The Black-Scholes Equation in Presence of Arbitrage

Simone Farinelli Core Dynamics GmbH Scheuchzerstrasse 43

CH-8006 Zurich

Email: simone@coredynamics.ch

and

Hideyuki Takada

Department of Information Science Narashino Campus, Toho University 2-2-1-Miyama, Funabashi-Shi

J-274-8510 Chiba

Email: hideyuki.takada@is.sci.toho-u.ac.jp

October 13, 2021

Abstract

We apply Geometric Arbitrage Theory to obtain results in Mathematical Finance, which do not need stochastic differential geometry in their formulation. First, for generic market dynamics given by a subclass of multidimensional Itô processes we specify and prove the equivalence between No-Free-Lunch-with-Vanishing-Risk (NFLVR) and expected utility maximization. As a by-product we provide a geometric characterization of the No-Unbounded-Profit-with-Bounded-Risk (NUPBR) condition given by the zero curvature (ZC) condition for this subclass of Itô processes. Finally, we extend the Black-Scholes partial differential equation to markets allowing arbitrage.

Keywords: NFLVR, NUPBR, Geometric Arbitrage Theory, Non linear Black Scholes PDE

AMS: 91G80, 53C07

Contents

1	Inti	roduction		2
2	Geo	ometric Arbitrage Theory Background		4
	2.1	The Classical Market Model		4
	2.2	Geometric Reformulation of the Market Model: Primitives		6
	2.3	Geometric Reformulation of the Market Model: Portfolios		7
	2.4	Arbitrage Theory in a Differential Geometric Framework		8
		2.4.1 Market Model as Principal Fibre Bundle		8
		2.4.2 Nelson Weak \mathcal{D} -Differentiable Market Model	. :	10
		2.4.3 Arbitrage as Curvature		12
3	Arb	oitrage and Utility	2	20
4	Ark	pitrage and Derivative Pricing	2	26
	4.1	The Black-Scholes PDE in the Presence of Arbitrage		26
	4.2	Approximate Solution of the Modified Black-Scholes PDE		29
5	Cor	nclusions	3	32
A	Ger	neralized Derivatives of Stochastic Processes		33

1 Introduction

This paper provides applications of a conceptual structure - called Geometric Arbitrage Theory (GAT in short) - to prove results in financial mathematics which are comprehensible without the use of stochastic differential geometry and extend well known classical facts. We expect therefore to make GAT accessible to a wider public in the mathematical finance community.

GAT rephrases classical stochastic finance in stochastic differential geometric terms in order to characterize arbitrage. The main idea of the GAT approach consists of modeling markets made of basic financial instruments together with their term structures as principal fibre bundles. Financial features of this market - like no arbitrage and equilibrium - are then characterized in terms of standard

differential geometric constructions - like curvature - associated to a natural connection in this fibre bundle. Principal fibre bundle theory has been heavily exploited in theoretical physics as the language in which laws of nature can be best formulated by providing an invariant framework to describe physical systems and their dynamics. These ideas can be carried over to mathematical finance and economics. A market is a financial-economic system that can be described by an appropriate principle fibre bundle. A principle like the invariance of market laws under change of numéraire can be seen then as gauge invariance.

The fact that gauge theories are the natural language to describe economics was first proposed by Malaney and Weinstein in the context of the economic index problem ([35], [43]). Ilinski (see [27] and [28]) and Young [44] proposed to view arbitrage as the curvature of a gauge connection, in analogy to some physical theories. Independently, [41] further developed [15] seminal work and utilized techniques from differential geometry to reduce the complexity of asset models before stochastic modeling.

Why is arbitrage modelling important? The no arbitrage condition is only an approximation and it is not fulfilled when we consider real markets. This is the case for non traded assets, traded assets when the frequency of the trades falls below 2 minutes (cf. [11]) or electricity markets, where we do not have the possibility of completely liquidating the portfolio at any given time, as we implicitly assume in mathematical finance. This has been recognized for a long time and in recent years the modelling of markets allowing for arbitrage beyond pathological cases has made a relevant progress (see f.i. [23, 38]). The benchmark approach to mathematical finance models markets by [21] allowing for arbitrage even if this is not explicitly mentioned.

This paper is structured as follows. Section 2 reviews classical stochastic finance and Geometric Arbitrage Theory, summarizing [12], where GAT has been given a rigorous mathematical foundation utilizing the formal background of stochastic differential geometry as in [40], [9], [10], [20], [42] and [24]. Arbitrage is seen as curvature of a principal fibre bundle representing the market which defines the quantity of arbitrage associated to it. The zero curvature condition is a weaker condition than No-Free-Lunch-with-Vanishing-Risk (NFLVR). It becomes equivalent under additional assumptions introduced for a guiding example, a market whose asset prices are Itô processes. In general, the zero curvature condition follows from the No-Unbounded-Profit-with-Bounded-Risk (NUPBR) condition, as we prove in Section 3, where we analyze the relationship between arbitrage and expected utility maximization. The equivalence is proved for a certain subclass of Itô processes. In Section 4, GAT is applied to prove an extension of the Black Scholes PDE in the case of markets allowing for arbitrage. Section 5 concludes, and Appendix A reviews Nelson's stochastic derivatives.

2 Geometric Arbitrage Theory Background

In this section we explain the main concepts of Geometric Arbitrage Theory introduced in [12], to which we refer for proofs and additional examples. Since the differential geometric thinking is not so widespread in the mathematical finance community, we explain in detail the reformulation of the asset model as principal fibre bundle with a connection, whose curvature can be seen as a measure of arbitrage. New results and more pedagogical results in comparison to [12] are provided.

2.1 The Classical Market Model

In this subsection we will summarize the classical set up, which will be rephrased in Section 2.4 in differential geometric terms. We basically follow [26] and the ultimate reference [7].

We assume continuous time trading and that the set of trading dates is $[0, +\infty[$. This assumption is general enough to embed the cases of finite and infinite discrete times as well as the one with a finite horizon in continuous time. Note that while it is true that in the real world trading occurs at discrete times only, these are not known a priori and can be virtually any points in the time continuum. This motivates the technical effort of continuous time stochastic finance.

The uncertainty is modelled by a filtered probability space $(\Omega, \mathcal{A}, \mathbb{P})$, where \mathbb{P} is the statistical (physical) probability measure, $\mathcal{A} = \{\mathcal{A}_t\}_{t \in [0, +\infty[}$ an increasing family of sub- σ -algebras of \mathcal{A}_{∞} and $(\Omega, \mathcal{A}_{\infty}, \mathbb{P})$ is a probability space. The filtration \mathcal{A} is assumed to satisfy the usual conditions, that is

- right continuity: $A_t = \bigcap_{s>t} A_s$ for all $t \in [0, +\infty[$.
- \mathcal{A}_0 contains all null sets of \mathcal{A}_{∞} .

The market consists of finitely many assets indexed by $j=1,\ldots,N$, whose nominal prices are given by the vector valued semimartingale $S:[0,+\infty[\times\Omega\to\mathbb{R}^N$ denoted by $(S_t)_{t\in[0,+\infty[}$ adapted to the filtration \mathcal{A} . The stochastic process $(S_t^j)_{t\in[0,+\infty[}$ describes the price at time t of the j-th asset in terms of unit of cash at time t=0. More precisely, we assume the existence of a 0-th asset, the cash, a strictly positive semimartingale, which evolves according to $S_t^0=\exp(\int_0^t du\,r_u^0)$, where the integrable semimartingale $(r_t^0)_{t\in[0,+\infty[}$ represents the continuous interest rate provided by the cash account: one always knows in advance what the interest rate on the own bank account is, but this can change from time to time. The cash account is therefore considered the locally risk less asset in contrast to the other assets, the risky ones. In the following we will mainly utilize discounted prices, defined as $\hat{S}_t^j:=S_t^j/S_t^0$, representing the asset prices in terms of current unit of cash.

We remark that there is no need to assume that asset prices are positive. But, there must be at

least one strictly positive asset, in our case the cash. If we want to renormalize the prices by choosing another asset instead of the cash as reference, i.e. by making it to our **numéraire**, then this asset must have a strictly positive price process. More precisely, a generic numéraire is an asset, whose nominal price is represented by a strictly positive stochastic process $(B_t)_{t \in [0,+\infty[}$, and which is a portfolio of the original assets j = 0, 1, 2, ..., N. The discounted prices of the original assets are then represented in terms of the numéraire by the semimartingales $\hat{S}_t^j := S_t^j/B_t$.

We assume that there are no transaction costs and that short sales are allowed. Remark that the absence of transaction costs can be a serious limitation for a realistic model. The filtration \mathcal{A} is not necessarily generated by the price process $(S_t)_{t\in[0,+\infty[}$: other sources of information than prices are allowed. All agents have access to the same information structure, that is to the filtration \mathcal{A} .

Let v be a positive real number. A v-admissible **strategy** $x = (x_t)_{t \in [0, +\infty[}$ is a S-integrable predictable process for which the Itô integral $\int_0^t x \cdot dS \ge -v$ a.s. for all $t \ge 0$ with $x_0 = 0$. A strategy is admissible if it is v-admissible for some $v \ge 0$.

Definition 1 (Arbitrage). Let the process $(S_t)_{[0,+\infty[}$ be a semimartingale and $(x_t)_{t\in[0,+\infty[}$ be admissible self-financing strategy. Let us consider trading up to time $T \leq \infty$. The portfolio wealth at time t is given by $V_t(x) := V_0 + \int_0^t x_u \cdot dS_u$, and we denote by K_0 the subset of $L^0(\Omega, \mathcal{A}_T, \mathbb{P})$ containing all such $V_T(x)$, where x is any admissible self-financing strategy. We define

- $C_0 := K_0 L^0_{\perp}(\Omega, \mathcal{A}_T, \mathbb{P}).$
- $C := C_0 \cap L^{\infty}(\Omega, \mathcal{A}_T, \mathbb{P}).$
- \bar{C} : the closure of C in L^{∞} with respect to the norm topology.
- $\mathcal{V}^{V_0} := \{(V_t)_{t \in [0,+\infty[} \mid V_t = V_t(x), \text{ where } x \text{ is } V_0\text{-admissible}\}.$
- $\mathcal{V}_T^{V_0} := \{V_T \mid (V_t)_{t \in [0,+\infty[} \in \mathcal{V}^{V_0}\}: \text{ terminal wealth for } V_0\text{-admissible self-financing strategies.}$

And let $L^{\infty}_{+}(\Omega, \mathcal{A}_{T}, \mathbb{P})$ be the set of positive random variables in $L^{\infty}(\Omega, \mathcal{A}_{T}, \mathbb{P})$. We say that S satisfies

- (NA), no arbitrage, if and only if $C \cap L^{\infty}_{\perp}(\Omega, \mathcal{A}_T, \mathbb{P}) = \{0\}.$
- (NFLVR), no-free-lunch-with-vanishing-risk, if and only if $\bar{C} \cap L_+^{\infty}(\Omega, \mathcal{A}_T, \mathbb{P}) = \{0\}.$
- (NUPBR), no-unbounded-profit-with-bounded-risk, if and only if $V_T^{V_0}$ is bounded in L^0 for some $V_0 > 0$.

The relationship between these three different types of arbitrage has been elucidated in [6] and in [30] with the proof of the following result.

Theorem 2.

$$(NFLVR) \Leftrightarrow (NA) + (NUPBR).$$

Remark 3. We recall that, as shown in [6, 30, 32, 31], (NUPBR) is equivalent to (NAA1), i.e. no asymptotic arbitrage of the 1st kind, and equivalent to (NA1), i.e. no arbitrage of the 1st kind.

2.2 Geometric Reformulation of the Market Model: Primitives

We are going to introduce a more general representation of the market model introduced in Section 2.1, which better suits to the arbitrage modeling task.

Definition 4. A gauge is an ordered pair of two A-adapted real valued semimartingales (D, P), where $D = (D_t)_{t \geq 0} : [0, +\infty[\times\Omega \to \mathbb{R} \text{ is called deflator and } P = (P_{t,s})_{t,s} : \mathcal{T} \times \Omega \to \mathbb{R}, \text{ which is called term structure, is considered as a stochastic process with respect to the time <math>t$, termed valuation date and $\mathcal{T} := \{(t,s) \in [0, +\infty[^2 \mid s \geq t\}. \text{ The parameter } s \geq t \text{ is referred as maturity date. The following properties must be satisfied almost surely for all <math>t,s$ such that $s \geq t \geq 0$; $P_{t,s} > 0$, $P_{t,t} = 1$.

Deflators and term structures can be considered *outside the context of fixed income*. An arbitrary financial instrument is mapped to a gauge (D, P) with the following economic interpretation:

- Deflator: D_t is the value of the financial instrument at time t expressed in terms of some numéraire. If we choose the cash account, the 0-th asset as numéraire, then we can set $D_t^j := \hat{S}_t^j = \frac{S_t^j}{S_t^0}$ $(j = 1, \dots N)$.
- Term structure: $P_{t,s}$ is the value at time t (expressed in units of deflator at time t) of a synthetic zero coupon bond with maturity s delivering one unit of financial instrument at time s. It represents a term structure of forward prices with respect to the chosen numéraire.

We point out that there is no unique choice for deflators and term structures describing an asset model. For example, if a set of deflators qualifies, then we can multiply every deflator by the same positive semimartingale to obtain another suitable set of deflators. Of course term structures have to be modified accordingly. The term "deflator" is clearly inspired by actuarial mathematics and was first introduced in [41]. In the present context it refers to an asset value up division by a strictly positive semimartingale (which can be the state price deflator if this exists and it is made to the numéraire). There is no need to assume that a deflator is a positive process. However, if we want to make an asset to our numéraire, then we have to make sure that the corresponding deflator is a strictly positive stochastic process.

2.3 Geometric Reformulation of the Market Model: Portfolios

We want now to introduce transforms of deflators and term structures in order to group gauges containing the same (or less) stochastic information. That for, we will consider *deterministic* linear combinations of assets modelled by the same gauge (e. g. zero bonds of the same credit quality with different maturities).

Definition 5. Let $\pi:[0,+\infty[\longrightarrow \mathbb{R}$ be a deterministic cashflow intensity (possibly generalized) function. It induces a **gauge transform** $(D,P)\mapsto \pi(D,P):=(D,P)^{\pi}:=(D^{\pi},P^{\pi})$ by the formulae

$$D_t^{\pi} := D_t \int_0^{+\infty} dh \, \pi_h P_{t,t+h}, \qquad P_{t,s}^{\pi} := \frac{\int_0^{+\infty} dh \, \pi_h P_{t,s+h}}{\int_0^{+\infty} dh \, \pi_h P_{t,t+h}}.$$

Remark 6. The cashflow intensity π specifies the bond cashflow structure. The bond value at time t expressed in terms of the market model numéraire is given by D_t^{π} . The term structure of forward prices for the bond future expressed in terms of the bond current value is given by $P_{t,s}^{\pi}$.

Proposition 7. Gauge transforms induced by cashflow vectors have the following property:

$$((D, P)^{\pi})^{\nu} = ((D, P)^{\nu})^{\pi} = (D, P)^{\pi * \nu}, \tag{1}$$

where * denotes the convolution product of two cashflow vectors or intensities respectively:

$$(\pi * \nu)_t := \int_0^t dh \, \pi_h \nu_{t-h}. \tag{2}$$

Proof. We can observe that

$$(D_t^{\pi})^{\nu} = D_t^{\pi} \int_0^{+\infty} dh \, \nu_h P_{t,t+h}^{\pi} = D_t \int_0^{+\infty} dh \, \nu_h \int_0^{+\infty} du \, \pi_u P_{t,t+h+u}.$$

By changing variables v := h + u, one has

$$(D_t^{\pi})^{\nu} = D_t \int_0^{+\infty} dv \left(\int_0^v dh \, \nu_h \pi_{v-h} \right) P_{t,t+v} = (D_t)^{\pi * \nu}$$

and this coincide with $(D_t^{\nu})^{\pi}$, proving the first component of (1). The second component can be derived similarly.

The convolution of two non-invertible gauge transform is non-invertible. The convolution of a non-invertible with an invertible gauge transform is non-invertible.

Definition 8. The term structure can be written as a functional of the instantaneous forward rate f defined as

$$f_{t,s} := -\frac{\partial}{\partial s} \log P_{t,s}, \quad P_{t,s} = \exp\left(-\int_t^s dh f_{t,h}\right),$$

and

$$r_t := \lim_{s \to t^+} f_{t,s} \tag{3}$$

is termed short rate.

Remark 9. Since $(P_{t,s})_{t,s}$ is a t-stochastic process (semimartingale) depending on a parameter $s \geq t$, the s-derivative can be defined deterministically, and the expressions above make sense pathwise in a both classical and generalized sense. In a generalized sense we will always have a \mathcal{D}' derivative for any $\omega \in \Omega$; this corresponds to a classic s-continuous derivative if $P_{t,s}(\omega)$ is a C^1 -function of s for any fixed $t \geq 0$ and $\omega \in \Omega$.

Remark 10. The special choice of vanishing interest rate $r \equiv 0$ or flat term structure $P \equiv 1$ for all assets corresponds to the classical model, where only asset prices and their dynamics are relevant.

2.4 Arbitrage Theory in a Differential Geometric Framework

Now we are in the position to rephrase the asset model presented in Subsection 2.1 in terms of a natural geometric language. Given N base assets we want to construct a portfolio theory and study arbitrage and thus we cannot a priori assume the existence of a risk neutral measure or of a state price deflator. In terms of differential geometry, we will adopt the mathematician's and not the physicist's approach. The market model is seen as a principal fibre bundle of the (deflator, term structure) pairs, discounting and portfolio rebalance (or foreign exchange) as a parallel transport, numéraire as global section of the gauge bundle, arbitrage as curvature. The no-unbounded-profit-with-bounded-risk condition is proved to imply a zero curvature condition.

2.4.1 Market Model as Principal Fibre Bundle

Let us consider -in continuous time- a market with N assets and a numéraire. A general portfolio at time t is described by the vector of nominals $x \in X$, for an open set $X \subset \mathbb{R}^N$. By nominals x^1, \ldots, x^N we mean the number of assets that we hold in our portfolio. Following Definition 4, the asset model consisting in N synthetic zero bonds is described by means of the gauges

$$(D^j, P^j) = ((D_t^j)_{t \in [0, +\infty[}, (P_{t,s}^j)_{s \ge t}),$$

where D^j denotes the deflator and P^j the term structure for j = 1, ..., N. More exactly: D_t^j is the value of the j-th financial instrument at time t expressed in terms of some numéraire, and $P_{t,s}^j$ is the value at time t (expressed in units of deflator D_t^j at time t) of the j-th synthetic zero coupon bond with maturity s delivering one unit of financial instrument at time s.

The term structure can be written as

$$P_{t,s}^{j} = \exp\left(-\int_{t}^{s} f_{t,u}^{j} du\right),\,$$

where f^j is the instantaneous forward rate process for the j-th asset and the corresponding short rate is given by $r^j_t := \lim_{u \to t^+} f^j_{t,u}$. For a portfolio with nominals $x \in X \subset \mathbb{R}^N$ we define

$$D_t^x := \sum_{j=1}^N x_j D_t^j \quad f_{t,u}^x := \sum_{j=1}^N \frac{x_j D_t^j}{\sum_{k=1}^N x_k D_t^k} f_{t,u}^j \quad P_{t,s}^x := \exp\left(-\int_t^s f_{t,u}^x du\right).$$

The short rate writes

$$r_t^x := \lim_{u \to t^+} f_{t,u}^x = \sum_{j=1}^N \frac{x_j D_t^j}{\sum_{k=1}^N x_k D_t^k} r_t^j.$$

The image space of all possible strategies reads

$$M := \{(x, t) \in X \times [0, +\infty[\}.$$

In Subsection 2.3 cashflow intensities and the corresponding gauge transforms were introduced. They have the structure of an Abelian semigroup

$$H := \mathcal{E}'([0, +\infty[, \mathbb{R}) = \{F \in \mathcal{D}'([0, +\infty[) \mid \operatorname{supp}(F) \subset [0, +\infty[\text{ is compact} \},$$

where the semigroup operation on distributions with compact support is the convolution (see [22], Chapter IV), which extends the convolution of regular functions as defined by formula (2).

Definition 11. The Market Fibre Bundle is defined as the fibre bundle of gauges

$$\mathcal{B} := \{ (D_t^x, P_{t, \cdot}^x)^{\pi} | (x, t) \in M, \pi \in G \}.$$

The cashflow intensities defining invertible transforms constitute an Abelian group

$$G := \{ \pi \in H | \text{ it exists } \nu \in H \text{ such that } \pi * \nu = \delta \} \subset \mathcal{E}'([0, +\infty[, \mathbb{R}).$$

where δ is Dirac delta function acts as an identity element. From Proposition 7 we obtain

Theorem 12. The market fibre bundle \mathcal{B} has the structure of a G-principal fibre bundle given by the action

$$\mathcal{B} \times G \longrightarrow \mathcal{B}$$

$$((D, P), \pi) \mapsto (D, P)^{\pi} = (D^{\pi}, P^{\pi})$$

The group G acts freely and differentiably on \mathcal{B} to the right.

The market fibre bundle repackages all the information concerning market dynamics of the asset futures and their underlyings. The principal bundle structure reflects the portfolio construction possibilities at a fixed time, as well as the synthetic bond construction possibilities for given cash flow patterns specified by the gauge transforms.

2.4.2 Nelson Weak \mathcal{D} -Differentiable Market Model

We continue to reformulate the classic asset model introduced in Subsection 2.1 in terms of stochastic differential geometry.

Definition 13. A Nelson weak \mathcal{D} -differentiable market model for N assets is described by N gauges which are Nelson weak \mathcal{D} -differentiable with respect to the time variable. More exactly, for all $t \in [0, +\infty[$ and $s \geq t$ there is an open time interval $I \ni t$ such that for the deflators $D_t := [D_t^1, \ldots, D_t^N]^{\dagger}$ and the term structures $P_{t,s} := [P_{t,s}^1, \ldots, P_{t,s}^N]^{\dagger}$, the latter seen as processes in t and parameter s, there exist a weak \mathcal{D} -derivative with respect to the time variable t (see Appendix s). The short rates are defined by s is s in s and s in s and s in s i

A strategy is a curve $\gamma: I \to X$ in the portfolio space parameterized by the time. This means that the allocation at time t is given by the vector of nominals $x_t := \gamma(t)$. We denote by $\bar{\gamma}$ the lift of γ to M, that is $\bar{\gamma}(t) := (\gamma(t), t)$. A strategy is said to be **closed** if it represented by a closed curve. A **weak** \mathcal{D} -admissible strategy is predictable and weak \mathcal{D} -differentiable.

Remark 14. We require weak \mathcal{D} -differentiability and not strong \mathcal{D} -differentiability because imposing a priori regularity properties on the trading strategies corresponds to restricting the class of admissible strategies with respect to the classical notion of Delbaen and Schachermayer. Every (no-)arbitrage consideration depends crucially on the chosen definition of admissibility. Therefore, restricting the class of admissible strategies may lead to the automatic exclusion of potential arbitrage opportunities, leading to vacuous statements for kinds of Fundamental Theorem of Asset Pricing. An admissible strategy in the classic sense (see Section 2) is weak \mathcal{D} -differentiable.

In general the allocation can depend on the state of the nature i.e. $x_t = x_t(\omega)$ for $\omega \in \Omega$.

Proposition 15. A weak \mathcal{D} -admissible strategy is self-financing if and only if

$$\mathcal{D}(x_t \cdot D_t) = x_t \cdot \mathcal{D}D_t - \frac{1}{2}\mathfrak{D}_* \langle x, D \rangle_t \quad or \quad \mathcal{D}x_t \cdot D_t = -\frac{1}{2}\mathfrak{D}_* \langle x, D \rangle_t \quad or \quad \mathfrak{D}x_t \cdot D_t = 0, \tag{4}$$

almost surely. The bracket $\langle \cdot, \cdot \rangle$ denotes the continuous part of the quadratic covariation.

Proof. The strategy is self-financing if and only if

$$x_t \cdot D_t = x_0 \cdot D_0 + \int_0^t x_u \cdot dD_u,$$

which is, symbolizing Itô's differential d, equivalent to

$$\mathfrak{D}(x_t \cdot D_t) = x_t \cdot \mathfrak{D}D_t, \tag{5}$$

or equivalent to

$$\mathfrak{D}x_t \cdot D_t = 0. \tag{6}$$

The self-financing condition can be expressed by means of the anticipative differential d_* as

$$x_t \cdot D_t = x_0 \cdot D_0 + \int_0^t x_u \cdot d_* D_u - \int_0^t d\langle x, D \rangle_u,$$

which is equivalent to

$$\mathfrak{D}_*(x_t \cdot D_t) = x_t \cdot \mathfrak{D}_* D_t - \mathfrak{D}_* \langle x, D \rangle_t. \tag{7}$$

By summing equations (5) and (7) we obtain

$$\mathcal{D}(x_t \cdot D_t) = \frac{1}{2} (\mathfrak{D} + \mathfrak{D}_*)(x_t \cdot D_t) = x_t \cdot \mathcal{D}D_t - \frac{1}{2} \mathfrak{D}_* \langle x, D \rangle_t.$$

To prove the second statement in expression (4) we consider the integration by parts formula for Itô's integral

$$\int_0^t x_u \cdot dD_u + \int_0^t D_u \cdot dx_u = x_t \cdot D_t - x_0 \cdot D_0 - \langle x, D \rangle_t,$$

which, expressed in terms of Stratonovich's integral, leads to

$$\int_0^t x_u \circ dD_u - \frac{1}{2} \langle x, D \rangle_t + \int_0^t D_u \circ dx_u - \frac{1}{2} \langle x, D \rangle_t = x_t \cdot D_t - x_0 \cdot D_0 - \langle x, D \rangle_t.$$

By taking Stratonovich's derivative \mathcal{D} on both side we get

$$\mathcal{D}(x_t \cdot D_t) = \mathcal{D}x_t \cdot D_t + x_t \cdot \mathcal{D}D_t,$$

which, together with the first statement in expression (4) proves the second one.

For the reminder of this paper unless otherwise stated we will deal only with weak \mathcal{D} -differentiable market models, weak \mathcal{D} -differentiable strategies, and, when necessary, with weak \mathcal{D} -differentiable state price deflators. All Itô processes are weak \mathcal{D} -differentiable, so that the class of considered admissible strategies is very large.

2.4.3 Arbitrage as Curvature

The Lie algebra of G is the function space of all real valued functions on $[0, +\infty[$ denoted by

$$\mathfrak{q}=\mathbb{R}^{[0,+\infty[}$$

and therefore commutative. Following Ilinski's idea proposed in [28], we motivate the choice of a particular g-valued connection 1-form by the fact that it allows to encode portfolio rebalance (or foreign exchange) and discounting as parallel transport.

Theorem 16. With the choice of connection

$$\chi(x,t,g).(\delta x,\delta t) := \left(\frac{D_t^{\delta x}}{D_t^x} - r_t^x \delta t\right) g,\tag{8}$$

the stochastic parallel transport in \mathcal{B} has the following financial interpretations:

- Parallel transport along the nominal directions (x-lines) corresponds to a multiplication by an exchange rate.
- Parallel transport along the time direction (t-line) corresponds to a division by a stochastic discount factor.

Proof. We refer to Theorem 28 in [12].

Recall that time derivatives needed to define the parallel transport along the time lines have to be understood in Stratonovich's sense. We see that the bundle is trivial, because it has a global trivialization, but the connection is not trivial. The connection χ writes as a linear combination of

basis differential forms as

$$\chi(x,t,g) = \left(\frac{1}{D_t^x} \sum_{j=1}^N D_t^j dx_j - r_t^x dt\right) g. \tag{9}$$

The \mathfrak{g} -valued curvature 2-form is defined as

$$R := d\chi + [\chi, \chi],\tag{10}$$

meaning by this, that for all $(x, t, g) \in \mathcal{B}$ and for all $\xi, \eta \in T_{(x,t)}M$

$$R(x,t,g)(\xi,\eta) := d\chi(x,t,g)(\xi,\eta) + [\chi(x,t,g)(\xi),\chi(x,t,g)(\eta)]. \tag{11}$$

Remark that, being the Lie algebra commutative, the Lie bracket $[\cdot, \cdot]$ vanishes. After some calculations we obtain

$$R(x,t,g) = \frac{g}{D_t^x} \sum_{j=1}^N D_t^j \left(r_t^x + \mathcal{D}\log(D_t^x) - r_t^j - \mathcal{D}\log(D_t^j) \right) dx_j \wedge dt, \tag{12}$$

summarized as the following.

Proposition 17 (Curvature Formula). Let R be the curvature. Then, the following equality holds:

$$R(x,t,g) = gdt \wedge d_x \left[\mathcal{D}\log(D_t^x) + r_t^x \right]. \tag{13}$$

The curvature represents the capacity of instantaneous arbitrage allowed by the market. Although the original proof can be found in Proposition 38 in [12], it is based on the physical concept such as the divergence and the current, which is not so familiar for mathematical finance, here we state afresh more straightforward proof.

Proof. Since the Lie bracket $[\cdot, \cdot]$ vanishes, and the exterior derivative d acts only for the first term of the right hand side of (9),

$$R(x,t,g) = d\chi(x,t,g) = g \cdot d\left(\sum_{i=1}^{N} \frac{\partial \log(D_t^x)}{\partial x_i} \cdot dx_i - r_t^x dt\right).$$

We note that the differential d acts as $d = d_x + d_t = d_x + \mathcal{D}$ for the first term, while the differential d acts as $d = d_x$ for the second term $(-r_t^x)dt$ because $dt \wedge dt = 0$ as bellow.

$$d(-r_t^x dt) = d_x(-r_t^x dt) = -\sum_i \frac{\partial}{\partial x_j} r_t^x dx_j \wedge dt = \sum_i \frac{\partial}{\partial x_j} r_t^x dt \wedge dx_j.$$

And then we have

$$R(x,t,g) = g \cdot \left(\sum_{i < j} \frac{\partial^2}{\partial x_i \partial x_j} \left(\log(D_t^x) \right) dx_i \wedge dx_j + \left(\sum_j \mathcal{D} \frac{\partial \log(D_t^x)}{\partial x_i} + \sum_j \frac{\partial}{\partial x_j} r_t^x \right) dt \wedge dx_j \right),$$

but the first term vanish because of the anticommutativity of the wedge product $dx_i \wedge dx_j = -dx_j \wedge dx_i$. Rearrange the order of $\frac{\partial}{\partial x_i}$ and \mathcal{D} , we can conclude that

$$R(x, t, g) = g \cdot \sum_{j} \frac{\partial}{\partial x_{j}} \Big(\mathcal{D} \log(D_{t}^{x}) + r_{t}^{x} \Big) dt \wedge dx_{j}$$
$$= g \cdot dt \wedge d_{x} \Big(\mathcal{D} \log(D_{t}^{x}) + r_{t}^{x} \Big).$$

We can prove following results which characterizes arbitrage as curvature.

Theorem 18 (No Arbitrage). The following assertions are equivalent:

- (i) The market model (consisting base assets and futures with discounted prices D and P) satisfies the no-free-lunch-with-vanishing-risk condition.
- (ii) There exists a positive local martingale $\beta = (\beta_t)_{t\geq 0}$ such that deflators and short rates satisfy for all portfolio nominals and all times the condition

$$r_t^x = -\mathcal{D}\log(\beta_t D_t^x). \tag{14}$$

(iii) There exists a positive local martingale $\beta = (\beta_t)_{t\geq 0}$ such that deflators and term structures satisfy for all portfolio nominals and all times the condition

$$P_{t,s}^x = \frac{\mathbb{E}_t[\beta_s D_s^x]}{\beta_t D_t^x}. (15)$$

Proof. We refer to Theorem 33 in [12].

This motivates the following definition.

Definition 19. The market model satisfies the **zero curvature** (**ZC**) if and only if the curvature vanishes a.s.

Therefore, we have following implication relying two different definitions of no-abitrage:

Corollary 20.

$$(NFLVR) \Rightarrow (ZC)$$
.

As an example to demonstrate how the most important geometric concepts of Section 2 can be applied we consider an asset model whose dynamics is given by a multidimensional Itô process. Let us consider a market consisting of N+1 assets labeled by $j=0,1,\ldots,N$, where the 0-th asset is the cash account utilized as a numéraire. Therefore, as explained in the introductory Subsection 2.1, it suffices to model the price dynamics of the other assets $j=1,\ldots,N$ expressed in terms of the 0-th asset. As vector valued semimartingales for the discounted price process $\hat{S}:[0,+\infty[\times\Omega\to\mathbb{R}^N]$ and the short rate $r:[0,+\infty[\times\Omega\to\mathbb{R}^N]$, we chose the multidimensional Itô processes given by

$$d\hat{S}_t = \hat{S}_t(\alpha_t dt + \sigma_t dW_t)$$

$$dr_t = a_t dt + b_t dW_t,$$
(16)

where

- $(W_t)_{t\in[0,+\infty[}$ is a standard \mathbb{P} -Brownian motion in \mathbb{R}^K , for some $K\in\mathbb{N}$,
- $(\sigma_t)_{t\in[0,+\infty[}, (\alpha_t)_{t\in[0,+\infty[}$ are $\mathbb{R}^{N\times K}$ -, and respectively, \mathbb{R}^N valued stochastic processes, σ_t has maximal rank, i.e. $\operatorname{rank}(\sigma_t) = K$, and,
- $(b_t)_{t\in[0,+\infty[}, (a_t)_{t\in[0,+\infty[} \text{ are } \mathbb{R}^{N\times K}$ -, and respectively, \mathbb{R}^N valued stochastic processes.

Proposition 21. Let the dynamics of a market model be specified by following Itô processes as in (16), where we additionally assume that the coefficients

• $(\alpha_t)_t, (\sigma_t)_t$, and $(r_t)_t$ satisfy

$$\lim_{s \to t^-} \mathbb{E}_s[\alpha_t] = \alpha_t, \quad \lim_{s \to t^-} \mathbb{E}_s[r_t] = r_t, \quad \lim_{s \to t^-} \mathbb{E}_s[\sigma_t] = \sigma_t,$$

- $(\sigma_t)_t$ is an Itô process,
- $(\sigma_t)_t$ and $(W_t)_t$ are independent processes.

Then, the market model satisfies the (ZC) condition if and only if

$$\alpha_t + r_t \in \text{Range}(\sigma_t).$$
 (17)

Remark 22. In the case of the classical model, where there are no term structures (i.e. $r \equiv 0$), the condition (17) reads as $\alpha_t \in \text{Range}(\sigma_t)$.

Proof. Let us consider the expression for Itô's integral with respect to Stratonovich's

$$\int_0^t \sigma_u dW_u = \int_0^t \sigma_u \circ dW_u - \frac{1}{2} \int_0^t d\langle \sigma, W \rangle_u,$$

and take Nelson's derivative corresponding to the Stratonovich's integral:

$$\mathcal{D} \int_{0}^{t} \sigma_{u} dW_{u} = \sigma_{t} \mathcal{D} W_{t} - \frac{1}{2} \mathcal{D} \langle \sigma, W \rangle_{t}.$$
(18)

Since

$$\mathcal{D}W_t = \frac{W_t}{2t} \tag{19}$$

and, because of the independence assumption for the two Itô processes $(\sigma_t)_t$ and $(W_t)_t$,

$$\langle \sigma, W \rangle_t \equiv 0,$$

we obtain

$$\mathcal{D} \int_0^t \sigma_u dW_u = \sigma_t \frac{W_t}{2t},$$

which, inserted into the asset dynamics

$$\hat{S}_t = \hat{S}_0 \exp\left(\int_0^t (\alpha_u - \frac{1}{2} \operatorname{diag}(\sigma_u \sigma_u^{\dagger})) du + \int_0^t \sigma_u dW_u\right),$$

leads to

$$\mathcal{D}\log \hat{S}_t = \alpha_t - \frac{1}{2}\operatorname{diag}(\sigma_t \sigma_t^{\dagger}) + \sigma_t \frac{W_t}{2t}.$$

By Proposition 17 the curvature vanishes if and only if for all $x \in \mathbb{R}^N$

$$\mathcal{D}\log\hat{S}_t^x + r_t^x = C_t,$$

for a real valued stochastic process $(C_t)_{t\geq 0}$, or, equivalently

$$\mathcal{D}\log\hat{S}_t + r_t = C_t e,$$

where $e := [1, \dots, 1]^{\dagger}$ or

$$\alpha_t + r_t - \frac{1}{2} \operatorname{diag}(\sigma_t \sigma_t^{\dagger}) + \sigma_t \frac{W_t}{2t} = C_t e. \tag{20}$$

Equation (20) is the formulation of the (ZC) condition for the market model (16). By taking on both sides of (20) $\lim_{h\to 0^+} \mathbb{E}_{t-h}[\cdot]$, and utilizing the independence assumption, from which

$$\mathbb{E}_{t-h}\left[\sigma_t \frac{W_t}{2t}\right] = \mathbb{E}_{t-h}\left[\sigma_t\right] \underbrace{\mathbb{E}_{t-h}\left[\frac{W_t}{2t}\right]}_{=0} = 0$$

follows, we obtain, using the continuity assumption for $(\alpha_t)_t$, $(\sigma_t)_t$, and $(r_t)_t$,

$$\alpha_t + r_t - \frac{1}{2} \operatorname{diag}(\sigma_t \sigma_t^{\dagger}) = \beta_t e,$$

where $\beta_t := \lim_{h\to 0^+} \mathbb{E}_{t-h}[C_t]$ is a predictable process. Therefore, equation (20) becomes

$$\sigma_t \frac{W_t}{2t} = (C_t - \beta_t)e, \tag{21}$$

and, thus

$$e \in \text{Range}(\sigma_t),$$
 (22)

the space spanned by the column vectors of σ_t . Since σ_t has maximal rank, the K column vectors of σ_t are linearly independent and $C_t - \beta_t \neq 0$.

Let P_{σ_t} , $P_{\sigma_t^{\perp}}$ denote the orthogonal projections onto Range (σ_t) and its orthogonal complement Range $(\sigma_t)^{\perp}$, respectively. Then, we can decompose

$$\alpha_t + r_t = P_{\sigma_t}(\alpha_t + r_t) + P_{\sigma_t^{\perp}}(\alpha_t + r_t), \tag{23}$$

and

$$P_{\sigma_t^{\perp}}(\alpha_t + r_t) = P_{\sigma_t^{\perp}}(C_t e) - P_{\sigma_t^{\perp}}\left(\sigma_t \frac{W_t}{2t}\right) + P_{\sigma_t^{\perp}}\left(\frac{1}{2}\operatorname{diag}(\sigma_t \sigma_t^{\dagger})\right). \tag{24}$$

Since e and $\sigma_t W_t$ lie in Range(σ_t), the first two addenda on the right hand side of (24) vanish. By Lemmata 23 and 24 the third one vanishes as well, so that $P_{\sigma_t^{\perp}}(\alpha_t + r_t) = 0$, i.e. $\alpha_t + r_t \in \text{Range}(\sigma_t)$. Conversely, if $\alpha_t + r_t \in \text{Range}(\sigma_t)$, then equation (20) holds true, and the proof of the equivalence between the (ZC) condition and (17) is completed.

Lemma 23. Let A be a linear operator on the euclidean \mathbb{R}^N . The vector

$$\operatorname{diag}(A) := \sum_{j=1}^{N} (Ae_j \cdot e_j)e_j$$

does not depend on the choice of the orthonormal basis $\{e_1, \ldots, e_n\}$ of \mathbb{R}^N and defines the **diagonal** of

A.

Proof. The coordinates of diag(A) with respect to the orthonormal basis $\{e_1, \ldots, e_N\}$ can be written as

$$[\operatorname{diag}(A)]_{\{e\}} = \sum_{j=1}^{N} ([e_j]_{\{e\}}^{\dagger} [A]_{\{e\}} [e_j]_{\{e\}}) [e_j]_{\{e\}}$$
(25)

Let us consider another orthonormal basis $\{f_1, \ldots, f_n\}$ of \mathbb{R}^N . This means that there exists an orthogonal linear operator U on \mathbb{R}^N such that $Ue_j = f_j$ for all $j = 1, \ldots, N$. Therefore we can write

$$[\operatorname{diag}(A)]_{\{e\}} = \sum_{j=1}^{N} \left(([U]_{\{f\}}^{\dagger}[f_j]_{\{f\}})^{\dagger}[A]_{\{e\}}[U]_{\{f\}}^{\dagger}[f_j]_{\{f\}} \right) [U]_{\{f\}}^{\dagger}[f_j]_{\{f\}}$$

$$= \sum_{j=1}^{N} \left([f_j]_{\{f\}}^{\dagger} \left([U]_{\{f\}}[A]_{\{e\}}[U]_{\{f\}}^{\dagger} \right) [f_j]_{\{f\}} \right) [U]_{\{f\}}^{\dagger}[f_j]_{\{f\}}$$

$$= [U]_{\{f\}}^{\dagger} \left(\sum_{j=1}^{N} [f_j]_{\{f\}}^{\dagger}[A]_{\{f\}}[f_j]_{\{f\}} \right)$$

$$= [U]_{\{f\}}^{\dagger}[\operatorname{diag}(A)]_{\{f\}}.$$
(26)

Therefore, the coordinates of the diagonal transforms like a vector during a change of basis, and hence the diagonal is well defined.

Lemma 24. Let σ be a $\mathbb{R}^{N \times K}$ real matrix of rank K and P the orthogonal projection onto the orthogonal complement to the subspace generated by the column vectors of σ . Then,

$$P \operatorname{diag}(\sigma \sigma^{\dagger}) = 0 \in \mathbb{R}^{N}.$$

Proof. The real symmetric matrix $\sigma \sigma^{\dagger} \in \mathbb{R}^{N \times N}$ induces via standard orthonormal basis a selfadjoint linear operator on \mathbb{R}^N , which by Lemma 23 has a well defined diagonal. Let us enlarge σ to an $\mathbb{R}^{N \times N}$ matrix, by adding N-K zero column vectors. The matrix $\sigma \sigma^{\dagger} \in \mathbb{R}^{N \times N}$ remains the same. Let us consider an orthonormal basis of \mathbb{R}^N , $\{f_1, \ldots, f_N\}$, where $\{f_1, \ldots, f_K\}$ is a basis of Range (σ) and $\{f_{K+1}, \ldots, f_N\}$ is a basis of its orthogonal complement, Range $(\sigma)^{\perp}$. The diagonal of $\sigma \sigma^{\dagger}$ reads

$$\operatorname{diag}(\sigma\sigma^{\dagger}) = \sum_{j=1}^{N} (\sigma\sigma^{\dagger}f_{j} \cdot f_{j})f_{j} = \sum_{j=1}^{N} (\sigma^{\dagger}f_{j} \cdot \sigma^{\dagger}f_{j})f_{j} = \sum_{j=1}^{K} (\sigma^{\dagger}f_{j} \cdot \sigma^{\dagger}f_{j})f_{j}, \tag{27}$$

because $\sigma^{\dagger} f_j = 0$ for $j = K + 1, \dots, N$, being f_j in the orthogonal complement of Range(σ). Therefore,

$$P\operatorname{diag}(\sigma\sigma^{\dagger}) = \sum_{j=1}^{K} (\sigma^{\dagger} f_j \cdot \sigma^{\dagger} f_j) Pf_j = 0,$$
(28)

because f_j is in Range (σ) for $j=1,\ldots,K$ and P is the projection onto Range $(\sigma)^{\perp}$.

Next, we show the equivalence of the (ZC) condition with (NFLVR) in the case of Itô's dynamics.

Proposition 25. Under the same assumptions as Proposition 21, the zero curvature condition for the market model specified by (16), that is

$$\mathcal{D}\log\hat{S}_t + r_t = C_t e,$$

is equivalent to the no-free-lunch-with-vanishing-risk condition if the positive stochastic process $(\beta_t)_{t\geq 0}$, defined as

 $\beta_t := \exp\left(-\int_0^t C_u du\right)$

is a local martingale.

Proof. By Proposition 17 the zero curvature (ZC) condition R = 0 is equivalent with the existence of a stochastic process $(C_t)_{t\geq 0}$ such that for all i = 1, ..., N the equation

$$\mathcal{D}\log \hat{S}_t^i + r_t^i = C_t$$

holds. This means that

$$\mathcal{D}\log \hat{S}_t^i = C_t - r_t^i$$

$$\log \frac{S_t^i}{S_0^i} = \int_0^t (C_u - r_u^i) du$$

$$S_t^i = S_0^i \exp\left(\int_0^t C_u du\right) \exp\left(-\int_0^t r_u^i du\right).$$

Therefore,

$$\mathcal{D}\log(\beta_t D_t^i) + r_t^i = 0$$

for all i = 1, ..., N. By Theorem 18, if $(\beta_t)_{t \geq 0}$ is a martingale, then we have proved (NFLVR).

We can reformulate the result of Proposition 21 as follows.

Corollary 26. Let $\{J_t^1, \ldots, J_t^B\}$ be an orthonormal basis of $\ker(\sigma_t) \subset \mathbb{R}^N$. Under the same assumptions as Proposition 21 the (ZC) condition for the market model (16), which is equivalent to (NFLVR), is equivalent to

$$\rho_t := J_t^{\dagger}(\alpha_t + r_t) \equiv 0 \in \mathbb{R}^B, \tag{29}$$

where $J_t := [J_t^1, ..., J_t^B].$

Remark 27 (Counterexamples). Let us consider a financial market with a cash account with $r_t^0 = 0$ and a single risky asset with (discounted) price given by

$$S_t = e^{X_t}, \text{ where } X_t := \int_0^t \frac{W_u}{u} + W_t,$$
 (30)

for a standard univariate Brownian motion $(W_t)_t$. By Itô's formula, it follows that

$$dS_t = S_t \left(\frac{1}{2} + \frac{W_t}{t}\right) + S_t dW_t \text{ and } S_0 = 1.$$

In the notation of Proposition 21, this corresponds to $\alpha_t = \frac{1}{2} + \frac{W_t}{t}$ and $r_t = 0$. The coefficient α_t does not satisfy the assumption $\lim_{s\to t^-} \mathbb{E}_s[\alpha_t] = \alpha_t$, because

$$\lim_{s \to t^{-}} \mathbb{E}_{s} \left[\frac{1}{2} + \frac{W_{t}}{t} \right] = \frac{1}{2} + \frac{1}{2} \lim_{s \to t^{-}} \underbrace{\mathbb{E}_{s}[W_{t}]}_{=0} \neq \alpha_{t}.$$

The process $(S_t)_t$ does not satisfy (NFLVR), since $\int_0^{\epsilon} \left(\frac{W_t}{t}\right)^2 dt > 0$ for all $\epsilon > 0$ as a consequence of Corollary 3.2 of [29]. In the terminology of Delbaen & Schachermayer, model (30) generates immediate arbitrage opportunities. Other simple counterexamples can be constructed from Brownian bridges, which provide well known examples of models admitting arbitrage (see [33]).

Moreover, Fontana & Runggaldier present asset models in [17] (Example 7.5) and [18] (page 59) based on Bessel processes, which do not fulfill the assumptions of Propositions 21 and 25. They are an example of dynamics satisfying (NUPBR) but not (NFLVR). The proof of the (NUPBR) property is based on its equivalence with the non-existence of arbitrage possibilities of the first kind.

3 Arbitrage and Utility

Let us now consider a utility function, that is a real C^2 -function of a real variable, which is strictly monotone increasing (i.e. u' > 0) and concave (i.e. u'' < 0). Typically, a market participant would like to maximize the expected utility of its wealth at some time horizon. Let us assume that he (or she) holds a portfolio of synthetic zero bonds delivering at maturity base assets and that the time horizon is

infinitesimally near, that is that the utility of the instantaneous total return has to be maximized. The portfolio values read as:

• At time t - h: $D_{t-h}^x P_{t-h,t+h}^x$.

• At time $t: D_t^x P_{t,t+h}^x$.

• At time t + h: D_{t+h}^x .

From now on we make the following

Assumptions:

(A1): The market filtration $(A_t)_{t\geq 0}$ is the coarsest filtration for which $(D_t)_{t\geq 0}$ is adapted.

(A2): The process $(D_t)_{t\geq 0}$ is Markov with respect to the filtration $(A_t)_{t\geq 0}$.

Proposition 28. Under the assumptions (A1) and (A2) the synthetic bond portfolio instantaneous return can be computed as:

$$Ret_{t}^{x} := \lim_{h \to 0^{+}} \mathbb{E}_{t} \left[\frac{D_{t+h}^{x} - D_{t-h}^{x} P_{t-h,t+h}^{x}}{2h D_{t-h}^{x} P_{t-h,t+h}^{x}} \right] = \mathcal{D}\log(D_{t}^{x}) + r_{t}^{x}.$$

Proof. Under the assumptions (A1) and (A2) the conditional expectations with respect to the market filtration $(\mathcal{A}_t)_{t\geq 0}$ are the same as those computed with respect to the present $(\mathcal{N}_t)_{t\geq 0}$, past $(\mathcal{P}_t)_{t\geq 0}$ and future $(\mathcal{F}_t)_{t>0}$ filtrations (see Appendix A). Therefore, we can develop the instantaneous return as

$$\begin{split} &\lim_{h \to 0^+} \mathbb{E}_t \left[\frac{D^x_{t+h} - D^x_{t-h} P^x_{t-h,t+h}}{2h D^x_{t-h} P^x_{t-h,t+h}} \right] \\ &= \lim_{h \to 0^+} \mathbb{E}_t \left[\frac{D^x_{t+h} - D^x_{t-h}}{2h D^x_{t-h} P^x_{t-h,t+h}} + \frac{1 - P^x_{t-h,t+h}}{2h P^x_{t-h,t+h}} \right] \\ &= \frac{1}{D^x_t} \mathcal{D}D^x_t + \lim_{h \to 0^+} \frac{\exp\left(\int_{t-h}^{t+h} ds \, f^x_{t-h,s}\right) - 1}{2h} = \mathcal{D} \log D^x_t + r^x_t. \end{split}$$

Remark 29. This portfolio of synthetic zero bonds in the theory corresponds to a portfolio of futures in practice. If the short rate vanishes, then the future corresponds to the original asset.

Definition 30 (Expected Utility of Synthetic Bond Portfolio Return). Let $t \geq s$ be fixed times. The expected utility maximization problem at time s for the horizon T for initial capital ξ writes

$$\sup_{\substack{x = \{x_h\}_{h \ge s} \\ D_s^{x_s} = \xi}} \mathbb{E}_s \left[u \left(\exp \left(\int_s^T dt \, \left(\mathcal{D} \log(D_t^{x_t}) + r_t^{x_t} \right) \right) D_s^{x_s} P_{s,T}^{x_s} \right) \right], \tag{31}$$

where the supremum is taken over all weak \mathcal{D} -differentiable self-financing admissible strategies $x = \{x_u\}_{u\geq 0}$.

Now we can formulate the first result of this subsection.

Theorem 31. Let us assume (A1) and (A2), and that $(D_t)_t$, $(r_t)_t$ are weakly \mathcal{D} -differentiable semimartingales. The market curvature vanishes if and only if the expected utility maximization problem can be solved for all times and horizons for a chosen utility function.

This result can be seen as the natural generalization of the corresponding result in discrete time, as Theorem 3.5 in [16], see also [37]. Compare with Bellini's, Frittelli's and Schachermayer's results for infinite dimensional optimization problems in continuous time, see Theorem 22 in [2] and Theorem 2.2 in [39]. Nothing is said about the fulfilment of the no-free-lunch-with-vanishing-risk condition: only the weaker zero curvature condition is equivalent to the maximization of the expected utility at all times for all horizons.

Proof. The optimization problem (31) into a standard problem of stochastic optimal theory in continuous time which can be solved by means of a fundamental solution of the Hamilton-Jacobi-Bellman partial differential equation.

However, there is a direct method, using Lagrange multipliers for Banach spaces (see [34] pages 239–270 and [45] Section 4.14, pages 270–271). First, remark that problem (30) is a concave optimization problem with convex domain and concave utility function and has therefore a unique solution corresponding to a global maximum. The Lagrange principal function corresponding to the maximum problem

$$\Phi(x,\lambda,\mu) := \mathbb{E}_s \left[u \left(\exp\left(\int_s^T dt \, \left(\mathcal{D} \log(D_t^{x_t}) + r_t^{x_t} \right) \right) D_s^{x_s} P_{s,T}^{x_s} \right) - \int_s^T dt \, \lambda_t \mathfrak{D} x_t \cdot D_t \right] - \mu(D_s^{x_s} - \xi).$$
(32)

Note that the Lagrange multiplier λ corresponding to the self-financing condition (6), expressed in terms of Nelson's derivative corresponding to Itô's differential, is a stochastic process $(\lambda_t)_{t\geq 0}$. This Lagrange multiplier is weak \mathcal{D} -differentiable as all process involved so far are. The Lagrange multiplier μ corresponding to the initial wealth is a real number. To solve the maximization problem for Φ with

respect to the processes (x_t) and (λ_t) we embed the optimal solution into a one parameter family as

$$\begin{cases} x_t(\epsilon) & := x_t + \epsilon \delta x_t \\ \lambda_t(\eta) & := \lambda_t + \eta \delta \lambda_t \\ \mu(\nu) & := \mu + \nu \delta \mu, \end{cases}$$

where ϵ , η and ν are real parameters defined in a neighborhood of 0, and δx_t , $\delta \lambda_t$ and $\delta \nu$ are arbitrary variations such that the boundary conditions

$$\begin{cases} x_s(\epsilon) \equiv x_s \\ x_T(\epsilon) \equiv x_T, \end{cases}$$
 (33)

are satisfied. The Lagrange principal equations associated to this maximization problem read

$$\begin{cases}
\frac{\partial \Phi}{\partial \epsilon} \Big|_{\epsilon=\eta=\nu:=0} &= \mathbb{E}_{s} \left[u' \left(\exp \left(\int_{s}^{T} dt \left(\mathcal{D} \log(D_{t}^{x_{t}}) + r_{t}^{x_{t}} \right) \right) D_{s}^{x_{s}} P_{s,T}^{x_{s}} \right) \\
&\cdot \exp \left(\int_{s}^{T} dt \left(\mathcal{D} \log(D_{t}^{x_{t}}) + r_{t}^{x_{t}} \right) \right) D_{s}^{x_{s}} P_{s,T}^{x_{s}} \\
&\cdot \int_{s}^{T} dt \frac{\partial}{\partial x} \left(\mathcal{D} \log(D_{t}^{x}) + r_{t}^{x_{t}} \right) \Big|_{x=x_{t}} \cdot \delta x_{t} \\
&- \int_{s}^{T} dt \lambda_{t} \mathfrak{D} \delta x_{t} \cdot D_{t} \Big] - \mu D_{s}^{\delta x_{s}} = 0
\end{cases}$$

$$\frac{\partial \Phi}{\partial \eta} \Big|_{\epsilon=\eta=\nu:=0} = -\int_{s}^{T} dt \, \delta \lambda_{t} \mathfrak{D} x_{t} \cdot D_{t} = 0 \\
\frac{\partial \Phi}{\partial \nu} \Big|_{\epsilon=\eta=\nu:=0} = -\delta \mu(D_{s}^{x_{s}} - \xi) = 0,$$
(34)

where, by Leibniz's theorem, we have interchanged differentiation with respect to ϵ or η with the integration with respect to t and the conditional expectation. The boundary condition implies $\delta x_s = 0$, and hence $\mu D_s^{\delta x_s} = 0$.

Integration by parts with respect to the time variable shows that

$$-\int_{s}^{T} dt \, \lambda_{t} \mathfrak{D} \delta x_{t} \cdot D_{t} = \int_{s}^{T} dt \, \mathfrak{D}(\lambda_{t} D_{t}) \cdot \delta x_{t},$$

which, inserted into the first equation of (34) leads to

$$\mathbb{E}_{s} \left[\int_{s}^{T} dt \left(M \frac{\partial}{\partial x} \left(\mathcal{D} \log(D_{t}^{x}) + r_{t}^{x} \right) \Big|_{x = x_{t}} + \mathfrak{D}(\lambda_{t} D_{t}) \right) \cdot \delta x_{t} \right] = 0, \tag{35}$$

where

$$M := u' \left(\exp\left(\int_s^T dt \, \left(\mathcal{D} \log(D_t^{x_t}) + r_t^{x_t} \right) \right) D_s^{x_s} P_{s,T}^{x_s} \right)$$
$$\cdot \exp\left(\int_s^T dt \, \left(\mathcal{D} \log(D_t^{x_t}) + r_t^{x_t} \right) \right) D_s^{x_s} P_{s,T}^{x_s}$$

is a strictly positive random variable. Since the variation δx_t is arbitrary we infer from (35)

$$M \frac{\partial}{\partial x} \left(\mathcal{D} \log(D_t^x) + r_t^x \right) \bigg|_{x=x_t} + \mathcal{D}(\lambda_t D_t) = 0 \quad \text{ for any } t \in [s, T],$$

and, thus, for the choice t:=s, it follows, being the initial condition $x_s\in\mathbb{R}^N$ arbitrary

$$\mathcal{D}\log(D_t^x) + r_t^x = -\frac{1}{M}\mathfrak{D}(\lambda_t D_t^j)x_j + C_t^j \text{ for all } j = 1, \dots, N,$$

for a stochastic process $(C_t^j)_{t\geq 0}$. Therefore

$$-\frac{1}{M}\mathfrak{D}(\lambda_t D_t^j)x_j + C_t^j = -\frac{1}{M}\mathfrak{D}(\lambda_t D_t^i)x_i + C_t^i \text{ for all } j \neq i,$$

which can hold true if and only if

$$\begin{cases}
\mathfrak{D}(\lambda_t D_t^j) = 0 \\
C_t^j = C_t
\end{cases}$$
(36)

for all j = 1, ..., N. Hence, for the optimal Lagrange multiplier,

$$\mathfrak{D}(\lambda_t D_t) = 0 \in \mathbb{R}^N.$$

and

$$\mathcal{D}\log(D_t^x) + r_t^x = C_t \text{ for all } j = 1, \dots, N.$$
(37)

Therefore, by Proposition 17, the curvature must vanish, which means that the existence of a solution to the maximization problem implies the vanishing of the curvature. The converse is also true, as it can be seen by following back the steps in this proof from (37) to (32). Hence, the equivalence between (ZC) and (31) holds true.

It turns out that the two weaker notions of arbitrage, the zero curvature and the no-unbounded-profitwith-bounded-risk are equivalent. **Theorem 32.** Let us assume (A1) and (A2), and that $(D_t)_t$, $(r_t)_t$ are semimartingales. Then,

$$(NUPBR) \Rightarrow (ZC)$$
.

Remark 33 (Counterexample). The model given by (30) satisfies (ZC), because it is one dimensional in the risky assets, but does not fulfill (NUPBR), because it allows for immediate arbitrage opportunities as shown in Remark 27.

Proof of Theorem 32. By Proposition 2.1 (4) in [25] the (NUPBR) is equivalent with the existence of a growth optimal portfolio. We apply the classic set up of portfolio optimization to the portfolio of futures under consideration, (which covers as a special case the portfolio of base assets). Since the value of the portfolio at time s is

$$D_s^{x_s} P_{s,T}^{x_s}$$
,

and the growth factor from s to T is

$$\exp\left(\int_{s}^{T} dt \, \left(\mathcal{D}\log(D_{t}^{x_{t}}) + r_{t}^{x_{t}}\right)\right),\,$$

the solution of the expected utility maximization for s := 0 and arbitrary T with utility function $u := \log$ must be equal to the optimal growth portfolio. Therefore, by Theorem 31 (ZC) follows.

Under what conditions is the converse of Theorem 32 true? The equivalence of expected utility maximization and (NFLVR) can be proved for a particular choice of a Markov dynamics. Namely, if the asset dynamics follows an Itô process, Proposition 25 and Theorem 31 lead to

Proposition 34. Let the dynamics of a market model be specified by following Itô processes as in (16), where we additionally assume that the coefficients

• $(\alpha_t)_t, (\sigma_t)_t, and (r_t)_t satisfy$

$$\lim_{s \to t^-} \mathbb{E}_s[\alpha_t] = \alpha_t, \quad \lim_{s \to t^-} \mathbb{E}_s[r_t] = r_t, \quad \lim_{s \to t^-} \mathbb{E}_s[\sigma_t] = \sigma_t,$$

- $(\sigma_t)_t$ is an Itô process,
- $(\sigma_t)_t$ and $(W_t)_t$ are independent processes.

Then, the (NFLVR) condition holds true if and only if the expected utility maximization problem can be solved for all times and horizons for a chosen utility function.

Corollary 35. Under the same assumptions of Proposition 34,

$$(ZC) \Rightarrow (NUPBR).$$

These last two results are in line with the well-known results of [1], [5], [31] and [25].

4 Arbitrage and Derivative Pricing

The (NFLVR) ia an equilibrium condition for financial markets, and the Black-Scholes PDE allows for a unique pricing of derivatives of the base assets of those financial markets. Even if the (ZC) is not fulfilled, the market forces determine an asset dynamics minimizing the total quantity of arbitrage allowed by the market, as it was shown in [12, 13]. The minimal arbitrage is an equilibrium condition as well, which generalizes the benchmark approach (e.g. [21]) leading to a probability measure equivalent to the statistical one, which is the best possible approximation for the risk neutral measure (cf. the forthcoming [14]). In this case too, a (non-linear) PDE allows for a unique pricing of derivatives of the base assets, in which the arbitrage measure explicitly appears.

4.1 The Black-Scholes PDE in the Presence of Arbitrage

For markets allowing for arbitrage we are in the position to derive the price dynamics of derivatives whose underlying following an Itô process. It is a non linear partial differential equation which coincides with the linear Black-Scholes partial differential equation as soon as the arbitrage vanishes.

Theorem 36. Let us consider a market consisting in a bank account, an asset and a derivative whose discounted prices X_t and $\Phi(t, X_t)$ follow an Itô's process. In particular

$$dX_t = X_t(\alpha_t dt + \sigma_t dW_t),$$

where $(\alpha_t)_{t\in[0,+\infty[}$ and $(\sigma_t)_{t\in[0,+\infty[}$ are real valued adapted processes, the latter with finite variation. Assuming that the pay-off function $\Phi = \Phi(t,x) \in C^{1,2}$, the derivative discounted price solves the PDE

$$\frac{\partial \Phi}{\partial t} + \frac{\sigma_t^2}{2} X_t^2 \frac{\partial^2 \Phi}{\partial x^2} = \rho_t \Phi \left(1 + \left(\frac{1}{\Phi} \frac{\partial \Phi}{\partial x} X_t \right)^2 \right)^{\frac{1}{2}}, \tag{38}$$

where ρ_t , defined in (29) measures the arbitrage allowed by the market.

Proof. We prove this theorem in the context of Corollary 26 with vanishing short rate r_t . By assumption,

choosing N := 2 and B := 1, the market dynamics reads

$$d\hat{S}_t = \hat{S}_t(\bar{\alpha}_t dt + \bar{\sigma}_t dW_t), \tag{39}$$

where

$$\hat{S}_t := \left[\begin{array}{c} X_t \\ \Phi(t, X_t) \end{array} \right], \quad \bar{\alpha}_t := \left[\begin{array}{c} \alpha_t \\ \beta_t \end{array} \right], \quad \bar{\sigma}_t := \left[\begin{array}{c} \sigma_t \\ \tau_t \end{array} \right].$$

for appropriate real valued predictable processes $(\beta_t)_{t\in[0,+\infty[}$ and $(\tau_t)_{t\in[0,+\infty[}$ characterizing the dynamics of the derivative. We apply Itô's Lemma to the second component of (39). By comparing deterministic and stochastic terms we obtain

$$\begin{cases}
\frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial x} X_t \alpha_t + \frac{\sigma_t^2}{2} \frac{\partial^2 \Phi}{\partial x^2} X_t^2 = \beta_t \Phi \\
\frac{\partial \Phi}{\partial x} X_t \sigma_t = \tau_t \Phi.
\end{cases} (40)$$

The one dimensional $\ker(\bar{\sigma}_t)$ is spanned by

$$J_t := (\sigma_t^2 + \tau_t^2)^{-\frac{1}{2}} \begin{bmatrix} -\tau_t \\ +\sigma_t \end{bmatrix}, \tag{41}$$

and the vector $\bar{\alpha}_t$ admits the decomposition

$$\bar{\alpha}_t = \lambda_t \bar{\sigma}_t + \rho_t J_t, \tag{42}$$

for reals λ_t and $\rho_t = \bar{\alpha}_t^{\dagger} J_t$. Now we can insert (42) into (40) and eliminate λ_t , since the λ_t terms cancel out. The first equation of (40) becomes

$$\frac{\partial \Phi}{\partial t} + \frac{\sigma_t^2}{2} X_t^2 \frac{\partial^2 \Phi}{\partial x^2} = \rho_t X_t \frac{\partial \Phi}{\partial x} \left(\frac{\sigma_t^2 + \tau_t^2}{\tau_t^2} \right)^{\frac{1}{2}}.$$
 (43)

The second equation of (40) can be written as

$$\frac{\sigma_t}{\tau_t} = \frac{\Phi}{X_t \frac{\partial \Phi}{\partial x}},$$

which, inserted into (43) gives (38).

Remark 37. In [11], utilizing another measure of arbitrage $\tilde{\rho}_t$, the PDE

$$\frac{\partial \Phi}{\partial t} + \frac{\sigma_t^2}{2} X_t^2 \frac{\partial^2 \Phi}{\partial x^2} = -\sqrt{2} \tilde{\rho}_t \Phi \left[1 + \frac{X_t^2}{\Phi^2} \left(\frac{\partial \Phi}{\partial x} \right)^2 - \frac{\partial \Phi}{\partial x} \right]^{\frac{1}{2}}, \tag{44}$$

was derived. After some computations, it turns out that

$$\tilde{\rho}_t = -\frac{1}{\sqrt{2}} \left(\frac{1 + \frac{X_t^2}{\Phi^2} \left(\frac{\partial \Phi}{\partial x} \right)^2}{1 - \frac{X_t}{\Phi} \frac{\partial \Phi}{\partial x} + \frac{X_t^2}{\Phi^2} \left(\frac{\partial \Phi}{\partial x} \right)^2} \right)^{\frac{1}{2}} \rho_t,$$

thus guaranteeing that both (40) and (44) are two representations of the same non linear Black-Scholes PDE for the price of a derivative in the presence of arbitrage.

Remark 38. If arbitrage possibilities are allowed, there is no risk neutral probability measure. Asset pricing can nevertheless be obtained as (conditional) expectation of discounted asset's cash flows with respect to the minimal arbitrage probability measure, as explained in [14].

It is possible to reformulate Theorem 36 directly in terms of prices and not discounted prices.

Corollary 39. Let us consider a market consisting in a bank account with constant instantaneous risk free rate r, an asset and a derivative whose prices S_t and $\Psi(t, S_t)$ follow an Itô process. In particular

$$dS_t = S_t(\alpha_t dt + \sigma_t dW_t),$$

where $(\alpha_t)_{t\in[0,+\infty[}$ and $(\sigma_t)_{t\in[0,+\infty[}$ are real valued adapted processes, the latter with finite variation. Assuming that the pay-off function $\Psi = \Psi(t,s) \in C^{1,2}$, the derivative price solves the PDE

$$\frac{\partial \Psi}{\partial t} + rS_t \frac{\partial \Psi}{\partial s} + \frac{\sigma_t^2}{2} S_t^2 \frac{\partial^2 \Psi}{\partial s^2} - r\Psi = \rho_t \Psi \left(1 + \left(\frac{1}{\Psi} \frac{\partial \Psi}{\partial s} S_t \right)^2 \right)^{\frac{1}{2}},\tag{45}$$

where ρ_t , defined in (29) measures the arbitrage allowed by the market.

Note that in the (ZC) case (45) becomes the celebrated linear Black-Scholes PDE well known from textbooks.

Proof. In the equation (38) we insert

$$\begin{cases} \Phi(t,x) = \Psi(t,s)e^{-rt} \\ x = e^{-rt}s, \end{cases}$$

and, taking into account that

$$\frac{\partial}{\partial x} = e^{rt} \frac{\partial}{\partial s} \qquad \frac{\partial^2}{\partial x^2} = e^{2rt} \frac{\partial^2}{\partial s^2} \qquad \frac{\partial s}{\partial t} = rs,$$

we obtain, after some algebra equation (45).

4.2 Approximate Solution of the Modified Black-Scholes PDE

In this subsection we derive a dependence relation between a call option price, the price of its underlying and the arbitrage measure ρ in an implicit form. For this purpose, we assume that the arbitrage measure $\rho_t \equiv \rho$ is constant during the period considered, typically between 0 and the derivative maturity T. As [11] discussed empirically, arbitrage measure is relatively small so we consider perturbations with respect to ρ and seek an approximate solution of the modified Black-Scholes PDE (38). We note that the non linear term of the modified Black-Scholes PDE (38) is multiplied by ρ linearly.

Theorem 40. For sufficiently small $\rho > 0$, an approximated solution of the modified Black-Scholes PDE (38) under the terminal condition $\Phi(T, X_T) = (X_T - K)^+$, where K is the strike price at time T on the discounted value of the underlying with constant volatility σ , is given by

$$\Phi(t, X_t) = K e^{\frac{1}{2} \log \frac{X_t}{K} - \frac{1}{8} \sigma^2(T - t)} u\left(\frac{1}{2} \sigma^2(T - t), \log \frac{X_t}{K}\right), \tag{46}$$

where

$$u(\tau, y) = u_0(\tau, y) + \rho U_1(\tau, y) + \rho^2 U_2(\tau, y) + O(\rho^3) \quad (\rho \to 0)$$

and $u_0(\tau, y)$ is the solution of $(\partial_{\tau} - \partial_y^2)u_0(\tau, y) = 0$ with the initial condition $u(0, y) = \max\{e^{\frac{y}{2}} - e^{-\frac{y}{2}}, 0\}$, and

$$f(v_{1}, v_{2}) := \frac{2K}{\sigma^{2}} \sqrt{\frac{5}{4}v_{1}^{2} + v_{1}v_{2} + v_{2}^{2}}$$

$$G(\tau, y; s, z) := \frac{1}{2\sqrt{\pi(\tau - s)}} \exp\left(-\frac{(y - z)^{2}}{4(\tau - s)}\right)$$

$$U_{1}(\tau, y) := \int_{0}^{\tau} ds \int_{-\infty}^{\infty} dz G(\tau, y; s, z) f(u_{0}(s, z), u'_{0}(s, z))$$

$$U_{2}(\tau, y) := \int_{0}^{\tau} ds \int_{-\infty}^{\infty} dz G(\tau, y; s, z) \left[f_{-1}(u_{0}(s, z), u'_{0}(s, z))U_{1}(s, z) + f_{-2}(u_{0}(s, z), u'_{0}(s, z))U'_{1}(s, z)\right].$$

$$(47)$$

The prime ' denotes the derivative with respect to the second argument and $f_{\cdot j}$ is the derivative of the function f with respect to the jth variable.

Proof. By means of the change of variables as $x=Ke^y, t=T-2\tau/\sigma^2$ and

$$\frac{\partial}{\partial t} = -\frac{\sigma^2}{2} \frac{\partial}{\partial \tau}, \quad \frac{\partial}{\partial x} = \frac{1}{x} \frac{\partial}{\partial y},$$

the modified Black-Scholes PDE (38) and the terminal condition $\Phi(T, X_T) = (X_T - K)^+$ are rewritten for the unknown function $v(\tau, y) := K^{-1}\Phi(t, x)$ as

$$\frac{\partial v(\tau, y)}{\partial \tau} = \frac{\partial^2 v(\tau, y)}{\partial y^2} - \frac{\partial v(\tau, y)}{\partial y} + \frac{2\rho K}{\sigma^2} \sqrt{v(\tau, y)^2 + \left(\frac{\partial v(\tau, y)}{\partial y}\right)^2}$$
$$v(0, y) = \max\{e^y - 1, 0\}.$$

By introducing the new unknown function $u = u(\tau, y)$ defined as $v(\tau, y) = e^{\frac{y}{2} - \frac{1}{4}\tau}u(\tau, y)$, we obtain the canonical form of diffusion equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial y^2} + \rho f\left(u(\tau, y), u'(\tau, y)\right).$$

Here the terminal condition is changed to $u(0,y) = \max\{e^{\frac{y}{2}} - e^{-\frac{y}{2}}, 0\}$. By introducing an unknown function $B(k,\tau)$, suppose that the solution of (48) has the form

$$u(\tau, y) = u_0(\tau, y) + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} B(k, \tau) e^{iky} dk, \tag{48}$$

where $u_0(\tau, y)$ is the solution for the case $\rho = 0$, i.e., $(\partial_{\tau} - \partial_y^2)u_0(\tau, y) = 0$. Thus,

$$u_0(\tau, y) = \int_{-\infty}^{\infty} G(\tau, y; 0, z) \max\{e^{\frac{z}{2}} - e^{-\frac{z}{2}}, 0\} dz.$$

Inserting the representation of $u_0(\tau, y)$ into (48) yields

$$(\partial_{\tau} - \partial_{y}^{2})u = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \left\{ \frac{\partial B(k,\tau)}{\partial \tau} + k^{2}B(k,\tau) \right\} e^{iky} = \rho f(u,u').$$

Via Fourier transform,

$$\frac{\partial B(k,\tau)}{\partial \tau} = -k^2 B(k,\tau) + \rho \tilde{f}(\tau,k), \tag{49}$$

where

$$\tilde{f}(\tau,k) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} f(u(\tau,y), u'(\tau,y)) e^{-iky} dy.$$

We solve (49) via variation of parameters. By introducing new function $\tilde{B}(k,\tau)$, we assume that the solution has the form

$$B(k,\tau) = e^{-k^2\tau} \tilde{B}(k,\tau). \tag{50}$$

Inserting this into (49) gives

$$e^{-k^2\tau} \frac{\partial \tilde{B}(k,\tau)}{\partial \tau} = \rho \tilde{f}(\tau,k),$$

which is equivalent to

$$\tilde{B}(k,\tau) = \rho \int_0^{\tau} e^{k^2 t} \, \tilde{f}(t,k) dt.$$

Consequently, the difference between the arbitrage solution u and the no arbitrage solution u_0 is

$$u(\tau, y) - u_0(\tau, y) =$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{iky} B(\tau, k) dk$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{iky} e^{-k^2 \tau} \left\{ \rho \int_{0}^{\tau} e^{k^2 t} \tilde{f}(t, k) dt \right\} dk$$

$$= \rho \frac{1}{2\pi} \int_{0}^{\tau} \left(\int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} e^{-k^2 (\tau - s) + ik(y - z)} f(u(s, z), u'(s, z)) dz \right\} dk \right) ds$$

$$= \rho \int_{0}^{\tau} \left(\int_{-\infty}^{\infty} G(\tau, y; s, z) f(u(s, z), u'(s, z)) dz \right) ds =: \rho F[u](\tau, y).$$
(51)

The non linear Black-Scholes PDE (38) with the terminal condition is therefore equivalent to the functional equation

$$G[u] := u - u_0 - \rho F[u] = 0, (52)$$

which can be solved by a Newton's approximation scheme. The first element of the approximation sequence of the solution u is u_0 . The second, u_1 is the solution of the linearization of (52) at u_0

$$G[u_0] + G^*[u_0].(u_1 - u_0) = 0, (53)$$

where the star denotes the Gâteaux derivative. The solution reads

$$u_1 = u_0 + \rho (\mathbf{1} - \rho F^*[u_0])^{-1} . F[u_0]$$

= $u_0 + \rho (\mathbf{1} + \rho F^*[u_0]) . F[u_0] + O(\rho^3)$
= $u_0 + \rho U_1 + \rho^2 F^*[u_0] . U_1 + O(\rho^3) \quad (\rho \to 0),$

where $U_1 := F[u_0]$ corresponds to (47). We now compute the Gâteaux derivative of F at u_0 as

$$F^*[u].U_1(\tau, y)$$

$$= \int_0^{\tau} ds \int_{-\infty}^{\infty} dz G(\tau, y; s, z) [f_{\cdot 1}(u_0(s, z), u'_0(s, z)) U_1(s, z) + f_{\cdot 2}(u_0(s, z), u'_0(s, z)) U'_1(s, z)].$$

We can now derive the second order approximate solution for u as

$$u(\tau, y) = u_0(\tau, y) + \rho \int_0^{\tau} ds \int_{-\infty}^{\infty} dz \, G(\tau, y; s, z) f(u_0(s, z), u'_0(s, z))$$
$$+ \rho^2 \int_0^{\tau} ds \int_{-\infty}^{\infty} dz \, G(\tau, y; s, z) \big[u_1(s, z) f_{\cdot 1}(u_0(s, z), u'_0(s, z))$$
$$+ u'_1(s, z) f_{\cdot 2}(u_0(s, z), u'_0(s, z)) \big]$$
$$+ O(\rho^3) \quad (\rho \to 0).$$

By tracing back of the change of variables in (38) we can obtain the solution $\Phi(t, X_t)$ as in (46).

5 Conclusions

We apply Geometric Arbitrage Theory to obtain results in Mathematical Finance, which do not need stochastic differential geometry in their formulation. First, we utilize the equivalence for a certain subclass of Itô processes between the no-unbounded-profit-with-bounded-risk condition and the expected utility maximization to prove the equivalence between the (NUPBR) condition with the (ZC) condition. Then, we generalize the Black-Scholes PDE to markets allowing arbitrage, computing an approximated solution for the non linear PDE for a call option with arbitrage.

Acknowledgements

We are grateful to the participants of the Quantitative Methods in Finance congress (Sydney, December 2017) and 10th World Congress of Bachelier Finance Society (Dublin, July 2018) for valuable discussions, especially for Stefan Tappe, suggesting the relation between (NUPBR) and (ZC). We would like to extend our gratitude to Claudio Fontana, who highlighted that for a previous incorrect version of Proposition 25 Bessel's processes, which satisfy (NUPBR) but not (NFLVR) as shown in [17] and in [18], would have been a counterexample, thus leading to the current corrected version.

A Generalized Derivatives of Stochastic Processes

In stochastic differential geometry one would like to lift the constructions of stochastic analysis from open subsets of \mathbb{R}^N to N dimensional differentiable manifolds. To that aim, chart invariant definitions are needed and hence a stochastic calculus satisfying the usual chain rule and not Itô's Lemma is required, (cf. [20], Chapter 7, and the remark in Chapter 4 at the beginning of page 200). That is why the papers about geometric arbitrage theory are mainly concerned in by stochastic integrals and derivatives meant in *Stratonovich*'s sense and not in $It\hat{o}$'s. Of course, at the end of the computation, Stratonovich integrals can be transformed into Itô's. Note that a fundamental portfolio equation, the self-financing condition cannot be directly formally expressed with Stratonovich integrals, but first with Itô's and then transformed into Stratonovich's, because it is a non-anticipative condition.

Definition 41. Let I be a real interval and $Q = (Q_t)_{t \in I}$ be a \mathbb{R}^N -valued stochastic process on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The process Q determines three families of σ -subalgebras of the σ -algebra \mathcal{A} :

- (i) "Past" \mathcal{P}_t , generated by the preimages of Borel sets in \mathbf{R}^N by all mappings $Q_s: \Omega \to \mathbf{R}^N$ for 0 < s < t.
- (ii) "Future" \mathcal{F}_t , generated by the preimages of Borel sets in \mathbf{R}^N by all mappings $Q_s: \Omega \to \mathbf{R}^N$ for 0 < t < s.
- (iii) "Present" \mathcal{N}_t , generated by the preimages of Borel sets in \mathbf{R}^N by the mapping $Q_s: \Omega \to \mathbf{R}^N$.

Let $Q = (Q_t)_{t \in I}$ be continuous. Assuming that the following limits exist, **Nelson's stochastic derivatives** are defined as

$$\mathfrak{D}Q_{t} := \lim_{h \to 0^{+}} \mathbb{E}\left[\frac{Q_{t+h} - Q_{t}}{h} \middle| \mathcal{P}_{t}\right] : forward \ derivative,$$

$$\mathfrak{D}_{*}Q_{t} := \lim_{h \to 0^{+}} \mathbb{E}\left[\frac{Q_{t} - Q_{t-h}}{h} \middle| \mathcal{F}_{t}\right] : backward \ derivative,$$

$$\mathcal{D}Q_{t} := \frac{\mathfrak{D}Q_{t} + \mathfrak{D}_{*}Q_{t}}{2} : mean \ derivative.$$
(54)

Let $S^1(I)$ the set of all processes Q such that $t \mapsto Q_t$, $t \mapsto \mathfrak{D}Q_t$ and $t \mapsto \mathfrak{D}_*Q_t$ are continuous mappings from I to $L^2(\Omega, \mathcal{A})$. Let $C^1(I)$ the completion of $S^1(I)$ with respect to the norm

$$||Q|| := \sup_{t \in I} \left(||Q_t||_{L^2(\Omega, \mathcal{A})} + ||\mathfrak{D}Q_t||_{L^2(\Omega, \mathcal{A})} + ||\mathfrak{D}_*Q_t||_{L^2(\Omega, \mathcal{A})} \right).$$
 (55)

Remark 42. The stochastic derivatives \mathfrak{D} , \mathfrak{D}_* and \mathcal{D} correspond to Itô's, to the anticipative and, respectively, to Stratonovich's integral (cf. [19]). The process space $\mathcal{C}^1(I)$ contains all Itô processes.

If Q is a Markov process, then the sigma algebras \mathcal{P}_t ("past") and \mathcal{F}_t ("future") in the definitions of forward and backward derivatives can be substituted by the sigma algebra \mathcal{N}_t ("present"), see Chapter 6.1 and 8.1 in ([19]).

Stochastic derivatives can be defined pointwise in $\omega \in \Omega$ outside the class \mathcal{C}^1 in terms of generalized functions.

Definition 43. Let $Q: I \times \Omega \to \mathbb{R}^N$ be a continuous linear functional in the test processes $\varphi: I \times \Omega \to \mathbb{R}^N$ for $\varphi(\cdot, \omega) \in C_c^{\infty}(I, \mathbb{R}^N)$. We mean by this that for a fixed $\omega \in \Omega$ the functional $Q(\cdot, \omega) \in \mathcal{D}(I, \mathbb{R}^N)$, the topological vector space of continuous distributions. We can then define **Nelson's generalized** stochastic derivatives:

$$\mathfrak{D}Q(\varphi_t) := -Q(\mathfrak{D}\varphi_t): \text{ forward generalized derivative,}$$

$$\mathfrak{D}_*Q(\varphi_t) := -Q(\mathfrak{D}_*\varphi_t): \text{ backward generalized derivative,}$$

$$\mathfrak{D}Q(\varphi_t) := -Q(\mathfrak{D}\varphi_t): \text{ mean generalized derivative.}$$
(56)

If the generalized derivative is regular, then the process has a derivative in the classic sense. This construction is nothing else than a straightforward pathwise lift of the theory of generalized functions to a wider class of stochastic processes which do not a priori allow for Nelson's derivatives in the strong sense.

References

- [1] Becherer, D., The Numéraire Portfolio for Unbounded Semimartingales. *Finance and Stochastics*, 2001, **5**, 327–341.
- [2] Bellini, F. and Frittelli, M., On the Existence of Minimax Martingale Measures. Mathematical Finance, 2002, 12, 1–21.
- [3] Björk, T. and Hult, H., A Note on Wick Products and the Fractional Black-Scholes Model. Finance and Stochastics, 2005, 9, 197–209.
- [4] Bleecker, D., Gauge Theory and Variational Principles. Addison-Wesley Publishing. (1981) (republished by Dover 2005).
- [5] Christensen, M. M. and Larsen, K., No Arbitrage and Growth Optimal Portfolio. Stochastic Analysis and Applications, 2007, 25, 255–280.

- [6] Delbaen, F. and Schachermayer, W., A General Version of the Fundamental Theorem of Asset Pricing. Mathematische Annalen, 1994, 300, 463–520.
- [7] Delbaen, F. and Schachermayer, W., The Mathematics of Arbitrage. Springer-Verlag Berlin Heidelberg, 2008.
- [8] Dellachérie, C., and Meyer, P. A., Probabilité et potentiel II Théorie des martingales Chapitres 5 à 8, Hermann, 1980.
- [9] Elworthy, K. D., Stochastic Differential Equations on Manifolds, London Mathematical Society Lecture Notes Series, 1982.
- [10] Eméry, M., Stochastic Calculus on Manifolds-With an Appendix by P. A. Meyer, Springer-Verlag Berlin Heidelberg, 1989.
- [11] Farinelli, S. and Vazquez, S., Gauge Invariance, Geometry and Arbitrage. The Journal of Investment Strategies, 2012, 1, 23–66.
- [12] Farinelli, S., Geometric Arbitrage Theory and Market Dynamics. Journal of Geometric Mechanics, 2015, 7, 431–471.
- [13] Farinelli, S., Geometric Arbitrage Theory and Market Dynamics reloaded. preprint, arXiv, 2021.
- [14] Farinelli, S., and Takada, H., Can You Hear the Shape of Market? Geometric Arbitrage and Spectral Theory, preprint, arXiv 1509.03264.
- [15] Flesaker, B. and Hughston, L., Positive Interest. Risk Magazine, 1996, 9, 46–49.
- [16] Föllmer, H. and Schied, A., Stochastic Finance: An Introduction In Discrete Time. Second Edition, De Gruyter Studies in Mathematics, 2004.
- [17] Fontana, C., Weak and Strong No-Arbitrage Conditions for Continuous Financial Markets. International Journal of Theoretical and Applied Finance, 2015, 18, 1–34.
- [18] Fontana, C. and Runggaldier, W. J., Diffusion-Based Models for Financial Markets without Martingale Measures. Chapter 4 In *Risk Measures and Attitudes* (eds. Biagini, F., Richter, A. and Schlesinger, H.), Springer-Verlag, London, 2013, 45–91.
- [19] Gliklikh, Y. E., Global and Stochastic Analysis with Applications to Mathematical Physics, Theoretical and Mathematical Physics, Springer-Verlag London, 2010.
- [20] Hackenbroch, W. and Thalmaier, A., Stochastische Analysis. Eine Einführung in die Theorie der stetigen Semimartingale, Teubner Verlag, 1994.

- [21] Heath, D. and Platen, E., A Benchmark Approach to Quantitative Finance, Springer, 2006.
- [22] Hörmander, L., The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis. Springer-Verlag Berlin Heidelberg, 2003.
- [23] Hugonnier, J. and Prieto, R., Asset Pricing with Arbitrage Activity. Journal of Financial Economics, 2015, 115, 411-428.
- [24] Hsu, E. P., Stochastic Analysis on Manifolds, Graduate Studies in Mathematics, 2002, 38, AMS.
- [25] Hulley, H. and Schweizer, M., M⁶ On Minimal Market Models and Minimal Martingale Measures. In Contemporary Quantitative Finance. Essays in Honour of Eckhard Platen (eds. Chiarella, C. and Novikov, A.), Springer-Verlag Berlin Heidelberg, 2010, 35–51.
- [26] Hunt, P. J. and Kennedy, J. E., Financial Derivatives in Theory and Practice, Wiley Series in Probability and Statistics, John Wiley & Sons, 2004.
- [27] Ilinski, K., Gauge Geometry of Financial Markets. J. Phys. A: Math. Gen. 2000, 33, 5-14.
- [28] Ilinski, K., Physics of Finance: Gauge Modelling in Non-Equilibrium Pricing, Wiley, 2001.
- [29] Jeulin, T. and Yor, M., *Inégalité de Hardy, semimartingales, et faux-amis*, Séminaire de Probabilités, XIII, 1979, 332–359.
- [30] Kabanov, Y. M., On the FTAP of Kreps-Delbaen-Schachermayer. In Statistics and Control of Stochastic Processes, The Liptser Festschrift Proceedings of Steklov Mathematical Institute Seminar, Moscow, Russia (eds. Kabanov, Y. M.), World Scientific, Singapore, 1997, 191–203.
- [31] Karatzas, I. and Kardaras, C., The Numéraire Portfolio in Semimartingale Financial Models. Finance and Stochastics, 2007, 11, 447–493.
- [32] Kabanov, Y. M. and Kramkov, D. O., Large Financial Markets: Asymptotic Arbitrage and Contiguity. Probab. Theory and Its Applications, 39, 1994, 222–229.
- [33] Loewenstein, M. and Willard, G.A., Local martingales, arbitrage, and viability. *Economic Theory*, 2000, **16**, 135–161.
- [34] Luenberger David G., Local Theory of Constrained Optimization: Optimization by Vector Space Methods, New York John Wiley & Sons, 1969.
- [35] Malaney, P. N., The Index Number Problem: A Differential Geometric Approach. Ph.D thesis, Harvard University Economics Department, 1996.

- [36] Protter, Ph. E., Stochastic Integration and Differential Equations: Version 2.1. Stochastic Modelling and Applied Probability, Springer, 2010.
- [37] Rogers, L. C. G., Equivalent Martingale Measures and No-Arbitrage. Stochastics, Stochastics Rep., 1994, 51, 41–49.
- [38] Ruf, J., Hedging under Arbitrage. Mathematical Finance, 2013, 23, 297–317.
- [39] Schachermayer, W., Optimal Investment in Incomplete Markets When Wealth May Become Negative. Annals of Applied Probability, 2001, 11, 694–734.
- [40] Schwartz, L., Semi-martingales sur des variétés et martingales conformes sur des variétés analytiques complexes. Springer Lecture Notes in Mathematics, Springer-Verlag Berlin Heidelberg, 1980.
- [41] Smith, A. and Speed, C. Gauge Transforms in Stochastic Investment. Proceedings of the 1998 AFIR Colloquim, Cambridge, England. 1998.
- [42] Stroock, D. W., An Introduction to the Analysis of Paths on a Riemannian Manifold. Mathematical Surveys and Monographs, 2000, 74, AMS.
- [43] Weinstein, E., Gauge Theory and Inflation: Enlarging the Wu-Yang Dictionary to a unifying Rosetta Stone for Geometry in Application. Talk given at Perimeter Institute, (2006).
- [44] Young, K., Foreign Exchange Market as a Lattice Gauge Theory. Am. J. Phys., 1999, 67, 862–868.
- [45] Zeidler E., Applied functional analysis: Variational Methods and Optimization, Applied Mathematical Sciences 109, New York, NY, Springer-Verlag, 1995.