PART-1

Improper Integrals, Beta and Gama Functions and their Properties.

CONCEPT OUTLINE

Improper Integrals : By definition of a regular (or proper) definite integral $\int_a^b f(x)dx$, it is assumed that the limits of integration are finite and that the integrand $f(x)$ is continuous for every value of x in the interval $a \le x \le b$. If at least one of these conditions is violated, then the integral is known as an improper integral (or singular or generalized or infinite integral).

Beta Function : The definite integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ is called the Beta function, where m and n are positive. Beta function is denoted by $\beta(m, n)$. Thus

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

Property 1 : $\beta(m, n) = \beta(n, m)$

Property 2 : Transformation of Beta function is

$$\beta(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$$

Gamma Function : Gamma function for a positive number n is denoted by $\lceil n$ and is given by

$$\lceil n = \int_0^\infty e^{-x}x^{n-1}dx, n > 0$$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 2.1. Evaluate $\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}dx$.

Answer

$$\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}dx = \int_0^\infty \frac{x^4dx}{(1+x)^{15}} + \int_0^\infty \frac{x^9}{(1+x)^{15}}dx$$

$$= \int_0^\infty \frac{x^{5-1}}{(1+x)^{5+10}}dx + \int_0^\infty \frac{x^{10-1}}{(1+x)^{10+5}}dx$$

$$= \beta(5, 10) + \beta(10, 5)$$

$$= 2\beta(5, 10) \qquad [\because \beta(m, n) = \beta(n, m)]$$

Que 2.2. To prove $\lceil n+1 = n\lceil n$.

Answer

We know that $\lceil n = \int_0^\infty e^{-x}x^{n-1}dx$

$$\lceil n+1 = \int_0^\infty e^{-x}x^{n+1-1}dx = \int_0^\infty e^{-x}x^n dx$$

Integrating by parts,

$$\lceil n+1 = \left[-x^n e^{-x}\right]_0^\infty - \int_0^\infty nx^{n-1}\left(-e^{-x}\right)dx$$

$$= 0 + n\int_0^\infty e^{-x}x^{n-1}dx$$

$$\lceil n+1 = n\lceil n$$

Que 2.3. Prove that $\beta(m, n) = \beta(m+1, n) + \beta(m, n+1)$.

Answer

$$\text{RHS} = \beta(m+1, n) + \beta(m, n+1)$$

$$= \int_0^1 x^{m+1-1}(1-x)^{n-1}dx + \int_0^1 x^{m-1}(1-x)^{n+1-1}dx$$

$$= \int_0^1 x^m(1-x)^{n-1}dx + \int_0^1 x^{m-1}(1-x)^n dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}(x+1-x)dx = \beta(m, n)$$

Que 2.4. Find the value of $\left\lceil \frac{1}{2}\right.$.

Answer

$$\lceil n = \int_0^\infty e^{-x}x^{n-1}dx$$

$$\left\lceil\frac{1}{2}\right. = \int_0^\infty e^{-x}\, x^{-1/2}\, dx$$

Let, $$x = y^2$$

$$dx = 2y\, dy = \int_0^\infty e^{-y^2} \frac{1}{y} 2y\, dy$$

$$\left\lceil\frac{1}{2}\right. = 2\int_0^\infty e^{-y^2}\, dy \qquad ...(2.4.1)$$

Similarly, $$\left\lceil\frac{1}{2}\right. = 2\int_0^\infty e^{-x^2}\, dy \qquad ...(2.4.2)$$

Multiplying eq. (2.4.1) and eq. (2.4.2), we get

$$\left(\left\lceil\frac{1}{2}\right.\right)^2 = 4\int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$$

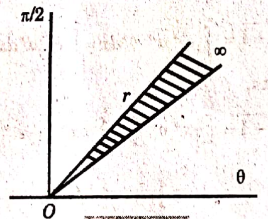

Fig. 2.4.1.

Changing this integral to polar coordinate by putting $x = r\cos\theta$, $y = r\sin\theta$ and $dx\, dy = r\, dr\, d\theta$.

Region of integration is the complete positive quadrant r will vary from 0 to ∞ and θ from 0 to $\pi/2$.

$$\left(\left\lceil\frac{1}{2}\right.\right)^2 = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\, r\, dr\, d\theta$$

$$= 4\int_0^{\pi/2}\left[-\frac{1}{2}e^{-r^2}\right]_0^\infty d\theta = 2\int_0^{\pi/2} d\theta = \pi$$

$$\left(\left\lceil\frac{1}{2}\right.\right)^2 = \pi$$

$$\left\lceil\frac{1}{2}\right. = \sqrt{\pi}$$

Que 2.5. **To prove that $\beta(m, n) = \int_0^1 \dfrac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\, dx$.**

Answer

Let, $$x = \frac{1}{1+y}$$

$$dx = \frac{-1}{(1+y)^2}\, dy$$

$$\beta(m, n) = \int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1}\left(\frac{y}{1+y}\right)^{n-1}\left(\frac{-1}{(1+y)^2}\right) dy$$

$$= \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\, dy$$

$$= \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}}\, dy + \int_1^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\, dy \qquad ...(2.5.1)$$

Now in the second integral,

Let, $$y = \frac{1}{t}$$

$$dy = -\frac{1}{t^2}\, dt$$

$$\int_1^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\, dy = \int_1^0 \frac{\left(\frac{1}{t}\right)^{n-1}}{\left(1+\frac{1}{t}\right)^{m+n}}\left(-\frac{1}{t^2}\right) dt$$

$$= \int_0^1 \frac{t^{m-1}}{(1+t)^{m+n}}\, dt = \int_0^1 \frac{y^{m-1}}{(1+y)^{m+n}}\, dy$$

From eq. (2.5.1),

$$\beta(m, n) = \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}}\, dy + \int_0^1 \frac{y^{m-1}}{(1+y)^{m+n}}\, dy = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\, dx$$

Que 2.6. **Prove that : $\beta(m, n) = \dfrac{\left\lceil m\right.\left\lceil n\right.}{\left\lceil(m+n)\right.}$, $m > 0, n > 0$.**

AKTU 2017-18, Marks 07

Answer

We know that, $$\left\lceil n\right. = k^n \int_0^\infty e^{-kx}\, x^{n-1}\, dx$$

Replacing k by z, $$\left\lceil n\right. = z^n \int_0^\infty e^{-zx}\, x^{n-1}\, dx$$

Multiplying both sides by $e^{-z}z^{m-1}$,

$$\Gamma n\, e^{-z} z^{n-1} = \int_0^\infty z^{n+m-1} e^{-z(1+x)} x^{n-1}\, dx$$

Integrating both sides w.r.t z from 0 to ∞,

$$\Gamma n \int_0^\infty e^{-z} z^{m-1} dz = \int_0^\infty x^{n-1} \left\{ \int_0^\infty e^{-z(1+x)} z^{m+n-1}\, dz \right\} dx$$

Let, $z(1+x) = y$

$$dz = \frac{dy}{1+x}$$

$$\Gamma n\, \Gamma m = \int_0^\infty x^{n-1} \int_0^\infty e^{-y} \frac{y^{m+n-1}}{(1+x)^{m+n}}\, dy\, dx$$

$$\Gamma n\, \Gamma m = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} \left\{ \int_0^\infty e^{-y} y^{m+n-1} dy \right\} dx$$

$$= \Gamma(m+n) \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

$$\Gamma n\, \Gamma m = \Gamma(m+n)\ \beta(m, n)$$

Thus, $$\beta(m, n) = \frac{\Gamma m\, \Gamma n}{\Gamma(m+n)}$$

Que 2.7. **Evaluate : $\int_0^\infty \cos x^2\, dx$**

Answer

We know that

$$\int_0^\infty e^{-ax} x^{n-1} \cos bx\, dx = \frac{\Gamma n \cos n\theta}{(a^2+b^2)^{n/2}}, \text{ where } \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

Put $$a = 0,\ \int_0^\infty x^{n-1} \cos bx\, dx = \frac{\Gamma n}{b^n} \cos \frac{n\pi}{2}$$

Also putting $x^n = z$ so that $x^{n-1} dx = \frac{dz}{n}$ and $x = z^{1/n}$

$\therefore$ $$\int_0^\infty \cos bz^{1/n}\, dz = \frac{n\Gamma n}{b^n} \cos \frac{n\pi}{2}$$

or $$\int_0^\infty \cos(bx^{1/n})\, dx = \frac{\Gamma(n+1)}{b^n} \cos \frac{n\pi}{2}$$

Here $b = 1$, $n = 1/2$

$\therefore$ $$\int_0^\infty \cos x^2\, dx = \Gamma(3/2) \cos \frac{\pi}{4} = \frac{\sqrt{\pi}}{2} \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

Que 2.8. **Prove that $\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \dfrac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}}$.**

Answer

We know that $\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx$

Putting $x = \sin^2\theta$

$dx = 2 \sin\theta \cos\theta\, d\theta$

$$= \int_0^{\pi/2} 2\sin^{2(m-1)}\theta \cos^{2(n-1)}\theta \sin\theta \cos\theta\, d\theta$$

$$\beta(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

$$\frac{\Gamma m\, \Gamma n}{\Gamma(m+n)} = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta \qquad \left[\because \beta(m, n) = \frac{\Gamma m\, \Gamma n}{\Gamma(m+n)}\right]$$

Let, $2m - 1 = p$ and $2n - 1 = q$, $m = p + 1/2$, $n = q + 1/2$

$\therefore$ $$\frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{\Gamma\frac{p+q+2}{2}} = 2\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta$$

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}}$$

Que 2.9. **State and prove the duplication formula.**

Answer

A. Duplication Formula :

$$\Gamma m\, \Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}} \Gamma 2m, \text{ where } m \text{ is positive.}$$

B. Proof : We know that

$$\beta(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

$$\frac{\Gamma m\, \Gamma n}{2\Gamma(m+n)} = \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta \qquad ...(2.9.1)$$

Let, $2n - 1 = 0$

$n = 1/2$

Now from eq. (2.9.1)

$$\int_0^{\pi/2} \sin^{2m-1}\theta\, d\theta = \frac{\Gamma m\, \Gamma\frac{1}{2}}{2\Gamma\left(m+\frac{1}{2}\right)} \qquad ...(2.9.2)$$

Again in eq. (2.9.1), let $n = m$

$$\frac{\Gamma m\,\Gamma m}{2\Gamma 2m} = \int_0^{\pi/2} (\sin\theta\cos\theta)^{2m-1}\,d\theta = \frac{1}{2^{2m-1}}\int_0^{\pi/2} (\sin 2\theta)^{2m-1}\,d\theta$$

Let,
$$2\theta = \phi$$
$$2d\theta = d\phi$$

$$= \frac{1}{2^{2m-1}}\int_0^{\pi} (\sin\phi)^{2m-1}\frac{d\phi}{2} = \frac{1}{2^{2m}}\int_0^{\pi} \sin^{2m-1}\phi\,d\phi$$

$$\frac{(\Gamma m)^2}{2\Gamma 2m} = \frac{2}{2^{2m}}\int_0^{\pi/2} \sin^{2m-1}\theta\,d\theta$$

[Using property of definite integral]

$$\frac{(\Gamma m)^2\, 2^{2m-1}}{2\Gamma 2m} = \int_0^{\pi/2} \sin^{2m-1}\theta\,d\theta \quad ...(2.9.3)$$

From eq. (2.9.2) and eq. (2.9.3),

$$\frac{\Gamma m\,\Gamma\frac{1}{2}}{2\Gamma\left(m+\frac{1}{2}\right)} = \frac{(\Gamma m)^2\, 2^{2m-1}}{2\Gamma 2m}$$

$$\Gamma m\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma 2m$$

Que 2.10. Prove that $\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta = \frac{\pi}{\sqrt{2}}$.

Answer

$$\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta = \int_0^{\pi/2}\sqrt{\tan\left(\frac{\pi}{2}-\theta\right)}\,d\theta$$

$$= \int_0^{\pi/2}\sqrt{\cot\theta}\,d\theta \qquad \left(\because \int_0^a f(x)\,dx = \int_0^a (a-x)\,dx\right)$$

$$= \int_0^{\pi/2}\cos^{1/2}\theta\,\sin^{-1/2}\theta\,d\theta = \frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)}{2\Gamma\left(\frac{\frac{1}{2}-\frac{1}{2}+2}{2}\right)}$$

$$= \frac{\Gamma\frac{3}{4}\,\Gamma\frac{1}{4}}{2\Gamma 1} = \frac{1}{2}\Gamma\frac{1}{4}\,\Gamma\left(1-\frac{1}{4}\right) = \frac{1}{2}\frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{\sqrt{2}}$$

$$\left(\because \Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right)$$

Que 2.11. Using Beta and Gamma functions, evaluate $\int_0^{\infty}\frac{dx}{1+x^4}$.

AKTU 2011-12, Marks 05

Answer

$$I = \int_0^{\infty}\frac{dx}{1+x^4}$$

Let,
$$x^2 = \tan\theta$$
$$2x\,dx = \sec^2\theta\,d\theta$$

$$dx = \frac{\sec^2\theta\,d\theta}{2\sqrt{\tan\theta}} = \frac{1}{2\sqrt{\sin\theta}\cos^{3/2}\theta}\,d\theta$$

$$dx = \frac{1}{2}\sin^{-1/2}\theta\cos^{-3/2}\theta\,d\theta$$

$$I = \int_0^{\pi/2}\frac{1}{2}\frac{\sin^{-1/2}\theta\cos^{-3/2}\theta}{\sec^2\theta}\,d\theta$$

$$I = \frac{1}{2}\int_0^{\pi/2}\sin^{-1/2}\theta\cos^{1/2}\theta\,d\theta$$

$$\left[\because \int_0^{\pi/2}\sin^p\theta\cos^q\theta\,d\theta = \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\Gamma\frac{p+q+2}{2}}\right]$$

$$I = \frac{1}{2}\frac{\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}}{2\Gamma 1} = \frac{1}{4}\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}$$

$$= \frac{1}{4}\frac{\pi}{\sin\frac{3\pi}{4}} = \frac{\pi}{4}\sqrt{2} \Rightarrow I = \frac{\pi}{2\sqrt{2}} \quad \left[\because \frac{\pi}{\sin n\pi} = \Gamma n\,\Gamma(1-n)\right]$$

Que 2.12. Using Beta and Gamma function, evaluate $\int_0^1\left(\frac{x^3}{1-x^3}\right)^{\frac{1}{2}}dx$.

AKTU 2014-15, Marks 3.5

Answer

Same as Q. 2.11, Page 2–9F, Unit-2.

$$\left(\textbf{Answer: } I = \frac{1}{3}\frac{\sqrt{\pi}\,\Gamma 5/6}{\Gamma 4/3}\right)$$

Que 2.13. For the Gamma function, show that

$$\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{2}{3}\right)} = (2)^{1/3}\sqrt{\pi}.$$

AKTU 2016-17, Marks 07

Answer

$$\text{LHS} = \frac{\Gamma\frac{1}{3}\,\Gamma\frac{5}{6}}{\Gamma\frac{2}{3}} = \frac{\Gamma\frac{1}{3}\,\Gamma\left(\frac{1}{3}+\frac{1}{2}\right)}{\Gamma\frac{2}{3}} = \frac{\sqrt{\pi}}{(2)^{2\cdot\frac{1}{3}-1}}\,\frac{\Gamma\left(2\times\frac{1}{3}\right)}{\Gamma\frac{2}{3}}$$

$$\left[\because \text{By Duplication formula, } \Gamma m\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma 2m\right]$$

$$= \frac{\sqrt{\pi}}{(2)^{\frac{2-3}{3}}} = (2)^{1/3}\sqrt{\pi} = \text{RHS}$$

PART-2

Dirichlet's Integral and its Applications.

CONCEPT OUTLINE

Dirichlet's Integral : Dirichlet's integral is given as,

$$\iint_D x^{l-1}y^{m-1}\,dx\,dy = \frac{\Gamma l\,\Gamma m}{\Gamma(l+m+1)}\,a^{l+m}$$

Where D is the domain $x \ge 0, y \ge 0$ and $x + y \le a$.

Dirichlet's Integral for Three Variables :

$$\iiint_D x^{l-1}y^{m-1}z^{n-1}\,dx\,dy\,dz = \frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n+1)}$$

Where D is the domain $x \ge 0, y \ge 0, z \ge 0$ and $x + y + z \le 1$.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 2.14. State and prove Dirichlet's integral for two variables.

Answer

A. **Dirichlet's Integral for Two Variables :** The Dirichlet's integral for two variables is given by,

$$\iint_D x^{l-1}y^{m-1}\,dx\,dy = \frac{\Gamma l\,\Gamma m}{\Gamma(l+m+1)}\,a^{l+m}$$

Where D is the domain $x \ge 0, y \ge 0$ and $x + y \le a$

B. **Proof :** Let,

$$x = aX$$
$$y = aY$$

Therefore, given integral becomes $\iint_{D'} (aX)^{l-1}(aY)^{m-1}a^2\,dX\,dY$

Where D' is the domain and $X \ge 0, Y \ge 0$ and $X + Y \le 1$

$$= a^{l+m}\iint_{D'} X^{l-1}Y^{m-1}\,dX\,dY$$

$$= a^{l+m}\int_0^1\int_0^{1-X} X^{l-1}Y^{m-1}\,dX\,dY = a^{l+m}\int_0^1 X^{l-1}\left[\frac{Y^m}{m}\right]_0^{1-X}dX$$

$$= \frac{a^{l+m}}{m}\int_0^1 X^{l-1}(1-X)^m\,dX$$

$$= \frac{a^{l+m}}{m}\int_0^1 X^{l-1}(1-X)^{m+1-1}\,dX = \frac{a^{l+m}}{m}\beta(l, m+1)$$

$$= \frac{a^{l+m}}{m}\,\frac{\Gamma l\,\Gamma(m+1)}{\Gamma(l+m+1)} = a^{l+m}\,\frac{\Gamma l\,\Gamma m}{\Gamma(l+m+1)}$$

Que 2.15. State and prove Dirichlet's integral for three variables.

Answer

A. **Dirichlet's Integral for Three Variables :**

$$\iiint_D x^{l-1}y^{m-1}z^{n-1}\,dx\,dy\,dz = \frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n+1)}$$

Where D is the domain $x \ge 0, y \ge 0, z \ge 0$ and $x + y + z \le 1$

B. **Proof :** $x + y + z \le 1$

$$y + z \le 1 - x = a \text{ (let)}$$

Therefore given integral becomes $\int_0^1\int_0^{1-x}\int_0^{1-x-y} x^{l-1}y^{m-1}z^{n-1}\,dz\,dy\,dx$

$$= \int_0^1 x^{l-1}\left[\int_0^a\int_0^{a-y} y^{m-1}z^{n-1}\,dz\,dy\right]dx$$

$$= \int_0 x^{l-1} \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)} a^{m+n}\, dx$$

$$= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)} \int_0 x^{l-1}(1-x)^{m+n+1-1}\, dx$$

$$= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)} \beta(l, m+n+1)$$

$$= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)} \frac{\Gamma(l)\Gamma(m+n+1)}{\Gamma(l+m+n+1)} = \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)}$$

Que 2.16. **Evaluate $\iiint_V (ax^2 + by^2 + cz^2)\, dx\, dy\, dz$ where V is the region bounded by $x^2 + y^2 + z^2 \le 1$.** **AKTU 2012-13, Marks 10**

Answer

$\iiint_V (ax^2 + by^2 + cz^2)\, dx\, dy\, dz$, Where $V = x^2 + y^2 + z^2 \le 1$

Let, $x^2 = u, y^2 = v, z^2 = w$

$\therefore \quad x = \sqrt{u}, \quad y = \sqrt{v}, \quad z = \sqrt{w}$

And, $\quad dx = \frac{1}{2\sqrt{u}}, \quad dy = \frac{1}{2\sqrt{v}}, \quad dz = \frac{1}{2\sqrt{w}}$

$$= \iiint_{V'} (au + bv + cw) \frac{1}{8\sqrt{uvw}}\, du\, dv\, dw, \text{ where } V' = u + v + w \le 1$$

$$= \iiint_{V'} \frac{a}{8} u^{1/2} v^{-1/2} w^{-1/2}\, du\, dv\, dw + \frac{b}{8} \iiint u^{-1/2} v^{1/2} w^{-1/2}\, du\, dv\, dw$$

$$+ \frac{c}{8} \iiint u^{-1/2} v^{-1/2} w^{1/2}\, du\, dv\, dw$$

$$= \frac{a}{8} \iiint_{V'} u^{\frac{3}{2}-1} v^{\frac{1}{2}-1} w^{\frac{1}{2}-1}\, du\, dv\, dw + \frac{b}{8} \iiint_{V'} u^{\frac{1}{2}-1} v^{\frac{3}{2}-1} w^{\frac{1}{2}-1}\, du\, dv\, dw$$

$$+ \frac{c}{8} \iiint_{V'} u^{\frac{1}{2}-1} v^{\frac{1}{2}-1} w^{\frac{3}{2}-1}\, du\, dv\, dw$$

$$= \frac{a}{8} \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{1}{2}+\frac{1}{2}+1\right)} + \frac{b}{8} \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{7}{2}\right)} + \frac{c}{8} \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{7}{2}\right)} = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{8\,\Gamma\left(\frac{7}{2}\right)} (a+b+c)$$

$$= \frac{1}{2} \frac{\pi\sqrt{\pi}}{8 \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi}} (a+b+c) = \frac{\pi}{30} (a+b+c)$$

Que 2.17. **Prove that : $\sqrt{\pi}\,\Gamma(2n) = 2^{2n-1}\,\Gamma(n)\,\Gamma\left(n + \frac{1}{2}\right)$, where n is not a negative integer or zero.** **AKTU 2012-13, Marks 10**

Answer

We know that $\dfrac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)} = \int_0^{\pi/2} \sin^p \theta \cos^q \theta\, d\theta$

Let, $q = p$

$$\therefore \quad \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{p+1}{2}\right)}{2\,\Gamma(p+1)} = \int_0^{\pi/2} (\sin \theta \cos \theta)^p\, d\theta$$

$$= \frac{1}{2^p} \int_0^{\pi/2} (\sin 2\theta)^p\, d\theta$$

Let, $2\theta = t$

$$= \frac{1}{2^{p+1}} \int_0^{\pi} \sin^p t\, dt = \frac{1}{2^p} \int_0^{\pi/2} \sin^p t\, dt = \frac{1}{2^p} \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\,\Gamma\left(\frac{p+2}{2}\right)}$$

$$\therefore \quad \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{p+1}{2}\right)}{2\,\Gamma(p+1)} = \frac{1}{2^p} \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(\frac{p+2}{2}\right)}$$

$$\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma(p+1)} = \frac{1}{2^p} \frac{\sqrt{\pi}}{\Gamma\left(\frac{p+2}{2}\right)}$$

Let, $\frac{p+1}{2} = n \quad$ or $\quad p = 2n - 1$

$$\frac{\Gamma(n)}{\Gamma(2n)} = \frac{1}{2^{2n-1}} \frac{\sqrt{\pi}}{\Gamma\left(\frac{2n+1}{2}\right)}$$

or $\quad \sqrt{\pi}\,\Gamma(2n) = 2^{2n-1}\,\Gamma(n)\,\Gamma\left(n + \frac{1}{2}\right)$

Que 2.18. **Find the volume and the mass contained in the solid region in the first octant of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, if the density at any point $\rho(x, y, z) = kxyz$.** **AKTU 2014-15, Marks 10**

Answer

Volume of the solid bounded by the ellipsoid $= 8\iiint_D dx\,dy\,dz$

Let, $\dfrac{x^2}{a^2} = u, \dfrac{y^2}{b^2} = v, \dfrac{z^2}{c^2} = w$

$$2x\,dx = a^2\,du$$

$$dx = \frac{a\,du}{2\sqrt{u}}$$

Similarly, $dy = \dfrac{b\,dv}{2\sqrt{v}}$

$$dz = \frac{c\,dw}{2\sqrt{w}}$$

Required volume,

$$V = 8\iiint_{D'} \frac{abc}{8\sqrt{u\,v\,w}}\,du\,dv\,dw$$

Where D' is the region when $u \geq 0, v \geq 0, w \geq 0$ and $u + v + w = 1$

$$= 8\frac{abc}{8}\iiint_{D'} u^{-1/2}\, v^{-1/2}\, w^{-1/2}\, du\,dv\,dw$$

$$= 8\frac{abc}{8}\iiint_{D'} u^{\frac{1}{2}-1}\, v^{\frac{1}{2}-1}\, w^{\frac{1}{2}-1}\, du\,dv\,dw$$

Using Dirichlet's integral,

$$= 8\frac{abc}{8}\frac{\Gamma\frac{1}{2}\Gamma\frac{1}{2}\Gamma\frac{1}{2}}{\Gamma\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+1\right)} = 8\frac{abc}{8}\frac{(\sqrt{\pi})^3}{\frac{3}{2}\frac{1}{2}\sqrt{\pi}}$$

$$= \frac{4}{3}\pi\,abc \text{ cubic unit}$$

Mass = Volume × Density $= \iiint_D kxyz\,dx\,dy\,dz$

Let, $\dfrac{x^2}{a^2} = u, \dfrac{y^2}{b^2} = v$ and $\dfrac{z^2}{c^2} = w$

$$x = a\sqrt{u} \text{ and } dx = \frac{a}{2\sqrt{u}}\,du$$

Similarly, $y = a\sqrt{v}$ and $dy = \dfrac{b}{2\sqrt{v}}\,dv$

$z = c\sqrt{w}$ and $dz = \dfrac{c}{2\sqrt{w}}\,dw$

$$\text{Mass} = \iiint_{D'} \frac{abc}{8} u^{-1/2}\, v^{-1/2}\, w^{-1/2}\, k\, a\sqrt{u}\, b\sqrt{v}\, c\sqrt{w}\, du\,dv\,dw$$

$$= \frac{k\,a^2b^2c^2}{8}\iiint_{D'} u^0\, v^0\, w^0\, du\,dv\,dw$$

Where D' is the domain,

$$u \geq 0, v \geq 0, w \geq 0, \; u + v + w = 1$$

$$= \frac{ka^2b^2c^2}{8}\iiint_{D'} u^{1-1}\, v^{1-1}\, w^{1-1}\, du\,dv\,dw$$

$$= \frac{ka^2b^2c^2}{8}\frac{\Gamma 1\,\Gamma 1\,\Gamma 1}{\Gamma(1+1+1+1)} = \frac{ka^2b^2c^2}{48}$$

Que 2.19. Find the mass of a solid $\left(\dfrac{x}{ab}\right)^p + \left(\dfrac{y}{b}\right)^q + \left(\dfrac{z}{c}\right)^r = 1$, the density at any point being $\rho = kx^{l-1}y^{m-1}z^{n-1}$, where x, y, z are all positive.

AKTU 2015-16, Marks 10

Answer

Let us take

$$\left(\frac{x}{ab}\right)^p = u \text{ or } \frac{x}{ab} = u^{1/p} \text{ or } x = ab\,u^{1/p}$$

$$\left(\frac{y}{b}\right)^q = v \text{ or } \frac{y}{b} = v^{1/q} \text{ or } y = bv^{1/q}$$

$$\left(\frac{z}{c}\right)^r = w \text{ or } \frac{z}{c} = w^{1/r} \text{ or } z = cw^{1/r}$$

Now $dx = \dfrac{ab}{p}u^{\left(\frac{1}{p}-1\right)}du$

$$dy = \frac{b}{q}v^{\left(\frac{1}{q}-1\right)}dv$$

$$dz = \frac{c}{r}w^{\left(\frac{1}{r}-1\right)}dw$$

$\therefore$ Volume $= \iiint dxdydz = \iiint \dfrac{ab}{p}u^{\left(\frac{1}{p}-1\right)}du\,\dfrac{b}{q}v^{\left(\frac{1}{q}-1\right)}\dfrac{c}{r}w^{\left(\frac{1}{r}-1\right)}dw\,dv$

$$= \iiint \frac{ab^2c}{pqr}u^{\left(\frac{1}{p}-1\right)}v^{\left(\frac{1}{q}-1\right)}w^{\left(\frac{1}{r}-1\right)}du\,dv\,dw$$

Mass = Volume × Density

$$= \iiint \frac{ab^2c}{pqr} u^{\left(\frac{1}{p}-1\right)} v^{\left(\frac{1}{q}-1\right)} w^{\left(\frac{1}{r}-1\right)} kx^{(l-1)} y^{(m-1)} z^{(n-1)} \, du \, dw \, dv$$

$$= \iiint \frac{ab^2c}{pqr} u^{\left(\frac{1}{p}-1\right)} v^{\left(\frac{1}{q}-1\right)} w^{\left(\frac{1}{r}-1\right)} k(ab)^{(l-1)}$$

$$u^{\left(\frac{l-1}{p}\right)} b^{(m-1)} v^{\left(\frac{m-1}{q}\right)} c^{(n-1)} w^{\left(\frac{n-1}{r}\right)} \, du \, dv \, dw$$

$$= \iiint \frac{ab^2c}{pqr} u^{\left(\frac{1}{p}-1\right)} v^{\left(\frac{1}{q}-1\right)} w^{\left(\frac{1}{r}-1\right)} k(ab)^{(l-1)}$$

$$b^{(m-1)} c^{(n-1)} u^{(l/p-1/p)} v^{(m/q-1/q)} w^{(n/r-1/r)} \, du \, dv \, dw$$

$$= \iiint \frac{ka^l b^{(l+m-1)} c^n}{pqr} u^{\left(\frac{1}{p}-1\right)} v^{\left(\frac{1}{q}-1\right)} w^{\left(\frac{1}{r}-1\right)} \, du \, dv \, dw$$

$$= \frac{ka^l b^{m+l} c^n}{pqr} \frac{\overline{|l/p} \, \overline{|m/q} \, \overline{|n/r}}{\overline{|l/p+m/q+n/r+1}}$$ (By using Dirichlet's integral unit)

Que 2.20. **Find the volume of the solid bounded by the co-ordinate planes and the surface $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}+\sqrt{\frac{z}{c}} = 1$.**

Answer

Put $\sqrt{\frac{x}{a}} = u, \sqrt{\frac{y}{b}} = v, \sqrt{\frac{z}{c}} = w$ then $u \geq 0, v \geq 0, w \geq 0$ and $u + v + w = 1$

Also, $dx = 2au \, du$, $dy = 2bv \, dv$, $dz = 2cw \, dw$

Required volume $= \iiint_D dx \, dy \, dz$

$= \iiint_D 8\,abc\,uvw \, du \, dv \, dw$, where $u + v + w = 1$

$= 8\,abc \iiint_D u^{2-1} v^{2-1} w^{2-1} \, du \, dv \, dw$

$= 8\,abc \dfrac{\overline{|2}\,\overline{|2}\,\overline{|2}}{\overline{|(2+2+2+1)}} = 8\,abc \cdot \dfrac{1.1.1}{\overline{|7}} = \dfrac{abc}{90}$

Que 2.21. **Find the mass of a plate which is formed by the coordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1$, the density is given by $\rho = kxyz$.**

AKTU 2017-18, Marks 3.5

AKTU 2011-12, Marks 05

Answer

Same as Q. 2.18, Page 2–13F, Unit-2.

Answer : $M = \dfrac{ka^2b^2c^2}{720}$

PART-3

Applications of Definite Integrals to Evaluate Surface Areas and Volume of Revolutions.

CONCEPT OUTLINE

Surface of the Solid of Revolution : The curved surface of the solid generated by the revolution, about the x-axis, of the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$, $x = b$ is

$$\int_{x=a}^{x=b} 2\pi\, y \, ds$$

Where s is the length of the arc of the curve measured from a fixed point on it to any point (x, y).

Three Practical Forms of Surface Formula :

i. **Surface Formula for Cartesian Equation :** The curved surface of the solid generated by the revolution about the x-axis, of the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$, $x = b$ is

$$\int_{x=a}^{x=b} 2\pi\, y \frac{ds}{dx} dx, \text{ where } \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

ii. **Surface Formula for Parametric Equation :** The curved surface of the solid generated by the revolution about the x-axis, of the area bounded by the curve $x = f(t)$, $y = \phi(t)$, the x-axis and the ordinates at the point, where $t = a$, $t = b$ is

$$\int_{x=a}^{x=b} 2\pi\, y \frac{ds}{dx} dt, \text{ where } \frac{ds}{dx} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dx}\right)^2}$$

iii. **Surface Formula for Polar Equation :** The curved surface of the solid generated by the revolution, about the initial line, of the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \alpha$, $\theta = \beta$ is

$$\int_{\theta=\alpha}^{\theta=\beta} 2\pi\, y \frac{ds}{dx} d\theta, \text{ where } \frac{ds}{d\theta} = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$$

and $y = r \sin\theta$.

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Revolution about y-axis : The curved surface of the solid generated by the revolution about the y-axis of the area bounded by the curve $x = f(y)$, the y-axis and the abscissa $y = a$, $y = b$ is

$$\int_{y=a}^{y=b} 2\pi x \, ds$$

Volume between Two Solids : The volume of the solid generated by the revolution about the x-axis, of the arc bounded by the curves $y = f(x)$, $y = \phi(x)$, and the ordinates $x = a$, $x = b$ is

$$\int_{b}^{a} \pi (y_1^2 - y_2^2) \, dx$$

Where y_1 is the 'y' of the upper curve and y_2 that of the lower curve.

Volume Formula for Parametric Equations :

i. The volume of the solid generated by the revolution about the x-axis, of the area bounded by the curve $x = f(t)$, $y = \phi(t)$, the x-axis and the ordinates, where $t = a$, $t = b$ is

$$\int_{a}^{b} \pi y^2 \frac{dx}{dt} \, dt$$

ii. The volume of the solid generated by the revolution about the y-axis, of the area bounded by the curves $x = f(t)$, $y = \phi(t)$, the y-axis and the abscissa at the points, where $t = a$, $t = b$ is

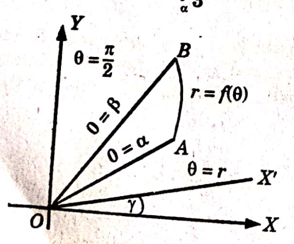
$$\int_{a}^{b} \pi x^2 \frac{dy}{dt} \, dt$$

Volume Formulae for Polar Curves : The volume of the solid generated by the revolution of the area bounded by the curves $r = f(\theta)$, and the radii vectors $\theta = \alpha$, $\theta = \beta$

i. About the initial line OX $(\theta = 0)$ is $\displaystyle\int_{\alpha}^{\beta} \frac{2}{3} \pi r^3 \sin\theta \, d\theta$

ii. About the line OY $\left(\theta = \dfrac{\pi}{2}\right)$ is $\displaystyle\int_{\alpha}^{\beta} \frac{2}{3} \pi r^3 \cos\theta \, d\theta$

iii. About any line OX' $(\theta = \gamma)$ is $\displaystyle\int_{\alpha}^{\beta} \frac{2}{3} \pi r^3 \sin(\theta - \gamma) \, d\theta$



Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 2.22. **Find the area of the surface formed by the revolution of the parabola $y^2 = 4ax$ about the x-axis by the arc from the vertex to one end of the latus rectum.**

Answer

The equation of the parabola is $y^2 = 4ax$ ...(2.22.1)

Differentiating wrt x, we get

$$2y\frac{dy}{dx} = 4a \text{ or } \frac{dy}{dx} = \frac{2a}{y}$$

$$\therefore \quad \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{4a^2}{y^2}} = \sqrt{1 + \frac{4a^2}{4ax}} = \sqrt{\frac{x+a}{x}}$$

For the arc from the vertex O to L, the end of the latus rectum, x varies from 0 to a.

$$\therefore \quad \text{Required surface} = \int_{x=0}^{a} 2\pi y \frac{ds}{dx} \, dx$$

$$= \int_{x=0}^{a} 2\pi \sqrt{4ax} \sqrt{\frac{x+a}{x}} \, dx$$

[$\because$ From eq. (2.22.1) $y = \sqrt{4ax}$]

$$= 4\pi\sqrt{a} \int_{x=0}^{a} (x+a)^{1/2} \, dx$$

$$= 4\pi\sqrt{a}\,\frac{2}{3}\left[(x+a)^{3/2}\right]_0^a = \frac{8\pi a^2}{3}(2\sqrt{2} - 1)$$

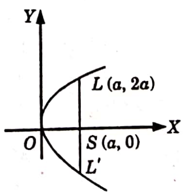


Fig. 2.22.1.

Que 2.23. The curve $r = a(1 + \cos\theta)$ revolves about the initial line. **Find the surface of the figure so formed.**

Answer

The equation of the cardioid is $r = a(1 + \cos\theta)$ ...(2.23.1)

The cardioid is symmetrical about the initial line and for the upper half of the curve, θ varies from 0 to π.

Now from eq. (2.23.1), $\frac{dr}{d\theta} = -a \sin\theta$

$$\therefore \quad \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

$$= \sqrt{a^2(1+\cos\theta)^2 + a^2 \sin^2\theta}$$

$$= a\sqrt{2(1+\cos\theta)} = 2a\cos\frac{\theta}{2}$$

$\therefore$ Required surface $= \int 2\pi y \frac{ds}{d\theta} d\theta$, where $y = r\sin\theta$

$$= 2\pi \int_0^{\pi} a\sin\theta(1+\cos\theta)\, 2a\cos\frac{\theta}{2}\, d\theta \quad (\because r = a(1+\cos\theta))$$

$$= 2\pi \int_0^{\pi} a2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\, 2\cos^2\frac{\theta}{2}\, 2a\cos\frac{\theta}{2}\, d\theta$$

$$= 16\pi a^2 \int_0^{\pi} \cos^4\frac{\theta}{2}\sin\frac{\theta}{2}\, d\theta$$

$$= 16\pi a^2 \left[\frac{-\cos^5\theta/2}{5\times\frac{1}{2}}\right]_0^{\pi}$$

$$= -\frac{32}{5}\pi a^2(0-1) = \frac{32}{5}\pi a^2$$

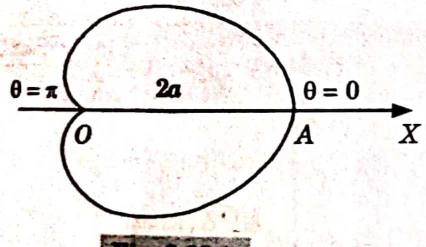


Fig. 2.23.1.

Que 2.24. The arc of the cardioid $r = a(1 + \cos\theta)$ included between $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ is rotated about the line $\theta = \frac{\pi}{2}$. Find the area of surface generated.

Answer

The cardioid is $r = a(1+\cos\theta)$ ...(2.24.1)

The arc CAB $\left(\text{from } \theta = -\frac{\pi}{2} \text{ to } \theta = \frac{\pi}{2}\right)$ revolves about the line $\theta = \frac{\pi}{2}$, *i.e.*, the y-axis.

Also the curve is symmetrical about the initial line or x-axis.

From eq. (2.24.1), $\frac{dr}{d\theta} = -a\sin\theta$

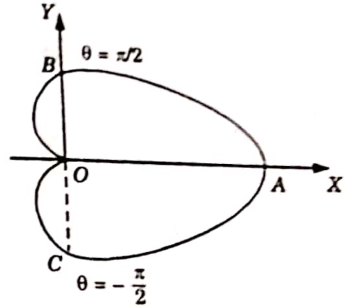


Fig. 2.24.1.

$$\therefore \quad \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}$$

$$= a\sqrt{2(1+\cos\theta)} = 2a\cos\frac{\theta}{2}$$

$\therefore$ Required surface area

$= 2 \times$ Surface generated by the revolution of arc AB

$$= 2\int_0^{\pi/2} 2\pi x \frac{ds}{d\theta} d\theta \quad [\because \text{ For the arc } AB, \theta \text{ varies from 0 to } \pi/2]$$

$$= 4\pi \int_0^{\pi/2} r\cos\theta\, 2a\cos\frac{\theta}{2}\, d\theta \quad (\because x = r\cos\theta)$$

$$= 8\pi a \int_0^{\pi/2} a(1+\cos\theta)\cos\theta\cos\frac{\theta}{2}\, d\theta$$

$$= 8\pi a^2 \int_0^{\pi/2} \left(2 - 2\sin^2\frac{\theta}{2}\right)\left(1 - 2\sin^2\frac{\theta}{2}\right)\cos\frac{\theta}{2}\, d\theta$$

Put $\sin\frac{\theta}{2} = T \quad \therefore \quad \frac{1}{2}\cos\frac{\theta}{2}\, d\theta = dt$

Now the limits are given as follows,

When $\theta = 0$, $t = 0$ and when $\theta = \pi/2$, $t = 1/\sqrt{2}$.

Now, surface area $= 16\pi a^2 \int_0^{1/\sqrt{2}} (1 - 3t^2 + 2t^4)\, 2dt$

$$= 32\pi a^2 \left[t - t^3 + \frac{2t^5}{5} \right]_0^{1/\sqrt{2}} = \frac{96}{5\sqrt{2}} \pi a^2$$

Que 2.25. **Find the volume of the solid generated by the revolution of $r = 2a \cos \theta$ about the initial line.**

Answer

The equation of the curve is

$$r = 2a \cos \theta \quad ...(2.25.1)$$

Eq. (2.25.1) is clearly a circle passing through the pole. The curve is symmetrical about the initial line and for the upper half of the circle θ varies from 0 to $\frac{\pi}{2}$.

$\therefore$ Required volume

$$= \int_0^{\pi/2} \frac{2}{3} \pi r^3 \sin \theta \, d\theta = \frac{2}{3} \pi \int_0^{\pi/2} (2a \cos \theta)^3 \sin \theta \, d\theta$$

$$= \frac{16}{3} \pi a^3 \int_0^{\pi/2} \cos^3 \theta \sin \theta \, d\theta$$

$$= -\frac{16}{3} \pi a^3 \left[\frac{\cos^4 \theta}{4} \right]_0^{\pi/2} = -\frac{4}{3} \pi a^3 (0 - 1) = \frac{4}{3} \pi a^3$$

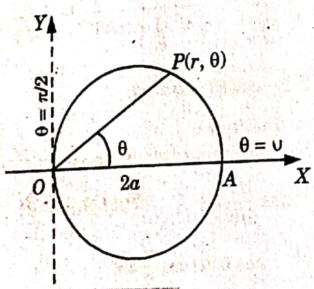


Fig. 2.25.1.

Que 2.26. **Show that the volume of the solid formed by the revolution of the curve $r = a + b \cos \theta \ (a > b)$ about the initial line is $\frac{4}{3} \pi a(a^2 + b^2)$.**

Answer

The equation of the curve is

$$r = a + b \cos \theta \ (a > b) \quad ...(2.26.1)$$

The curve is symmetrical about the initial line and for the upper half of the curve θ varies from 0 to π.

$\therefore$ Required volume

$$= \int_0^{\pi} \frac{2}{3} \pi r^3 \sin \theta \, d\theta$$

$$= \frac{2}{3} \pi \int_0^{\pi} (a + b \cos \theta)^3 \sin \theta \, d\theta$$

$$= -\frac{2}{3} \frac{\pi}{b} \int_0^{\pi} (a + b \cos \theta)^3 (-b \sin \theta \, d\theta)$$

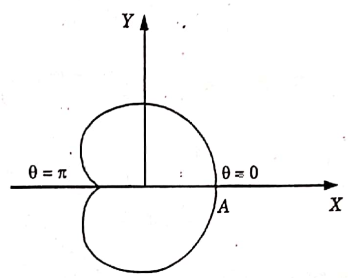


Fig. 2.26.1.

$$= -\frac{2}{3} \frac{\pi}{b} \left[\frac{(a + b \cos \theta)^4}{4} \right]_0^{\pi}$$

$$= -\frac{2\pi}{3b} \left[\frac{(a-b)^4}{4} - \frac{(a+b)^4}{4} \right]$$

$$= \frac{\pi}{6b} [(a+b)^4 - (a-b)^4] = \frac{4}{3} \pi a (a^2 + b^2)$$

Que 2.27. **Find the volume of the solid formed by the revolution of the cissoid $y^2 (2a - x) = x^3$ about its asymptote.**

Answer

The equation of the curve is $y^2(2a-x) = x^3$ or $y^2 = \dfrac{x^3}{2a-x}$ ...(2.27.1)

The curve is symmetrical about the x-axis and the asymptote is the line

$$2a - x = 0 \text{ or } x = 2a.$$

If $P(x, y)$ be any point on the curve and $PM \perp$ on the asymptote (the axis of revolution), and $PN \perp OX$.

Then $PM = NA = OA - ON = 2a - x$ and $AM = NP = y$,

where A is the point of intersection of the asymptote and the x-axis.

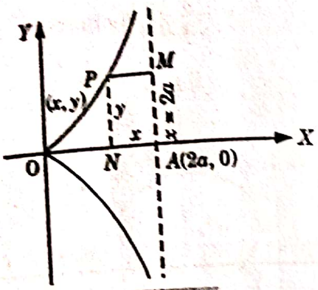


Fig. 2.27.1.

$\therefore$ Required volume $= 2\int \pi(PM)^2\, d(AM)$ ...(2.27.2)

Now, $\quad AM = y = \dfrac{x^{3/2}}{\sqrt{2a-x}}$ [From eq. (2.27.1)]

$\therefore \quad d(AM) = dy$

$$= \frac{(2a-x)^{1/2}\,\frac{3}{2}x^{1/2} - x^{3/2}\,\frac{1}{2}(2a-x)^{-1/2}(-1)}{2a-x}\,dx$$

$$= \frac{3x^{1/2}(2a-x) + x^{3/2}}{2(2a-x)^{3/2}}\,dx = \frac{\sqrt{x}\,(3a-x)}{(2a-x)^{3/2}}\,dx$$

From eq. (2.27.2), we get

$\therefore$ Required volume

$$= 2\pi\int_0^{2a}(2a-x)^2\,\frac{\sqrt{x}(3a-x)}{(2a-x)^{3/2}}\,dx$$

$$= 2\pi\int_0^{2a}(3a-x)^2\,\sqrt{x}\,\sqrt{2a-x\,dx}$$

Put $x = 2a\sin^2\theta \quad \therefore \quad dx = 4a\sin\theta\cos\theta\, d\theta$

Now the limits of the integral are given as follows,

When $x = 0$, $\theta = 0$, and when $x = 2a$, $\theta = \dfrac{\pi}{2}$

Now, required volume

$$= 2\pi\int_0^{\pi/2}(3a - 2a\sin^2\theta)\,\sqrt{2a\sin^2\theta}\,\sqrt{2a(1-\sin^2\theta)}\;4a\sin\theta\cos\theta\, d\theta$$

$$= 16\pi a^3\int_0^{\pi/2}(3 - 2\sin^2\theta)\sin^2\theta\cos^2\theta\, d\theta$$

$$= 16\pi a^3\int_0^{\pi/2}(3\sin^2\theta\cos^2\theta - 2\sin^4\theta\cos^2\theta)\, d\theta$$

$$= 16\pi a^3\left[3\cdot\frac{1.1}{4.2}\cdot\frac{\pi}{2} - 2\cdot\frac{3.1.1}{6.4.2}\cdot\frac{\pi}{2}\right] = 16\pi a^3\left[\frac{3\pi}{16} - \frac{\pi}{16}\right] = 2\pi^2 a^3$$

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