Computational Methods in Finance

Ali Hirsa



Contents

Li	st of	Symbo	ols and Acronyms	xv
Li	st of	Figure	es	xvii
Li	st of	Tables	i de la companya de	xxi
Pı	reface	e		xxv
A	cknov	vledgn	nents	xxix
Ι	Pr	icing a	and Valuation	1
1	Sto	chastic	Processes and Risk-Neutral Pricing	3
	1.1	Charac	cteristic Function	3
		1.1.1	Cumulative Distribution Function via Characteristic Function	4
		1.1.2	Moments of a Random Variable via Characteristic Function	5
		1.1.3	Characteristic Function of Demeaned Random Variables	5
		1.1.4	Calculating Jensen's Inequality Correction	6
		1.1.5	Calculating the Characteristic Function of the Logarithmic of a Mar-	
			tingale	6
		1.1.6	Exponential Distribution	7
		1.1.7	Gamma Distribution	8
		1.1.8	Lévy Processes	8
		1.1.9	Standard Normal Distribution	8
		1.1.10	Normal Distribution	9
	1.2	Stocha	astic Models of Asset Prices	10
		1.2.1	Geometric Brownian Motion — Black–Scholes	10
			1.2.1.1 Stochastic Differential Equation	10
			1.2.1.2 Black-Scholes Partial Differential Equation	11
			1.2.1.3 Characteristic Function of the Log of a Geometric Brownian	
			Motion	
		1.2.2	Local Volatility Models — Derman and Kani	
			1.2.2.1 Stochastic Differential Equation	
			1.2.2.2 Generalized Black–Scholes Equation	
			1.2.2.3 Characteristic Function	
		1.2.3	Geometric Brownian Motion with Stochastic Volatility — Heston	
			Model	12
			1.2.3.1 Heston Stochastic Volatility Model — Stochastic Differential	
			Equation	12

viii Contents

			1.2.3.2	Heston Model — Characteristic Function of th Price	_
		1.2.4	Miving 1	Model — Stochastic Local Volatility (SLV) Mode	
		1.2.4 $1.2.5$		ric Brownian Motion with Mean Reversion —	
		1.2.0		eck Process	
			1.2.5.1	Ornstein-Uhlenbeck Process — Stochastic	
				Equation	
			1.2.5.2	Vasicek Model	
		1.2.6	Cox-Ing	gersoll–Ross Model	
			1.2.6.1	Stochastic Differential Equation	
			1.2.6.2	Characteristic Function of-Integral	
		1.2.7	Variance	e Gamma Model	
			1.2.7.1	Stochastic Differential Equation	
			1.2.7.2	Characteristic Function	
		1.2.8	CGMY	Model	
			1.2.8.1	Characteristic Function	
		1.2.9	Normal	Inverse Gaussian Model	
			1.2.9.1	Characteristic Function	
		1.2.10	Variance	e Gamma with Stochastic Arrival (VGSA) Mode	1
			1.2.10.1	Stochastic Differential Equation	
			1.2.10.2	Characteristic Function	
	1.3	Valuin	g Derivat	tives under Various Measures	
		1.3.1	Pricing	under the Risk-Neutral Measure	
		1.3.2	Change	of Probability Measure	
		1.3.3	Pricing	under Forward Measure	
			1.3.3.1	Floorlet/Caplet Price	
		1.3.4	Pricing	under Swap Measure	
	1.4	Types	of Deriva	atives	
	Pro			· · · · · · · · · · · · · · · · · · ·	
;	Der	rivative	s Pricin	g via Transform Techniques	
	0.1	D:	4: D.:	single the Deat Descion Theorem	
	2.1			cing via the Fast Fourier Transform	
		2.1.1	_	tion Pricing via the Fourier Transform	
		2.1.2	_	tion Pricing via the Fourier Transform	
		2.1.3		ing the Pricing Integral	
			2.1.3.1	Numerical Integration	
		0.1.4	2.1.3.2	Fast Fourier Transform	
		2.1.4		entation of Fast Fourier Transform	
	0.0	2.1.5		g factor α	
	2.2			Fourier Transform	
		2.2.1		on of Fractional FFT	
	0.0	2.2.2		entation of Fractional FFT	
	2.3			cing via the Fourier-Cosine (COS) Method	
		2.3.1		ethod	
			2.3.1.1	Cosine Series Expansion of Arbitrary Functions	
			2.3.1.2	Cosine Series Coefficients in Terms of Characte	
				tion	

Contents	ix

			2.3.1.3 COS Option Pricing					57
		2.3.2	COS Option Pricing for Different Payoffs					57
			2.3.2.1 Vanilla Option Price under the COS Method					58
			2.3.2.2 Digital Option Price under the COS Method					59
		2.3.3	Truncation Range for the COS method					59
		2.3.4	Numerical Results for the COS Method					59
			2.3.4.1 Geometric Brownian Motion (GBM)					59
			2.3.4.2 Heston Stochastic Volatility Model					60
			2.3.4.3 Variance Gamma (VG) Model					61
			2.3.4.4 CGMY Model					62
	2.4	Cosine	e Method for Path-Dependent Options					63
		2.4.1	Bermudan Options					63
		2.4.2	Discretely Monitored Barrier Options					65
			2.4.2.1 Numerical Results — COS versus Monte Carlo					65
	2.5	Saddle	epoint Method					66
	2.0	2.5.1	Generalized Lugannani–Rice Approximation					67
		2.5.2	Option Prices as Tail Probabilities					68
		2.5.2	Lugannani–Rice Approximation for Option Pricing					70
		2.5.4	Implementation of the Saddlepoint Approximation					71
		2.5.4 $2.5.5$	Numerical Results for Saddlepoint Methods					73
		2.0.0	2.5.5.1 Geometric Brownian Motion (GBM)					73
			2.5.5.2 Heston Stochastic Volatility Model					73
			2.5.5.3 Variance Gamma Model					74
	0.0	D	2.5.5.4 CGMY Model					$\frac{75}{76}$
			Option Pricing via the Fourier Transform					76
	Proi	$_{ m lems}$		• •	•	•	•	78
3	Intr	oducti	ion to Finite Differences					83
	3.1	Taylor	Expansion					83
	3.2	·	Difference Method					85
	0.2	3.2.1	Explicit Discretization					87
		0.2.1	3.2.1.1 Algorithm for the Explicit Scheme					89
		3.2.2	Implicit Discretization					89
		0.2.2	3.2.2.1 Algorithm for the Implicit Scheme					91
		3.2.3	Crank-Nicolson Discretization					92
		0.2.0	3.2.3.1 Algorithm for the Crank–Nicolson Scheme					95
		3.2.4	Multi-Step Scheme					96
		0.2.4	3.2.4.1 Algorithm for the Multi-Step Scheme					98
	3.3	Ctob:1						99
	ა.ა		· ·					
		3.3.1	Stability of the Explicit Scheme					102
		3.3.2	Stability of the Implicit Scheme					103
		3.3.3	Stability of the Crank-Nicolson Scheme					103
	0.4	3.3.4	Stability of the Multi-Step Scheme					104
	3.4		ative Approximation by Finite Differences: Generic Approach					104
	3.5		x Equations Solver					106
		3.5.1	Tridiagonal Matrix Solver					106
		3.5.2	Pentadiagonal Matrix Solver					108

x Contents

				110 113
4	Der	ivative	Pricing via Numerical Solutions of PDEs	115
	4.1	Option	n Pricing under the Generalized Black–Scholes PDE	117
		4.1.1	Explicit Discretization	117
		4.1.2	Implicit Discretization	119
		4.1.3	Crank–Nicolson Discretization	120
	4.2		lary Conditions and Critical Points	121
		4.2.1	Implementing Boundary Conditions	121
		,	4.2.1.1 Dirichlet Boundary Conditions	122
			4.2.1.2 Neumann Boundary Conditions	122
		4.2.2	Implementing Deterministic Jump Conditions	125
	4.3		niform Grid Points	126
		4.3.1	Coordinate Transformation	127
		_ `	4.3.1.1 Black-Scholes PDE after Coordinate Transformation	129
	4.4		asion Reduction	130
	4.5		g Path-Dependent Options in a Diffusion Framework	131
		4.5.1	Bermudan Options	131
		4.5.2	American Options	133
			4.5.2.1 Bermudan Approximation	133
			4.5.2.2 Black–Scholes PDE with a Synthetic Dividend Process	134
			4.5.2.3 Brennan–Schwartz Algorithm	135
		4.5.3	Barrier Options	138
			4.5.3.1 Single Knock-Out Barrier Options	140
			4.5.3.2 Single Knock-In Barrier Options	141
		_	4.5.3.3 Double Barrier Options	141
	4.6		rd PDEs	141
		4.6.1	Vanilla Calls	142
		4.6.2	Down-and-Out Calls	143
		4.6.3	Up-and-Out Calls	143
	4.7		Differences in Higher Dimensions	146
		4.7.1	Heston Stochastic Volatility Model	146
		4.7.2	Options Pricing under the Heston PDE	148
		4 7 0	4.7.2.1 Implementation of the Boundary Conditions	
		4.7.3	Alternative Direction Implicit (ADI) Scheme	156
			4.7.3.1 Derivation of the Craig-Sneyd Scheme for the Heston PDE	158
		4.7.4	Heston PDE	161
	ъ.	4.7.5	Numerical Results and Conclusion	161
				164
	Cas	e Studie	es	168
5	Der	rivative	e Pricing via Numerical Solutions of PIDEs	171
	5.1	Nume	rical Solution of PIDEs (a Generic Example)	171
	J.1	5.1.1	Derivation of the PIDE	172
			Discretization	176

Contents	xi
----------	----

		5.1.3	Evaluation of the Integral Term	178
		5.1.4	Difference Equation	180
			5.1.4.1 Implementing Neumann Boundary Conditions	183
	5.2	Ameri	can Options	184
		5.2.1	Heaviside Term – Synthetic Dividend Process	187
		5.2.2	Numerical Experiments	188
	5.3	PIDE	Solutions for Lévy Processes	190
	5.4	Forwa	rd PIDEs	191
		5.4.1	American Options	191
		5.4.2	Down-and-Out and Up-and-Out Calls	194
	5.5	Calcul	ation of g_1 and g_2	198
	Prob	olems		199
	Case	e Studie	es	200
6	Sim	ulatio	n Methods for Derivatives Pricing	203
	C 1	D 1	N v l · Committee	905
	6.1		om Number Generation	
	c o	6.1.1	Standard Uniform Distribution	
	6.2	•	es from Various Distributions	
		6.2.1	Inverse Transform Method	
		6.2.2	Acceptance—Rejection Method	
			6.2.2.1 Standard Normal Distribution via Acceptance—Rejection .	
			6.2.2.2 Poisson Distribution via Acceptance–Rejection	
			6.2.2.3 Gamma Distribution via Acceptance—Rejection	
			6.2.2.4 Beta Distribution via Acceptance–Rejection	
		6.2.3	Univariate Standard Normal Random Variables	
			6.2.3.1 Rational Approximation	
			6.2.3.2 Box–Muller Method	
			6.2.3.3 Marsaglia's Polar Method	
		6.2.4	Multivariate Normal Random Variables	
		6.2.5	Cholesky Factorization	
			6.2.5.1 Simulating Multivariate Distributions with Specific Corre-	
	4.0		lations	
	6.3		s of Dependence	
		6.3.1	Full Rank Gaussian Copula Model	
		6.3.2	correcting compensation in a variation desired respective	
		622	tation	
	6.4	6.3.3	nian Bridge	
	6.5		e Carlo Integration	
	0.5		•	
		6.5.1	Quasi-Monte Carlo Methods	
	6.6	6.5.2	Latin Hypercube Sampling Methods	
	6.6		Euler Scheme	
		6.6.1	Milstein Scheme	
		6.6.2		
	67	6.6.3	Runge-Kutta Scheme	
	6.7		ating SDEs under Different Models	
		0.1.1	Geometric Diominali Monon	∠o⊥

xii Contents

		6.7.2	Ornstein-U	Uhlenbeck Process			
		6.7.3	CIR Proce	SS			. 232
		6.7.4	Heston Sto	ochastic Volatility Model			. 232
			6.7.4.1 F	full Truncation Algorithm			. 233
		6.7.5	Variance C	Gamma Process			. 234
		6.7.6	Variance C	Gamma with Stochastic Arrival (VGSA) Process			. 236
	6.8	Outpu	t/Simulatio	on Analysis			. 240
	6.9	Varian	ce Reductio	on Techniques			. 241
		6.9.1	Control Va	ariate Method			. 241
		6.9.2	Antithetic	Variates Method			. 243
		6.9.3	Conditiona	al Monte Carlo Methods			. 244
		,	6.9.3.1 A	Algorithm for Conditional Monte Carlo Simulation	n		. 245
		6.9.4	Importanc	e Sampling Methods			. 247
			6.9.4.1 V	Variance Reduction via Importance Sampling			. 248
		6.9.5		Sampling Methods			
			6.9.5.1 F	Findings and Observations			. 251
				Algorithm for Stratified Sampling Methods			
		6.9.6	Common I	Random Numbers			. 253
	Prob	lems .					. 254
II	C	alibra	tion and	Estimation			2 59
7	Mod	lel Cal	ibration				261
				1		`	
	7.1	Calibra	ation Form	$ulation \dots \dots \dots \dots \dots$. 263
		7.1.1	General Fo	ormulation			. 264
		7.1.2	Weighted !	Least-Squares Formulation			. 264
		7.1.3	Regularize	ed Calibration Formulations			. 264
	7.2	Calibra	ation of a S	Single Underlier Model			. 265
		7.2.1	Black-Sch	oles Model			. 265
		7.2.2	Local Vola	tility Model			. 266
				Forward Partial Differential Equations for Europe			
				Options			
				Construction of the Local Volatility Surface			
		7.2.3		Elasticity of Variance (CEV) Model			
		7.2.4		ochastic Volatility Model			
		7.2.5	Mixing Mo	odel — Stochastic Local Volatility (SLV) Model			. 275
		7.2.6	Variance (Gamma Model			. 276
		7.2.7	CGMY M	odel			. 277
		7.2.8	Variance (Gamma with Stochastic Arrival Model			. 277
		7.2.9	Lévy Mod	els			. 281
	7.3	Interes	t Rate Moo	dels			. 282
		7.3.1	Short Rate	e Models			. 285
			7.3.1.1 \	Vasicek Model			. 285
			7.3.1.2 F	Pricing Swaptions with the Vasicek Model			. 287
			7.3.1.3 A	Alternative Vasicek Model Calibration			. 288
			7.3.1.4	CIR Model			. 289
			7.3.1.5 F	Pricing Swaptions with the CIR Model			. 292

Contents	xiii
Contents	XIII

			7.3.1.0	Alternative CIR Model Calibration		293
			7.3.1.7	Ho–Lee Model		294
			7.3.1.8	Hull-White (Extended Vasicek) Model		297
		7.3.2	Multi-Fa	actor Short Rate Models		297
			7.3.2.1	Multi-Factor Vasicek Model		298
			7.3.2.2	Multi-Factor CIR Model		
			7.3.2.3	CIR Two-Factor Model Calibration		299
			7.3.2.4	Pricing Swaptions with the CIR Two-Factor Model .		
			7.3.2.5	Alternative CIR Two-Factor Model Calibration		
			7.3.2.6	Findings		
		7.3.3		erm Structure Models		
	,	7.3.4		Rate (HJM) Models		
		1.0.1	7.3.4.1	Discrete-Time Version of HJM		
			7.3.4.2	Factor Structure Selection		
		7.3.5		Market Models		
	7.4			ve Models		
	7.5	Model	_			
	7.6			ad Optimization Methodology		
	1.0	7.6.1		arch		
		7.6.2				
		7.6.2		Mead Simplex Method		
				9		
		7.6.4		n, Fletcher, and Powell (DFP) Method		
		7.6.5		Method		
		7.6.6	_	nconstrained Optimization for Linear Constrained Input		
		7.6.7		egion Methods for Constrained Problems		
		7.6.8		tion-Maximization (EM) Algorithm		
	7.7			the Discount Curve		
		7.7.1		Yield Instruments		
			7.7.1.1	Simple Interest Rates to Discount Factors		
			7.7.1.2	Forward Rates to Discount Factors		
			7.7.1.3	Swap Rates to Discount Factors		
		7.7.2		cting the Yield Curve		
			7.7.2.1	Construction of the Short End of the Curve		
			7.7.2.2	Construction of the Long End of the Curve		
		7.7.3	•	nial Splines for Constructing Discount Curves		
			7.7.3.1	Hermite Spline		327
			7.7.3.2	Natural Cubic Spline		328
			7.7.3.3	Tension Spline		328
	7.8	Arbitr	age Restr	rictions on Option Premiums		331
	7.9	Interes	st Rate D	efinitions		331
	Prob	$_{ m lems}$				333
	Case	Studie	es			333
8	Filt	ering a	and Para	ameter Estimation		341
	8.1	Filteri	ng			343
		8.1.1		ction of $p(\mathbf{x}_k \mathbf{z}_{1:k})$		
	8.2			etion $\ldots \ldots \ldots \ldots$		
					- '	0

xiv Contents

Index	,	409
Refere	ces	395
Pro	ems	394
8.9	Markov Chain Monte Carlo (MCMC)	393
	3.8.3 Problem of Resampling in Particle Filter and Possible Panaceas	392
	3.8.2 Sampling Importance Resampling (SIR) Particle Filtering	382
	8.8.1 Sequential Importance Sampling (SIS) Particle Filtering	381
8.8	Particle Filter	380
8.7	Square Root Unscented Kalman Filter (SR_UKF)	376
	3.6.3 Implementation of Unscented Kalman Filter (UKF)	364
	3.6.2 Update	363
	8.6.1 Predict	362
8.6	Unscented Kalman Filter	362
8.5	Extended Kalman Filter	359
8.4	Non-Linear Filters	359
	terpretation of the Optimal Kalman Gain	356
	3.3.2 Posterior Estimate Covariance under Optimal Kalman Gain and In-	001
0.0	3.3.1 Underlying Model	351
8.3	Kalman Filter	-351