4 UNIT

Complex Variable Differentiation

CONTENTS

PART-1

Limit, Continuity and Differentiability.

CONCEPT OUTLINE

Limit : The function $f(x, y)$ tends to the limit l as $x \to a$ and $y \to b$ if and only if the limit l is independent of the path followed by the point (x, y) as $x \to a$ and $y \to b$. Then

$$\lim_{\substack{x\to a\\ y\to b}} f(x, y) = l$$

The function $f(x, y)$ in region R tends to the limit l as $x \to a$ and $y \to b$ if and only if corresponding to a positive number $\in (a, b)$, there exists another positive number δ such that

$$|f(x, y) - l| < \in \text{ for } 0 < (x - a)^2 + (y - b)^2 < \delta^2$$

for every point (x, y) in R.

Continuity : A function $f(x, y)$ is said to be continuous at the point (a, b) if $\lim\limits_{\substack{x\to a\\ y\to b}} f(x, y) = f(a, b)$ irrespective of the path along with $x \to a$, $y \to b$.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 4.1. Evaluate $\lim\limits_{\substack{x\to 1\\ y\to 2}} \dfrac{3x^2 y}{x^2 + y^2 + 5}$.

Answer

$$\lim_{\substack{x\to 1\\ y\to 2}} \frac{3x^2 y}{x^2 + y^2 + 5} = \lim_{x\to 1}\left[\lim_{y\to 2} \frac{3x^2 y}{x^2 + y^2 + 5}\right] = \lim_{x\to 1} \frac{3x^2 (2)}{x^2 + (2)^2 + 5}$$

$$= \lim_{x\to 1} \frac{6x^2}{x^2 + 9} = \frac{6}{1+9} = \frac{3}{5}$$

Que 4.2. Evaluate $\lim\limits_{\substack{x\to \infty\\ y\to 2}} \dfrac{xy + 4}{x^2 + 2y^2}$.

Answer

$$\lim_{\substack{x\to \infty\\ y\to 2}} \frac{xy + 4}{x^2 + 2y^2} = \lim_{x\to\infty}\left[\lim_{y\to 2} \frac{xy + 4}{x^2 + 2y^2}\right] = \lim_{x\to\infty}\left[\frac{x(2) + 4}{x^2 + 2(2)^2}\right] = \lim_{x\to\infty} \frac{2x + 4}{x^2 + 8}$$

$$= \lim_{x\to\infty} \frac{2+\frac{4}{x}}{x+\frac{8}{x}} = \frac{2+0}{\infty+0} = 0$$

Que 4.3. **Show that the function $f(x, y) = x - y$ is continuous for all $(x, y) \in R^2$.**

Answer

Let $(a, b) \in R^2$ then $f(a, b) = a - b$

$$\therefore \quad |f(x, y) - f(a, b)| = |(x - y) - (a - b)|$$
$$= |(x - a) + (b - y)|$$
$$\le |x - a| + |y - b| \qquad [\because |x| = |-x|] \qquad ...(4.3.1)$$

Let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{2}$ then for $|x - a| < \delta$ and $|y - b| < \delta$, we have from eq. (4.3.1)

$$|f(x, y) - f(a, b)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence, the function $f(x, y) = x - y$ is continuous for all $(a, b) \in R^2$. But (a, b) is an arbitrary element of R^2, so $f(x, y) = x - y$ is continuous for all $(x, y) \in R^2$.

Que 4.4. **If $f(x, y) = \frac{x^3 - y^3}{x^2 + y^2}$ when $x \ne 0, y \ne 0$ and $f(x, y) = 0$ when $x = 0, y = 0$, find out whether the function $f(x, y)$ is continuous at origin.**

Answer

First calculate the limit of the function :

I. $$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3 - y^3}{x^2 + y^2} = \lim_{y\to 0}\left(\frac{-y^3}{y^2}\right) = \lim_{y\to 0}(-y) = 0$$

II. $$\lim_{\substack{y\to 0\\ x\to 0}} \frac{x^3 - y^3}{x^2 + y^2} = \lim_{x\to 0}\left(\frac{x^3}{x^2}\right) = \lim_{x\to 0}(x) = 0$$

III. $$\lim_{\substack{y\to mx\\ x\to 0}} \frac{x^3 - y^3}{x^2 + y^2} = \lim_{x\to 0} \frac{x^3 - m^3x^3}{x^2 + m^2x^2} = \lim_{x\to 0} \frac{(1 - m^3)}{(1 + m^2)}\, x = 0$$

IV. $$\lim_{\substack{y\to mx^2\\ x\to 0}} \frac{x^3 - y^3}{x^2 + y^2} = \lim_{x\to 0} \frac{x^3 - m^3 - m^3x^6}{x^2 + m^2x^4} = \lim_{x\to 0} \frac{x^3(1 - m^3x^3)}{x^2(1 + m^2x^2)} \lim_{x\to 0} \frac{(1 - m^3x^2)}{(1 + m^2x^2)}x = 0$$

Since the limit along any path is same, the limit exists and equal to zero which is the value of the function $f(x, y)$ at the origin. Hence, the function f is continuous at the origin.

PART-2

Functions of Complex Variable, Analytic Functions, Cauchy Riemann Equations (Cartesian and Polar Form).

CONCEPT OUTLINE

Cauchy-Riemann or C-R Equation :

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

and $$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 4.5. **Define analytic function and state the necessary and sufficient condition for function to be analytic.**

Answer

A. **Analytic Function :** A function $f(z)$ is said to be analytic at a point z_0 if it is one valued and differentiable not only at z_0 but at every point of some neighbourhood of z_0.

B. **Necessary and Sufficient Conditions for $f(z)$ to be Analytic :** The necessary and sufficient conditions for the function

$$w = f(z) = u(x, y) + iv(x, y)$$

to be analytic in a region R, are

i. $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous functions of x and y in the region R.

ii. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

The conditions in (ii) are known as Cauchy-Riemann equations or briefl[y] C-R equations.

Que 4.6. **Define analytic function. Discuss the analyticity of $f(z) = \text{Re}(z^3)$ in the complex plane.** **AKTU 2013-14 (III), Marks 05**

Answer

A. Analytic Function : Refer Q. 4.5, Page 4–4F, Unit-4.

B. Numerical :

$$z = (x + iy)$$
$$z^3 = (x + iy)^3 = x^3 - iy^3 + 3xiy\,(x + iy)$$
$$= (x^3 - 3xy^2) + (3x^2y - y^3)i$$
$$u = x^3 - 3xy^2$$
$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2, \quad \frac{\partial u}{\partial y} = -6xy$$
$$v = (3x^2y - y^3)$$
$$\frac{\partial v}{\partial x} = 6xy, \quad \frac{\partial v}{\partial y} = 3x^2 - 3y^2$$
$$\because \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$
$$\therefore \quad f(z) = \text{Re}\,(z^3) \text{ is analytic function.}$$

Que 4.7. **Show that $f(z) = \log z$ is analytic everywhere in the complex plane except at the origin.** **AKTU 2013-14 (IV), Marks 05**

Answer

Here $f(z) = u + iv = \log z = \log(x + iy)$ $\quad (\because \; z = x + iy)$

Let $x = r\cos\theta$ and $y = r\sin\theta$ so that

$x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$$\log(x + iy) = \log(r\,e^{i\theta}) = \log r + i\theta = \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$$

Separating real and imaginary parts, we get

$$u = \frac{1}{2}\log(x^2 + y^2) \text{ and } v = \tan^{-1}\left(\frac{y}{x}\right)$$

Now, $$\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2}, \quad \frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2}$$

And $$\frac{\partial v}{\partial x} = \frac{-y}{x^2 + y^2}, \quad \frac{\partial v}{\partial y} = \frac{x}{x^2 + y^2}$$

We observe that the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are satisfied except when $x^2 + y^2 = 0$ *i.e.*, when $x = 0, y = 0$
Hence, the function $f(z) = \log z$ is analytic everywhere in the complex plane except at the origin.

Que 4.8. **Find the values of c_1 and c_2 such that the function**

$$f(z) = x^2 + c_1y^2 - 2xy + i\,(c_2x^2 - y^2 + 2xy)$$

is analytic. Also find $f'(z)$. **AKTU 2016-17 (III), Marks 05**

Answer

$$f(z) = x^2 + c_1y^2 - 2xy + i\,(c_2x^2 - y^2 + 2xy)$$
$$u + iv = x^2 + c_1y^2 - 2xy + i\,(c_2x^2 - y^2 + 2xy)$$

Comparing real and imaginary parts, we get

$$u = x^2 + c_1y^2 - 2xy$$

And $$v = c_2x^2 - y^2 + 2xy$$

$$\frac{\partial u}{\partial x} = 2x - 2y \text{ and } \frac{\partial v}{\partial x} = 2c_2x + 2y$$
$$\frac{\partial u}{\partial y} = 2c_1y - 2x \text{ and } \frac{\partial v}{\partial y} = -2y + 2x$$

C-R equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$2x - 2y = -2y + 2x \qquad \text{...(4.8.1)}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$
$$2c_1y - 2x = -2c_2x - 2y \qquad \text{...(4.8.2)}$$

From eq. (4.8.1) and eq. (4.8.2), equating the coefficient of x and y, we get

$$2c_1 = -2 \Rightarrow c_1 = -1$$
$$-2 = -2c_2 \Rightarrow c_2 = 1$$

Now, $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = (2x - 2y) + i\,(2x + 2y)$$
$$= (2x - 2y) + i\,(2x + 2y)$$
$$= 2[x + ix + (-y + iy) = 2[(1 + i)\,x + i\,(1 + i)y]$$
$$= 2(1 + i)\,(x + iy) = 2(1 + i)z$$

Que 4.9. **Find p such that the function $f(z)$ expressed in polar coordinates as $f(z) = r^2\cos 2\theta + ir^2\sin p\theta$ in analytic.**

Answer

Let $f(z) = u + iv$, then $u = r^2\cos 2\theta$, $v = r^2\sin p\theta$

$$\frac{\partial u}{\partial r} = 2r \cos 2\theta, \quad \frac{\partial v}{\partial r} = 2r \sin p\theta$$

$$\frac{\partial u}{\partial \theta} = -2r^2 \sin 2\theta, \quad \frac{\partial v}{\partial \theta} = pr^2 \cos p\theta$$

For $f(z)$ to be analytic, $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

$\therefore$ $2r \cos 2\theta = pr \cos p\theta$ and $2r \sin p\theta = 2r \sin 2\theta$

Both these equations are satisfied if $p = 2$.

Que 4.10. **Show that the function defined by $f(z) = \sqrt{|xy|}$ is not regular at the origin, although Cauchy-Riemann equations are satisfied.**

AKTU 2016-17 (IV), Marks 05

Answer

$f(z) = u(x, y) + iv(x, y) = \sqrt{|xy|}$ then $u(x, y) = \sqrt{|xy|}$, $v(x, y) = 0$

At the origin (0, 0), we have

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x, 0) - u(0, 0)}{x} = \lim_{x \to 0} \frac{0 - 0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0, y) - u(0, 0)}{y} = \lim_{y \to 0} \frac{0 - 0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x \to 0} \frac{v(x, 0) - v(0, 0)}{x} = \lim_{x \to 0} \frac{0 - 0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y \to 0} \frac{v(0, y) - v(0, 0)}{y} = \lim_{y \to 0} \frac{0 - 0}{y} = 0$$

Clearly, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Hence, C-R equations are satisfied at the origin.

Now $$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{z \to 0} \frac{\sqrt{|xy|} - 0}{x + iy}$$

If $z \to 0$ along the line $y = mx$, we get

$$f'(0) = \lim_{x \to 0} \frac{\sqrt{|mx^2|}}{x(1 + im)} = \lim_{x \to 0} \frac{\sqrt{|m|}}{1 + im}$$

Now this limit is not unique since it depends on m. Therefore, $f'(0)$ does not exist.

Hence, the function $f(z)$ is not regular at the origin.

Que 4.11. **Prove that the function sinh z is analytic and find its derivation.**

Answer

Here $f(z) = u + iv = \sinh z = \sinh (x + iy) = \sinh x \cos y + i \cosh x \sin y$

$$\frac{\partial u}{\partial x} = \cosh x \cos y, \quad \frac{\partial u}{\partial y} = -\sinh x \sin y$$

$$\frac{\partial v}{\partial x} = \sinh x \sin y, \quad \frac{\partial v}{\partial y} = \cosh x \cos y$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Thus C-R equations are satisfied.

Since $\sinh x$, $\cosh x$, $\sin y$ and $\cos y$ are continuous function, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are also continuous functions satisfying C-R equations.

Hence $f(z)$ is analytic everywhere.

Now, $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$
$$= \cosh x \cos y + i \sinh x \sin y$$
$$= \cosh (x + iy) = \cosh z.$$

Que 4.12. **Using C – R equations show that $f(z) = |z|^2$ is not analytical at any point.**

AKTU 2014-15 (IV), Marks 05

Answer

Let $$w = f(z) = u + iv = |z|^2$$
$$u + iv = x^2 + y^2$$

Comparing both sides,

$$u = x^2 + y^2, \quad \frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y$$

$$v = 0, \quad \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$$

Using C-R equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow 2x = 0 \Rightarrow x = 0$$

And $$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow 2y = 0 \Rightarrow y = 0$$

At (0, 0) C-R equations are satisfied and the function is differentiable.

Hence, the function is not analytic anywhere except at origin.

Que 4.13. If $f(z) = \dfrac{x^3y(y-ix)}{x^6+y^2}$ when $z \neq 0$

$= 0$ when $z = 0$

Prove that $\dfrac{f(z)-f(0)}{z} \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ in any manner.

AKTU 2012-13 (III), Marks 05

Answer

$$f(z) = u + iv = \frac{x^3y(y-ix)}{x^6+y^2}, \; z \neq 0$$

$$u = \frac{x^3y^2}{x^6+y^2}, \; v = \frac{-x^4y}{x^6+y^2}$$

$$\frac{\partial u}{\partial x} = \lim_{h\to 0}\frac{u(0+h,0)-u(0,0)}{h} = \lim_{h\to 0}\frac{0/h^6}{h} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{k\to 0}\frac{u(0,0+k)-u(0,0)}{k} = \lim_{k\to 0}\frac{0/k^2}{k} = 0$$

Similarly, $\dfrac{\partial v}{\partial x} = 0$ and $\dfrac{\partial v}{\partial y} = 0$

Thus $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \; \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

Hence, C-R equations are satisfied at origin.

Now, $$\frac{f(z)-f(0)}{z} = \left[\frac{x^3y(y-ix)}{x^6+y^2} - 0\right]\frac{1}{x+iy}$$

$$= \frac{x^3y(y-ix)}{x^6+y^2}\,\frac{1}{(x+iy)} = \frac{-ix^3y}{x^6+y^2}$$

Let $z \to 0$ along radius vector $y = mx$, then

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{x\to 0}\frac{-ix^3(mx)}{x^6+m^2x^2} = \lim_{x\to 0}\frac{-imx^2}{x^4+m^2} = 0$$

Hence $\dfrac{f(z)-f(0)}{z} \to 0$ as $z \to 0$ along any radius vector

Let $z \to 0$ along $y = x^3$ then,

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{x\to 0}\frac{-ix^3x^3}{x^6+x^6} = \frac{-i}{2}$$

Thus $f'(0)$ does not exist, hence $f(z)$ is not analytic at $z = 0$.

Que 4.14. **Examine the nature of the function**

$$f(z) = \frac{x^2y^5(x+iy)}{x^4+y^{10}}; \; z \neq 0, \; f(0) = 0$$

In the region including the origin. **AKTU 2015-16 (III), Marks 10**

Answer

Same as Q. 4.13, Page 4–9F, Unit-4.

(**Answer :** $f'(0)$ does not exist. Hence, $f(z)$ is not analytic at origin).

Que 4.15. **Prove that the function $f(z)$ defined by**

$$f(z) = \begin{cases} \dfrac{x^3(1+i)-y^3(1-i)}{x^2+y^2} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

is continuous and the Cauchy-Riemann equations are satisfied at the origin, yet $f'(0)$ does not exist. **AKTU 2016-17 (IV), Marks 10**

Answer

$$f(z) = \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2} = u + iv$$

where, $$u = \frac{x^3-y^3}{x^2+y^2}, \; v = \frac{x^3+y^3}{x^2+y^2}$$

$\because$ The value of $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ at (0, 0) we get $\dfrac{0}{0}$, so we apply first principle method.

At the origin,

$$\frac{\partial u}{\partial x} = \lim_{h\to 0}\frac{u(0+h,0)-u(0,0)}{h} = \lim_{h\to 0}\left(\frac{h^3}{h^2}\right)/h = 1$$

$$\frac{\partial u}{\partial y} = \lim_{k\to 0}\frac{u(0,0+k)-u(0,0)}{k} = \lim_{k\to 0}\left(\frac{-k^3}{k^2}\right)/k = -1$$

$$\frac{\partial v}{\partial x} = \lim_{h\to 0}\frac{v(0+h,0)-v(0,0)}{h} = \lim_{h\to 0}\left(\frac{h^3}{h^2}\right)/h = 1$$

$$\frac{\partial v}{\partial y} = \lim_{k\to 0}\frac{v(0,0+k)-v(0,0)}{k} = \lim_{k\to 0}\left(\frac{k^3}{k^2}\right)/k = 1$$

$\therefore$ $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

Hence, C-R equations are satisfied at origin.

Now $f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{z \to 0} \left[\frac{x^3 - y^3 + i(x^3 + y^3)}{x^2 + y^2} \cdot \frac{1}{x + iy} \right]$

Now let $z \to 0$ along $y = mx$, then

$$f'(0) = \lim_{x \to 0} \left[\frac{x^3 - m^3x^3 + i(x^3 + m^3x^3)}{x^2 + m^2x^2} \cdot \frac{1}{x + imx} \right]$$

$$= \lim_{x \to 0} \left[\frac{1 - m^3 + i(1 + m^3)}{(1 + m^2)(1 + im)} \right] = \frac{m^3(-1 + i) + (1 + i)}{(1 + m^2)(1 + im)}$$

$\because$ The value of $f'(0)$ depends on m, therefore $f'(0)$ is not unique.
Hence, the function is not analytic at $z = 0$.

PART-3

Harmonic Function, Method to Find Analytic Functions.

CONCEPT OUTLINE

Harmonic Function : A function of (x, y) which possesses continuous partial derivatives of the first and second orders and satisfies Laplace equation is called a harmonic function.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 4.16. If $f(z)$ is a harmonic function of z, show that

$$\left\{ \frac{\partial}{\partial x} |f(z)| \right\}^2 + \left\{ \frac{\partial}{\partial y} |f(z)| \right\}^2 = |f'(z)|^2$$

Answer

We have, $f(z) = u + iv$ ...(4.16.1)

$\therefore$ $|f(z)| = \sqrt{u^2 + v^2}$ ...(4.16.2)

Partially differentiating eq. (4.16.2) w.r.t x and y, we get

$$\frac{\partial}{\partial x} |f(z)| = \frac{1}{2}(u^2 + v^2)^{-1/2} \left(2u \frac{\partial u}{\partial x} + 2v \frac{\partial v}{dx} \right)$$

$$= \frac{u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial x}}{|f(z)|} \quad ...(4.16.3)$$

Similarly, $\frac{\partial}{\partial y} |f(z)| = \frac{u \frac{\partial u}{\partial y} + v \frac{\partial v}{\partial y}}{|f(z)|}$ ...(4.16.4)

Squaring and adding eq. (4.16.3) and eq. (4.16.4), we get

$$\left\{ \frac{\partial}{\partial x} |f(z)| \right\}^2 + \left\{ \frac{\partial}{\partial y} |f(z)| \right\}^2 = \frac{\left(u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial x} \right)^2 + \left(u \frac{\partial u}{\partial y} + v \frac{\partial v}{\partial y} \right)^2}{|f(z)|^2}$$

$$= \frac{\left(u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial x} \right)^2 + \left(-u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial x} \right)^2}{|f(z)|^2}$$

(Using C-R equation)

$$= \frac{(u^2 + v^2)\left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 \right]}{|f(z)|^2}$$

$$= \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 \quad (\because |f(z)|^2 = u^2 + v^2)$$

$$= |f'(z)|^2 \quad \left(\because f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right)$$

Que 4.17. Verify that the function on $u_1(x, y) = xy$ is harmonic and find its conjugate harmonic function. Express $u + iv$ as an analytic function $f(z)$.

$$u = x^2 - y^2 - y$$

AKTU 2015-16 (III), Marks 05

Answer

$$u(x, y) = xy$$

$\frac{\partial u}{\partial x} = y$ $\quad \therefore \frac{\partial^2 u}{\partial x^2} = 0$

$\frac{\partial u}{\partial y} = x$ $\quad \therefore \frac{\partial^2 u}{\partial y^2} = 0$

For a function to be harmonic, it must satisfy Laplace equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Hence, function $u(x, y)$ is harmonic.
Using Cauchy-Riemann equation,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Total differentiation of v is given as,

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -xdx + ydy$$

$$\therefore \quad v = \frac{-x^2}{2} + \frac{y^2}{2} + c$$

u and v are said complex conjugate.

Again, $u = x^2 - y^2 - y$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y - 1$$

Using Cauchy-Riemann equation,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and, } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\because \quad dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = (2y+1)\,dx + 2xdy = d(2xy + x)$$

$$\therefore \quad v = 2xy + x + c$$

Then, $f(x, y) = u + iv = (x^2 - y^2 - y) + i(2xy + x + c)$

Que 4.18. **Show that $v(x, y) = e^{-x}(x \cos y + y \sin y)$ is harmonic. Find its harmonic conjugate.**

AKTU 2013-14 (III), Marks 05

Answer

$$v(x, y) = e^{-x}(x \cos y + y \sin y)$$

$$\frac{\partial v}{\partial x} = -e^{-x}(x \cos y + y \sin y) + e^{-x}(\cos y)$$

$$\frac{\partial v}{\partial y} = e^{-x}(-x \sin y + y \cos y + \sin y)$$

$$\frac{\partial^2 v}{\partial x^2} = -[-e^{-x}(x \cos y + y \sin y) + e^{-x}(\cos y)] - e^{-x}(\cos y)$$

$$\frac{\partial^2 v}{\partial y^2} = e^{-x}[-x \cos y + (\cos y - y \sin y) + \cos y]$$

$$= e^{-x}[2 \cos y - y \sin y - x \cos y]$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = e^{-x}[x \cos y + y \sin y - \cos y - \cos y] + e^{-x}[2 \cos y - y \sin y + x \cos y]$$

$$= e^{-x}[x \cos y + y \sin y - 2 \cos y + 2 \cos y - y \sin y - x \cos y]$$

$$= 0$$

Since, v satisfies the Laplace equation hence v is harmonic function.

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \left(\frac{\partial v}{\partial y}\right)dx + \left(\frac{-\partial v}{\partial x}\right)dy$$

$$du = [e^{-x}(-x \sin y + y \cos y + \sin y)]\,dx + e^{-x}[x \cos y + y \sin y - \cos y]\,dy$$

$$u = \int_{y=\text{cont}} e^{-x}(-x \sin y + y \cos y + \sin y)dx + \int_{x=\text{cont}} e^{-x}(x \cos y + y \sin y - \cos y)dy$$

$$u = -\int e^{-x} x \sin y\, dx + y \cos y \int e^{-x} dx + \sin y \int e^{-x} dx + xe^{-x}\int \cos y\, dy + e^{-x}\int y \sin y\, dy - e^{-x}\int \cos y\, dy$$

$$u = -(-2x\,e^{-x}) \sin y - e^{-x} y \cos y - e^{-x} \sin y + x\,e^{-x} \sin y + e^{-x}(-y \cos y - y \sin y) - e^{-x} \sin y$$

$$u = 2x\,e^{-x} \sin y - e^{-x} y \cos y - e^{-x} \sin y + x\,e^{-x} \sin y - e^{-x} y \cos y - e^{-x} y \sin y - e^{-x} \sin y$$

$$u = 3x\,e^{-x} \sin y - 2e^{-x} y \cos y - e^{-x} y \sin y - 2e^{-x} \sin y$$

Here u is the harmonic conjugate of v.

Que 4.19. **Find an analytic function whose imaginary part is $e^{-x}(x \cos y + y \sin y)$.**

AKTU 2013-14 (IV), Marks 05

Answer

Let $f(z) = u + iv$ be the required analytic function.

Here, $v = e^{-x}(x \cos y + y \sin y)$

$$\frac{\partial v}{\partial y} = e^{-x}(-x \sin y + y \cos y + \sin y) = \psi_1(x, y)$$

$$\frac{\partial v}{\partial x} = e^{-x} \cos y - e^{-x}(x \cos y + y \sin y) = \psi_2(x, y)$$

$$\therefore \quad \psi_1(z, 0) = 0, \; \psi_2(z, 0) = e^{-z} - e^{-z} z = (1 - z)\,e^{-z}$$

By Milne's Thomson method,

$$f(z) = \int [\psi_1(z, 0) + i\,\psi_2(z, 0)]\,dz + c = i\int (1 - z)\,e^{-z}\,dz + c$$

$$= i\left[(1 - z)(-e^{-z}) - \int (-1)(-e^{-z})dz\right] + c$$

$$= i\,[(z - 1)\,e^{-z} + e^{-z}] + c$$

$$f(z) = ize^{-z} + c$$

Que 4.20. **Find the analytic function whose real part is $e^{2x}(x \cos 2y - y \sin 2y)$.**

AKTU 2014-15 (IV), Marks 05

Answer

Let, $u = e^{2x}(x \cos 2y - y \sin 2y)$

$$\frac{\partial u}{\partial x} = e^{2x}(\cos 2y) + 2e^{2x}(x \cos 2y - y \sin 2y) = \phi_1(x, y)$$

$$\frac{\partial u}{\partial y} = e^{2x}[-2x \sin 2y - 2y \cos 2y - \sin 2y] = \phi_2 (x, y)$$

On replacing x by z and y by 0,

$$f(z) = \int \phi_1 (z, 0)\, dz - i \int \phi_2 (z, 0)\, dz + c$$

$$= \int [e^{2z} \cos 0 + 2e^{2z} (z)]\, dz - \int 0\, dz + c$$

$$= \int (e^{2z} + 2ze^{2z})\, dz + c = \frac{1}{2} e^{2z} + 2\left[z\frac{e^{2z}}{2} - \frac{e^{2z}}{4}\right] + c$$

$$f(z) = z\, e^{2z} + c$$

Que 4.21. **If $u = 3x^2y - y^3$ find the analytic function $f(z) = u + iv$.**

AKTU 2012-13 (III), Marks 05

Answer

$$u = 3x^2y - y^3$$

$$\frac{\partial u}{\partial x} = 6xy, \quad \frac{\partial u}{\partial y} = 3x^2 - 3y^2$$

Now,
$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = \left(-\frac{\partial u}{\partial y}\right) dx + \left(\frac{\partial u}{\partial x}\right) dy$$

$$dv = (-(3x^2 - 3y^2))dx + 6xy\, dy$$

On integrating,
$$v = \underset{(y \text{ as constant})}{\int M\, dx} + \underset{(\text{Ignoring terms of } x)}{\int N\, dy}$$

$\therefore$
$$v = \int (3y^2 - 3x^2)dx + 0 = 3xy^2 - x^3 + c$$

Now,
$$u + iv = 3x^2y - y^3 + i(3xy^2 - x^3 + c)$$

$$= [3x^2y - y^3 + i(3xy^2 - x^3)] + ic$$

$$= -i(x^3 - iy^3 - 3xy^2 + 3ix^2y) + ic$$

$$= -i(x + iy)^3 + ic = -iz^3 + ic$$

$$u + iv = -i(z^3 - c)$$

Que 4.22. **Show that $e^x \cos y$ is harmonic function, find the analytic function of which it is real part.**

Answer

Let,
$$u = e^x \cos y$$

$\therefore$
$$\frac{\partial u}{\partial x} = e^x \cos y \Rightarrow \frac{\partial^2 u}{\partial x^2} = e^x \cos y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y \Rightarrow \frac{\partial^2 u}{\partial y^2} = -e^x \cos y$$

Since $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, therefore u is a harmonic function.

Let
$$d_v = \frac{\partial u}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$= \left(-\frac{\partial u}{\partial x}\right) dx + \left(\frac{\partial v}{\partial y}\right) dy \quad \text{(By C-R equation)}$$

$$= e^x \sin y\, dx + e^x \cos y\, dy$$

$$= d\,(e^x \sin y)$$

Integration yields,
$$v = e^x \sin y + c$$

Hence
$$f(z) = u + iv = e^x \cos y + i(e^x \sin y + c)$$

$$= e^x(\cos y + i \sin y) + c_1 \quad (\text{where } c_1 = ic)$$

$$= e^{x + iy} + c_1 = e^z + c_1.$$

Que 4.23. **Show that the function $u = \dfrac{1}{2} \log (x^2 + y^2)$ is harmonic. Find the harmonic conjugate of u.** **AKTU 2014-15 (III), Marks 05**

Answer

$$u = \frac{1}{2} \log (x^2 + y^2)$$

$$\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2}, \quad \frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{x^2 + y^2 - 2y^2}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \text{ hence } u \text{ is harmonic.}$$

Now,
$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

$$dv = \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy$$

$$dv = \frac{x\, dy - y\, dx}{x^2 + y^2} = d\left[\tan^{-1}\left(\frac{y}{x}\right)\right]$$

Integration yields, $v = \tan^{-1}\left(\dfrac{y}{x}\right) + c$

This is the required harmonic conjugate function of u.

Que 4.24. **If $f(z) = u + iv$ is analytic function and $u - v = e^x (\cos y - \sin y)$, find $f(z)$ in terms of z.**

AKTU 2015-16 (III), Marks 05

Answer

$$u + iv = f(z)$$
$$i(u + iv) = if(z) \quad \text{...(4.24.1)}$$
$$iu - v = if(z) \quad \text{...(4.24.2)}$$

On adding eq. (4.24.1) and eq. (4.24.2),

$$u - v + i(u + v) = (1 + i) f(z)$$
$$U + iV = F(z)$$

Where, $U = u - v = e^x (\cos y - \sin y)$

$$V = u + v$$
$$(1 + i) f(z) = F(z)$$

Now using Milne's Thomson method,

$$\frac{\partial U}{\partial x} = \phi_1 = e^x (\cos y - \sin y)$$

So, $\phi_1(z, 0) = e^z (\cos 0 - \sin 0)$

$$\phi_1(z, 0) = e^z$$

Now $\dfrac{\partial U}{\partial y} = \phi_2 = e^x (-\sin y - \cos y)$

$$\phi_2(z, 0) = e^z (-\sin 0 - \cos 0)$$
$$\phi_2(z, 0) = -e^z$$

According to Milne's Thomson method,

$$F(z) = \int \{\phi_1(z, 0) - i\,\phi_2(z, 0)\}\, dz + c$$
$$= \int (e^z + ie^z)\, dz + C = \int e^z (1 + i)\, dz + c$$
$$F(z) = (1 + i) e^z + c$$

or $(1 + i) f(z) = (1 + i) e^z + c$

$$f(z) = e^z + \frac{c}{1 + i}$$

Que 4.25. **Determine an analytic function $f(z)$ in term of z if**

$$u + v = 2\frac{\sin 2x}{e^{2y}} + e^{2y} - 2\cos 2x.$$

AKTU 2017-18 (IV), Marks 07

Answer

Let $f(z) = u + iv$

$\therefore$ $if(z) = iu - v$

$$(1 + i) f(z) = (u - v) + i(u + v)$$
$$F(z) = U + IV$$

Where, $U = (u - v)$ and $V = u + v$

Hence, $V = u + v = \dfrac{2\sin 2x}{e^{2y}} + e^{2y} - 2\cos 2x$

Now, $\dfrac{\partial V}{\partial x} = \dfrac{4\cos 2x}{e^{2y}} + 4\sin 2x = \psi_2(x, y)$

and $\dfrac{\partial V}{\partial y} = \dfrac{-4\sin 2x}{e^{2y}} + 2e^{2y} = \psi_1(x, y)$

$\therefore$ $\psi_1(z, 0) = -4\sin 2z + 2$

$$\psi_2(z, 0) = 4\cos 2z + 4\sin 2z$$

By Milne's Thomson method,

$$F(z) = \int \{\psi_1(z, 0) + i\,\psi_2(z, 0)\}\, dz + c$$
$$= \int \{(-4\sin 2z + 2) + i(4\cos 2z + 4\sin 2z)\}\, dz + c$$
$$= \left(\frac{4\cos 2z}{2} + 2z\right) + i\left(\frac{4\sin 2z}{2} - \frac{4\cos 2z}{2}\right) + c$$
$$= (2\cos 2z + 2z) + i(2\sin 2z - 2\cos 2z) + c$$

or $(1 + i) f(z) = (2\cos 2z + 2z) + i(2\sin 2z - 2\cos 2z) + c$

$$\text{or } f(z) = \frac{2(\cos 2z + z)}{(1 + i)} + \frac{2i(\sin 2z - \cos 2z)}{(1 + i)} + \frac{c}{(1 + i)}$$

Multiply and divide by $(1 - i)$ on RHS, we get

$$f(z) = \frac{2(\cos 2z + z)}{(1 + i)}\left(\frac{1 - i}{1 - i}\right) + \frac{2i(\sin 2z - \cos 2z)}{(1 + i)}\left(\frac{1 - i}{1 - i}\right) + \frac{c}{(1 + i)}\left(\frac{1 - i}{1 - i}\right)$$

$$= \frac{2(1 - i)(z + \cos 2z)}{1^2 - i^2} + \frac{2i(1 - i)(\sin 2z - \cos 2z)}{1^2 - i^2} + c_1$$

{Where, c_1 = Constant}

$$= \frac{2(1 - i)(z + \cos 2z)}{2} + \frac{2i(1 - i)(\sin 2z - \cos 2z)}{2} + c_1 \quad (\because i^2 = -1)$$

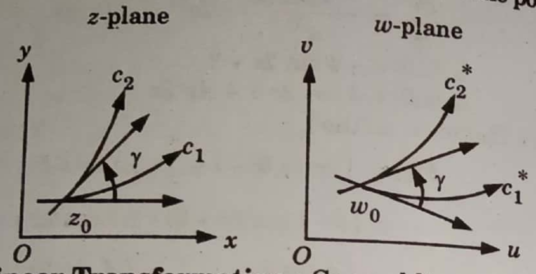
$$= (z + \cos 2z) - i(z + \cos 2z) + (i + 1)(\sin 2z - \cos 2z)$$
$$= (z + \cos 2z + \sin 2z - \cos 2z) + i(-z - \cos 2z + \sin 2z - \cos 2z)$$
$$f(z) = (z + \sin 2z) + i(\sin 2z - 2\cos 2z - z)$$

PART-4

Conformal Mapping.

CONCEPT OUTLINE

Conformal Mapping : A mapping $w = f(z)$ is said to be conformal if the angle between any two smooth curves c_1, c_2 in the z-plane intersecting at the point z_0 is equal in magnitude and sense to the angle between their images c_1^*, c_2^* in the w-plane at the point $w_0 = f(z_0)$



General Linear Transformation : General linear transformation or simply linear transformation defined by the function

$$w = f(z) = az + b \quad ...(1)$$

($a \neq 0$, and b are arbitrary complex constants) maps conformally the extended complex z-plane onto the extended w-plane, since this function is analytic and $f'(z) = a \neq 0$ for any z. If $a = 0$, eq. (1) reduces to a constant function.

Special Cases of Linear Transformation :

i. **Identity Transformation :** In this, $w = z$ for $a = 1, b = 0$, which maps a point z onto itself.

ii. **Translation :** In this, $w = z + b$ for $a = 1$, which translates (shifts) z through a distance $|b|$ in the direction of b.

iii. **Rotation :** In this, $w = e^{i\theta_0} + z$ for $a = e^{i\theta_0}$, $b = 0$ which rotates (the radius vector of point) z through a scalar angle θ_0 (counterclockwise if $\theta_0 > 0$, while clockwise of $\theta_0 < 0$).

iv. **Stretching (Scaling) :** In this, $w = az$ for 'a' real stretches if $a > 1$ (contracts if $0 < a < 1$) the radius sector by a factor 'a'.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 4.26. State and prove condition for conformality.

Answer

Statement : A mapping $w = f(z)$ is conformal at each point z_0 where $f(z)$ is analytic and $f'(z_0) \neq 0$.

Proof : Since f is analytic, f' exists and since $f' \neq 0$, we have at a point z_0

$$R_0 e^{i\theta_0} = f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left(\left| \frac{\Delta w}{\Delta z} \right| + i \arg \frac{\Delta w}{\Delta z} \right)$$

So $$\theta_0 = \lim_{\Delta z \to 0} \left(\arg \frac{\Delta w}{\Delta z} \right)$$

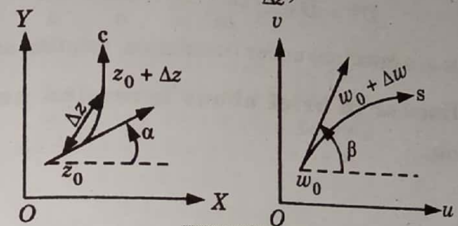


Fig. 4.26.1.

Since $$\Delta w = \frac{\Delta w}{\Delta z} \Delta z$$

$$\arg \Delta w = \arg \frac{\Delta w}{\Delta z} + \arg \Delta z$$

As $\Delta z \to 0$, $\quad \beta = \theta_0 + \alpha$

Thus the directed tangent to curve c at z_0 is rotated through an angle $\theta_0 = \arg f'(z_0)$, which is same for all curves through z_0. Let α_1, α_2 be angles of inclination of two curves c_1 and c_2 and β_1 and β_2 be the corresponding angles for their images S_1 and S_2.

Then $\quad \beta_1 = \alpha_1 + \theta_0 \quad$ and $\quad \beta_2 = \alpha_2 + \theta_0$

Thus $\quad \beta_2 - \beta_1 = \alpha_2 - \alpha_1 = \gamma$

Hence, the angle γ between the curves c_1 and c_2 and their images S_1 and S_2 is same both in magnitude and sense.

Que 4.27. Show that circles are invariant under translation, rotation and stretching.

Answer

Linear transformation preserves circles *i.e.*, a circle in the z-plane under linear transformation maps to a circle in the w-plane.

Consider any circle in the z-plane

$$A(x^2 + y^2) + Bx + Cy + D = 0 \quad ...(4.27.1)$$

Let $\quad w = f(z) = az + b$

From above $\quad u + iv = w = az + b = a(x + iy) + (b_1 + ib_2)$

or $\quad u = ax + b_1, v = ay + b_2$

or
$$x = \frac{u - b_1}{a}, y = \frac{v - b_2}{a}, a \neq 0 \quad ...(4.27.2)$$

Substituting the value of x and y from eq. (4.27.2) in eq. (4.27.1), we get
$$A^*(u^2 + v^2) + B^* u + C^* v + D^* = 0 \quad ...(4.27.3)$$

Which is circle in the w-plane.

Where,
$$A^* = \frac{A}{a^2}, B^* = \frac{B - 2Ab_1}{a}, C^* = \frac{C - 2Ab_2}{a}$$

and
$$D^* = D + A\left(\frac{b_1^2 + b_2^2}{a^2}\right) - \frac{Bb_1}{a} - \frac{Cb_2}{a}$$

Thus circles are invariant under translation, rotation and stretching.

Que 4.28. **Discuss in brief about inversion and reflection transformation.**

Answer

Consider,
$$w = \frac{1}{z} \quad \text{for } z \neq 0 \quad ...(4.28.1)$$

In polar coordinates,
$$Re^{i\phi} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$$

So $R = \frac{1}{r}$, $\phi = -\theta$. Thus this transformation consists of an inversion in the unit circle ($Rr = 1$) followed by a mirror reflection about the real axis.

Also $|w| = \frac{1}{|z|}$. So the unit circle $|z| = 1$ maps onto the unit circle $|w| = \frac{1}{1} = 1$. Further the interior of the unit circle $|z| = 1$ (point lying within $|z| = 1$) are transformed to the exterior of the unit circle $|w| = 1$ (points lying outside $|w| = 1$) or vice versa (Fig. 4.28.1).

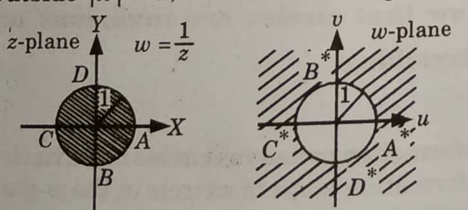


Fig. 4.28.1.

Que 4.29. **Find and plot the image of triangular region with vertices at (0, 0), (1, 0), (0, 1) under the transformation $w = (1 - i)z + 3$ (Fig. 4.29.1).**

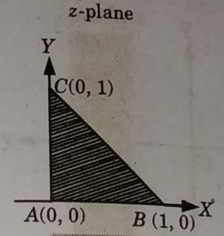


Fig. 4.29.1.

Answer

Here,
$$u + iv = w = (1 - i)(x + iy) + 3$$
$$= x + iy - ix + y + 3$$

So
$$u(x, y) = x + y + 3, v(x, y) = y - x$$

At AB, $y = 0$, $u = x + 3$, $v = -x$

or
$$u = -v + 3$$

$\therefore$
$$v = 3 - u \text{ gives } A^* B^*$$

At AC, $x = 0$, $u = y + 3$, $v = y$,

or
$$u = v + 3$$

$\therefore$
$$v = u - 3 \text{ gives } A^*C^*$$

At BC, $x + y = 1$, or substituting $u = (x + y) + 3$
$$= 1 + 3 = 4,$$
$$u = 4 \text{ gives } B^*C^*$$

So the image is the triangular region with vertices at $A^*(3, 0)$, $B^*(4, -1)$, $C^*(4, 1)$. Let $D\left(\frac{1}{4}, \frac{1}{4}\right)$ be any interior point of ABC. Its image is $D^*(3, 5, 0)$ which is also an interior point of $A^*B^*C^*$.

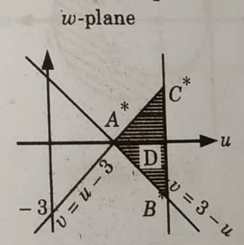


Fig. 4.29.2.

Que 4.30. **Find the graph for the strip $1 < x < 2$ under the mapping $w = \frac{1}{z}$ (Fig. 4.30.1).**

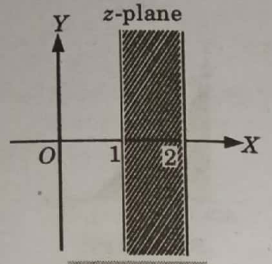


Fig. 4.30.1.

Answer

Here, $$u + iv = w = \frac{1}{z} = \frac{x - iy}{x^2 + y^2}$$

So $$x = \frac{u}{u^2 + v^2}, \quad y = \frac{-v}{u^2 + v^2}$$

Since $1 < x < 2$ so $1 < \dfrac{u}{u^2 + v^2}, < 2$

or $u^2 + v^2 - u < 0$ and $2(u^2 + v^2) - u > 0$

Rewriting $\left(u - \dfrac{1}{2}\right)^2 + v^2 < \dfrac{1}{4}$ and $\left(u - \dfrac{1}{4}\right)^2 + v^2 > \dfrac{1}{16}$

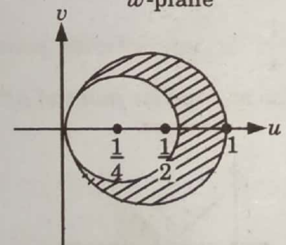


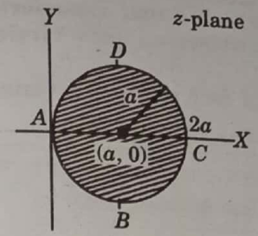
Fig. 4.30.2.

or $\left|w - \dfrac{1}{2}\right| < \dfrac{1}{2}$ and $\left|w - \dfrac{1}{4}\right| > \dfrac{1}{4}$

i.e., interior of the circle with centre at $\left(\dfrac{1}{2}, 0\right)$ and radius $\dfrac{1}{2}$ and exterior of the circle with centre at $\left(\dfrac{1}{4}, 0\right)$ and radius $\dfrac{1}{4}$.

Thus the infinite strip maps to the region shaded in the w-plane.

Que 4.31. **Determine and graph the image of $|z - a| = a$ under $w = z^2$ (Fig. 4.31.1).**



$|z - a| = a$

Fig. 4.31.1.

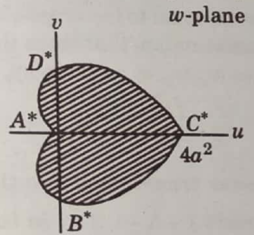
Answer

The given region is a circle in the z-plane with centre at $(a, 0)$ and radius a, *i.e.*,

$$z - a = ae^{i\theta} \quad \text{or} \quad z = a + ae^{i\theta} = a(1 + e^{i\theta})$$

So
$$\begin{aligned} w &= z^2 = a^2(1 + e^{i\theta})^2 = a^2(1 + \cos\theta + i\sin\theta)^2 \\ &= 2a^2(\cos^2\theta + \cos\theta + i\sin\theta\cos\theta + i\sin\theta) \\ Re^{i\phi} = w &= 2a^2(1 + \cos\theta)(\cos\theta + i\sin\theta) \\ &= 2a^2(1 + \cos\theta)e^{i\theta} \end{aligned}$$

Thus
$$\begin{aligned} R &= 2a^2(1 + \cos\theta) \\ &= 2a^2(1 + \cos\phi) \quad (\because \phi = \theta) \end{aligned}$$



$R = 2a^2(1 + \cos\phi)$ cardioid

Fig. 4.31.2.

PART-5

Mobius Transformation and their Properties.

CONCEPT OUTLINE

Mobius Transformation : It is also known as bilinear transformation. Bilinear transformation is the function w of a complex variable z of the form

$$w = f(z) = \frac{az + b}{cz + d}$$

Where a, b, c, d are complex or real constants subject to $ad - bc \neq 0$.

Properties of Mobius or Bilinear Transformation :
1. Circles are transformed into circles under bilinear transformation.
2. The cross-ratio of four points is invariant under a bilinear transformation.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 4.32. How could you determine the bilinear transformation ?

Answer

1. A bilinear transformation can be uniquely determined by three given conditions. To find the unique bilinear transformation which maps three given distinct points z_1, z_2, z_3 onto three distinct images w_1, w_2, w_3, consider w which is the image of a general point z under this transformation.
2. Now by theorem 2 which states that the cross-ratio of four points is invariant under a bilinear transformation, the cross-ratio of the four point w_1, w_2, w_3, w must be equal to the cross-ratio of z_1, z_2, z_3, z. Hence the unique bilinear transformation that maps three given point z_1, z_2, z_3 on to three given images w_1, w_2, w_3 is given by,

$$\frac{(w_1 - w_2)(w_3 - w)}{(w_1 - w)(w_3 - w_2)} = \frac{(z_1 - z_2)(z_3 - z)}{(z_1 - z)(z_3 - z_2)}$$

Que 4.33. Find the bilinear transformation that maps the point 0, 1, i in z-plane onto the points $1 + i, - i, 2 - i$ in the w-plane.

Answer

The required bilinear transformation is

$$\frac{(w_1 - w_2)(w_3 - w)}{(w_1 - w)(w_3 - w_2)} = \frac{(z_1 - z_2)(z_3 - z)}{(z_1 - z)(z_3 - z_2)}$$

$$\frac{(1+i+i)(2-i-w)}{(1+i-w)(2-i+i)} = \frac{(0-1)(i-z)}{(0-z)(i-z_1)}$$

$$\frac{(1+2i)}{2}\frac{(2-i-w)}{(1+i-w)} = (i-1)\left(\frac{i-z}{z}\right)$$

$$\frac{2-i-w}{1+i-w} = \frac{2(3i+1)}{5}\left(\frac{i-z}{z}\right)$$

Solving for w,

$$5z(2-i-w) = 2(3i+1)(1+i-w)(i-z)$$

$$w = \frac{(6i+2)(1+i)(i-z)-(2-i)5z}{-5z+(6i+2)(i-z)}$$

or

$$w = \frac{z(6+3i)+(8+4i)}{z(7+6i)+(6-2i)}$$

Que 4.34. Determine the Mobius transformation having 1 and i as fixed (invariant) points and maps 0 to – 1.

Answer

The Mobius transformation having α and β as fixed points is given by

$$w = \frac{\gamma z - \alpha\beta}{z - \alpha - \beta + \gamma}$$

For $\alpha = 1, \beta = i$, we have

$$w = \frac{\gamma z - i}{z - 1 - i + \gamma}$$

Since $z = 0$ is mapped to $w = -1$,

$$-1 = \frac{0 - i}{0 - 1 - i + \gamma}$$

or

$$\gamma = 2i + 1$$

Thus the required transformation is

$$w = \frac{(2i+1)z - i}{z + i}$$

Que 4.35. Find a bilinear transformation which maps the upper half of the z-plane into the interior of a unit circle in the w-plane. Verify the transformation (Fig. 4.35.1).

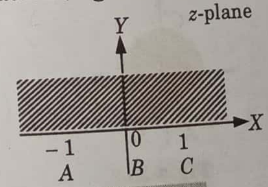


Fig. 4.35.1.

Answer

Suppose any three points in the upper half of z-plane say $A : -1, B : 0, C = 1$ gets mapped to any three points in the interior of the circle $|w| = 1$ in the w-plane, say $A^* : -i, B^* : 1, C^* : i$. Thus the required bilinear transformation is the one which maps $-1, 0, 1$ from z-plane to $-i, 1, i$ in the w-plane.

Now according to cross-ratio property,

$$\frac{(z_1 - z_2)(z_3 - z)}{(z_1 - z)(z_3 - z_2)} = \frac{(w_1 - w_2)(w_3 - w)}{(w_1 - w)(w_3 - w_2)}$$

$$\frac{(-1-0)(1-z)}{(-1-z)(1-0)} = \frac{(-i-1)(i-w)}{(-i-w)(i-1)}$$

or $$\frac{1-z}{1+z} = \frac{1+iw}{i+w}$$

On solving, $$w = \frac{i-z}{i+z}$$

Verification : $$|w| = \left|\frac{i-z}{i+z}\right| \le 1$$

or $$|i - z| \le |i + z|$$

$$\sqrt{x^2 + (1-y)^2} \le \sqrt{x^2 + (1+y)^2}$$

$$4y \ge 0$$

Thus the bilinear transformation $w = \dfrac{i-z}{i+z}$ transforms interior of unit circle in w-plane onto the upper half plane in z-plane.

Also, $$|w| = \left|\frac{i-z}{i+z}\right| = \sqrt{\frac{x^2 + (1-y)^2}{x^2 + (1+y)^2}}$$

For $y = 0$, $|w| = \sqrt{\dfrac{x^2 + 1}{x^2 + 1}} = 1$. Thus the real axis ($y = 0$) gets mapped to the unit circle $|w| = 1$.

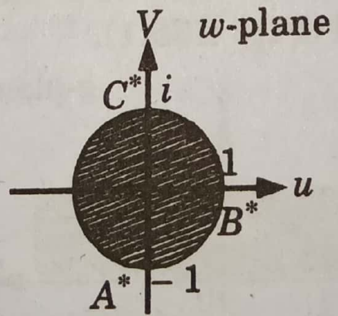


Fig. 4.35.2.

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