5

UNIT

Complex Variable Integration

CONTENTS

PART-1

Complex Integrals, Contour Integrals, Cauchy-Goursat Theorem, Cauchy Integral Formula.

CONCEPT OUTLINE

Contour Integral : If the initial point and final point coincide so that C is a closed curve then this integral is called contour integral and is denoted by $\oint_c f(z)\,dz$.

If $\quad f(z) = u(x,y) + iv(x,y)$

since $\quad dz = dx + idy$

$$\int_c f(z)dz = \int_c (u+iv)(dx+idy) = \int_c (u\,dx - v\,dy) + i\int_c (v\,dx + u\,dy)$$

which shows that the evaluation of line integral of a complex function can be reduced to the evaluation of two line integrals of real functions.

Cauchy's Integral Theorem : If $f(z)$ is an analytic function and $f'(z)$ is continuous at each point within a simple closed curve C, then

$$\oint_C f(z)dz = 0$$

For multiple connected regions,

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$$

when integral along each curve is taken in anticlockwise direction.

Cauchy's-Goursat Theorem : Cauchy's theorem without the assumption that $f'(z)$ is continuous is known as Cauchy's-Goursat theorem.

Cauchy's Integral Formula : If $f(z)$ is analytic within and on a closed curve C and if a is any point within C, then

$$f(a) = \frac{1}{2\pi i}\oint_z \frac{f(z)}{(z-a)}dz$$

Cauchy's Integral Formula for Derivative of an Analytic Function : If a function $f(z)$ is analytic in a domain D, then at any point $z = a$ of D, $f(z)$ has derivatives of all orders, all of which are again analytic functions in D and are given by

$$f'(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$$

Where C is any closed curve in D surrounding the point $z = a$.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 5.1. **State Cauchy's Integral theorem and derive it.**

Answer

A. Statement : If $f(z)$ is an analytic function and $f'(z)$ is continuous at each point within and on a simple closed curve C, then

$$\oint_C f(z)dz = 0$$

B. Proof : Let R be the region bounded by the curve C.

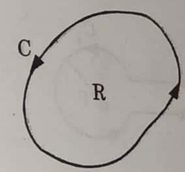

Fig. 5.1.1.

Let, $\quad f(z) = u(x, y) + iv(x, y)$, then

$$\oint_C f(z)dz = \oint_C (u+iv)(dx+idy)$$

$$= \oint_C (udx - vdy) + i\oint_C (vdx + udy) \qquad \text{...(5.1.1)}$$

Since $f'(z)$ is continuous, the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous in R. Hence by Green's theorem, we have

$$\oint_C f(z)dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dxdy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dxdy \qquad \text{...(5.1.2)}$$

Now $f(z)$ being analytic at each point of the region R, by Cauchy-Riemann equations, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Thus, the two double integrals in eq. (5.1.2) vanish.

Hence $\quad \oint_C f(z)dz = 0$

Que 5.2. **State and prove Cauchy's integral formula.**

AKTU 2017-18 (III), Marks 10

Answer

A. **Statement :** If $f(z)$ is analytic within and on a closed curve C and a is any point within C, then

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$$

B. **Proof :** Consider the function $\frac{f(z)}{z-a}$, which is analytic at every point within C except at $z = a$. Draw a circle C_1 with a as centre and radius ρ such that C_1 lies entirely inside C. Thus $\frac{f(z)}{z-a}$ is analytic in the region between C and C_1.

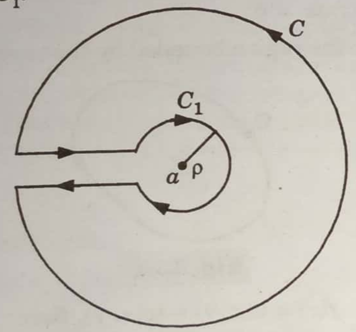

Fig. 5.2.1.

$\therefore$ By Cauchy's theorem, we have

$$\oint_C \frac{f(z)}{z-a}\,dz = \oint_{C_1} \frac{f(z)}{z-a}\,dz \qquad ...(5.2.1)$$

Now, the equation of circle C_1 is $|z-a| = \rho$ or $z - a = \rho e^{i\theta}$

So that $\qquad dz = i\rho e^{i\theta}\,d\theta$

$$\therefore \oint_{C_1} \frac{f(z)}{z-a}\,dz = \int_0^{2\pi} \frac{f(a+\rho e^{i\theta})}{\rho e^{i\theta}}\, i\rho e^{i\theta}\,d\theta = i\int_0^{2\pi} f(a+\rho e^{i\theta})d\theta$$

Hence by eq. (5.2.1), we have

$$\oint_C \frac{f(z)}{z-a}\,dz = i\int_0^{2\pi} f(a+\rho e^{i\theta})d\theta \qquad ...(5.2.2)$$

In the limiting form, as the circle C_1 shrinks to the point a, *i.e.*, $\rho \to 0$, then from eq. (5.2.2),

$$\oint_C \frac{f(z)}{z-a}\,dz = i\int_0^{2\pi} f(a)d\theta = if(a)\int_0^{2\pi} d\theta = 2\pi i f(a)$$

Hence $\qquad f(a) = \dfrac{1}{2\pi i}\oint_C \dfrac{f(z)}{z-a}\,dz$

Que 5.3. **State Cauchy integral theorem for an analytic function. Verify this theorem by integrating the function $z^3 + iz$ along the boundary of the rectangle with vertices $1, -1, i, -i$.**

AKTU 2014-15 (III), Marks 05

Answer

A. **Cauchy's Integral Theorem :** Refer Q. 5.1, Page 5-3F, Unit-5.

B. **Numerical :**

$$\int_C f(z)\,dz = \int_{AB} f(z)\,dz + \int_{BC} f(z)\,dz + \int_{CD} f(z)\,dz + \int_{DA} f(z)\,dz$$

$$\int_{AB} f(z)\,dz = \int (x+iy)^3 + i(x+iy)\{dx + i\,dy\} = 0 \qquad ...(5.3.1)$$

$$\int_{BC} f(z)\,dz = \int ((x+(x-1))^3 + i(2x-1)\{2\,dx\}$$

$$= 2\int_1^0 \{(2x-1)^3 + i(2x-1)\}\,dx = -i \qquad ...(5.3.2)$$

$$\int_{CD} f(z)\,dz = \int\left[\{x + i(+ix+i)\}^3 + i(-ix+i)\right](0) = 0 \qquad ...(5.3.3)$$

$$\int_{DA} f(z)\,dz = 2\int\left[\{x - i(ix+i)\}^3 + i(2x+1)\right]dx$$

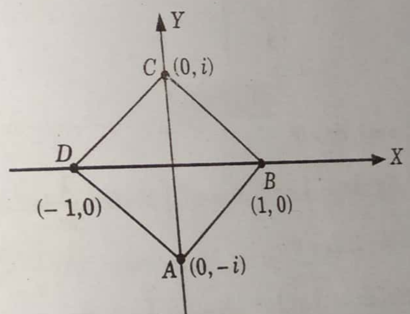

Fig. 5.3.1.

$$= 2\int [(2x+1)^3 + i(2x+1)]\,dx$$

$$= 2\left[\frac{(2x+1)^4}{8} + i\frac{(2x+1)}{2}\right]_{-1}^{0} = i \qquad ...(5.3.4)$$

From eq. (5.3.1), eq. (5.3.2), eq. (5.3.3) and eq. (5.3.4), we have

$$\int_C f(z)\,dz = -i + 0 + 0 + i = 0 \text{ (Hence proved)}$$

Que 5.4. **Verify Cauchy's theorem by integrating e^{iz} along the boundary of the triangle with the vertices at the points $1+i, -1+i$ and $-1-i$.**

AKTU 2017-18 (II), Marks 10

Answer

$$\oint_C f(z)\,dz = \int_{AB} f(z)\,dz + \int_{BC} f(z)\,dz + \int_{CA} f(z)\,dz \quad ...(5.4.1)$$

Along AB, $y = x,\ dy = dx$

$f(z) = e^{iz} = e^{i(x+iy)}$

$f(x) = e^{i(1+i)x}$

$$\int_{AB} f(z)\,dz = \int_{-1}^{1} e^{i(1+i)x}\,(dx + i dx)$$

$$= (1+i)\left[\frac{e^{i(1+i)x}}{i(1+i)}\right]_{-1}^{1} = \frac{(i+1)}{(i-1)}\,[e^{i-1} - e^{-i+1}]$$

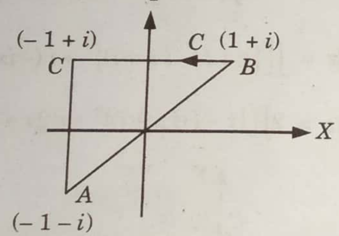

Fig. 5.4.1.

Along BC, $y = 1$, $dy = 0$

$$\int_{BC} f(z)\,dz = \int_{1}^{-1} e^{i(x+i)}\,dx = e^{-1}\int_{1}^{-1} e^{ix}\,dx = \frac{1}{ie}(e^{-i} - e^{i})$$

Along CA, $x = -1$, $dx = 0$

$$\int_{CA} f(z)\,dz = \int_{1}^{-1} e^{i(-1+iy)}\,i dy = ie^{-i}\int_{+1}^{-1} e^{-y}\,dy$$

$$= -ie^{-i}(e^{+1} - e^{-1}) = -ie^{-i}(e - e^{-1})$$

From eq. (5.4.1)

$$\oint_C f(z)\,dz = \frac{(i+1)^2}{-2}\left[\frac{e^{i}}{e} - ee^{-i}\right] + \frac{e^{-i}}{ie} - \frac{e^{i}}{ie} - ie^{-i}e + \frac{ie^{-i}}{e}$$

$$= -\frac{ie^{i}}{e} + iee^{-i} + \frac{e^{-i}}{ie} - \frac{e^{i}}{ie} - ie^{-i}e + \frac{ie^{-i}}{e}$$

$$= -ie^{i-1} + ie^{-i+1} - ie^{-i-1} + ie^{i-1} - ie^{-i+1} + ie^{-i-1}$$

$\therefore$ $\oint_C f(z)\,dz = 0$ (Hence proved)

Que 5.5. **State Cauchy's integral formula. Hence, Evaluate $\oint_C \frac{dz}{z^2(z^2-4)e^z}$, where C is $|z| = 1$**

AKTU 2012-13 (IV), Marks 05

Answer

A. **Cauchy's Integral Formula :** Refer Q. 5.2, Page 5–3F, Unit-5.

B. **Numerical :**

Let,
$$I = \oint_C \frac{dz}{z^2(z^2-4)e^z},\ C = |z| = 1$$

$$I = \oint_C \frac{e^{-z}dz}{z^2(z^2-4)}$$

Poles are $z = 0$ (of order 2), $z = \pm 2$

$z = 0$ is the only pole which lie inside C.

$\therefore$
$$I = \oint_C \frac{e^{-z}/(z^2-4)}{z^2}\,dz = 2\pi i\left[\frac{d}{dz}\left(\frac{e^{-z}}{z^2-4}\right)\right]_{z=0}$$

$$I = 2\pi i\left[\frac{-(z^2-4)e^{-z} - 2ze^{-z}}{(z^2-4)^2}\right]_{z=0}$$

$$I = -2\pi i\left[\frac{-4+0}{16}\right]$$

$$I = \frac{\pi i}{2}$$

Thus
$$\oint_C \frac{1}{z^2(z^2-4)e^z}\,dz = \frac{\pi i}{2}$$

Que 5.6. **State Cauchy's integral formula. Hence evaluate :**

$$\int_C \frac{2z+1}{z^2+z}\,dz, \text{ where C is } |z| = \frac{1}{2}.$$

AKTU 2014-15 (IV), Marks 05

Answer

A. **Cauchy Integral Formula :** Refer Q. 5.2, Page 5–3F, Unit-5.

B. **Numerical :**

Poles are given by $z^2 + z = 0$, $z = 0, -1$

$|z| = \frac{1}{2}$ is a circle with centre at origin and radius $\frac{1}{2}$. Pole $z = 0$ enclosed in $|z| = \frac{1}{2}$.

$$\int_C \frac{2z+1}{z(z+1)} dz = \int_C \frac{\frac{2z+1}{z+1}}{z} = 2\pi i \left[\frac{2z+1}{z+1}\right]_{z=0}$$

$$\int_C \frac{2z+1}{z(z+1)} dz = 2\pi i$$

Que 5.7. **Use Cauchy's integral formula to show that**

$$\int_C \frac{e^{zt}}{z^2+1} dz = 2\pi i \sin t \text{ if } t > 0 \text{ and } C \text{ is the circle } |z| = 3.$$

AKTU 2013-14 (IV), Marks 05

Answer

Poles of the integrand are given by

$$z^2 + 1 = 0, z = \pm i \text{(order 1)}$$

The circle $|z| = 3$ has centre at $z = 0$ and radius 3. It encloses both the singularities $z = i$ and $z = -i$.

$$\text{Now}, \int_C \frac{e^{zt}}{z^2+1} dz = \int_C \frac{e^{zt}}{(z+i)(z-i)} dz$$

$$= \int_{C_1} \frac{\left(\frac{e^{zt}}{z-i}\right)}{z+i} dz + \int_{C_2} \frac{\left(\frac{e^{zt}}{z+i}\right)}{z-i} dz = 2\pi i \left(\frac{e^{zt}}{z+i}\right)\Bigg|_{z=i} + 2\pi i \left(\frac{e^{zt}}{z-i}\right)\Bigg|_{z=-i}$$

$$= \pi(e^{it} - e^{-it}) = 2\pi i \sin t$$

Que 5.8. **Evaluate by Cauchy's integral formula**

$$\oint_C \frac{z^2-2z}{(z+1)^2(z^2+4)} dz, \text{ where } C \text{ is the circle } |z| = 3.$$

AKTU 2015-16 (III), Marks 05

Answer

Here, we have $\int_C \frac{z^2-2z}{(z+1)^2(z^2+4)} dz$

The poles are determined by putting the denominator equal to zero.

$$(z+1)^2(z^2+4) = 0$$

$$z = -1, -1 \text{ and } z = \pm 2i$$

The circle $|z| = 3$ with centre at origin and radius = 3 encloses a pole at $z = -1$ of second order and simple poles $z = \pm 2i$.

Let the given integral $= I_1 + I_2 + I_3$ ...(5.8.1)

$$I_1 = \int_{C_1} \frac{z^2-2z}{(z+1)^2(z^2+4)} dz = \int_{C_1} \frac{\frac{z^2-2z}{z^2+4}}{(z+1)^2} dz$$

$$= 2\pi i \left[\frac{d}{dz} \frac{z^2-2z}{z^2+4}\right]_{z=-1}$$

$$= 2\pi i \left[\frac{(z^2+4)(2z-2) - (z^2-2z)2z}{(z^2+4)^2}\right]_{z=-1}$$

$$= 2\pi i \left[\frac{(1+4)(-2-2) - (1+2)2(-1)}{(1+4)^2}\right]$$

$$= 2\pi i \left(-\frac{14}{25}\right) = \frac{-28\pi i}{25}$$

$$I_2 = \int_{C_2} \frac{\frac{z^2-2z}{(z+1)^2(z+2i)}}{(z-2i)} dz = 2\pi i \left[\frac{z^2-2z}{(z+1)^2(z+2i)}\right]_{z=2i}$$

$$= 2\pi i \left[\frac{-4-4i}{(2i+1)^2(2i+2i)}\right] = 2\pi i \frac{(1+i)}{4+3i}$$

$$I_3 = \int_{C_3} \frac{\frac{z^2-2z}{(z+1)^2(z-2i)}}{(z+2i)} dz = 2\pi i \left[\frac{z^2-2z}{(z+1)^2(z-2i)}\right]_{z=-2i}$$

$$= 2\pi i \left[\frac{-4+4i}{(-2i+1)^2(-2i-2i)}\right] = 2\pi i \frac{(i-1)}{(3i-4)}$$

Now putting the value of I_1, I_2 and I_3 in eq. (5.8.1), we get

$$\int_C \frac{z^2-2z}{(z+1)^2(z^2+4)} dz = \frac{-28\pi i}{25} + 2\pi i \left(\frac{1+i}{4+3i}\right) + 2\pi i \left(\frac{i-1}{3i-4}\right)$$

$$= 2\pi i \left[\frac{-14}{25} + \frac{1+i}{(4+3i)} + \frac{(i-1)}{(3i-4)}\right]$$

$$= 2\pi i \left[\frac{-14}{25} + \frac{(1+i)(3i-4) + (i-1)(4+3i)}{(-9-16)}\right]$$

$$= \frac{2\pi i}{-25}[14 + (3i - 4 - 3 - 4i) + (4i - 3 - 4 - 3i)]$$

$$= 0$$

Que 5.9. **Evaluate the integral $\int \frac{e^{2z}}{(z+1)^5}\, dz$, around the boundary of the circle $|z| = 2$.**

AKTU 2014-15 (III), Marks 05

Answer

Poles are $z = -1$ of order 5 will lie in $|z| = 2$

Using cauchy integral formula, we get

$$\int \frac{e^{2z}}{(z+1)^5}\, dz = \frac{2\pi i}{4!}\left[\frac{d^4}{dz^4}(e^{2z})\right]_{z=-1}$$

$$= \frac{2\pi i}{4!}(16e^{2z})_{z=-1} = \frac{32\pi i}{24} \times e^{-2} = \frac{4\pi i}{3e^2}$$

Que 5.10. **Using Cauchy's integral formula evaluate $\int_C \frac{e^{2z}}{(z+1)^4}\, dz$, where C is the circle $|z| = 3$.**

AKTU 2016-17 (IV), Marks 10

Answer

Same as Q. 5.9, Page 5–10F, Unit-5. $\left(\textbf{Answer :} \frac{8\pi i}{3e^2}\right)$.

Que 5.11. **Evaluate $\int_C \frac{(1+z)\sin z}{(2z-3)^2}\, dz$, where C is the circle $|z - i| = 2$ counter clockwise.**

AKTU 2013-14 (III), Marks 05

Answer

The given integral is $\int \frac{(1+z)\sin z}{(2z-3)^2}\, dz$

Poles of integrand,

$$(2z-3)^2 = 0$$

$$z = \frac{3}{2}, \frac{3}{2}$$

$\therefore$ Pole lie inside the circle of radius 2.

By Cauchy's integral formula,

$$\int_c \frac{(1+z)\sin z}{(2z-3)^2}\, dz = 2\pi i\left[\frac{d}{dz}(1+z)\sin z\right]_{z=3/2}$$

$$= 2\pi i\,[(1+z)\cos z + \sin z]_{z=3/2}$$

$$= 2\pi i\left(\frac{5}{2}\cos\frac{3}{2} + \sin\frac{3}{2}\right)$$

Que 5.12. **Evaluate $\int_0^{3+i} (\bar{z})^2\, dz$, along the real axis from $z = 0$ to $z = 3$ and then along a line parallel to imaginary axis from $z = 3$ to $z = 3 + i$.**

AKTU 2012-13 (IV), Marks 05

Answer

$$\int_0^{3+i} (\bar{z})^2\, dz = \int_0^{3+i} (x - iy)^2\,(dx + idy) = \int_{OA} x^2\, dx + \int_{AB} (3 - iy)^2\, idy$$

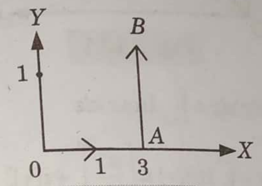

Fig. 5.12.1.

Along OA, $y = 0$, $dy = 0$, x varies 0 to 3

Along AB, $x = 3$, $dx = 0$, and y varies 0 to 1

$$\therefore \int_0^{3+i} (\bar{z})^2\, dz = \int_0^3 x^2\, dx + \int_0^1 (3 - iy)^2\, idy = \left[\frac{x^3}{3}\right]_0^3 + i\int_0^1 (9 - 6iy - y^2)dy$$

$$= \left[\frac{x^3}{3}\right]_0^3 + i\left[9y - 3iy^2 - \frac{y^3}{3}\right]_0^1 = \frac{27}{3} + i\left[\frac{26}{3} - 3i\right] = 12 + \frac{26i}{3}$$

Que 5.13. **Integrate $f(z) = \text{Re}(z)$ from $z = 0$ to $z = 1 + 2i$, (i) along straight line joining $z = 0$ to $z = 1 + 2i$, (ii) along the real axis from $z = 0$ to $z = 1$ and then along a line parallel to imaginary axis from $z = 1$ to $z = 1 + 2i$.**

AKTU 2013-14 (III), Marks 05

Answer

i. $$\int_0^{1+2i} f(z)\, dz = \int_0^{1+2i} \text{Re}(z)\, dz$$

Equation of OB is,

$$y - 0 = \frac{2-0}{1-0}(x - 0)$$

$$y = 2x$$

$$dy = 2dx$$

$$z = x + iy$$

$$dz = dx + idy = dx + i2dx$$

$$\int_0^{1+2i} \text{Re}(z)dz = \int_0^1 x(dx + idy)$$

$$= \int_0^1 x(dx + 2idx) = (1 + 2i) \int_0^1 x dx = (1 + 2i)\left[\frac{x^2}{2}\right]_0^1 = \frac{1+2i}{2}$$

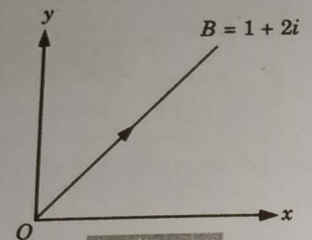

Fig. 5.13.1.

ii. $$\int f(z)dz = \int_{OA} \text{Re}(z)dz + \int_{AB} \text{Im}(z)dz$$

$$= \int_0^1 x\,dx + \int_0^2 1(idy) = \left[\frac{x^2}{2}\right]_0^1 + i[y]_0^2 = \frac{1}{2} + 2i = \frac{1+4i}{2}$$

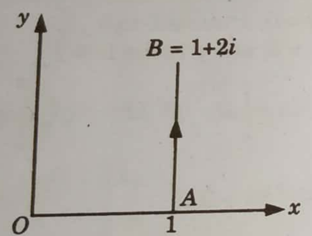

Fig. 5.13.2.

Que 5.14. **Evaluate :** $\int_0^\infty \frac{\sin mx}{x} dx,\ m > 0.$

AKTU 2017-18 (IV), Marks 10

Answer

Consider the integral $\int_C \frac{e^{miz}}{z} dz = \int_C f(z)\,dz$ where C consists of

i. The real axis from r to R.
ii. The upper half of the circle $C_R : |z| = R$,
iii. The real axis $-R$ to $-r$,
iv. The upper half of the circle $C_r : |z| = r$ (Fig. 5.14.1)
Since $f(z)$ has no singularity inside C (its only singular point being a simple pole at $z = 0$ which has been deleted by drawing C_r), we have by Cauchy's theorem :

$$\int_r^R f(x)dx + \int_{C_R} f(z)dz + \int_{-R}^{-r} f(x)dx + \int_{C_r} f(z)dz = 0 \quad ...(5.14.1)$$

Now $\int_{C_R} f(z)\,dz = \int_0^\pi \frac{e^{imR(\cos\theta + i\sin\theta)}}{Re^{i\theta}} Rie^{\theta}\,d\theta$ $[\because\ z = Re^{i\theta}]$

$$= i\int_0^\pi e^{imR(\cos\theta + i\sin\theta)}\,d\theta$$

Since $\left|e^{imR(\cos\theta + i\sin\theta)}\right| = \left|e^{-mR\sin\theta + imR\cos\theta}\right| = e^{-mR}\sin\theta$

$\therefore$ $\left|\int_{C_R} f(z)\,dz\right| \le \int_0^\pi e^{-mR\sin\theta}\,d\theta = 2\int_0^{\pi/2} e^{-mR\sin\theta}\,d\theta$

$= 2\int_0^{\pi/2} e^{-2mR\theta/\pi}\,d\theta$ $[\because$ for $0 \le \theta \le \pi/2$, $\sin\theta/\theta \ge 2/\pi]$

$= \frac{\pi}{mR}(1 - e^{-mR})$ which $\to 0$ as $R \to \infty$,

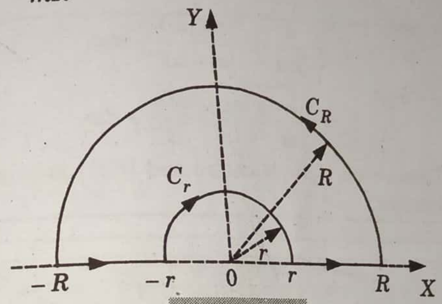

Fig. 5.14.1.

Also $\int_{C_r} f(z)\,dz = i\int_\pi^0 e^{imr(\cos\theta + i\sin\theta)}\,d\theta \to i\int_\pi^0 d\theta$ i.e. $-i\pi$ as $r \to 0$

Hence as $r \to 0$ and $R \to \infty$, we get from eq. (5.14.1).

$$\int_0^\infty f(x)dx + 0 + \int_{-\infty}^0 f(x)dx - i\pi = 0$$

or $\int_{-\infty}^\infty f(x)dx = i\pi$ i.e. $\int_{-\infty}^\infty \frac{e^{imx}}{x}dx = i\pi$ ...(5.14.2)

Equating imaginary parts from both sides,

$$\int_{-\infty}^\infty \frac{\sin mx}{x}\,dx = x$$

Hence $$\int_0^\infty \frac{\sin mx}{x}\,dx = \frac{\pi}{2}$$

PART-2

Taylor's Series, Laurent's Series, Liouville's Theorem.

CONCEPT OUTLINE

Taylor's Series : A function $f(z)$ which is analytic at all points within a circle C with centre at a can be represented uniquely as a convergent power series known as Taylor's series.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

Where,
$$a_n = \frac{f^n(a)}{n!}.$$

Laurent's Series : If $f(z)$ is analytic inside and on the boundary of the annular (ring shaped) region R bounded by two concentric circles C_1 and C_2 of radii r_1 and r_2 $(r_1 > r_2)$ respectively having centre at a, then for all z in R

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

Where,
$$a_n = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^{n+1}}\, dw$$

and
$$b_n = \frac{1}{2\pi i}\oint_{C_2} \frac{f(w)}{(w-a)^{-n+1}}\, dw$$

Liouville's Theorem : If $f(z)$ is entire and $|f(z)|$ is bounded for all z, then $f(z)$ is constant.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 5.15. Expand $\dfrac{1}{z^2 - 3z + 2}$ in the region $1 < |z| < 2$.

AKTU 2014-15 (IV), Marks 05

Answer

$$f(z) = \frac{1}{z^2 - 3z + 2} = \frac{1}{(z-2)} - \frac{1}{(z-1)} = -\frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{2}\left[1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} \ldots\right] - \frac{1}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \ldots\right]$$

After rearranging, we get,

$$f(z) = \ldots\ldots - z^{-3} - z^{-2} - z^{-1} - \frac{1}{2} - \frac{1}{4}z - \frac{1}{8}z^2 - \frac{1}{16}z^3 \ldots\ldots$$

Que 5.16. Obtain the Taylor's series expansion of $f(z) = \dfrac{1}{z^2 - 4z + 3}$ about the point $z = 4$. Find its region of convergence.

AKTU 2013-14 (IV), Marks 05

Answer

If the centre of the circle is at $z = 4$, then the distances of the singularities $z = 1$ and $z = 3$ from centre are 3 and 1.

Hence if a circle is drawn with centre at $z = 4$ and radius 1 then within circle $|z - 4| = 1$, the given function $f(z)$ is analytic hence it can be expanded in Taylor's series within the circle $|z - 4| = 1$ which is therefore the region of convergence.

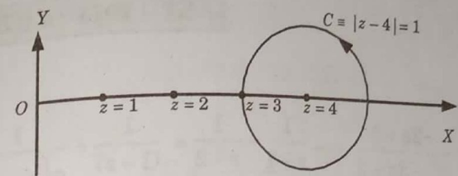

Fig. 5.16.1.

$$f(z) = \frac{1}{(z-1)(z-3)} = \frac{1}{2}\left[\frac{1}{z-3} - \frac{1}{z-1}\right] = \frac{1}{2}\left[\frac{1}{z-4+1} - \frac{1}{z-4+3}\right]$$

$$= \frac{1}{2}\left[\{1 + (z-4)\}^{-1} - \frac{1}{3}\left\{1 + \left(\frac{z-4}{3}\right)\right\}^{-1}\right]$$

$$f(z) = \frac{1}{2}\left[\sum_{n=0}^{\infty} (-1)^n (z-4)^n - \frac{1}{3}\sum_{n=0}^{\infty} (-1)^n \left(\frac{z-4}{3}\right)^n\right]$$

Que 5.17. Find the Taylor series expansion of the function $\tan^{-1} z$ about the point $z = \pi/4$.

AKTU 2014-15 (III), Marks 05

Answer

$$f(z) = \tan^{-1} z$$

$$f'(z) = \frac{1}{1 + z^2}$$

$$f''(z) = \frac{-2z}{(1 + z^2)^2}$$

$$f'''(z) = -2\left[\frac{(1+z^2)^2 - 4z^2(1+z^2)}{(1+z^2)^4}\right] = -2\left[\frac{1 + z^2 - 4z^2}{(1+z^2)^3}\right] = \frac{2(3z^2 - 1)}{(1+z^2)^3}$$

$$f'\left(\frac{\pi}{4}\right) = \tan^{-1}\left(\frac{\pi}{4}\right) = 0.6658,\ f'\left(\frac{\pi}{4}\right) = 0.6185$$

$$f''\left(\frac{\pi}{4}\right) = \frac{-2\,(0.785)}{2.6142} = -0.60087$$

Thus,

$$\tan^{-1} z = 0.6658 + \left(z - \frac{\pi}{4}\right)(0.6185) + \frac{\left(z-\frac{\pi}{4}\right)^2}{2!}(-0.60087) + ...$$

Que 5.18. **Find all Taylor and Laurent series expansion of the following function about $z = 0$**

$$f(z) = \frac{-2z+3}{z^2-3z+2}$$

AKTU 2013-14 (III), Marks 05

Answer

$$f(z) = \frac{-2z+3}{z^2-3z+2} = -\frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{(1-z)} + \frac{1}{2\left(1-\frac{z}{2}\right)} \quad ...(5.18.1)$$

$$= (1-z)^{-1} + \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

Now expanding by binomial expansion

$$f(z) = (1 + z + z^2 + z^3 + ...) + \frac{1}{2}\left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + ...\right]$$

or
$$f(z) = \sum_{n=0}^{n} (1)^n z^n + \frac{1}{2}\sum_{n=0}^{n}(1)^n\left(\frac{z}{2}\right)^n$$

This is the Taylor's series expansion of given function.

Eq. (5.18.1) can also be written as,

$$f(z) = -\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

Now expanding by binomial expansion we get

$$f(z) = -\frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+....\right] - \frac{1}{z}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\left(\frac{2}{z}\right)^3+....\right]$$

$$f(z) = -\frac{1}{z}\sum_{n=0}^{n}(1)^n\frac{1}{z^n} - \frac{1}{z}\sum_{n=0}^{n}(1)^n\left(\frac{2}{z}\right)^n$$

This is the Laurent's series expansion of given function.

Que 5.19. **Find the Laurent series for the function**

$$f(z) = \frac{7z^2+9z-18}{z^3-9z}$$, **z is complex variable valid for the regions**

i. $0 < |z| < 3$ ii. $|z| > 3$

AKTU 2015-16 (IV), Marks 10

AKTU 2012-13 (IV), Marks 05

Answer

$$f(z) = \frac{7z^2+9z-18}{z^3-9z}$$

Using partial fraction,

$$\frac{7z^2+9z-18}{z^3-9z} = \frac{A}{z} + \frac{B}{z-3} + \frac{C}{z+3}$$

$$A = \left.\frac{7z^2+9z-18}{(z-3)(z+3)}\right|_{z=0} = \frac{-18}{-3\times 3} = 2$$

$$B = \left.\frac{7z^2+9z-18}{z(z+3)}\right|_{z=3} = 4$$

$$C = \left.\frac{7z^2+9z-18}{z(z-3)}\right|_{z=-3} = 1$$

i. $0 < |z| < 3$

Rearrangement of function $f(z)$,

$$f(z) = \frac{2}{z} - \frac{4}{3\left(1-\frac{z}{3}\right)} + \frac{1}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$f(z) = \frac{2}{z} - \frac{4}{3}\left(1-\frac{z}{3}\right)^{-1} + \frac{1}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$\therefore \quad f(z) = \frac{2}{z} - \frac{4}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n + \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

ii. $|z| > 3$

$$f(z) = \frac{2}{z} + \frac{4}{z\left(1-\frac{3}{z}\right)} + \frac{1}{z\left(1+\frac{3}{z}\right)} = \frac{2}{z} + \frac{4}{z}\left(1-\frac{3}{z}\right)^{-1} + \frac{1}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$\therefore \quad f(z) = \frac{2}{z} + \frac{4}{z}\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^n + \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{z}\right)^n$$

Que 5.20. **Expand $f(z) = \frac{z}{(z-1)(2-z)}$ in Laurent series valid for**

i. $|z-1| > 1$ and ii. $0 < |z-2| < 1$.

AKTU 2012-13 (III), Marks 05

Answer

$$f(z) = \frac{z}{(z-1)(2-z)}$$

$$f(z) = \frac{1}{z-1} - \frac{2}{z-2}$$

i. $|z-1|>1$

$$f(z) = \frac{1}{z-1} - \frac{2}{(z-1)-1} = \frac{1}{z-1} - \frac{2}{(z-1)}\left[1-\frac{1}{z-1}\right]^{-1}$$

$$= \frac{1}{z-1} - \frac{2}{(z-1)}\left[1+\frac{1}{z-1}+\frac{1}{(z-1)^2}+\ldots\right]$$

$$f(z) = \frac{1}{z-1} - 2\sum_{n=0}^{\infty}\frac{1}{(z-1)^{n+1}}$$

ii. $0<|z-2|<1$

$$f(z) = \frac{1}{(z-2)+1} - \frac{2}{z-2} = [1+(z-2)]^{-1} - \frac{2}{z-2}$$

$$f(z) = [1-(z-2)+(z-2)^2-(z-2)^3+\ldots] - \frac{2}{z-2}$$

$$f(z) = -\left(\frac{2}{z-2}\right) + \sum_{n=0}^{\infty}(-1)^n (z-2)^n$$

Que 5.21. **Find the Laurent series expansion of**

$f(z) = \dfrac{7z-2}{z(z+1)(z+2)}$ **in the region $1<|z+1|<3$.**

AKTU 2016-17 (III), Marks 05

Answer

Same as Q. 5.20, Page 5–17F, Unit-5.

$$\left(\textbf{Answer : } f(z) = -\sum_{n=0}^{\infty}\frac{1}{(z+1)^{n+1}} + \frac{9}{z+1} - 8\sum_{n=0}^{\infty}\frac{(-1)^n}{(z+1)^{n+1}}\right)$$

PART-3

Singularities, Classification of Singularities, Zeros of Analytic Function.

CONCEPT OUTLINE

Singularity : A singularity of a function $f(z)$ is a point at which the function ceases to be analytic.

Types of Singularities :

i. **Isolated Singularity :** If $z = a$ is a singularity of $f(z)$ such that $f(z)$ is analytic at each point in its neighbourhood (*i.e.*, there exists a circle with centre a which has no other singularity), then $z = a$ is called an isolated singularity.

In such a case, $f(z)$ can be expanded in a Laurent's series around $z = a$, giving

$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \ldots\ldots(1)$

For example, $f(z) = \cot(\pi/z)$ is not analytic where as $\tan(\pi/z) = 0$ *i.e.*, at the points $\pi/z = 4\pi$ or $z = 1/n$ $(n = 1, 2, 3,\ldots)$.

Thus $z = 1, 1/2, 1/3,\ldots$ are all isolated singularities as there is no other singularity in their neighbourhood.

But when n is large, $z = 0$ is such a singularity that there are infinite number of other singularities in its neighbourhood.

Thus $z = 0$ is the non-isolated singularity of $f(z)$.

ii. **Removable singularity :** If all the negative powers of $(z-a)$ in eq. (1) are zero, then $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$. Here the singularity can be removed by defining $f(z)$ at $z = a$ in such a way that it becomes analytic at $z = a$. Such a singularity is called removable singularity.

Thus if $\lim_{z\to 0}$ exists finitely, then $z = a$ is a removable singularity.

iii. **Poles :** If all the negative powers of $(z-a)$ in eq. (1) after the n^{th} are missing, then the singularity at $z = a$ is called a pole of order n. A pole of first order is called a simple pole.

iv. **Essential Singularity :** If the number of negative powers of $(z-a)$ in eq. (1) is infinite, then $z = a$ is called an essential singularity. In this case, $\lim_{z\to a} f(z)$ does not exist.

Zeros of an Analytic Function : A zero of an analytic function $f(z)$ is that value for z for which $f(z) = 0$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 5.22. **Find the nature and location of singularities of the following functions :**

i. $\dfrac{z - \sin z}{z^2}$

ii. $(z+1)\sin\dfrac{1}{z-2}$

iii. $\dfrac{1}{\cos z - \sin z}$

Answer

i. Here, $z = 0$ is a singularity.

Also, $\frac{z - \sin z}{z^2} = \frac{1}{z^2}\left\{z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + ...\right)\right\} = \frac{z}{3!} - \frac{z^3}{5!} + \frac{z^5}{7!} .....$

Since there are no negative powers of z in the expansion, $z = 0$ is a removable singularity.

ii. $(z+1)\sin\frac{1}{z-2} = (t+2+1)\sin\frac{1}{t}$ Where, $t = z - 2$

$$= (t+3)\left\{\frac{1}{t} - \frac{1}{3!\,t^3} + \frac{1}{5!\,t^5} - ...\right\}$$

$$= \left(1 - \frac{1}{3!\,t^2} + \frac{1}{5!\,t^4} - ...\right) + \left(\frac{3}{t} - \frac{1}{2t^3} + \frac{3}{5!\,t^5} - ...\right)$$

$$= 1 + \frac{3}{t} - \frac{1}{6t^2} - \frac{1}{2t^3} + \frac{1}{120t^4} - .....$$

$$= 1 + \frac{3}{z-2} - \frac{1}{6(z-2)^2} - \frac{1}{2(z-2)^3} + ...$$

Since there are infinite number of terms in the negative powers of $(z-2)$, $z = 2$ is an essential singularity.

iii. Poles of $f(z) = \frac{1}{\cos z - \sin z}$ are given by equating the denominator to zero, *i.e.*, $\cos z - \sin z = 0$ or $\tan z = 1$ or $z = \pi/4$. Clearly $z = \frac{\pi}{4}$ is a simple pole of $f(z)$.

PART-4

Residues, Methods of Finding Residues, Cauchy Residue Theorem.

CONCEPT OUTLINE

Residues : The coefficient of $(z-a)^{-1}$ in the expansion of $f(z)$ around an isolated singularity is called the residue of $f(z)$ at that point. Thus in the Laurent's series expansion of $f(z)$ around $z = a$ *i.e.*, $f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + ... + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + ...$, the residue of $f(z)$ at $z = a$ is a_{-1}.

$\therefore$ $$\text{Res } f(a) = \frac{1}{2\pi i}\oint_C f(z)dz$$

i.e., $$\oint_C f(z)dz = 2\pi i \text{ Res } f(a)$$

Cauchy's Residue Theorem or Theorem of Residues :
If a function $f(z)$ is analytic, except at a finite number of poles within a closed contour C and continuous on the boundary C, then

$$\oint_C f(z)dz = 2\pi i\,\Sigma R$$

$$= 2\pi i\begin{Bmatrix}\text{Sum of residues of } f(z) \text{ at its}\\ \text{poles within } C\end{Bmatrix}$$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 5.23. Find the residues of $f(z) = \frac{z-3}{z^2+2z+5}$ at its poles. Hence or otherwise evaluate $\oint_C \frac{z-3}{z^2+2z+5}$, where C is the circle $|z+1-i| = 2$.

AKTU 2012-13 (IV), Marks 05

Answer

The poles of $f(z) = \frac{z-3}{z^2+2z+5}$ are given by

$$z^2 + 2z + 5 = 0 \Rightarrow z = -1 \pm 2i$$

Only the pole $z = -1 + 2i$ lies inside the circle $|z+1-i| = 2$

Residue of $f(z)$ at $z = -1 + 2i$ is

$$= \lim_{z\to -1+2i}(z+1-2i)f(z)$$

$$= \lim_{z\to -1+2i}\frac{(z-\alpha)(z-3)}{z^2+2z+5}, \text{ where } \alpha = -1+2i\left(\text{Form}\frac{0}{0}\right)$$

$$= \lim_{z\to\alpha}\frac{(z-\alpha)+(z-3)}{2z+2} \quad \text{(By L' Hospital's Rule)}$$

$$= \frac{\alpha-3}{2\alpha+2} = \frac{-1+2i-3}{-2+4i+2} = \frac{i-2}{2i}$$

$\therefore$ By Cauchy' residue theorem,

$$\oint_C \frac{z-3}{z^2+2z+5}dz = 2\pi i\left(\frac{i-2}{2i}\right) = \pi(i-2)$$

Que 5.24. Determine the poles and residues at each pole for $f(z) = \frac{z-1}{(z+1)^2(z-2)}$ and hence evaluate $\oint_C f(z)dz$ where C is the circle $|z-i| = 2$.

AKTU 2013-14 (IV), Marks 05

Answer

Poles of $f(z)$ are given by

$$(z+1)^2(z-2) = 0, z = -1 \text{ (Order 2)}, 2 \text{ (Simple pole)}$$

Residue of $f(z)$ at $z = -1$ is,

$$R_1 = \frac{1}{(2-1)!}\left[\frac{d}{dz}\left\{(z+1)^2 \frac{z-1}{(z+1)^2(z-2)}\right\}\right]_{z=-1}$$

$$= \left[\frac{d}{dz}\left(\frac{z-1}{z-2}\right)\right]_{z=-1} = \left[\frac{-1}{(z-2)^2}\right]_{z=-1} = \frac{-1}{9}$$

Residue of $f(z)$ at $z = 2$ is,

$$R_2 = \lim_{z\to 2}(z-2)\frac{z-1}{(z+1)^2(z-2)} = \lim_{z\to 2}\frac{z-1}{(z+1)^2} = \frac{1}{9}$$

The given curve $C = |z - i| = 2$ is a circle whose centre is at $z = i$ [i.e., at (0, 1)] and radius is 2. Clearly, only the pole $z = -1$ lies inside the curve C.

Hence, by Cauchy's residue theorem

$$\oint_C f(z)\,dz = 2\pi i\,(R_1) = 2\pi i\left(\frac{-1}{9}\right) = -\frac{2\pi i}{9}$$

Que 5.25. **Determine the poles of the following function and residue at each pole :**

$$f(z) = \frac{z^2}{(z-1)^2(z+2)} \text{ and hence evaluate}$$

$$\int_C f(z)\,dz, \text{ where } C : |z| = 3.$$

AKTU 2014-15 (IV), Marks 05

Answer

Same as Q. 5.24, Page 5–21F, Unit-5. (**Answer :** $2\pi i$)

Que 5.26. **Find the poles (with its order) and residue at each poles of the following function :**

$$f(z) = \frac{1-2z}{z(z-1)(z-2)^2}$$

AKTU 2016-17 (III), Marks 05

Answer

Same as Q. 5.24, Page 5–21F Unit-5. $\left(\textbf{Answer :} \text{ Residues are } -\frac{1}{4}, -1, \frac{5}{4}\right)$

Que 5.27. **Evaluate** $\int_C \frac{e^z}{\cos \pi z}dz$**, where C is the unit circle** $|z| = 1$**.**

Answer

Here, $f(z) = \dfrac{e^z}{\cos \pi z} = \dfrac{e^z}{\left[1 - \frac{(\pi z)^2}{2!} + \frac{(\pi z)^4}{4!} - \ldots\right]}$

It has simple poles at $z = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots$, of which only $z = \pm\frac{1}{2}$ lie inside the circle $|z| = 1$.

Residue of $f(z)$ at $z = \frac{1}{2}$ is

$$\lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)f(z) = \lim_{z\to\frac{1}{2}}\frac{\left(z-\frac{1}{2}\right)e^z}{\cos \pi z} \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{z\to\frac{1}{2}}\frac{\left(z-\frac{1}{2}\right)e^z}{-\pi \sin \pi z} \qquad \text{[By L' Hospital's Rule]}$$

$$= \frac{e^{1/2}}{-\pi}$$

Similarly, residue of $f(z)$ at $z = \frac{1}{-2}$ is $\frac{e^{-1/2}}{\pi}$

$\therefore$ By residue theorem,

$$\oint_C \frac{e^z}{\cos \pi z}dz = 2\pi i \text{ (Sum of residues)}$$

$$= 2\pi i\left(-\frac{e^{1/2}}{\pi} + \frac{e^{-1/2}}{\pi}\right) = -4i\left(\frac{e^{1/2} - e^{-1/2}}{2}\right) = -4i \sinh\frac{1}{2}$$

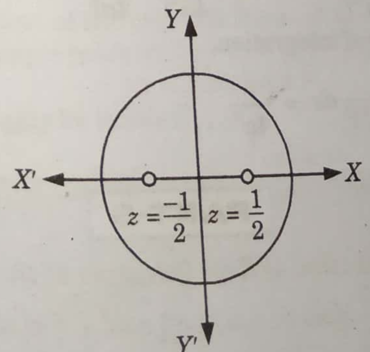

Fig. 5.27.1.

Que 5.28. Using calculus of residue, evaluate the following integral

$$\int_0^\infty \frac{dx}{(a^2+x^2)^2}.$$

AKTU 2015-16 (IV), Marks 10

Answer

Let,
$$I = \int_0^\infty \frac{dx}{(x^2+a^2)^2}$$

$$f(x) = \frac{1}{(x^2+a^2)^2}$$

Poles, $(x^2+a^2)^2 = 0$

$\Rightarrow$ $x^2+a^2 = 0$

$\therefore$ $x = \pm ai$

Only one pole but repeated nature.

Residue at $x = ai$

$$= \frac{1}{(2-1)!}\left[\frac{d}{dx}\left\{(x-ai)^2 \times \frac{1}{(x-ai)^2(x+ai)^2}\right\}\right]_{x=ai}$$

$$= \frac{1}{(2-1)!}\left[\frac{d}{dx}\left(\frac{1}{x+ai}\right)^2\right]_{x=ai} = \left[\frac{-2}{(x+ai)^3}\right]_{x=ai}$$

$$= \frac{-1}{-4a^3 i} = \frac{1}{4a^3 i}$$

Using Cauchy's Residue theorem,

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \text{ [Sum of residue]}$$

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2}dx = 2\pi i \times \frac{1}{4a^3 i} = \frac{\pi}{2a^3}$$

Using property of integration,

$$\int_0^\infty \frac{1}{(x^2+a^2)^2}dx = \frac{\pi}{4a^3}$$

PART-5

Evaluation of Real Integrals of the Type $\int_{-\infty}^{\infty} f(\cos\theta, \sin\theta)\, d\theta$ and $\int_{-\infty}^{\infty} f(x)\, dx$.

CONCEPT OUTLINE

Evaluation of Real Integral of Type $\int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta$, $\int_{-\infty}^{+\infty} f(x)dx$:

Integrals of the type $\int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta$, where $f(\cos\theta, \sin\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$.

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta = \oint_C f\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)\frac{dz}{iz}$$

where C is a unit circle of $|z| = 1$.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 5.29. Use calculus of residue to evaluate the following integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}dx$$

AKTU 2016-17 (III), Marks 10

Answer

We consider $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}dx = \int_C f(z)dz$

Where C is the contour consisting of the semi-circle C_R of radius R together with the part of the real axis from $-R$ to $+R$.

The integral has simple poles at

$$z = \pm ai, z = \pm bi$$

of which $z = ai, bi$ only lie inside C.

The residue (at $z = ai$) $= \displaystyle\lim_{z\to ai}(z-ai)\frac{\cos z\, dz}{(z^2+a^2)(z^2+b^2)}$

$$= \lim_{z\to ai}(z-ai)\frac{\cos z\, dz}{(z-ai)(z+ai)(z^2+b^2)} = \lim_{z\to ai}\frac{\cos z\, dz}{(z+ai)(z^2+b^2)}$$

$$= \left[\frac{\cos ai}{(ai+ai)((ai)^2+b^2)}\right] = \frac{\cos ai}{2ai(b^2-a^2)}$$

The residue (at $z = bi$) $= \lim\limits_{z \to bi} (z - bi) \dfrac{\cos z\, dz}{(z^2 + a^2)(z - bi)(z + bi)}$

$$= \lim_{z \to bi} \frac{\cos z\, dz}{(z^2 + a^2)(z + bi)} = \left[\frac{\cos bi}{((bi)^2 + a^2)(bi + bi)}\right] = \frac{\cos bi}{(a^2 - b^2)2bi}$$

Sum of Residues $(R) = \dfrac{\cos ai}{2ai(b^2 - a^2)} + \dfrac{\cos bi}{(a^2 - b^2)2bi}$

$$= \frac{1}{2i}\left[\frac{\cos ai}{a(b^2 - a^2)} + \frac{\cos bi}{b(a^2 - b^2)}\right] = \frac{1}{2i}\left[-\frac{\cos ai}{a(a^2 - b^2)} + \frac{\cos bi}{b(a^2 - b^2)}\right]$$

$$= \frac{1}{2i}\left[\frac{\cos bi}{b(a^2 - b^2)} - \frac{\cos ai}{a(a^2 - b^2)}\right] = \frac{1}{2i(a^2 - b^2)}\left[\frac{\cos bi}{b} - \frac{\cos ai}{a}\right]$$

$\therefore$ Using Cauchy's Residue theorem,

$$\int_{-\infty}^{\infty} \frac{\cos x\, dx}{(x^2 + a^2)(x^2 + b^2)} = 2\pi i \frac{1}{2i(a^2 - b^2)}\left[\frac{\cos bi}{b} - \frac{\cos ai}{a}\right]$$

$$= \text{Re}\left[\frac{\pi}{(a^2 - b^2)}\left(\frac{\cos bi}{b} - \frac{\cos ai}{a}\right)\right]$$

Que 5.30. **Using complex integration method, evaluate** $\displaystyle\int_0^{\pi} \frac{1}{3 + \sin^2\theta}\, d\theta$.

AKTU 2012-13 (IV), Marks 05

Answer

$$I = \int_0^{\pi} \frac{1}{3 + \sin^2\theta}\, d\theta = \int_0^{\pi} \frac{1}{3 + \frac{1}{2}(1 - \cos 2\theta)}\, d\theta$$

$$\left[\because \sin^2\theta = \frac{1 - \cos 2\theta}{2}\right]$$

$$= 2\int_0^{\pi} \frac{1}{7 - \cos 2\theta}\, d\theta$$

Put $\quad 2\theta = \phi,\ d\theta = \dfrac{d\phi}{2}$

$$= \int_0^{2\pi} \frac{1}{7 - \cos\phi}\, d\phi = \int_0^{2\pi} \frac{1}{7 - \frac{(e^{i\phi} + e^{-i\phi})}{2}}\, d\phi \quad \left[\because \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}\right]$$

$$= \int_0^{2\pi} \frac{2}{14 - (e^{i\phi} + e^{-i\phi})}\, d\phi \qquad \text{...(5.30.1)}$$

But $z = e^{i\phi}$ so that $d\phi = \dfrac{dz}{iz}$ then eq. (5.30.1) reduces to,

$$I = 2\int_C \frac{1}{14 - \left(z + \frac{1}{z}\right)} \frac{dz}{iz} = \frac{2}{i}\int_C \frac{dz}{14z - z^2 - 1}$$

$$= 2i\int_C \frac{dz}{z^2 - 14z + 1} = \int_C \frac{2i}{(z - \alpha)(z - \beta)}\, dz$$

$[\because\ z^2 - 14z + 1 = 0]$

Where, $\alpha = 7 + 4\sqrt{3}$ and $\beta = 7 - 4\sqrt{3}$

Here $\beta < 1$, so only β lies inside C.

Residue at $(z = \beta) = \lim\limits_{z \to \beta} (z - \beta) \times \dfrac{2i}{(z - \alpha)(z - \beta)}$

$$= \frac{2i}{\beta - \alpha} = \frac{2i}{7 - 4\sqrt{3} - 7 - 4\sqrt{3}} = -\frac{i}{4\sqrt{3}}$$

By Cauchy Residue theorem,

$$\int_0^{\pi} \frac{1}{3 + \sin^2\theta}\, d\theta = 2\pi i\left(\frac{-i}{4\sqrt{3}}\right) = \frac{2\pi}{4\sqrt{3}} = \frac{\pi}{2\sqrt{3}}$$

Que 5.31. **Use contour integral to evaluate** $\displaystyle\int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta}$.

AKTU 2012-13 (III), 2013-14 (III); Marks 05

Answer

Here; $\quad I = \displaystyle\int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta}$

$= \displaystyle\oint_C f(z)\, dz,$ where C is the unit circle $|z| = 1$.

We know that, $\quad z = e^{i\theta}$ and $d\theta = \dfrac{dz}{iz}$,

$$\therefore \quad I = \oint_C \frac{1}{3 - \left(\frac{z^2 + 1}{z}\right) + \left(\frac{z^2 - 1}{2iz}\right)} \frac{dz}{iz} \quad \left[\begin{array}{l}\because \cos\theta = \dfrac{e^{\theta i} + e^{-\theta i}}{2} = \dfrac{z^2 + 1}{2z} \\ \text{and } \sin\theta = \dfrac{e^{\theta i} - e^{-\theta i}}{2i} = \dfrac{z^2 - 1}{2iz}\end{array}\right]$$

$$I = \oint_C \frac{1}{3 - z - \frac{1}{z} - \frac{zi}{2} + \frac{i}{2z}} \frac{dz}{iz} = \frac{2}{i}\oint_C \frac{dz}{6z - 2z^2 - 2 - iz^2 + i}$$

$$= -\frac{2}{i}\oint_C \frac{dz}{(i + 2)z^2 - 6z - (i - 2)} = -\frac{2}{i}\oint_C \frac{dz}{(i + 2)z^2 - 5z - z - (i - 2)}$$

$$= -\frac{2}{i}\oint_C \frac{dz}{z[(i + 2)z - 5] - 1[z + (i - 2)]}$$

$$= -\frac{2}{i}\oint_C \frac{dz}{z[(i + 2)z - 5] - \frac{1}{(i + 2)}[(i + 2)z + (i - 2)(i + 2)]}$$

$$= -\frac{2}{i}\oint_C \frac{dz}{[z(i+2)-5]\left[z-\frac{1}{i+2}\right]} = -\frac{2}{i}\oint_C \frac{dz}{[z(i+2)-5]\left[z+\frac{i-2}{5}\right]}$$

Poles are $(2-i)$ and $\left(\frac{2-i}{5}\right)$. The only pole which lie inside C is $z = \frac{2-i}{5}$.

Residue at $z = \frac{2-i}{5} = \lim_{z \to \frac{2-i}{5}} \left(z + \frac{i-2}{5}\right) f(z)$

$$= \lim_{z \to \left(\frac{2-i}{5}\right)} \left(-\frac{2}{i}\,\frac{1}{z(i+2)-5}\right) = \frac{1}{2i}$$

By Cauchy's residue theorem,

$$\oint_C f(z)\, dz = 2\pi i \text{ (Sum of all residues)}$$

$$\int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta} = 2\pi i\left(\frac{1}{2i}\right) = \pi$$

Que 5.32. **Evaluate $\int_0^{2\pi} \frac{\cos 3\theta}{5+4\cos\theta}\, d\theta$.**

AKTU 2013-14 (IV), Marks 05

Answer

Let $I = \int_0^{2\pi} \frac{\cos 3\theta}{5+4\cos\theta}\, d\theta$ = Real part of $\int_0^{2\pi} \frac{e^{3i\theta}}{5+2(e^{i\theta}+e^{-i\theta})}\, d\theta$

$$= \text{Real part of } \oint_C \left|\frac{z^3}{5+2\left(z+\frac{1}{z}\right)}\right| \frac{dz}{iz} \quad \left(\text{Writing } e^{i\theta} = z, d\theta = \frac{dz}{iz}\right)$$

$$= \text{Real part of } \frac{1}{i}\int_C \frac{z^3}{(2z+1)(z+2)}\, dz$$

Singularities of $f(z)$ are given by, $(2z+1)(z+2) = 0$

$$z = -\frac{1}{2}, -2$$

Only, $z = -\frac{1}{2}$ lies within the unit circle $|z| = 1$.

$\therefore$ Residue of $f(z)$ $\left(\text{at } z = -\frac{1}{2}\right) = \lim_{z \to -\frac{1}{2}} \left(z+\frac{1}{2}\right) \times \frac{z^3}{i(2z+1)(z+2)}$

$$= \lim_{z \to -\frac{1}{2}} \frac{z^3}{2i(z+2)} = \frac{1}{2i}\left(\frac{-1}{8}\right)\times\left(\frac{2}{3}\right) = \frac{-1}{24i}$$

Hence by Cauchy's Residue theorem

$$I = \oint_C f(z)\, dz = 2\pi i\left(\frac{-1}{24i}\right) = -\frac{\pi}{12}$$

Que 5.33. **Evaluate the integral $\int_0^{\pi} \frac{\cos^2 3\theta}{5-4\cos 2\theta}\, d\theta$.**

AKTU 2014-15 (III), Marks 05

Answer

Same as Q. 5.32, Page 5–28F, Unit-5. $\left(\textbf{Answer}: \frac{3\pi}{32}\right)$

Que 5.34. **Evaluate : $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta}$ if $a > |b|$**

AKTU 2016-17 (IV), Marks 05

Answer

Consider the integration round a unit circle $C = |z| = 1$

So that $z = e^{i\theta} \quad \therefore \quad d\theta = \frac{dz}{iz}$

Also, $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$

Then the given integral reduces to

$$I = \oint_C \frac{1}{\left[a+\frac{b}{2i}\left(z-\frac{1}{z}\right)\right]}\left(\frac{dz}{iz}\right) = \oint_C \frac{2iz}{bz^2+2iaz-b}\left(\frac{dz}{iz}\right)$$

$$= \frac{2}{b}\oint_C \frac{dz}{z^2+\frac{2ia}{b}z-1}$$

Poles are given by, $z^2 + \frac{2ia}{b}z - 1 = 0$

$$z = \frac{-\frac{2ia}{b} \pm \sqrt{-\frac{4a^2}{b^2}+4}}{2} = \frac{-ia}{b} \pm \frac{\sqrt{b^2-a^2}}{b}$$

$$= \frac{-ia}{b} \pm \frac{i\sqrt{a^2-b^2}}{b} = \alpha, \beta \text{ (simple poles)}$$

where, $\alpha = \frac{-ia}{b} + \frac{i\sqrt{a^2-b^2}}{b}$ and $\beta = \frac{-ia}{b} - \frac{i\sqrt{a^2-b^2}}{b}$

Clearly, $|\beta| > 1$

But $\alpha\beta = -1$

$\therefore$ $|\alpha\beta| = 1$

$|\alpha||\beta| = 1$

$|\alpha| < 1$

Hence $z = \alpha$ is the only pole which lies inside circle $C = |z| = 1$.

Residue of $f(z)$ at $(z = \alpha)$ is

$$R = \lim_{z\to\alpha}(z-\alpha) \times \frac{2}{b(z-\alpha)(z-\beta)} = \frac{2}{b(\alpha-\beta)}$$

$$= \frac{2}{b\left(\frac{2i\sqrt{a^2-b^2}}{b}\right)} = \frac{1}{i\sqrt{a^2-b^2}}$$

$\therefore$ By Cauchy's Residue theorem,

$$I = 2\pi i(R) = 2\pi i\left(\frac{1}{i\sqrt{a^2-b^2}}\right) = \frac{2\pi}{\sqrt{a^2-b^2}}$$

$$\therefore \quad \int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

Que 5.35. **Evaluate the integral :** $\int_0^{2\pi} \frac{d\theta}{5 - 3\cos\theta}$

AKTU 2014-15 (IV), Marks 05

Answer

Same as Q. 5.34, Page 5–29F, Unit-5. $\left(\textbf{Answer : } \frac{\pi}{2}\right)$

Que 5.36. **Using complex variable techniques evaluate the real integral** $\int_0^{2\pi} \frac{\sin 2\theta}{5 - 4\cos\theta}\,d\theta$

AKTU 2017-18 (III), Marks 10

Answer

The given integral, $I = \int_0^{2\pi} \frac{\sin 2\theta}{5 - 4\cos\theta}\,d\theta$ ...(5.36.1)

$$\sin 2\theta = \frac{1}{2i}(e^{2i\theta} - e^{-2i\theta})$$

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

Putting $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$ in eq. (5.36.1), we get

$$I = \oint_C \frac{\frac{1}{2i}\left(z^2 - \frac{1}{z^2}\right)}{5 - 4 \times \frac{1}{2}\left(z + \frac{1}{z}\right)} \frac{dz}{iz}$$

$$= \frac{1}{2i}\oint_C \frac{z^4 - 1}{z^2\left(5 - 2\left(\frac{z^2+1}{z}\right)\right)} \frac{dz}{iz}$$

$$= \frac{1}{2i^2}\oint_C \frac{z^4-1}{z^2\,\frac{(5z - 2z^2 - 2)}{z}} \frac{dz}{z}$$

$$= \frac{1}{2}\oint_C \frac{z^4-1}{z^2\,(2z^2 - 5z + 2)}\,dz$$

$$= \frac{1}{2}\oint_C \frac{z^4-1}{z^2\,(2z-1)(z-2)}\,dz$$

$$= \frac{1}{2}\oint_C f(z)dz, \text{ where C is the unit circle } |z| = 1.$$

Now $f(z)$ has a pole of order 2 at $z = 0$ and simple poles at $z = 1/2$ and $z = 2$, of these only $z = 0$ and $z = 1/2$ lie within the circle.

$$\therefore \quad \text{Res}\, f\left(\frac{1}{2}\right) = \lim_{z\to 1/2}\left(z - \frac{1}{2}\right)\frac{(z^4-1)}{z^2\,(2z-1)(z-2)}$$

$$= \lim_{z\to 1/2}\left[\frac{z^4-1}{2z^2\,(z-2)}\right]$$

$$= \frac{\frac{1}{16} - 1}{2 \times \frac{1}{4}\left(\frac{1}{2} - 2\right)} = \frac{\frac{-15}{16}}{\frac{1}{2} \times \left(\frac{-3}{2}\right)} = \frac{5}{4}$$

$$\text{Res}\, f(0) = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-0)^n f(z)]\right\}_{z=0}$$

$$= \frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}}\left[(z-0)^2 \times \frac{z^4-1}{z^2(2z-1)(z-2)}\right]_{z=0}$$

$(\because\ n = 2)$

$$= \left[\frac{d}{dz} \times z^2\,\frac{(z^4-1)}{z^2\,(2z-1)(z-2)}\right]_{z=0}$$

$$= \left[\frac{d}{dz}\frac{z^4 - 1}{(2z-1)(z-2)}\right]_{z=0}$$

$$= \left\{\frac{(2z-1)(z-2)(4z^3) - (z^4 - 1)[(2z-1) + (z-2)2]}{[(2z-1)(z-2)]^2}\right\}_{z=0}$$

$$= \frac{0 - (-1)(-1-4)}{[-1(-2)]^2} = \frac{-5}{4}$$

Hence $I = \frac{1}{2}\{2\pi i\,[\text{Res}\,f\,(1/2) + \text{Res}\,f\,(0)]\} = 2i\left(\frac{5}{4} - \frac{5}{4}\right) = 0$

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