1}{6}\left[\frac{d^2}{dz^2}\left\{1 - \frac{27}{(z-3)^2} + \frac{8}{(z-2)^2}\right\}\right]_{z=1}$$

$$= \frac{1}{6}\left[\frac{d}{dz}\left\{\frac{54}{(z-3)^3} - \frac{16}{(z-2)^3}\right\}\right]_{z=1} = \frac{1}{6}\left[\frac{-162}{(z-3)^4} + \frac{48}{(z-2)^4}\right]_{z=1}$$

$$= \frac{1}{6}\left[\frac{-162}{16} + 48\right] = 8 - \frac{27}{16} = \frac{101}{16}.$$

The given curve $C \equiv |z| = 5/2$ is a circle with centre at (0, 0) and radius 5/2. Clearly, only the poles $z = 1$ and $z = 2$ lie inside this circle.

Hence, By Cauchy's Residue theorem,

$$\int_C f(z)\, dz = 2\pi i\,(R_3 + R_1) = 2\pi i\left(\frac{101}{16} - 8\right) = 2\pi i\left(\frac{-27}{16}\right) = -\frac{27\pi i}{8}.$$

Example 11. (*i*) *Find the value of* $\oint_C ze^{1/z}\, dz$ *around the unit circle.*

(*ii*) *Using Residue theorem, evaluate* $\dfrac{1}{2\pi i}\displaystyle\int_C \frac{e^{zt}}{z^2(z^2+2z+2)}\, dz$*, where C is the circle* $|z| = 3$. (U.P.T.U. 2009)

Sol. (*i*) The only singularity of $ze^{1/z}$ is at the origin. Expanding $e^{1/z}$, we have

$$ze^{1/z} = z\left[1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \ldots\ldots\right] = z + 1 + \frac{1}{2z} + \frac{1}{6z^2} + \ldots\ldots$$

Residue at origin = coefficient of $\dfrac{1}{z} = \dfrac{1}{2}$.

Hence, the required integral $= 2\pi i\left(\dfrac{1}{2}\right) = \pi i$.

(*ii*) Singularities are given by

$$z^2(z^2 + 2z + 2) = 0 \quad\Rightarrow\quad z = 0, -1 \pm i$$

$z = 0$ is a pole of order 2. $z = -1 \pm i$ are simple poles. All these poles lie inside the circle $|z| = 3$.

Residue (at $z = 0$) is

$$R_1 = \frac{1}{(2-1)!}\left[\frac{d}{dz}\left\{z^2 \cdot \frac{e^{zt}}{2\pi i\, z^2(z^2+2z+2)}\right\}\right]_{z=0} = \left[\frac{d}{dz}\left\{\frac{e^{zt}}{2\pi i\,(z^2+2z+2)}\right\}\right]_{z=0}$$

$$= \frac{1}{2\pi i}\left[\frac{(z^2+2z+2)\,t\,e^{zt} - e^{zt}\,(2z+2)}{(z^2+2z+2)^2}\right]_{z=0} = \frac{1}{2\pi i}\left(\frac{t-1}{2}\right)$$

Let $-1 + i = \alpha$ and $-1 - i = \beta$ then

Residue at $(z = \alpha = -1 + i)$ is

$$R_2 = \underset{z\to\alpha}{\text{Lt}}\,(z-\alpha)\cdot\frac{1}{2\pi i}\,\frac{e^{zt}}{z^2\,(z-\alpha)\,(z-\beta)} = \frac{1}{2\pi i}\,\frac{e^{\alpha t}}{\alpha^2(\alpha-\beta)} = \frac{1}{2\pi i}\left[\frac{1}{4}\,e^{(-1+i)t}\right]$$

Residue at $(z = \beta = -1 - i)$ is

$$R_3 = \frac{1}{2\pi i}\left[\frac{1}{4}\,e^{(-1-i)t}\right]$$

By Residue theorem,

$$\frac{1}{2\pi i}\int_C \frac{e^{zt}}{z^2(z^2+2z+2)}\,dz = 2\pi i\left[\frac{1}{2\pi i}\left(\frac{t-1}{2}\right)+\frac{1}{2\pi i}\left\{\frac{e^{(-1+i)t}+(e^{(-1-i)t}}{4}\right\}\right]$$

$$= \frac{t-1}{2}+\frac{e^{-t}}{2}\cos t.$$

Example 12. *Obtain Laurent's expansion for the function* $f(z) = \dfrac{1}{z^2 \sinh z}$ *at the isolated singularity and hence evaluate* $\oint_C \dfrac{1}{z^2 \sinh z}\,dz$, *where C is the circle* $|z-1| = 2$.

Sol. Here, $f(z) = \dfrac{1}{z^2\sinh z} = \dfrac{2}{z^2(e^z - e^{-z})}$

$$= \frac{2}{z^2\left[\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\ldots\ldots\right)-\left(1-z+\frac{z^2}{2!}-\frac{z^3}{3!}+\ldots\ldots\right)\right]}$$

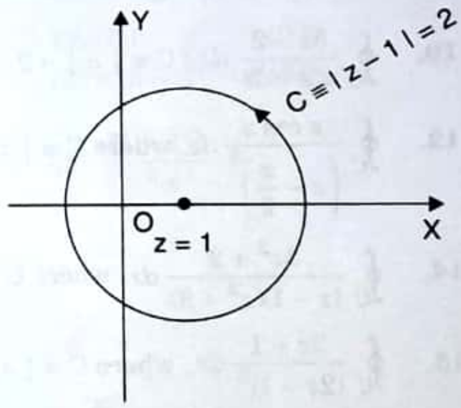
$$= \frac{2}{z^2\left(2z+\frac{2z^3}{3!}+\frac{2z^5}{5!}+\ldots\ldots\right)}$$

$$= \frac{1}{z^3\left(1+\frac{z^2}{3!}+\frac{z^4}{5!}+\ldots\ldots\right)}$$

$$= z^{-3}\left[1+\left(\frac{z^2}{6}+\frac{z^4}{120}\right)+\ldots\ldots\right]^{-1}$$

$$= z^{-3}\left(1-\frac{z^2}{6}-\frac{z^4}{120}+\frac{z^4}{36}+\ldots\ldots\right)$$

$$= \frac{1}{z^3}-\frac{1}{6z}+\frac{7}{360}z+\ldots\ldots$$



Only pole $z = 0$ of order two lies inside the circle $C \equiv |z-1| = 2$.

Residue of $f(z)$ at $(z = 0)$ is = coeff. of $\dfrac{1}{z}$ in the Laurent's expansion of $f(z) = -\dfrac{1}{6}$.

By Cauchy's Residue theorem,

$$\oint_C \frac{dz}{z^2\sinh z} = 2\pi i\left(-\frac{1}{6}\right) = -\frac{\pi i}{3}.$$

TEST YOUR KNOWLEDGE

Determine the poles of the following functions and the residue at each pole:

1. $\dfrac{2z+1}{z^2-z-2}$ **2.** $\dfrac{z+1}{z^2(z-2)}$ **3.** $\dfrac{e^z}{z^2+\pi^2}$.

Evaluate the following integrals using Cauchy's residue theorem:

4. $\oint_C \left[\dfrac{\cos \pi z^2 + \sin \pi z^2}{(z+1)(z+2)}\right] dz$; $C \equiv |z| = 3$

5. $\oint_C \left[\dfrac{3z^2+z+1}{(z^2-1)(z+3)}\right] dz$, where C is the circle $|z| = 2$.

6. $\oint_C \dfrac{z^2+2z-2}{z-4}\, dz$, where C is a closed curve containing the point $z = 4$ in its interior.

7. $\oint_C \dfrac{1-2z}{z(z-1)(z-2)}\, dz$, where C is the circle $|z| = 1.5$.

8. $\oint_C \dfrac{z}{(z-1)(z-2)^2}\, dz$, where C is the circle $|z-2| = \dfrac{1}{2}$.

9. $\oint_C \dfrac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)}\, dz$, where C is the circle $|z| = 3$.

10. $\oint_C \dfrac{5z-2}{z(z-1)}\, dz$; $C \equiv |z| = 2$

11. $\oint_C \dfrac{e^z-1}{z(z-i)^2(z-1)}\, dz$; $C \equiv |z| = 1/2$

12. $\oint_C \dfrac{z\cos z}{\left(z-\dfrac{\pi}{2}\right)^3}\, dz$, where $C \equiv |z-1| = 1$

13. $\oint_C \dfrac{dz}{(z^2+4)^2}$, where $C \equiv |z-i| = 2$

14. $\oint_C \dfrac{3z^2+2}{(z-1)(z^2+9)}\, dz$, where $C \equiv |z-2| = 2$

15. $\oint_C \dfrac{2z+1}{(2z-1)^2}\, dz$, where $C \equiv |z| = 1$

16. $\oint_C \dfrac{dz}{(z^2+1)(z^2-4)}\, dz$, where $C \equiv |z| = 1.5$

17. (*i*) $\oint_C \dfrac{4z^2-4z+1}{(z-2)(4+z^2)}\, dz$, where $C \equiv |z| = 1$.

(*ii*) $\oint_C \dfrac{24z-7}{(z-1)^2(2z+3)}\, dz$, where C is the circle of radius 2 with centre at the origin.

(*M.T.U. 2012*)

18. $\oint_C \dfrac{z^2+4}{z(z^2+2z+2)}\, dz$, where C is

(*i*) $|z| = 1$ (*ii*) $|z+1-i| = 1$

(*iii*) $|z+1+i| = 1$ (*iv*) $|z-1| = 5$

19. $\oint_C \dfrac{e^{-z}}{z^2}\, dz$; $C \equiv |z| = 1$

20. $\oint_C z^2 e^{1/z}\, dz$; $C \equiv |z| = 1$

21. $\oint_C \dfrac{1}{z^2 \sin z}\, dz$, where C is the triangle with vertices (0, 1), (2 – 2) and (7, 1). *(G.B.T.U. 2012)*

22. Determine the poles and residues at each pole of the function $f(z) = \dfrac{z}{z^2 - 3z + 2}$ and hence evaluate $\oint_C f(z)\, dz$ where C is the circle $|z - 2| = \dfrac{1}{2}$. *(G.B.T.U. 2011)*

23. Find the poles (with its order) and residue at each pole of the following function

$$f(x) = \frac{1 - 2z}{z(z-1)(z-2)^2}.$$ *(A.K.T.U. 2017)*

Answers

1. $z = -1, 2 ; \frac{1}{3}, \frac{5}{3}$ **2.** $z = 0, 2 ; -\frac{3}{4}, \frac{3}{4}$ **3.** $z = \pm \pi i ; \pm \frac{i}{2\pi}$

4. $-4\pi i$ **5.** $-\frac{\pi i}{4}$ **6.** $44\pi i$

7. $3\pi i$ **8.** $-2\pi i$ **9.** $4\pi i(\pi + 1)$

10. $10\pi i$ **11.** 0 **12.** $-2\pi i$

13. $\pi/16$ **14.** πi

15. πi **16.** 0 **17.** (*i*) 0 (*ii*) 0

18. (*i*) $4\pi i$ (*ii*) $-\pi(3 + i)$ (*iii*) $\pi(3 - i)$ (*iv*) $2\pi i$

19. $-2\pi i$ **20.** $\frac{\pi i}{3}$. **21.** $\frac{2i(-1)^n}{n^2 \pi}$

22. $z = 1, 2 ; -1, 2 ; 4\pi i$.

23. Poles : $z = 0$ (order 1), $z = 1$ (order 1), $z = 2$ (order 2)

Residues : $-\frac{1}{4}, -1, \frac{5}{4}$.

5.26 CONTOUR INTEGRATION

We take a closed curve C, find the poles of $f(z)$ within C and calculate residue at these poles. Then by Cauchy's residue theorem

$$\int_C f(z)\, dz = 2\pi i \text{ [sum of the residues of } f(z) \text{ at the poles within C]}$$

The curve is called a contour.

The process of integration along a contour is called *contour integration.*

5.27 APPLICATION OF RESIDUE THEOREM TO EVALUATE REAL INTEGRALS

The residue theorem provides a simple and elegant method for evaluating many important definite integrals of real variables. Some of these are illustrated below.

5.27.1 Integrals of the Type $\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$, where $F(\cos\theta, \sin\theta)$ is a Rational Function of $\cos\theta$ and $\sin\theta$.

Such integrals can be reduced to complex line integrals by the substitution $z = e^{i\theta}$, so that

$$dz = ie^{i\theta}\, d\theta, \text{ i.e., } d\theta = \frac{dz}{iz}.$$

Also,
$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{1}{2i}\left(z - \frac{1}{z}\right).$$

As θ varies from 0 to 2π, z moves once round the unit circle in the anti-clockwise direction.

$$\therefore \quad \boxed{\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta = \oint_C F\left(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2i}\right)\frac{dz}{iz}}$$

where C is the unit circle $|z| = 1$.

ILLUSTRATIVE EXAMPLES

Example 1. *Using contour integration, evaluate* $\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta}$ *where* $a > |b|$

Hence or otherwise evaluate (U.K.T.U. 2010)

(*i*) $\int_0^{2\pi} \frac{d\theta}{\sqrt{2} - \cos\theta}$ (*ii*) $\int_0^{\pi} \frac{d\theta}{a + b\cos\theta}$; $a > |b|$

Sol. Consider the integration round a unit circle $C \equiv |z| = 1$ so that $z = e^{i\theta}$

$$\therefore \quad dz = ie^{i\theta} d\theta = iz\, d\theta \;\Rightarrow\; d\theta = \frac{dz}{iz}$$

Also,
$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

Then the given integral reduces to

$$I = \oint_C \frac{1}{\left[a + \frac{b}{2}\left(z + \frac{1}{z}\right)\right]}\left(\frac{dz}{iz}\right) = \oint_C \frac{2z}{bz^2 + 2az + b}\left(\frac{dz}{iz}\right)$$

$$= \frac{2}{ib}\oint_C \frac{dz}{z^2 + \frac{2a}{b}z + 1} = \frac{2}{ib}\oint_C \frac{dz}{(z-\alpha)(z-\beta)}$$

where,
$$\alpha = -\frac{a}{b} + \frac{\sqrt{a^2 - b^2}}{b} \quad \text{and} \quad \beta = -\frac{a}{b} - \frac{\sqrt{a^2 - b^2}}{b}$$

Poles are given by $(z-\alpha)(z-\beta) = 0 \quad \Rightarrow \quad z = \alpha, \beta$

Both are simple poles.

Since $\quad a > |b| \quad \therefore \quad |\beta| > 1$

Since $\quad \alpha\beta = 1$

$\therefore \quad |\alpha\beta| = 1$

$|\alpha|\,|\beta| = 1$

$\Rightarrow \quad |\alpha| < 1 \qquad | \because \ |\beta| > 1$

Hence $z = \alpha$ is the only pole which lies inside the circle $C \equiv |z| = 1$.

Residue of $f(z)$ at $(z = \alpha)$ is

$$R = \underset{z\to\alpha}{\text{Lt}}\ (z-\alpha) f(z) = \underset{z\to\alpha}{\text{Lt}}\ (z-\alpha) . \frac{2}{ib(z-\alpha)(z-\beta)}$$

$$= \frac{2}{ib(\alpha-\beta)} = \frac{2(b)}{ib(2\sqrt{a^2-b^2})} = \frac{1}{i\sqrt{a^2-b^2}}$$

By Cauchy's Residue theorem,

$$I = 2\pi i(R) = 2\pi i\left(\frac{1}{i\sqrt{a^2-b^2}}\right)$$

$$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}} \qquad ...(1)$$

(*i*) Putting $a = \sqrt{2}$ and $b = -1$ in (1), we get

$$\int_0^{2\pi} \frac{d\theta}{\sqrt{2}-\cos\theta} = \frac{2\pi}{\sqrt{2-1}} = 2\pi$$

(*ii*) From (1),

$$2\int_0^{\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}} \qquad | \text{ Using prop. of definite integrals}$$

$$\Rightarrow \quad \int_0^{\pi} \frac{d\theta}{a+b\cos\theta} = \frac{\pi}{\sqrt{a^2-b^2}}.$$

Example 2. *Evaluate by contour integration:* $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta}$, *where* $a > |b|$

(A.K.T.U. 2017)

Hence or otherwise evaluate $\int_0^{2\pi} \frac{d\theta}{1-2a\sin\theta+a^2}$, $0 < a < 1$.

Sol. Consider the integration round a unit circle $C \equiv |z| = 1$

so that $\quad z = e^{i\theta} \quad \therefore \quad d\theta = \frac{dz}{iz}$.

Also, $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$

Then the given integral reduces to

$$I = \oint_C \frac{1}{\left[a + \frac{b}{2i}\left(z - \frac{1}{z}\right)\right]}\left(\frac{dz}{iz}\right) = \oint_C \frac{2iz}{bz^2 + 2iaz - b}\left(\frac{dz}{iz}\right)$$

$$= \frac{2}{b}\oint_C \frac{dz}{z^2 + \frac{2ia}{b}z - 1}$$

Poles are given by

$$z^2 + \frac{2ia}{b}z - 1 = 0$$

$$\Rightarrow \quad z = \frac{-\frac{2ia}{b} \pm \sqrt{-\frac{4a^2}{b^2} + 4}}{2} = \frac{-ia}{b} \pm \frac{\sqrt{b^2 - a^2}}{b}$$

$$= \frac{-ia}{b} \pm \frac{i\sqrt{a^2 - b^2}}{b} = \alpha, \beta \text{ (simple poles)}$$

where, $\alpha = \frac{-ia}{b} + \frac{i\sqrt{a^2 - b^2}}{b}$ and $\beta = \frac{-ia}{b} - \frac{i\sqrt{a^2 - b^2}}{b}$

Clearly, $|\beta| > 1$

But $\alpha\beta = -1$

$\therefore$ $|\alpha\beta| = 1 \Rightarrow |\alpha|\,|\beta| = 1 \Rightarrow |\alpha| < 1$

Hence $z = \alpha$ is the only pole which lies inside circle $C \equiv |z| = 1$.

Residue of $f(z)$ at $(z = \alpha)$ is

$$R = \underset{z \to \alpha}{\text{Lt}} (z - \alpha) \cdot \frac{2}{b(z - \alpha)(z - \beta)} = \frac{2}{b(\alpha - \beta)}$$

$$= \frac{2}{b\left(\frac{2i\sqrt{a^2 - b^2}}{b}\right)} = \frac{1}{i\sqrt{a^2 - b^2}}$$

$\therefore$ By Cauchy's Residue theorem,

$$I = 2\pi i\,(R) = 2\pi i\left(\frac{1}{i\sqrt{a^2 - b^2}}\right) = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

$$\therefore \quad \int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}} \qquad ...(1)$$

If we replace a by $1 + a^2$ and b by $-2a$, then

$$\int_0^{2\pi} \frac{d\theta}{(1 + a^2) - 2a\sin\theta} = \frac{2\pi}{\sqrt{(1 + a^2)^2 - 4a^2}} = \frac{2\pi}{\sqrt{1 + a^4 - 2a^2}} = \frac{2\pi}{1 - a^2}.$$

Example 3. *Use contour integration method to evaluate the following integral:*

$$\int_0^{\pi} \frac{a\,d\theta}{a^2 + \sin^2\theta}, \; (a > 0).$$

Sol.

$$I = \int_0^{\pi} \frac{a\,d\theta}{a^2 + \dfrac{(1-\cos 2\theta)}{2}}$$

$$= 2a \int_0^{\pi} \frac{d\theta}{(2a^2+1) - \cos 2\theta} \qquad \Big|\; \text{Put } 2\theta = \phi,\ d\theta = \frac{d\phi}{2}$$

$$= a \int_0^{2\pi} \frac{d\phi}{(2a^2+1) - \cos\phi}$$

$$I = 2a \int_0^{2\pi} \frac{d\phi}{(4a^2+2) - (e^{i\phi} + e^{-i\phi})} \qquad \ldots(1)$$

But $z = e^{i\phi}$ so that $d\phi = \dfrac{dz}{iz}$ then (1) reduces to

$$I = 2a \int_C \frac{1}{(4a^2+2) - \left(z + \dfrac{1}{z}\right)} \cdot \frac{dz}{iz} = \frac{2a}{i} \int_C \frac{dz}{4a^2z + 2z - z^2 - 1}$$

$$= 2ai \int_C \frac{dz}{z^2 - 2z(1+2a^2) + 1} = 2ai \int_C \frac{dz}{(z-\alpha)(z-\beta)}$$

where, $\alpha = (1 + 2a^2) + 2a\sqrt{1+a^2}$ and $\beta = (1 + 2a^2) - 2a\sqrt{1+a^2}$

Clearly, $|\alpha| > 1$

$\because$ $|\alpha\beta| = 1$ $\quad\therefore\quad |\beta| < 1$

$\therefore$ Only β lies inside C.

Residue (at $z = \beta$) is $\displaystyle = \lim_{z\to\beta} (z-\beta) \cdot \frac{2ai}{(z-\alpha)(z-\beta)} = \frac{2ai}{\beta - \alpha} = \frac{2ai}{-4a\sqrt{1+a^2}} = \frac{-i}{2\sqrt{1+a^2}}.$

By Cauchy Residue theorem,

$$I = 2\pi i \left(\frac{-i}{2\sqrt{1+a^2}}\right) = \frac{\pi}{\sqrt{1+a^2}}.$$

Example 4. *Apply Calculus of residues to prove that:*

$$\int_0^{2\pi} \frac{d\phi}{(a + b\cos\phi)^2} = \frac{2\pi a}{(a^2 - b^2)^{3/2}}; \; \textit{where } a > 0,\ b > 0,\ a > b.$$

Sol. Let,

$$I = \int_0^{2\pi} \frac{d\phi}{(a + b\cos\phi)^2} = \int_0^{2\pi} \frac{d\phi}{\left\{a + \dfrac{b}{2}(e^{i\phi} + e^{-i\phi})\right\}^2} \qquad \ldots(1)$$

Put $e^{i\phi} = z$ so that $d\phi = \dfrac{dz}{iz}$ then, From (1), $$I = \oint_C \frac{1}{\left\{a + \frac{b}{2}\left(z + \frac{1}{z}\right)\right\}^2} \frac{dz}{iz} = \oint_C \frac{-4iz\,dz}{(bz^2 + 2az + b)^2}$$

$$= -\frac{4i}{b^2} \oint_C \frac{z\,dz}{\left(z^2 + \frac{2az}{b} + 1\right)^2}$$

Poles are given by,

$$\left(z^2 + \frac{2az}{b} + 1\right)^2 = 0 \Rightarrow (z-\alpha)^2(z-\beta)^2 = 0 \text{ where, } \alpha + \beta = -\frac{2a}{b} \text{ and } \alpha\beta = 1.$$

Also, $$\alpha = \frac{-\frac{2a}{b} + \sqrt{\frac{4a^2}{b^2} - 4}}{2} = \frac{-a + \sqrt{a^2 - b^2}}{b}$$

$$\beta = \frac{-\frac{2a}{b} - \sqrt{\frac{4a^2}{b^2} - 4}}{2} = \frac{-a - \sqrt{a^2 - b^2}}{b}$$

There are two poles, at $z = \alpha$ and at $z = \beta$ each of order 2.

Since, $|\alpha\beta| = 1$

or $|\alpha|\,|\beta| = 1$

But $|\beta| > 1 \quad \therefore \quad |\alpha| < 1$ $\quad | \because \; a > b$

$\therefore$ Only $z = \alpha$ lies inside the unit circle $|z| = 1$.

Residue of $f(z)$ at the double pole $z = \alpha$ is

$$= \frac{1}{(2-1)!}\left[\frac{d}{dz}\left\{(z-\alpha)^2 \cdot \frac{-4iz}{b^2(z-\alpha)^2\,(z-\beta)^2}\right\}\right]_{z=\alpha}$$

$$= \left[\frac{d}{dz}\left\{\frac{-4iz}{b^2(z-\beta)^2}\right\}\right]_{z=\alpha} = -\frac{4i}{b^2} \cdot \left[\frac{(z-\beta)^2 \,.\, 1 - z \,.\, 2(z-\beta)}{(z-\beta)^4}\right]_{z=\alpha}$$

$$= -\frac{4i}{b^2}\left[\frac{(-\beta - z)}{(z-\beta)^3}\right]_{z=\alpha} = -\frac{4i}{b^2}\,\frac{(-\alpha-\beta)}{(\alpha-\beta)^3} = \frac{4i}{b^2}\,\frac{(\alpha+\beta)}{(\alpha-\beta)^3}$$

$$= \frac{4i}{b^2} \cdot \frac{\left(-\frac{2a}{b}\right)}{\left(\frac{2}{b}\sqrt{a^2 - b^2}\right)^3} = -\frac{ia}{(a^2 - b^2)^{3/2}}$$

$\therefore$ $$I = 2\pi i\left[\frac{-ia}{(a^2 - b^2)^{3/2}}\right] = \frac{2\pi a}{(a^2 - b^2)^{3/2}}.$$

Example 5. *Apply Calculus of residues to prove that:*

$$\int_0^{\pi} \frac{\cos 2\theta \, d\theta}{1 - 2p\cos\theta + p^2} = \frac{\pi p^2}{1 - p^2} \quad (0 < p < 1).$$

Sol.
$$I = \int_0^{\pi} \frac{\cos 2\theta \, d\theta}{1 - 2p\cos\theta + p^2} = \frac{1}{2}\int_0^{2\pi} \frac{\cos 2\theta \, d\theta}{1 - p(e^{i\theta} + e^{-i\theta}) + p^2}$$

$$= \frac{1}{2} \text{ real part of } \int_0^{2\pi} \frac{e^{2i\theta}}{(1 - pe^{i\theta})(1 - pe^{-i\theta})} d\theta$$

$$= \frac{1}{2} \text{ real part of } \oint_C \frac{z^2}{(1 - pz)\left(1 - \frac{p}{z}\right)} \frac{dz}{iz} \qquad \Big|\ \text{writing } e^{i\theta} = z,\ d\theta = \frac{dz}{iz}$$

$$= \frac{1}{2} \text{ real part of } \oint_C \frac{-iz^2}{(1 - pz)(z - p)} dz$$

$$= \frac{1}{2} \text{ real part of } \oint_C f(z)\, dz \qquad \Big|\ \text{where, } f(z) = \frac{-iz^2}{(1 - pz)(z - p)}$$

Poles of $f(z)$ are given by $(1 - pz)(z - p) = 0$.

Thus $z = \frac{1}{p}$ and $z = p$ are the simple poles. Only $z = p$ lies within the unit circle C as $p < 1$.

The residue of $f(z)$ at $z = p$ is

$$= \lim_{z \to p} (z - p) f(z) = \lim_{z \to p} (z - p) \frac{-iz^2}{(1 - pz)(z - p)} = -\frac{ip^2}{1 - p^2}.$$

Hence by Cauchy's residue theorem, we have

$$\oint_C f(z)\, dz = 2\pi i \times [\text{Sum of residues within the contour}]$$

$$= 2\pi i \left(-\frac{ip^2}{1 - p^2}\right) = \frac{2\pi p^2}{1 - p^2} \text{ which is purely real.}$$

Hence, $$I = \frac{1}{2} \text{ real part of } \oint_C f(z)\, dz = \frac{\pi p^2}{1 - p^2}.$$

Example 6. *Use Complex integration method to prove that*

$$\int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta} d\theta = \frac{2\pi}{b^2}\left(a - \sqrt{a^2 - b^2}\right), \text{ where } 0 < b < a.$$

Sol. Let
$$I = \int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta} d\theta = \int_0^{2\pi} \frac{1 - \cos 2\theta}{2(a + b\cos\theta)} d\theta$$

$$= \text{Real part of } \int_0^{2\pi} \frac{1 - e^{2i\theta}}{2a + 2b\cos\theta} d\theta$$

Put $z = e^{i\theta}$ so that $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$ and $d\theta = \frac{dz}{iz}$

Then $\int_0^{2\pi} \frac{1 - e^{2i\theta}}{2a + 2b\cos\theta}\, dz = \oint_C \frac{1 - z^2}{2a + b\left(z + \frac{1}{z}\right)}\left(\frac{dz}{iz}\right) = \oint_C \frac{1 - z^2}{i(bz^2 + 2az + b)}\, dz$

where C is the circle $|z| = 1$.

The poles of the integrand are the roots of $bz^2 + 2az + b = 0$, *viz.*

$$z = \frac{-2a \pm \sqrt{4a^2 - 4b^2}}{2b} = \frac{-a \pm \sqrt{a^2 - b^2}}{b}$$

Let $$\alpha = \frac{-a + \sqrt{a^2 - b^2}}{b} \quad \text{and} \quad \beta = \frac{-a - \sqrt{a^2 - b^2}}{b}$$

Clearly, $|\beta| > 1$ so that $z = \alpha$ is the only simple pole inside C.

Also, $\quad bz^2 + 2az + b = b(z - \alpha)(z - \beta)$

Residue at $z = \alpha$ is

$$\underset{z \to \alpha}{\text{Lt}}\ (z - \alpha) . \frac{1 - z^2}{ib(z - \alpha)(z - \beta)} = \underset{z \to \alpha}{\text{Lt}}\ \frac{1 - z^2}{ib(z - \beta)} = \frac{1 - \alpha^2}{ib(\alpha - \beta)}$$

$$= \frac{\alpha\left(\frac{1}{\alpha} - \alpha\right)}{ib(\alpha - \beta)} = \frac{\alpha(\beta - \alpha)}{ib(\alpha - \beta)} \qquad | \because \alpha\beta = 1$$

$$= -\frac{\alpha}{ib} = \frac{a - \sqrt{a^2 - b^2}}{ib^2}$$

$\therefore$ By Residue theorem,

$$\oint_C \frac{1 - z^2}{i(bz^2 + 2az + b)}\, dz = 2\pi i . \frac{a - \sqrt{a^2 - b^2}}{ib^2} = \frac{2\pi}{b^2}\left(a - \sqrt{a^2 - b^2}\right)$$

Hence I = Real part of $\oint_C \frac{1 - z^2}{i(bz^2 + 2az + b)}\, dz = \frac{2\pi}{b^2}\left(a - \sqrt{a^2 - b^2}\right)$.

Example 7. *Using complex integration method, evaluate* $\int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\cos\theta}\, d\theta$.

(M.T.U. 2012, G.B.T.U. 2010)

Sol. Let $\quad$ I = Real part of $\int_0^{2\pi} \frac{e^{2i\theta}}{5 + 2(e^{i\theta} + e^{-i\theta})}\, d\theta$

$$= \text{Real part of} \oint_C \frac{z^2}{5 + 2\left(z + \frac{1}{z}\right)}\left(\frac{dz}{iz}\right) \qquad \begin{array}{|l} \text{writing } e^{i\theta} = z \\ \therefore\ d\theta = \frac{dz}{iz} \end{array}$$

$$= \text{Real part of } \frac{1}{i}\oint_C \frac{z^2}{2z^2 + 5z + 2}\, dz$$

Singularities are given by

$$2z^2 + 5z + 2 = 0 \quad \Rightarrow \quad z = -\frac{1}{2}, -2$$

$z = -\frac{1}{2}$ is the only pole which lies inside the unit circle $C \equiv |z| = 1$.

Residue of $f(z)$ at $\left(z = -\frac{1}{2}\right)$ is

$$R = \underset{z \to -\frac{1}{2}}{\mathrm{Lt}} \left(z + \frac{1}{2}\right) \cdot \frac{z^2}{i(2z+1)(z+2)} = \underset{z \to -\frac{1}{2}}{\mathrm{Lt}} \frac{z^2}{2i(z+2)} = \frac{1}{2i}\left(\frac{1}{4}\right)\left(\frac{2}{3}\right) = \frac{1}{12i}$$

Hence by Cauchy's Residue theorem,

$$I = \oint_C f(z)\, dz = 2\pi i \left(\frac{1}{12i}\right) = \frac{\pi}{6}.$$

Example 8. *Evaluate:* $\int_0^{\pi} \frac{1 + 2\cos\theta}{5 + 4\cos\theta}\, d\theta$.

Sol. Let

$$I = \int_0^{2\pi} \frac{1 + 2\cos\theta}{5 + 4\cos\theta}\, d\theta = \text{Real part of } \int_0^{2\pi} \frac{1 + 2e^{i\theta}}{5 + 4\cos\theta}\, d\theta$$

$$= \text{Real part of } \oint_C \frac{1 + 2z}{5 + 2\left(z + \frac{1}{z}\right)} \left(\frac{dz}{iz}\right) \qquad \Bigg|\ \text{Putting } e^{i\theta} = z \quad \therefore \quad d\theta = \frac{dz}{iz}$$

$$= \text{Real part of } \frac{1}{i} \oint_C \frac{1 + 2z}{2z^2 + 5z + 2}\, dz$$

Poles are given by

$$(2z + 1)(z + 2) = 0 \quad \Rightarrow \quad z = -\frac{1}{2}, -2 \quad \text{(simple poles)}$$

$z = -\frac{1}{2}$ lies inside unit circle $C \equiv |z| = 1$

$$\text{Residue}\left(\text{at } z = -\frac{1}{2}\right) = \underset{z \to -\frac{1}{2}}{\mathrm{Lt}} \left(z + \frac{1}{2}\right) \cdot \frac{1}{i} \frac{1 + 2z}{(2z+1)(z+2)} = \left(\frac{1}{2i}\right) \underset{z \to -\frac{1}{2}}{\mathrm{Lt}} \frac{1 + 2z}{z + 2} = 0$$

Hence by Cauchy's Residue theorem,

$$I = 2\pi i\,(0) = 0$$

$$\int_0^{2\pi} \frac{1 + 2\cos\theta}{5 + 4\cos\theta}\, d\theta = 0$$

$$\Rightarrow \quad \int_0^{\pi} \frac{1 + 2\cos\theta}{5 + 4\cos\theta}\, d\theta = 0 \qquad \Big|\ \text{Using property of definite integrals}$$

Example 9. *Evaluate by Contour integration:* $\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta - n\theta)\, d\theta$.

Sol. Let
$$I = \int_0^{2\pi} e^{\cos\theta} [\cos(\sin\theta - n\theta) + i \sin(\sin\theta - n\theta)]\, d\theta$$
$$= \int_0^{2\pi} e^{\cos\theta} . e^{i(\sin\theta - n\theta)}\, d\theta = \int_0^{2\pi} e^{e^{i\theta}} . e^{-in\theta}\, d\theta \quad ...(1)$$

Put $e^{i\theta} = z$ so that $d\theta = \frac{dz}{iz}$ then,
$$I = \int_C e^z . \frac{1}{z^n} . \frac{dz}{iz} = -i \int_C \frac{e^z}{z^{n+1}}\, dz$$

Poles are given by
$$z = 0 \quad \text{[of order } (n+1)]$$

It lies inside the unit circle.

Residue of $f(z)$ at $z = 0$ is
$$R = \frac{1}{(n+1-1)!}\left[\frac{d^n}{dz^n}\left\{z^{n+1} . \frac{-ie^z}{z^{n+1}}\right\}\right]_{z=0} = \frac{-i}{n!}\left[\frac{d^n}{dz^n}(e^z)\right]_{z=0} = \frac{-i}{n!}$$

$\therefore$ By Cauchy's Residue theorem,
$$I = 2\pi i\left(\frac{-i}{n!}\right) = \frac{2\pi}{n!}$$

Comparing real parts, we have
$$\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta - n\theta)\, d\theta = \frac{2\pi}{n!}.$$

Example 10. *Evaluate the integral:* $\int_0^{\pi} \frac{\cos^2 3\theta}{5 - 4\cos 2\theta}\, d\theta$. (U.P.T.U. 2015)

Sol. Let,
$$I = \frac{1}{2}\int_0^{2\pi} \frac{\cos^2 3\theta}{5 - 4\cos 2\theta}\, d\theta = \frac{1}{4}\int_0^{2\pi} \frac{1 + \cos 6\theta}{5 - 4\cos 2\theta}\, d\theta \quad ...(1)$$

Consider the integration round a unit circle $c \equiv |z| = 1$ so that $z = e^{i\theta}$

$$\therefore \quad dz = ie^{i\theta}\, d\theta = iz\, d\theta \Rightarrow d\theta = \frac{dz}{iz}$$

Also,
$$\cos 2\theta = \frac{1}{2}(e^{2i\theta} + e^{-2i\theta}) = \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right)$$

and
$$\cos 6\theta = \frac{1}{2}\left(z^6 + \frac{1}{z^6}\right)$$

Then the given integral (1) reduces to
$$I = \frac{1}{4}\oint_c \frac{\left(1 + \frac{z^{12}+1}{2z^6}\right)}{5 - 2\left(\frac{z^4+1}{z^2}\right)} \cdot \frac{dz}{iz} = -\frac{1}{16i}\oint_c \frac{z^{12} + 2z^6 + 1}{z^5\left(z^4 - \frac{5}{2}z^2 + 1\right)}\, dz$$

Singularities are, $z = 0$ (order 5), $z = \pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}$ (order 1)

Clearly, $z = 0$ and $z = \pm\frac{1}{\sqrt{2}}$ lie inside C.

Now we will find residues at $z = 0$ and $z = \pm\frac{1}{\sqrt{2}}$.

Let
$$f(z) = \frac{z^{12} + 2z^6 + 1}{z^5\left(z^4 - \frac{5}{2}z^2 + 1\right)} = \frac{z^{12} + 2z^6 + 1}{z^5}\left[1 - \left(\frac{5}{2}z^2 - z^4\right)\right]^{-1}$$

$$= \frac{(z^6+1)^2}{z^5}\left[1 + \frac{5}{2}z^2 - z^4 + \frac{25}{4}z^4 + z^8 - 5z^6 + \ldots\right]$$

Residue of $f(z)$ at $z = 0$ is the coefficient of $\frac{1}{z}$ in this laurent series expansion. Hence,

$$R_1 = \text{Residue of } f(z) \text{ at } z = 0 = -1 + \frac{25}{4} = \frac{21}{4}$$

$$R_2 = \text{Residue of } f(z) \text{ at } z = \frac{1}{\sqrt{2}}$$

$$= \underset{z \to \frac{1}{\sqrt{2}}}{\text{Lt}} \left(z - \frac{1}{\sqrt{2}}\right) \cdot \frac{(z^6+1)^2}{z^5\,(z^2-2)\left(z - \frac{1}{\sqrt{2}}\right)\left(z + \frac{1}{\sqrt{2}}\right)}$$

$$= \underset{z \to \frac{1}{\sqrt{2}}}{\text{Lt.}} \frac{(z^6+1)^2}{z^5\,(z^2-2)\left(z + \frac{1}{\sqrt{2}}\right)} = -\frac{27}{8}$$

$$R_3 = \text{Residue of } f(z) \text{ at } z = -\frac{1}{\sqrt{2}}$$

$$= \underset{z \to -\frac{1}{\sqrt{2}}}{\text{Lt}} \left(z + \frac{1}{\sqrt{2}}\right) \cdot \frac{(z^6+1)^2}{z^5\,(z^2-2)\left(z - \frac{1}{\sqrt{2}}\right)\left(z + \frac{1}{\sqrt{2}}\right)}$$

$$= \underset{z \to -\frac{1}{\sqrt{2}}}{\text{Lt}} \frac{(z^6+1)^2}{z^5\,(z^2-2)\left(z - \frac{1}{\sqrt{2}}\right)} = -\frac{27}{8}$$

Now, by Cauchy-Residue theorem,

$$\text{I} = -\frac{1}{16i}\left[2\pi i\,(\text{R}_1 + \text{R}_2 + \text{R}_3)\right] = -\frac{\pi}{8}\left(\frac{21}{4} - \frac{27}{4}\right) = \frac{3\pi}{16}.$$

TEST YOUR KNOWLEDGE

Evaluate the following integrals by using contour integration:

1. (i) $\int_0^{2\pi} \frac{d\theta}{5 - 3\cos\theta}$ (U.K.T.U. 2011) (ii) $\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta}$ (U.P.T.U. 2015)

2. (i) $\int_0^{\pi} \frac{d\theta}{5 + 4\cos\theta}$ (ii) $\int_0^{\pi} \frac{d\theta}{17 - 8\cos\theta}$

(iii) $\int_0^{\pi} \frac{a\,d\theta}{1 + 2a^2 - \cos 2\theta}$ [G.B.T.U. (C.O.) 2011]

3. (i) $\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta}$ (G.B.T.U. 2011) (ii) $\int_0^{\pi} \frac{d\theta}{3 + \sin^2\theta}$ (G.B.T.U. 2013)

4. (i) $\int_0^{2\pi} \frac{d\theta}{(5 - 3\cos\theta)^2}$ (ii) $\int_0^{\pi} \frac{d\theta}{(a + \cos\theta)^2}$

5. (i) $\int_0^{\pi} \frac{a\,d\phi}{a^2 + \cos^2\phi}$ $(a > 0)$ (ii) $\int_0^{2\pi} \frac{\cos 2\theta}{1 - 2p\cos\theta + p^2}\,d\theta,\ 0 < p < 1$

6. (i) $\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta}\,d\theta$ (ii) $\int_0^{\pi} \frac{\cos 3\theta}{5 - 4\cos\theta}\,d\theta$

(iii) $\int_0^{2\pi} \frac{\cos 3\theta}{5 + 4\cos\theta}\,d\theta$ (iv) $\int_0^{\pi} \frac{\cos 2\theta}{5 + 4\cos\theta}\,d\theta$ (U.P.T.U. 2014)

7. $\int_{-\pi}^{\pi} \frac{a\cos\theta}{a + \cos\theta}\,d\theta$; $a > 1$ **8.** $\int_0^{2\pi} \frac{\sin^2\theta}{5 - 4\cos\theta}\,d\theta$

9. $\int_0^{2\pi} e^{-\cos\theta}\cos(n\theta + \sin\theta)\,d\theta$; $n \in I$

10. (i) $\int_0^{2\pi} \cos^{2n}\theta\,d\theta$; $n \in I$ (ii) $\int_0^{\pi} \sin^4\theta\,d\theta$ (A.K.T.U. 2018)

11. $\int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta}$. [G.B.T.U. 2013; U.P.T.U. 2014]

Answers

1. (i) $\frac{\pi}{2}$ (ii) $\frac{2\pi}{\sqrt{3}}$

2. (i) $\frac{\pi}{3}$ (ii) $\frac{\pi}{15}$ (iii) $\frac{\pi}{2\sqrt{1 + a^2}}$

3. (i) $\frac{2\pi}{3}$ (ii) $\frac{\pi}{2\sqrt{3}}$

4. (i) $\frac{5\pi}{32}$ (ii) $\frac{\pi a}{(a^2 - 1)^{3/2}}$ **5.** (i) $\frac{\pi}{\sqrt{1 + a^2}}$ (ii) $\frac{2\pi p^2}{1 - p^2}$

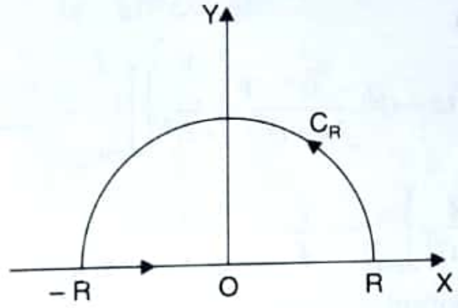
6. (i) $\frac{\pi}{12}$ (ii) $\frac{\pi}{24}$ (iii) $-\frac{\pi}{12}$ (iv) $\frac{\pi}{12}$

7. $2\pi a\left(1-\frac{a}{\sqrt{a^2-1}}\right)$ **8.** $\frac{\pi}{4}$ **9.** $\frac{2\pi}{n!}(-1)^n$

10. (i) $\frac{\pi(2n)!}{(2)^{2n-1}(n!)^2}$ (ii) $\frac{3\pi}{8}$ **11.** π.

5.27.2 Integrals of the Type $\int_{-\infty}^{\infty}\frac{f(x)}{F(x)}dx$, where $f(x)$ and $F(x)$ are polynomials in x such that $\frac{xf(x)}{F(x)} \to 0$ as $x \to \infty$ and $F(x)$ has no zeros on the real axis.

Consider the integral $\oint_C \frac{f(z)}{F(z)}dz$ over the closed contour C consisting of the real axis from – R to R and the semi-circle C_R of radius R in the upper half plane.



We take R large enough so that all the poles of $\frac{f(z)}{F(z)}$ in the upper half plane lie within C.

By residue theorem, we have

$$\oint_C \frac{f(z)}{F(z)}dz = 2\pi i\left[\text{sum of the residues of } \frac{f(z)}{F(z)} \text{ in the upper half plane}\right]$$

$$\oint_{C_R} \frac{f(z)}{F(z)}dz + \int_{-R}^{R}\frac{f(x)}{F(x)}dx = 2\pi i\left[\text{sum of the residues of } \frac{f(z)}{F(z)} \text{ in the upper half plane}\right]$$

...(1) ($\because$ on the real axis, $z = x$)

If we put $z = Re^{i\theta}$ in the first integral on the left side, then R is constant on C_R and as z moves along C_1, θ varies from 0 to π.

$$\therefore \quad \int_{C_R}\frac{f(z)}{F(z)}dz = \int_0^{\pi}\frac{f(Re^{i\theta})}{F(Re^{i\theta})}Re^{i\theta}\,i\,d\theta$$

For large R, $\left|\int_0^{\pi}\frac{f(Re^{i\theta})}{F(Re^{i\theta})}Re^{i\theta}\,id\theta\right| \to 0$ $\quad\Big|$ is of the order $\frac{Rf(R)}{F(R)}$

$$\therefore \quad \int_0^{\pi}\frac{f(Re^{i\theta})}{F(Re^{i\theta})}Re^{i\theta}\,id\theta \to 0 \text{ when } R \to \infty$$

Hence from (1), we have

$$\int_{-\infty}^{\infty}\frac{f(x)}{F(x)}dx = 2\pi i\left[\text{sum of the residues of } \frac{f(z)}{F(z)} \text{ in the upper half plane}\right]$$

ILLUSTRATIVE EXAMPLES

Example 1. *Using contour integration, prove that* $\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^2}=\frac{\pi}{2}$.

Hence or otherwise evaluate $\int_0^{\infty}\frac{dx}{(1+x^2)^2}$.

Sol. Consider the integral $\int_C f(z)\,dz$ where $f(z)=\frac{1}{(1+z^2)^2}$ taken round the closed contour C consisting of the semi-circle C_R which is upper half of a large circle $|z|=R$ and the part of real axis from $-R$ to R.

For poles, $(1+z^2)^2=0$

$\Rightarrow\quad z^2=-1$

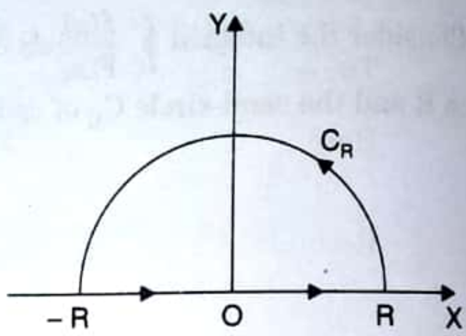
$\Rightarrow\quad z=\pm i$ (Poles of order 2)

$z=-i$ is outside C.

So, $z=i$ is the only pole inside C and is of order 2.

Residue of $f(z)$ at $z=i$ is

$$=\left[\frac{d}{dz}\left\{(z-i)^2\cdot\frac{1}{(z-i)^2(z+i)^2}\right\}\right]_{z=i}$$



$$=\left[\frac{-2}{(z+i)^3}\right]_{z=i}=-\frac{i}{4}$$

By Cauchy's residue theorem,

$$\int_{-R}^{R}\frac{dx}{(1+x^2)^2}+\int_{C_R}\frac{dz}{(1+z^2)^2}=2\pi i\left(\frac{-i}{4}\right)=\frac{\pi}{2}$$

Taking limit as $R\to\infty$

$$\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^2}+\lim_{R\to\infty}\int_{C_R}\frac{dz}{(1+z^2)^2}=\frac{\pi}{2}\qquad\text{...(1)}$$

Now, $$\left|\int_{C_R}\frac{dz}{(1+z^2)^2}\right|\le\int_{C_R}\frac{|dz|}{|1+z^2|^2}$$

$$\le\int_{C_R}\frac{|dz|}{\{|z|^2-1\}^2}$$

$$=\int_0^{\pi}\frac{R\,d\theta}{(R^2-1)^2}$$

$\because\ |z|=R$ on C_R and $|dz|=R\,d\theta$ also, $0<\theta<\pi$

$$=\frac{\pi R}{(R^2-1)^2}\to 0\text{ as }R\to\infty$$

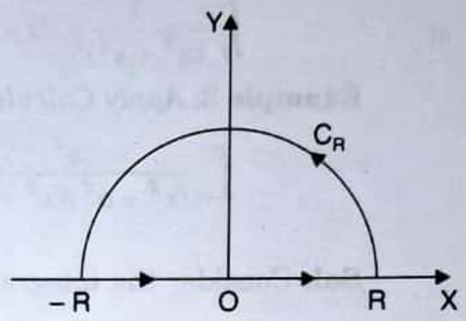
Hence, $$\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^2}=\frac{\pi}{2}$$

Now, $$2\int_0^{\infty}\frac{dx}{(1+x^2)^2}=\frac{\pi}{2}\Rightarrow\int_0^{\infty}\frac{dx}{(1+x^2)^2}=\frac{\pi}{4}.$$

Example 2. *Apply calculus of residues to prove that*

$$\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}\ ;\ a > 0. \qquad \text{(A.K.T.U. 2016)}$$

Sol. Consider the integral $\int_C f(z)\,dz$ where $f(z) = \frac{1}{(a^2+z^2)^2}$ taken round the closed contour C consisting of the semi-circle c_R which is upper half of a large circle $|z| = R$ and the part of real axis from $-R$ to R.



Poles of $f(z)$ are given by

$$(a^2+z^2)^2 = 0$$

i.e., $$a^2+z^2 = 0$$

or $z = \pm\, ai$ each repeated twice.

The only pole within the contour is $z = ai$, and is of the order 2.

Here, $$f(z) = \frac{1}{(z-ai)^2(z+ai)^2}.$$

Residue (at $z = ai$) is $$= \frac{1}{(2-1)!}\left[\frac{d}{dz}\left\{(z-ai)^2 \cdot \frac{1}{(z-ai)^2(z+ai)^2}\right\}\right]_{z=ai}$$

$$= \left[\frac{d}{dz}\left\{\frac{1}{(z+ai)^2}\right\}\right]_{z=ai} = \left[\frac{-2}{(z+ai)^3}\right]_{z=ai}$$

$$= \frac{-2}{(2ai)^3} = \frac{-1}{4a^3 i^3} = \frac{1}{4a^3 i}.$$

Hence by Cauchy's residue theorem, we have

$$\int_C f(z)\,dz = 2\pi i \text{ (Sum of residues within C)}$$

i.e., $$\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i\left(\frac{1}{4a^3 i}\right)$$

or $$\int_{-R}^{R} \frac{1}{(a^2+x^2)^2}\,dx + \int_{C_R} \frac{1}{(a^2+z^2)^2}\,dz = \frac{\pi}{2a^3} \qquad \text{...(1)}$$

Now, $$\left|\int_{C_R} \frac{1}{(a^2+z^2)^2}\,dz\right| \le \int_{C_R} \frac{|dz|}{|a^2+z^2|^2}$$

$$\le \int_{C_R} \frac{|dz|}{(|z|^2-a^2)^2}$$

$$= \int_0^\pi \frac{R\,d\theta}{(R^2-a^2)^2}, \text{ since } z = Re^{i\theta}$$

$$= \frac{\pi R}{(R^2-a^2)^2} \to 0 \text{ as } R \to \infty$$

Hence taking $R \to \infty$, relation (1) becomes,

$$\int_{-\infty}^{\infty} \frac{1}{(a^2+x^2)^2}\, dx = \frac{\pi}{2a^3}$$

or
$$\int_{0}^{\infty} \frac{1}{(a^2+x^2)^2}\, dx = \frac{\pi}{4a^3}.$$

Example 3. *Apply Calculus of residues to prove that*

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\, dx = \frac{\pi}{a+b} \quad (a > 0,\ b > 0).$$

Sol. Consider the integral $\int_C f(z)\, dz$ where $f(z) = \dfrac{z^2}{(z^2+a^2)(z^2+b^2)}$ taken round the closed contour C consisting of the semi-circle C_R which is upper half of a large circle $|z| = R$ and the part of real axis from $-R$ to R.

The poles of $f(z) = \dfrac{z^2}{(z^2+a^2)(z^2+b^2)}$ are $z = \pm ia, \pm ib$. Of these, $z = ia$ and $z = ib$ lie in the upper half of the z-plane.

Residue of $f(z)$ at $z = ia$ is

$$= \lim_{z \to ia} (z - ia) \frac{z^2}{(z^2+a^2)(z^2+b^2)}$$

$$= \lim_{z \to ia} \frac{z^2}{(z+ia)(z^2+b^2)} = -\frac{a^2}{2ia(-a^2+b^2)} = \frac{a}{2i(a^2-b^2)}.$$

Residue of $f(z)$ at $z = ib$ is

$$= \lim_{z \to ib} (z - ib) \frac{z^2}{(z^2+a^2)(z^2+b^2)}$$

$$= \lim_{z \to ib} \frac{z^2}{(z^2+a^2)(z+ib)} = \frac{-b^2}{(-b^2+a^2)(2ib)} = \frac{-b}{2i(a^2-b^2)}.$$

By Cauchy's residue theorem,

$$\int_{-R}^{R} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\, dx + \int_{C_R} \frac{z^2}{(z^2+a^2)(z^2+b^2)}\, dz = 2\pi i \left[\frac{a}{2i(a^2-b^2)} - \frac{b}{2i(a^2-b^2)}\right]$$

Taking limit as $R \to \infty$

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\, dx + \underset{R \to \infty}{\text{Lt}} \int_{C_R} \frac{z^2}{(z^2+a^2)(z^2+b^2)}\, dz = \pi \left[\frac{a-b}{a^2-b^2}\right]$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\, dx + \underset{R \to \infty}{\text{Lt}} \int_{C_R} \frac{z^2}{(z^2+a^2)(z^2+b^2)}\, dz = \frac{\pi}{a+b} \quad \ldots(1)$$

Now, $$\left| \int_{C_R} \frac{z^2}{(z^2+a^2)(z^2+b^2)}\, dz \right| \le \int_{C_R} \frac{|z|^2}{|z^2+a^2|\,|z^2+b^2|} |dz|$$

$$\le \int_{C_R} \frac{|z|^2}{(|z|^2-a^2)(|z|^2-b^2)} |dz|$$

$$= \frac{R^2}{(R^2 - a^2)(R^2 - b^2)} \int_0^{\pi} R \, d\theta \qquad | \because |z| = R \text{ on } C_R$$

$$= \frac{\pi R^3}{(R^2 - a^2)(R^2 - b^2)} \to 0 \text{ as } R \to \infty$$

$\therefore$ From (1), $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + a^2)(x^2 + b^2)} dx = \frac{\pi}{a+b}$.

Example 4. (*i*) *Apply Calculus of residues to prove that*

$$\int_0^{\infty} \frac{dx}{x^4 + a^4} = \frac{\pi\sqrt{2}}{4a^3}, \; (a > 0).$$

(*ii*) *Using contour integration, evaluate* $\displaystyle\int_0^{\infty} \frac{dx}{1 + x^4}$.

Sol. (*i*) Consider the integral $\displaystyle\int_C f(z)\, dz$ where $f(z) = \dfrac{1}{z^4 + a^4}$.

The poles of $f(z)$ are given by

$$z^4 + a^4 = 0 \Rightarrow z^4 = -a^4 = a^4 e^{\pi i} = a^4 e^{2n\pi i + \pi i}$$

or
$$z = ae^{(2n+1)\pi i/4}; \; n = 0, 1, 2, 3.$$

Since there is no pole on the real axis, therefore, we may take the closed contour C consisting of the upper half C_R of a large circle $|z| = R$ and the part of real axis from $-R$ to R.

$\therefore$ By Cauchy's residue theorem, we have

$$\int_{-R}^{R} f(x)\, dx + \int_{C_R} f(z)\, dz = \int_C f(z)\, dz$$

or
$$\int_{-R}^{R} \frac{1}{x^4 + a^4} dx + \int_{C_R} \frac{1}{z^4 + a^4} dz = 2\pi i \sum R^+ \qquad \text{...(1)}$$

| where $\sum R^+$ = sum of residues of $f(z)$ at poles within C.

The poles $z = ae^{\frac{\pi i}{4}}$ and $z = ae^{3\pi i/4}$ are the only two poles which lie within the contour C.

Let α denote any one of these poles, then

$$\alpha^4 + a^4 = 0 \quad \Rightarrow \quad \alpha^4 = -a^4.$$

Residue of $f(z)$ (at $z = \alpha$) is $= \left[\dfrac{1}{\dfrac{d}{dz}(z^4 + a^4)}\right]_{z=\alpha} = \dfrac{1}{4\alpha^3} = \dfrac{\alpha}{-4a^4}$

$\therefore$ Residue at $z = ae^{\pi i/4}$ is $= -\dfrac{1}{4a^3} e^{\pi i/4}$

and residue at $z = ae^{3\pi i/4}$ is $= -\dfrac{1}{4a^3} e^{3\pi i/4} = \dfrac{e^{-\pi i/4}}{4a^3}$

$\therefore$ Sum of residues $= -\dfrac{1}{2a^3}\left[\dfrac{e^{i\pi/4} - e^{-i\pi/4}}{2}\right] = -\dfrac{1}{2a^3} i \sin\dfrac{\pi}{4} = -\dfrac{i}{2\sqrt{2}\, a^3} = \sum R^+$.

$\therefore$ From (1),

$$\int_{-R}^{R} \frac{dx}{x^4 + a^4} + \int_{C_R} \frac{dz}{z^4 + a^4} = 2\pi i \left(\frac{-i}{2\sqrt{2}\, a^3} \right) = \frac{\pi\sqrt{2}}{2a^3} \qquad \ldots(2)$$

Now,
$$\left| \int_{C_R} \frac{1}{z^4 + a^4}\, dz \right| \le \int_{C_R} \frac{|dz|}{|z^4 + a^4|} \le \int_{C_R} \frac{|dz|}{|z|^4 - |a^4|}$$

$$= \int_0^{\pi} \frac{R d\theta}{R^4 - a^4} \qquad | \because |z| = R \text{ on } C_R$$

$$= \frac{\pi R}{R^4 - a^4} \to 0 \text{ as } R \to \infty.$$

Hence when $R \to \infty$, relation (2) becomes

$$\int_{-\infty}^{\infty} \frac{dx}{x^4 + a^4} = \frac{\pi\sqrt{2}}{2a^3} \quad \text{or} \quad \int_0^{\infty} \frac{dx}{x^4 + a^4} = \frac{\pi\sqrt{2}}{4a^3}.$$

(*ii*) Consider the integral $\int_C f(z)\, dz$ where $f(z) = \dfrac{1}{1+z^4}$ taken round a closed contour C, consisting of the semi-circle C_R which is upper half of a large circle $|z| = R$ and the part of real axis from $-R$ to R.

The poles of $f(z) = \dfrac{1}{z^4 + 1}$ are obtained by solving $z^4 + 1 = 0$.

Now $\qquad z^4 + 1 = 0$

$\Rightarrow \qquad z = (-1)^{1/4} = (\cos \pi + i \sin \pi)^{1/4} = [\cos (2n\pi + \pi) + i \sin (2n\pi + \pi)]^{1/4}$

$$= \cos \frac{(2n+1)\pi}{4} + i \sin \frac{(2n+1)\pi}{4} \quad \text{where } n = 0, 1, 2, 3.$$

| By De Moivre's theorem

When $n = 0$, $\quad z = \cos \dfrac{\pi}{4} + i \sin \dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}} + i \dfrac{1}{\sqrt{2}}$

When $n = 1$, $\quad z = \cos \dfrac{3\pi}{4} + i \sin \dfrac{3\pi}{4} = -\dfrac{1}{\sqrt{2}} + i \dfrac{1}{\sqrt{2}}$

When $n = 2$, $\quad z = \cos \dfrac{5\pi}{4} + i \sin \dfrac{5\pi}{4} = -\dfrac{1}{\sqrt{2}} - i \dfrac{1}{\sqrt{2}}$

When $n = 3$, $\quad z = \cos \dfrac{7\pi}{4} + i \sin \dfrac{7\pi}{4} = \dfrac{1}{\sqrt{2}} - i \dfrac{1}{\sqrt{2}}$

Of these, only the poles corresponding to $n = 0, 1$, *viz*, $z = e^{i\frac{\pi}{4}}$ and $z = e^{3i\frac{\pi}{4}}$ lie in the upper half of z-plane.

Residue of $f(z)$ at $z = e^{i\frac{\pi}{4}}$ is $\displaystyle \operatorname*{Lt}_{z \to e^{i\frac{\pi}{4}}} \frac{z - e^{i\frac{\pi}{4}}}{z^4 + 1}$ $\qquad \left| \text{Form } \dfrac{0}{0} \right.$

$$= \operatorname*{Lt}_{z \to e^{i\frac{\pi}{4}}} \frac{1}{4z^3} \qquad | \text{ By L' Hospital's rule}$$

$$= \frac{1}{4e^{3i\frac{\pi}{4}}} = \frac{1}{4} e^{-3i\frac{\pi}{4}}$$

Similarly, residue of $f(z)$ at $z = e^{3i\frac{\pi}{4}}$ is $\frac{1}{4}e^{-9i\frac{\pi}{4}}$

Sum of residues $= \frac{1}{4} \left(e^{-3i\pi/4} + e^{-9i\pi/4}\right)$

$$= \frac{1}{4}\left[\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4} + \cos\frac{9\pi}{4} - i\sin\frac{9\pi}{4}\right]$$

$$= \frac{1}{4}\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) = \frac{-i\sqrt{2}}{4}$$

By Cauchy Residue theorem

$$\int_{-R}^{R} \frac{dx}{1+x^4} + \int_{C_R} \frac{dz}{1+z^4} = 2\pi i\left(\frac{-i\sqrt{2}}{4}\right) = \frac{\pi\sqrt{2}}{2}$$

Taking Limit $R \to \infty$,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} + \underset{R\to\infty}{\text{Lt}} \int_{C_R} \frac{dz}{1+z^4} = \frac{\pi\sqrt{2}}{2}. \quad \text{...(1)}$$

Now, $\left|\int_{C_R} \frac{dz}{1+z^4}\right| \le \int_{C_R} \frac{|dz|}{|z^4+1|} \le \int_{C_R} \frac{|dz|}{|z|^4 - 1}$

$$= \frac{R}{R^4 - 1}\int_0^{\pi} d\theta \qquad | \because \quad |z| = R \text{ on } C_R$$

$$= \frac{\pi R}{R^4 - 1} \to 0 \text{ as } R \to \infty$$

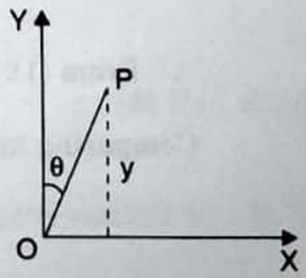
$\therefore$ From (1), $\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi\sqrt{2}}{2}$ or $\int_0^{\infty} \frac{dx}{1+x^4} = \frac{\pi\sqrt{2}}{4}$.

Note. The above method can also be applied to some cases where $f(x)$ contains trigonometric functions also.

5.27.2 (a) Jordan's Inequality

Consider the relation $y = \cos\theta$. As θ increases, $\cos\theta$ decreases and therefore y decreases.

The mean ordinate between 0 and $\theta = \frac{1}{\theta}\int_0^{\theta} \cos\theta \, d\theta = \frac{\sin\theta}{\theta}$



when $\theta = 0$, ordinate is $\cos 0$ *i.e.* 1

when $\theta = \frac{\pi}{2}$, mean ordinate is $\frac{\sin \pi/2}{\pi/2}$ *i.e.* $\frac{2}{\pi}$

Hence, when $0 < \theta < \pi/2$,

Mean ordinate lies between 1 and $\frac{2}{\pi}$

i.e., $\frac{2}{\pi} < \frac{\sin\theta}{\theta} < 1$

This is known as **Jordan's Inequality.**

5.27.2 (b) Jordan's Lemma

If $f(z) \to 0$ uniformly as $|z| \to \infty$, then $\underset{R \to \infty}{\text{Lt}} \int_{C_R} e^{imz} f(z)\, dz = 0, (m > 0)$

where C_R denotes the semi-circle $|z| = R$, $I(z) > 0$.

Example 5. *Apply calculus of residues to evaluate* $\int_0^\infty \frac{x \sin x}{x^2 + a^2} dx, a > 0.$

(G.B.T.U. 2010)

Sol. Consider the integral $\int_C f(z)\, dz$ where $f(z) = \frac{z e^{iz}}{z^2 + a^2}$ taken round a closed contour C, consisting of a semi-circle C_R which is upper half of a large circle $|z| = R$ and the part of real axis from $-R$ to R.

For poles, $z^2 + a^2 = 0$

$\Rightarrow$ $z = \pm ai$

$z = ai$ is the only pole which lie inside C.

$\therefore$ Residue of $f(z)$ at $(z = ai) = \lim_{z \to ai} (z - ai) \cdot \frac{z e^{iz}}{(z - ai)(z + ai)} = \frac{ai e^{-a}}{2ai} = \frac{e^{-a}}{2}$

$\therefore$ By Cauchy Residue theorem,

$$\int_{-R}^{R} \frac{x e^{ix}}{x^2 + a^2} dx + \int_{C_R} \frac{z e^{iz}}{z^2 + a^2} dz = 2\pi i \left(\frac{e^{-a}}{2}\right) = \pi i e^{-a}$$

Taking limit as $R \to \infty$,

$$\underset{R \to \infty}{\text{Lt}} \int_{-R}^{R} \frac{x e^{ix}}{x^2 + a^2} dx + \underset{R \to \infty}{\text{Lt}} \int_{C_R} \frac{z e^{iz}}{z^2 + a^2} dz = \pi i e^{-a}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2 + a^2} dx + \underset{R \to \infty}{\text{Lt}} \int_{C_R} \frac{z e^{iz}}{z^2 + a^2} dz = \pi i e^{-a} \quad ...(1)$$

Since $\frac{z}{z^2 + a^2} \to 0$ as $|z| \to \infty$, therefore by Jordan's Lemma,

$$\underset{R \to \infty}{\text{Lt}} \int_{C_R} \frac{z e^{iz}}{z^2 + a^2} dz = 0$$

$\therefore$ From (1), $\int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2 + a^2} dx = \pi i e^{-a}$

Comparing imaginary parts,

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \pi e^{-a}$$

or $$\int_0^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \frac{\pi}{2} e^{-a}.$$

Example 6. *Evaluate by using Contour integration* $\int_0^\infty \frac{\cos ax}{x^2+1}\,dx\ ;\ a \ge 0.$

(G.B.T.U. 2011)

Sol. Consider the integral $\int_C f(z)\,dz$ where $f(z) = \frac{e^{iaz}}{z^2+1}$ taken round a closed contour C, consisting of a semi-circle C_R which is upper half of a large circle $|z| = R$ and the part of real axis from $-R$ to R.

For poles, $z^2 + 1 = 0$

$\Rightarrow$ $z = \pm i$

$z = i$ is the only pole which lies inside C.

$$\therefore \quad \text{Res.}\ (z = i) = \underset{z \to i}{\text{Lt}}\ (z - i)\cdot\frac{e^{iaz}}{(z-i)(z+i)} = \frac{e^{-a}}{2i}$$

$\therefore$ By Cauchy Residue theorem,

$$\int_{-R}^{R} \frac{e^{iax}}{x^2+1}\,dx + \int_{C_R} \frac{e^{iaz}}{z^2+1}\,dz = 2\pi i\left(\frac{e^{-a}}{2i}\right) = \pi e^{-a}$$

Taking Limit as $R \to \infty$,

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}\,dx + \underset{R\to\infty}{\text{Lt}} \int_{C_R} \frac{e^{iaz}}{z^2+1}\,dz = \pi\, e^{-a} \qquad \text{...(1)}$$

Since $\frac{1}{z^2+1} \to 0$ as $|z| \to \infty$, therefore by Jordan's Lemma,

$$\underset{R\to\infty}{\text{Lt}} \int_{C_R} \frac{e^{iaz}}{z^2+1}\,dz = 0 \qquad (a > 0)$$

$$\therefore \quad \text{From (1),} \qquad \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}\,dx = \pi\, e^{-a}$$

Equating real parts, we get

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}\,dx = \pi e^{-a}$$

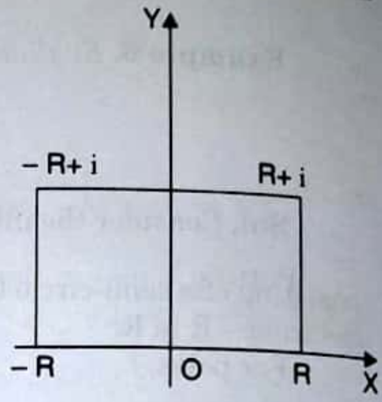
or

$$\int_0^{\infty} \frac{\cos ax}{x^2+1}\,dx = \frac{\pi e^{-a}}{2}.$$

Example 7. *Apply calculus of residues to prove that*

$$\int_0^\infty \frac{\cosh ax}{\cosh \pi x}\,dx = \frac{1}{2}\sec\frac{a}{2},\ -\pi < a < \pi. \qquad (M.T.U.\ 2014)$$

Sol. Consider $\int_c f(z)\,dz$ where $f(z) = \frac{e^{az}}{\cosh \pi z}$, c is the rectangle with vertices at $-R$, R, $R + i$ and $-R + i$.



$f(z)$ has simple poles given by

$$\cosh \pi z = 0$$

or $$e^{\pi z} + e^{-\pi z} = 0$$

or $$e^{\pi z} = -e^{-\pi z} = e^{(2n+1)\pi i - \pi z}$$

whence, $z = \dfrac{(2n+1)i}{2}$; $n = 0, \pm 1, \pm 2$. Of these poles, only $z = \dfrac{i}{2}$ lies inside c.

$$\text{Residue}\left(\text{at } z = \frac{i}{2}\right) = \left[\frac{e^{az}}{\frac{d}{dz}(\cosh \pi z)}\right]_{z=\frac{i}{2}}$$

$$= \frac{e^{ia/2}}{\pi \sinh \frac{i\pi}{2}} = \frac{e^{ia/2}}{\pi i \sin \frac{\pi}{2}} = \frac{1}{\pi i} e^{ia/2}$$

By residue theorem, we get

$$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_0^1 f(R+iy)\,.\,idy + \int_R^{-R} f(x+i)\,dx + \int_1^0 f(-R+iy)\,.\,idy$$

$$= 2\pi i \,.\, \frac{1}{\pi i} e^{ia/2} = 2e^{ia/2} \qquad \ldots(1)$$

or $$I_1 + I_2 + I_3 + I_4 = 2e^{ia/2}.$$

Now, $$|\,I_2\,| = \left|\int_0^1 \frac{e^{a(R+iy)}}{\cosh \pi(R+iy)}\, idy\right|$$

$$\leq \int_0^1 \frac{2e^{aR}\,|\,e^{aiy}\,||\,i\,|\,dy}{|\,e^{\pi(R+iy)} + e^{-\pi(R+iy)}\,|}$$

$$= \int_0^1 \frac{2e^{aR}\,dy}{e^{\pi R} - e^{-\pi R}} = \frac{2e^{aR}}{e^{\pi R} - e^{-\pi R}} \to 0 \text{ as } R \to \infty \qquad |\text{ since } -\pi < a < \pi.$$

In the same way $I_4 \to 0$. Hence when $R \to \infty$, we get from (1),

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh \pi x}\,dx + \int_{\infty}^{-\infty} \frac{e^{a(x+i)}}{\cosh \pi(x+i)}\,dx = 2e^{ia/2}$$

or $$\int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh \pi x}\,dx - \int_{-\infty}^{\infty} \frac{e^{ax}\,.\,e^{ai}}{-\cosh \pi x}\,dx = 2e^{ia/2} \qquad [\because \cosh \pi(x+i) = -\cosh \pi x]$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{(1+e^{ia})\,.\,e^{ax}}{\cosh \pi x}\,dx = 2e^{ia/2}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{(e^{ia/2} + e^{-ia/2})\,e^{ax}}{\cosh \pi x}\,dx = 2$$

or
$$\int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh \pi x}\, dx = \frac{1}{\cos \frac{a}{2}}$$

or
$$\int_{-\infty}^{0} \frac{e^{ax}}{\cosh \pi x}\, dx + \int_{0}^{\infty} \frac{e^{ax}}{\cosh \pi x}\, dx = \frac{1}{\cos \frac{a}{2}} \qquad ...(2)$$

Putting $x = -t$ in the first integral, we get
$$\int_{-\infty}^{0} \frac{e^{ax}}{\cosh \pi x}\, dx = -\int_{\infty}^{0} \frac{e^{-at}}{\cosh \pi t}\, dt = \int_{0}^{\infty} \frac{e^{-ax}}{\cosh \pi x}\, dx$$

$\therefore$ From (2),
$$\int_{0}^{\infty} \frac{e^{ax} + e^{-ax}}{\cosh \pi x}\, dx = \frac{1}{\cos \frac{a}{2}}$$

or
$$\int_{0}^{\infty} \frac{\cosh ax}{\cosh \pi x}\, dx = \frac{1}{2 \cos \frac{a}{2}} = \frac{1}{2} \sec \frac{a}{2}.$$

Example 8. *Using contour integration, prove that:* $\int_0^{\infty} \frac{\log (1 + x^2)}{1 + x^2}\, dx = \pi \log_e 2.$

Sol. Consider the integral $\int_C f(z)\, dz$ where $f(z) = \frac{\log (z + i)}{z^2 + 1}$ taken round a closed contour C which consists of semi-circle C_R, the upper half of a large circle $|z| = R$ and the part of real axis from $-R$ to R.

For poles, $z^2 + 1 = 0 \Rightarrow z = \pm i$

Only the pole $z = i$ lies inside C.

$$\text{Res. } (z = i) = \operatorname*{Lt}_{z \to i} (z - i) . \frac{\log (z + i)}{(z - i)(z + i)} = \frac{\log (2i)}{2i} = \frac{\log 2 + i \frac{\pi}{2}}{2i}$$

$\therefore$ By Cauchy Residue theorem,
$$\operatorname*{Lt}_{R \to \infty} \int_{-R}^{R} \frac{\log (x + i)}{1 + x^2}\, dx + \operatorname*{Lt}_{R \to \infty} \int_{C_R} \frac{\log (z + i)}{1 + z^2}\, dz = \frac{2\pi i}{2i} \left(\log 2 + i \frac{\pi}{2} \right) = \pi \left(\log 2 + \frac{\pi i}{2} \right) \qquad ...(1)$$

Now, $$\operatorname*{Lt}_{z \to \infty} \frac{z \log (z + i)}{z^2 + 1} = \operatorname*{Lt}_{z \to \infty} \left[\frac{z}{z - i} . \frac{\log (z + i)}{z + i} \right]$$
$$= \operatorname*{Lt}_{z \to \infty} \frac{z}{z - i} \operatorname*{Lt}_{z \to \infty} \frac{\log (z + i)}{z + i} = 1.0 = 0$$

Hence, $$\operatorname*{Lt}_{z \to \infty} \int_{C_R} \frac{z \log (z + i)}{1 + z^2}\, dz = 0$$

$$\Rightarrow \qquad \operatorname*{Lt}_{z \to \infty} \int_{C_R} \frac{\log (z + i)}{1 + z^2}\, dz = 0$$

From (1), $$\int_{-\infty}^{\infty} \frac{\log (x + i)}{1 + x^2}\, dx = \pi \left(\log 2 + \frac{i\pi}{2} \right)$$

Equating real parts, we get

$$\int_{-\infty}^{\infty} \frac{1}{2} \cdot \frac{\log (x^2+1)}{1+x^2}\, dx = \pi \log 2$$

$$\Rightarrow \qquad \int_{0}^{\infty} \frac{\log (1+x^2)}{1+x^2}\, dx = \pi \log 2.$$

TEST YOUR KNOWLEDGE

Evaluate the following integrals using Contour integration:

1. (i) $\displaystyle\int_0^{\infty} \frac{dx}{1+x^2}$ (ii) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3}$

   (iii) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2}\, dx$ (G.B.T.U. 2012) (iv) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^3}\, dx;\ a > 0$

2. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)};\ a > b > 0$

3. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\, dx$

4. $\displaystyle\int_0^{\infty} \frac{x^2}{(x^2+9)(x^2+4)^2}\, dx$

5. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}\, dx$

6. (i) $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2+1)(x^2+2x+2)}\, dx$ (ii) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2(x^2+2x+2)}\, dx$

7. $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2+4x+13)^2}\, dx$

8. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^6+1}$

9. (i) $\displaystyle\int_0^{\infty} \frac{\cos mx}{x^2+a^2}\, dx;\ (m \ge 0,\ a > 0)$ (ii) $\displaystyle\int_0^{\infty} \frac{\cos mx}{(a^2+x^2)^2}\, dx;\ m \ge 0,\ a > 0$

10. (i) $\displaystyle\int_0^{\infty} \frac{x \sin ax}{x^2+k^2}\, dx;\ a > 0$ (ii) $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^2+2x+5}\, dx$

11. (i) $\displaystyle\int_0^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\, dx\ (a > b > 0)$ (ii) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\, dx,\ (a > b > 0)$ [A.K.T.U. 2017]

    (iii) $\displaystyle\int_0^{\infty} \frac{\sin x}{(x^2+a^2)(x^2+b^2)}\, dx\ (a > b > 0)$

12. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x^2+4x+5}\, dx$

13. $\displaystyle\int_0^{\infty} \frac{x^3 \sin x}{(x^2+a^2)(x^2+b^2)}\, dx;\ (a > 0,\ b > 0)$

14. If $a > 0$, prove that

    (i) $\displaystyle\int_{-\infty}^{\infty} \frac{a\cos x + x \sin x}{x^2+a^2}\, dx = 2\pi e^{-a}$ (ii) $\displaystyle\int_{-\infty}^{\infty} \frac{x \cos x - a \sin x}{x^2+a^2}\, dx = 0$

[**Hint:** Consider $f(z) = \dfrac{e^{iz}}{z-ia}$. At last multiply both N_r and D_r by $x + ia$ and separate real and imaginary parts]

15. Prove that $\int_0^\infty \frac{\cos^2 x}{(1+x^2)^2}\,dx = \frac{\pi}{2}\left(1+\frac{3}{e^2}\right)$ $\left[\textbf{Hint: } \text{Take } f(z) = \frac{1+e^{2iz}}{(1+z^2)^2} \text{ so that } f(x) = \frac{1+\cos 2x}{(1+x^2)^2}\right]$

16. By integrating e^{-z^2} round the rectangle whose vertices are $0, R, R+ia, ia$, show that

(*i*) $\int_0^\infty e^{-x^2}\cos 2ax\,dx = \frac{e^{-a^2}}{2}\sqrt{\pi}$ and (*ii*) $\int_0^\infty e^{-x^2}\sin 2ax\,dx = e^{-a^2}\int_0^a e^{y^2}\,dy$.

17. Apply calculus of residues to prove that $\int_0^\infty \sin x^2\,dx = \int_0^\infty \cos x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$.

Answers

1. (*i*) $\pi/2$ (*ii*) $3\pi/8$ (*iii*) $\frac{\pi}{2}$ (*iv*) $\frac{\pi}{8a^3}$

2. $\frac{\pi}{ab(a+b)}$ **3.** $\pi/3$ **4.** $\frac{\pi}{200}$ **5.** $\frac{5\pi}{12}$

6. (*i*) $-\frac{\pi}{5}$ (*ii*) $\frac{7\pi}{50}$ **7.** $-\frac{\pi}{27}$ **8.** $\pi/3$

9. (*i*) $\frac{\pi}{2a}e^{-ma}$ (*ii*) $\frac{\pi e^{-am}(am+1)}{4a^3}$ **10.** (*i*) $\frac{\pi}{2}e^{-ak}$ (*ii*) $-\pi e^{-2\pi}$

11. (*i*) $\frac{\pi}{2(a^2-b^2)}\left(\frac{e^{-b}}{b}-\frac{e^{-a}}{a}\right)$ (*ii*) $\frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b}-\frac{e^{-a}}{a}\right)$ (*iii*) 0

12. $-\frac{\pi}{e}\sin 2$ **13.** $\frac{\pi}{2}\cdot\frac{(a^2e^{-a}-b^2e^{-b})}{(a^2-b^2)}$

5.27.3 Integrals of the Type $\int_{-\infty}^{\infty} \frac{f(x)}{F(x)}\,dx$, when F(x) has Zeros on the Real Axis.

When the poles of $f(z)$ lie on the real axis and also within the semi-circular region, then those which lie on the real axis can be avoided by drawing small semi-circles C_r, C_r' etc. about those poles as centres and small radii r and r' in the upper half of the plane.

This method is said to be **'indenting the semi-circular contour'**.

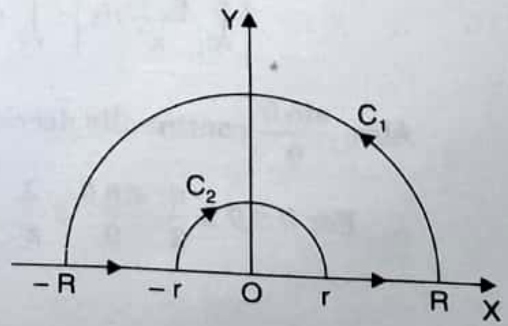
When the semi-circle of radius R has been indented then $f(z)$ is analytic along this modified contour C and the integral $\int_C f(z)\,dz$ can be evaluated by Cauchy-Residue theorem.

Example. *Evaluate* $\int_0^\infty \frac{\sin mx}{x}\,dx,\ m>0.$

Sol. Since $\sin mx$ is the imaginary part of e^{imx}, we consider the function

$$\phi(z) = \frac{e^{imz}}{z}.$$

This has a simple pole at $z = 0$, which lies on the real axis. Enclose this singularity by a small semi-circle $C_2 : |z| = r$. Evaluate the function $\phi(z)$ over the contour C shown in the figure consisting of parts of the real axis from $-R$ to $-r$ and r to R, the small semi-circle C_2 and the large semi-circle C_1. Since the function has no singularity within this contour, by Cauchy's theorem, we have



$$\oint_C \phi(z)\, dz = 0$$

$$\Rightarrow \quad \int_{-R}^{-r} \phi(x)\, dx + \int_{C_2} \phi(z)\, dz + \int_r^R \phi(x)\, dx + \int_{C_1} \phi(z)\, dz = 0$$

$$\Rightarrow \quad \int_{-R}^{-r} \frac{e^{imx}}{x}\, dx + \int_{C_2} \frac{e^{imz}}{z}\, dz + \int_r^R \frac{e^{imx}}{x}\, dx + \int_{C_1} \frac{e^{imz}}{z}\, dz = 0 \quad \text{...(1)}$$

Substituting $-x$ for x in the first integral and combining it with the third integral, we get

$$\int_r^R \frac{e^{imx} - e^{-imx}}{x}\, dx + \int_{C_2} \frac{e^{imz}}{z}\, dz + \int_{C_1} \frac{e^{imz}}{z}\, dz = 0$$

or
$$2i \int_r^R \frac{\sin mx}{x}\, dx + \int_{C_2} \frac{e^{imz}}{z}\, dz + \int_{C_1} \frac{e^{imz}}{z}\, dz = 0 \quad \text{...(2)}$$

Now,
$$\int_{C_2} \frac{e^{imz}}{z}\, dz = \int_{C_2} \frac{1}{z}\, dz + \int_{C_2} \frac{e^{imz} - 1}{z}\, dz \quad \text{...(3)}$$

On C_2, $z = re^{i\theta}$

$$\therefore \quad \int_{C_2} \frac{1}{z}\, dz = \int_\pi^0 \frac{re^{i\theta}\, i\, d\theta}{re^{i\theta}} = -\int_0^\pi i\, d\theta = -i\pi$$

Also,
$$\left| \int_{C_2} \frac{e^{imz} - 1}{z}\, dz \right| \le M \int_{C_2} \frac{|dz|}{|z|} = \pi M$$

where M is the maximum value on C_2 of $|e^{imz} - 1| = |e^{imr(\cos\theta + i\sin\theta)} - 1|$

Clearly, $M \to 0$ as $r \to 0$

$\therefore$ From (3), $\displaystyle\int_{C_2} \frac{e^{imz}}{z}\, dz = -i\pi$

Putting $z = Re^{i\theta}$ in the integral over C_1, we get ...(4)

$$\int_{C_1} \frac{e^{imz}}{z}\, dz = \int_0^\pi \frac{e^{imR(\cos\theta + i\sin\theta)}}{Re^{i\theta}} Re^{i\theta}\, i\, d\theta = i\int_0^\pi e^{imR\cos\theta} \,.\, e^{-mR\sin\theta}\, d\theta$$

Since, $\quad |e^{imR\cos\theta}| \le 1$

$$\therefore \quad \left| \int_{C_1} \frac{e^{imz}}{z}\, dz \right| \le \int_0^\pi e^{-mR\sin\theta}\, d\theta = 2\int_0^{\pi/2} e^{-mR\sin\theta}\, d\theta$$

Also, $\dfrac{\sin\theta}{\theta}$ continually decreases from 1 to $\dfrac{2}{\pi}$ as θ increases from 0 to $\dfrac{\pi}{2}$.

$\therefore$ For $0 \le \theta \le \dfrac{\pi}{2}, \dfrac{\sin\theta}{\theta} \ge \dfrac{2}{\pi}$ or $\sin\theta \ge \dfrac{2\theta}{\pi}$

$\therefore$ $$\left| \int \frac{e^{imz}}{z} dz \right| \le 2 \int_0^{\pi/2} e^{-2mR\,\theta/\pi}\, d\theta = \left[-\frac{\pi}{mR} e^{-2mR\theta/\pi} \right]_0^{\pi/2} = \frac{\pi}{mR}(1 - e^{-mR})$$

As $R \to \infty$, $\frac{\pi}{mR}(1 - e^{-mR}) \to 0$

$\therefore$ $$\int_{C_1} \frac{e^{imz}}{z} dz = 0$$

Hence from (2), on taking the limit as $r \to 0$ and $R \to \infty$, we get

$$2i \int_0^{\infty} \frac{\sin mx}{x} dx - i\pi = 0$$

or $$\int_0^{\infty} \frac{\sin mx}{x} dx = \frac{\pi}{2}.$$

TEST YOUR KNOWLEDGE

Apply calculus of residues to prove that:

1. (i) $\int_0^{\infty} \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin \pi p}, 0 < p < 1$ (*M.T.U. 2013*) (ii) $\int_0^{\infty} \frac{x^{a-1}}{1-x} dx = \pi \cot a\pi, 0 < a < 1$

   (iii) $\int_0^{\infty} \frac{x^{a-1}}{1+x^2} dx = \frac{\pi}{2} \operatorname{cosec}\left(\frac{\pi a}{2}\right), 0 < a < 2$

2. (i) $\int_0^{\infty} \frac{\cos x}{x} dx = 0$ (ii) $\int_{-\infty}^{\infty} \frac{\cos x}{a^2 - x^2} dx = \frac{\pi}{a} \sin a, (a > 0)$

3. (i) $\int_0^{\infty} \frac{\sin mx}{x(x^2 + a^2)} dx = \frac{\pi}{2a^2}(1 - e^{-ma}); a > 0$ (ii) $\int_0^{\infty} \frac{\sin \pi x}{x(1 - x^2)} dx = \pi$

4. (i) $\int_0^{\infty} \frac{\log x}{(1+x^2)^2} dx = -\frac{\pi}{4}$ (ii) $\int_0^{\infty} \frac{(\log x)^2}{1+x^2} dx = \frac{\pi^2}{8}$

5. (i) $\int_0^{\infty} \frac{x^a}{(1+x^2)^2} dx = \frac{\pi}{4}(1-a) \sec\left(\frac{\pi a}{2}\right); -1 < a < 3$

   (ii) $\int_0^{\infty} \frac{x^a}{x^2 - x + 1} dx = \frac{2\pi}{\sqrt{3}} \sin\left(\frac{2a\pi}{3}\right) \operatorname{cosec}(a\pi); -1 < a < 1$

6. $\int_0^{\infty} \frac{\cos 2ax - \cos 2bx}{x^2} dx = \pi(b - a)$ if $a \ge b \ge 0$.

ASSIGNMENT- V

1. Evaluate $\int_0^{1+i} z^2\, dz$.

2. Evaluate the integral $\int_C \frac{e^{iz}}{z^3} dz$ where C : $|z| = 1$. (M.T.U. 2013)

3. (i) Define isolated and non-isolated singular points. (M.T.U. 2012)
   (ii) Define removable and essential singular points with example. (M.T.U. 2012)

4. (i) Define singular point of an analytic function. Find nature and location of the singularity of
$$f(z) = \frac{z - \sin z}{z^2}.$$ (M.T.U. 2013)

   (ii) Find the nature of singularity of $f(z) = \frac{z - \sin z}{z^3}$ at $z = 0$.

5. Evaluate $\oint_C \frac{dz}{z-2}$ around the circle $|z - 2| = 4$.

6. (i) State Cauchy's integral theorem. (A.K.T.U. 2016)
   (ii) Evaluate $\oint_C (5z^4 - z^3 + 2)\, dz$ around the unit circle $|z| = 1$.

7. If $F(\alpha) = \oint_C \frac{5z^2 - 4z + 3}{z - \alpha} dz$ which C is the ellipse $16x^2 + 9y^2 = 144$, then find F(2).

8. Evaluate $\oint_C \frac{dz}{z^2 + 9}$ where C is $|z - 3i| = 4$.

9. Find residue of $f(z) = \left(\frac{z+1}{z-1}\right)^3$ at $z = 1$.

10. Find the residue of $f(z) = \cot z$ at its pole. (A.K.T.U. 2017)

11. (i) Find residue of $f(z) = \frac{z^2}{(z^2 + 3z + 2)^2}$ at the pole $z = -1$.

    (ii) Find residue of $f(z) = \frac{z^2}{z^2 + 3z + 2}$ at the pole -1. (U.P.T.U. 2014)

    (iii) Find residue of $f(z) = \frac{2z + 1}{z^2 - z - 2}$ at the pole $z = -1$. (M.T.U. 2014)

12. Evaluate $\oint_C \frac{4 - 3z}{z^2 - z} dz$, where C is any simple closed path such that $1 \in C$, $0 \notin C$.

13. Write the statement of generalized Cauchy's integral formula for n^{th} derivative of an analytic function at the point $z = z_0$. (A.K.T.U. 2016)

14. Evaluate $\oint_C \frac{z - 3}{z^2 + 2z + 5} dz$ when $C \equiv |z| = 1$.

15. Let $I = \int_C \frac{f(z)}{(z-1)(z-2)}\,dz$ where $f(z) = \sin\frac{\pi z}{2} + \cos\frac{\pi z}{2}$ and C is the curve $|z| = 3$ oriented anti-clockwise. Find the value of I.

16. Let $\sum_{n=-\infty}^{\infty} b_n z^n$ be the Laurent's series expansion of the function $\frac{1}{z\sinh z}$, $0 < |z| < \pi$, then find b_{-2}, b_0 and b_2.

17. Let $f(z) = \sum_{n=0}^{15} z^n$ for $z \in \mathbb{C}$. If $\mathbb{C} : |z - i| = 2$, then evaluate $\oint_C \frac{f(z)}{(z-i)^{15}}\,dz$.

18. Let $u(x, y)$ be the real part of an entire function $f(z) = u(x, y) + i\,v(x, y)$ for $z = x + iy \in$ C. If C is the positively oriented boundary of a rectangular region R in R^2 then evaluate $\oint_C \left(\frac{\partial u}{\partial y}dx - \frac{\partial u}{\partial x}dy\right)$.

19. Consider the function $f(z) = \frac{e^{iz}}{z(z^2+1)}$. Find the residue of f at the isolated singular point in the upper half plane $\{z = x + iy \in \mathbb{C} : y > 0\}$.

20. Let S be the positively oriented circle given by $|z - 3i| = 2$. Then evaluate $\int_S \frac{dz}{z^2+4}$.

21. Let $f(z)$ be an analytic function. Then evaluate $\int_0^{2\pi} f(e^{it}) \cos(t)\,dt$.

22. Let $f(z) = \frac{1}{z^2 - 3z + 2}$, then find the coefficient of $\frac{1}{z^3}$ in the Laurent's series expansion of $f(z)$ for $|z| > 2$.

23. Evaluate: $\int_C \frac{z^2+1}{z^2-1}\,dz$, where C is the circle $|z| = \frac{3}{2}$. *(A.K.T.U. 2016)*

24. Expand $\frac{1}{(z+1)(z+3)}$ in the region $|z| < 1$. *(A.K.T.U. 2016)*

25. Evaluate: $\int_{|z|=\frac{1}{2}} \frac{e^z}{z^2+1}\,dz$. *(A.K.T.U. 2017)*

26. Evaluate: $\int_C \frac{e^z}{z+1}\,dz$, where C is the circle $|z| = 2$. *(A.K.T.U. 2017)*

27. Let $C = \{z \in \mathbb{C};\ |z - i| = 2\}$. Then evaluate: $\frac{1}{2\pi}\oint_C \frac{z^2-4}{z^2+4}\,dz$.

28. Let $\gamma = \{z \in \mathbb{C} : |z| = 2\}$ be oriented in the counter-clockwise directions. Let $I = \frac{1}{2\pi i}\oint_\gamma z^7 \cos\left(\frac{1}{z^2}\right)dz$, then find the value of I.

29. Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} 4^{(-1)^n \cdot n} z^{2n}.$$

$$\left[\textbf{Hint: } a_n = \begin{cases} 4^n, & n = 2k \\ 0, & n = 2k-1 \end{cases} \text{ where } k = 1, 2, 3, \ldots\right]$$

30. Consider the power series $\sum_{n=0}^{\infty} a_n z^n$ where $a_n = \begin{cases} \dfrac{1}{3^n}, & \text{if } n \text{ is even} \\ \dfrac{1}{5^n}, & \text{if } n \text{ is odd} \end{cases}$

what is the radius of convergence of the series?

31. Find the coefficient of $(z-\pi)^2$ in the Taylor's series expansion of

$$f(z) = \begin{cases} \dfrac{\sin z}{z-\pi}, & \text{if } z \neq \pi \\ -1, & \text{if } z = \pi \end{cases} \text{ around } \pi.$$

32. If $\sum_{n=-\infty}^{\infty} a_n (z-2)^n$ is the Laurent series of the function $f(z) = \dfrac{z^4+z^3+z^2}{(z-2)^3}$ for $z \in \dfrac{C}{\{2\}}$, then find a_{-2}.

Answers

1. $-\frac{2}{3}+\frac{2}{3}i$ **2.** $-\pi i$

4. (*i*) removable singularity at $z = 0$ (*ii*) removable singularity

5. $2\pi i$ **6.** (*ii*) 0 **7.** $30\pi i$ **8.** $\frac{\pi}{3}$

9. 6 **10.** 1 **11.** (*i*) – 4 (*ii*) 1 (*iii*) $\frac{1}{3}$

12. $2\pi i$ **13.** $f^n(z_0) = \dfrac{\lfloor n}{2\pi i} \int_C \dfrac{f(z)}{(z-z_0)^{n+1}} dz.$ **14.** 0

15. $-4\pi i$ **16.** $b_{-2} = 1, b_0 = -1/6, b_2 = 7/360$ **17.** $2\pi i(1+15i)$

18. 0 **19.** $-\frac{1}{2e}$ **20.** $\frac{\pi}{2}$ **21.** $\pi f'(0)$

22. 3 **23.** 0 **24.** $\frac{1}{2}\left[\sum_{n=0}^{\infty}(-1)^n z^n - \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n\right]$

25. 0 **26.** $\frac{2\pi i}{e}$ **27.** – 2 **28.** $\frac{1}{24}$

29. 0.50 **30.** 3 **31.** $\frac{1}{6}$ **32.** 48.

EXAMINATION PAPERS

B. Tech. [SEMESTER-II] **RAS–203**

THEORY EXAMINATION, 2017-2018

ENGINEERING MATHEMATICS–II

(PAPER ID: 199223)

Time: 3 Hours *Total Marks: 70*

Note: Attempt all sections. If require any missing data, then choose suitably.

SECTION–A

1. Attempt **all** questions in brief: **(2 × 7 = 14)**

(*a*) Determine the differential equation whose set of independent solutions is

$$\{e^x, xe^x, x^2 e^x\}$$

(*b*) Solve: $(D + 1)^3 y = 2e^{-x}$.

(*c*) Prove that: $P_n(-x) = (-1)^n P_n(x)$.

(*d*) Find inverse Laplace transform of $\dfrac{s+8}{s^2+4s+5}$.

(*e*) If $L\{F\sqrt{t}\} = \dfrac{e^{-1/s}}{s}$, find $L\{e^{-1} F(3\sqrt{t})\}$.

(*f*) Solve: $(D + 4D' + 5)^2 z = 0$, where $D = \dfrac{\partial}{\partial x}$, $D' = \dfrac{\partial}{\partial y}$.

(*g*) Classify the equation: $z_{xx} + 2x\, z_{xy} + (1 - y^2) z_{yy} = 0$.

SECTION–B

2. Attempt **any three** of the following: **(7 × 3 = 21)**

(*a*) Solve $(D^2 - 2D + 4)y = e^x \cos x + \sin x \cos 3x$,

(*b*) Prove that: $J_{5/2}(x) = \sqrt{\dfrac{2}{\pi x}}\left[\left(\dfrac{3-x^2}{x^2}\right)\sin x - \dfrac{3\cos x}{x}\right]$.

(*c*) Draw the graph and find the Laplace transform of the triangular wave function of period 2π given by

$$F(t) = \begin{cases} t, & 0 < t \le \pi \\ 2\pi - t, & \pi < t < 2\pi \end{cases}$$

(*d*) Obtain half range cosine series for the function $(t) = \begin{cases} 2t, & 0 < t < 1 \\ 2(2-t), & 1 < t < 2 \end{cases}$

(*e*) Solve by method of separation of variables: $\dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x} - 2u$; $u(x, 0) = 10e^{-x} - 6e^{-4x}$.

SECTION-C

3. Attempt **any one** part of the following: **(7 × 1 = 7)**

(*a*) Solve the simultaneous differential equations:

$$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = y \text{ and } \frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 25x + 16e^t.$$

(*b*) Use variation of parameter method to solve the differential equation $x^2 y^2 + xy' - y = x^2 e^x$.

SECTION-D

4. Attempt **any one** part of the following: **(7 × 1 = 7)**

(*a*) State and prove Rodrigue's formula for Legendre's polynomial.

(*b*) Solve in series: $2x(1 - x)y'' + (1 - x)y' + 3y = 0$.

5. Attempt **any one** part of the following: **(7 × 1 = 7)**

(*a*) State convolution theorem and hence find inverse Laplace transform of

$$\frac{s^2}{(s^2 + a^2)(s^2 + b^2)}.$$

(*b*) Solve the following differential equation using Laplace transform

$$\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - y = t^2e^t. \text{ where } y(0) = 1, y'(0) = 0 \text{ and } y''(0) = -2.$$

6. Attempt **any one** part of the following: **(7 × 1 = 7)**

(*a*) Obtain Fourier series for the function $f(x) = \begin{cases} x, & -\pi < x \le 0 \\ -x, & 0 < x < \pi \end{cases}$ and hence show that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

(*b*) Solve the linear partial differential equation: $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x \partial y} = \sin x \cos 2y$.

7. Attempt **any one** part of the following: **(7 × 1 = 7)**

(*a*) A string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form $y = A \sin\dfrac{\pi x}{l}$ from which it is released at time $t = 0$. Find the displacement of any point at a distance x from one end at any time t.

(*b*) A rectangular plate with insulated surface is 8 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along one short edge $y = 0$ is given by $u(x, 0) = 100 \sin\dfrac{\pi x}{8}$, $0 < x < 8$ while two long edges $x = 0$ and $x = 8$ as well as the other short edge are kept at 0° C. Find the temperature $u(x, y)$ at any point in steady state.

B. Tech. [SEMESTER-II] RAS–203

THEORY EXAMINATION, 2016-2017

MATHEMATICS-II

Time: 3 Hours *Total Marks: 70*

Note: *Be precise in your answer. In case of numerical problem assume data whereever not provided.*

SECTION-A

1. Attempt **any seven** parts for the following: **(7 × 2 = 14)**

(*a*) Solve the differential equation $\frac{d^2y}{dx^2} = -12x^2 + 24x - 20$ with the condition $x = 0$, $y = 5$ and $x = 0$. $y' = 21$ and hence find the value of y at $x = 1$.

(*b*) For a differential equation $\frac{d^2y}{dx^2} + 2\alpha\frac{dy}{dx} + y = 0$, find the value of α for which the differential equation characteristic equation has equal number.

(*c*) For a Legendre polynominal prove that $P_n(1) = 1$ and $P_n(-1) = (-1)^n$.

(*d*) For the Bessel's function $J_n(x)$ prove the following identities:
$J_{-n}(x) = (-1)^n J_n(x)$ and $J_{-n}(-x) = (-1)^n J_n(x)$.

(*e*) Evaluate the Laplace transform of Integral of a function $L\left\{\int f(t)\,dt\right\}$.

(*f*) Evaluate the value of integral $\int_0^\infty t\,.\,e^{-2t} \cos t\,dt$.

(*g*) Find the Fourier coefficient for the function $f(x) = x^2$, $0 < x < 2\pi$.

(*h*) Find the partial differential equation of all sphere whose centre lie on Z-axis.

(*i*) Formulate the PDE by eliminating the arbitrary function from $\phi(x^2 + y^2, y^2 + z^2) = 0$.

(*j*) Specify with suitable example the clarification Partial Differential Equation (PDE) for elliptic, parabolic and hyperbolic differential equation.

SECTION-B

2. Attempt **any three** parts of the following questions: **(7 × 3 = 21)**

(*a*) A function $n(x)$ satisfies the differential equation $\frac{d^2n(x)}{dx^2} - \frac{n(x)}{n^2} = 0$, where L is a constant. The boundary conditions are $n(0) = c$ and $n(\infty) = 0$. Find the solution to this equation.

(*b*) Find the series solution by Fobenius method for the differential equation $(1 - x^2)y'' - 2xy' + 20y = 0$.

(c) Determine the response of damped mass – spring system under a square wave given by the differential equation

$$y'' + 3y' + 2y = u(t-1) - u(t-2),\ y(0) = 0,\ y'(0) = 0.$$

Using the Laplace transform.

(d) Obtain the Fourier expansion of $f(x) = x \sin x$ as cosine series in $(0, \pi)$ and hence show that:

$$\frac{1}{1\times 3} - \frac{1}{3\times 5} + \frac{1}{5\times 7} - \dots\dots\dots = \left(\frac{\pi - 2}{4}\right).$$

(e) Solve by method of separation of variable for PDE

$$x\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0,\ u(x, 0) = 4e^{-x}.$$

SECTION-C

*Attempt **all parts** of the following questions:* **(7 × 5 = 35)**

3. Attempt **any two** of the following questions:

(a) Find the particular solution of the differential equation

$$\frac{d^2y}{dx^2} + a^2 = \sec ax.$$

(b) If $y = y_1(x)$ and $y = y_2(x)$ are two solutions of the equation $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$,

then show that $y_1\left(\frac{dy_2}{dx}\right) - y_2\left(\frac{dy_1}{ax}\right) = ce^{-\int P dx}$, where c is constant.

(c) Solve by method of variation of Parameter for the differential equation:

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + qy = \left(\frac{e^{2x}}{x^2}\right).$$

4. Attempt **any two** parts of the following:

(a) Prove that $\sqrt{\frac{\pi x}{2}} \cdot J_{3/2}(x) = \left(\frac{1}{2}\sin x - \cos x\right)$.

(b) Show that Legendre polynomials are orthogonal on the interval [–1, 1].

(c) Prove that: $\int_{-1}^{+1} xP_n(x)P_{n-1}(x)\, dx = \frac{2n}{4n^2 - 1}$.

5. Attempt **any two** parts of the following:

(a) Find the Laplace transform of saw – tooth wave function $F(t) = Kt$ in $0 < t < 1$ with period 1.

(*b*) Use Convolution theorem to find the inverse of function $F(s) = \frac{4}{s^2 + 2s + 5}$.

(*c*) Solve the simultaneous differential equation, using Laplace tranformation

$$-\frac{dy}{dt} + 2x = \sin 2t; \quad \frac{dy}{dt} - 2y = \cos 2t, \text{ where } x(0) = 1, y(0) = 0.$$

6. Attempt **any two** parts of the following:

(*a*) If $f(x) = \left[\frac{\pi - x}{2}\right]^2$, $0 < x < 2\pi$ then show that $f(x) = \frac{n^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$.

(*b*) Find the complete solution of PDE

$(D^2 + 7DD' + 12D'^2)z = \sinh x$, where symbols have their usual meaning.

(*c*) Solve the PDE $p + 3q = 5z + \tan(y - 3x)$.

7. Attempt **any one** part of the following:

(*a*) A square plate is bounded by lines $x = 0, y = 0, x = 20, y = 20$. Its faces are isolated. The temperature along the upper horizontal edge is given at $u(x, 20) = x(20 - x)$ when $0 < x < 20$ while the upper three edges are kept at 0° C. Find the steady state temperature.

(*b*) A bar of 10 cm long with insulated sides A and B are kept 20° C and 40° C respectively until steady state conditions prevail. The temperature at A is then suddenly varies to 50° C and the same instant at B lowered at 10° C. Find the subsequent temperature at any point of the bar at any time.

B. Tech. [SEMESTER-II]

EAS–203

THEORY EXAMINATION, 2016–2017
ENGINEERING MATHEMATICS–II

Time: 3 Hours *Total Marks: 100*

Note: *Be precise in your answer. In case of numerical problem assume data wherever not provided.*

SECTION–A

1. Explain the following: **(10 × 2 = 20)**

(*a*) Show that the differential equation $y\,dx - 2x\,dy = 0$ represents a family of parabolas.

(*b*) Classify the partial differential equation

$$(1-x^2)\frac{\partial^2 z}{\partial x^2} - 2xy\frac{\partial^2 z}{\partial y \partial x} + (1-y^2)\frac{\partial^2 z}{\partial y^2} = 2z.$$

(*c*) Find the particular integral of $(D-\alpha)^2 y = e^{\alpha x} f''(x)$.

(*d*) Write the Dirichlet's conditions for the Fourier series.

(*e*) Prove that $J'_0(x) = -J_1(x)$.

(*f*) Prove that $L[e^{at} f(t)] = F(s-a)$.

(*g*) Find the Laplace transform of $f(t) = \dfrac{\sin at}{t}$.

(*h*) Write one and two dimensional wave equations.

(*i*) Find the constant term when $f(x) = |x|$ is expanded in Fourier series in the interval $(-2, 2)$.

(*j*) Write the generating function for Legendre polynominal $P_n(x)$.

SECTION–B

2. Attempt **any five** of the following questions: **(5 × 10 = 50)**

(*a*) Solve the differential equation

$$(D^2 + 2D + 2)y = e^{-x}\sec^3 x, \text{ where } D = \frac{d}{dx}.$$

(*b*) Prove that $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$.
where $P_n(x)$ is the Legendre's function.

(*c*) Find the series solution of the differential equation

$$2x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x+1)y = 0.$$

(*d*) Using Laplace tranform, solve the differential equation

$$\frac{d^2y}{dt^2} + 9y = \cos 2t\ ;\ y(0) = 1,\ y\left(\frac{\pi}{2}\right) = -1.$$

(e) Obtain the Fourier series of the function,

$$f(t) = t, \quad -\pi < t < 0$$
$$= -t, \quad 0 < t < \pi.$$

Hence, deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\ldots = \frac{\pi^2}{8}$.

(f) Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ under the conditions $u(0, y) = 0$,

$u(l, y) = 0$, $u(x, 0) = 0$ and $u(x, a) = \sin\frac{n\pi x}{l}$.

(g) Solve the partial differential equation:

$$(D^3 - 4D^2D' + 5D\,D'^2 - 2D'^3)z = e^{y+2x} + \sqrt{y+x}.$$

(h) Using convolution theorem, find $L^{-1}\left[\frac{1}{(s+1)(s^2+1)}\right]$.

SECTION–C

Attempt **any two** of the following questions: **(2 × 15 = 30)**

3. (a) Solve the differential equation $(D^2 - 2D + 1)y = e^x \sin x$.

(b) Solve the equation by Laplace transform method:

$$\frac{dy}{dt} + 2y + \int_0^t y\,dt = \sin t,\ y(0) = 1.$$

(c) Solve the partial differential equation

$$(y^2 + z^2)\,p - xyq + zx = 0, \text{ where } p = \frac{\partial z}{\partial x} \text{ and } q = \frac{\partial z}{\partial y}.$$

4. (a) Find the Laplace transform of $\frac{\cos at - \cos bt}{t}$.

(b) Express $f(x) = 4x^3 - 2x^2 - 3x + 8$ in terms of Legendre's polynomial.

(c) Expand $f(x) = 2x - 1$ as a cosine series in $0 < x < 2$.

5. (a) Show that $J_3(x) = \left(\frac{8}{x^2} - 1\right)J_1(x) - \frac{4}{x}J_0(x)$.

(b) Solve the $2\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} + 5z = 0$; $z(0, y) = 2e^{-y}$ by the method of separation of variables.

(c) A tightly stretched string with fixed end $x = 0$ and $x = l$ is initially in a position given by $y = a\sin\frac{\pi x}{l}$. If it is released from rest from this position, find the displacement $y(x, t)$.

ABOUT THE BOOK

The underlying object of this book is to provide the reader with the thorough understanding of topics included in the syllabus of Mathematics for engineering students.

The salient features of the book are as follows:

- It exactly covers the prescribed syllabus. Nothing undesirable has been included and nothing essential has been left.
- Its approach is explanatory and language is lucid and communicable.
- The exposition of the subject matter is systematic and the students are better prepared to solve the problems.
- All fundamentals of the included topics are explained with a micro-analysis.
- Sufficient number of solved examples have been given to let the students understand the various skills necessary to solve the problems. These examples are well-graded.
- Unsolved exercises of multi-varieties have been given in a well-graded style. Attempting these on their own, will enable a student to create confidence and independence in him/her regarding the understanding of the subject.
- Daily life problems and practical applications have been incorporated in the body of the text.
- A large number of attractive and accurate figures have been drawn which enable a student to grasp the subject in an easier way.
- All the answers have been checked and verified.

ABOUT THE AUTHORS

N.P. Bali is a prolific author of over 100 books for degree and engineering students. He has been writing books for more than forty years.

His books on the following topics are well known for their easy comprehension and lucid presentation: Algebra, Trigonometry, Differential Calculus, Integral Calculus, Real Analysis, Co-ordinate Geometry, Statics, Dynamics etc.

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ISBN:978-93-80386-78-2

9789380386782-0455

M.R.P. ₹ 455.00

UEM-9344-455-ENGG MATH II (UP)-GOY

UNIVERSITY SCIENCE PRESS

(An Imprint of Laxmi Publications Pvt. Ltd.)

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