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Tomas Björk • Mariana Khapko • Agatha Murgoci

Time-Inconsistent Control Theory with Finance Applications

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Tomas Björk
Stockholm School of Economics
Stockholm, Sweden

Mariana Khapko
University of Toronto
Toronto, ON, Canada

Agatha Murgoci
Ørsted
Hellerup, Denmark

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Preface

The purpose of this book is to present an overview of, and introduction to, the time-inconsistent control theory developed by the authors during the last decade. The theory is developed for discrete as well as continuous time, and the kernel of the content is drawn mainly from our journal articles Björk and Murgoci (2014), Björk et al. (2017), and Björk et al. (2014). Starting from these articles, we have included more examples and substantially simplified the exposition of the discrete-time theory compared with that of Björk and Murgoci (2014). Moreover, we have extended our framework to study time-inconsistent stopping problems, including stopping problems with non-exponential discounting, mean-variance objective, and distorted probabilities. Alongside our own results we have included discussions of recent developments in the field. In order to make the text more self-contained we have also added a brief recapitulation of optimal control and stopping in discrete and continuous time.

It is important to recognize that the analysis of time inconsistency has a long history in economics and finance literature. The idea of time consistency was alluded to by Samuelson (1937) when introducing the most commonly used time-discounting method in economics, exponential discounting. Strotz (1955) in his seminal paper pointed out that any other choice of discounting function, apart from the exponential case, will lead to a dynamically inconsistent problem. More generally, in order to achieve dynamic consistency and analyze the agent's problem with a standard toolkit, one needs to make specific and often restrictive assumptions about the objective functional the agent is maximizing (or minimizing). If the agent's objective does not conform to these assumptions (see Remark 1.1), time consistency fails to hold and the usual concept of optimality does not apply. Loosely speaking, this means that the agent's tastes change over time, so that a plan for some future period deemed optimal today is not necessarily optimal when that future period actually arrives.

How can one handle time-inconsistent problems? One approach is to look for a solution that is optimal today, ignoring the time inconsistency. Strotz (1955) refers to an agent who fails to recognize the time inconsistency issue and adopts such an approach as "spendthrift." The term later coined in the literature is "naïve."

The naïve agent's strategies are myopic and constantly changing. Strotz (1955) also outlines two approaches for modeling a "sophisticated" agent who is aware of the time inconsistency of their tastes: the strategy of pre-commitment and that of consistent planning. In the former case, the agent decides on a plan of action that is optimal today and commits to it, ignoring the incentives to revise it in the future. In the latter case, the agent internalizes the incentives to deviate and treats them as a constraint, thus aiming to arrive at a deviation-proof solution. Importantly, in both cases the agent recognizes that their "today self" and their "future selves" may have conflicting tastes.

In our work we choose to follow the consistent planning approach of Strotz (1955) in that an agent's strategies are taken to be the outcome of an intrapersonal game whose players are successive incarnations of the same agent. Essentially, we replace the usual concept of optimality with a more general concept of intrapersonal equilibrium and look for Nash subgame-perfect equilibrium points (Selten, 1965). Using this game-theoretic approach, we present an extension of the standard dynamic programming equation, in the form of a system of nonlinear equations, for determining the intrapersonal equilibrium strategy. This extended system, loosely speaking, accounts for the incentives to deviate as time evolves and an agent's tastes change. This means that, for a general Markov process and a fairly general objective functional, we obtain a plan that the agent will actually follow. We fully acknowledge that, while our focus here is on the game-theoretic approach, the other approaches that have been studied in the literature—the problem of a naïve agent who reoptimizes as time goes by or that of a sophisticated agent who is able to pre-commit—are both interesting and relevant.

The structure of the book is as follows. Following an introductory chapter, in Part I we start by providing a brief review of optimal stochastic control in discrete time. We present the standard results of discrete-time dynamic programming theory and illustrate them by solving a standard linear quadratic regulator problem and a simple discrete-time equilibrium model. Part II contains the main results for stochastic time-inconsistent control problems in discrete time, originally developed in Björk and Murgoci (2014), together with extensions and applications. We first give an account of time-inconsistent control theory[1] and present a number of interesting extensions, including the generalization of the additively separable expected utility model. The rest of the chapters in Part II discuss concrete examples of the general theory. The applications we present include control problems with non-exponential discounting and with mean-variance objective, time-inconsistent regulator problems, and a time-inconsistent version of the simple equilibrium model. In Part III, we summarize the continuous-time optimal control theory.

[1]The term "time-inconsistent" control was coined in the literature to emphasize the contrast with optimal control theory, which deals with time-consistent problems. This terminology may seem a bit confusing because, while the problem itself is inherently time inconsistent, the controls that we aim to find are deviation-proof, meaning that they are time consistent. To be as precise as we can be, we are studying time-consistent behavior of non-committed sophisticated agents who are maximizing (or minimizing) a time-inconsistent objective functional.

We first give a brief introduction to standard dynamic programming results in continuous time and then proceed to illustrate the theory with a number of examples. In Part IV, we build on results developed in Björk et al. (2017) for a class of continuous-time stochastic control problems that, in various ways, are time inconsistent. The structure of chapters in this part intentionally parallels that of Part II, reflecting the fact that the discrete-time setting serves as a natural starting point for the limiting arguments we use in the continuous-time case. In Part V, we briefly summarize the standard optimal stopping theory. Then, in Part VI,[2] we extend our methods in order to tackle time-inconsistent stopping problems in discrete and continuous time, including stopping problems under prospect-type distorted probabilities. Examples studied in this last part include a time-inconsistent version of the simple secretary problem, costly procrastination, and the problem of selling an asset (or investing in a project) that becomes time inconsistent if we allow for non-exponential discounting or mean-variance preferences. Finally, we review some basic concepts from arbitrage theory in the appendix.

This text is intended for graduate students and researchers in finance and economics who are interested in the issues of time inconsistency that prevail in many dynamic choice problems. In this book we aim to give the main arguments on how to handle time-inconsistent problems, outline the guiding intuition, and illustrate the general theory with a number of examples that are relevant in finance. While the continuous-time applications are likely to be the main focus of mathematical finance researchers, the discrete-time examples largely target the economics readership. Our focus on presenting main arguments and ideas means that we often go lightly on some of the more technical issues, so measurability and integrability issues are at times swept under the carpet.

Since the book is intended to be self-contained, it contains a brief summary of optimal control and stopping in discrete and continuous time. The reader comfortable with these standard results is welcome to skip the summary chapters and proceed to the more complicated time-inconsistent framework directly. We acknowledge that there are many excellent textbooks on optimal stopping and control. This is why we keep our discussion of the standard theory brief and refer the reader to the extensive literature on the subject for further information. The summary of the standard results is included for completeness as well as to allow for comparing and contrasting the "intrapersonal equilibrium" results with the standard optimal results in concrete applications.

It is also worth noting that a number of open problems remain for future research. First, we note that existence and/or uniqueness remain to be proved for solutions of the extended Bellman system in a number of settings. Second, the present theory depends critically on the Markovian structure. It is intriguing to follow the new

[2]Note that Part VI was unfortunately finalized without Tomas Björk. The results presented in this last part of the book are the product of numerous discussions between the authors over the last few years. However, any remaining errors or omissions in this part are the responsibility of Mariana Khapko and Agatha Murgoci.

developments in the literature that operate without this assumption. Third, in this book we present extensions of the standard dynamic programming results for time-inconsistent problems, and it would be very interesting to see whether there exists an efficient martingale formulation for these problems. Further open research problems are discussed in Björk and Murgoci (2014) and Björk et al. (2017).

Notes on the literature can be found at the end of most chapters. These notes provide discussions of the relevant literature, emphasizing new developments and alternative approaches. They provide the reader with opportunities to explore each topic further. We have tried to keep the reference list as complete as possible, including both the work that has influenced us and also the new papers where our methodology has been used. Any serious omission is unintentional.

Dedication

Here we would like to pay our tribute to Tomas Björk, without whom this book would not exist. Tomas was an internationally recognized figure in financial mathematics, a brilliant scholar and teacher, and a caring colleague. But to us he was so much more. He was our role model, our mentor, and a close friend. We are forever grateful for all the time, guidance, support, and encouragement he so generously bestowed upon us. We miss him, dearly, every day.

In its obituary for Tomas, the Bachelier Finance Society rightly remarked that he "was still active in his beloved mathematics up to the last day." Indeed, this book, putting together a decade of his interest in time-inconsistent problems, was among the last things Tomas was working on. We very much hope that we have been able to complete it in accordance with his vision.

Toronto, ON, Canada
Hellerup, Denmark
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Mariana Khapko
Agatha Murgoci

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Contents