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Joint Control of Emissions Permit Trading and Production Involving Fixed and Variable Transaction Costs

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The use of permit markets to mitigate harmful emissions is on the rise. When participating in such a market, an emitting firm needs to acquire from it permits that cover emissions resulting from production. Thus, it has to simultaneously cope with fluctuating permit prices and random demand, and also juggle between the activities of permit trading and permit-consuming production. We shed light on this complex dynamic control problem, while confronting difficulties brought on by fixed as well as variable transaction costs associated with permit trading. We exploit K-convexity variants that are suitable for two-dimensional control, and achieve the partial characterization of optimal control policies. When the selling of permits is prohibited, we prescribe an (s, S)-type permit purchasing policy. For the more general case involving two-way trading, we find it optimal to carry out trading in a three-interval fashion. Heuristics, including one based on the uncoupling of trading and production activities, are introduced. Their effectiveness has been illustrated in computational studies.

Key words: emissions permit trading; fixed and variable transaction costs; dynamic programming; (s, S) policy; production planning
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1. Introduction

The ever-increasing emissions of harmful substances, ranging from greenhouse gases such as carbon dioxide (CO_2) to mercury-contaminated waste, if left unchecked, would threat humanity's future survival and prosperity. Emissions permits were introduced to lure profit-seeking corporations into the fight against ominous trends in rising emissions levels. A tight supply of permits on the permit market will force firms to bid up their prices and shift production to those able to extract the most benefits out of their emissions. This will in the long run incentivize firms to improve efficiencies and reduce emissions.

Industrial implementations of systems that support the issuing and trading of emissions permits include the European Union Emissions Trading Scheme (EU ETS), the Acid Rain Program in the United States, the REgional CLean Air Incentives Market (RECLAIM) in California, and the New Zealand Emissions Trading Scheme. Under the EU ETS cap-and-trade scheme for CO_2 emissions permits, for instance, a firm must cover its CO_2 emissions that exceed the cap over a year with an equivalent quantity of permits acquired from the exchange market.

Emissions permits constitute a factor of production much like commodity inputs, however, with the distinction that they have to be balanced only by the end of a certain period, say a year. Thus, firms dealing with emissions permits have maximum freedom in trading on the market while fulfilling their production obligations. Effects of an emitting firm's permit trading on production decisions have not been well understood. We thus emphasize the dynamic coordination between production and permit trading over a given time horizon. As for the firm, it can be imagined as a cement, lime, or glass producer, steel or paper mill, oil refinery, or aluminum smelter. All aforementioned sectors are energy-intensive and according to Heindl and Lutz (2012) and Jaraite and Kazukauskas (2015), contain many firms that participate in EU ETS. In fact, From 2013 onwards, EU has required all CO_2 emitters in these sectors to balance all or significant portions of their emissions from the market.

The following is a brief account of our modeling details. Being a price taker, our small- to medium-sized manufacturing firm faces a permit price that fluctuates from one period to another in a Markovian fashion. Besides trading on the market, the emitting firm also produces to meet random demand. In every period, the price of emissions permits is first observed. Then, the initial inventory level and the number of on-hand permits are reviewed. Next, the firm decides on the number of emissions permits to buy or sell and the quantity of product items to produce, with the latter action leading to a corresponding quantity of emissions release and permit consumption. Finally, demand is realized and inventory costs are assessed. The firm only needs to fulfill the outstanding emissions obligation by the end of a given planning horizon.

In the permit market, transaction costs that recur with trades should not be ignored. They result from search and information collection, bargaining and decision, and monitoring and enforcements (Hahn and Stavins 2011), and pose as major impediments to the economic performances of emissions trading systems (Krutilla and Krause 2010). According to Stavins (1995), such costs often exhibit economies of scale, and hence can be approximated by the fixed-plus-variable form. The fixed portion can come from administrative requirements, search and bargaining, along with writing and enforcing of contracts; for instance, under the EPA's Emissions Trading Program for criteria air pollutants, buyers often pay substantial fees to consultants who assist in the search for permits. Meanwhile, the variable portion can represent brokerage fees. Fixed and variable transaction costs in permit trading were treated in Singh and Weninger (2017), who focused on the interplay between production capacity and permit utilization.

The prominence of transaction costs and especially their fixed-price portions vary from one emissions trading scheme to another. Wisconsin's scheme dealing with the rights to discharge water pollutants into the Fox River failed due largely to exorbitant transaction costs; the latter were in turn caused mainly by high administrative requirements. Also, fixed transaction costs in the REgional CLean Air Incentives Market (RECLAIM) launched in 1994 were significant, prompting the non-participation of many firms; see Gangadharan (2000). From a survey of German firms that participated in EU ETS, Heindl (2012) found that 19.6% of the overall trading costs were attributable to transactions for the period of 2009–2010; all the while, average transaction costs reached 0.76 Euros per ton for ten-thousand-ton-per-year emitters. However, the Lead Permit Trading Program aimed at reducing lead levels in gasoline products experienced high trading levels thanks to low transaction costs; see Jaraite et al. (2010). Regardless of the specific situations being dealt with, by considering non-negligible fixed as well as variable transaction costs, our models would offer more flexibility and generality.

Production consumes rather than generates permits. While the option of buying permits is essential, that of selling might only be a small boost to a firm's profitability. Speculators, such as big banks and hedge funds, are active in large markets for emissions permits; see Colla et al. (2012). However, our small- to medium-sized emitting firm, whose core business competence lies in the manufacturing of its products, is certainly no match for these professional investors pursuing sustainable profits from speculation on permits. Thus, it will most likely treat trading on the emissions market as a necessity for life rather than a main source of profit.

The number of manufacturing firms that now participate in emissions permit markets is vast, with many small firms trading permits in secondary markets while using spot permit prices as benchmarks in transactions. Also, larger firms are more likely to develop abatement technologies. This makes them more likely to have surpluses of permits. As the entire market tends to be in balance by the self-adjusting permit price, shortages are more likely to accrue at smaller firms. For the latter, purchasing will likely be easier than selling—competition for the former activity is among peers while that for the latter is with titans. The extreme case would have selling becoming prohibitively expensive for our small-to-medium-sized production firm. Through computation, however, this purchase-only case can be shown to approximate less extreme cases well. Hence, we pay particular attention to the model without selling.

From the managerial perspective, it is clear that a full coordination of production and permit purchase decisions requires the integration of operations and treasury management. Then, an important issue is whether a separate or decentralized decision-making heuristic could appear competitive against the optimal joint decision of production and permit purchase. Using the CO_2 permit prices from EU market data, our numerical study shows that a heuristic allowing disjoint trading and production decisions may offer a simple and yet effective recourse, especially when the firm can monitor the emissions permit market at a high enough frequency.

Overall, we contribute to the understanding of simultaneous management of permit trading and production in a dynamic environment. We depart from existing literature on the realistic consideration of fixed as well as variable transaction costs involved in permit trading. Difficulties inherent with our non-convex multi-dimensional control problems are partially overcome by two new concepts, namely, $\mathcal{C}^2(K)$ for the purchase-only case and $C^2_{-}(K^B, K^S, q)$ for the more general two-way trading case. Preservations of these properties over periods lead to the optimality of policies that are relatively easy to implement. Indeed, the first concept results in an (s, S)-type permit purchase policy, while the second is responsible for a three-interval trading policy. Even though partially characterized, these policies are based on the echelon inventory level that we define for the problem and the policy parameters vary with finished product inventory levels. Thus, it is practically helpful that we also provide even simpler heuristics involving separately managed trading and production activities. They can serve as competitive alternatives when integrated/coordinated decisions are hard to reach.

The remainder of this study is organized as follows. We discuss the related literature in section 2 and introduce the problem formulation in section 3. We devote section 4 to the analysis of the purchase-only case. In section 5, we derive the structure of an optimal trading policy for the general two-way trading case. Heuristics are proposed in section 6. We present numerical results in section 7 and conclude the study in section 8.

2. Literature Review

This work overlaps with three research areas: the effects that emissions concerns have on supply chain management, joint considerations of commodity trading and operations management, and models that describe emissions permit price processes.

There is an emerging research stream studying the effects of emissions concerns on supply chains. Cachon (2011) discussed how efforts made in the reduction of carbon footprints would affect supply chain operations in a retailing environment. Benjaafar et al. (2013) highlighted issues having to do with the incorporation of carbon footprint considerations into supply chain management. They also identified methods for reducing carbon emissions at less or no additional costs. Caro et al. (2013) investigated how a product's total CO_2 emissions should be allocated among supply chain members.

Operations decisions under emissions trading have also been studied. In a deterministic two-period setting, Laffont and Tirole (1996) analyzed the effects of spot and futures markets for tradable pollution permits on firms' decisions concerning pollution abatement and production. Assuming that permit prices are known over a planning horizon, Dobos (2007) used the Arrow–Karlin inventory model to study the effect that emissions trading would have on production decisions. In addition, Gong and Zhou (2013) researched a firm's production planning and emissions trading under two production technologies, with the more costly option resulting in milder emissions. While they exploited convexity and modularity properties, we are compelled by the consideration of fixed as well as variable transaction costs in two-dimensional settings to identify new properties rooted in the traditional one-dimensional concept of K-convexity.

To the best of our knowledge, the two-dimensional concept $\mathcal{C}^2(K)$ for the purchase-only case has not appeared in literature. On the other hand, our derivation on the two-way trading case is related to the recently developed notion of weak (K^B, K^S)-convexity. The two-dimensional functions in $\mathcal{C}^2_{-}(K^B, K^S, q)$ have one dimension that satisfies a stronger version of this property. In their study of a capacity management problem in which fixed costs are involved in both the purchase and selling of capacities, Ye and Duenyas (2007) introduced the notion of (K^B, K^S)-concavity/convexity. Semple (2007) discovered that a weaker version, the weak (K^B, K^S)-concavity/convexity, could perform almost the same trick when it comes to policy characterization.

Given the freedom to balance permits only at the end of a horizon, permit trading for an emitter is notably different from commodity trading for a manufacturer who constantly requires a positive stock of some commodity for input. Still, the literature on production/operations models involving commodity procurement decisions provides certain technical insights into the analysis of the problem that we study here. There is a growing body of works on commodity procurement with uncertainties in both purchase price and demand; e.g., Haksoz and Seshadri (2007), Ding et al. (2007), Yang and Xia (2009), Secomandi (2010), and Berling and Martinez-de-Albeniz (2011).

The last relevant pocket of literature comprises studies on emissions permit prices. Permits can be sold on spot markets or via auctions. Clearing prices from auctions have been observed to be almost identical to spot prices during auctions in recent years; e.g., UK Page (2016). Benz and Trück (2009) tested several Markov switching and AR-GARCH models based on the EU trading data. Similar studies were also conducted by Paolella and Taschini (2008) and Seifert et al. (2008).

Through a multi-firm emissions trading and abatement model, Carmona et al. (2009) showed that the equilibrium permit price evolves as a martingale process under a suitable measure $\mathbb{Q}$. But this measure is generally different from the empirical measure $\mathbb{P}$ that can be estimated using statistical tools. Carmona and Hinz (2011) studied the calibration of emissions permit prices and the valuation of derivatives based on such prices. For more on the principles and practices of emissions/pollutants trading markets, the reader may turn to Hansjürgens (2005) and Tietenberg (2006).

3. The Purchase-Only Case

We now set up the problem involving permit purchase only. As alluded to in section 1, such a model is a good approximation to reality. The more general case involving two-way trading is discussed in section 5.

3.1. Problem Description

Our control problem covers regular periods $1, 2, \ldots, T$ and a terminal period $T + 1$. The firm faces random discrete customer demand D_t in each of the periods $t = 1, 2, \ldots, T$. The discrete-item assumption has alleviated us of the need to worry about continuity issues pertaining to the two-dimensional properties that will appear later. Note that continuous-item models involving the traditional single-dimensional K-convexity property have to concern themselves with continuities or even differentiabilities of cost-to-go functions. Here, such properties would not simplify our derivation.

Being a small-to-medium-sized price taker, the firm has no way of influencing the global permit market through its actions. We suppose that market-wise permit prices P_t for periods $t = 1, 2, \ldots, T, T + 1$ form an exogenous Markov process. When conditioned on P_t being any particular p, the two random variables P_{t+1} and D_t are independent of each other. Thus, while demand D_t might be contingent on permit price P_t, it does not interfere in the transition from P_t to P_{t+1}. The potential dependence of D_t on P_t allows P_t to pose as an economic indicator and D_t to be driven by the economy.

At the beginning of each period, the price p of emissions permits is observed and the number of on-hand permits u and the initial inventory level x are reviewed. If the period is non-terminal, the firm decides how many emissions permits to buy and how many product items to produce. At the end of each period, customer demand is realized and inventory costs are assessed. If the period is terminal, the firm uses production to bring its product inventory level back to zero and uses purchase to fulfill its permit obligations. Without loss of generality, we suppose that each product consumes one emissions permit. We allow the permit position u to be of either sign. A positive u means credit in permits and a negative u means permits owed in production.

The firm's permits purchase activities involve both fixed and variable transaction costs. First, it will shoulder a fixed transaction cost $K^B \geq 0$ in any period with purchase. Second, the firm will shell out some $q^B \geq 0$ for every purchased permit on top of the price p paid to the seller; e.g., in the major emissions exchanges house ecc.de, such a clearing fee is charged. To achieve a post-purchase permit level w, the total permit purchase cost will be $K^B\delta(w - u) + (p + q^B)(w - u)$, in which $\delta(x) = 1$ when $x \geq 1$ and $\delta(x) = 0$ when $x \leq 0$. In the current purchase-only case, the positivity of q^B will not make any structural difference as $p + q^B$ can be treated as the "effective" price. However, the positivity of K^B will necessitate innovations on structural properties.

Demand is fulfilled to the maximum extent by on-hand inventory after production, and any unmet demand is backlogged. Let $I(y, p)$ be the expected one-period inventory holding/backlogging cost if the firm produces up to y in a period with price p:

$$I(y,p) = \mathbb{E}[h(y - D_t)^+ + b(D_t - y)^+ | P_t = p].$$

The function is assumed to be convex in y. This covers the case where there are unit holding and backlogging cost rates h and b, respectively. We also let c be the unit production cost. When the post-production inventory level is y for the period, the firm will experience production cost $c(y - x)$. Let $\alpha \in [0, 1]$ be the discount rate per period.

Throughout the study, cost parameters and price–demand dynamics can be time-varying, with the general understanding that the discounted fixed and variable transaction costs be decreasing over time. Later, part (a) of both Lemmas 1 and 4 will help preserve desired properties for this case. However, for notational simplicity, we suppress the time subscript t whenever confusion shall not arise from the context.

3.2. Initial Formulation

The firm's goal is to control post-purchase permit levels and post-production inventory levels so that the expected total discounted cost is minimized. To this end, we denote the minimum level of such cost from period t to $T + 1$ by $\tilde{V}_t(u, x, p)$, when in period t, u is the initial permit level, x the starting inventory level, and p the current permit price.

At the beginning of the terminal period $T + 1$, we suppose that the firm must restore its permit level to the non-negative territory. Also, each backlogged demand must be produced. This c-costing process also consumes a permit. Thus, with x^- as its production quantity, the firm needs to make a purchase of $(x^- - u)^+$ permits. In a regular period $t = 1, \ldots, T$, an item backlogged costs the firm a delay penalty b for that period, with the understanding that the firm will eventually fulfill the production obligation, in either a later regular period or the terminal period.

In the terminal period $T + 1$, no further deferment is possible. So the firm has to eke out some $c + \mu$ per unit to finish production, with $\mu \geq 0$ standing for the extra cost associated with expedited production. We also suppose that any leftover product item is worth some $c - v$, with $v \geq 0$ reflecting the difference between a unit's cost and its salvage value. For leftover permits, we let their total value be the firm's earning through market liquidation whenever it is profitable. As selling is prohibited right now, the leftovers will be effectively worthless. Therefore,

$$\tilde{V}_{T+1}(u,x,p) = K^B\delta(x^- - u) + (p+q^B)(x^- - u)^+ - cx + \mu x^- + vx^+. \tag{1}$$

In Equation (1), we have utilized the facts that $\delta((x^- - u)^+) = \delta(x^- - u)$ and $cx^- - cx^+ = -cx$. Also, note that production costs $(c + \mu)x^-$ and salvage brings in $(c - v)x^+$.

For periods $t = T, T - 1, \ldots, 1$, however,

$$\begin{aligned}\tilde{V}_t(u,x,p) = \min_{w\geq u, y\geq x}\{&K^B\delta(w-u) + (p+q^B)(w-u)\\ &+ c(y-x) + I(y,p) + \alpha\mathbb{E}[\tilde{V}_{t+1}(w-(y-x),\\ &y - D_t, P_{t+1})|P_t = p]\}.\end{aligned} \tag{2}$$

The part $(w - (y - x), y - D_t, P_{t+1})$ of Equation (2) inside $\tilde{V}_{t+1}(\cdot)$ is due to the consumption of $y - x$ permits via production, inventory depletion by demand D_t, and time evolution of the permit price to P_{t+1}. If permits were exactly the same as ordinary input factors, we would have requested w to be above $y - x$, that there be enough post-purchase permits to cover production. Here, though, we only require $w \geq u$ due to the prohibition on selling. During a regular period, the permit level could go negative.

3.3. A Transformation

Both decision variables w and y appear in the state transition $u' = w - (y - x)$, where u' stands for the next-period starting permit level. This can be simplified by letting

$$e = u + x, \quad \text{and} \quad m = w + x. \tag{3}$$

Here, e can be understood as the "echelon" inventory level if the permit level is viewed as an inventory level at the upper stream of the two-stage production inventory system; similarly, m can be viewed as the post-purchase "echelon" inventory level. With this transformation, $m - e$ will be the purchase quantity in a period.

Let us define $V_t(e, x, p)$ by

$$V_t(e,x,p) = cx + \tilde{V}_t(e-x,x,p), \tag{4}$$

where the new cost-to-go function can be understood as having roughly captured the cost of building up the initial product inventory position x. Then, after going through the transformation involving Equations (3) and (4), the dynamic program (DP) comprising Equations (1) and (2) can be re-expressed as

$$V_{T+1}(e,x,p) = K^B\delta(x^+ - e) + (p+q^B)(x^+ - e)^+ + \mu x^- + vx^+; \tag{5}$$

and, for $t = T, T - 1, \ldots, 1$,

$$V_t(e,x,p) = -(p+q^B)e + \min_{m\geq e, y\geq x}\left[K^B\delta(m-e) + G_t(m,y,p)\right], \tag{6}$$

where

$$G_t(m,y,p) = (p+q^B)m + H(y,p) + \alpha\mathbb{E}[V_{t+1}(m-D_t, y-D_t, P_{t+1})|P_t = p]. \tag{7}$$

In the above, we have let

$$H(y,p) = \alpha c\mathbb{E}[D_t|P_t = p] + (1-\alpha)cy + I(y,p).$$

To see how Equations (6) and (7) come into being, note that

$$\begin{aligned}V_t(e,x,p) &= cx + \tilde{V}_t(e-x,x,p)\\ &= cx + \min_{w\geq e-x, y\geq x}\{K\delta(w-e+x) + (p+q^B)(w-e+x)\\ &\quad + c(y-x) + I(y,p) + \alpha\mathbb{E}[\tilde{V}_{t+1}(w-(y-x),\\ &\quad y-D_t, P_{t+1})|P_t = p]\} = -(p+q^B)e +\\ &\quad \min_{m\geq e, y\geq x}\{K\delta(m-e) + (p+q^B)m + cy + I(y,p)\\ &\quad + \alpha\mathbb{E}[\tilde{V}_{t+1}(m-y, y-D_t, P_{t+1})|P_t = p]\}\\ &= -(p+q^B)e + \alpha c\mathbb{E}[D_t|P_t = p]\\ &\quad + \min_{m\geq e, y\geq x}\{K\delta(m-e) + (p+q^B)m + (1-\alpha)cy\\ &\quad + I(y,p) + \alpha\mathbb{E}[V_{t+1}(m-D_t, y-D_t, P_{t+1})|P_t = p]\},\end{aligned}$$

where the first equality comes from Equation (4), the second equality uses Equation (2) with $u = e - x$, the third equality is due to Equation (3), and the fourth equality relies on Equation (4) again.

To see the last point, note Equation (4) is equivalent to $\tilde{V}_t(u, x, p) = -cx + V_t(u + x, x, p)$. But this will lead to

$$\begin{aligned}&\alpha\mathbb{E}[\tilde{V}_{t+1}(m-y, y-D_t, P_{t+1})|P_t = p]\\&= -\alpha cy + \alpha c\mathbb{E}[D_t|P_t = p] + \alpha\mathbb{E}[V_{t+1}\\&\quad(m-D_t, y-D_t, P_{t+1})|P_t = p],\end{aligned}$$

where we have utilized the fact that $m - D_t = (m - y) + (y - D_t)$. In Equation (6), there would have been the requirement that $m = w + x \geq y = (y - x) + x$ were permits to act like regular input factors that were immediately required at production. The absence of such a complicating requirement is one of the enablers for our ensuing structural results. Another simplifying factor is that the term $G_t(m, y, p)$ extracted out for Equation (6), as detailed in Equation (7), is independent of both state variables e and x.

We may lexicographically denote the largest optimal solution to Equation (6) at period t and state (e, x, p) by $(m_t^*(e, x, p), y_t^*(e, x, p))$. Among all optimal solutions, $m_t^*(e, x, p)$ has the highest post-purchase echelon inventory level and among all optimal solutions with that post-purchase echelon inventory level, $y_t^*(e, x, p)$ achieves the highest post-production inventory level. Our main task is to characterize $m_t^*(\cdot, \cdot, p)$ and $y_t^*(\cdot, \cdot, p)$.

The use of echelon levels to simplify the essentially two-installation setup certainly reminds one of the legendary work of Clark and Scarf (1960) on multi-echelon inventory systems. Actually, our problem is simpler in regular periods due to the absence of the requirement $e \geq x$; yet, it is more complex in the terminal period. The entanglement of e and x in period T+1 as shown in Equation (5) will be passed through backward induction all the way to period 1. This renders optimal policies not as clean-cut as one might expect from our formulation's resemblance to the earlier work, which achieved almost complete decomposition between different echelons.

Indeed, we have numerical examples on the following: (i) that $m_t^*(e, x, p)$ can depend on x in non-monotone ways, (ii) that $y_t^*(e, \cdot, p)$ is not always base-stock, and (iii) that $y_t^*(e, x, p)$ can depend on e in non-monotone ways. This being said, we do manage to identify practically important characteristics for both $m_t^*(\cdot, \cdot, p)$ and $y_t^*(\cdot, \cdot, p)$. A new property on cost-to-go functions turns out to be essential to this task.

4. Purchase-Only Analysis

We now introduce a two-dimensional concept that fuses together both the traditional K-convexity and pure convexity notions. Using it, we show that $m_t^*(\cdot, \cdot, p)$ exhibits (s, S)-like traits and that $y_t^*(\cdot, \cdot, p)$ is partially characterizable as base-stock.

4.1. The $\mathcal{C}^2(K)$ Class of Functions

Due to the fixed setup cost K^B for purchase, properties like monotonicity, pure convexity, and supermodularity are not preserved in our setting. So the general work of Smith and McCardle (2002) on stochastic dynamic programming would not help here. Existing multi-dimensional extensions of K-convexity, such as the ones studied by Johnson (1967), Kalin (1980), and Gallego and Sethi (2005), are not known to lead to (s, S)-type policies. So we propose a new function class $\mathcal{C}^2(K)$ and prove its relevant preservation properties.

Define subset Σ_1 of $\mathbb{Z}^2$, so that

$$\Sigma_1 = \{(e, x) \in \mathbb{Z}^2 | e \geq x^+\}.$$

On top of that, define another subset of $\mathbb{Z}^2$:

$$\Sigma_2 = \Sigma_1 \cup (\mathbb{Z}_-)^2 = \{(e, x) \in \mathbb{Z}^2 | e^+ \geq x\},$$

where $\mathbb{Z}_-$ is the set of non-positive integers. As complements to Σ_1 and Σ_2, we have

$$\begin{aligned}\Sigma_1^c &= \{(e, x) \in \mathbb{Z}^2 | e \leq x^+ - 1\},\\ \Sigma_2^c &= \{(e, x) \in \mathbb{Z}^2 | e^+ \leq x - 1\}.\end{aligned}$$

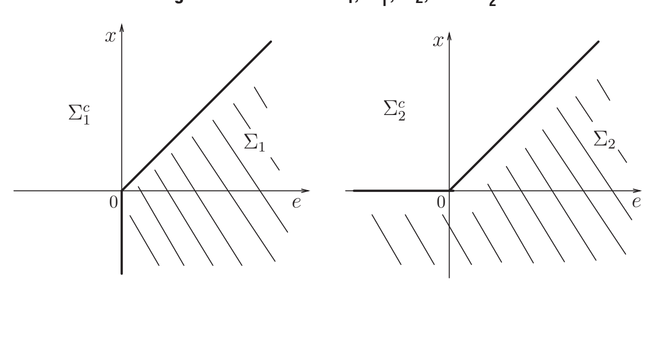
Figure 1 offers a depiction of these subsets. We now come to the definition of function class $\mathcal{C}^2(K)$. In it, both (e_1, x_1) and (e_2, x_2) are assumed to come from $\mathbb{Z}^2$. In addition, Z_{++} represents the set of strictly positive integers.

Definition 1. *Let $\mathcal{C}^2(K)$ be the set of functions $f: \mathbb{Z}^2 \to \mathbb{R}$ that satisfy the following:*

(i) When $\lambda \in \mathbb{Z}_{++}$, $e_1 \leq e_2$, and either $(e_1, x_1) \in \Sigma_1^c$ or $(e_2, x_2) \in \Sigma_1$,

$$f(e_1+1, x_1) - f(e_1, x_1) \leq \frac{f(e_2+\lambda, x_2) - f(e_2, x_2) + K}{\lambda}.$$

Figure 1 Subsets Σ_1, Σ_1^c, Σ_2, and Σ_2^c



(ii) When $x_1 \le x_2$ and $(e_1, x_1+1) \in \Sigma_2$,

$$f(e_1,x_1+1)-f(e_1,x_1) \le f(e_2,x_2+1)-f(e_2,x_2).$$

(iii) When $\lambda \in \mathbb{Z}_{++}$, $e_1 \le e_2$, $(e_1, x_1) \in \Sigma_1$, and $(e_2, x_2) \in \Sigma_1^c$,

$$f(e_1+1,x_1+1)-f(e_1,x_1) \le \frac{f(e_2+\lambda,x_2+\lambda)-f(e_2,x_2)+K}{\lambda}.$$

Functions within $\mathcal{C}^2(K)$ may be understood as possessing "projected" K-convexity and convexity properties. Property (i) is related to K-convexity being projected in the direction of permit levels: when $x_1 = x_2$, it is K-convexity in e. Meanwhile, property (ii) is about convexity being projected in the direction of inventory levels: when $e_1 = e_2$, it merely expresses convexity in x. Finally, property (iii) is related to K-convexity being projected in the direction of simultaneous echelon and product inventory moves.

There can be a geometric understanding to (i). Let components of the three-dimensional real space $\Re^3$ be denoted as $(\underline{e}, \underline{x}, \underline{g})$. At any fixed integers e_1, x_1, and e_2 satisfying $e_2 \ge e_1$, consider the two-dimensional surface

$$\mathbf{S}(e_1,x_1,e_2) \equiv \Big\{(\underline{e},\underline{x},\underline{g}) \in \Re^3 | \underline{g} = f(e_2,\underline{x}) + [f(e_1+1,x_1) - f(e_1,x_1)](\underline{e}-e_2)\Big\}.$$

At each fixed x_2, the entity $\mathbf{W}(x_2) \equiv \{(\underline{e}, \underline{x}, \underline{g}) \in \Re^3 | \underline{x} = x_2\}$ defines a "vertical wall whose face separates front from back." The intersection $\mathbf{S}(e_1, x_1, e_2) \cap \mathbf{W}(x_2)$ between the surface and the wall, meanwhile, is a straight line say $\mathbf{L}(e_1, x_1, e_2, x_2)$ that passes through $(e_2, x_2, f(e_2, x_2))$ with gradient $f(e_1+1, x_1) - f(e_1, x_1)$ in the "left-right" $\underline{e}$-direction. Now move any λ distance to the "right" of (e_2, x_2) and reach $(e_2+\lambda, x_2)$. As long as $f(e_2+\lambda, x_2)$ is given a K-sized lift to $f(e_2+\lambda, x_2)+K$, property (i) says that the point $(e_2+\lambda, x_2, f(e_2+\lambda, x_2)+K)$ will be "above" the line $\mathbf{L}(e_1, x_1, e_2, x_2)$ on the wall when either $(e_1, x_1) \in \Sigma_1^c$ or $(e_2, x_2) \in \Sigma_1$.

Properties (ii) and (iii) have similar interpretations. For the former, we need a vertical wall to separate "left" from "right," but no more lifting; for the latter, we need a vertical wall that is half way between the previous two and still the K-sized lifting. They also have their own peculiar requirements on (e_1, x_1) and (e_2, x_2). Typical functions $f \in \mathcal{C}^2(K)$ include $f(e, x) = K\delta(\max\{e, x^+\} - e)$, $f(e, x) = K\delta(e) + \max\{e, x^+\}$, $f(e,x) = \max\{e+x^-, |x|\}$, and $f(e,x) = K\delta(e^+ - e) + |e|^l - x$ for $l \in (1, +\infty)$. Here are some useful results pertaining to the $\mathcal{C}^2(K)$ class of functions.

Lemma 1. *The following properties are all true.*

(a) $\mathcal{C}^2(K) \subseteq \mathcal{C}^2(K')$ for $K \le K'$; in particular, $\mathcal{C}^2(0) \subseteq \mathcal{C}^2(K)$ for any $K \ge 0$.
(b) If $f \in \mathcal{C}^2(K)$, $f' \in \mathcal{C}^2(K')$, and $\beta, \beta' \ge 0$, then $\beta f + \beta' f' \in \mathcal{C}^2(\beta K + \beta' K')$.
(c) If $f \in \mathcal{C}^2(K)$ and Ψ is a positive- and integer-valued random variable, then for $g(e,x) = \mathbb{E}[f(e-\Psi, x-\Psi)]$, we have $g \in \mathcal{C}^2(K)$ provided that $\mathbb{E}[|f(e-\Psi, x-\Psi)|] < +\infty$.
(d) Functions in $\mathcal{C}^2(K)$ are separable on Σ_2. That is, for any $f \in \mathcal{C}^2(K)$, there exist functions $f^1(\cdot)$ and $f^2(\cdot)$, so that $f(e, x) = f^1(e) + f^2(x)$ for $(e, x) \in \Sigma_2$.

In Lemma 1, parts (a) to (c) are the "usual" K-convexity preservation properties. Part (d) stems completely from the nature of the Σ_2 set and property (ii) of the $\mathcal{C}^2(K)$-membership. When $(e_1, x_1+1) \in \Sigma_2$ and $e_2 \ge e_1$, all three points (e_1, x_1), (e_2, x_1), and (e_2, x_1+1) are also in Σ_2. Then, property (ii) would imply that, within Σ_2, f's difference with respect to e (or x) is independent of x (or e). When $(e, x) \in \Sigma_2$, property (i) would ensure that $f^1(e)$ is K-convex and property (ii) would ensure that $f^2(x)$ is convex. We caution that $\mathcal{C}^2(0)$ has left out many jointly convex functions. For instance, $f(e, x) = (e+x)^2$ is jointly convex, but nowhere separable.

The minimization involved in Equation (6) can be abstracted into

$$g(e,x) = \min_{m \ge e, y \ge x} [K\delta(m-e) + f(m,y)]. \tag{8}$$

Our most technical result is concerned the above optimization.

Lemma 2. *If $f \in \mathcal{C}^2(K)$, then g as defined in Equation (8) belongs to $\mathcal{C}^2(K)$ as well.*

Lemmas 1 and 2 would together lead to structural results on cost-to-go functions involved in the optimization problem (5)–(7). The $\mathcal{C}^2(K)$-preservation result of Lemma 2 under Equation (8)-like optimization might also find applications elsewhere.

4.2. (s, S)-Type Policy Implications

Let $\mathcal{Z}$ be the subset of $(\mathbb{Z}^2)^{\mathbb{Z}^2}$ that contains all functions $(m, y) \equiv (m(e, x), y(e, x) | (e, x) \in \mathbb{Z}^2)$ from $\mathbb{Z}^2$ to $\mathbb{Z}^2$, such that $m(e, x) \ge e$ and $y(e, x) \ge x$. The thus defined $\mathcal{Z}$ is related to joint control policies for Equation (8) and by association, those for Equation (6). Also, let $\mathcal{Z}^B$ be the subset of policies (m, y) in $\mathcal{Z}$ that never allow $e+1 \le m(e, x) \le (y(e, x))^+ - 1$ to happen for any $(e, x) \in \mathbb{Z}^2$. A policy $(m, y) \in \mathcal{Z}^B$ always has $m(e, x) \ge (y(e, x))^+$ whenever $m(e, x) \ge e+1$. That is, whenever the purchase of permits occurs, such a policy ensures that it is done adequately to uphold the positivity of both the echelon inventory and post-production permit levels.

Consider $f \in \mathcal{C}^2(K)$. Let $(m^*(e, x), y^*(e, x))$ be lexicographically the largest optimal solution to Equation (8) at each (e, x), and let $(m^*, y^*) \in \mathcal{Z}$ be the policy $(m^*(e, x), y^*(e, x)|(e, x) \in \mathbb{Z}^2)$ thus formed. Through the properties of $\mathcal{C}^2(K)$ previously derived, we can achieve the following result about the joint purchase-production strategy.

LEMMA 3. *Let $f \in \mathcal{C}^2(K)$ and $(m^*, y^*) \in \mathcal{Z}$ be defined as above. Suppose furthermore, that $(m^*, y^*) \in \mathcal{Z}^B$, then $m^*(e, x)$ must be of the (s, S)-type at a fixed x: there exist $s(x)$ and $S(x)$ with $s(x) \le S(x) - 1$, such that $m^*(e, x) = S(x)$ for $e \le s(x)$ and $m^*(e, x) = e$ otherwise. Let (e^0, x^0) be lexicographically the largest minimum point of f. Then, it follows that $e^0 = S(x^0)$ and*

$$(m^*(e,x), y^*(e,x)) = \begin{cases} (e^0 \text{which is } S(x^0), x^0), & \text{if } e \le s(x) \text{ and } x \le x^0, \\ (S(x), x), & \text{if } e \le s(x) \text{ and } x \ge x^0 + 1, \\ (e, y^*(e,x)), & \text{if } e \ge s(x) + 1. \end{cases}$$

Suppose (m^*, y^*) is indeed in $\mathcal{Z}^B$. Then according to Lemma 3, $m^*(e, x)$ is of the (s, S)-type at each fixed x, so that for $e \le s(x)$, we would have $m^*(e, x) = S(y^*(e, x))$ where $y^*(e, x) = x \vee x^0$ is of the base-stock type; at this time, the post-purchase echelon level is greater than $(y^*(e, x))^+ = (x \vee x^0)^+$ and further x^+. Therefore, for any x, we have $(S(x), x) \in \Sigma_1$. As $e^0 = S(x^0)$, the point (e^0, x^0) belongs to Σ_1 as well.

Because $(e^0, x^0) = (S(x^0), x^0)$ constitutes a global minimum for f, it is a natural target for the joint purchase–production effort when the starting echelon inventory level e is low enough and the starting product inventory level x is below x^0. At a higher product inventory level x but still sufficiently low echelon level, production will be halted with purchase re-aimed at an x-adaptive target $S(x)$. When the echelon level e rises above the low bar $s(x)$, however, purchase will be stopped. For convenience, we call the policy (m^*, y^*) specified by Lemma 3 an $(s(\cdot), S(\cdot), x^0)$-policy. Here, the production portion $y^*(e, x)$ for $e \ge s(x) + 1$ is left unspecified.

4.3. Control Problem at Hand

We now present findings on the joint control problem at hand. First, we show that $V_{T+1}(\cdot, \cdot, p)$ as defined by Equation (5) is a member of $\mathcal{C}^2(K^B)$. Due to the transformation (3) and (4), this certainly has much to do with Equation (1) for the initial $\tilde{V}_{T+1}(\cdot, \cdot, p)$.

PROPOSITION 1. *At each price level p, it follows that $V_{T+1}(\cdot, \cdot, p) \in \mathcal{C}^2(K^B)$.*

With Proposition 1 as a starting point, we can use mathematical induction on the relationships (6) and (7) to verify the $\mathcal{C}^2(K^B)$-membership of all the remaining cost-to-go functions for $t = T, T-1, \ldots, 1$.

THEOREM 1. *At each price level p, the cost-to-go functions $G_t(\cdot, \cdot, p)$ and $V_t(\cdot, \cdot, p)$ for $t = T, T-1, \ldots, 1$ are all members of $\mathcal{C}^2(K^B)$.*

The key step of the proof is to show that $\mathcal{C}^2(K^B)$-membership is preserved under the recursive process of Equations (6) and (7) that is traceable back to Equation (2). It is no surprising that Lemma 1 on rudimentary $\mathcal{C}^2(K^B)$-preservation properties can be relied on. Due to Equation (6)'s abstraction into Equation (8), Lemma 2 will provide the most critical support for preservation.

With cost-to-go functions' $\mathcal{C}^2(K^B)$-membership now settled, we move on to the optimal policy, namely $(m_t^*(\cdot, \cdot, p), y_t^*(\cdot, \cdot, p))$ at each fixed permit price p, as defined in section 3.3. To take advantage of Lemma 3, it is important that the policy's membership in the subset $\mathcal{Z}^B$, first defined in section 4.2, be established. To this end, define functions $\pi_t^B(\cdot)$ iteratively, so that $\pi_{T+1}^B(p) = p + q^B$ and for $t = T, T-1, \ldots, 1$,

$$\pi_t^B(p) = \min\{p + q^B, \alpha\mathbb{E}[\pi_{t+1}(P_{t+1})|P_t = p]\}.$$

Note that $\pi_t^B(p)$ can be seen as the expectation of the best price to buy a permit from period t to period T. Its definition helps to characterize an intermediate result.

PROPOSITION 2. *The following properties are all true. (U^B) For $t = T+1, T, \ldots, 1$, $V_t(e, x, p) - V_t(e + \lambda, x, p) \le K^B + \lambda\pi_t^B(p)$ for $\lambda \in \mathbb{Z}_{++}$. ($U'^B$) For $t = T, T-1, \ldots, 1$, $m_t^*(e, x, p) \ge e + 1$ would imply*

$$p + q^B \le \alpha\mathbb{E}[\pi_{t+1}^B(P_{t+1})|P_t = p].$$

(L^B) For $t = T+1, T, \ldots, 1$, $V_t(e, x, p) - V_t(e + 1, x, p) \ge \pi_t^B(p)$ for $e \le x^+ - 1$. At each fixed p, let $(m_t^*(p), y_t^*(p))$ be the function from $\mathbb{Z}^2$ to $\mathbb{Z}^2$ that is formed from the optimal decisions $(m_t^*(e, x, p), y_t^*(e, x, p))$ at various (e,x) locations. A consequence of the above is that $(m_t^*(p), y_t^*(p)) \in \mathcal{Z}^B$.

We may understand Proposition 2 as stating the upper and lower bounds on the marginal value of emissions permits. In the result, property (U^B) offers an upper bound on the marginal value for all occasions. This bound is then responsible for property (U'^B), which says that the firm will not be tempted to purchase additional permits unless the current effective price is below the average best effective price in the future.

Since $\pi^B_{t+1}(p) \le p + q^B$, the necessary condition will lead to $p + q^B \le \alpha(\mathbb{E}[P_{t+1}|P_t = p] + q^B)$. Therefore, permit purchase will occur at price p in a non-terminal period t only if the just-stated more necessary condition is true. One extreme consequence is that, when the discounted effective price process forms a strict super-martingale so that $p + q^B > \alpha(\mathbb{E}[P_{t+1}|P_t = p] + q^B)$ all the time, the firm will wait till the terminal period to make all its permit purchases.

In result (L^B), the condition $e < x^+$ describes the situation where the firm's emissions credits are either not able to cover past production ($e < x$ and $x \ge 0$) or are in the negative territory along with the inventory level ($e < 0$ and $x < 0$). The property naturally says that when short on emissions credits, the firm would value emissions permits highly, indeed higher than the average best prices to come in the future.

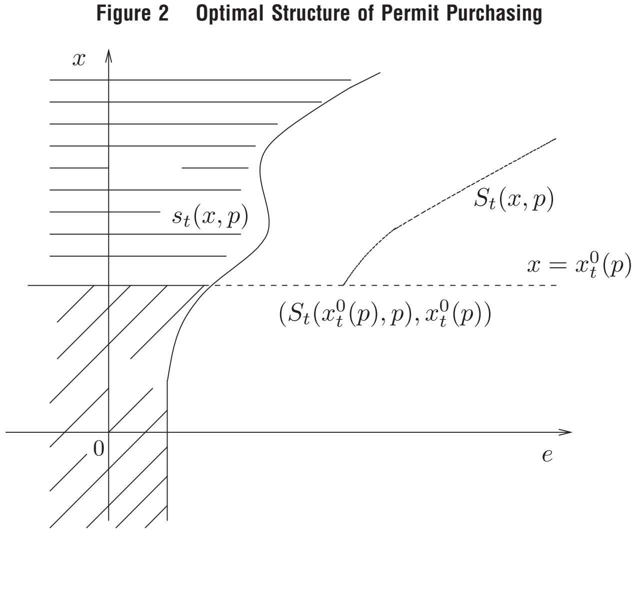
With $G_t(\cdot, \cdot, p) \in \mathcal{C}^2(K^B)$ settled by Theorem 1 and $(m^*_t(p), y^*_t(p)) \in \mathcal{Z}^B$ established at Proposition 2, we can call on Lemma 3 to partially characterize one of the firm's optimal joint purchase and production control policies.

Theorem 2. *At each $t = T, T-1, \ldots, 1$ and permit price p, an optimal policy $(m^*_t(p), y^*_t(p))$ for the control problem (6) can be described as follows. There exist in the $\mathbb{Z}^2$-plane curves $e = s_t(x, p)$ and $e = S_t(x, p)$, with $s_t(x, p) \le S_t(x, p) - 1$ and $(S_t(x, p), x) \in \Sigma_1$ for every $x \in \mathbb{Z}$, and point $x^0_t(p) \in \mathbb{Z}$, so that $(m^*_t(p), y^*_t(p))$ is an $(s_t(\cdot, p), S_t(\cdot, p), x^0_t(p))$-policy as defined in Lemma 3.*

The above partially depicts the firm's t- and p-dependent joint permit purchase and production policy, which is illustrated in Figure 2. When the starting echelon level e is below some x-dependent permit repurchase triggering point $s_t(x, p)$, the production policy $y^*_t(e, \cdot, p)$ is of the base-stock type with some e-independent $x^0_t(p)$ serving as the base level: $y^*_t(e, x, p) = x \vee x^0_t(p)$. At this time, purchase would target some echelon level $S_t(x \vee x^0_t(p), p)$. Note this conveys more information than merely setting the target echelon level at some $S'_t(x, p)$—we can be sure that the target level $S_t(x^0_t(p), p)$ would be fixed for any $x \le x^0_t(p)$. On the other hand, when the starting echelon level e is above $s_t(x, p)$, there would be no permit purchase; however, the production policy from such a starting state is thus far unspecified.

Computation has shown that Theorem 2 is hard to be further improved: $s_t(x, p)$ is not necessarily monotone let alone flat in x; $y^*_t(e, x, p)$ used in the third case of Lemma 3 is not necessarily monotone let alone flat in e; its dependence on x is not necessarily of the base-stock type either. Thus far, it remains unanswered as to whether the target echelon level $S_t(x, p)$ is monotone in x. All of our numerical examples confirm that such is the case, and yet we have not been able to prove it as a fact.

Figure 2 Optimal Structure of Permit Purchasing



5. The Case with Two-Way Trading

We now suppose that trading can occur both ways, effectively allowing finite fixed transaction cost K^S and variable transaction cost q^S for selling. Effective prices experienced by our firm would fluctuate along with the market price, with $P_t + q^B$ understandable as the effective purchase price and $P_t - q^S$ the effective selling price. This makes our variable-cost/price portion not much different from what has been assumed by other authors say Gong and Zhou (2013). Indeed, if P_t as a process forms a martingale, then

$$\begin{aligned} p + q^B &= \alpha^\tau \mathbb{E}[P_{t+\tau}|P_t \\ &= p] + q^B \ge \alpha^\tau \mathbb{E}[P_{t+\tau} - q^S|(P_t + q^B, P_t - q^S) \\ &= (p + q^B, p - q^S)], \end{aligned}$$

and

$$\begin{aligned} p - q^S &= \alpha^\tau \mathbb{E}[P_{t+\tau}|P_t \\ &= p] - q^S \le \alpha^\tau \mathbb{E}[P_{t+\tau} + q^B|(P_t + q^B, P_t - q^S) \\ &= (p + q^B, p - q^S)], \end{aligned}$$

just because q^B and q^S are positive. But these correspond to enabling inequalities for Gong and Zhou (2013); see their (1). Even without the martingale assumption, we would verify the effectiveness of a three-interval policy that advises purchasing at low echelon inventory levels, staying put at medium levels, and selling at high echelon levels.

5.1. Problem Formulation

As earlier, in the terminal period $T + 1$, any backlog has to be cleared at the production cost of $c + \mu$ per unit; meanwhile, any leftover product item has a salvage value of $c - v$. If the permit level in period $T+1$ is insufficient to cover the end-of-horizon production, that is, if $u \le x^- - 1$, the firm will face the same situation as in the purchase-only case. If there are leftover permits, however, the firm will now have the option of selling them if doing so is profitable. Therefore,

$$\begin{aligned}\tilde{V}_{T+1}(u,x,p) = {} & K^B(x^- - u) + (p+q^B)(x^- - u)^+ \\ & - \left[(p-q^S)(u-x^-)^+ - K^S\delta(u-x^-)\right]^+ \\ & - cx + \mu x^- + vx^+. \end{aligned} \tag{9}$$

Note that $(p - q^S)(u - x^-)^+ - K^S$ is the potential profit the firm can make by selling permits after catering to backlog-induced production. When K^S approaches $+\infty$, the term involving K^S and q^S will disappear, and Equation (9) will revert back to the purchase-only case of (1). For periods $t = T, T-1, \ldots, 1$, we have $\tilde{V}_t(u, x, p)$ being equal to

$$\begin{aligned}\min_{w \text{ free}, y \ge x} & \{K^B\delta(w-u) + K^S\delta(u-w) + (p+q^B)(w-u)^+ \\ & + c(y-x) - (p-q^S)(u-w)^+ + I(y,p) \\ & + \alpha\mathbb{E}[\tilde{V}_{t+1}(w-(y-x), y-D_t, P_{t+1})|P_t = p]\}. \end{aligned} \tag{10}$$

Here, w is free due to the possibility of two-way trading. Besides the fixed transaction costs, the firm needs to pay $p + q^B$ for every additional permit purchased and effectively earns $p - q^S$ for every additional permit sold. When K^S approaches $+\infty$, any choice with $w \le u - 1$ will become utterly unattractive and Equation (10) will revert back to Equation (2). Going through the same transformation involving Equations (3) and (4) on Equations (9) and (10), we obtain

$$\begin{aligned}V_{T+1}(e,x,p) = {} & K^B\delta(x^+ - e) + (p+q^B)(x^+ - e)^+ \\ & - \left[(p-q^S)(e-x^+)^+ - K^S\delta(e-x^+)\right]^+ \\ & + \mu x^- + vx^+; \end{aligned} \tag{11}$$

also, for $t = T, T-1, \ldots, 1$,

$$\begin{aligned}V_t(e,x,p) = {} & -(p+q^B)e + \min_{m \text{ free}}[K^B\delta(m-e) + K^S\delta(e-m) \\ & + (q^B+q^S)(e-m)^+ + J_t(m,x,p)], \end{aligned} \tag{12}$$

where

$$J_t(m,x,p) = \min_{y \ge x} G_t(m,y,p), \tag{13}$$

while $G_t(m, y, p)$ can still be defined through Equation (7). Here, $m - e$, which can be of either sign, is the trading volume rather than merely purchase quantity. Besides the range of m, the optimization involved in Equations (12) and (13) is different from that of Equation (6) in the extra term of $K^S\delta(e - m) + (q^B + q^S)(e - m)^+$. Other than $\mathcal{C}^2(K)$, we need a new concept.

5.2. The $\mathcal{C}^2_-(K^B, K^S, q)$ Class of Functions

To face up to challenges posed by the new DP of Equation (7), as well as Equations (11)–(13), we introduce the $\mathcal{C}^2_-(K^B, K^S, q)$ class of functions. The concept is an adaptation of the weak (K^B, K^S)-convexity (Semple 2007; a weaker version of one proposed by Ye and Duenyas 2007) to the current two-dimensional case with variable transaction costs. In the following definition, both (e^S, x^S) and (e^B, x^B) are assumed to come from $\mathbb{Z}^2$.

DEFINITION 2. *Given positive constants K^B, K^S, and q, let $\mathcal{C}^{\pm}2_-(K^B, K^S, q)$ be the set of functions $f : \mathbb{Z}^2 \to \mathbb{R}$ that, for $e^S \le e^B$ and any $\lambda^B, \lambda^S \in \mathbb{Z}_{++}$,*

$$\frac{f(e^S,x^S) - f(e^S - \lambda^S, x^S) - K^S}{\lambda^S} - \frac{f(e^B + \lambda^B, x^B) - f(e^B, x^B) + K^B}{\lambda^B} \le q.$$

When $q = 0$, functions within $\mathcal{C}^2_-(K^B, K^S, 0)$ may be understood as being slightly more than weak (K^B, K^S)-convex in the e-direction. For those $f \in \mathcal{C}^2_-(K^B, K^S, 0)$ that are x-independent, the defining inequality at $e = e^S = e^B$ would amount to

$$f((1-\eta)e^0 + \eta e^1) \le (1-\eta)[K^S + f(e^0)] + \eta[K^B + f(e^1)],$$

where $\eta = \lambda^S/(\lambda^B + \lambda^S)$, $e^0 = e - \lambda^S$, and $e^1 = e + \lambda^B$. But this is exactly the defining inequality for weak (K^B, K^S)-convexity due to Semple (2007). A cost-to-go function f within $\mathcal{C}^2_-(K^B, K^S, q^B + q^S)$ tends to be against buying at high e-levels, because by Definition 2, the cost difference $f(e^B + \lambda^B, x^B) + K^B + q^B\lambda^B - f(e^B, x^B)$ between buying λ^B units and staying put is bounded from below by

$$\lambda^B \times \max_{e^S \le e^B}\left[\max_{\lambda^S = 1,2,\ldots,x^S \text{free}} \frac{f(e^S,x^S) - f(e^S - \lambda^S, x^S) - K^S - q^S\lambda^S}{\lambda^S}\right],$$

which is increasing in e^B. Symmetrically, the same function also tends to be against selling at low e-levels.

At the core of our DP involving Equations (7), (12), and (13) lies the optimization problem

$$\begin{aligned}g(e,x) = \min_{m \text{ free}} & [K^B\delta(m-e) + K^S\delta(e-m) + q(e-m)^+ \\ & + j(m,x)], \end{aligned} \tag{14}$$

where

$$j(m,x) = \min_{y \geq x} f(m,y). \tag{15}$$

The $\mathcal{C}_-^2(K^B, K^S, q)$ class of functions have some important properties.

LEMMA 4. *The following properties are all true.*

(*a*) *$\mathcal{C}_-^2(K^B, K^S, q) \subseteq \mathcal{C}_-^2(K'^B, K'^S, q')$ for $K^B \leq K'^B$, $K^S \leq K'^S$, and $q \leq q'$; in particular, both $\mathcal{C}_-^2(0, 0, q)$ and $\mathcal{C}_-^2(K^B, K^S, 0)$ are subsets of $\mathcal{C}_-^2(K^B, K^S, q)$.*

(*b*) *If $f \in \mathcal{C}_-^2(K^B, K^S, q)$, $f' \in \mathcal{C}_-^2(K'^B, K'^S, q')$, and $\beta, \beta' \geq 0$, then $\beta f + \beta' f' \in \mathcal{C}_-^2(\beta K^B + \beta' K'^B, \beta K^S + \beta' K'^S, \beta q + \beta' q')$.*

(*c*) *If $f \in \mathcal{C}_-^2(K^B, K^S, q)$ and Ψ as well as Φ are integer-valued random variables, then for $g(e,x) = \mathbb{E}[f(e+\Psi, x+\Phi)]$, we have $g \in \mathcal{C}_-^2(K^B, K^S, q)$ provided that $\mathbb{E}[|f(e+\Psi, x+\Phi)|] < +\infty$ for every $(e,x) \in \mathbb{Z}^2$.*

(*d*) *If $j \in \mathcal{C}_-^2(K^B, K^S, q)$, then g as defined in Equation (14) belongs to $\mathcal{C}_-^2(K^B, K^S, q)$.*

(*e*) *If $f \in \mathcal{C}_-^2(K^B, K^S, q)$, then j as defined in Equation (15) belongs to $\mathcal{C}_-^2(K^B, K^S, q)$.*

Parts (a)–(c) of Lemma 4 are, again, routine properties like those of Lemma 1. Part (d) is our other most technical result second only to Lemma 2; along with part (e), it serve as a critical stepping stone for our structural derivation.

5.3. Optimization Involving $\mathcal{C}_-^2(K^B, K^S, q)$

We now make a few definitions for a given $j \in \mathcal{C}_-^2(K^B, K^S, q)$. If we let

$$\begin{cases} g^B(e,x) = \min_{m \geq e}\left[K^B\delta(m-e) + j(m,x)\right], \\ g^S(e,x) = \min_{m \leq e}\left[K^S\delta(e-m) + q(e-m) + j(m,x)\right], \end{cases}$$

then, for g defined through Equation (14),

$$g(e,x) = \min\{g^B(e,x), g^S(e,x)\}.$$

Also, with the definition

$$\begin{cases} m^B(e,x) = \min\arg\min_{m \geq e+1} j(m,x), \\ m^S(e,x) = \max\arg\min_{m \leq e-1}[q(e-m) + j(m,x)], \end{cases} \tag{16}$$

we have

$$\begin{cases} g^B(e,x) = \min\{j(e,x), K^B + j(m^B(e,x),x)\}, \\ g^S(e,x) = \min\{j(e,x), K^S + q(e-m^S(e,x)) + j(m^S(e,x),x)\}, \\ g(e,x) = \min\{j(e,x), K^B + j(m^B(e,x),x), \\ \qquad K^S + q(e-m^S(e,x)) + j(m^S(e,x),x)\}. \end{cases} \tag{17}$$

Because $j \in \mathcal{C}_-^2(K^B, K^S, q)$, it will be impossible to have both $K^S + q(e - m^S(e,x)) + j(m^S(e,x), x)$ and $K^B + j(m^B(e,x), x)$ below $j(e,x)$ with at least one inequality being strict. Otherwise, it would happen that

$$\begin{aligned} & K^B + j(m^B(e,x),x) - j(e,x) < (\leq) 0 \\ & \quad \leq (<) j(e,x) - K^S - q(e - m^S(e,x)) - j^S(m^S(e,x),x), \end{aligned}$$

and hence, since both $m^B(e,x) - e$ and $e - m^S(e,x)$ are at least 1,

$$\frac{K^B + j(m^B(e,x),x) - j(e,x)}{m^B(e,x) - e} < \frac{j(e,x) - K^S - j(m^S(e,x),x)}{e - m^S(e,x)} - q.$$

But this contradicts the proclaimed membership of j. Therefore, while $g^B(e,x)$ and $g^S(e,x)$ are both below $j(e,x)$, they cannot be strictly so simultaneously.

Thus, at each x, we can partition the entire $\mathbb{Z}$ into three sets $E^B(x)$, $E^S(x)$, and $\mathbb{Z} \setminus (E^B(x) \cup E^S(x))$:

$$\begin{cases} E^B(x) = \{e \in \mathbb{Z} | g^B(e,x) < j(e,x) = g^S(e,x)\}, \\ E^S(x) = \{e \in \mathbb{Z} | g^B(e,x) = j(e,x) > g^S(e,x)\}, \\ \mathbb{Z} \setminus (E^B(x) \cup E^S(x)) = \{e \in \mathbb{Z} | g^B(e,x) = j(e,x) = g^S(e,x)\}. \end{cases} \tag{18}$$

Suppose we define $m^*(e,x)$ so that

$$m^*(e,x) = \begin{cases} m^B(e,x) \geq e+1, & \text{when } e \in E^B(x), \\ m^S(e,x) \leq e-1, & \text{when } e \in E^S(x), \\ e, & \text{when } e \in \mathbb{Z} \setminus (E^B(x) \cup E^S(x)), \end{cases} \tag{19}$$

then it would constitute an optimal m for Equation (14).

Define $e^B(x)$ and $e^S(x)$ so that

$$\begin{cases} e^B(x) = \max\{e \in \mathbb{Z} | e \in E^B(x)\}, \\ e^S(x) = \min\{e \in \mathbb{Z} | e \in E^S(x)\}, \end{cases} \tag{20}$$

where we take the convention $-\infty = \max\varnothing$ and $+\infty = \min\varnothing$. The relative positioning of the two sets and a policy form can be treated subsequently.

LEMMA 5. *For the $j \in \mathcal{C}_-^2(K^B, K^S, q)$, we have $E^B(x^B)$ lying strictly to the left of $E^S(x^S)$ for any $x^B, x^S \in \mathbb{Z}$, that is, $\max_{x\ free} e^B(x) < \min_{x\ free} e^S(x)$. Consequently, the policy $m^*(e,x)$ can be characterized as follows:*

$$m^*(e,x) = \begin{cases} m^B(e,x) \text{ or } e, & \text{when } e \leq e^B(x) - 1, \\ m^B(e,x), & \text{when } e = e^B(x), \\ e, & \text{when } e^B(x) + 1 \leq e \leq e^S(x) - 1, \\ m^S(e,x), & \text{when } e = e^S(x), \\ e \text{ or } m^S(e,x), & \text{when } e \geq e^S(x) + 1. \end{cases}$$

Lemma 5 spells out the general trend of buying at low echelon inventory levels and selling at high echelon inventory levels. That there is an x-independent demarcation line between purchasing and selling indicates that, between e and x, the former is the dominating factor in determining the type of trade to conduct. This is largely so because $e = u + x$ contains much more aggregate information on permit needs than x alone.

5.4. Three-Interval Policy Characterization

For convenience, we call the policy in Lemma 5 an $(e^B(\cdot), m^B(\cdot,\cdot), e^S(\cdot), m^S(\cdot,\cdot))$-policy. This policy divides the e-line into three x-dependent intervals, with the left interval $(-\infty, e^B(x)]$ hesitating between purchasing and staying put, the middle interval $[e^B(x)+1, e^S(x)-1]$ settled on staying put, and the right interval $[e^S(x), +\infty)$ hesitating between selling and staying put. In addition, there is definitely purchasing up to target $m^B(e, x)$ at the left boundary $e^B(x)$ and definitely selling down to target $m^S(e, x)$ at the right boundary $e^S(x)$. If we define $e^B_L(x)$ and $e^S_H(x)$ by

$$\begin{cases} e^B_L(x) = \max\{e \in \mathbb{Z} | g^B(z,x) < g^S(z,x), \text{ for any } z \le e\}, \\ e^S_H(x) = \min\{e \in \mathbb{Z} | g^B(z,x) > g^S(z,x), \text{ for any } z \ge e\}, \end{cases}$$

we can further claim that $e^B_L(x) \le e^B(x) < e^S(x) \le e^S_H(x)$, that $m^*(e, x) = m^B(e, x)$ when $e \le e^B_L(x)$, and that $m^*(e, x) = m^S(e, x)$ when $e \ge e^S_H(x)$. However, as the derived property is highly dependent on the definition of the new critical numbers, we do not emphasize this aspect of the policy.

When $e^B_L(x) = e^B(x)$, or equivalently, $E^B(x) = (-\infty, e^B(x)] \cap \mathbb{Z}$, the firm would, instead of wavering between purchasing and staying put, delve into the former activity whenever $e \le e^B(x) - 1$. However, this seems to demand that $j(\cdot, x)$ be quasi K^B-convex, which is unlikely. Indeed, we have a counter example detailed in the Appendix with $e^B_L(x) \le e^B(x) - 1$. Symmetrically, there is no guarantee on $e^S_H(x) = e^S(x)$ either. Besides, no other characterization seems likely, as our computational study has uncovered no clear trend on $e^{B(S)}(\cdot)$ or $m^{B(S)}(\cdot,\cdot)$.

In order to utilize Lemma 5 to prescribe policies for the DP comprising Equation (7), as well as Equations (11)–(13), we need to establish that $J_t(\cdot,\cdot,p)$ as defined at Equation (13) is a member of $\mathcal{C}^2_-(K^B, K^S, q^B + q^S)$ at every permit price p. First, though, we show that $V_{T+1}(\cdot,\cdot,p)$ as defined by Equation (11) is a member of $\mathcal{C}^2_-(K^B, K^S, q^B + q^S)$. Due again to the transformation (3) and (4), this depends very much on the initial $\tilde{V}_{T+1}(\cdot,\cdot,p)$ delineated at Equation (9).

PROPOSITION 3. *At each price level p, it follows that $V_{T+1}(\cdot,\cdot,p) \in \mathcal{C}^2_-(K^B, K^S, q^B + q^S)$.*

With Proposition 3 as a starting point, we can use mathematical induction on the relationships (7), (12), and (13), to verify the $\mathcal{C}^2_-(K^B, K^S, q^B + q^S)$-membership of all the remaining cost-to-go functions for $t = T, T-1, \ldots, 1$.

THEOREM 3. *At each price level p, the cost-to-go functions $G_t(\cdot,\cdot,p)$, $J_t(\cdot,\cdot,p)$, and $V_t(\cdot,\cdot,p)$ for $t = T, T-1, \ldots, 1$ are all members of $\mathcal{C}^2_-(K^B, K^S, q^B + q^S)$.*

To show that $\mathcal{C}^2_-(K^B, K^S, q^B + q^S)$-membership is preserved under the recursive process of Equations (7), (12), and (13) that is the transformed version of Equation (10), we call on Lemma 5 because this time, Equations (7), (12), and (13) have been abstracted into Equations (14) and (15). With $J_t(\cdot,\cdot,p) \in \mathcal{C}^2_-(K^B, K^S, q^B + q^S)$ as provided by Theorem 3, we can use Lemma 5 to reach a partial characterization of the trading policy.

THEOREM 4. *At each $t = T, T-1, \ldots, 1$ and permit price p, there exist curves $e^B_t(\cdot, p)$, $e^S_t(\cdot, p)$ satisfying*

$$\max_{x \text{ free}} e^B_t(x,p) < \min_{x \text{ free}} e^S_t(x,p),$$

and curves $m^B_t(\cdot,\cdot,p)$, $m^S_t(\cdot,\cdot,p)$ satisfying

$$m^B_t(e,x,p) \ge e+1, m^S_t(e,x,p) \le e-1,$$

such that the resulting $(e^B_t(\cdot,p), m^B_t(\cdot,\cdot,p), e^S_t(\cdot,p), m^S_t(\cdot,\cdot,p))$-policy for trading, as defined in Lemma 5, is optimal.

The above suggests that permit trading should follow a (t, p)-dependent three-interval policy. Its recommended behavior is very intuitive: purchasing at low echelon inventory levels, staying put at medium levels, and selling at high echelon levels. At this point, there is not much we can say about the optimal production policy. This will change in one of the heuristics in which trading and production decisions are made less entangled.

6. Heuristics and a Lower Bound

The optimal policy obtained in section 4.3 requires a certain degree of coordination between the activities of permit purchase and production, which might be difficult to come by in real practice. Moreover, the optimal policy identified in section 5.4 only covers the trading portion and is computationally expensive to obtain.

Thus, we seek heuristic solutions. Comparing the formulation (5)–(7) for the purchase-only case with the formulation (7), as well as Equations (11)–(13) for the two-way trading case, we can see that the former can be treated as a special case of the latter with K^S and q^S set at $+\infty$. We thus focus on the latter case. We propose three heuristics, with the first also being able to produce a lower bound to the total expected discounted cost.

6.1. A Heuristic and Lower Bound

The first heuristic takes advantage of the problem structure and disentangles trading and production decisions. This uncoupling heuristic (U) would simultaneously generate lower and upper bounds for the optimal cost.

From the DP comprising Equation (7), as well as Equations (11)–(13), we see that the trading and production decisions are intertwined due only to the terminal-period peculiarities. We can get an easier-to-compute lower bound by starting with a decomposed lower bound to Equation (11) and then going through the same DP machinery as suggested by Equations (7), (12), and (13). Corresponding to Equation (11), define

$$\begin{cases} V^1_{T+1}(e,p) = K^B\delta(-e) + (p+q^B)(-e)^+ - (p-q^S)^+e^+, \\ V^2_{T+1}(x,p) = \mu x^- + \nu x^+; \end{cases} \tag{21}$$

for $t = T, T-1, \ldots, 1$, let

$$\begin{cases} V^1_t(e,p) = -(p+q^B)e + \min_{m \text{ free}}\{K^B\delta(m-e) \\ \quad +K^S\delta(e-m) + (q^B+q^S)(e-m)^+ + (p+q^B)m \\ \quad +\alpha\mathbb{E}[V^1_{t+1}(m-D_t, P_{t+1})|P_t=p]\}, \\ V^2_t(x,p) = \min_{y\geq x}\{H(y,p) + \alpha\mathbb{E}[V^2_{t+1}(y-D_t, P_{t+1})| \\ \quad P_t=p]\}. \end{cases} \tag{22}$$

Note the combined dynamics for $V^1_t(e, p) + V^2_t(x, p)$ is the same for that experienced by the earlier $V_t(e, x, p)$ through Equations (7), (12), and (13). So for $t = T + 1, T, \ldots, 1$, we let

$$V^L_t(e,x,p) = V^1_t(e,p) + V^2_t(x,p). \tag{23}$$

Note that V^1_t is not a function of x, nor is V^2_t a function of e. It will be easy to verify that $V^L_t(e, x, p)$ offers a lower bound to the optimal cost $V_t(e, x, p)$.

Suppose that $m^1_t(e, p)$ is an optimal solution to the trading portion of the DP comprising Equations (21) and (22), and that $y^2_t(x, p)$ is the same for the production portion of the DP. These solutions constitute our first, uncoupling heuristic (U). The heuristic's performance, $V^U_t(e, x, p)$, offers an upper bound for $V_t(e, x, p)$. Whenever at state (e, x, p) in period t, the $m^1_t(e, p)$ is used for trading and the $y^2_t(x, p)$ is used for production. Let us use $V^U_t(e, x, p)$ to denote the cost-to-go function for the heuristic. Then, $V^U_{T+1}(e, x, p) = V_{T+1}(e, x, p)$ as given in Equation (11), and for $t = T, T-1, \ldots, 1$,

$$\begin{aligned} V^U_t(e,x,p) =& K^B\delta(m^1_t(e,p)-e) + K^S\delta(e-m^1_t(e,p)) \\ &+(p+q^B)(m^1_t(e,p)-e) + (q^B+q^S)(e-m^1_t(e,p))^+ \\ &+H_t(y^2_t(x,p),p) + \alpha\mathbb{E}[V^U_{t+1}(m^1_t(e,p) \\ &-D_t, y^2_t(x,p)-D_t, P_{t+1})|P_t=p]. \end{aligned} \tag{24}$$

In Equation (24), we merely assess the expected total discounted cost when the uncoupled policy $(m^1_t(\cdot, p), y^2_t(\cdot, p))$ is used by the firm. The following affirms the lower-bound status of $V^L_t(e, x, p)$ and the upper-bound status of $V^U_t(e, x, p)$, both for $V_t(e, x, p)$.

PROPOSITION 4. *For every $t = T + 1, T, \ldots, 1$ and state (e, x, p), we have*

$$V^L_t(e,x,p) \leq V_t(e,x,p) \leq V^U_t(e,x,p).$$

6.2. Analysis of the Uncoupling Heuristic

When compared with Equations (12) and (13) in the original DP, the heuristic's optimization portion at Equation (22) is considerably simpler. Before, the processes for achieving optimal trading and production decisions are layered and inter-dependent, so that both $m^*_t(e, x, p)$ and $y^*_t(e, x, p)$ are functions of both e and x. Now in heuristic (U), the process of achieving the $m^1_t(e, p)$'s and $y^2_t(x, p)$'s is uncoupled, so that the former depends on e only and the latter on x only. For this reason, we can get more out of the joint heuristic trading-production policy. The other portions of the heuristic, involving Equations (21) and (24), are computationally less expensive than its optimization portion.

Now, we analyze the trading and production portions of the heuristic separately. For the former, we define $\mathcal{C}_-(K^B, K^S, q)$ to be the set of functions $f: \mathbb{Z} \to \mathbb{R}$ that will guarantee $f' \in \mathcal{C}^2_-(K^B, K^S, q)$ when $f'(e, x) = f(e)$ at every $(e, x) \in \mathbb{Z}^2$. As shown in section 5.2, functions in $\mathcal{C}_-(K^B, K^S, 0)$ are slightly more than weak (K^B, K^S)-convex. The latter concept and its variant have been used by Semple (2007) and Ye and Duenyas (2007) to tackle capacity management problems. The newly proposed notion of $\mathcal{C}_-(K^B, K^S, q)$ with $e^S \leq e^B$ instead of $e^S = e^B$ can be used on similar problems.

We now use the new concept on our current purposes. Just like Theorems 3 and 4, we can first establish the membership of the trading-related cost-to-go function in $\mathcal{C}_-(K^B, K^S, q^B + q^S)$ and then verify the form of an optimal policy.

PROPOSITION 5. *At each price level p, the cost-to-go function $V^1_t(\cdot, p) \in \mathcal{C}_-(K^B, K^S, q^B + q^S)$ for $t = T + 1,$*

$T, \ldots, 1$. Consequently, at each $t = T, T-1, \ldots, 1$ and permit price p, an $(e_t^{1B}(p), m_t^{1B}(\cdot, p), e_t^{1S}(p), m_t^{1S}(\cdot, p))$-policy will be optimal for trading. That is, one optimal $m_t^1(\cdot, p)$ is three-interval.

Regarding the production portion of the heuristic and the lower-bounding DP, we obviously have that each function $V_t^2(\cdot, p)$ is convex and as a consequence, an optimal $y_t^2(x, p) = \max\{x, x_t^2(p)\}$ for some $x_t^2(p)$ level. This together with Proposition 5 would allow us to summarize heuristic (U) as one that uses an x-independent three-interval policy for trading and an e-independent base-stock policy for production.

6.3. Other Heuristics

In addition to (U), we have two other heuristics. Our second heuristic simply lets the firm trade permits only in the terminal period $T + 1$. It has the potential to save costs on redundant buying and selling. In particular, we let production during periods 1 through T be done optimally, but without paying attention to concurrent permit levels. For this carefree heuristic (C), we still have $V_{T+1}^C(e, x, p) = V_{T+1}(e, x, p)$ as given in Equation (11), and for $t = T, T-1, \ldots, 1$,

$$V_t^C(e,x,p) = H(y_t^2(x,p),p) + \alpha\mathbb{E}[V_{t+1}^C(e - D_t, y_t^2(x,p) - D_t, P_{t+1})|P_t = p], \tag{25}$$

where $y_t^2(x, p)$ is the disjoint production policy derived from the production portion of the lower-bounding DP at Equation (22). In the above, the $(e - D_t)$-term in $V_{t+1}^C(\cdot, \cdot, \cdot)$ signifies the firm's no-trading stance in this heuristic.

The third heuristic ensures the sufficiency of permit levels for production at all times, whereas production during periods 1 through T is again done as best as can without consulting the concurrent permit levels. It strives to save on selling but not buying in order to reduce the overall cost. For this nervous heuristic (N), we still have $V_{T+1}^N(e, x, p) = V_{T+1}(e, x, p)$ as given in Equation (11) and for $t = T, T-1, \ldots, 1$,

$$V_t^N(e,x,p) = K^B\delta(y_t^2(x,p) - e) + (p + q^B)(y_t^2(x,p) - e)^+ + H(y_t^2(x,p),p) + \alpha\mathbb{E}[V_{t+1}^N(\max\{e, y_t^2(x,p)\} - D_t, y_t^2(x,p) - D_t, P_{t+1})|P_t = p], \tag{26}$$

where $y_t^2(x, p)$ has the same meaning as in the above. There are $e - x$ permits available before purchasing and the number of permits to be consumed by production is $y_t^2(x, p) - x$. Post-production, the permit level would drop to $(e - x) - (y_t^2(x, p) - x) = e - y_t^2(x, p)$ if there were no replenishment. So (N) will prompt a permit purchase quantity of $(y_t^2(x, p) - e)^+$. Also, $m - e = (y_t^2(x, p) - e)^+$ will certainly lead to $m = \max\{e, y_t^2(x, p)\}$.

Heuristics (U), (C), and (N) all have to use the second recursive relation in Equation (22) to compute the disjoint production policy $y_t^2(x, p)$. Heuristic (U) has to additionally deal with the first recursive relation in Equation (22) and then use Equation (24) to compute its performances. In contrast, heuristics (C) and (N) are computationally less involved as they have to go through, respectively, the simpler steps (25) and (26) which involve no optimization.

7. A Numerical Study

We conduct a numerical study to showcase the importance of both the transaction-cost considerations and the purchase-only model, to assess merits of the various heuristics, and to gain a deeper understanding of the joint trading-production control problem.

7.1. Computational Setup

Let horizon length $T = 12$, thus implying month-long periods and a yearly re-balancing of permits. We set the discount rate at $\alpha = 0.998$, reflecting an annual discount around 2.5%. Let us model the price process through a binomial tree. The latter is parameterized by two price multipliers u and d, as well as a transition probability π. Between any two consecutive periods, the probability of moving from price p up to the higher alternative $u \times p$ is π and that of moving down to the lower alternative $d \times p$ is $1 - \pi$.

We fix $\bar{p} = 8$ as the starting price for period 1, $u = 1.06$, and $d = 1/u \simeq 0.9434$. Note that $\pi^0 = (1 - \alpha d)/(\alpha u - \alpha d) \simeq 0.503$ would satisfy

$$\alpha[\pi^0 u + (1 - \pi^0)d] \simeq 1.$$

When $\pi \simeq \pi^0$, the discounted permit price process would be roughly a martingale with no discernible increasing or decreasing pattern. When $\pi < \pi^0$, it would be a super-martingale with a declining trend; whereas, when $\pi > \pi^0$, it would be a sub-martingale with a rising trend. Our choices on $\bar{p}$, u, d, along with a π within ± 0.1 of the current $\pi^0 \simeq 0.503$ can be typical of a price process seen in real life. For instance, we can download daily EU CO_2 allowance (EUA) prices from the Bloomberg site of http://www.bloomberg.com/quote/MO1:COM. The monthly average of these real data can be computed. Next, we can find the median price as $\bar{p}$ (Euros per ton), and identify drift μ and volatility σ for the Brownian motion (BM) that best approximates the price process; see chapter 4 of Fouque et al. (2000). Finally, by making the binomial-tree price process as close to the Brownian motion (see chapter 11 of Luenberger 1997) as possible, we can obtain u, d, and π. For data from June 1, 2010 to May 30, 2012, we can obtain $\bar{p} \simeq 12.5$, $u \simeq 1.061$, $d \simeq 0.943$, and $\pi \simeq 0.513$.

We let the demand process be independent of the price process, and let it be uniform on $\{0, 1, \ldots, 6\}$ for each period. As default values, we let unit production cost $c = 50$, holding cost rate $h = 0.5$, and backlogging cost rate $b = 2$. We have condensed demand into small numbers for ease of computation. In reality, each of the demand units may represent hundreds of product items. When such is the case, all fixed costs per trade should undergo the same scaling as demand units; nevertheless, all variable costs per item or per item per period should stay intact. As their values within reasonable ranges do not much affect conclusions, we let terminal-period parameters μ and v both be zero.

For the selection of transaction costs, we consider six "representative" scenarios. While supply and demand tend to be balanced on the whole by the always-adjusting permit price, imbalance can exist between the segment of large firms and that of small- to medium-sized firms like the one under our consideration.

- Scenario #1 ($1/K^S \gg 1/K^B$): when shortages concentrate in large firms and surpluses concentrate in average small- to medium-sized emitters. Since smaller firms are less accessible than larger firms, our firm will have a much tougher time buying than selling. Let us set $K^B = 100$, $q^B = 5$, $K^S = 1$, and $q^S = 0.05$.
- Scenario #2 ($1/K^S > 1/K^B$): when shortages are more common among larger firms and surpluses are more common among smaller emitters. Let us set $K^B = 20$, $q^B = 1$, $K^S = 5$, and $q^S = 0.25$.
- Scenario #3 ($1/K^S = 1/K^B$): when large firms are as likely as average emitters to have either surplus or shortage. Let us set $K^B = K^S = 10$ and $q^B = q^S = 0.5$.
- Scenario #4 ($1/K^S < 1/K^B$): when surpluses are more common among larger firms and shortages are more common among smaller emitters. Let us set $K^B = 5$, $q^B = 0.25$, $K^S = 20$, and $q^S = 1$.
- Scenario #5 ($1/K^S \ll 1/K^B$): when surpluses concentrate in large firms and shortages concentrate in average small- to medium-sized emitters. Let us set $K^B = 1$, $q^B = 0.05$, $K^S = 100$, and $q^S = 5$.
- Scenario #6 ($1/K^S = 0$): the limiting purchase-only case in which our firm has no access to selling. Let us set $K^B = 0.8$, $q^B = 0.04$, and $K^S = q^S = +\infty$.

We plan to test on these scenarios for the sake of completeness. Since as we have argued surpluses in permits tend to go to large firms and shortages tend to concentrate in small-to medium-sized firms, the latter scenarios, especially the more moderate ones #3 and #4, will be more likely to occur on our small- to medium-sized firm.

7.2. Preliminary Examinations

In order to study the consequences of misrepresenting fixed transaction costs in a variety of circumstances, we introduce a heuristic class $(M(k^B, k^S))$. At each given (k^B, k^S)-pair, the heuristic $(M(k^B, k^S))$ strives to solve the original problem, however, with k^B mistaken as whatever original K^B there might be and k^S mistaken as whatever original K^S there might be. For instance, $(M(0, 0))$ will mistake both fixed costs as 0, $(M(0, -))$ will treat just the fixed cost for purchase as 0, and $(M(-, \infty))$ will treat the fixed cost for selling as ∞. The latter will effectively solve the purchase-only model and then apply the solution to the original problem where selling might after all be allowed. We introduce these heuristics here rather than in section 6.3, because none of them is "a heuristic in the traditional sense," that is, one that saves on computational efforts.

To evaluate the heuristics, we define η^H for $H = U, C, N, M(k^B, k^S)$, so that

$$\eta^H = 100 \times \frac{V_1^H(0,0,\bar{p}) - V_1(0,0,\bar{p})}{V_1(0,0,\bar{p})}\%,$$

where $V_1^H(0, 0, \bar{p})$ stands for the period-1, state-$(0, 0, \bar{p})$ cost-to-go resulting from the application of heuristic (H) on the firm, and $V_1(0, 0, \bar{p})$ is its optimal counterpart. Each η^H measures the percentage cost increase brought in by (H)'s sub-optimality.

Now in Tables 1, 2, and 3, we present η^H values for heuristics (H) under various scenarios on transaction costs when, respectively, $\pi = 0.403$, 0.503, and 0.603.

From Tables 1–3, we find that heuristic (U) performs consistently well in almost all circumstances. It is less than 5% from optimality even in extreme cases. Also, heuristic (C) performs well except for the case where $1/K^S \ll 1/K^B$. In its current mild range, the changing π does not seem to affect the performance of (C) much. Nevertheless, heuristic (N) performs poorly across the board. It may have suggested too frequent a purchasing schedule in the face of reasonably large K^B values.

The heuristic $(M(0, -))$, where 0 is mistaken as K^B, performs extremely well when price is on the rise; see Table 3. When the price is on the rise, the firm may have a tendency to purchase most of its needs at the beginning, regardless of the size of the fixed transaction cost K^B. However, when price has a declining or stagnant trend, $(M(0, -))$ can perform poorly,

Table 1 Heuristics' Performances when Price Tends to Decline

Scenario/Heuristic	(*U*)	(*C*)	(*N*)	(*M*(0, −))	(*M*(−, 0))	(*M*(−, ∞))
#1: $1/K^S \gg 1/K^B$	4.76%	4.91%	50.66%	48.03%	0%	73.98%
#2: $1/K^S > 1/K^B$	2.39%	4.92%	32.36%	8.13%	0.01%	0.46%
#3: $1/K^S = 1/K^B$	2.07%	4.89%	30.59%	3.06%	0.04%	0.03%
#4: $1/K^S < 1/K^B$	1.94%	3.89%	29.97%	0.78%	0.48%	0.001%
#5: $1/K^S \ll 1/K^B$	3.29%	57.25%	119.10%	0%	74.63%	0%
#6: $1/K^S = 0$	0.80%	4.04%	55.40%	0%	—	0%

Table 2 Heuristics' Performances when Price is Roughly Stagnant

Scenario/Heuristic	(*U*)	(*C*)	(*N*)	(*M*(0, −))	(*M*(−, 0))	(*M*(−, ∞))
#1: $1/K^S \gg 1/K^B$	4.56%	4.56%	56.38%	15.94%	0%	81.13%
#2: $1/K^S > 1/K^B$	2.11%	2.11%	35.86%	5.43%	0.00001%	0.20%
#3: $1/K^S = 1/K^B$	1.79%	1.81%	33.95%	2.77%	0.23%	0.19%
#4: $1/K^S < 1/K^B$	1.67%	1.88%	43.76%	1.36%	0.21%	0.001%
#5: $1/K^S \ll 1/K^B$	3.13%	66.05%	143.43%	0%	82.30%	0%
#6: $1/K^S = 0$	0.59%	0.26%	55.61%	0%	—	0%

Table 3 Heuristics' Performances when Price Tends to Rise

Scenario/Heuristic	(*U*)	(*C*)	(*N*)	(*M*(0, −))	(*M*(−, 0))	(*M*(−, ∞))
#1: $1/K^S \gg 1/K^B$	4.44%	4.44%	61.51%	0.01%	0%	71.47%
#2: $1/K^S > 1/K^B$	1.93%	1.93%	41.83%	00008%	0%	0.23%
#3: $1/K^S = 1/K^B$	1.60%	1.62%	40.07%	0.0003%	0.02%	0.01%
#4: $1/K^S < 1/K^B$	1.47%	1.74%	41.06%	0.0005%	0.27%	0%
#5: $1/K^S \ll 1/K^B$	2.99%	75.55%	167.83%	0%	72.57%	0%
#6: $1/K^S = 0$	0.49%	1.72%	58.26%	0%	—	0%

especially when K^B is substantial; see Tables 1 and 2. So firms will have to ignore K^B at their own peril.

As expected, ($M(-, 0)$), where 0 is mistaken as K^S, performs well when the actual K^S is low and poorly when the actual K^S is high. Clearly, ignoring the fixed transaction cost for selling could have severe consequences as well.

Moreover, ($M(-, \infty)$), where selling is treated as non-existent no matter how low K^S really is, performs poorly when the actual K^S is low and well when the actual K^S is high. However, when K^S reaches above 5, ($M(-, \infty)$) will already perform reasonably well, with errors below 0.5%. This indicates that the purchase-only case studied in sections 3 and 4, where policy derivation is more thorough, can offer very accurate guidance even in situations where selling is still far from being prohibitively expensive.

7.3. Performance of Heuristics

We further our examination of the computation-easing heuristics (*U*), (*C*), and (*N*). Note that (*U*) is also associated with a lower bound. To assess it, we define ζ so that

$$\zeta = 100 \times \frac{V_1(0,0,\bar{p}) - V_1^L(0,0,\bar{p})}{V_1(0,0,\bar{p})}\%,$$

where $V_1^L(e, x, p)$ stands for the period-1 lower bound defined from Equation (21) to (23). Note that ζ gives the percentage cost reduction when terminal-period relaxation is facilitated.

To focus, we now narrow from six down to two scenarios on transaction costs, namely, #3 ($1/K^S = 1/K^B$) and #4 ($1/K^S < 1/K^B$). The former is a balanced situation and the latter reflects a mild oversupply of permits in large firms and a mild shortage in small- to medium-sized firms. Very likely, larger firms can afford more than smaller firms to invest in abatement technologies and then reap the benefits. Hence, they are more likely than smaller firms to have surpluses. For Scenario #3 with $1/K^S = 1/K^B$ and the case with all other parameters at default values, we have $\eta^U \simeq 1.79\%$, $\eta^C \simeq 1.81\%$, $\eta^N \simeq 33.95\%$, and $\zeta \simeq 0.25\%$. The first two heuristics perform almost equally well and the lower bound is fairly tight. Because (*N*) wastes fixed purchase costs too readily, it is not competitive at all. For Scenario #4 with $1/K^S < 1/K^B$ and the case with all other parameters at default values, we have $\eta^U \simeq 1.67\%$, $\eta^C \simeq 1.88\%$, $\eta^N \simeq 43.76\%$, and $\zeta \simeq 0.25\%$. The picture is very similar. When we let parameters vary mildly, the relative standings of the three heuristics do not alter much. The ability to make opportune permit trades thus brings some benefits to the firm, while the freedom of not having to balance permits with production every period makes a significant difference. The above also suggests that, with a sizable K^B, heuristic (*N*) need not be considered in ordinary situations.

For the two likely scenarios #3 ($1/K^S = 1/K^B$) and #4 ($1/K^S < 1/K^B$), we present in Figures 3 and 4 the η^H values for $H = U, C$, and N and $-\zeta$ as π varies.

From Figures 3 and 4, we see that ζ is consistently small. This confirms the high quality of the lower bound. With a consistently small performance gap, (*U*) appears as an all-weather policy. Also, (*C*) is in contention when the fixed costs for purchase and selling are both mild and the price is on the rise; see Figure 4. When price increases in general, it will not hurt much by waiting till the last moment to purchase. However, as shown in Figure 3, (*C*)'s performance under a lower K^S can worsen as the upward trend of price becomes more pronounced. The heuristic may have missed opportunities to take advantage of the not-so-expensive selling options. The performance of (*N*) worsens as π increases. The more the price rises, the worse it will be to purchase too early.

We have allowed parameters like K^B, q^B, K^S, and q^S to vary in slightly broader ranges; still, patterns seen in Tables 1–3 and Figures 3 and 4 remain valid. For example, we display in Figure 5 two variations to Figure 4 on Scenario #4 ($K^B = 5$, $q^B = 0.25$, $K^S = 20$, and $q^S = 1$): in (a) $K^S = 10$ and in (b) $q^S = 0.3$. Evidently, the two new plots look nearly identical to Figure 4.

7.4. Effect of Monitoring Frequency

We know from section 6.1 that (*U*) differs from the optimal policy only in its treatment of the terminal period. Presumably, as the common horizon of the concerned DP formulations lengthens, the heuristic's performance relative to the optimal policy should improve. This can be verified for a setting with an actual horizon length of $T = 60$.

Each period can be thought of as a week. The weekly discount rate is set at $\alpha = 0.999$. We fix $\bar{p} = 8$, $u = 1.02$, $d = 1/d \simeq 0.98$, and let $\pi = 0.525$. This is almost $\pi^0 = (1 - \alpha d)/(\alpha u - \alpha d) \simeq 0.525$. The weekly demand process is again independent of the price process. Let it be uniform on $\{0, 1, 2\}$ in each week. For other parameters, we let $c = 50$, $h = 0.2$, and $b = 0.5$. Again, let the terminal-period parameters $\mu = \nu = 0$. We still consider the two likely scenarios #3 ($1/K^S = 1/K^B$) and #4 ($1/K^S < 1/K^B$).

We can use different frequencies of monitoring to elicit different DP lengths from the same actual horizon length T. Suppose that trading and production are conducted once every M periods, with $L \equiv T/M$ still being an integer. We can then treat the problem as an L-period super problem (M). For this problem, we can use $\alpha_M = (\alpha)^M$ as the effective discount factor, $h_M = h(1 - \alpha_M)/(1 - \alpha)$ as the effective holding cost rate, and $b_M = b(1 - \alpha_M)/(1 - \alpha)$ as the effective backlogging cost rate. Demand $D_{M,l}$ for the l-th super period, for $l = 1, 2, \ldots, L$, should be the M-term summation $\sum_{t=M(l-1)+1}^{Ml} D_t$, whose distribution can be obtained through convolution.

We can use a binomial tree comprising the super periods to approximate the same underlying BM process. Each single super period is then ($1/L \equiv$

Figure 3 Performances of Heuristics at Different π's in Scenario #3 ($1/K^S = 1/K^B$) [Color figure can be viewed at wileyonlinelibrary.com]

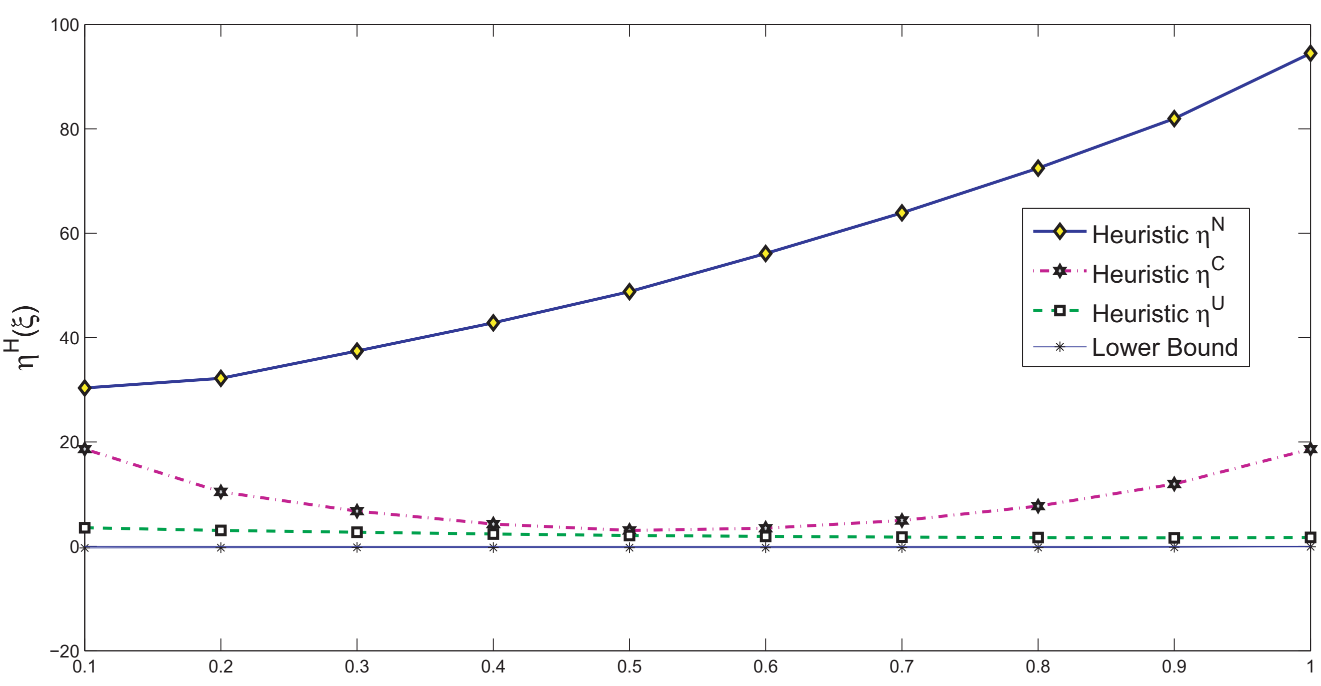
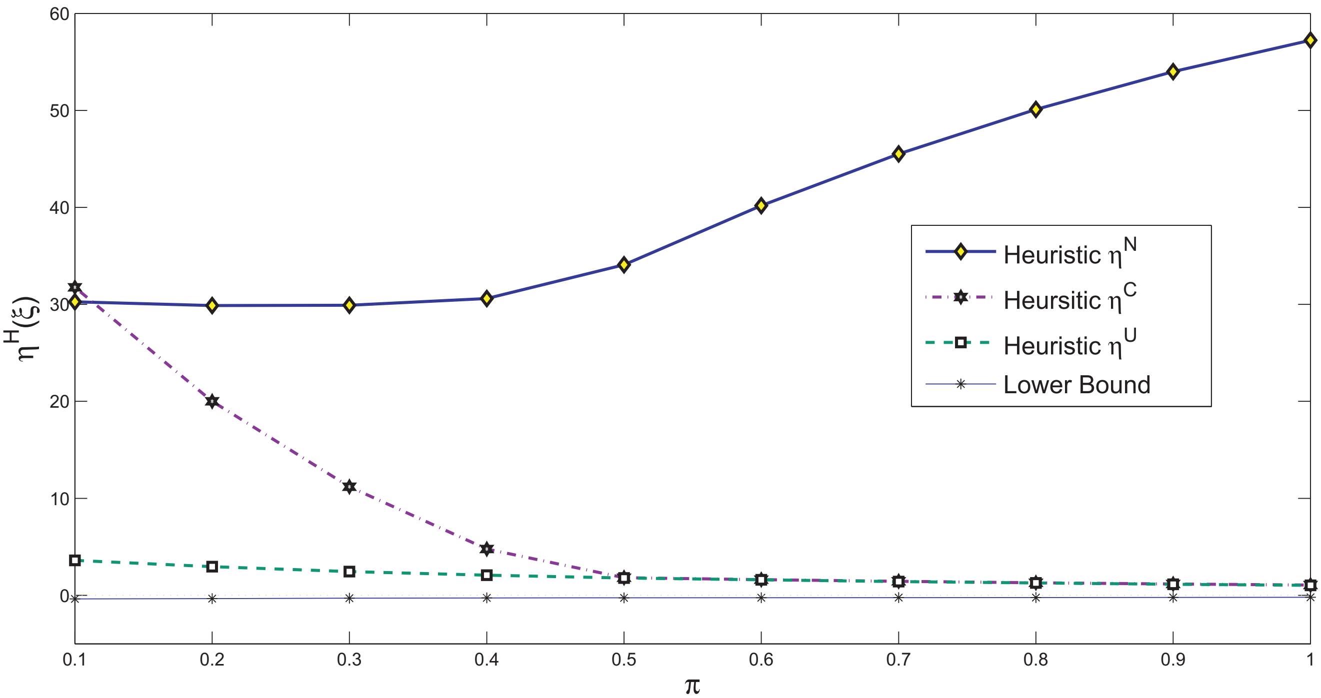


Figure 4 Performances of Heuristics at Different π's in Scenario #4 ($1/K^S < 1/K^B$) [Color figure can be viewed at wileyonlinelibrary.com]



M/T)-long. From one super period to another, we should allow price p to transition into $u_M \times p$ at probability π_M and into $d_M \times p$ at probability $1 - \pi_M$, with $u_M = \exp(\sqrt{\sigma(M/T) + \mu(M/T)^2})$, $d_M = 1/u_M$, and $\pi_M = (1/\alpha_M - d_M)/(u_M - d_M)$; see, e.g., Luenberger (1997).

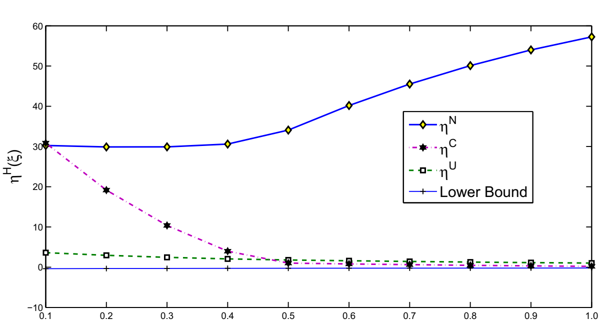
In Figure 6, for both likely scenarios of #3 ($1/K^S = 1/K^B$) and #4 ($1/K^S < 1/K^B$), we show η^U and $-\zeta$ as the monitoring interval M is in the descending order of 60, 30, 20, 15, 12, 10, 6, 2, and 1 and the monitoring frequency $1/M \equiv L/T$ is in the ascending order of 1/60, 1/30, 1/20, 1/15, 1/12, 1/10, 1/6, 1/2, and 1.

From Figure 6, we see that the heuristic's performance improves with monitoring frequency for both scenarios. Indeed, the pictures at other π values, such as 0.425 and 0.625, all look similar. Thus, if the firm can monitor the permit market at a reasonable frequency and make trading and production decisions accordingly, a simple disjoint decision-making mechanism may cause little loss in efficiency. When such is not the case, however, it is worth following a more complex joint trading and production policy.

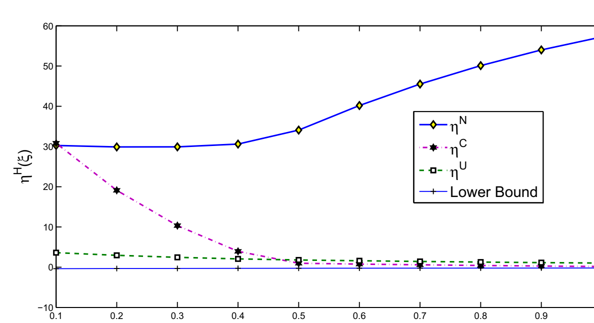
8. Concluding Remarks

We have partially characterized optimal joint control policies for a firm that trades permits in a spot market in order to cover its production needs. The consideration of fixed as well as variable transaction costs for

Figure 5 Performances of Heuristics at Different π's: Variations to Figure 4 [Color figure can be viewed at wileyonlinelibrary.com]

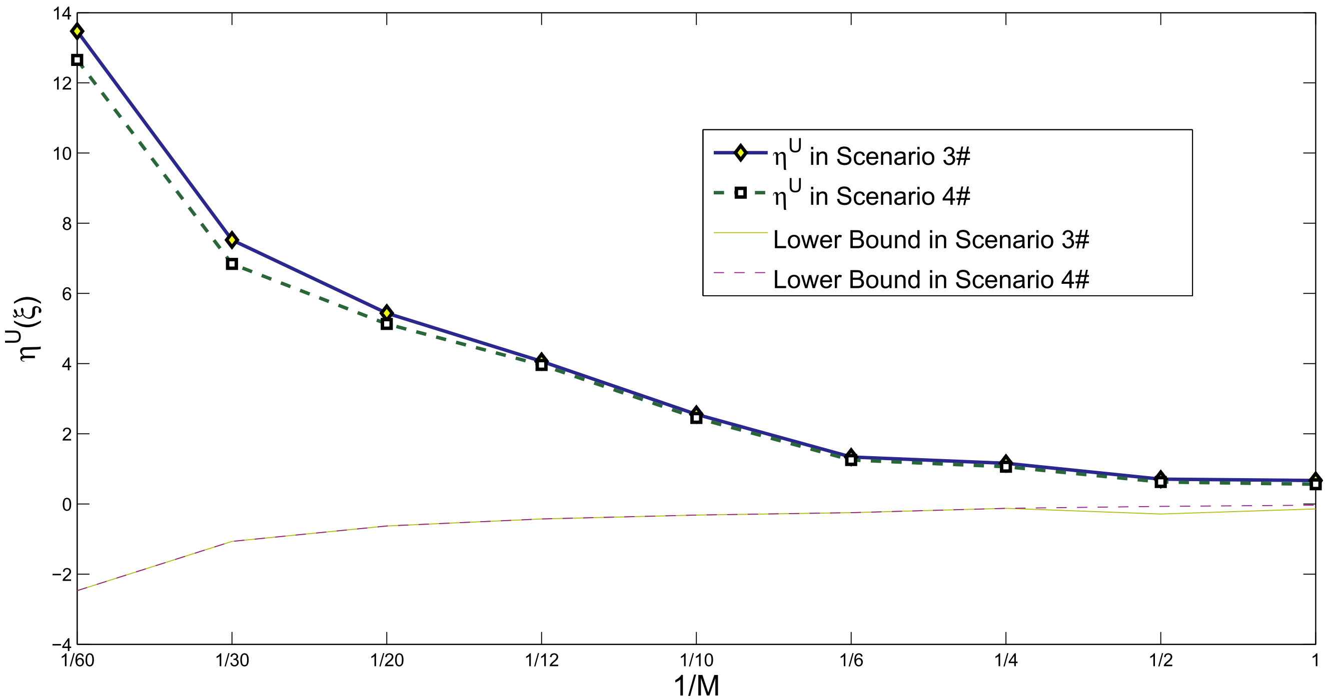


(a) $K^S = 10$



(b) $q^S = 0.3$

Figure 6 Heuristic (*U*)'s Performances when Price is Roughly Stagnant [Color figure can be viewed at wileyonlinelibrary.com]



trading results in very complex DP formulations. These in turn motivate the introduction of two-dimensional variants of K-convex functions. The $\mathcal{C}^2(K)$ and $\mathcal{C}^2_-(K^B, K^S, q)$ concepts are associated with (s, S)-type purchase policies, three-interval trading policies, and base-stock-type production policies that all help ease real operations. They also have potential uses in other occasions involving fixed setups and joint controls.

A numerical study has confirmed that our policy characterization is nearly complete, and indicated occasions when disjoint decisions may or may not cause substantial losses. When the firm monitors the permit market sufficiently frequently, we have found relative strength in an easy-to-implement heuristic involving uncoupled decision making on permit trading and production. This work can still be expanded in a few directions. First, the case where the fixed transaction costs are time-varying in arbitrary fashions await to be explored. Second, aside from the backlogging case, it would be interesting to further consider the lost-sales model. Third, there is a futures market for CO_2 permits. How to combine it into the current model can turn into a worthy research project.

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Appendix A. Technical Details in Section 4

Proof of Lemma 1. As parts (a) and (b) are obvious, we just concentrate on parts (c) and (d). For part (c), $g(e, x) = E[f(e - \Psi, x - \Psi)]$ is well defined due to the integrability hypothesis. To see $g \in \mathcal{C}^2(K)$, it suffices to prove that $f(e - \psi, x - \psi)$ satisfies properties (i) to (iii) of Definition 1 for any fixed $\psi \in \mathbb{Z}_+$.

(i) Suppose $e_1 \le e_2$ and either $(e_1, x_1) \in \Sigma_1^c$ or $(e_2, x_2) \in \Sigma_1$. By definition of Σ_1, we know either $e_1 \le (x_1)^+ - 1$ or $e_2 \ge (x_2)^+$. If $\psi \ge e_1 + 1$, then $e_1 - \psi < 0 \le (x_1 - \psi)^+$; if $\psi \le e_1$ and $e_1 \le (x_1)^+ - 1$, then $(x_1 - \psi)^+ \ge (x_1)^+ - \psi > e_1 - \psi \ge 0$; if $\psi \le e_1$ and $e_2 \ge (x_2)^+$, then $e_2 - \psi \ge e_1 - \psi \ge 0$ and $e_2 \ge x_2$, implying $e_2 - \psi \ge (x_2 - \psi)^+$. In all these cases we know either $(e_1 - \psi, x_1 - \psi) \in \Sigma_1^c$ or $(e_2 - \psi, x_2 - \psi) \in \Sigma_1$ from the definition of Σ_1. By property (i) of $f(e, x)$, it concludes that for any $\lambda \in \mathbb{Z}_{++}$,

$$f(e_1 - \psi + 1, x_1 - \psi) - f(e_1 - \psi, x_1 - \psi) \le [f(e_2 - \psi + \lambda, x_2 - \psi) - f(e_2 - \psi, x_2 - \psi) + K]/\lambda.$$

This means that $f(e - \psi, x - \psi)$ satisfies property (i).

(ii) Suppose $x_1 \le x_2$ and $(e_1, x_1 + 1) \in \Sigma_2$. Since $(e_1)^+ \ge x_1 + 1$ by the definition of Σ_2, we know $(e_1 - \psi)^+ \ge (e_1)^+ - \psi \ge x_1 - \psi + 1$ implying $(e_1 - \psi, x_1 - \psi + 1) \in \Sigma_2$. Because $x_1 - \psi \le x_2 - \psi$ and $f(e, x)$ satisfies property (ii), it concludes that

$$f(e_1 - \psi, x_1 - \psi + 1) - f(e_1 - \psi, x_1 - \psi) \le f(e_2 - \psi, x_2 - \psi + 1) - f(e_2 - \psi, x_2 - \psi).$$

This means that $f(e - \psi, x - \psi)$ satisfies property (ii).

(iii) Suppose $e_1 \le e_2$, $(e_1, x_1) \in \Sigma_1$, and $(e_2, x_2) \in \Sigma_1^c$, where $e_1 \ge (x_1)^+$ and $e_2 \le (x_2)^+ - 1$ by the definition of Σ_1. We distinguish between two cases depending on whether or not $e_1 \ge \psi$.
Case 1: When $e_1 \ge \psi$, we have $e_1 \ge x_1$ by $e_1 \ge 0$, implying $e_1 - \psi \ge (x_1 - \psi)^+$ and hence $(e_1 - \psi, x_1 - \psi) \in \Sigma_1$. Moreover, $e_2 - \psi < (x_2)^+ - \psi \le (x_2 - \psi)^+$ and hence $(e_2, x_2) \in \Sigma_1^c$. As $e_1 - \psi \le e_2 - \psi$ and $f(e, x)$ satisfies property (iii), it leads to

$$f(e_1 - \psi + 1, x_1 - \psi + 1) - f(e_1 - \psi, x_1 - \psi) \le [f(e_2 - \psi + \lambda, x_2 - \psi + \lambda) - f(e_2 - \psi, x_2 - \psi) + K]/\lambda, \quad \text{(A.1)}$$

and hence $f(e - \psi, x - \psi)$ satisfies property (iii).
Case 2: When $e_1 \le \psi - 1$, we have $e_1 - \psi < 0 \le (x_1 - \psi)^+$ and hence $(e_1 - \psi, x_1 - \psi) \in \Sigma_1^c$. Moreover, $(e_1 - \psi + 1)^+ = 0 > e_1 - \psi \ge (x_1)^+ - \psi \ge x_1 - \psi$ and hence $(e_1 - \psi + 1, x_1 - \psi + 1) \in \Sigma_2$. By property (i) of $f(e, x)$, we have

$$f(e_1 - \psi + 1, x_1 - \psi) - f(e_1 - \psi, x_1 - \psi) \le [f(e_2 - \psi + \lambda, x_2 - \psi) - f(e_2 - \psi, x_2 - \psi) + K]/\lambda. \quad \text{(A.2)}$$

Meanwhile, by property (ii) of $f(e, x)$, we have

$$f(e_1 - \psi + 1, x_1 - \psi + 1) - f(e_1 - \psi + 1, x_1 - \psi) \le f(e_2 - \psi + \lambda, x_2 - \psi + \eta + 1) - f(e_2 - \psi + \lambda, x_2 - \psi + \eta),$$

for $\eta = 0, 1, \ldots, \lambda - 1$, which, after being summed over and then divided by λ, become

$$f(e_1 - \psi + 1, x_1 - \psi + 1) - f(e_1 - \psi + 1, x_1 - \psi) \le [f(e_2 - \psi + \lambda, x_2 - \psi + \lambda) - f(e_2 - \psi + \lambda, x_2 - \psi)]/\lambda. \quad \text{(A.3)}$$

Adding Equations (A.2) and (A.3), we obtain Equation (A.1) which indicates again that $f(e - \psi, x - \psi)$ also satisfies property (iii).

For part (d), suppose $e_2 \ge e_1$, $(e_1, x_1 + 1) \in \Sigma_2$ and hence $(e_1)^+ \ge x_1$. Then $(e_2)^+ \ge (e_1)^+ \ge x_1 > x_1 - 1$, hence the three other points (e_1, x_1), (e_2, x_1), and $(e_2, x_1 + 1)$ belong to Σ_2 as well. According to property (ii), we have not only $f(e_1, x_1 + 1) - f(e_1, x_1) \le f(e_2, x_1 + 1) - f(e_2, x_1)$ because $x_1 \le x_1$ and $(e_1, x_1 + 1) \in \Sigma_2$, but also $f(e_2, x_1 + 1) - f(e_2, x_1) \le f(e_1, x_1 + 1) - f(e_1, x_1)$ because $x_1 \le x_1$ and $(e_2, x_1 + 1) \in \Sigma_2$. The two inequalities result in

$$f(e_2, x_1) - f(e_1, x_1) = f(e_2, x_1 + 1) - f(e_1, x_1 + 1).$$

This implies that in Σ_2, f's difference with respect to e (or x) is independent of x (or e). We can denote the difference $f(e, x) - f(e - 1, x)$ for any $(e - 1, x) \in \Sigma_2$ by $\Delta f^1(e)$, and the difference $f(e, x) - f(e, x - 1)$ for any $(e, x) \in \Sigma_2$ by $\Delta f^2(x)$. Now define

$$f^1(e) = \begin{cases} f(0,0) + \sum_{e'=1}^{e} \Delta f^1(e'), & \text{when } e \geq 0, \\ f(0,0) - \sum_{e'=e+1}^{0} \Delta f^1(e'), & \text{when } e \leq -1. \end{cases}$$

Similarly, define $f^2(\cdot)$, so that

$$f^2(x) = \begin{cases} \sum_{x'=1}^{x} \Delta f^2(x'), & \text{when } x \geq 0, \\ -\sum_{x'=x+1}^{0} \Delta f^2(x'), & \text{when } x \leq -1. \end{cases}$$

We can easily verify that $f(e,x) = f^1(e) + f^2(x)$ for $(e, x) \in \Sigma_2$. □

Proof of Lemma 2. We divide the proof into steps (i) to (iii) corresponding to the three properties of Definition 1 that g is supposed to satisfy.

(i) Suppose $\lambda \in \mathbb{Z}_{++}$, $e_1 \leq e_2$, $(m_1^*, y_1^*) \in \arg\min_{m \geq e_1, y \geq x_1}[K\delta(m - e_1) + f(m, y)]$, and $(m_2^*, y_2^*) \in \arg\min_{m \geq e_2+\lambda, y \geq x_2}[K\delta(m - e_2 - \lambda) + f(m, y)]$. First suppose $m_1^* \geq e_1 + 1$. Here, we have

$$g(e_1+1,x_1) - g(e_1,x_1) \leq [K\delta(m_1^* - e_1 - 1) + f(m_1^*,y_1^*)] - [K\delta(m_1^* - e_1) + f(m_1^*,y_1^*)] \leq 0,$$

and

$$g(e_2+\lambda,x_2) - g(e_2,x_2) \geq [K\delta(m_2^* - e_2 - \lambda) + f(m_2^*,y_2^*)] - [K\delta(m_2^* - e_2) + f(m_2^*,y_2^*)] \geq -K.$$

Therefore, $g(e_1 + 1, x_1) - g(e_1, x_1) \leq (g(e_2 + \lambda, x_2) - g(e_2, x_2) + K)/\lambda$, and hence g satisfies property (i). Then suppose $m_1^* = e_1$. We have a few cases to consider.

Case 1: $(e_1, x_1) \in \Sigma_1^c$, so that $e_1 + 1 \leq (x_1)^+ \leq (y_1^*)^+$ and hence $(e^1, y_1^*) \in \Sigma_1^c$. Then,

$$g(e_1+1,x_1) - g(e_1,x_1) \leq f(e_1+1,y_1^*) - f(e_1,y_1^*), \tag{A.4}$$

and

$$\begin{aligned} g(e_2+\lambda,x_2) - g(e_2,x_2) \geq [K\delta(m_2^* - e_2 - \lambda) + f(m_2^*,y_2^*)] - [K\delta(m_2^* - \lambda - e_2) \\ + f(m_2^* - \lambda, y_2^*)] = f(m_2^*,y_2^*) - f(m_2^* - \lambda, y_2^*). \end{aligned} \tag{A.5}$$

Because $m_2^* - \lambda \geq e_2 \geq e_1$ and $(e_1, y_1^*) \in \Sigma_1^c$, we can use f's satisfaction of (i) to reach

$$g(e_1+1,x_1) - g(e_1,x_1) \leq \frac{g(e_2+\lambda,x_2) - g(e_2,x_2) + K}{\lambda}. \tag{A.6}$$

But this exactly means that g satisfies property (i).

Case 2:
$(e_2, x_2) \in \Sigma_1$ and hence $e_2 \geq (x_2)^+$.

Subcase 2.1:
$(e^1, y_1^*) \in \Sigma_1^c$ or $(m_2^* - \lambda, y_2^*) \in \Sigma_1$. Then, Equations (A.4) and (A.5) still hold. Also because $m_2^* - \lambda \geq e_2 \geq e_1$, we can again use f's satisfaction of property (i) to derive (A.6) and hence g's satisfaction of the same property.

Subcase 2.2:
$(e_1, y_1^*) \in \Sigma_1$, $(m_2^* - \lambda, y_2^*) \in \Sigma_1^c$, and $(m_2^*, y_2^*) \in \Sigma_1$. Noting that $(e_2, x_2) \in \Sigma_1$, we have $m_2^* - \lambda \geq e_2 \geq (x_2)^+ \geq x_2$. Then,

$$\begin{aligned} g(e_2+\lambda,x_2)-g(e_2,x_2) &\geq [K\delta(m_2^*-e_2-\lambda)+f(m_2^*,y_2^*)]-[K\delta(m_2^*-\lambda-e_2)+f(m_2^*-\lambda,m_2^*-\lambda)] \\ &= f(m_2^*,y_2^*)-f(m_2^*-\lambda,m_2^*-\lambda)=f(m_2^*,y_2^*)-f(m_2^*,m_2^*-\lambda)+f(m_2^*,m_2^*-\lambda)-f(m_2^*-\lambda,m_2^*-\lambda). \end{aligned} \tag{A.7}$$

By the optimality of (m_2^*, y_2^*), it holds that

$$f(m_2^*,y_2^*)-f(m_2^*,m_2^*-\lambda)\leq 0. \tag{A.8}$$

On the other hand, we have

$$g(e_1+1,x_1)-g(e_1,x_1)\leq[f(e^1+1,y_1^*+1)]-[f(e^1,y_1^*)]=f(e^1+1,y_1^*+1)-f(e^1+1,y_1^*)+f(e^1+1,y_1^*)-f(e^1,y_1^*). \tag{A.9}$$

Since $(e_1, y_1^*)\in\Sigma_1$, it follows that $m_2^*-\lambda\geq e_2\geq e_1\geq y_1^*$ and $(e_1+1, y_1^*+1)\in\Sigma_2$. From $(m_2^*-\lambda, y_2^*)\in\Sigma_1^c$, we have $m_2^*-\lambda\leq(y_2^*)^+-1$. But $m_2^*-\lambda\geq(x_2)^+\geq 0$, and hence $y_2^*=(y_2^*)^+\geq 1$. Thus, as f holds property (ii), we have

$$f(e^1+1,y_1^*+1)-f(e^1+1,y_1^*)\leq\frac{f(m_2^*,y_2^*)-f(m_2^*,m_2^*-\lambda)}{y_2^*-(m_2^*-\lambda)}. \tag{A.10}$$

Because $(m_2^*,y_2^*)\in\Sigma_1$, it is true that $y_2^*-(m_2^*-\lambda)\leq\lambda$. In view of Equation (A.8), the right-hand side of Equation (A.10) is further smaller than $[f(m_2^*, y_2^*)-f(m_2^*, m_2^*-\lambda)]/\lambda$. Note that $m_2^*-\lambda\geq e_2\geq e_1\geq(y_1^*)^+\geq 0$, and hence $(m_2^*-\lambda, m_2^*-\lambda)\in\Sigma_1$. Since f satisfies property (i), we have

$$f(e^1+1,y_1^*)-f(e^1,y_1^*)\leq\frac{f(m_2^*,m_2^*-\lambda)-f(m_2^*-\lambda,m_2^*-\lambda)+K}{\lambda}. \tag{A.11}$$

Combining Equation (A.7) to Equation (A.11), we reach Equation (A.6) and g's satisfaction of property (i).

Subcase 2.3:
$(e_1, y_1^*)\in\Sigma_1$, $(m_2^*-\lambda, y_2^*)\in\Sigma_1^c$, and $(m_2^*, y_2^*)\in\Sigma_1^c$. In this case, we have $e_1\geq(y_1^*)^+\geq 0$ and $m_2^*\geq e_2+\lambda\geq e_1+\lambda\geq\lambda\geq 1$. Because $(m_2^*, y_2^*)\in\Sigma_1^c$ and $(e_2, x_2)\in\Sigma_1$, we further have $y_2^*-1\geq m_2^*\geq e_2+\lambda\geq x_2+\lambda$. Therefore, $g(e_1+1, x_1)-g(e_1, x_1)\leq f(e_1+1, y_1^*+1)-f(e_1, y_1^*)$, and

$$\begin{aligned} g(e_2+\lambda,x_2)-g(e_2,x_2) &\geq [K\delta(m_2^*-e_2-\lambda)+f(m_2^*,y_2^*)]-[K\delta(m_2^*-\lambda-e_2)+f(m_2^*-\lambda,y_2^*-\lambda)] \\ &= f(m_2^*,y_2^*)-f(m_2^*-\lambda,y_2^*-\lambda). \end{aligned}$$

Note that $(e_1, y_1^*)\in\Sigma_1$, $(m_2^*-\lambda, y_2^*-\lambda)\in\Sigma_1^c$, and $m_2^*-\lambda\geq e_2\geq e_1$. Since f satisfies property (iii), we have

$$f(e_1+1,y_1^*+1)-f(e_1,y_1^*)\leq\frac{f(m_2^*,y_2^*)-f(m_2^*-\lambda,y_2^*-\lambda)+K}{\lambda}.$$

Combine the above three and we can obtain Equation (A.6), indicating g's satisfaction of (i).

(ii) Suppose $x_1\leq x_2$, $(e_1,x_1+1)\in\Sigma_2$ and hence $(e_1)^+\geq x_1+1$, $(m_1^*,y_1^*)\in\arg\min_{m\geq e_1,y\geq x_1}[K\delta(m-e_1)+f(m,y)]$, and $(m_2^*,y_2^*)\in\arg\min_{m\geq e_2,y\geq x_2+1}[K\delta(m-e_2)+f(m,y)]$. There are two cases.

Case 1:
$y_1^*\geq x_1+1$. In this case, we have

$$g(e_1,x_1+1)-g(e_1,x_1)\leq[K\delta(m_1^*-e_1)+f(m_1^*,y_1^*)]-[K\delta(m_1^*-e_1)+f(m_1^*,y_1^*)]=0,$$

and

$$g(e_2,x_2+1)-g(e_2,x_2)\geq[K\delta(m_2^*-e_2)+f(m_2^*,y_2^*)]-[K\delta(m_2^*-e_2)+f(m_2^*,y_2^*)]=0.$$

Therefore,

$$g(e_1,x_1+1)-g(e_1,x_1)\leq g(e_2,x_2+1)-g(e_2,x_2), \tag{A.12}$$

and hence g satisfies property (ii).

Case 2:
$y_1^* = x_1$. In this case, we have

$$g(e_1, x_1+1) - g(e_1, x_1) \le [K\delta(m_1^* - e_1) + f(m_1^*, x^1+1)] - [K\delta(m_1^* - e_1) + f(m_1^*, x^1)],$$

which is equal to $f(m_1^*, x^1 + 1) - f(m_1^*, x^1)$; also,

$$g(e_2, x_2+1) - g(e_2, x_2) \ge [K\delta(m_2^* - e_2) + f(m_2^*, y_2^*)] - [K\delta(m_2^* - e_2) + f(m_2^*, y_2^* - 1)],$$

which is equal to $f(m_2^*, y_2^*) - f(m_2^*, y_2^* - 1)$. Since $m_1^* \ge e_1$ and $(e_1, x_1 + 1) \in \Sigma_2$, we have $(m_1^*, x_1 + 1) \in \Sigma_2$. Also, $y_2^* \ge x^2 + 1 \ge x^1 + 1$. Then, we can use f's satisfaction of property (ii) to reach Equation (A.12) and hence g's satisfaction of the same property.

(iii) Suppose $\lambda \in \mathbb{Z}_{++}$, $e_1 \le e_2$, $(e_1, x_1) \in \Sigma_1$ and hence $e_1 \ge (x_1)^+$, and $(e_2, x_2) \in \Sigma_1^c$ and hence $e_2 \le (x_2)^+ - 1$. Also, let $(m_1^*, y_1^*) \in \arg\min_{m \ge e_1, y \ge x_1} [K\delta(m - e_1) + f(m, y)]$, and $(m_2^*, y_2^*) \in \arg\min_{m \ge e_2+\lambda, y \ge x_2+\lambda} [K\delta(m - e_2 - \lambda) + f(m, y)]$. We consider a few cases.

Case 1:
$y_1^* \ge x_1 + 1$ and $m_1^* \ge e_1 + 1$. In this case, we have

$$g(e_1+1, x_1+1) - g(e_1, x_1) \le [K\delta(m_1^* - e_1 - 1) + f(m_1^*, y_1^*)] - [K + f(m_1^*, y_1^*)] \le 0,$$

and

$$g(e_2+\lambda, x_2+\lambda) - g(e_2, x_2) \ge [K\delta(m_2^* - e_2 - \lambda) + f(m_2^*, y_2^*)] - [K\delta(m_2^* - e_2) + f(m_2^*, y_2^*)] = -K.$$

Therefore,

$$g(e_1+1, x_1+1) - g(e_1, x_1) \le \frac{g(e_2+\lambda, x_2+\lambda) - g(e_2, x_2) + K}{\lambda}, \tag{A.13}$$

which means g's satisfaction of property (iii).

Case 2:
$y_1^* = x_1$ and $m_1^* \ge e_1 + 1$. In this case,

$$g(e_1+1, x_1+1) - g(e_1, x_1) \le [K\delta(m_1^* - e_1 - 1) + f(m_1^*, x^1+1)] - [K + f(m_1^*, x^1)],$$

which is below $f(m_1^*, x^1 + 1) - f(m_1^*, x^1)$; also,

$$g(e_2+\lambda, x_2+\lambda) - g(e_2, x_2) \ge [K\delta(m_2^* - e_2 - \lambda) + f(m_2^*, y_2^*)] - [K\delta(m_2^* - e_2) + f(m_2^*, y_2^* - \lambda)],$$

which is above $-K + f(m_2^*, y_2^*) - f(m_2^*, y_2^* - \lambda)$. From $(e_1, x_1) \in \Sigma_1$, we have $(x_1 + 1)^+ \le (x_1)^+ + 1 \le e_1 + 1 \le m_1^*$, and hence $(m_1^*, x_1 + 1) \in \Sigma_1 \subset \Sigma_2$. Also, we have $y_2^* - \lambda \ge x_2 \ge x_1$. Since f holds property (ii), we have

$$f(m_1^*, x^1+1) - f(m_1^*, x^1) \le \frac{f(m_2^*, y_2^*) - f(m_2^*, y_2^* - \lambda)}{\lambda}.$$

Combine the above three inequalities and we can obtain Equation (A.13), that g holds (iii).

Case 3:
$y_1^* \ge x_1$ and $m_1^* = e_1$.

Subcase 3.1:
$(e_1, y_1^*) \in \Sigma_1^c$, and hence $(y_1^*)^+ \ge e_1 + 1$. Both $(y_1^*)^+ \ge e_1 + 1$ and $e_1 \ge (x_1)^+$ imply $e_1 \ge 0$ and $y_1^* = (y_1^*)^+ \ge e_1 + 1 \ge x_1 + 1$. Therefore, $g(e_1 + 1, x_1 + 1) - g(e_1, x_1) \le f(e_1 + 1, y_1^*) - f(e_1, y_1^*)$. In the mean time,

$$g(e_2+\lambda,x_2+\lambda)-g(e_2,x_2)\geq[K\delta(m_2^*-e_2-\lambda)+f(m_2^*,y_2^*)]-[K\delta(m_2^*-\lambda-e_2)+f(m_2^*-\lambda,y_2^*)],$$

which is equal to $f(m_2^*, y_2^*) - f(m_2^* - \lambda, y_2^*)$. Because $m_2^* - \lambda \geq e_2 \geq e_1$ and f satisfies property (i), it concludes that g satisfies Equation (A.13) and hence property (iii).

Subcase 3.2:
$(e_1, y_1^*) \in \Sigma_1$ and hence $(y_1^*)^+ \leq e_1$. Then,

$$g(e_1+1,x_1+1)-g(e_1,x_1)\leq f(e_1+1,y_1^*+1)-f(e_1,y_1^*), \tag{A.14}$$

and

$$\begin{aligned}g(e_2+\lambda,x_2+\lambda)-g(e_2,x_2)\geq&[K\delta(m_2^*-e_2-\lambda)+f(m_2^*,y_2^*)]-[K\delta(m_2^*-\lambda-e_2)\\&+f(m_2^*-\lambda,y_2^*-\lambda)]\geq f(m_2^*,y_2^*)-f(m_2^*-\lambda,y_2^*-\lambda).\end{aligned} \tag{A.15}$$

If $(m_2^*, y_2^*) \in \Sigma_1^c$, then $(y_2^*)^+ \geq m_2^*+1$ and hence $(y_2^* - \lambda)^+ \geq (y_2^*)^+ - \lambda \geq m_2^* - \lambda + 1$. This will lead to $(m_2^* - \lambda, y_2^* - \lambda) \in \Sigma_1^c$. Since $m_2^* - \lambda \geq e_2 \geq e_1$ and f satisfies property (iii), it follows that g satisfies Equation (A.13) and hence property (iii). Otherwise, $(m_2^*, y_2^*) \in \Sigma_1$, and hence $(y_2^*)^+ \leq m_2^*$. Because $m_2^* - \lambda \geq e_2 \geq e_1 \geq 0$, we have $(y_2^* - \lambda)^+ \leq m_2^* - \lambda$ and hence $(m_2^* - \lambda, y_2^* - \lambda) \in \Sigma_1$. Since f holds property (i),

$$f(e_1+1,y_1^*)-f(e_1,y_1^*)\leq\frac{f(m_2^*,y_2^*-\lambda)-f(m_2^*-\lambda,y_2^*-\lambda)+K}{\lambda}. \tag{A.16}$$

Recall that $(e_1 + 1, y_1^* + 1) \in \Sigma_1 \subset \Sigma_2$. Also, from $(e_2, x_2) \in \Sigma_1^c$, we get $(x_2)^+ - 1 \geq e_2 \geq e_1 \geq 0$, $x_2 \geq 1$ and $x_2 > e_2$; while from $(e_1, y_1^*) \in \Sigma_1$, we have $e_1 \geq (y_1^*)^+ \geq y_1^*$. Hence, $y_2^* - \lambda \geq x_2 \geq e_2 \geq e_1 \geq y_1^*$. Since f satisfies property (ii), we have

$$f(e_1+1,y_1^*+1)-f(e_1+1,y_1^*)\leq\frac{f(m_2^*,y_2^*)-f(m_2^*,y_2^*-\lambda)}{\lambda}. \tag{A.17}$$

Summing up (A.16) and (A.17) and taking into account (A.14) and (A.15), we can get (A.13) and hence g's satisfaction of property (iii).

This thus wraps up our proof. □

Proof of Lemma 3. We first prove the result that $m^*(e + 1, x) \geq e + 2$ would lead to $m^*(e, x) \geq e + 1$. To this end, we use the simplified notation $(\bar{m}, \bar{y}) = (m^*(e + 1, x), y^*(e + 1, x))$.

For an arbitrary $y \geq x$, we consider two cases.
Case 1: $(e, y) \in \Sigma_1^c$ or $(e + 1, \bar{y}) \in \Sigma_1$. Then, since $f \in \mathcal{C}^2(K)$ and hence holds property (i),

$$f(e+1,y)-f(e,y)\leq\frac{K+f(\bar{m},\bar{y})-f(e+1,\bar{y})}{\bar{m}-(e+1)}.$$

On the other hand, due to the optimality of $(\bar{m}, \bar{y})$, we have $K + f(\bar{m}, \bar{y}) \leq f(e + 1, \bar{y})$. Therefore,

$$K+f(\bar{m},\bar{y})\leq f(e+1,y)\leq f(e,y). \tag{A.18}$$

Case 2: $(e, y) \in \Sigma_1$ and $(e + 1, \bar{y}) \in \Sigma_1^c$. Hence, by the definition of Σ_1, we have $e \geq y^+$ and $e + 1 < (\bar{y})^+$ we have $\bar{y} = (\bar{y})^+ > e + 1 \geq y^+ + 1 > 0$, and $\bar{y} \geq e + 2$. Moreover, $(e + 1)^+ \geq e + 1 \geq y^+ + 1 \geq y + 1 > y$, and hence $(e + 1, y + 1) \in \Sigma_2$. It follows from the optimality of $(\bar{m}, \bar{y})$ that

$$f(\bar{m},\bar{y})\leq f(\bar{m},e+1). \tag{A.19}$$

Since $f \in \mathcal{C}^2(K)$ and hence satisfies property (i),

$$f(e+1,y) - f(e,y) \leq \frac{K + f(\bar{m}, e+1) - f(e+1, e+1)}{\bar{m} - (e+1)}. \tag{A.20}$$

As f satisfies property (ii) as well, we have

$$f(e+1,y+1) - f(e+1,y) \leq \frac{f(\bar{m},\bar{y}) - f(\bar{m},e+1)}{\bar{y} - (e+1)}. \tag{A.21}$$

By the hypotheses that $(m^*, e^*) \in \mathcal{Z}^B$ and $\bar{m} \geq e + 2$, we obtain $(\bar{m}, \bar{y}) \in \Sigma_1$. This then leads to $\bar{m} \geq (\bar{y})^+ \geq \bar{y}$. Note that we also have Equation (A.19), the above Equation (A.21) further leads to

$$f(e+1,y+1) - f(e+1,y) \leq \frac{f(\bar{m},\bar{y}) - f(\bar{m},e+1)}{\bar{m} - (e+1)}. \tag{A.22}$$

Adding Equations (A.20) and (A.22), we have

$$f(e+1,y+1) - f(e,y) \leq \frac{K + f(\bar{m},\bar{y}) - f(e+1,e+1)}{\bar{m} - (e+1)},$$

which is negative due to the optimality of $(\bar{m}, \bar{y})$. As a result,

$$K + f(\bar{m},\bar{y}) \leq f(e+1,y+1) \leq f(e,y). \tag{A.23}$$

By Equations (A.18) and (A.23), and the arbitrariness of $y \geq x$, we see that $K + f(\bar{m}, \bar{y}) \leq f(e, y)$, and hence $m^*(e, x) \geq e + 1$.

We now prove that $m^*(e, x)$ is of the (s, S)-type at a fixed x. Because $m^*(e + 1, x) \geq e + 2$ leads to $m^*(e, x) \geq e + 1$, there exists $s(x) = \max\{e|m^*(e, x) \geq e + 1\}$ so that $m^*(e, x) \geq e + 1$ whenever $e \leq s(x)$ and $m^*(e, x) = e$ whenever $e \geq s(x) + 1$. We can let $s(x) = -\infty$ when $m^*(e) = e$ all the time and $s(x) = +\infty$ when $m^*(e) \geq e + 1$ all the time. When $s(x) > -\infty$, consider any $e \leq s(x) - 1$. Note (a) $m^*(e, x) \geq e + 1$ and (b) $m^*(e + 1) \geq e + 2$. Due to (a),

$$\min_{m \geq e, y \geq x}[K\delta(m-e) + f(m,y)] = \min_{m \geq e+1, y \geq x}[K\delta(m-e) + f(m,y)] = K + \min_{m \geq e+1, y \geq x} f(m,y);$$

meanwhile, due to (b),

$$\min_{m \geq e+1, y \geq x}[K\delta(m-e-1) + f(m,y)] = K + \min_{m \geq e+1, y \geq x} f(m,y).$$

In view of the identical right-hand sides and the meanings of the two optimal levels, we can obtain $m^*(e, x) = m^*(e + 1, x)$. Repeat this and we will get $m^*(e, x) = m^*(e + 1, x) = \cdots = m^*(s(x), x)$. Then, we can define $S(x) = m^*(s(x),x) \geq s(x)+1$. Of course this implies $S(x) = +\infty$ when $s(x) = +\infty$. We can simply let $S(x) = -\infty$ when $s(x) = -\infty$.

Let (e^0, x^0) be lexicographically the largest minimum of f, we show that $S(x^0)$ must be e^0, that is, the largest minimum of $f(\cdot, x^0)$. Since $S(x^0) = m^*(s(x^0), x^0)$ and x^0 minimizes $f(S(x^0), \cdot)$, it must be the case that

$$f(S(x^0), x^0) = \min_{y \geq x^0} f(S(x^0), y) = \min_{m \geq s(x^0), y \geq x^0} f(m,y) \leq \min_{m \geq s(x^0)} f(m, x^0).$$

We are done if $\min_{m \leq s(x^0)-1} f(m, x^0) \geq f(S(x^0), x^0)$, because this means the largest minimum of $f(\cdot, x^0)$, namely e^0, is at least $s(x^0)$. From the above, it will happen that $e^0 = S(x^0)$. Suppose, otherwise, $\min_{m \leq s(x^0)-1} f(m, x^0) < f(S(x^0), x^0)$. But this means $e^0 \leq s(x^0) - 1$ and $m^*(e^0, x^0) = e^0$ which is in contradiction of $s(x^0)$'s definition.

Now define three regions as follows:

$$R_1 = \{(e,x) \in \mathbb{Z}^2 | e \leq s(x), x \leq x^0\}, \tag{A.24}$$

$$R_2 = \{(e,x) \in \mathbb{Z}^2 | e \leq s(x), x \geq x^0 + 1\}, \tag{A.25}$$

$$R_3 = \{(e,x) \in \mathbb{Z}^2 | e \geq s(x) + 1\}. \quad (A.26)$$

The three regions R_1, R_2, and R_3 form an exact partition of $\mathbb{Z}^2$.

Consider $(e, x) \in R_1$, which according to the definition in Equation (A.24), satisfies $e \leq s(x)$ and $x \leq x^0$. We first prove $s(x) \leq S(x^0)$. Otherwise, suppose $s(x) \geq S(x^0) + 1$, then $m^*(S(x^0), x) = S(x) \geq s(x) + 1 \geq S(x^0) + 2$. This is a contradiction because $x \leq x^0$ and $(S(x^0), x^0)$ offers lexicographically the largest optimal solution for f. Now since $e \leq s(x) \leq S(x^0)$ and $x \leq x^0$, it must follow that $(m^*(e, x), y^*(e, x)) = (S(x^0), x^0)$.

Consider $(e, x) \in R_2$, which according to the definition in Equation (A.25), satisfies $e \leq s(x)$ and $x \geq x^0 + 1$. Since $(S(x^0), x^0) \in \Sigma_1 \subset \Sigma_2$, we have from Lemma 1's part (e) that

$$f(S(x^0), x^0) = f^1(S(x^0)) + f^2(x^0).$$

By the definition of $(S(x^0), x^0)$, we know x^0 is the largest optimal solution for $\min_{x' free} f^2(x')$. From property (ii) of a $\mathcal{C}^2(K)$ function, we can also see that $f^2(\cdot)$ is convex. Therefore, for $y^*(e, x) \geq x \geq x^0 + 1$, it must be true that $f^2(y^*(e, x)) \geq f^2(x)$.

As $(m^*, x^*) \in \mathcal{Z}^B$ and hence $m^*(\cdot, x)$ is of the (s, S)-type, it must be true that $m^*(e, x) = S(x)$. By the definition of $\mathcal{Z}^B$, this also leads to $(S(x), y^*(e, x)) \in \Sigma_1$. For the equally separable $f(S(x), y^*(e, x)) = f^1(S(x)) + f^2(y^*(e, x))$, the optimal choice for $y^*(e, x)$ must be x. That is, we must have $(m^*(e, x), y^*(e, x)) = (S(x), x)$.

Let $(e, x) \in R_3$, which due to the definition in Equation (A.26), satisfies $e \geq s(x) + 1$. By the definition of $s(x)$, we have $m^*(e, x) = e$ for $e \geq s(x) + 1$. Thus, $(m^*(e, x), y^*(e, x)) = (e, y^*(e, x))$. □

PROOF OF PROPOSITION 1. We focus on the definition (5). Suppose $e_1 \leq e_2$ and $\lambda \in \mathbb{Z}_{++}$ for the sake of property (i) of Definition 1.

First, suppose $(e_1, x_1) \in \Sigma_1^c$ and hence $e_1 \leq (x_1)^+ - 1$. Then,

$$V_{T+1}(e_1 + 1, x_1, p) - V_{T+1}(e_1, x_1, p) = K^B[\delta((x_1)^+ - e_1 - 1) - 1] - (p + q^B) \leq -(p + q^B),$$

because $\delta(y) \leq 1$ for any y. On the other hand,

$$V_{T+1}(e_2 + \lambda, x_2, p) - V_{T+1}(e_2, x_2, p) = K^B[\delta((x_2)^+ - e_2 - \lambda) - \delta((x_2)^+ - e_2)] - (p + q^B)\lambda,$$

which is gretar than $-K^B - (p + q^B)\lambda$ because $\delta(y) - \delta(z) \geq -1$ for any y, z. Next, suppose $(e_2, x_2) \in \Sigma_1$ and hence $e_2 \geq (x_2)^+$. Then, $V_{T+1}(e_2 + \lambda, x_2, p) = V_{T+1}(e_2, x_2, p) = 0$. Yet, $V_{T+1}(e_1 + 1, x_1, p) - V_{T+1}(e_1, x_1, p) \leq 0$. Combining the two cases, we always have property (i) holds.

To prove for property (ii) of Definition 1, suppose $x_1 \leq x_2$ and $(e_1)^+ \geq x_1 + 1$. Then,

$$\begin{aligned} V_{T+1}(e_1, x_1 + 1, p) - \mu(x_1 + 1)^- - v(x_1 + 1)^+ &= V_{T+1}(e_1, x_1, p) - \mu x_1^- - v x_1^+ \\ &= K^B\delta((e_1)^+ - e_1) + (p + q^B)((e_1)^+ - e_1). \end{aligned}$$

At the same time, $V_{T+1}(e_2, x_2 + 1, p) - \mu(x_2 + 1)^- - v(x_2 + 1)^+ \geq V_{T+1}(e_2, x_2, p) - \mu x_2^- - v x_2^+$. Due to the convexity of $\mu x^- + v x^+$ and the hypothesis that $x_1 \leq x_2$, property (ii) is true.

To prove for Property (iii) of Definition 1, suppose $e_1 \leq e_2$, $e_1 \geq (x_1)^+$, $e_2 \leq (x_2)^+ - 1$, and $\lambda \in \mathbb{Z}_{++}$. Then, $e_1 + 1 \geq (x_1)^+ + 1 \geq (x_1 + 1)^+$ and

$$V_{T+1}(e_1 + 1, x_1 + 1, p) - \mu(x_1 + 1)^- - v(x_1 + 1)^+ = V_{T+1}(e_1, x_1, p) - \mu x_1^- - v x_1^+ = 0.$$

From $(x_2)^+ - 1 \geq e_2 \geq e_1 \geq (x_1)^+ \geq 0$, we get $x_2 - 1 \geq e_2 \geq 0$. Then,

$$V_{T+1}(e_2 + \lambda, x_2 + \lambda, p) - \mu(x_2 + \lambda)^- - v(x_2 + \lambda)^+ = V_{T+1}(e_2, x_2, p) - \mu x_2^- - v x_2^+ = K^B + (p + q^B)(x_2 - e_2).$$

Note that $x_1^+ \leq e_1 \leq e_2 \leq x_2^+ - 1$ and hence $x_1 \leq x_2$. By the convexity of $\mu x^- + v x^+$, property (iii) holds. From the above, we see that $V_{T+1}(\cdot, \cdot, p) \in \mathcal{C}^2(K)$ under definition (5). □

Proof of Theorem 1. First, Proposition 1 has guaranteed $V_{T+1}(\cdot,\cdot,p)$'s membership in $\mathcal{C}^2(K^B)$ for any p. Then, for any $t = T, T-1, \ldots, 1$, we show that $V_{t+1}(\cdot,\cdot,p) \in \mathcal{C}^2(K^B)$ for any p would lead to $V_t(\cdot,\cdot,p) \in \mathcal{C}^2(K^B)$ for any p. First, note the definition of $G_t(m, y, p)$ from $V_{t+1}(e, x, p)$ via Equation (7). Thus, from (a), (b), and (c) of Lemma 1, we can see that $G_t(\cdot,\cdot,p) \in \mathcal{C}^2(K^B)$. Furthermore, note the definition of $V_t(e, x, p)$ from $G_t(m, y, p)$ via Equation (6). By (a) and (d) of Lemma 1 and Lemma 2, we obtain $V_t(\cdot,\cdot,p) \in \mathcal{C}^2(K^B)$. □

Proof of Proposition 2. To prove (U^B) under definition (5), we first note that

$$\begin{aligned} V_{T+1}(e,x,p) - V_{T+1}(e+\lambda,x,p) = {} & K^B[\delta(\max\{e,x^+\}-e) - \delta(\max\{e+\lambda,x^+\}-e-\lambda)] \\ & + (p+q^B)[\max\{e,x^+\} - \max\{e+\lambda,x^+\}+\lambda], \end{aligned} \tag{A.27}$$

which equals

$$\begin{cases} (p+q^B)\lambda, & \text{when } e \le x^+ - \lambda - 1, \\ K^B + (p+q^B)(x^+-e), & \text{when } x^+ - \lambda \le e \le x^+ - 1, \\ 0, & \text{when } e \ge x^+. \end{cases}$$

It is easy to check that the left-hand side of Equation (A.27) is less than $K^B + (p + q^B)\lambda = K^B + \lambda\pi^B_{T+1}(p)$. Hence ($\mathrm{U}^B$) is true for period $T + 1$. Now suppose (U^B) holds for period $t + 1$ for some $t = T, T-1, \ldots, 1$. For period t, we have from Equation (6),

$$\begin{aligned} V_t(x,e,p) &= -(p+q^B)e + \min_{m\ge e, y \ge x}[K^B\delta(m-e) + G_t(m,y,p)] \le K^B + (p+q^B)\lambda - (p+q^B)(e+\lambda) \\ &\quad + \min_{m \ge e+\lambda, y\ge x}[K^B\delta(m-e-\lambda) + G_t(m,y,p)] \\ &= K^B + (p+q^B)\lambda + V_t(e+\lambda,x,p). \end{aligned} \tag{A.28}$$

On the other hand, it follows from Equation (7) and the induction hypothesis that

$$\begin{aligned} & G_t(m-\lambda,y,p) - G_t(m,y,p) = -(p+q^B)\lambda \\ & \quad + \alpha\mathbb{E}[V_{t+1}(m-\lambda-D_t,y-D_t,P_{t+1}) - V_{t+1}(m-D_t,y-D_t,P_{t+1})|P_t = p] \\ & \le -(p+q^B)\lambda + \alpha K^B + \alpha\lambda\mathbb{E}[\pi^B_{t+1}(P_{t+1})|P_t=p]. \end{aligned}$$

The above in turn leads to

$$\begin{aligned} V_t(e,x,p) &= -(p+q^B)e + \min_{m\ge e,y\ge x}[K^B\delta(m-e) + G_t(m,y,p)] \\ &= -(p+q^B)e + \min_{m'\ge e+\lambda, y \ge x}[K^B\delta(m'-\lambda-e) + G_t(m'-\lambda,y,p)] \\ &\le \alpha(K^B + \lambda\mathbb{E}[\pi^B_{t+1}(P_{t+1})|P_t=p]) - (p+q^B)(e+\lambda) + \min_{m\ge e+\lambda,y\ge x}[K^B\delta(m-e-\lambda) + G_t(m,y,p)] \\ &\le K^B + \alpha\lambda\mathbb{E}[\pi^B_{t+1}(P_{t+1})|P_t=p] + V_t(e+\lambda,x,p). \end{aligned}$$

Combining this with Equation (A.28), we obtain

$$V_t(e,x,p) - V_t(e+\lambda,x,p) \le K^B + \lambda\min\{p+q^B, \alpha\mathbb{E}[\pi^B_{t+1}(P_{t+1})|P_t=p]\},$$

which is equal to $K^B + \lambda\pi^B_t(p)$. We have thus completed the induction for (U^B).

To prove (U'^B), we use the short-hand notation (m^*_t, y^*_t) for $(m^*_t(e, x, p), y^*_t(e, x, p))$. Suppose $m^*_t \ge e + 1$. Then, from Equation (7), we have

$$\begin{aligned} & G_t(e,y^*_t,p) - [K^B\delta(m^*_t-e) + G_t(m^*_t,y^*_t,p)] = -K^B - (p+q^B)(m^*_t-e) \\ & \quad + \alpha\mathbb{E}[V_{t+1}(e-D_t,y^*_t-D_t,P_{t+1}) - V_{t+1}(m^*_t-D_t,y^*_t-D_t,P_{t+1})|P_t=p], \end{aligned}$$

which by (U'^B) is less than

$$\begin{aligned}&-K^B-(p+q^B)(m_t^*-e)+\alpha\mathbb{E}[K^B+(m_t^*-e)\pi_{t+1}^B(P_{t+1})|P_t=p]\\&\quad=-(K^B-\alpha K^B)-(p+q^B-\alpha\mathbb{E}[\pi_{t+1}^B(P_{t+1})|P_t=p])(m_t^*-e).\end{aligned}$$

When $p+q^B>\alpha\mathbb{E}[\pi_{t+1}^B(P_{t+1})|P_t=p]$, the above would be strictly negative. But by Equation (6), this would mean that $m=e$ is a better choice than the current $m_t^*\geq e+1$. Therefore, for $m_t^*\geq e+1$, it is necessary that $p+q^B\leq\alpha\mathbb{E}[\pi_{t+1}^B(P_{t+1})|P_t=p]$.

To prove (L^B), note that Equation (5) under $e\leq x^+-1$ would lead to

$$\begin{aligned}V_{T+1}(e,x,p)-V_{T+1}(e+1,x,p)&=K^B[\delta(\max\{e,x^+\}-e)-\delta(\max\{e+1,x^+\}\\&\quad-e-1)]+p+q^B(\max\{e,x^+\}-\max\{e+1,x^+\}+1),\end{aligned}\tag{A.29}$$

which equals

$$\begin{cases}p+q^B & \text{when } e\leq x^+-2,\\ K^B+p+q^B & \text{when } e=x^+-1.\end{cases}$$

Thus, (L^B) is true for period $T+1$. Now suppose (L^B) holds for period $t+1$ for some $t=T, T-1, \ldots, 1$. For period t, let us again use the short-hand notation (m_t^*, y_t^*) for $(m_t^*(e, x, p), y_t^*(e, x, p))$. If $m_t^*\geq e+1$, we have, according to Equation (6),

$$\begin{aligned}V_t(e+1,x,p)&=-(p+q^B)(e+1)+\min_{m\geq e+1,y\geq x}[K\delta(m-e-1)+G_t(m,y,p)]\\&\leq-(p+q^B)(e+1)+K+G_t(m_t^*,y_t^*,p)=-p-q^B+V_t(e,x,p).\end{aligned}$$

For $m_t^*=e$, we have, by Equations (6) and (7),

$$\begin{aligned}V_t(e+1,x,p)-V_t(e,x,p)&\leq[-(p+q^B)(e+1)+G_t(e+1,y_t^*,p)]-[-(p+q^B)e+G_t(e,y_t^*,p)]\\&=\alpha\mathbb{E}[V_{t+1}(e+1-D_t,y_t^*-D_t,P_{t+1})-V_{t+1}(e-D_t,y_t^*-D_t,P_{t+1})|P_t=p].\end{aligned}\tag{A.30}$$

Note that

$$(y_t^*-D_t)^+\geq(y_t^*)^+-D_t\geq x^+-D_t\geq e-D_t+1.$$

Hence the induction hypothesis would lead to the right-hand side of Equation (A.30) being less than $-\alpha\mathbb{E}[\pi_{t+1}^B(P_{t+1})|P_t=p]$. Combining the two cases, we see that

$$V_t(e,x,p)-V_t(e+1,x,p)\geq\min\{p+q^B,\alpha\mathbb{E}[\pi_{t+1}^B(P_{t+1})|P_t=p]\}=\pi_t^B(p).$$

We have thus completed the induction process for property (L^B).

For convenience, we use the short-hand notation (m_t^*, y_t^*) for $(m_t^*(e, x, p), y_t^*(e, x, p))$. Now in contrary, suppose $m_t^*\geq e+1$ and yet $m_t^*\leq(y_t^*)^+-1$. Then, from Equation (7) we have

$$G_t(m_t^*+1,y_t^*,p)-G_t(m_t^*,y_t^*,p)=p+q^B+\alpha\mathbb{E}[V_{t+1}(m_t^*+1-D_t,y_t^*-D_t,P_{t+1})-V_{t+1}(m_t^*-D_t,y_t^*-D_t,P_{t+1})|P_t=p],$$

which is less than $p+q^B-\alpha\mathbb{E}[\pi_{t+1}^B(P_{t+1})|P_t=p]$ due to the fact that $m_t^*-D_t\leq(y_t^*)^+-1-D_t\leq(y_t^*-D_t)^+-1$ and property (L^B) of Proposition 2. The latter is also negative by the hypothesis that $m_t^*\geq e+1$ and (U'^B) of Proposition 2. This means (m_t^*+1, y_t^*) is as good as (m_t^*, y_t^*). But this contradicts m_t^*'s definition as the largest optimal post-purchase emissions level. □

Proof of Theorem 2. We can take $(m_t^*(p), y_t^*(p))$ to be lexicographically the largest solution to Equation (6), which is in $\mathcal{Z}$. Then, by Proposition 2, we have $(m_t^*(p), y_t^*(p))\in\mathcal{Z}^B$. Also, we know from Theorem 1 that $G_t(\cdot, \cdot, p)\in\mathcal{C}^2(K^B)$. These facts along with Lemma 3 would together lead to the desired policy shape.

Appendix B. Technical Details in Section 5

Proof of Lemma 4. As parts (a) and (b) are obvious, we concentrate on parts (c)–(e).

For part (c), we know $g(e, x) = E[f(e + \Psi, x + \Phi)]$ is well defined due to the integrability hypothesis. Suppose $e^S \le e^B$, λ^B, $\lambda^S \in \mathbb{Z}_{++}$, $e^S + \psi \le e^B + \psi$. Because $f \in \mathcal{C}^2_-(K^B, K^S, q)$,

$$[f(e^S+\psi,x^S+\phi)-f(e^S+\psi-\lambda^S,x^S+\phi)-K^S]/\lambda^S-[f(e^B+\psi+\lambda^B,x^B+\phi)+K^B-f(e^B+\psi,x^B+\phi)]/\lambda^B\le q.$$

But by taking average over (Ψ, Φ), this leads to

$$\frac{g(e^S,x^S)-g(e^S-\lambda^S,x^S)-K^S}{\lambda^S}-\frac{g(e^B+\lambda^B,x^B)+K^B-g(e^B,x^B)}{\lambda^B}\le q.$$

According to Definition 2, this means $g \in \mathcal{C}^2_-(K^B, K^S, q)$.

For part (d), suppose $\lambda^B \in \mathbb{Z}_{++}$, $\lambda^S \in \mathbb{Z}_{++}$, and $e^S \le e^B$. Let

$$m^{S^*}\in\arg\min_{m\text{ free}}[K^B\delta(m-e^S+\lambda^S)+K^S\delta(e^S-\lambda^S-m)+q(e^S-\lambda^S-m)^++j(m,x)],$$

and

$$m^{B^*}\in\arg\min_{m\text{ free}}[K^B\delta(m-e^B-\lambda^B)+K^S\delta(e^B+\lambda^B-m)+q(e^B+\lambda^B-m)^++j(m,x)].$$

Then,

$$\begin{aligned}g(e^S,x^S)-g(e^S-\lambda^S,x^S)\le{}&[K^B\delta(m^{S^*}-e^S)+K^S\delta(e^S-m^{S^*})+q(e^S-m^{S^*})^++j(m^{S^*},x^S)]\\&-[K^B\delta(m^{S^*}-e^S+\lambda^S)+K^S\delta(e^S-\lambda^S-m^{S^*})+q(e^S-\lambda^S-m^{S^*})^++j(m^{S^*},x^S)],\end{aligned}$$

which is less than $K^S + q\lambda^S$ due to the monotonicity of the $\delta(\cdot)$ function, and

$$\begin{aligned}g(e^B+\lambda^B,x^B)-g(e^B,x^B)\ge{}&[K^B\delta(m^{B^*}-e^B-\lambda^B)+K^S\delta(e^B+\lambda^B-m^{B^*})+q(e^B+\lambda^B-m^{B^*})^++j(m^{B^*},x^B)]\\&-[K^B\delta(m^{B^*}-e^B)+K^S\delta(e^B-m^{B^*})+q(e^B-m^{B^*})^++j(m^{B^*},x^B)],\end{aligned}$$

which is greater than $-K^B$ again due to the monotonicity of the $\delta(\cdot)$ function. Hence, in light of Definition 2, we have $g \in \mathcal{C}^2_-(K^B, K^S, q)$.

For part (e), suppose $\lambda^B \in \mathbb{Z}_{++}$, $\lambda^S \in \mathbb{Z}_{++}$, and $m^S \le m^B$. Let

$$\begin{cases}y^{S*}\in\arg\min_{y\ge x^S}f(m^S-\lambda^S,y),\\ y^{B*}\in\arg\min_{y\ge x^B}f(m^B+\lambda^B,y).\end{cases}$$

Then,

$$\begin{cases}j(m^S-\lambda^S,x^S)=f(m^S-\lambda^S,y^{S*}),\\ j(m^S,x^S)\le f(m^S,y^{S*}),\\ j(m^B+\lambda^B,x^B)=f(m^B+\lambda^B,y^{B*}),\\ j(m^B,x^B)\le f(m^B,y^{B*}).\end{cases}$$

Together with $f \in \mathcal{C}^2_-(K^B, K^S, q)$, the above leads to

$$\begin{aligned}[j(m^S,x^S)-j(m^S-\lambda^S,x^S)-K^S]/\lambda^S&\le[f(m^S,y^{S*})-f(m^S-\lambda^S,y^{S*})-K^S]/\lambda^S\\&\le[f(m^B+\lambda^B,y^{B*})+K^B-f(m^B,y^{B*})]/\lambda^B+q\le[j(m^B+\lambda^B,x^B)+K^B-j(m^B,x^B)]/\lambda^B+q.\end{aligned}$$

Thus, $j \in \mathcal{C}^2_-(K^B, K^S, q)$ as well. □

Proof of Lemma 5. Let $y^*(e, x)$ be the largest optimal y for Equation (15) after the optimal $m^*(e, x)$ to Equation (14) has been settled by Equation (19). For

$$y^0(m,x)=\max argmin_{y\ge x}f(m,y), \tag{B.1}$$

we have $j(m, x) = f(m, y^0(m, x))$ and $y^*(e, x) = y^0(m^*(e, x), x)$.

Suppose, on the contrary, that for some x^B, $x^S \in \mathbb{Z}$, $\tilde{e}^B \in E^B(x^B)$, and $\tilde{e}^S \in E^S(x^S)$, we have $\tilde{e}^B \geq \tilde{e}^S$. Note that, by Equations (14) and (19),

$$\begin{cases} E^B(x^B) = \{e \in \mathbb{Z} | m^*(e, x^B) = m^B(e, x^B) \geq e + 1\}, \\ E^S(x^S) = \{e \in \mathbb{Z} | m^*(e, x^S) = m^S(e, x^S) \leq e - 1\}. \end{cases} \tag{B.2}$$

So the above hypothesis would result with $m^*(\tilde{e}^B, x^B) \geq \tilde{e}^B + 1 \geq \tilde{e}^S + 1 \geq m^*(\tilde{e}^S, x^S) + 2$. On the other hand,

$$K^B + f(m^*(\tilde{e}^B, x^B), y^*(\tilde{e}^B, x^B)) < f(\tilde{e}^B, y^0(\tilde{e}^B, x^B)) \leq f(\tilde{e}^B, y^0(m^*(\tilde{e}^B, x^B), x^B)) = f(\tilde{e}^B, y^*(\tilde{e}^B, x^B)),$$

and

$$K^S + q(\tilde{e}^S - m^*(\tilde{e}^S, x^S)) + f(m^*(\tilde{e}^S, x^S), y^*(\tilde{e}^S, x^S)) < f(\tilde{e}^S, y^0(\tilde{e}^S, x^S)) \leq f(\tilde{e}^S, y^0(m^*(\tilde{e}^S, x^S), x^S)) = f(\tilde{e}^S, y^*(\tilde{e}^S, x^S)).$$

These lead to

$$\begin{aligned} &[f(\tilde{e}^S, y^*(\tilde{e}^S, x^S)) - f(m^*(\tilde{e}^S, x^S), y^*(\tilde{e}^S, x^S)) - K^S]/[\tilde{e}^S - m^*(\tilde{e}^S, x^S)] - q \\ &> 0 > [f(m^*(\tilde{e}^B, x^B), y^*(\tilde{e}^B, x^B)) + K^B - f(\tilde{e}^B, y^*(\tilde{e}^B, x^B))]/[m^*(\tilde{e}^B, x^B) - \tilde{e}^B]. \end{aligned}$$

But this contradicts the hypothesis that $f \in \mathcal{C}^2_-(K^B, K^S, q)$.

When $e \leq e^B(x)$, we must have $e \notin E^S(x)$ and hence $g^B(e, x) \leq g^S(e, x)$. Thus, depending on whether $e \in E^B(x)$ or $\mathbb{Z} \setminus (E^B(x) \cup E^S(x))$, we have $m^*(e, x) = m^B(e, x)$ or e. When $e = e^B(x)$, however, the fact that $e \in E^B(x)$ prompts $m^*(e, x) = m^B(e, x)$. When $e^B(x) + 1 \leq e \leq e^S(x) - 1$, we have $e \in \mathbb{Z} \setminus (E^B(x) \cup E^S(x))$ and hence $m^*(e, x) = e$. When $e \geq e^S(x)$, we have a case symmetrical to the first one; so symmetrically, we have $m^*(e, x) = e$ or $m^S(e, x)$.

A Counter Example with $j \in \mathcal{C}^2_-(K^B, K^S, q)$ **but** $e^B_L(x) < e^B(x)$: In this example, $K^B = 5$, $K^S = 10$, and $q = 0$. Also, we let

$$j(e, x) = \begin{cases} -5e + 20 & e \leq 0, \\ 10 & 1 \leq e \leq 10, \\ 0 & e \geq 11. \end{cases}$$

which is a member of $\mathcal{C}^2_-(5, 10, 0)$. However, we find $e^B_L(0) = 0 < 10 = e^B(0)$. □

Proof of Proposition 3. We show that $V_{T+1}(\cdot, \cdot, p) \in \mathcal{C}^2_-(K^B, K^S, q^B + q^S)$ for any p, where V_{T+1} is given by Equation (11). To this end, we show that

$$V_{T+1}(e^S, x^S, p) - V_{T+1}(e^S - \lambda^S, x^S, p) \leq K^S - (p - q^S)\lambda^S, \tag{B.3}$$

and

$$V_{T+1}(e^B + \lambda^B, x^B, p) - V_{T+1}(e^B, x^B, p) \geq -K^B - (p + q^B)\lambda^B, \tag{B.4}$$

for any $e^S \leq e^B$ and λ^S, $\lambda^B \in \mathbb{Z}_{++}$. The two inequalities would together lead to

$$\begin{aligned} &\frac{V_{T+1}(e^S, x^S, p) - V_{T+1}(e^S - \lambda^S, x^S, p) - K^S}{\lambda^S} - \frac{V_{T+1}(e^B + \lambda^B, x^B, p) - V_{T+1}(e^B, x^B, p) + K^B}{\lambda^B} \\ &\leq -(p - q^S) + (p + q^B) = q^B + q^S. \end{aligned}$$

In either Equation (B.3) or Equation (B.4), the left-hand side is the difference between two V_{T+1}-terms with the same x argument: one with $x = x^S$ and the other $x = x^B$. In either case, the term $\mu x^- + \nu x^+$ of Equation (11) will be canceled out and contribute nothing in the end. Hence, we might as well just pretend $\mu = \nu = 0$ in ensuing derivations. We now first prove Equation (B.3) by discussing three cases.

Case 1: $(x^S)^+ \geq e^S + 1$. In this case, $V_{T+1}(e^S, x^S, p) = K^B + (p + q^B)[(x^S)^+ - e^S]$ and $V_{T+1}(e^S - \lambda^S, x^S, p) = K^B + (p + q^B)[(x^S)^+ - e^S + \lambda^S]$. Then,

$$V_{T+1}(e^S, x^S, p) - V_{T+1}(e^S - \lambda^S, x^S, p) = -(p + q^B)\lambda^S \leq K^S - (p + q^B)\lambda^S.$$

Case 2: $e^S - \lambda^S + 1 \le (x^S)^+ \le e^S$. In this case,

$$V_{T+1}(e^S, x^S, p) = -\{K^S\delta((x^S)^+ - e^S) + (p - q^S)[(x^S)^+ - e^S]\}^- \le -\{K^S + (p - q^S)[(x^S)^+ - e^S]\}^-,$$

and $V_{T+1}(e^S - \lambda^S, x^S, p) = K^B + (p + q^B)[(x^S)^+ - e^S + \lambda^S]$.
If $K^S + (p - q^S)[(x^S)^- - e^S] < 0$, then $V_{T+1}(e^S, x^S, p) \le K^S + (p - q^S)[(x^S)^- - e^S]$. Thus,

$$\begin{aligned} V_{T+1}(e^S, x^S, p) - V_{T+1}(e^S - \lambda^S, x^S, p) &\le K^S + (p - q^S)[(x^S)^+ - e^S] - K^B - (p + q^B)[(x^S)^+ - e^S + \lambda^S] \\ &= K^S - K^B - (q^S + q^B)[(x^S)^+ - e^S] - (p + q^B)\lambda^S \le K^S - K^B + (q^S + q^B)\lambda^S - (p + q^B)\lambda^S \\ &= K^S - K^B - (p - q^S)\lambda^S \le K^S - (p - q^S)\lambda^S. \end{aligned}$$

If $K^S + (p - q^S)[(x^S)^+ - e^S] \ge 0$, we have $V_{T+1}(e^S, x^S, p) \le 0$. The inequality $K^S + (p - q^S)[(x^S)^+ - e^S] \ge 0$ implies that $-(p - q^S)[(x^S)^+ - e^S] \le K^S$. Then,

$$\begin{aligned} V_{T+1}(e^S, x^S, p) - V_{T+1}(e^S - \lambda^S, x^S, p) &= -K^B - (p + q^B)[(x^S)^+ - e^S + \lambda^S] \\ &= -K^B - (p - q^S)[(x^S)^+ - e^S + \lambda^S] - (q^S + q^B)[(x^S)^+ - e^S + \lambda^S] \\ &\le -K^B - (p - q^S)[(x^S)^+ - e^S + \lambda^S] = -K^B - (p - q^S)\lambda^S - (p - q^S)[(x^S)^+ - e^S] \\ &\le -K^B - (p - q^S)\lambda^S + K^S \le K^S - (p - q^S)\lambda^S. \end{aligned}$$

Case 3: $(x^S)^+ \le e^S - \lambda^S$. In this case,

$$V_{T+1}(e^S - \lambda^S, x^S, p) = -[K^S\delta(e^S - \lambda^S - (x^S)^+) + (p - q^S)[(x^S)^+ - e^S + \lambda^S]]^-,$$

and $V_{T+1}(e^S, x^S, p) = -[K^S + (p - q^S)[(x^S)^+ - e^S]]^-$.
For ease of notation, we write $\delta(e^S - \lambda^S - (x^S)^+)$ simply as δ_0. If $p > q^S$,

$$\begin{aligned} &V_{T+1}(e^S, x^S, p) - V_{T+1}(e^S - \lambda^S, x^S, p) \\ &\le -[K^S + (p - q^S)[(x^S)^+ - e^S]]^- + [K^S\delta_0 + (p - q^S)[(x^S)^+ - e^S + \lambda^S]]^- \\ &\le K^S + (p - q^S)[(x^S)^+ - e^S] + [K^S\delta_0 + (p - q^S)[(x^S)^+ - e^S + \lambda^S]]^- \\ &= -\lambda^S(p - q^S) + K^S(1 - \delta_0) + K^S\delta_0 + (p - q^S)[(x^S)^+ - e^S + \lambda^S] \\ &\quad + [K^S + (p - q^S)[(x^S)^+ - e^S + \lambda^S]]^- = -\lambda^S(p - q^S) + K^S(1 - \delta_0) + [K^S\delta_0 + (p - q^S)[(x^S)^+ - e^S + \lambda]]^+ \\ &\le -\lambda^S(p - q^S) + K^S(1 - \delta_0) + K^S\delta_0 = K^S - \lambda^S(p - q^S), \end{aligned}$$

where the last inequality holds because $(x^S)^+ - e^S + \lambda^S < 0$. If $p \le q^S$, both $V_{t+1}(e^S - \lambda, x^S, p)$ and $V_{T+1}(e^S, x^S, p)$ are equal to zero. Thus,

$$V_{T+1}(e^S, x^S, p) - V_{t+1}(e^s - \lambda, x^s, p) = 0 \le K^S - (p - q^S)\lambda^S.$$

Next, we prove Equation (B.4) again by discussing three cases.
Case 1: $(x^B)^+ \ge e^B + \lambda^B$. In this case, $V_{T+1}(e^B + \lambda^B, x^B, p) = K^B\delta((x^B)^+ - e^B - \lambda^B) + (p + q^B)[(x^B)^+ - e^B - \lambda^B]$ and $V_{T+1}(e^B, x^B, p) = K^B + (p + q^B)[(x^B)^+ - e^B]$. Thus,

$$V_{T+1}(e^B + \lambda^B, x^B, p) - V_{T+1}(e^B, x^B, p) = K^B\delta((x^B)^+ - e^B - \lambda^B) - K^B - \lambda^B(p + q^B) \ge -K^B - \lambda^B(p + q^B).$$

Case 2: $e^B + 1 \le (x^B)^+ \le e^B + \lambda^B - 1$. In this case, $V_{T+1}(e^B + \lambda^B, x^B, p) = -\{K^S + (p - q^S)[(x^B)^+ - e^B - \lambda^B]\}^-$ and $V_{T+1}(e^B, x^B, p) = K^B\delta((x^B)^+ - e^B) + (p + q^B)[(x^B)^+ - e^B] \le K^B + (p + q^B)[(x^B)^+ - e^B]$. If $K^S + (p - q^S)[(x^B)^+ - e^B - \lambda^B] < 0$, we have

$$\begin{aligned} &V_{T+1}(e^B + \lambda^B, x^B, p) - V_{T+1}(e^B, x^B, p) \ge K^S + (p - q^S)[(x^B)^+ - e^B - \lambda^B] - K^B - (p + q^B)[(x^B)^+ - e^B] \\ &= K^S - K^B - (q^S + q^B)[(x^B)^+ - e^B] - (p - q^S)\lambda^B \ge K^S - K^B - (q^S + q^B)\lambda^B - (p - q^S)\lambda^B = K^S - K^B - (q^B + p)\lambda^B. \end{aligned}$$

If $K^S + (p - q^S)[(x^B)^+ - e^B - \lambda^B] \ge 0$, we have $V_{T+1}(e^B + \lambda^B, x^B, p) = 0$. Since $(x^B)^+ - e^B \le \lambda^B - 1$, we have

$$V_{T+1}(e^B + \lambda^B, x^B, p) - V_{T+1}(e^B, x^B, p) \ge -K^B - (p + q^B)[(x^B)^+ - e^B] \ge -K^B - (p + q^B)\lambda^B.$$

Case 3: $(x^B)^+ \le e^B$. In this case, $V_{T+1}(e^B+\lambda^B, x^B, p) = -\{K^S + (p-q^S)[(x^B)^+ - e^B - \lambda^B]\}^-$ and $V_{T+1}(e^B, x^B, p) = -\{K^S + (p-q^S)[(x^B)^+ - e^B]\}^-$.

If $K^S + (p-q^S)[(x^B)^+ - e^B - \lambda^B] < 0$, we have

$$V_{T+1}(e^B+\lambda^B,x^B,p) - V_{T+1}(e^B,x^B,p) = K^S + (p-q^S)[(x^B)^+ - e^B - \lambda^B] + \left\{K^S + (p-q^S)[(x^B)^+ - e^B]\right\}^- \ge K^S + (p-q^S)[(x^B)^+ - e^B - \lambda^B] - \left\{K^S + (p-q^S)[(x^B)^+ - e^B]\right\} \ge -(p-q^S)\lambda^B \ge -K^B - (p-q^S)\lambda^B \ge -K^B - (p+q^B)\lambda^B.$$

If $K^S + (p-q^S)[(x^B)^+ - e^B - \lambda^B] \ge 0$, we have

$$V_{T+1}(e^B+\lambda^B,x^B,p) - V_{T+1}(e^B,x^B,p) = 0 + \left\{K^S + (p-q^S)[(x^B)^+ - e^B]\right\}^- \ge 0 \ge -K^B - (p+q^B)\lambda^B.$$

Thus, $V_{T+1}(\cdot,\cdot,p) \in \mathcal{C}^2_-(K^B, K^S, q^B+q^S)$ for any p under definition (11). □

Proof of Theorem 3. For $V_{T+1}(\cdot,\cdot,p)$ defined through Equation (11), Proposition 3 has guaranteed its membership in $\mathcal{C}^2_-(K^B, K^S, q^B+q^S)$ for any p. Now for $t = T, T-1, \ldots, 1$, note the definition of $G_t(m, y, p)$ from $V_{t+1}(e, x, p)$ via Equation (7). Thus, from (a), (b), and (c) of Lemma 4, we can see that $G_t(\cdot,\cdot,p) \in \mathcal{C}^2_-(K^B, K^S, q^B+q^S)$. Also, note the definition of $J_t(m, x, p)$ from $G_t(m, y, p)$ via Equation (13). Then, by (e) of Lemma 4, we obtain $J_t(\cdot,\cdot,p) \in \mathcal{C}^2_-(K^B, K^S, q^B+q^S)$. Now come to the definition of $V_t(e, x, p)$ from $J_t(m, x, p)$ via Equation (12). Due to (d) of Lemma 4, we obtain that $V_t(\cdot,\cdot,p) \in \mathcal{C}^2_-(K^B, K^S, q^B+q^S)$ as well. □

Proof of Theorem 4. Note that Equation (12) can be rewritten as

$$V_t(e,x,p) = -(p+q^B)e + \min\{V_t^B(e,x,p), V_t^S(e,x,p)\},$$

where

$$\begin{cases} V_t^B(e,x,p) = \min_{m\ge e}[K^B\delta(m-e) + J_t(m,x,p)], \\ V_t^S(e,x,p) = \min_{m\le e}[K^S\delta(e-m) + (q^B+q^S)(e-m) + J_t(m,x,p)]. \end{cases}$$

Let us define $m_t^B(e, x, p)$ and $m_t^S(e, x, p)$ so that

$$\begin{cases} m_t^B(e,x,p) = \min\arg\min_{m\ge e+1} J_t(m,x,p), \\ m_t^S(e,x,p) = \max\arg\min_{m\le e-1}[(q^B+q^S)(e-m) + J_t(m,x,p)]. \end{cases}$$

As shown by Theorem 3, $J_t(\cdot,\cdot,p) \in \mathcal{C}^2_-(K^B, K^S, q^B+q^S)$. So inspired by Equation (18), we let

$$\begin{cases} E_t^B(x,p) = \{e\in\mathbb{Z} | V_t^B(e,x,p) < J_t(e,x,p) = V_t^S(e,x,p)\}, \\ E_t^S(x,p) = \{e\in\mathbb{Z} | V_t^B(e,x,p) = J_t(e,x,p) > V_t^S(e,x,p)\}. \end{cases} \tag{B.5}$$

Also, following Equation (19), we know that

$$m_t^*(e,x,p) = \begin{cases} m_t^B(e,x,p), & \text{when } e\in E_t^B(x,p), \\ m_t^S(e,x,p), & \text{when } e\in E_t^S(x,p), \\ e, & \text{when } e\in \mathbb{Z}\setminus(E_t^B(x,p)\cup E_t^S(x,p)), \end{cases} \tag{B.6}$$

would constitute an optimal m for Equation (12). As pointed out in Equation (20), we can define $e_t^B(x, p)$ and $e_t^S(x, p)$ so that

$$\begin{cases} e_t^B(x,p) = \max\{e\in\mathbb{Z} | e\in E_t^B(x,p)\}, \\ e_t^S(x,p) = \min\{e\in\mathbb{Z} | e\in E_t^S(x,p)\}, \end{cases} \tag{B.7}$$

and then claim the policy form as depicted in Lemma 5.

Appendix C. Technical Details in Section 6

Proof of Proposition 4. We prove by induction. By a comparison between Equations (11), (21), and (23), and the fact that $V^U_{T+1}(e, x, p) = V_{T+1}(e, x, p)$, we can decide that

$$V^L_{T+1}(e,x,p) \le V_{T+1}(e,x,p) \le V^U_{T+1}(e,x,p). \tag{C.1}$$

Suppose for some $t = T, T-1, \ldots, 1$, an inequality in the fashion of Equation (C.1) is true for period $t+1$. Then, by comparing Equations (12), (22), and (23), we have

$$V^L_t(e,x,p) \le V_t(e,x,p).$$

By comparing Equations (12) and (24), while noting the feasibility of $m^1_t(e, p)$ and $y^2_t(x, p)$, we see

$$V_t(e,x,p) \le V^U_t(e,x,p).$$

We have thus completed the induction process.

A Counter Example related to Weak (K^B, K^S, q)**-convexity**: Here, $K^B = 5$, $K^S = 2$, and $q = 0$. Define $f: \mathbb{Z} \to \mathbb{R}$ so that $f(e) = 4$ when $e \le -1$, $f(e) = 11$ when $0 \le e \le 2$, $f(e) = 6$ when $e = 3$, and $f(e) = 0$ when $e \ge 4$. We can check that f is weak (5, 2, 1, 2)-convex. Now consider $g: \mathbb{Z} \to \mathbb{R}$ satisfying

$$g(e) = \min_{m \text{ free}} [K^B \delta(m-e) + K^S \delta(e-m) + f(m)].$$

We can compute to obtain $g(e) = 4$ when $e \le -1$, $g(e) = 6$ when $0 \le e \le 1$, $g(e) = 11$ when $e = 2$, $g(e) = 5$ when $e = 3$, and $g(e) = 0$ when $e \ge 4$. Then, for $e = 2$, $\lambda^B = 1$, and $\lambda^S = 1$, we have

$$\frac{g(2) - g(1) - 2}{1} = 3 > -1 = \frac{g(3) + 5 - g(2)}{1}.$$

Thus, while f is weak (5, 2, 0)-convex, g is not. □

Proof of Proposition 5. First, we can establish that $V^1_{T+1}(\cdot, p) \in \mathcal{C}(K^B, K^S, q^B + q^S)$ for any p, where $V^1_{T+1}(e, p)$ is given at Equation (21). Consider three cases.

Case 1: $e^B + 1 \le 0 \le e^B + \lambda^B$. In this case, $V^1_{T+1}(e^B + \lambda^B, p) = -(p - q^S)^+(e^B + \lambda^B)$ and $V_{T+1}(e^B, p) = K^B - (p + q^B)e^B$. Then,

$$\begin{aligned} V^1_{T+1}(e^B + \lambda^B, p) - V_{T+1}(e^B, p) &= -(p - q^S)^+(e^B + \lambda^B) + (p + q^B)e^B - K^B \\ &= -\lambda^B(p + q^B) + (e^B + \lambda^B)\left[(p + q^B) - (p - q^S)^+\right] - K^B. \end{aligned}$$

Since $e^S \le e^B \le -1$, we have $V^1_{T+1}(e^S, p) = K^B - e^S(p + q^B)$ and $V^1_{T+1}(e^S - \lambda^S, p) = K^B - (e^S - \lambda^S)(p + q^B)$. Then, $V^1_{T+1}(e^S, p) - V^1_{T+1}(e^S - \lambda^S, p) = -\lambda^S(p + q^B)$. Thus,

$$\begin{aligned} &\frac{V^1_{T+1}(e^S, p) - V^1_{T+1}(e^S - \lambda^S, p) - K^S}{\lambda^S} - \frac{V^1_{T+1}(e^B + \lambda^B, p) - V_{T+1}(e^B, p) + K^B}{\lambda^B} \\ &= -(p + q^B) - \frac{K^S}{\lambda^S} + (p + q^B) - \frac{(e^B + \lambda^B)\left[(p + q^B) - (p - q^S)^+\right]}{\lambda^B} \\ &= -\frac{K^S}{\lambda^S} - \frac{(e^B + \lambda^B)\left[(p + q^B) - (p - q^S)^+\right]}{\lambda^B} \le 0 \le q^B + q^S. \end{aligned}$$

Case 2: $e^B \ge 0$. In this case, $V^1_{T+1}(e^B, p) = -(p - q^S)^+ e^B$ and $V^1_{T+1}(e^B + \lambda^B, p) = -(p - q^S)^+(e^B + \lambda^B)$. It follows that $V^1_{T+1}(e^B + \lambda^B, p) - V^1_{T+1}(e^B, p) = -(p - q^S)^+ \lambda^B$. If $e^S \le -1$, then $V^1_{T+1}(e^S, p) = K^B - e^S(p + q^B)$ and $V^1_{T+1}(e^S - \lambda^S, p) = K^B - (e^S - \lambda^S)(p + q^B)$. Thus,

$$V^1_{T+1}(e^S, p) - V^1_{T+1}(e^S - \lambda, p) = -\lambda^S(p + q^B) \le -\lambda^S(p - q^S)^+.$$

If $e^S - \lambda^S + 1 \le 0 \le e^S$, then $V^1_{T+1}(e^S, p) = -(p - q^S)^+ e^S$ and $V^1_{T+1}(e^S - \lambda^S, p) = K^B - (e^S - \lambda^S)(p + q^B)$. We have

$$\begin{aligned}V^1_{T+1}(e^S,p) - V^1_{T+1}(e^S - \lambda^S, p) &= -(p-q^S)^+ e^S + (e^S - \lambda^S)(p+q^B) - K^B \\ &= -(p-q^S)^+\lambda^S + (e^S - \lambda^S)[p + q^B - (p-q^S)^+] - K^B \le -(p-q^S)^+\lambda^S.\end{aligned}$$

If $e^S - \lambda^S \ge 0$, then $V^1_{T+1}(e^S, p) = -(p - q^S)^+ e^S$ and $V^1_{T+1}(e^S - \lambda^S, p) = -(p - q^S)^+(e^S - \lambda^S)$. We have $V^1_{T+1}(e^S, p) - V^1_{T+1}(e^S - \lambda^S, p) = -\lambda^S(p - q^S)^+$. Thus,

$$\begin{aligned}&\frac{V^1_{T+1}(e^S,p) - V^1_{T+1}(e^S - \lambda^S,p) - K^S}{\lambda^S} - \frac{V^1_{T+1}(e^B + \lambda^B,p) - V_{T+1}(e^B,p) + K^B}{\lambda^B} \\ &\le -(p-q^S)^+ - \frac{K^S}{\lambda^S} + (p - q^S) - \frac{K^B}{\lambda^B} \le 0 \le q^B + q^S.\end{aligned}$$

Case 3: $e^B + \lambda^B \le -1$. In this case, $V^1_{T+1}(e^B, p) = K^B - (e^B + \lambda^B)(p + q^B)$ and $V^1_{T+1}(e^B + \lambda^B, p) = K^B - e^B(p + q^B)$. Then, $V^1_{T+1}(e^B + \lambda^B, p) - V^1_{T+1}(e^B, p) = -\lambda^B(p + q^B)$. As $e^S \le e^B \le e^B + \lambda^B - 1 \le -1$, we have $V^1_{T+1}(e^S, p) - V^1_{T+1}(e^S - \lambda^S, p) = -\lambda^S(p + q^B)$. Thus,

$$\begin{aligned}&\frac{V^1_{T+1}(e^S,p) - V^1_{T+1}(e^S - \lambda^S,p) - K^S}{\lambda^S} - \frac{V^1_{T+1}(e^B + \lambda^B,p) - V_{T+1}(e^B,p) + K^B}{\lambda^B} \\ &= -(p+q^B)^+ - \frac{K^S}{\lambda^S} + (p + q^B) - \frac{K^B}{\lambda^B} \le 0 \le q^B + q^S.\end{aligned}$$

Combining all these, we see that $V^1_{T+1}(\cdot, p)$ as defined in Equation (21) is in $\mathcal{C}_-(K^B, K^S, q^B + q^S)$.

Next, using x-independent versions of arguments employed in the proof of Lemma 4, we can establish x-independent versions of the parts (a)–(e) of that lemma. Following the lemma's use in the last paragraph of Theorem 3's proof, we can inductively prove that $V^1_t(\cdot, p) \in \mathcal{C}_-(K^B, K^S, q^B + q^S)$ for $t = T, T - 1, \ldots, 1$.

Now we can define entities that describe an x-independent optimal trading policy. Let

$$\begin{cases} V^{1B}_t(e,p) = \min_{m \ge e}[K^B\delta(m-e) + G^1_t(m,p)], \\ V^{1S}_t(e,p) = \min_{m \le e}[K^S\delta(e-m) + (q^B + q^S)(e-m) + G^1_t(m,p)], \end{cases}$$

where

$$G^1_t(m,p) = (p + q^B)m + \alpha\mathbb{E}[V^1_{t+1}(m - D_t, P_{t+1})|P_t = p].$$

Also, let

$$\begin{cases} m^{1B}_t(e,p) = \min\arg\min_{m \ge e+1} G^1_t(m,p), \\ m^{1S}_t(e,p) = \max\arg\min_{m \le e-1}[(q^B + q^S)(e-m) + G^1_t(m,p)]. \end{cases}$$

Similarly to Equations (B.5) and (B.7), we can define

$$\begin{cases} E^{1B}_t(p) = \{e \in \mathbb{Z}|V^{1B}_t(e,p) < G^1_t(e,p) = V^{1S}_t(e,p)\}, \\ E^{1S}_t(p) = \{e \in \mathbb{Z}|V^{1B}_t(e,p) = G^1_t(e,p) > V^{1S}_t(e,p)\}, \end{cases}$$

and

$$\begin{cases} e^{1B}_t(p) = \max\{e \in \mathbb{Z}|e \in E^{1B}_t(p)\}, \\ e^{1S}_t(p) = \min\{e \in \mathbb{Z}|e \in E^{1S}_t(p)\}. \end{cases}$$

Finally, following an x-independent version of the proof for Lemma 5, we can arrive to the following. At each $t = T, T - 1, \ldots, 1$ and permit price p, we have $e^B_t(p) < e^S_t(p)$; consequently, one optimal $m^1_t(e, p)$ for the trading portion of the lower-bounding DP is an $(e^{1B}_t(p), m^{1B}_t(\cdot, p), e^{1S}_t(p), m^{1S}_t(\cdot, p))$-one.

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