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Continuous facility location with backbone network costs

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Appendix. Online supplement to "Continuous facility location with backbone network costs"

A. Analysis of the Archimedes heuristic

We consider here the limiting behavior of the "Archimedes heuristic" in which facilities are located on an Archimedean spiral with equation given in polar coordinates by $r = a\theta$ for some appropriately chosen a. Suppose that the service region C is a circle with radius $r = 1/\sqrt{\pi}$ (i.e. with area 1). Suppose that we distribute an infinite number of facilities X on an Archimedean spiral with $a = \sqrt{\phi/\psi}/\pi$. Using the arc length formula for Archimedes' spiral, given by

$$s(\theta) = \frac{a}{2}\left[\theta\sqrt{1+\theta^2} + \log\left(\theta + \sqrt{1+\theta^2}\right)\right],$$

it is not hard to show that the travelling salesman tour of these facilities (i.e. the length of the spiral plus the trip back to the center) satisfies

$$\mathrm{TSP}(X) \sim \frac{\sqrt{\psi/\phi}}{2}$$

as $\psi \to \infty$ (the trip back to the center is a constant, roughly equal to $r = 1/\sqrt{\pi}$ and is dwarfed by the other terms). As Figure 4 of this paper suggests, it is clear that the facility-to-customer transportation cost $\mathrm{Dir}(X, C)$ is the same as the Fermat-Weber value of a collection of facilities distributed on a line of length

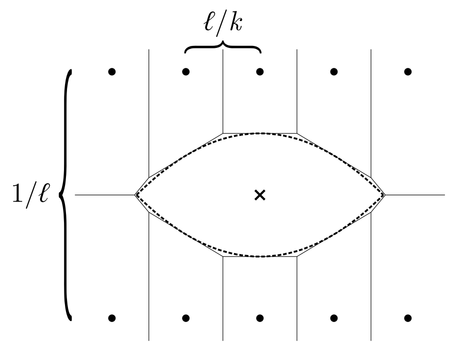

Figure 1 **The stolen region $S(p|X)$ when k points are placed equidistantly along the spiral (we assume that ℓ is sufficiently large that the curvature of the spiral does not contribute significantly). The dashed lines indicate the stolen region $S_\infty(p|X)$ when infinitely many facilities are placed along the spiral. We can think of $S(p|X)$ as simply being a piecewise linear approximation of $S_\infty(p|X)$, which will have a relative error of $\mathcal{O}((\ell^2/k)^2) = \mathcal{O}(\ell^4/k^2)$.**

$\sqrt{\psi/\phi}/2$ which is embedded in the middle of a rectangle of dimensions $\sqrt{\psi/\phi}/2 \times 2\sqrt{\phi/\psi}$, which evaluates to

$$\int_{-\sqrt{\phi/\psi}}^{\sqrt{\phi/\psi}} \int_0^{\sqrt{\psi/\phi}/2} |x_2|\, dx_1\, dx_2 = \sqrt{\phi/\psi}/2\,.$$

Thus, we have $\mathrm{TSP}(X) \sim \sqrt{\psi/\phi}/2$ and $\mathrm{Dir}(X,C) \sim \sqrt{\phi/\psi}/2$, whence $\phi\,\mathrm{TSP}(X) + \psi\,\mathrm{Dir}(X) \sim \sqrt{\phi\psi}$ as desired.

B. Competitive location when $\gamma_k \in o(k^{-1/2})$

In this section we consider the problem

$$\underset{X}{\text{minimize}}\ \gamma_{|X|} \cdot |X| + (\phi = 1)\,\mathrm{TSP}(X) + \psi \max_{p \in C} S(p\,|X\,)$$

when $\gamma_k \in o(k^{-1/2})$ and the facilities X are placed equidistantly along an Archimedean spiral of length $\ell = 18^{1/3}\psi^{1/3}/3$. From Figure 1, it is clear that as $\psi \to \infty$ (and thus as $\ell \to \infty$), the maximum area that the attacking facility p can steal is

$$\frac{1}{3\ell^2}\left(1 + \mathcal{O}\left(\frac{\ell^4}{k^2}\right)\right) = \frac{1}{3\ell^2} + \mathcal{O}\left(\frac{\ell^2}{k^2}\right)$$

where $k = |X|$ and thus, plugging in our desired value of ℓ, we find that the objective function is then given by

$$\gamma_k \cdot k + (\phi = 1)\ell + \frac{\psi}{3\ell^2} + \mathcal{O}\left(\frac{\psi\ell^2}{k^2}\right) \le \gamma_k \cdot k + c_0\frac{\psi^{5/3}}{k^2} + c\psi^{1/3}$$

where c_0 is some constant and $c\psi^{1/3} \approx 1.310\psi^{1/3}$ is the conjectured optimal cost to problem (4) when facility costs are ignored. Thus, it will suffice to show that if $\gamma_k \in o(k^{-1/2})$, then

$$\min_k \gamma_k \cdot k + c_0\frac{\psi^{5/3}}{k^2} \in o(\psi^{1/3})$$

as $\psi \to \infty$. This proof is basically identical to the proof of Claim 2.

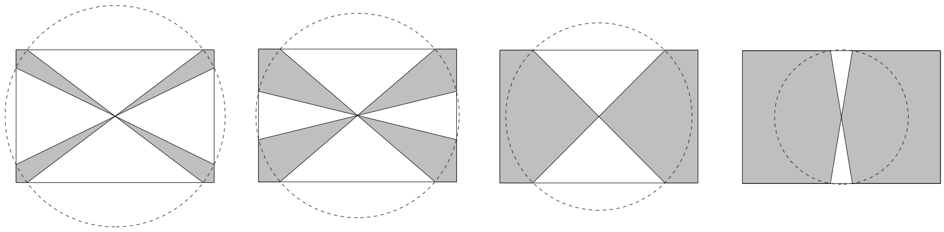

Figure 2 The worst-case regions C^* in a given rectangle R, for increasing values of A.

C. Proof of Theorem 4

DEFINITION 1. A region C is said to be *star convex at the point* p if the line segment from p to any point $x \in C$ is itself contained in C. Similarly, the *star convex hull* of a region S at the point p is the smallest star-convex region at the point p that contains S (i.e. the union of all segments between points $x \in S$ and p).

LEMMA 1. *Let R be a rectangle of dimensions $w \times h$ centered at the origin. The region C^* that solves the infinite-dimensional optimization problem*

$$\begin{aligned} \underset{C}{\text{maximize}}\, \mathrm{Dir}\,(C) \qquad & s.t. \\ C &\subseteq R \\ \mathrm{Area}\,(C) &= A \\ C &\ni (0,0) \\ C &\text{ is star convex at } (0,0) \end{aligned} \tag{1}$$

is the star convex hull of $R \backslash D$, where D is an appropriately chosen disk centered at the origin, as indicated in Figure 2. Furthermore for fixed w and h, the function $\Phi(A) = \mathrm{Dir}\,(C^*)$ (i.e. the maximal value of (1)) is monotonically increasing and concave.

Proof sketch. This follows from a standard argument where we consider the integer (or linear) program obtained by discretizing problem (1) using polar coordinates. See Figure 3a. Concavity of $\Phi(A)$ follows by observing that we build our optimal solution by adding sectors containing points that are strictly closer than the points in the sector that preceded them. □

In order to prove Theorem 4 we consider the infinite-dimensional optimization problem of choosing the worst-case convex region C that solves the problem

$$\begin{aligned} \underset{C}{\text{maximize}}\, \mathrm{Dir}\,(C) \qquad & s.t. \\ C &\subseteq R \\ \mathrm{Area}\,(C) &= A \\ C &\ni (0,0) \\ C &\text{ is convex.} \end{aligned}$$

By relaxing the convexity constraint with star convexity about the origin, the problem becomes equivalent to problem (1); we can use it to determine an upper bound on $\mathrm{Dir}(C)$.

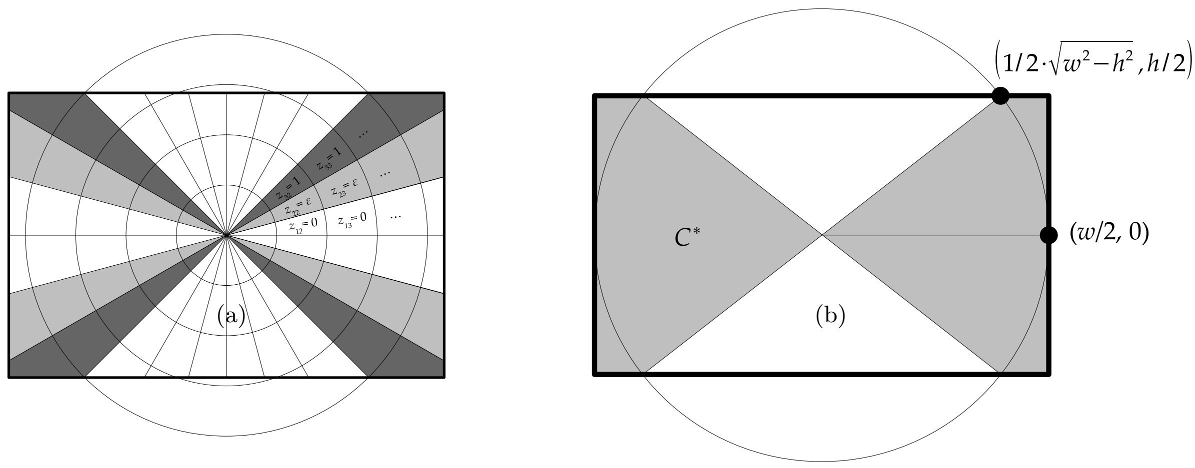

Figure 3 **In the discretization shown in 3a, our variables are set up in such a way that the star convexity constraint is equivalent to setting $z_{i(j+1)} \leq z_{ij}$ for all j. Since we are finding an upper bound of the Fermat-Weber value of a star-convex object in the given box, our objective is to maximize $\sum_{i,j} d_{ij} z_{ij}$ subject to the constraints that $\sum_{i,j} a_{ij} z_{ij} = A$, $z_{i(j+1)} \leq z_{ij}\, \forall i,j$, and $z_{ij} \geq 0 \forall i,j$, where d_{ij} denotes the distance from the origin to cell ij and a_{ij} denotes the area of cell ij. By the nature of the constraints it is clear that we may assume that $z^*_{i(j+1)} = z^*_{ij}$ at optimality since the distance from cell ij to the origin increases with j. The diagram above suggests a linear programming formulation, where the lighter regions indicate fractional solutions. Figure 3b shows the necessary value of A for which the region C^* consists of only two regions instead of four; the area of the shaded region is $wh - \frac{h}{2}\sqrt{w^2-h^2}$.**

Following Lemma 1 we see that the worst-case star-convex region C^* takes the form shown in Figure 2. If $A \geq wh - \frac{h}{2}\sqrt{w^2-h^2}$, then the optimal solution consists of two components (rather than 4) as shown in Figure 3b. The bound given in Theorem 4 is precisely the Fermat-Weber value $\iint_{C^*} ||x||\, dA$ obtained by analytic integration. We can prove Remark 7 by taking the Fermat-Weber values of C^* under the ℓ_1 and ℓ_∞ norms instead (which have a much simpler closed form) and observing that

$$\iint_{C^*} ||x||_1\, dA \sim \frac{2}{3}Aw - \frac{1}{12}w^2 h - \frac{1}{3}\cdot\frac{A^2}{h}$$

and

$$\iint_{C^*} ||x||_\infty\, dA \sim \frac{2}{3}Aw - \frac{1}{12}w^2 h - \frac{1}{3}\cdot\frac{A^2}{h}$$

from which (12) holds by the squeeze theorem.

D. Proof of Claim 3

To prove Claim 3 it is sufficient to show that the following lemma holds:

Lemma 2. *Suppose that $\tilde{R} \subseteq \square C$ is an intermediate rectangle obtained throughout Algorithm 1, which is further subdivided into $\tilde{R}'$ and $\tilde{R}''$. Then:*

1. If $\mathrm{AR}(\tilde{R}) > 3$, then

$$\mathrm{AR}(\tilde{R}'), \mathrm{AR}(\tilde{R}'') \leq \mathrm{AR}(\tilde{R}).$$

2. If $\mathrm{AR}(\tilde{R}) \leq 3$, then

$$\mathrm{AR}(\tilde{R}'), \mathrm{AR}(\tilde{R}'') \leq 3\,.$$

Proof. Claim 1 is trivial. To prove Claim 2 we assume that $\mathrm{AR}(\tilde{R}) \leq 3$. Assume without loss of generality that $\mathrm{width}(\tilde{R}) \geq \mathrm{height}(\tilde{R})$, so that $\mathrm{height}(\tilde{R}') = \mathrm{height}(\tilde{R})$. Since $\tilde{R}$ is always divided into proportions as close as 1/2 as possible, we have

$$\mathrm{width}(\tilde{R})/3 \leq \mathrm{width}(\tilde{R}') \leq 2\,\mathrm{width}(\tilde{R})/3$$

and, dividing by $\mathrm{height}(\tilde{R})$, we find that

$$\frac{\mathrm{width}(\tilde{R})}{3\,\mathrm{height}(\tilde{R})} \leq \frac{\mathrm{width}(\tilde{R}')}{\mathrm{height}(\tilde{R}')} = \frac{\mathrm{width}(\tilde{R}')}{\mathrm{height}(\tilde{R})} \leq \frac{2\,\mathrm{width}(\tilde{R})}{3\,\mathrm{height}(\tilde{R})} \leq 2$$

so that $\mathrm{width}(\tilde{R}')/\mathrm{height}(\tilde{R}') \leq 2$. Taking the reciprocal of this expression and observing that $3 \geq 3\,\mathrm{height}(\tilde{R})/\mathrm{width}(\tilde{R})$ since $\mathrm{width}(\tilde{R}) \geq \mathrm{height}(\tilde{R})$, we have

$$3 \geq \frac{3\,\mathrm{height}(\tilde{R})}{\mathrm{width}(\tilde{R})} \geq \frac{\mathrm{height}(\tilde{R}')}{\mathrm{width}(\tilde{R}')} = \frac{\mathrm{height}(\tilde{R})}{\mathrm{width}(\tilde{R}')} \geq \frac{3\,\mathrm{height}(\tilde{R})}{2\,\mathrm{width}(\tilde{R})}$$

so that $3 \geq \mathrm{height}(\tilde{R}')/\mathrm{width}(\tilde{R}')$. This same argument clearly applies to $\tilde{R}''$ as well, which completes Claim 2. $\square$

E. Proof of Theorem 5

In this section we give a sketch of the complete proof of Theorem 5 by showing that Algorithm 3 is a factor 3.93 approximation algorithm for minimizing objective function (1) when $\{\gamma_k\}$ is nonzero. Recall that Algorithm 3 merely iterates Algorithm 2 through potential values of $|X|$ from 1 to $K := \max\{\lfloor {}^{\alpha}/_{2\phi}\rceil, \lfloor 1/h^2\rceil\}$ and then selects the X with the best found objective value $F(X)$. Let X^* denote the *true* optimal solution that minimizes (1), and note that (depending on $\{\gamma_k\}$) we may have $|X^*| = \infty$. If $|X^*| > K$, then we claim that Algorithm 3 is guaranteed to have the same factor 3.93 approximation as in the case where $\gamma_k = 0$ everywhere. This is because, if we let $\bar{X}$ denote the best-found solution from Algorithm 3, then we have

$$\begin{aligned}
\frac{F(\bar{X})}{F(X^*)} = \frac{\gamma_{|\bar{X}|}\cdot|\bar{X}| + \phi\,\mathrm{TSP}\left(\bar{X}\right) + \mathrm{Dir}\left(\bar{X},C\right)}{\gamma_{|X^*|}\cdot|X^*| + \phi\,\mathrm{TSP}\left(X^*\right) + \mathrm{Dir}\left(X^*,C\right)} &\leq \frac{\gamma_{|X^*|}\cdot|X^*| + \phi\,\mathrm{TSP}\left(\bar{X}\right) + \mathrm{Dir}\left(\bar{X},C\right)}{\gamma_{|X^*|}\cdot|X^*| + \phi\,\mathrm{TSP}\left(X^*\right) + \mathrm{Dir}\left(X^*,C\right)} \\
&\leq \frac{\phi\,\mathrm{TSP}\left(\bar{X}\right) + \mathrm{Dir}\left(\bar{X},C\right)}{\phi\,\mathrm{TSP}\left(X^*\right) + \mathrm{Dir}\left(X^*,C\right)} \\
&= \frac{f(\bar{X})}{f(X^*)} \leq 3.93
\end{aligned}$$

where we have used the fact that $\gamma_{|X^*|}\cdot|X^*| \geq \gamma_{|\bar{X}|}\cdot|\bar{X}|$ because $\gamma_k \cdot k$ is an increasing sequence. Thus, it will suffice to consider the case where $|X^*| \leq K$. More specifically, letting $k^* = |X^*|$, we will consider the problem

$$\underset{X}{\mathrm{minimize}}\, f(X) := \phi\,\mathrm{TSP}(X) + \mathrm{Dir}(X,C) \qquad s.t.\, |X| \leq k^*$$

and show that Algorithm 3 will always produce a solution $\bar{X}$ whose objective value $f(\bar{X})$ is within a factor of 3.93 of the objective value $f(X^*)$. This will prove our claim because

$$\frac{\gamma_{|\bar{X}|}\cdot|\bar{X}| + f(\bar{X})}{\gamma_{k^*}\cdot k^* + f(X^*)} \leq \frac{\gamma_{k^*}\cdot k^* + f(\bar{X})}{\gamma_{k^*}\cdot k^* + f(X^*)} \leq \frac{f(\bar{X})}{f(X^*)} \leq 3.93\,.$$

As in Section 6, we will consider at most three candidate values of k, namely $k = 1$, $k = \min\{k^*, \lfloor 1/h^2\rceil\}$, and $k = \min\{k^*, \lfloor {}^{\alpha}/_{2\phi}\rceil\}$.

E.1. Proving bounds

In this section we will show that, if we apply Algorithm 2 with $k = 1$, $k = \min\{k^*, \lfloor 1/h^2 \rceil\}$, and $k = \min\{k^*, \lfloor {}^{\alpha}\!/_{2\phi} \rceil\}$, then we are guaranteed to find a solution $\bar{X}$ whose objective value $f(\bar{X})$ is within a factor of 3.93 of the optimal objective value $f(X^*)$ as in the previous section. We begin by altering the lower bounds (8) and (9) slightly, incorporating our assumption that $\psi = 1$ and omitting the contributions of the form $\gamma_k \cdot k$ (which we are safe in doing, since we are considering the ratio $f(\bar{X})/f(X^*)$ which does not depend on these contributions)

$$\Phi^1_{\mathrm{LB}}(A,\phi,k) = \begin{cases} \frac{3\sqrt{A\pi}}{8}(\sqrt{k}-1)\phi + \frac{2A^{3/2}\psi^{3/2}}{3\sqrt{\pi k}} & \text{if } \phi \le \frac{16A}{9\pi k} \\ A\sqrt{\phi} - \frac{3\phi\sqrt{\pi A}}{8} & \text{if } \frac{16A}{9\pi k} < \phi \le \frac{16A}{9\pi} \\ \frac{2A^{3/2}}{3\sqrt{\pi}} & \text{otherwise} \end{cases}$$

$$\Phi^2_{\mathrm{LB}}(A,\phi,h,k) = \begin{cases} \frac{2(1+1/\sqrt{k})A}{h}\phi + \frac{A^2}{4hk} & \text{if } \phi \le \frac{A}{4\sqrt{k}} \\ \frac{2\phi A}{h} - \frac{4\phi^2}{h} & \text{if } \frac{A}{4\sqrt{k}} < \phi \le \frac{A}{4} \\ \frac{A^2}{4h} & \text{otherwise} \end{cases}$$

It is obvious that the lower bounds above are decreasing in k. It is also clear that our upper bounds are only affected (i.e. they differ from the upper bounds in Section 6) when either $k^* < \lfloor {}^{\alpha}\!/_{2\phi} \rceil$ or $k^* < \lfloor 1/h^2 \rceil$. Since k^* is an integer, we may drop the rounding terms in the conditions $k^* < \lfloor {}^{\alpha}\!/_{2\phi} \rceil$ and $k^* < \lfloor 1/h^2 \rceil$ to obtain the equivalent constraints that $\phi < {}^{\alpha}\!/_{2k^*}$ and $h < 1/\sqrt{k^*}$ as shown in Figure 4a. We may therefore restrict our attention only to the subdomains marked (i) through (vi) in Figure 4b, since the upper bounding function is unaffected outside those subdomains (and the lower bounding function can only *increase* as a result of incorporating bounds in k). We can address these using precisely the same technique as in Section 6 and we omit the case-by-case study for brevity.

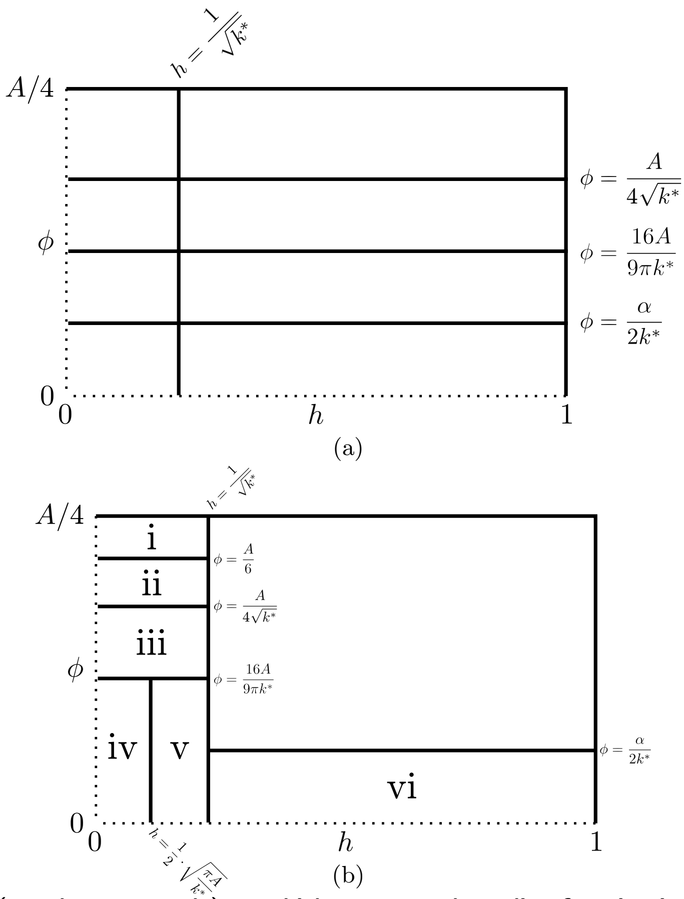

Figure 4 **Subdomains (not drawn to scale) on which our upper bounding function is affected because $\phi < \alpha/2k^*$ or $h < 1/\sqrt{k^*}$.**