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Robust state-dependent mean-variance portfolio selection: a closed-loop approach

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Abstract This paper studies a class of robust mean-variance portfolio selection problems with state-dependent risk aversion. Model uncertainty, in the sense of considering alternative dominated models, is introduced to the problem to reflect the investor's uncertainty-averse preference. To characterise the robust portfolios, we consider closed-loop equilibrium control and spike variation approaches. Moreover, we show that the closed-loop equilibrium strategy exists and is unique under some technical conditions. That partially addresses the open problem left in Björk et al. (2017, *Finance Stoch.*) and Pun (2018, *Automatica*). By using the necessary and sufficient condition for the equilibrium, we manage to derive the analytical form of the equilibrium strategy via the unique solution to a nonlinear ordinary differential equation system. To validate the proposed closed-loop control framework, we show that when there is no uncertainty, our equilibrium strategy is reduced to the strategy in Björk et al. (2014, *Math. Finance*), which cannot be deduced under the open-loop control framework.

Keywords Closed-loop control · Robust Mean-Variance portfolio selection · State-dependence · Time-inconsistency · Model uncertainty

Mathematics Subject Classification (2010) 49N90 · 91A80 · 91G10 · 91G80

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1 Introduction

Markowitz [27] pioneers the use of mean and variance as proxies of reward and risk of a portfolio and strikes the tradeoff between them, which build the foundation of modern portfolio theory and inspire tens of thousands of studies in portfolio selection. Portfolio selection is closely related to stochastic control theory. The dynamic programming/Hamilton–Jacobi–Bellman (HJB) equation and the backward stochastic differential equation (BSDE) approaches are commonly employed in characterising the optimal portfolios; see Yong and Zhou [38, Chap. 3 and 4]. For example, Li and Ng [24], Zhou and Li [39] derived the optimal mean-variance (MV) portfolio in dynamic settings by embedding the problem into a linear-quadratic (LQ) control problem that is solvable with the dynamic programming approach; Lim and Zhou [25] subsequently extended the study to random parameters using the BSDE approach.

The presence of variance operator in the objective functional prohibits the direct use of the dynamic programming and thus leads to time-inconsistency of the optimal control. Basak and Chabakauri [2] called the controls in the aforementioned studies as pre-commitment controls since they are (global) optimal by investigating the problem at the very beginning time but may not be (local) optimal when time evolves. In contrast, Basak and Chabakauri [2] introduced a new formulation of dynamic MV portfolio in the spirit of consistent planning originated in Strotz [33]. The problem is more involved when we consider state-dependent risk aversion for the investor (see Björk et al. [5]) since the state-dependence of the objective functional adds extra source of time-inconsistency of the controls. A general approach to time-inconsistency uses the concept of subgame perfect equilibrium to define a time-consistent control policy, which is initiated in Ekeland and Lazrak [6] for deterministic problems. Björk et al. [3], Björk and Murgoci [4], Björk et al. [5] extended it to the stochastic environment and established an extended HJB system framework for general time-inconsistent control problems. However, as admitted in Björk et al. [3], the existence and uniqueness of the equilibrium control or the solution to the extended HJB system remains an open problem, while these mathematical properties are core for the well-posedness of the problem. Recently, Hu et al. [17, 18] have used the BSDE approach to study a class of time-inconsistent LQ control problems, which nests mean-variance portfolio selection as a special case, and provided the existence and the uniqueness of the equilibrium policy.

From the statistical perspective, time-consistent controls are more desirable than the pre-committed controls, because local optimal control adopts online estimates of the model parameters. However, the model uncertainty still poses a serious concern in practice. The history of model uncertainty dates back to the early 20th century when Knight [23, Chap. VII] clarified the subtle difference between risk and uncertainty. Later, the famous Ellsberg paradox [7] revealed that most people are Knightian uncertainty (or ambiguity)-averse. The most widely used characterisation of such aversion to uncertainty is due to Gilboa and Schmeidler [12], which axiomatised uncertainty preferences with multiple priors. The control theory literature generally adopted the worst-case philosophy out of concerns for stability and the control that makes the system stable in the face of perturbations is called robust control. Robust control is closely related to Knightian uncertainty while the later is a human's behavior or preference and the former is a means to deal with the later. As illustrated in Gilboa and Schmeidler [12, Sect. 2], decision making with multiple priors can be represented as a maxmin expected utility problem and it is equivalent to Wald's minimax loss criterion in Wald [35] and Wald [36, Chap. 3]. Subsequently, Anderson et al. [1], Hansen and Sargent [14], Hansen et al. [16] formally established the links between robust control and model misspecification. Their prominent and influential approach to robust control and calibration of the model uncertainty set are summarised in Hansen and Sargent [15, Part III]. An early review on robust control can be found in Williams [37]. In the context of robust portfolio selection with time-consistent objective functional, Maenhout [26] and Pun and Wong [32] derived the general robust controls under alternative *dominated models* (i.e. with the assumption of mutual absolute continuity of the priors). Owing to the mathematical tractability of the mutually absolutely continuous multiple priors, the aforementioned studies are focused on this type of Knightian uncertainty. We term it as *model uncertainty* in order to distinguish it from another cluster of recent studies on the Knightian uncertainty modeled by multiple priors with singular measures. We term the uncertainty with *non-dominated models* (i.e. without the assumption of mutual absolute continuity of the priors) as *ambiguity* and the related literature include Epstein and Ji [9, 10], Fouque et al. [11], Epstein and Halevy [8], Ismail and Pham [22], Pun [31], which considered the ambiguity in either volatility or correlation. However, there are very few studies on the robust controls for the time-inconsistent objective functional. Ismail and Pham [22] considered robust MV portfolio selection with model uncertainty in the covariance matrix, but still in a pre-commitment sense. Specifically for the MV criterion, since it was conventionally formulated as a (risk) minimisation problem, the incorporation of robustness into the controller design naturally yields minimax formulation, which is indeed aligned with the framework of Hansen and Sargent [15, Part III]. Hence, this problem of our interest is also referred to as robust control problem, given that a minimax formulation can be converted to a maximin formulation.

Recently, Pun [30] has proposed an extended Hamilton–Jacobi–Bellman–Isaacs (HJBI) equations framework to handle both (*dominated*) model uncertainty and time-inconsistency using the concept of perfect equilibrium of "games in subgames." Although some worked-out examples of MV portfolio selection are solved in Pun [30], the mathematical properties of the robust control are still blurred. To make a response, Han et al. [13] studied a class of robust time-inconsistent LQ control problems using the BSDE with open-loop control approach and showed the existence and uniqueness of the robust time-consistent MV portfolio when the investor's risk aversion is constant. Though, there still remains an open problem in Pun [30] for the case of state-dependent risk aversion. This paper aims to fill this research gap. The Knightian uncertainty considered in this paper is *model uncertainty* with dominated models in accordance with Pun [30], Han et al. [13].

As stated in Hu et al. [17], the open-loop control framework (i.e. does not consider the effect of feedback) has its own limitation even for the case without robustness as their results show that the open-loop equilibrium control is inconsistent with the one found by using the extended HJB framework in Björk et al. [5] for the case of state-dependent risk aversion. Under the BSDE framework, a remedy to this inconsistency is to consider a closed-loop control formulation (i.e. consider the effect of feedback); see Huang et al. [20]. Indeed, open-loop and closed-loop controls have essential differences. In the optimal control literature, Sun et al. [34] pointed out that the existence of closed-loop optimal strategies is sufficient to guarantee the existence of open-loop optimal controls, but not vice versa. Loosely speaking, these two approaches to control formulation will generate different "optimal" controls in different senses. Moreover, the study of robust controls under the closed-loop control framework presents a lot of differences in terms of mathematical derivation from that under the open-loop control framework. Therefore, we are motivated to adopt the closed-loop control approach to study the robust MV portfolio selection with state-dependent risk aversion, in order to answer the open problem of existence and uniqueness of the equilibrium control in Björk et al. [3], Pun [30].

The distinct features and contributions of this paper are illustrated as follows. First, our closed-loop control formulation is innovative and different from that in Huang et al. [20]. To simplify the illustration, we present the difference when robustness is not incorporated. Similarly, both our paper and Huang et al. [20] adopt the pre-specified form of the optimal control $u_s^* = \alpha_s^* X_s^*$. However, while we consider spike variation of α_s^* and thus our perturbed control is of the form

$$u_s^{t,\varepsilon,v} = (\alpha_s^* + v\mathbf{1}_{s\in[t,t+\varepsilon)})X_s^{t,\varepsilon,v}, \tag{1.1}$$

where $(X_s^{t,\varepsilon,v})_{s\in[t,T]}$ is the state with $(u_s^{t,\varepsilon,v})_{s\in[t,T]}$. Huang et al. [20] considered

$$u_s^{t,\varepsilon,v} = \alpha_s^* X_s^{t,\varepsilon,v} + v\mathbf{1}_{s\in[t,t+\varepsilon)}. \tag{1.2}$$

Note that $(X_s^{t,\varepsilon,v})_{s\in[t,T]}$ in (1.1) and (1.2) are wealth processes under the corresponding perturbed controls, respectively, and thus are different. We adopt the form (1.1) for the following two main reasons:

- (1.1) ensures the positivity of the wealth process under the perturbed controls, given that the initial wealth $x_0 > 0$. However, (1.2) may violate this property when $\varepsilon > 0$. In literature, it is sensible to study the control problem under the positive wealth constraint and particularly, Björk et al. [5] also implicitly imposed it by proposing their ansatz of the controls.
- Heuristically, (1.1) only perturbs α_s^*, while (1.2) perturbs $\alpha_s^* X_s^{t,\varepsilon,v}$ as a whole. Our proposed closed-loop control form facilitates the mathematical definition of robust equilibrium control using a similar two-step approach as in Han et al. [13]. More specifically, the worst-case scenario characterised by ${}^t h^*$ depends on ${}^t\alpha^*$ implicitly (see (3.9) below), which poses analytical challenges in deriving the condition (4.5) for ${}^t\alpha^*$. Hereafter, we adopt a convention that a superscript t on the left denotes a process starting at t. Fortunately, with our proposed form (1.1), certain terms characterising the dependence of ${}^t h^*$ on ${}^t\alpha^*$ turn out to be of second order and thus are negligible; see Lemma 4.2 below. (1.1) allows us to get rid of the difficulty and yield the solutions.

Second, we characterise the robust closed-loop equilibrium control under our framework. The asymptotic results related to the equivalent conditions are presented (see Lemma 3.2 and 4.2 below) and simplified compared to the existing literature. One crucial assumption is deterministic risk premium θ, which is also imposed in Björk et al. [5], Huang et al. [20]. The Girsanov kernel h^* is also assumed to be deterministic since h^* appears in a similar position of θ. Mathematically, the *stochastic* Lebesgue differentiation theorem that was proved and applied in Hu et al. [18] is not needed under the closed-loop control formulation and the deterministic parameters assumption. Moreover, the uniqueness of the equilibrium control is converted to the uniqueness of the solutions to some ordinary differential equations (ODEs) ((3.10) and (4.7) below).

Third, we prove the existence and uniqueness of the solution to the nonlinear ODEs (3.10) and (4.7) in Lemmas 3.3 and 4.4. Lemma 4.4 uses truncation method to get rid of the nonlinearity of the ODE system. Moreover, we provide generic sufficient conditions for these mathematical results. The proofs of Lemmas 3.3 and 4.4 are inspirational and extendable for other models and problems.

The remainder of this paper is organised as follows. Section 2 formulates the problem of our interest. Sections 3 and 4 characterise the worst-case measure and the robust control with the necessary and sufficient conditions for the equilibrium. Moreover, the uniqueness of the worst-case measure and robust control is proved. Section 5 presents numerical results of our robust controls (portfolios) and their financial interpretation. Section 6 concludes. Some lengthy proofs are placed in the Appendix.

For readers' convenience, we summarise the notations that are frequently used in this paper in the followings. Let k and p be generic positive integers. We denote by

- A' the transpose of a matrix A, $|A| = \sqrt{\sum_{i,j} a_{ij}^2}$ the Frobenius norm of a matrix A,
- $A \succeq 0$ if A is a positive-semi-definite matrix, $A \succ 0$ if A is a positive-definite matrix,
- $\mathbb{S}^k$ the set of all $k \times k$ real symmetric matrices,
- $C(t, T; \mathcal{H})$ the set of all $\mathcal{H}$-valued deterministic continuous functions on time interval $[t, T]$,
- $L^{p,\mathcal{Q}}_{\mathcal{F}}(\Omega;\, \mathcal{H})$ the set of all $\mathcal{H}$-valued $\mathcal{F}$-measurable random variables X with $\sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}}[|X|^p] < \infty$,
- $L^{\infty,\mathcal{Q}}_{\mathcal{F}}(\Omega;\, \mathcal{H})$ the set of all $\mathcal{H}$-valued $\mathcal{F}$-measurable random variables X that are essentially bounded on any space $\mathbb{Q} \in \mathcal{Q}$,
- $L^{p,\mathcal{Q}}_{\mathcal{F}}(t,\, T;\, \mathcal{H})$ the set of all $(\mathcal{F}_s)_{s\in[t,T]}$-progressively measurable $\mathcal{H}$-valued stochastic processes $(X_s)_{s\in[t,T]}$ with $\sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}}[(\int_t^T |X_s|^2\, ds)^{p/2}] < \infty$,
- $L^{\infty,\mathcal{Q}}_{\mathcal{F}}(t,\, T;\, \mathcal{H})$ the set of all $(\mathcal{F}_s)_{s\in[t,T]}$-progressively measurable $\mathcal{H}$-valued stochastic processes that are essentially bounded on any space $\mathbb{Q} \in \mathcal{Q}$,
- $L^{p,\mathcal{Q}}_{\mathcal{F}}(\Omega;\, C(t,\, T;\, \mathcal{H}))$ the set of all continuous $(\mathcal{F}_s)_{s\in[t,T]}$-progressively measurable $\mathcal{H}$-valued stochastic processes $(X_s)_{s\in[t,T]}$ such that $\sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}}[\sup_{s\in[t,T]} |X_s|^p] < \infty$.

In this paper, for controls α, u, and Girsanov kernels h on $[0,T]$, ${}^t\alpha$, ${}^tu$, and th denote the processes starting at $t \in [0,T)$, i.e., the restrictions on $[t,T]$. Moreover, we adopt a convention that α, u, h without superscripts on the left are on $[0,T]$.

2 Problem formulation

We consider the mean-variance (MV) portfolio selection with state-dependent risk aversion/tolerance over a finite time horizon $T > 0$, subject to model uncertainty. Following the Hansen-Sargent framework of robust control [15, Part III], we start with a reference (nominal) model of the financial market. Then, the model uncertainty set consists of models, whose relative entropy or Kullback-Leibler divergence from the reference model is bounded by a specified value. In measure theory, the alternative models under our consideration are equivalent to each other and they are difficult to distinguish statistically from the reference model. By the Lagrange multiplier theorem, the entropy constraint is typically converted to a penalty term on perturbations from the reference model in addition to the original MV criterion. The Lagrange multiplier is then interpreted as the uncertainty aversion/tolerance coefficient. We are interested in the case that the risk-aversion coefficient and uncertainty-aversion coefficient depend on the state (wealth) process.

Hereafter, we will interchangeably use the terms: model and measure, and label the models by their probability measures, since the models we consider in this paper are only different in their probability measures.

2.1 The financial model under the reference measure

The reference model is defined on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, where $(\mathcal{F}_t)_{t\in[0,T]}$ is the filtration generated by the d-dimensional standard $\mathbb{P}$-Brownian motion $(W_t^{\mathbb{P}})_{t\in[0,T]}$.

We consider a complete market with one risk-free asset at risk-free rate $r_t > 0$ with $r \in C(0,T;\mathbb{R})$ and d risky assets, whose price vector $(S_t)_{t\in[0,T]}$ is driven by the following dynamics

$$dS_t = \mathrm{diag}(S_t)\big(m_t dt + \sigma_t dW_t^{\mathbb{P}}\big) \quad \text{with } S_0 = s_0, \tag{2.1}$$

where $\mathrm{diag}(S_t)$ is the diagonal matrix with the elements of S_t and $m \in C(0,T;\mathbb{R}^d)$ and $\sigma \in C(0,T;\mathbb{R}^{d\times d})$ are the expected return vector and the volatility matrix of risky assets, respectively. Throughout this paper, we impose a regularity condition that σ is invertible.

Denote by $\pi \in \mathbb{R}^d$ the control vector of money amounts invested in the d risky assets. By the self-financing principle of our investment, the wealth process $(X_s)_{s\in[t,T]}$ with any initial time $t \in [0,T)$ is driven by the following dynamics

$$dX_s = \big(r_s X_s + (m_s - r_s\mathbf{1})'\pi_s\big)ds + \pi_s'\sigma_s dW_s^{\mathbb{P}} = (r_s X_s + \theta_s' u_s)ds + u_s' dW_s^{\mathbb{P}}$$

with $X_t = x$, where $\mathbf{1} := (1,\dots,1)' \in \mathbb{R}^d$, $u_s = \sigma_s'\pi_s$ is a transformed control vector, and $\theta_s = \sigma_s^{-1}(m_s - r_s\mathbf{1})$ is the Sharpe ratio vector.

2.2 The financial models under alternative measures

Following the Hansen-Sargent framework [15, Part III], the investors consider perturbed models around the reference model (2.1) and look for a robust control. Similar to Maenhout [26], Pun [30], Pun and Wong [32], we consider the set of alternative probability measures that are equivalent to $\mathbb{P}$, denoted by $\mathcal{Q} := \{\mathbb{Q} : \mathbb{Q} \sim \mathbb{P}\}$. An alternative model is then defined as the financial market under an alternative measure. The specification of the model uncertainty set is directly linked to the robust control formulation. Our consideration is motivated by the well-known mean-blur problem (see Ingersoll [21, Chap. 16.12]) in empirical finance, i.e., estimating the expected return vector (m_s) is notoriously difficult. To see this relevance, note that our alternative measure is solely characterised by the relative entropy from the reference model or the Girsanov kernel, which will only perturb the drift part of the wealth process. To complete the formulation of the model uncertainty set, we ought to set bounds on the magnitude of the perturbation, which will be elaborated in the next subsection. Therefore, our formulation does not require an accurate estimate of the expected return vector but a range for it. In other words, the desiderata of our robust control is that the system remains stable in the face of perturbations on the expected return vector.

More specifically, by the Girsanov theorem, for each $\mathbb{Q} \in \mathcal{Q}$, there exists a d-dimensional $(\mathcal{F}_s)_{s\in[0,T]}$-progressively measurable stochastic process (also known as Girsanov kernel) $(h_s)_{s\in[0,T]}$ such that for $t \in [0,T]$, the Radon-Nikodým derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ is given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \exp\left(\int_0^t h_s' dW_s^{\mathbb{P}} - \frac{1}{2}\int_0^t h_s' h_s ds\right).$$

In this paper, Girsanov kernels considered in Definition 2.1 are essentially bounded and thus the Novikov condition is satisfied automatically. Then the density process is a positive $\mathbb{P}$-martingale and $W_t^{\mathbb{Q}}$, driven by $dW_t^{\mathbb{Q}} = dW_t^{\mathbb{P}} - h_t dt$, is a d-dimensional $\mathbb{Q}$-Brownian motion defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{Q})$. By the one-to-one correspondence between $\mathbb{Q}$ and $(h_t)_{t\in[0,T]}$, the selection of an alternative model is equivalent to the determination of the process $(h_t)_{t\in[0,T]}$, which is solely governed by the definition of the robustness for the control problem.

For $t \in [0,T)$, the dynamics of the wealth process $(X_s)_{s\in[t,T]}$ with $({}^t u, {}^t h) = (u_s, h_s)_{s\in[t,T]}$ under the alternative measure $\mathbb{Q}$ is given as follows.

$$dX_s = \big(r_s X_s + (\theta_s + h_s)' u_s\big) ds + u_s' dW_s^{\mathbb{Q}} \quad \text{with } X_t = x. \tag{2.2}$$

We remark that for any $({}^t u, {}^t h) \in L^{2,\mathcal{Q}}_{\mathcal{F}}(t, T; \mathbb{R}^d) \times L^{\infty,\mathcal{Q}}_{\mathcal{F}}(t, T; \mathbb{R}^d)$, there exists a unique strong solution of $X \in L^{2,\mathcal{Q}}_{\mathcal{F}}(\Omega; C(t, T; \mathbb{R}))$ to (2.2); see Yong and Zhou [38, Theorem 6.16, Chap. 1]. Under the alternative measure $\mathbb{Q}$, the Sharpe ratio (risk premium) vector is perturbed as $\theta_s + h_s$ (from θ_s).

2.3 Mean-Variance portfolio selection and closed-loop equilibrium control

Suppose that the investor considers the following robust MV criterion at every time $t \in [0, T)$ with $X_t = x$:

$$\min_{^t u} \max_{^t h} \frac{\phi(x)}{2} \mathrm{Var}_t^{\mathbb{Q}}(X_T) - \mathbb{E}_t^{\mathbb{Q}}[X_T] - \frac{1}{\xi} \mathbb{E}_t^{\mathbb{Q}} \Big[\int_t^T \frac{h_s' h_s}{2\varphi(x, X_s)} ds \Big], \tag{2.3}$$

where the first two terms account for the efficiency of the portfolio and the third term is a penalty on the perturbations deduced from the Lagrange multiplier theorem that converts the relative-entropy constraint: $\mathbb{E}_t^{\mathbb{Q}}[\int_t^T \frac{h_s' h_s}{2\varphi(x,X_s)} ds] \le \zeta$ for a constant $\zeta \ge 0$, while the Lagrange multiplier is $1/\xi \ge 0$. The third term can also be viewed as a generalised Kullback-Leibler divergence (relative entropy) between $\mathbb{Q}$ and $\mathbb{P}$. Here, $\phi \ge 0$ and $\varphi \ge 0$ represent the risk aversion and uncertainty aversion levels of the investor, respectively. The introduction of φ and its dependence on X_s are due to the pioneer work of Maenhout [26] on robust portfolio rules, which provides detailed economic implications of φ. Mathematically, the control problem with relative-entropy constraint (hard-constraint formulation) and the multiplier robust-control problem (soft-constraint formulation) give rise to the same optimal controls via the correspondence between ξ and ζ; see Peterson et al. [29] and Hansen and Sargent [14, Proposition 1]. We adopt the latter due to its mathematical tractability. The distinct feature of our study is that both ϕ and φ depend on the current wealth (state) x and thus we are dealing with a robust MV portfolio selection problem with state-dependent uncertainty aversion and risk aversion (SURA).

Among various choices of ϕ, we follow the studies in Björk et al. [5] and specify $\phi(x) = 1/(\mu_1 x)$, where $\mu_1 \ge 0$ is interpreted as the risk tolerance coefficient. Such a state-dependent risk aversion function has drawn a lot of attention; see Han et al. [13], Huang et al. [20], Pun [30]. In addition, we specify the uncertainty aversion function as $\varphi(x, X_s) = \mu_1 x / X_s^2$ and $\xi \ge 0$ in (2.3) is interpreted as the uncertainty aversion coefficient. Such a specification is consistent with the consideration in Maenhout [26] and Pun [30], where there is an affine map between uncertainty aversion and risk aversion. With these two choices, the objective functional of the MV investor with SURA at time t can be rewritten as follows:

$$\begin{aligned} J(t, x; {}^t u, {}^t h) =& \frac{1}{2} \mathrm{Var}_t^{\mathbb{Q}}(X_T) - \mu_1 x \mathbb{E}_t^{\mathbb{Q}}[X_T] - \frac{1}{2\xi} \mathbb{E}_t^{\mathbb{Q}} \Big[\int_t^T h_s' h_s X_s^2 ds \Big] \\ =& \frac{1}{2} \left(\mathbb{E}_t^{\mathbb{Q}}[X_T^2] - (\mathbb{E}_t^{\mathbb{Q}}[X_T])^2 \right) \\ & \underbrace{- \mu_1 x \mathbb{E}_t^{\mathbb{Q}}[X_T]}_{\text{Risk seeking}} \underbrace{- \frac{1}{2\xi} \mathbb{E}_t^{\mathbb{Q}} \Big[\int_t^T h_s' h_s X_s^2 ds \Big]}_{\text{Uncertainty aversion}}, \end{aligned} \tag{2.4}$$

where we have multiplied the original objective functional by $\mu_1 x > 0$ that would not affect the optimisation's nature but simplify the mathematical presentations. When $\xi \downarrow 0$, we must have $h_s \equiv 0$ or equivalently $\mathbb{Q} \equiv \mathbb{P}$ (i.e. no robustness) and our study is degenerated to the worked-out examples in Björk et al. [5], Hu et al. [17, 18]. Thus, we extend their studies to the robust counterparts.

In summary, we justify our choice of the risk and uncertainty aversion functions from the mathematical and economic perspectives as follows:

- We conduct a similar dimensional analysis as in Björk et al. [5, Sect. 4.1] to select proper risk and uncertainty aversion functions. Note that $\mathrm{Var}_t^{\mathbb{Q}}(X_T)$ has the dimension of $(\text{dollar})^2$. Hence, our choices of risk and uncertainty aversion functions introduce $\mu_1 x$ and X_s^2 to the second and the third terms in (2.4) such that they have the same dimension of $(\text{dollar})^2$.
- Homotheticity is imposed for economic reasons; see Maenhout [26, Sect. 2]. Consequently, the specification of (2.4) enables us to obtain a robust strategy ${}^t u^*$ that is proportional to the wealth and therefore homothetic. We remark that there could be other choices for risk and uncertainty aversion functions that can address the concerns above. However, our specification facilitates the mathematical tractability of the problem. Moreover, our numerical studies in Sect. 5 justify the out-of-sample outperformance of the robust strategy over the non-robust counterpart.

Our aim is to find a time-consistent strategy in the sense of consistent planning in Strotz [33] that minimises the objective (2.4) under the worst-case scenario of the alternative measures.

It is well-known that MV portfolio selection is a time-inconsistent control problem. Specifically, the control problem of (2.4) violates the Bellman's optimality principle by its state-dependence and nonlinear dependence on $\mathbb{E}_t^{\mathbb{Q}}[X_T]$; see Björk et al. [3], Björk and Murgoci [4], Pun [30]. As mentioned in the Introduction section, to address the time-inconsistency on both controls and alternative measures, we introduce a closed-loop equilibrium control pair as follows.

Definition 2.1 Given $\alpha^*_\cdot \in C(0,T;\mathbb{R}^d)$, which is a candidate equilibrium control. For any $t \in [0,T)$, $v \in L^{\infty,\mathcal{Q}}_{\mathcal{F}_t}(\Omega;\, \mathbb{R}^d)$, and $\varepsilon > 0$, define a perturbed control as

$$\alpha_s^{t,\varepsilon,v} = \alpha_s^* + v\mathbf{1}_{s\in[t,t+\varepsilon)}. \tag{2.5}$$

Then we further introduce a set as

$$\mathcal{A}_t := \{({}^s\alpha, s) \;:\; \alpha : [t,T] \to \mathbb{R}^d \text{ is measurable},\; s \in [t,T]\}$$

and denote $\mathcal{A} := \cup_{t\in[0,T)}\mathcal{A}_t$. Consider a map $\tilde{h}^*(\cdot,\cdot) : \mathcal{A} \to \mathbb{R}^d$ with the following properties:

1. $t \mapsto \tilde{h}^*({}^t\alpha^*, t) \in C(0,T;\mathbb{R}^d)$;
2. $s \mapsto \tilde{h}^*({}^s\alpha^{t,\varepsilon,v}, s) \in L^{\infty,\mathcal{Q}}_{\mathcal{F}}(t,\,T;\mathbb{R}^d)$ and $\tilde{h}^*({}^s\alpha^{t,\varepsilon,v}, s)$ is $\mathcal{F}_t$-measurable for $s \in [t,T]$.

For any $t \in [0,T)$, $v, \eta \in L^{\infty,\mathcal{Q}}_{\mathcal{F}_t}(\Omega;\, \mathbb{R}^d)$, and $\varepsilon > 0$, we define

$$\begin{aligned}
({}^t u^{t,\varepsilon,v,\eta}, {}^t h^{t,\varepsilon,v,\eta}) &= (u_s^{t,\varepsilon,v,\eta}, h_s^{t,\varepsilon,v,\eta})_{s\in[t,T]},\\
({}^t \tilde{u}^{t,\varepsilon,v,*}, {}^t \tilde{h}^{t,\varepsilon,v,*}) &= (\tilde{u}_s^{t,\varepsilon,v,*}, \tilde{h}_s^{t,\varepsilon,v,*})_{s\in[t,T]},\\
({}^t u^*, {}^t h^*) &= (u_s^*, h_s^*)_{s\in[t,T]},
\end{aligned}$$

where for $s \in [t, T]$,

$$
\begin{aligned}
&u_s^* = \alpha_s^* X_s^*, \quad h_s^* = \tilde{h}^*({}^s\alpha^*, s), \\
&u_s^{t,\varepsilon,v,\eta} = \alpha_s^{t,\varepsilon,v} X_s^{t,\varepsilon,v,\eta}, \quad \tilde{u}_s^{t,\varepsilon,v,*} = \alpha_s^{t,\varepsilon,v} \tilde{X}_s^{t,\varepsilon,v,*}, \\
&\tilde{h}_s^{t,\varepsilon,v,*} = \tilde{h}^*({}^s\alpha^{t,\varepsilon,v}, s), \quad h_s^{t,\varepsilon,v,\eta} = \tilde{h}_s^{t,\varepsilon,v,*} + \eta \mathbf{1}_{s\in[t,t+\varepsilon)},
\end{aligned} \tag{2.6}
$$

in which $(X_s^{t,\varepsilon,v,\eta})_{s\in[t,T]}$, $(\tilde{X}_s^{t,\varepsilon,v,*})_{s\in[t,T]}$, and $(X_s^*)_{s\in[t,T]}$ are the strong solutions to (2.2) with the policy pairs $({}^tu^{t,\varepsilon,v,\eta}, {}^th^{t,\varepsilon,v,\eta})$, $({}^t\tilde{u}^{t,\varepsilon,v,*}, {}^t\tilde{h}^{t,\varepsilon,v,*})$, and $({}^tu^*, {}^th^*)$, respectively.

Note that the randomness of $\tilde{h}_s^{t,\varepsilon,v,*} = \tilde{h}^*({}^s\alpha^{t,\varepsilon,v}, s)$ only comes from the $\mathcal{F}_t$-measurable random variable v, while $h_s^{t,\varepsilon,v,\eta}$ has randomness from v and η. Therefore, we have

$$
\begin{aligned}
&({}^tu^{t,\varepsilon,v,\eta}, {}^th^{t,\varepsilon,v,\eta}) \in L_{\mathcal{F}}^{2,\mathcal{Q}}(t, T; \mathbb{R}^d) \times L_{\mathcal{F}}^{\infty,\mathcal{Q}}(t, T; \mathbb{R}^d), \\
&({}^t\tilde{u}^{t,\varepsilon,v,*}, {}^t\tilde{h}^{t,\varepsilon,v,*}) \in L_{\mathcal{F}}^{2,\mathcal{Q}}(t, T; \mathbb{R}^d) \times L_{\mathcal{F}}^{\infty,\mathcal{Q}}(t, T; \mathbb{R}^d), \\
&X^{t,\varepsilon,v,\eta},\ \tilde{X}^{t,\varepsilon,v,*} \in L_{\mathcal{F}}^{2,\mathcal{Q}}\big(\Omega; C(t, T; \mathbb{R})\big), \\
\text{and} \quad &(u^*, h^*) \in L_{\mathcal{F}}^{2,\mathcal{Q}}(0, T; \mathbb{R}^d) \times C(0, T; \mathbb{R}^d),\ X^* \in L_{\mathcal{F}}^{2,\mathcal{Q}}\big(\Omega; C(0, T; \mathbb{R})\big).
\end{aligned}
$$

The assumption of α^*, h^*, and θ being deterministic is crucial to the proofs of Lemmas 3.1, 3.4, and 4.1, and the proposal of the ansatz (3.4). Even though we assume α^* and h^* to be deterministic, the perturbations v and η at time t are essentially bounded $\mathcal{F}_t$-measurable random variables, which allow us to consider the feasible pair of $({}^tu, {}^th)$ in $L_{\mathcal{F}}^{2,\mathcal{Q}}(t, T; \mathbb{R}^d) \times L_{\mathcal{F}}^{\infty,\mathcal{Q}}(t, T; \mathbb{R}^d)$. It makes an essential difference from Huang and Huang [19] that considers only deterministic th (or f in their notation). Another key assumption is deterministic risk premium θ. Under this assumption, it is well-known that the equilibrium strategy is deterministic when there is no uncertainty (see Huang et al. [20], Björk et al. [5]) and this assumption facilitates our later analyses. When θ is a stochastic process, the problem of our interest remains an open problem. The major difficulty stems from the interaction term $\theta_s' u_s$ in the state process (2.2) and the fact that u_s^* depends on X_s^* in the closed-loop control formulation. The same reason is also applicable to explain why we need to assume α^* and h^* to be deterministic. ~~Besides, a deterministic h^*, together with our specified state-dependent uncertainty aversion function $\varphi(x, X_s)$, agrees with the previous dimensional analysis, which is economically meaningful.~~

The specification of the controls (2.6) is inspired by the results of the worked-out examples in Björk et al. [5], Pun [30], from which we learned that u_t^* is linear in the wealth with deterministic coefficient function in t and h_t^* is a deterministic function in t. Thus, our later analyses can answer the open questions left in them regarding the existence and uniqueness of the robust controls. Moreover, this paper is distinct from the related literature, such as Han et al. [13], Huang et al. [20], Sun et al. [34], in terms of the closed-loop form (2.6) and the additional layer of robustness.

Using the Definition 2.1, we define the closed-loop equilibrium control pair (u^*, h^*) in the similar fashion of that in Han et al. [13], Pun [30].

Definition 2.2 With the policy pairs in Definition 2.1, the policy pair (u^*, h^*) is called a closed-loop equilibrium control pair if for any $t \in [0, T)$, $v, \eta \in L^{\infty,\mathcal{Q}}_{\mathcal{F}_t}(\Omega; \mathbb{R}^d)$, we have

$$\limsup_{\varepsilon \downarrow 0} \frac{J(t, \tilde{X}^{t,\varepsilon,v,*}_t; {}^t u^{t,\varepsilon,v,\eta}, {}^t h^{t,\varepsilon,v,\eta}) - J(t, \tilde{X}^{t,\varepsilon,v,*}_t; {}^t \tilde{u}^{t,\varepsilon,v,*}, {}^t \tilde{h}^{t,\varepsilon,v,*})}{\varepsilon} \leq 0 \tag{2.7}$$

$$\text{and} \quad \liminf_{\varepsilon \downarrow 0} \frac{J(t, X^*_t; {}^t \tilde{u}^{t,\varepsilon,v,*}, {}^t \tilde{h}^{t,\varepsilon,v,*}) - J(t, X^*_t; {}^t u^*, {}^t h^*)}{\varepsilon} \geq 0. \tag{2.8}$$

Heuristically, Definition 2.2 characterises a time-consistent (local optimal) solution of the set of problems indexed by time point $t \in [0, T)$:

$$\inf_{{}^t u} \sup_{{}^t h} J(t, x; {}^t u, {}^t h) \tag{2.9}$$

The preference ordering of (2.9) that reflects the formulation of robust controls is preserved with the use of two separate inequalities (2.7) and (2.8). Hence, in Sect. 3, we will first identify $\tilde{h}^*$ as a functional of ${}^s\alpha$, whose resulting policy pair satisfies (2.7). Then in Sect. 4, we find the robust control policy u^* via the characterisation of α^*, which satisfies (2.8) with the founded $\tilde{h}^*$.

To highlight the mathematical contribution of this paper, we outline our methodology in characterising the equilibrium controls here. The main idea is similar to the proof of the stochastic maximum principle (MP) in Yong and Zhou [38, Chap. 3] and Hu et al. [17]. We consider the Taylor expansions of cost functionals up to the second order in spike variation. Compared with the dynamic programming approach, the advantage of our approach is that we can prove the necessity and sufficiency of the first-order optimality conditions for equilibrium controls. Moreover, we can prove the uniqueness of equilibrium strategy. These are the open questions left in Björk et al. [5, 3], Pun [30], where an extended dynamic programming approach is adopted. In addition, since we adopt the closed-loop (instead of open-loop) control formulation, our robust solutions will be consistent with the considerations in Björk et al. [5], Pun [30]. The proofs are innovative in the characterisation of the equilibrium policy pair, especially in the treatment of the additional layer of robustness.

~~To ease the notational burden, we suppress some redundant time superscripts such as~~ $\cancel{\tilde{h}^*({}^s\alpha^*, s) = \tilde{h}^*({}^s\alpha^*, s)}$ ~~and~~ $\cancel{\tilde{h}^*({}^s\alpha^{t,\varepsilon,v}, s) = \tilde{h}^*({}^s\alpha^{t,\varepsilon,v}, s)}$~~. Under this convention, the functions as arguments are over the interval determined by the last argument of time. Thus, the interval is~~ $\cancel{[s, T]}$ ~~in these examples.~~ We denote by $\tilde{\mathbb{Q}}^*$ and $\mathbb{Q}^*$ the measures corresponding to ${}^t\tilde{h}^{t,\varepsilon,v,*}$ and ${}^t h^*$, respectively. We also read the Brownian motion and the conditional expectation operator under $\tilde{\mathbb{Q}}^*$ (resp. $\mathbb{Q}^*$) as $\tilde{W}^*_t$ and $\tilde{\mathbb{E}}^*_t[\cdot] := \mathbb{E}^{\tilde{\mathbb{Q}}^*}[\cdot|\mathcal{F}_t]$ (resp. W^*_t and $\mathbb{E}^*_t[\cdot] := \mathbb{E}^{\mathbb{Q}^*}[\cdot|\mathcal{F}_t]$).

3 Characterisation of $\tilde{h}^*$

First, we define the first-order adjoint process $(p^\varepsilon(\cdot;t), k^\varepsilon(\cdot;t)) \in L^{2,\mathcal{Q}}_{\mathcal{F}}(t,T;\mathbb{R}) \times L^{2,\mathcal{Q}}_{\mathcal{F}}(t,T;\mathbb{R}^d)$, which satisfies the backward stochastic differential equation (BSDE)

$$\begin{cases} dp^\varepsilon(s;t) = -\Big(\big(r_s + (\theta_s + \tilde{h}_s^{t,\varepsilon,v,*})'\alpha_s^{t,\varepsilon,v}\big)p^\varepsilon(s;t) + (\alpha_s^{t,\varepsilon,v})'k^\varepsilon(s;t) \\ \qquad\qquad -\frac{1}{\xi}(\tilde{h}_s^{t,\varepsilon,v,*})'\tilde{h}_s^{t,\varepsilon,v,*}\tilde{X}_s^{t,\varepsilon,v,*}\Big)ds + k^\varepsilon(s;t)'d\tilde{W}_s^*, \quad s \in [t,T], \\ p^\varepsilon(T;t) = \tilde{X}_T^{t,\varepsilon,v,*} - \tilde{\mathbb{E}}_t^*[\tilde{X}_T^{t,\varepsilon,v,*}] - \mu_1 x. \end{cases} \tag{3.1}$$

Similar to proving stochastic maximum principle under different settings in Han et al. [13], Hu et al. [17], Yong and Zhou [38, Chap. 3], we can obtain the perturbation results as follows. However, different from the classical results, the major difficulty of our problem is due to the functionals in (2.7), $J(t, \tilde{X}_t^{t,\varepsilon,v,*}; {}^t u^{t,\varepsilon,v,\eta}, {}^t h^{t,\varepsilon,v,\eta})$ and $J(t, \tilde{X}_t^{t,\varepsilon,v,*}; {}^t\tilde{u}^{t,\varepsilon,v,*}, {}^t\tilde{h}^{t,\varepsilon,v,*})$, which are under different measures. To overcome this difficulty, we construct a new process $L^{t,\varepsilon,v,\eta}$ under measure $\tilde{\mathbb{Q}}^*$. It enables us to proceed under the same measure. The idea is new and interesting in its own right. Besides, (3.3) relies on the specified $h_s^{t,\varepsilon,v,\eta}$, $\alpha_s^{t,\varepsilon,v}$ in Definition 2.1, and deterministic r_s, θ_s. (3.3) does not hold for general random parameters.

Lemma 3.1 *For any $t \in [0,T)$, $\varepsilon > 0$, $\eta \in L^{\infty,\mathcal{Q}}_{\mathcal{F}_t}(\Omega;\, \mathbb{R}^d)$, we have*

$$\begin{aligned} &J(t, \tilde{X}_t^{t,\varepsilon,v,*}; {}^t u^{t,\varepsilon,v,\eta}, {}^t h^{t,\varepsilon,v,\eta}) - J(t, \tilde{X}_t^{t,\varepsilon,v,*}; {}^t\tilde{u}^{t,\varepsilon,v,*}, {}^t\tilde{h}^{t,\varepsilon,v,*}) \\ &= \tilde{\mathbb{E}}_t^* \int_t^{t+\varepsilon} \Big[\Lambda^\varepsilon(s;t)'\eta - \frac{1}{2\xi}(\tilde{X}_s^{t,\varepsilon,v,*})^2\eta'\eta\Big]ds + o(\varepsilon), \end{aligned} \tag{3.2}$$

where $\Lambda^\varepsilon(s;t) = \alpha_s^{t,\varepsilon,v}\tilde{X}_s^{t,\varepsilon,v,*}p^\varepsilon(s;t) - \frac{1}{\xi}(\tilde{X}_s^{t,\varepsilon,v,*})^2\tilde{h}_s^{t,\varepsilon,v,*}$.

Proof Denote by $\tilde{\mathbb{Q}}^\eta$ the measure corresponding to ${}^t h^{t,\varepsilon,v,\eta}$ and by $\tilde{W}_t^\eta$ and $\tilde{\mathbb{E}}_t^\eta[\cdot] := \mathbb{E}^{\tilde{\mathbb{Q}}^\eta}[\cdot|\mathcal{F}_t]$ the Brownian motion and the conditional expectation operator under $\tilde{\mathbb{Q}}^\eta$, respectively. Recall that

$$\begin{cases} dX_s^{t,\varepsilon,v,\eta} = \big(r_s + (\theta_s + h_s^{t,\varepsilon,v,\eta})'\alpha_s^{t,\varepsilon,v}\big)X_s^{t,\varepsilon,v,\eta}ds + (\alpha_s^{t,\varepsilon,v})'X_s^{t,\varepsilon,v,\eta}d\tilde{W}_s^\eta, \\ X_t^{t,\varepsilon,v,\eta} \;\; = x. \end{cases}$$

Under measure $\tilde{\mathbb{Q}}^*$, define $L^{t,\varepsilon,v,\eta}$ as the solution to

$$\begin{cases} dL_s^{t,\varepsilon,v,\eta} = \big(r_s + (\theta_s + h_s^{t,\varepsilon,v,\eta})'\alpha_s^{t,\varepsilon,v}\big)L_s^{t,\varepsilon,v,\eta}ds + (\alpha_s^{t,\varepsilon,v})'L_s^{t,\varepsilon,v,\eta}d\tilde{W}_s^*, \\ L_t^{t,\varepsilon,v,\eta} \;\; = x. \end{cases}$$

Since $X_s^{t,\varepsilon,v,\eta}$ and $L_s^{t,\varepsilon,v,\eta}$ have the same distribution, then

$$\tilde{\mathbb{E}}_t^\eta[(X_s^{t,\varepsilon,v,\eta})^2] = \tilde{\mathbb{E}}_t^*[(L_s^{t,\varepsilon,v,\eta})^2],\ \tilde{\mathbb{E}}_t^\eta[X_s^{t,\varepsilon,v,\eta}] = \tilde{\mathbb{E}}_t^*[L_s^{t,\varepsilon,v,\eta}],\ s \in [t,T]. \tag{3.3}$$

We deduce

$$
\begin{aligned}
& J(t,\, \tilde{X}_t^{t,\varepsilon,v,*};\, {}^t u^{t,\varepsilon,v,\eta},\, {}^t h^{t,\varepsilon,v,\eta}) \\
=\ & \frac{1}{2}\Big(\tilde{\mathbb{E}}_t^\eta[(X_T^{t,\varepsilon,v,\eta})^2] - (\tilde{\mathbb{E}}_t^\eta[X_T^{t,\varepsilon,v,\eta}])^2\Big) - \mu_1 x \tilde{\mathbb{E}}_t^\eta[X_T^{t,\varepsilon,v,\eta}] \\
& -\frac{1}{2\xi}\tilde{\mathbb{E}}_t^\eta\Big[\int_t^T (h_s^{t,\varepsilon,v,\eta})' h_s^{t,\varepsilon,v,\eta} (X_s^{t,\varepsilon,v,\eta})^2 ds\Big] \\
=\ & \frac{1}{2}\Big(\tilde{\mathbb{E}}_t^*[(L_T^{t,\varepsilon,v,\eta})^2] - (\tilde{\mathbb{E}}_t^*[L_T^{t,\varepsilon,v,\eta}])^2\Big) - \mu_1 x \tilde{\mathbb{E}}_t^*[L_T^{t,\varepsilon,v,\eta}] \\
& -\frac{1}{2\xi}\tilde{\mathbb{E}}_t^*\Big[\int_t^T (h_s^{t,\varepsilon,v,\eta})' h_s^{t,\varepsilon,v,\eta} (L_s^{t,\varepsilon,v,\eta})^2 ds\Big] \\
=:\ & \tilde{\mathcal{L}}(t, L_t^{t,\varepsilon,v,\eta}).
\end{aligned}
$$

Define $(Z_s^{t,\varepsilon,v,\eta})_{s\in[t,T]}$ as the solution to the following SDE

$$
\begin{cases}
dZ_s^{t,\varepsilon,v,\eta} = \Big(\big(r_s + (\theta_s + \tilde{h}_s^{t,\varepsilon,v,*})'\alpha_s^{t,\varepsilon,v}\big) Z_s^{t,\varepsilon,v,\eta} + \eta' \alpha_s^{t,\varepsilon,v} \tilde{X}_s^{t,\varepsilon,v,*} \mathbf{1}_{s\in[t,t+\varepsilon)}\Big) ds \\
\qquad\qquad + (\alpha_s^{t,\varepsilon,v} Z_s^{t,\varepsilon,v,\eta})' d\tilde{W}_s^*, \\
Z_t^{t,\varepsilon,v,\eta} = 0.
\end{cases}
$$

According to Yong and Zhou [38, Theorem 4.4, Chap. 3], we have the following moment estimates,

$$
\begin{aligned}
\tilde{\mathbb{E}}_t^*\Big[\sup_{s\in[t,T]} |Z_s^{t,\varepsilon,v,\eta}|^2\Big] &= O(\varepsilon^2), \\
\tilde{\mathbb{E}}_t^*\Big[\sup_{s\in[t,T]} |L_s^{t,\varepsilon,v,\eta} - \tilde{X}_s^{t,\varepsilon,v,*} - Z_s^{t,\varepsilon,v,\eta}|^2\Big] &= o(\varepsilon^2).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
& J(t, \tilde{X}_t^{t,\varepsilon,v,*}; {}^t u^{t,\varepsilon,v,\eta}, {}^t h^{t,\varepsilon,v,\eta}) - J(t, \tilde{X}_t^{t,\varepsilon,v,*}; {}^t \tilde{u}^{t,\varepsilon,v,*}, {}^t \tilde{h}^{t,\varepsilon,v,*}) \\
& = \tilde{\mathcal{L}}(t, L_t^{t,\varepsilon,v,\eta}) - J(t, \tilde{X}_t^{t,\varepsilon,v,*}; {}^t \tilde{u}^{t,\varepsilon,v,*}, {}^t \tilde{h}^{t,\varepsilon,v,*}) \\
& = \tilde{\mathbb{E}}_t^*[(\tilde{X}_T^{t,\varepsilon,v,*} - \tilde{\mathbb{E}}_t^*[\tilde{X}_T^{t,\varepsilon,v,*}] - \mu_1 x) Z_T^{t,\varepsilon,v,\eta}] \\
& \quad - \tilde{\mathbb{E}}_t^* \int_t^T \Big[\frac{1}{\xi}(\tilde{h}_s^{t,\varepsilon,v,*})' \tilde{h}_s^{t,\varepsilon,v,*} \tilde{X}_s^{t,\varepsilon,v,*} Z_s^{t,\varepsilon,v,\eta} \\
& \qquad\qquad + \frac{1}{\xi}\eta' \tilde{h}_s^{t,\varepsilon,v,*} (\tilde{X}_s^{t,\varepsilon,v,*})^2 \mathbf{1}_{s\in[t,t+\varepsilon)} \\
& \qquad\qquad + \frac{1}{2\xi}(\tilde{X}_s^{t,\varepsilon,v,*})^2 \eta'\eta \mathbf{1}_{s\in[t,t+\varepsilon)}\Big] ds \\
& \quad + o(\varepsilon).
\end{aligned}
$$

By Itô's lemma, the first term can be rewritten as

$$
\tilde{\mathbb{E}}_t^*[(\tilde{X}_T^{t,\varepsilon,v,*} - \tilde{\mathbb{E}}_t^*[\tilde{X}_T^{t,\varepsilon,v,*}] - \mu_1 x) Z_T^{t,\varepsilon,v,\eta}]
$$

$$
\begin{aligned}
&= \tilde{\mathbb{E}}_t^* \int_t^T \Big[-\Big(\big(r_s + (\theta_s + \tilde{h}_s^{t,\varepsilon,v,*})' \alpha_s^{t,\varepsilon,v} \big) p^\varepsilon(s;t) + (\alpha_s^{t,\varepsilon,v})' k^\varepsilon(s;t) \\
&\qquad\qquad - \frac{1}{\xi} (\tilde{h}_s^{t,\varepsilon,v,*})' \tilde{h}_s^{t,\varepsilon,v,*} \tilde{X}_s^{t,\varepsilon,v,*} \Big) Z_s^{t,\varepsilon,v,\eta} \\
&\qquad + p^\varepsilon(s;t) \Big(\big(r_s + (\theta_s + \tilde{h}_s^{t,\varepsilon,v,*})' \alpha_s^{t,\varepsilon,v} \big) Z_s^{t,\varepsilon,v,\eta} \\
&\qquad\qquad + \eta' \alpha_s^{t,\varepsilon,v} \tilde{X}_s^{t,\varepsilon,v,*} \mathbf{1}_{s \in [t,t+\varepsilon)} \Big) \\
&\qquad + k^\varepsilon(s;t)' \alpha_s^{t,\varepsilon,v} Z_s^{t,\varepsilon,v,\eta} \Big] ds \\
&= \tilde{\mathbb{E}}_t^* \int_t^T \Big[\frac{1}{\xi} (\tilde{h}_s^{t,\varepsilon,v,*})' \tilde{h}_s^{t,\varepsilon,v,*} \tilde{X}_s^{t,\varepsilon,v,*} Z_s^{t,\varepsilon,v,\eta} \\
&\qquad + p^\varepsilon(s;t) \eta' \alpha_s^{t,\varepsilon,v} \tilde{X}_s^{t,\varepsilon,v,*} \mathbf{1}_{s \in [t,t+\varepsilon)} \Big] ds.
\end{aligned}
$$

Therefore, (3.2) follows. □

To match the boundary condition of $p^\varepsilon(s;t)$, we consider the following ansatz, which is inspired by Hu et al. [17, Eqn. (5.5)]:

$$
\begin{aligned}
p^\varepsilon(s;t) = &M({}^s\alpha^{t,\varepsilon,v}, {}^s\tilde{h}^{t,\varepsilon,v,*}, s) \tilde{X}_s^{t,\varepsilon,v,*} - \Gamma({}^s\alpha^{t,\varepsilon,v}, {}^s\tilde{h}^{t,\varepsilon,v,*}, s) x \\
&- N({}^s\alpha^{t,\varepsilon,v}, {}^s\tilde{h}^{t,\varepsilon,v,*}, s) \tilde{\mathbb{E}}_t^*[\tilde{X}_s^{t,\varepsilon,v,*}],
\end{aligned} \tag{3.4}
$$

where M, N, Γ satisfy the following ODE system

$$
\begin{cases}
\frac{\partial}{\partial s} M({}^s\alpha^{t,\varepsilon,v}, {}^s\tilde{h}^{t,\varepsilon,v,*}, s) = -F_M, & M({}^T\alpha^{t,\varepsilon,v}, {}^T\tilde{h}^{t,\varepsilon,v,*}, T) = 1, \\
\frac{\partial}{\partial s} N({}^s\alpha^{t,\varepsilon,v}, {}^s\tilde{h}^{t,\varepsilon,v,*}, s) = -F_N, & N({}^T\alpha^{t,\varepsilon,v}, {}^T\tilde{h}^{t,\varepsilon,v,*}, T) = 1, \\
\frac{\partial}{\partial s} \Gamma({}^s\alpha^{t,\varepsilon,v}, {}^s\tilde{h}^{t,\varepsilon,v,*}, s) = -F, & \Gamma({}^T\alpha^{t,\varepsilon,v}, {}^T\tilde{h}^{t,\varepsilon,v,*}, T) = \mu_1,
\end{cases} \tag{3.5}
$$

in which F_M, F_N, and F are determined by applying Itô's lemma to (3.4) and comparing it with (3.1), with $\tilde{h}_s^{t,\varepsilon,v,*}$, $\alpha_s^{t,\varepsilon,v}$ specified in Definition 2.1 and deterministic r_s, θ_s:

$$
\begin{cases}
F_M = 2\big(r_s + (\theta_s + \tilde{h}_s^{t,\varepsilon,v,*})' \alpha_s^{t,\varepsilon,v} \big) M({}^s\alpha^{t,\varepsilon,v}, {}^s\tilde{h}^{t,\varepsilon,v,*}, s) \\
\qquad + |\alpha_s^{t,\varepsilon,v}|^2 M({}^s\alpha^{t,\varepsilon,v}, {}^s\tilde{h}^{t,\varepsilon,v,*}, s) - \frac{1}{\xi} |\tilde{h}_s^{t,\varepsilon,v,*}|^2, \\
F_N = 2\big(r_s + (\theta_s + \tilde{h}_s^{t,\varepsilon,v,*})' \alpha_s^{t,\varepsilon,v} \big) N({}^s\alpha^{t,\varepsilon,v}, {}^s\tilde{h}^{t,\varepsilon,v,*}, s), \\
F = \big(r_s + (\theta_s + \tilde{h}_s^{t,\varepsilon,v,*})' \alpha_s^{t,\varepsilon,v} \big) \Gamma({}^s\alpha^{t,\varepsilon,v}, {}^s\tilde{h}^{t,\varepsilon,v,*}, s).
\end{cases}
$$

With the discussion on $p^\varepsilon(s;t)$ above, we can further simplify the result in Lemma 3.1 as follows. The proof is analogous to Lebesgue differentiation theorem while it relies on the assumption of uniform continuity property of (M, N, Γ) ((3.6) below).

Lemma 3.2 *For any $t \in [0,T)$, $\eta \in L^{\infty,\mathcal{Q}}_{\mathcal{F}_t}(\Omega; \mathbb{R}^d)$, suppose M, N, Γ satisfy ODEs (3.5), and*

$$\sup_{s\in[t,T]} |(M,N,\Gamma)({}^s\alpha^{t,\varepsilon,v}, {}^s\tilde{h}^{t,\varepsilon,v,*}, s) - (M,N,\Gamma)({}^s\alpha^*, {}^s h^*, s)|_1 \to 0 \quad as \quad \varepsilon \to 0, \tag{3.6}$$

with

$$\begin{aligned}
&|(M,N,\Gamma)({}^s\alpha^{t,\varepsilon,v}, {}^s\tilde{h}^{t,\varepsilon,v,*}, s) - (M,N,\Gamma)({}^s\alpha^*, {}^s h^*, s)|_1 \\
&:= |M({}^s\alpha^{t,\varepsilon,v}, {}^s\tilde{h}^{t,\varepsilon,v,*}, s) - M({}^s\alpha^*, {}^s h^*, s)| + |N({}^s\alpha^{t,\varepsilon,v}, {}^s\tilde{h}^{t,\varepsilon,v,*}, s) - N({}^s\alpha^*, {}^s h^*, s)| \\
&\quad + |\Gamma({}^s\alpha^{t,\varepsilon,v}, {}^s\tilde{h}^{t,\varepsilon,v,*}, s) - \Gamma({}^s\alpha^*, {}^s h^*, s)|.
\end{aligned}$$

Then

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\tilde{\mathbb{E}}^*_t \int_t^{t+\varepsilon} [\Lambda^\varepsilon(s;t)'\eta] ds \\
&= \Big((\alpha^*_t + v)\big(M({}^t\alpha^*, {}^t h^*, t) - N({}^t\alpha^*, {}^t h^*, t) - \Gamma({}^t\alpha^*, {}^t h^*, t)\big)x^2 \\
&\quad - \frac{1}{\xi}\tilde{h}^*({}^t\alpha^{t,0,v}, t)x^2\Big)'\eta,
\end{aligned} \tag{3.7}$$

where ${}^t\alpha^{t,0,v} = (\alpha^{t,0,v}_s)_{s\in[t,T]}$ and $\alpha^{t,0,v}_s = \alpha^*_s + v\mathbf{1}_{\{s=t\}}$.

By Lemmas 3.1 and 3.2 and noting that $-\frac{1}{2\xi}(\tilde{X}^{t,\varepsilon,v,*}_s)^2\eta'\eta \le 0$ and x, η are arbitrary, we can deduce that h^* is an equilibrium control if and only if

$$(\alpha^*_t + v)\big(M({}^t\alpha^*, {}^t h^*, t) - N({}^t\alpha^*, {}^t h^*, t) - \Gamma({}^t\alpha^*, {}^t h^*, t)\big) - \frac{1}{\xi}\tilde{h}^*({}^t\alpha^{t,0,v}, t) = 0. \tag{3.8}$$

(3.8) holds a.s., a.e. $t \in [0,T]$. Notice that from ODEs (3.5), we observe that M, N, Γ will not change if ${}^t\alpha^*$ varies in terms of a zero-measured set. Thus, for given $\alpha^*_\cdot \in C(0,T;\mathbb{R}^d)$, by setting $v = 0$, (3.8) defines

$$h^*_t = \tilde{h}^*({}^t\alpha^*, t) = \xi\alpha^*_t\big(M({}^t\alpha^*, {}^t h^*, t) - N({}^t\alpha^*, {}^t h^*, t) - \Gamma({}^t\alpha^*, {}^t h^*, t)\big). \tag{3.9}$$

(3.9) holds for every $t \in [0,T]$, thanks to the deterministic and continuous assumption.

We denote $M({}^t\alpha^*, t) = M({}^t\alpha^*, \tilde{h}^*({}^t\alpha^*, t), t)$, $N({}^t\alpha^*, t) = N({}^t\alpha^*, \tilde{h}^*({}^t\alpha^*, t), t)$, and $\Gamma({}^t\alpha^*, t) = \Gamma({}^t\alpha^*, \tilde{h}^*({}^t\alpha^*, t), t)$. We introduce

$$\Delta({}^t\alpha^*, t) = M({}^t\alpha^*, t) - N({}^t\alpha^*, t) - \Gamma({}^t\alpha^*, t).$$

Then the F's in ODEs (3.5) become

$$\begin{cases}
F_M = 2\big(r_s + \theta'_s\alpha^*_s + \xi|\alpha^*_s|^2\Delta({}^s\alpha^*, s)\big)M({}^s\alpha^*, s) + |\alpha^*_s|^2 M({}^s\alpha^*, s) \\
\qquad -\xi|\alpha^*_s|^2\Delta^2({}^s\alpha^*, s), \\
F_N = 2\big(r_s + \theta'_s\alpha^*_s + \xi|\alpha^*_s|^2\Delta({}^s\alpha^*, s)\big)N({}^s\alpha^*, s), \\
F = \big(r_s + \theta'_s\alpha^*_s + \xi|\alpha^*_s|^2\Delta({}^s\alpha^*, s)\big)\Gamma({}^s\alpha^*, s).
\end{cases} \tag{3.10}$$

Next, we derive the following result with respect to the existence and uniqueness of the solution to ODEs (3.5) with (3.10). The main idea stems from the observation that (3.10) is a coupled Riccati equation system. Moreover, we verify the assumption (3.6) in Lemma 3.2 under a more practical assumption (3.11).

Lemma 3.3 *Denote*

$$a := \sup_{t\in[0,T]} 6\xi|\alpha_t^*|^2,\ b := \sup_{t\in[0,T]} \big\{2|r_t + \theta_t'\alpha_t^*| + |\alpha_t^*|^2\big\},\ K_0 := \max\{1, \mu_1\}.$$

If

$$T < \frac{1}{b}\ln\Big(1 + \frac{b}{aK_0}\Big), \tag{3.11}$$

then ODE system (3.10) admits a unique solution (M, N, Γ). For every $t \in [0, T]$,

$$\max\{|M({}^t\alpha^*, t)|, |N({}^t\alpha^*, t)|, |\Gamma({}^t\alpha^*, t)|\} \le \frac{b}{a}\cdot\frac{Ce^{b(T-t)}}{1 - Ce^{b(T-t)}},\ \textit{with } C := \frac{aK_0}{aK_0 + b}. \tag{3.12}$$

Moreover, (M, N, Γ) is uniformly continuous in α^* in the sense that for any α^{*1}, α^{*2} with almost everywhere continuous functions α_t^{*1} and α_t^{*2}, respectively, satisfying (3.11)*, we have*

$$\sup_{s\in[0,T]} |(M, N, \Gamma)({}^s\alpha^{*1}, s) - (M, N, \Gamma)({}^s\alpha^{*2}, s)|_1 \to 0,\ \textit{as} \int_0^T |\alpha_t^{*1} - \alpha_t^{*2}|dt \to 0,$$

where we adopted the notation

$$\begin{aligned}
&|(M, N, \Gamma)({}^s\alpha^{*1}, s) - (M, N, \Gamma)({}^s\alpha^{*2}, s)|_1 \\
&:= |M({}^s\alpha^{*1}, s) - M({}^s\alpha^{*2}, s)| + |N({}^s\alpha^{*1}, s) - N({}^s\alpha^{*2}, s)| + |\Gamma({}^s\alpha^{*1}, s) - \Gamma({}^s\alpha^{*2}, s)|.
\end{aligned}$$

Proof A direct calculation shows (arguments and time s are suppressed)

$$\begin{cases}
F_M = \xi|\alpha^*|^2M^2 - \xi|\alpha^*|^2N^2 - \xi|\alpha^*|^2\Gamma^2 - 2\xi|\alpha^*|^2N\Gamma \\
\qquad + \big(2(r + \theta'\alpha^*) + |\alpha^*|^2\big)M, \\
F_N = -2\xi|\alpha^*|^2N^2 + 2\xi|\alpha^*|^2MN - 2\xi|\alpha^*|^2N\Gamma + 2(r + \theta'\alpha^*)N, \\
F = -\xi|\alpha^*|^2\Gamma^2 + \xi|\alpha^*|^2M\Gamma - \xi|\alpha^*|^2N\Gamma + (r + \theta'\alpha^*)\Gamma.
\end{cases}$$

With the same idea in Papavassilopoulos and Cruz [28], for every t, consider $Q_t := \max\{|M({}^t\alpha^*, t)|, |N({}^t\alpha^*, t)|, |\Gamma({}^t\alpha^*, t)|\}$, then

$$\max\Big\{\Big|\frac{\partial M({}^t\alpha^*, t)}{\partial t}\Big|, \Big|\frac{\partial N({}^t\alpha^*, t)}{\partial t}\Big|, \Big|\frac{\partial \Gamma({}^t\alpha^*, t)}{\partial t}\Big|\Big\} \le aQ_t^2 + bQ_t.$$

By part 2) of the proposition in Papavassilopoulos and Cruz [28], we have the claim in (3.11)-(3.12).

From the previous proof, we know that the solution (M, N, Γ) with either α^{*1} or α^{*2} is bounded. Hence, the ODE system (3.5) with (3.10) is Lipschitz continuous on (M, N, Γ) on any bounded sets. Therefore,

$$\begin{aligned}&|(M,N,\Gamma)({}^s\alpha^{*1},s)-(M,N,\Gamma)({}^s\alpha^{*2},s)|_1\\&\leq K_1\int_s^T|\alpha_t^{*1}-\alpha_t^{*2}|dt+K_2\int_s^T|(M,N,\Gamma)({}^t\alpha^{*1},t)-(M,N,\Gamma)({}^t\alpha^{*2},t)|_1dt,\end{aligned}$$

where K_1, K_2 depend on (M, N, Γ). By Gronwall's inequality, we have

$$\begin{aligned}|(M,N,\Gamma)({}^s\alpha^{*1},s)-(M,N,\Gamma)({}^s\alpha^{*2},s)|_1&\\ \leq K_1e^{K_2(T-s)}\int_s^T|\alpha_t^{*1}-\alpha_t^{*2}|dt.&\end{aligned}$$

Thus, the uniform continuity of (M, N, Γ) follows. □

To proceed, we first introduce $(p(s;t), k(s,t))$ that satisfy the BSDE (3.13) below. The solution pair $(p(s;t), k(s,t))$ is equal to $(p^\varepsilon(s;t), k^\varepsilon(s,t))$ with $v = 0$.

$$\begin{cases}dp(s;t)=-\Big(\big(r_s+(\theta_s+h_s^*)'\alpha_s^*\big)p(s;t)+(\alpha_s^*)'k(s;t)-\frac{1}{\xi}(h_s^*)'h_s^*X_s^*\Big)ds\\ \qquad\qquad +k(s;t)'dW_s^*,\quad s\in[t,T],\\ p(T;t)=\ X_T^*-\mathbb{E}_t^*[X_T^*]-\mu_1x.\end{cases}\tag{3.13}$$

Then we have the following Lemma 3.4 on the continuity property, which is an important tool for the later analyses. Han et al. [13, Lemma 3.4] present a similar result except that they derived under the same measure. Since we only consider closed-loop control and deterministic parameters, we manage to remove several technical assumptions. Our proof relies on the stability results of BSDE and we need to get rid of the difficulty in unifying different measures.

Lemma 3.4 *Consider any* $t \in [0, T)$ *and* $\eta \in L^{\infty,\mathbb{Q}}_{\mathcal{F}_t}(\Omega; \mathbb{R}^d)$. *Suppose* (3.11) *in Lemma 3.3 holds for* α^* *and* $\alpha^{t,\varepsilon,v}$*, then*

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\tilde{\mathbb{E}}_t^*\int_t^{t+\varepsilon}[\Lambda^\varepsilon(s;t)'\eta]ds=\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathbb{E}_t^*\int_t^{t+\varepsilon}[\Lambda(s;t)'\eta]ds,$$

where

$$\Lambda(s;t)=\alpha_s^*X_s^*p(s;t)-\frac{1}{\xi}(X_s^*)^2h_s^*.$$

As mentioned in the Introduction section, the uniqueness of closed-loop control can be proved in the same spirit of the proof of Huang et al. [20, Theorem 3.2].

Theorem 3.5 *If* α^* *satisfies* (3.11) *in Lemma 3.3, then* h^* *is unique.*

Proof Given α^*, we define $\alpha^{t,\varepsilon,v}$ as in (2.5). Suppose that there are two equilibrium strategies $\tilde{h}_1^*({}^s\alpha^{t,\varepsilon,v}, s)$, $\tilde{h}_2^*({}^s\alpha^{t,\varepsilon,v}, s)$ for the same $\alpha^{t,\varepsilon,v}$. We can deduce the BSDE for $\tilde{h}_1^*({}^s\alpha^{t,\varepsilon,v}, s)$ and $\tilde{h}_2^*({}^s\alpha^{t,\varepsilon,v}, s)$ of the same form (3.1). Notice that by Lemmas 3.1 and 3.2, equation (3.9) holds for both $\tilde{h}_1^*({}^s\alpha^*, s)$ and $\tilde{h}_2^*({}^s\alpha^*, s)$. Moreover, they satisfy the same ODEs (3.5). However, Lemma 3.3 shows the uniqueness of the solution to (3.5). Therefore, $\tilde{h}_1^*({}^s\alpha^*, s) = \tilde{h}_2^*({}^s\alpha^*, s)$ and h^* is unique. □

4 Characterisation of α^*

In Sect. 3, we have identified the map $\tilde{h}^*$ in Definition 2.1 as a functional of ${}^s\alpha$ in (3.8) and (3.9). The existence and uniqueness are addressed in Lemma 3.3 and Theorem 3.5 with a given candidate equilibrium α^*. The first inequality (2.7) in Definition 2.2 is then satisfied. Next, since $u_t^* = \alpha_t^* X_t^*$ by Definition 2.1, we can characterise u^* via α^* satisfying the second inequality (2.8) with the founded $\tilde{h}^*$. It reflects the preference ordering in the objective (2.9) and the closed-loop formulation. In this section, we adopt the founded $\tilde{h}^*$ in Sect. 3 and thus (2.7) is satisfied automatically.

Since the map $\tilde{h}^*$ is given, then the control is α only. Similar to the BSDE (3.1) in Sect. 3, we need the first-order adjoint process for stochastic MP. Remarkably, the process is the same as (3.13) with h_s^* given by (3.9). (3.13) has been solved already. Thus, for any $t \in [0, T)$, consider the solution $(p(\cdot;t), k(\cdot;t)) \in L_{\mathcal{F}}^{2,\mathcal{Q}}(t, T; \mathbb{R}) \times L_{\mathcal{F}}^{2,\mathcal{Q}}(t, T; \mathbb{R}^d)$ given in (4.1) and (4.2). Moreover, define the second-order adjoint process $P(\cdot;t) \in L_{\mathcal{F}}^{2,\mathcal{Q}}(t, T; \mathbb{R})$ in (4.3) for later use.

$$
\begin{aligned}
p(s;t) &= M({}^s\alpha^*, s)X_s^* - \Gamma({}^s\alpha^*, s)x - N({}^s\alpha^*, s)\mathbb{E}_t^*[X_s^*], && (4.1)\\
k(s;t) &= \alpha_s^* M({}^s\alpha^*, s)X_s^*, && (4.2)\\
P(s;t) &= e^{\int_s^T 2(r_u + (\theta_u + \tilde{h}^*({}^u\alpha^*, u))'\alpha_u^*) + |\alpha_u^*|^2 du} \\
&\quad - \int_s^T \frac{1}{\xi} |\tilde{h}^*({}^v\alpha^*, v)|^2 e^{\int_s^v 2(r_u + (\theta_u + \tilde{h}^*({}^u\alpha^*, u))'\alpha_u^*) + |\alpha_u^*|^2 du} dv. && (4.3)
\end{aligned}
$$

Similar to Lemma 3.1, we have the following perturbation result for cost functional with respect to α^*.

Lemma 4.1 *For any $t \in [0, T)$, $\varepsilon > 0$ and $v \in L_{\mathcal{F}_t}^{\infty,\mathcal{Q}}(\Omega; \mathbb{R}^d)$, we define $\alpha^{t,\varepsilon,v}$ by* (2.5)*. Then*

$$
\begin{aligned}
&J(t, X_t^*; {}^t\tilde{u}^{t,\varepsilon,v,*}, {}^t\tilde{h}^{t,\varepsilon,v,*}) - J(t, X_t^*; {}^tu^*, {}^th^*) && (4.4)\\
&= \mathbb{E}_t^* \int_t^T \left[\delta H(s, {}^s\alpha^*, X^*) + \frac{1}{2} P(s;t)(X_s^*)^2 v'v \mathbf{1}_{s\in[t,t+\varepsilon)} \right] ds + o(\varepsilon),
\end{aligned}
$$

where

$$
\begin{aligned}
\delta H(s, {}^s\alpha^*, X^*) &= \delta f^* + \delta\mu^* p(s;t) + X_s^* k(s;t)' v \mathbf{1}_{s\in[t,t+\varepsilon)},\\
\delta f^* &= -\frac{1}{2\xi}\big(|\tilde{h}^*({}^s\alpha^{t,\varepsilon,v}, s)|^2 - |\tilde{h}^*({}^s\alpha^*, s)|^2\big)(X_s^*)^2,\\
\delta\mu^* &= \big(\theta_s + \tilde{h}^*({}^s\alpha^{t,\varepsilon,v}, s)\big)'\alpha_s^{t,\varepsilon,v} X_s^* - \big(\theta_s + \tilde{h}^*({}^s\alpha^*, s)\big)'\alpha_s^* X_s^*.
\end{aligned}
$$

Proof By the L defined in (A.1), we have

$$\tilde{\mathbb{E}}_t^*[(\tilde{X}_s^{t,\varepsilon,v,*})^2] = \mathbb{E}_t^*[(L_s^{t,\varepsilon,v})^2], \quad \tilde{\mathbb{E}}_t^*[\tilde{X}_s^{t,\varepsilon,v,*}] = \mathbb{E}_t^*[L_s^{t,\varepsilon,v}], \quad s \in [t,T],$$

and

$$\begin{aligned}
&J(t, X_t^*; {}^t\tilde{u}^{t,\varepsilon,v,*}, {}^t\tilde{h}^{t,\varepsilon,v,*}) \\
=\ &\frac{1}{2}\Big(\tilde{\mathbb{E}}_t^*[(\tilde{X}_T^{t,\varepsilon,v,*})^2] - (\tilde{\mathbb{E}}_t^*[\tilde{X}_T^{t,\varepsilon,v,*}])^2\Big) - \mu_1 x \tilde{\mathbb{E}}_t^*[\tilde{X}_T^{t,\varepsilon,v,*}] \\
&-\frac{1}{2\xi}\tilde{\mathbb{E}}_t^*\Big[\int_t^T \tilde{h}^*({}^s\alpha^{t,\varepsilon,v}, s)'\tilde{h}^*({}^s\alpha^{t,\varepsilon,v}, s)(\tilde{X}_s^{t,\varepsilon,v,*})^2 ds\Big] \\
=\ &\frac{1}{2}\left(\mathbb{E}_t^*[(L_T^{t,\varepsilon,v})^2] - (\mathbb{E}_t^*[L_T^{t,\varepsilon,v}])^2\right) - \mu_1 x \mathbb{E}_t^*[L_T^{t,\varepsilon,v}] \\
&-\frac{1}{2\xi}\mathbb{E}_t^*\Big[\int_t^T \tilde{h}^*({}^s\alpha^{t,\varepsilon,v}, s)'\tilde{h}^*({}^s\alpha^{t,\varepsilon,v}, s)(L_s^{t,\varepsilon,v})^2 ds\Big] \\
=:\ &\mathcal{L}(t, L_t^{t,\varepsilon,v}).
\end{aligned}$$

Define $Y = Y^{t,\varepsilon,v}$ and $Z = Z^{t,\varepsilon,v}$ by

$$\begin{cases}
dY_s = \Big(r_s + \big(\theta_s + \tilde{h}^*({}^s\alpha^*, s)\big)'\alpha_s^*\Big)Y_s ds \\
\qquad\quad +(\alpha_s^* Y_s + X_s^* v \mathbf{1}_{s\in[t,t+\varepsilon)})' dW_s^*, \quad s \in [t,T], \\
Y_t = 0;
\end{cases}$$

$$\begin{cases}
dZ_s = \Big(r_s + \big(\theta_s + \tilde{h}^*({}^s\alpha^*, s)\big)'\alpha_s^*\Big)Z_s ds + (\delta\mu^* + \delta\mu_x^* Y_s)ds \\
\qquad\quad +(\alpha_s^* Z_s + Y_s v \mathbf{1}_{s\in[t,t+\varepsilon)})' dW_s^*, \qquad s \in [t,T], \\
Z_t = 0,
\end{cases}$$

where $\delta\mu_x^* = (\theta_s + \tilde{h}^*({}^s\alpha^{t,\varepsilon,v}, s))'\alpha_s^{t,\varepsilon,v} - (\theta_s + \tilde{h}^*({}^s\alpha^*, s))'\alpha_s^*$. Then we have the following moment estimates,

$$\begin{aligned}
&\mathbb{E}_t^*\Big[\sup_{s\in[t,T]}|Y_s|^2\Big] = O(\varepsilon), \quad \mathbb{E}_t^*\Big[\sup_{s\in[t,T]}|Z_s|^2\Big] = O(\varepsilon^2), \\
&\mathbb{E}_t^*\Big[\sup_{s\in[t,T]}|L_s^{t,\varepsilon,v} - X_s^* - Y_s^{t,\varepsilon,v} - Z_s^{t,\varepsilon,v}|^2\Big] = o(\varepsilon^2).
\end{aligned}$$

Furthermore, since $r, {}^th^*, \theta, {}^t\alpha^*$ are deterministic, we take conditional expectation on both sides of the SDE for Y, then $\mathbb{E}_t^*[Y_s]$ satisfies an ODE with 0 as its unique solution. Therefore, $\mathbb{E}_t^*[Y_s] = 0,\ s \in [t,T]$.

Then, we obtain the following

$$\begin{aligned}
&J(t, X_t^*; {}^t\tilde{u}^{t,\varepsilon,v,*}, {}^t\tilde{h}^{t,\varepsilon,v,*}) - J(t, X_t^*; {}^tu^*, {}^th^*) \\
=&\mathcal{L}(t, L_t^{t,\varepsilon,v}) - J(t, X_t^*; {}^tu^*, {}^th^*) \\
=&-\frac{1}{2\xi}\mathbb{E}_t^*\int_t^T [2|\tilde{h}^*({}^s\alpha^{t,\varepsilon,v}, s)|^2 X_s^*(Y_s + Z_s) + |\tilde{h}^*({}^s\alpha^{t,\varepsilon,v}, s)|^2(Y_s + Z_s)^2] ds
\end{aligned}$$

$$
\begin{aligned}
&+\mathbb{E}_t^*\int_t^T[\delta f^*]ds+\mathbb{E}_t^*[X_T^*(Y_T+Z_T)]+\frac{1}{2}\mathbb{E}_t^*[(Y_T+Z_T)^2]\\
&-(\mathbb{E}_t^*[X_T^*]+\mu_1 x)\mathbb{E}_t^*[Y_T+Z_T]-\frac{1}{2}(\mathbb{E}_t^*[Y_T+Z_T])^2+o(\varepsilon)\\
=&-\frac{1}{2\xi}\mathbb{E}_t^*\int_t^T[2|\tilde{h}^*({}^s\alpha^{t,\varepsilon,v},s)|^2X_s^*(Y_s+Z_s)+|\tilde{h}^*({}^s\alpha^{t,\varepsilon,v},s)|^2(Y_s+Z_s)^2]ds\\
&+\mathbb{E}_t^*\int_t^T[\delta f^*]ds+\mathbb{E}_t^*\big[(X_T^*-\mathbb{E}_t^*[X_T^*]-\mu_1 x)(Y_T+Z_T)\big]\\
&+\frac{1}{2}\mathbb{E}_t^*[(Y_T+Z_T)^2]+o(\varepsilon)\\
=&-\frac{1}{2\xi}\mathbb{E}_t^*\int_t^T[2|\tilde{h}^*({}^s\alpha^*,s)|^2X_s^*(Y_s+Z_s)+|\tilde{h}^*({}^s\alpha^*,s)|^2(Y_s+Z_s)^2]ds\\
&+\mathbb{E}_t^*\int_t^T[\delta f^*]ds+\mathbb{E}_t^*\big[(X_T^*-\mathbb{E}_t^*[X_T^*]-\mu_1 x)(Y_T+Z_T)\big]\\
&+\frac{1}{2}\mathbb{E}_t^*[(Y_T+Z_T)^2]+o(\varepsilon).
\end{aligned}
$$

Using the definition of $(p(\cdot;t),k(\cdot;t))$ and $P(\cdot;t)$ in (4.1), (4.2) and (4.3) and applying Itô's lemma to the last two terms, we have

$$
\begin{aligned}
&\mathbb{E}_t^*\big[(X_T^*-\mathbb{E}_t^*[X_T^*]-\mu_1 x)(Y_T+Z_T)\big]\\
=&-\mathbb{E}_t^*\int_t^T\Big[\Big(\big(r_s+(\theta_s+\tilde{h}^*({}^s\alpha^*,s))'\alpha_s^*\big)p(s;t)+(\alpha_s^*)'k(s;t)\\
&\qquad-\frac{1}{\xi}|\tilde{h}^*({}^s\alpha^*,s)|^2X_s^*\Big)(Y_s+Z_s)\Big]ds\\
&+\mathbb{E}_t^*\int_t^T\Big[p(s;t)\Big(\big(r_s+(\theta_s+\tilde{h}^*({}^s\alpha^*,s))'\alpha_s^*\big)(Y_s+Z_s)+\delta\mu^*+\delta\mu_x^*Y_s\Big)\Big]ds\\
&+\mathbb{E}_t^*\int_t^T\Big[k(s;t)'\big(\alpha_s^*(Y_s+Z_s)+Y_sv\mathbf{1}_{s\in[t,t+\varepsilon)}+X_s^*v\mathbf{1}_{s\in[t,t+\varepsilon)}\big)\Big]ds\\
=&\mathbb{E}_t^*\int_t^T\Big[\frac{1}{\xi}|\tilde{h}^*({}^s\alpha^*,s)|^2X_s^*(Y_s+Z_s)+\delta\mu^*p(s;t)+X_s^*k(s;t)'v\mathbf{1}_{s\in[t,t+\varepsilon)}\Big]ds\\
&+o(\varepsilon),
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}_t^*\big[(Y_T+Z_T)^2\big]\\
&=\mathbb{E}_t^*\int_t^T\Big[\frac{1}{\xi}|\tilde{h}^*({}^s\alpha^*,s)|^2(Y_s+Z_s)^2+P(s;t)(X_s^*)^2|v|^2\mathbf{1}_{s\in[t,t+\varepsilon)}\Big]ds+o(\varepsilon).
\end{aligned}
$$

Then (4.4) follows. □

The following lemma is analogous to Lemma 3.2 and also similar to Lebesgue differentiation theorem. Note that ${}^th^*$ is an implicit functional of ${}^s\alpha^*$ and $\delta H(s,{}^s\alpha^*,X^*)$

contains the second order terms of v. However, we managed to derive the explicit solutions, thanks to the design of our perturbation (closed-loop control) formulation in Definition 2.1.

Lemma 4.2 *Suppose that* (3.11) *in Lemma 3.3 holds for* α^* *and* $\alpha^{t,\varepsilon,v}$*, then for any* $t \in [0,T)$ *and* $v \in L^{\infty,\mathcal{Q}}_{\mathcal{F}_t}(\Omega;\mathbb{R}^d)$*,*

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathbb{E}^*_t\int_t^T\Big[\delta H(s,{}^s\alpha^*,X^*)+\frac{1}{2}P(s;t)(X^*_s)^2v'v\mathbf{1}_{s\in[t,t+\varepsilon)}\Big]ds\\
&=\Big(\big(\theta_t+\tilde{h}^*({}^t\alpha^*,t)\big)p(t;t)x+k(t;t)x\Big)'v\\
&\quad+\frac{\xi}{2}\big(M({}^t\alpha^*,t)-N({}^t\alpha^*,t)-\Gamma({}^t\alpha^*,t)\big)^2|v|^2x^2+P(t;t)\frac{|v|^2}{2}x^2.
\end{aligned}$$

Using Lemma 4.2, we can derive the following result.

Corollary 4.3 *Suppose that* $P(s;t)$ *is positive and* (3.11) *in Lemma 3.3 holds for* α^* *and* $\alpha^{t,\varepsilon,v}$*, then* u^* *is an equilibrium strategy in the sense of Definition 2.2 if and only if*

$$\big(\theta_t+\tilde{h}^*({}^t\alpha^*,t)\big)p(t;t)+k(t;t)=0. \tag{4.5}$$

Proof By Lemma 4.1 and Lemma 4.2, we have

$$\begin{aligned}
&\liminf_{\varepsilon\downarrow 0}\frac{J(t,X^*_t;{}^t\tilde{u}^{t,\varepsilon,v,*},{}^t\tilde{h}^{t,\varepsilon,v,*})-J(t,X^*_t;{}^tu^*,{}^th^*)}{\varepsilon}\\
&=\Big(\big(\theta_t+\tilde{h}^*({}^t\alpha^*,t)\big)p(t;t)x+k(t;t)x\Big)'v\\
&\quad+\frac{\xi}{2}\big(M({}^t\alpha^*,t)-N({}^t\alpha^*,t)-\Gamma({}^t\alpha^*,t)\big)^2|v|^2x^2+P(t;t)\frac{|v|^2}{2}x^2.
\end{aligned}$$

Its non-negativity for any x and v yields the equivalent condition (4.5). □

By (4.5), we have $u^*_t=\alpha^*_tX^*_t$ and

$$\alpha^*_t=-\frac{\Delta({}^t\alpha^*,t)}{\xi\Delta^2({}^t\alpha^*,t)+M({}^t\alpha^*,t)}\theta_t. \tag{4.6}$$

Moreover, the $F's$ in ODEs (3.5) are reduced to

$$\begin{cases}
F_M = 2\big(r-\frac{M\Delta}{(\xi\Delta^2+M)^2}|\theta|^2\big)M+\frac{\Delta^2}{(\xi\Delta^2+M)^2}|\theta|^2M-\frac{\xi\Delta^4}{(\xi\Delta^2+M)^2}|\theta|^2,\\
F_N = 2\big(r-\frac{M\Delta}{(\xi\Delta^2+M)^2}|\theta|^2\big)N,\\
F\;\; = \big(r-\frac{M\Delta}{(\xi\Delta^2+M)^2}|\theta|^2\big)\Gamma,
\end{cases} \tag{4.7}$$

where we suppress the arguments $({}^s\alpha^*,s)$ and the subscript time s. $M({}^t\alpha^*,t)$, $N({}^t\alpha^*,t)$, and $\Gamma({}^t\alpha^*,t)$ become simply functions of time. With slightly abuse of notations, we denote them by $M_t=M({}^t\alpha^*,t)$, $N_t=N({}^t\alpha^*,t)$, $\Gamma_t=\Gamma({}^t\alpha^*,t)$, and by $\Delta_t=M_t-N_t-\Gamma_t$. Then $-F_M=\dot{M}$, $-F_N=\dot{N}$, and $-F=\dot{\Gamma}$, the derivative with respect to time.

When $\xi = 0$, i.e., no robustness, the existence and uniqueness of solutions to ODEs (4.7) are given by Björk et al. [5, Theorem 4.7]. In the general robustness case where $\xi > 0$, we show the existence and uniqueness result with the following lemma. Loosely speaking, ODEs (4.7) are even more complex than the counterpart in Björk et al. [5, Proposition 4.5]. We adopt the truncation method instead of the Picard iteration in Björk et al. [5].

Lemma 4.4 *For a given constant $c > 0$ and $\xi > 0$, suppose that for any $s \in [0, T]$,*

$$e^{\int_s^T 2r_t dt} - \int_s^T \frac{1}{\xi}|\theta_t|^2 e^{\int_s^t 2r_u du} dt > c > 0 \tag{4.8}$$

$$\text{and } e^{\int_s^T (2r_t + \frac{|\theta_t|^2}{4\xi c} + \frac{3}{8}\sqrt{\frac{3}{\xi c}}|\theta_t|^2) dt} - e^{\int_s^T 2r_t dt} - \mu_1 e^{\int_s^T r_t dt} < 0. \tag{4.9}$$

Then ODEs (4.7) *admit a unique positive solution (M, N, Γ). Furthermore, $M > c$.*

Proof We define $M^c = M \vee c$ and $\Delta^- = \max\{-\Delta, 0\}$. Since N, Γ also depend on M^c, we use notations N^c and Γ^c to highlight this implicit dependence on c. The ODE system (4.7) becomes ODEs (4.10) for (M^c, N^c, Γ^c), with $F_M^c = -\dot{M}^c$, $F_N = -\dot{N}^c$, and $F = -\dot{\Gamma}^c$.

$$\begin{cases} F_M^c := 2\big(r + \frac{M^c \Delta^-}{(\xi|\Delta^-|^2 + M^c)^2}|\theta|^2\big)M^c + \frac{|\Delta^-|^2}{(\xi|\Delta^-|^2+M^c)^2}|\theta|^2 M^c \\ \qquad - \frac{\xi|\Delta^-|^4}{(\xi|\Delta^-|^2+M^c)^2}|\theta|^2, \\ F_N^c := 2\big(r + \frac{M^c\Delta^-}{(\xi|\Delta^-|^2+M^c)^2}|\theta|^2\big)N^c, \\ F^c := \big(r + \frac{M^c\Delta^-}{(\xi|\Delta^-|^2+M^c)^2}|\theta|^2\big)\Gamma^c. \end{cases} \tag{4.10}$$

Since

$$\frac{M^c\Delta^-}{(\xi|\Delta^-|^2 + M^c)^2} \le \frac{3}{16}\sqrt{\frac{3}{\xi c}}, \ \frac{|\Delta^-|^2}{(\xi|\Delta^-|^2 + M^c)^2} \le \frac{1}{4\xi c}, \ \frac{\xi|\Delta^-|^4}{(\xi|\Delta^-|^2 + M^c)^2} \le \frac{1}{\xi},$$

we have

$$\begin{aligned} 2rM^c - \frac{1}{\xi}|\theta|^2 \le F_M^c &\le \Big(2r + \frac{3}{8}\sqrt{\frac{3}{\xi c}}|\theta|^2 + \frac{1}{4\xi c}|\theta|^2\Big)M^c, \\ 2rN^c \le F_N^c &\le \Big(2r + \frac{3}{8}\sqrt{\frac{3}{\xi c}}|\theta|^2\Big)N^c, \\ r\Gamma^c \le F^c &\le \Big(r + \frac{3}{16}\sqrt{\frac{3}{\xi c}}|\theta|^2\Big)\Gamma^c. \end{aligned}$$

Note that (4.10) is locally Lipschitz and has linear growth on (M^c, N^c, Γ^c). Therefore, the ODEs (4.10) admit a unique solution (M^c, N^c, Γ^c). Furthermore, by the comparison principle of ODEs, we have

$$M_s^c \ge e^{\int_s^T 2r_t dt} - \int_s^T \frac{1}{\xi}|\theta_t|^2 e^{\int_s^t 2r_u du} dt > c,$$

$$-\Delta_s^- \leq e^{\int_s^T \left(2r_t + \frac{|\theta_t|^2}{4\xi c} + \frac{3}{8}\sqrt{\frac{3}{\xi c}}|\theta_t|^2\right)dt} - e^{\int_s^T 2r_t dt} - \mu_1 e^{\int_s^T r_t dt} < 0,$$

which imply that the truncation boundary is not binding and we have $(M, N, \Gamma) = (M^c, N^c, \Gamma^c)$. □

Assumptions (4.8)-(4.9) are tight under certain cases. For example, if risk premium parameter $\theta = 0$, then $M_s = e^{\int_s^T 2r_t dt}$. The second inequality (4.9) is satisfied automatically, while the first inequality (4.8) is exactly $M > c > 0$, which is the conclusion obtained for the general case. If we take S&P 500 index as an example, the absolute value of annualised risk premium is usually between 0.1 and 0.5. Therefore, $|\theta|^2$ will not be too large. Moreover, the left hand sides of these two inequalities are continuous in T. Note they are satisfied automatically when $T = 0$. Then for any parameter setting (r, θ, ξ, μ_1), there always exists $T > 0$ such that these assumptions are satisfied.

Theorem 4.5 *Suppose $P(s;t)$ is positive and assumptions in Lemmas 3.3 and 4.4 are satisfied, then the closed-loop equilibrium control pair (α^*, h^*) or (u^*, h^*) is unique.*

Proof Theorem 3.5 has shown the uniqueness of h^* given α^*. If there are two equilibrium strategies $(\alpha_1^*, \tilde{h}^*({}^t\alpha_1^*, t))$ and $(\alpha_2^*, \tilde{h}^*({}^t\alpha_2^*, t))$ with the same $\tilde{h}^*$, then Lemmas 4.1 and 4.2 show that both of α_1^* and α_2^* have to satisfy equation (4.6). However, Lemma 4.4 proves the uniqueness of the solutions (M, N, Γ). Therefore, $\alpha_1^* = \alpha_2^*$. □

When there is no uncertainty, that is, $\xi = 0$, our solution is degenerated to the solution in Björk et al. [5], Huang et al. [20]. More specifically,

$$\alpha_t^* = -\frac{\Delta_t}{M_t}\theta_t = -\theta_t + \theta_t e^{-\int_t^T |\alpha_s^*|^2 ds} + \mu_1 \theta_t e^{\int_t^T (-r_s - \theta_s' \alpha_s^* - |\alpha_s^*|^2) ds}, \tag{4.11}$$

which is identical to the integral equation found in Björk et al. [5, Proposition 4.5] or Huang et al. [20, Theorem 3.1].

5 Numerical studies

In this section, we illustrate our robust control policy with numerical studies. We first provide some financial insights of our robust strategy. Combining (3.9) and (4.6), we derive the worst-case drift as

$$h_t^* = -\frac{\xi \Delta^2({}^t\alpha^*, t)}{\xi \Delta^2({}^t\alpha^*, t) + M({}^t\alpha^*, t)}\theta_t.$$

Thus the perturbed Sharpe ratio in this case or the so-called worst-case Sharpe ratio is given by

$$\theta_t + h_t^* = \frac{M({}^t\alpha^*, t)}{\xi \Delta^2({}^t\alpha^*, t) + M({}^t\alpha^*, t)}\theta_t. \tag{5.1}$$

Since $M > c > 0$ by Lemma 4.4 under mild conditions, the worst-case Sharpe ratio is inversely proportional to ξ, the uncertainty aversion coefficient. Therefore, the uncertainty-averse investor should shrink her estimate of Sharpe ratio by the scaling factor suggested in (5.1) to constitute her conservative strategy.

To demonstrate the effects of robustness, we conduct both simulation and empirical studies. The simulation study elucidates that even with constant parameters and enough data (1000 months), it is still hard to distinguish two models. The robust equilibrium strategy improves the Sharpe ratio. In contrast, for the empirical study with much fewer data, the robust counterpart improves both the Sharpe ratio and the terminal wealth.

We consider a simulation study for model (2.1) with one-dimensional constant parameters, i.e., the geometric Brownian motion model. We follow Anderson et al. [1], Maenhout [26] to select the uncertainty aversion parameter ξ using detection-error probabilities. Specifically, we conduct likelihood ratio tests to choose between two potential models based on the available data. Two models are difficult to distinguish if the probability of rejecting one model mistakenly in favor of the another is high. Such a detection-error probability depends on ξ and sample horizon T, denoted by $\varepsilon_T(\xi)$. As in Maenhout [26, Sect. 3.2.2], it is easy to show that

$$\varepsilon_T(\xi) = \mathbb{P}\left[Z < -\frac{1}{2}\sqrt{\int_0^T (h_s^*)^2 ds}\right], \quad Z \sim N(0,1).$$

Anderson et al. [1] proposes to set 10% as the threshold of $\varepsilon_T(\xi)$ and finds the corresponding ξ. This threshold is satisfied by all (μ_1, ξ) pairs we considered in Table 5.1. We also verified (4.8) and (4.9) hold under all the parameter settings in Table 5.1. We adopt the method of rolling window with size of 1000 (months) to estimate m and σ and compute $\hat{\theta}$, the estimator of θ. To ease computational burden, we actually use the ground truth θ to solve ODEs (4.7). However, $\hat{\theta}$ is used to implement robust strategy (4.6) and non-robust strategy (4.11).

From Table 5.1, when $\xi = 0$, the investor ignores uncertainty. Larger μ_1 corresponds to more risk tolerance (seeking). The first row of Table 5.1 shows the non-robust strategy with a larger μ_1 achieves a higher terminal wealth but with a larger standard deviation (SD). We calculate the ratio of excess wealth over SD, i.e. Sharpe ratio, in Table 5.2. Non-robust strategy with larger μ_1 has a lower ratio. When $\xi > 0$, the investor takes into account the uncertainty. Under the same risk tolerance level, the investor who is more uncertainty-averse obtains a lower terminal wealth with a lower SD in general. However, Table 5.2 shows that the robustness significantly improves the performance of the strategy in terms of Sharpe ratio. It is also noteworthy that the values of Sharpe ratios across the row of Table 5.2 for the cases of $\xi > 0$ do not vary significantly in contrast with the case of $\xi = 0$.

For empirical study, we use the same dataset as in Han et al. [13]. Specifically, we download S&P 500 index prices from CRSP dataset and risk-free interest rate (one-month US Treasury bill rate) from the dataset of Fama-French Research Portfolios, while both are supported by Wharton Research Data Services. The data are collected on a monthly basis. We adopt the rolling-window estimation method with

μ_1	1	2	3	5	8
$\xi = 0$	0.503 (1.033)	0.791 (2.052)	0.927 (3.055)	1.101 (5.344)	1.183 (6.354)
$\xi = 1$	0.357 (0.631)	0.287 (0.477)	0.203 (0.328)	0.126 (0.193)	0.078 (0.120)
$\xi = 2$	0.235 (0.380)	0.147 (0.229)	0.101 (0.156)	0.062 (0.094)	0.037 (0.058)
$\xi = 3$	0.173 (0.271)	0.100 (0.155)	0.068 (0.104)	0.040 (0.062)	0.024 (0.039)
$\xi = 5$	0.109 (0.166)	0.060 (0.091)	0.039 (0.061)	0.023 (0.037)	0.013 (0.023)
$\xi = 8$	0.070 (0.107)	0.037 (0.057)	0.023 (0.038)	0.013 (0.023)	0.007 (0.014)

Table 5.1 Out-of-sample mean and standard deviation (in parentheses) of the excess terminal wealth over the risk-free asset with the initial wealth of 1, i.e., $x_0 = 1$, using the strategy (4.6) under different values of (μ_1, ξ). For each pair of (μ_1, ξ), we simulate 30000 stock paths of model (2.1) with stock return $m = 0.0062$ and stock volatility $\sigma = 0.0436$ on a monthly basis. The length of each path is 1080 (months). We adopt the method of rolling window with size of 1000 (months) to estimate m and σ and compute $\hat{\theta}$, the estimator of θ. Therefore, the investment (test) horizon is 80 (months). The risk-free rate is set as $r = 0.0027$. The results of $\xi = 0$ correspond to the case where the investor ignores uncertainty.

μ_1	1	2	3	5	8
$\xi = 0$	0.487	0.385	0.303	0.206	0.186
$\xi = 1$	0.566	0.600	0.620	0.654	0.648
$\xi = 2$	0.618	0.641	0.651	0.654	0.633
$\xi = 3$	0.638	0.645	0.657	0.650	0.618
$\xi = 5$	0.659	0.655	0.650	0.625	0.560
$\xi = 8$	0.656	0.650	0.615	0.562	0.488

Table 5.2 Sharpe ratios of the strategies considered in Table 5.1.

the window's length of 12 (months) to estimate the next month's S&P 500 index return and volatility. Figure 5.1 compares the empirical performance of the dynamic MV strategies with or without robustness and the S&P 500 index. The robust strategy realises a higher terminal wealth and a higher Sharpe ratio than its non-robust counterpart. It may be interesting to compare ours with the open-loop strategy in Han et al. [13] as we use the same dataset and comparable implementation. Compared to the open-loop strategies for the case of constant risk aversion, our closed-loop strategy has a lower profit and a lower Sharpe ratio. However, due to the construction of the closed-loop control, our closed-loop strategy guarantees a nonnegative wealth, which is not satisfied by the open-loop strategy in Han et al. [13].

6 Conclusions

In this paper, we apply a two-step closed-loop equilibrium control approach to study the robust mean-variance portfolio selection problem with state-dependent risk aversion. The proposed framework is new to the literature and the analytical solution to the robust equilibrium control is provided in (4.6) via the solution to ODEs (3.5) with (4.7). The existence and uniqueness results for the closed-loop equilibrium control pair are established. Our results address the open problem left in the worked-out example in Björk et al. [3, 5], Pun [30]. The significance of the uniqueness of the

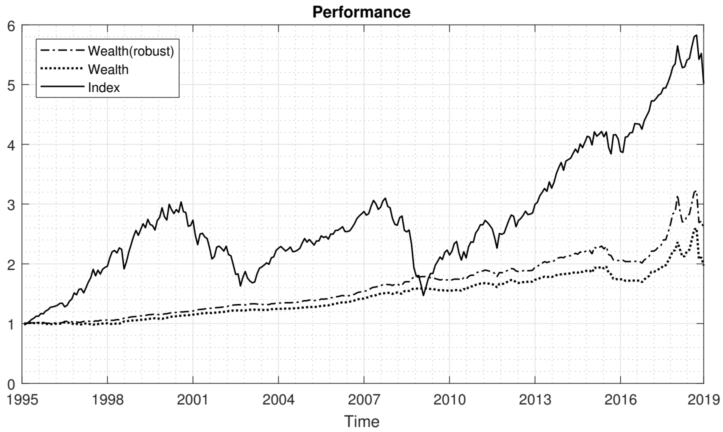

Fig. 5.1 Empirical performance on trading S&P 500 index from January 1995 to December 2018. The index (solid line) plots S&P 500 index prices with a scaling factor of 1/500. We set $\xi = 10$ and $\mu_1 = 0.2$. The initial wealth for both strategies is 1. Non-robust strategy (dotted line) achieves a terminal wealth 1.9720, with average excess monthly return 0.000636 and SD 0.018346 for excess monthly returns. Therefore, annualised Sharpe ratio is 0.1202. Robust strategy (dash-dotted line) obtains a terminal wealth 2.5908 and average excess monthly return 0.001594 with SD 0.018847. The annualised Sharpe ratio is 0.2930 and doubles the non-robust counterpart.

equilibrium control is to eliminate the doubts about using it in practice. ~~which makes an impact in financial economics.~~

At last, we remark that the results in Sections 3 and 4, except for Lemma 4.4, can be extended to the multidimensional case with a linear-quadratic objective as in Hu et al. [17] without much extra efforts. However, for Lemma 4.4, the truncation method we adopted is not readily extendable. While we can show the local existence by the Peano existence theorem, the global existence is still an open problem for a multi-dimensional case. It is noteworthy that the similar problem is also not addressed in Hu et al. [17] when there is no uncertainty, which makes it an interesting future research topic.

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A Proofs of results

A.1 Proof of Lemma 3.2

Proof By definition of Λ^ε in Lemma 3.1 and the ansatz (3.4), we have

$$\tilde{\mathbb{E}}_t^* \int_t^{t+\varepsilon} [\Lambda^\varepsilon(s;t)'\eta] ds$$

$$= \tilde{\mathbb{E}}_t^* \int_t^{t+\varepsilon} \Big[\Big(\alpha_s^{t,\varepsilon,v} \tilde{X}_s^{t,\varepsilon,v,*} \big(M({}^s\alpha^{t,\varepsilon,v}, {}^s\tilde{h}^{t,\varepsilon,v,*}, s) \tilde{X}_s^{t,\varepsilon,v,*} - \Gamma({}^s\alpha^{t,\varepsilon,v}, {}^s\tilde{h}^{t,\varepsilon,v,*}, s) x$$

$$- N({}^s\alpha^{t,\varepsilon,v}, {}^s\tilde{h}^{t,\varepsilon,v,*}, s)\tilde{\mathbb{E}}_t^*[\tilde{X}_s^{t,\varepsilon,v,*}]\Big) - \frac{1}{\xi}(\tilde{X}_s^{t,\varepsilon,v,*})^2\tilde{h}_s^{t,\varepsilon,v,*}\Big)'\eta\Big]ds.$$

Then we deal with these terms separately in the followings. As $\varepsilon \downarrow 0$,

$$\begin{aligned}
&\frac{1}{\varepsilon}\Big|\tilde{\mathbb{E}}_t^* \int_t^{t+\varepsilon} \Big[\alpha_s^{t,\varepsilon,v}\big(M({}^s\alpha^{t,\varepsilon,v}, {}^s\tilde{h}^{t,\varepsilon,v,*}, s)|\tilde{X}_s^{t,\varepsilon,v,*}|^2 - M({}^s\alpha^*, {}^sh^*, s)x^2\big)\Big]ds\Big| \\
&\le \frac{1}{\varepsilon}\int_t^{t+\varepsilon} |\alpha_s^{t,\varepsilon,v}|\big|M({}^s\alpha^{t,\varepsilon,v}, {}^s\tilde{h}^{t,\varepsilon,v,*}, s)\tilde{\mathbb{E}}_t^*\big[|\tilde{X}_s^{t,\varepsilon,v,*}|^2\big] - M({}^s\alpha^*, {}^sh^*, s)x^2\big|ds \\
&\le \sqrt{\frac{1}{\varepsilon}\int_t^{t+\varepsilon} \Big|\alpha_s^{t,\varepsilon,v}M({}^s\alpha^{t,\varepsilon,v}, {}^s\tilde{h}^{t,\varepsilon,v,*}, s) - \alpha_s^{t,\varepsilon,v}M({}^s\alpha^*, {}^sh^*, s)\Big|^2 ds}\sqrt{\frac{1}{\varepsilon}\int_t^{t+\varepsilon} x^4 ds} \\
&\quad + \sqrt{\frac{1}{\varepsilon}\int_t^{t+\varepsilon} \Big(\tilde{\mathbb{E}}_t^*\big[|\tilde{X}_s^{t,\varepsilon,v,*}|^2\big] - x^2\Big)^2 ds} \\
&\qquad \times \sqrt{\frac{1}{\varepsilon}\int_t^{t+\varepsilon} \Big|\alpha_s^{t,\varepsilon,v}M({}^s\alpha^{t,\varepsilon,v}, {}^s\tilde{h}^{t,\varepsilon,v,*}, s)\Big|^2 ds} \\
&\to 0,
\end{aligned}$$

where the last equality is due to (3.6) and continuity of $|\tilde{\mathbb{E}}_t^*[|\tilde{X}_s^{t,\varepsilon,v,*}|^2] - x^2|^2$ with respect to time s.

Similarly, we have

$$\begin{aligned}
&\frac{1}{\varepsilon}\Big|\tilde{\mathbb{E}}_t^* \int_t^{t+\varepsilon} \Big[\alpha_s^{t,\varepsilon,v}\big(\Gamma({}^s\alpha^{t,\varepsilon,v}, {}^s\tilde{h}^{t,\varepsilon,v,*}, s)\tilde{X}_s^{t,\varepsilon,v,*}x - \Gamma({}^s\alpha^*, {}^sh^*, s)x^2\big)\Big]ds\Big| \\
&\le \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \Big|\alpha_s^{t,\varepsilon,v}\Gamma({}^s\alpha^{t,\varepsilon,v}, {}^s\tilde{h}^{t,\varepsilon,v,*}, s)\tilde{\mathbb{E}}_t^*[\tilde{X}_s^{t,\varepsilon,v,*}]x - \alpha_s^{t,\varepsilon,v}\Gamma({}^s\alpha^*, {}^sh^*, s)x^2\Big|ds \\
&\le x\sqrt{\frac{1}{\varepsilon}\int_t^{t+\varepsilon} \Big|\alpha_s^{t,\varepsilon,v}\Gamma({}^s\alpha^{t,\varepsilon,v}, {}^s\tilde{h}^{t,\varepsilon,v,*}, s)x - \alpha_s^{t,\varepsilon,v}\Gamma({}^s\alpha^*, {}^sh^*, s)x\Big|^2 ds} \\
&\quad + x\sqrt{\frac{1}{\varepsilon}\int_t^{t+\varepsilon} \Big(\tilde{\mathbb{E}}_t^*[|\tilde{X}_s^{t,\varepsilon,v,*}|] - x\Big)^2 ds} \\
&\qquad \times \sqrt{\frac{1}{\varepsilon}\int_t^{t+\varepsilon} \Big|\alpha_s^{t,\varepsilon,v}\Gamma({}^s\alpha^{t,\varepsilon,v}, {}^s\tilde{h}^{t,\varepsilon,v,*}, s)\Big|^2 ds} \\
&\to 0
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\varepsilon}\Big|\tilde{\mathbb{E}}_t^* \int_t^{t+\varepsilon} \Big[\alpha_s^{t,\varepsilon,v}\big(N({}^s\alpha^{t,\varepsilon,v}, {}^s\tilde{h}^{t,\varepsilon,v,*}, s)\tilde{X}_s^{t,\varepsilon,v,*}\tilde{\mathbb{E}}_t^*[\tilde{X}_s^{t,\varepsilon,v,*}] - N({}^s\alpha^*, {}^sh^*, s)x^2\big)\Big]ds\Big| \\
&\le \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \Big|\alpha_s^{t,\varepsilon,v}N({}^s\alpha^{t,\varepsilon,v}, {}^s\tilde{h}^{t,\varepsilon,v,*}, s)(\tilde{\mathbb{E}}_t^*[\tilde{X}_s^{t,\varepsilon,v,*}])^2 - \alpha_s^{t,\varepsilon,v}N({}^s\alpha^*, {}^sh^*, s)x^2\Big|ds \\
&\le x^2\sqrt{\frac{1}{\varepsilon}\int_t^{t+\varepsilon} \Big|\alpha_s^{t,\varepsilon,v}N({}^s\alpha^{t,\varepsilon,v}, {}^s\tilde{h}^{t,\varepsilon,v,*}, s) - \alpha_s^{t,\varepsilon,v}N({}^s\alpha^*, {}^sh^*, s)\Big|^2 ds}
\end{aligned}$$

$$
\begin{aligned}
&+\sqrt{\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\Big((\tilde{\mathbb{E}}_t^*[\tilde{X}_s^{t,\varepsilon,v,*}])^2-x^2\Big)^2 ds}\\
&\quad\times\sqrt{\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\Big|\alpha_s^{t,\varepsilon,v}N({}^s\alpha^{t,\varepsilon,v},{}^s\tilde{h}^{t,\varepsilon,v,*},s)\Big|^2 ds}\\
\to\ &0.
\end{aligned}
$$

Noting that η is essentially bounded. Therefore, (3.7) follows. □

A.2 Proof of Lemma 3.4

Proof Under measure $\mathbb{Q}^*$, we define

$$
\begin{cases}
dL_s^{t,\varepsilon,v}=\big(r_s+(\theta_s+\tilde{h}^*({}^s\alpha^{t,\varepsilon,v},s))'\alpha_s^{t,\varepsilon,v}\big)L_s^{t,\varepsilon,v}ds\\
\qquad\qquad +(\alpha_s^{t,\varepsilon,v}L_s^{t,\varepsilon,v})'dW_s^*,\\
L_t^{t,\varepsilon,v}=x,
\end{cases}
\tag{A.1}
$$

and

$$
\begin{cases}
dl^\varepsilon(s;t)=-\Big(\big(r_s+(\theta_s+\tilde{h}_s^{t,\varepsilon,v,*})'\alpha_s^{t,\varepsilon,v}\big)l^\varepsilon(s;t)+(\alpha_s^{t,\varepsilon,v})'z^\varepsilon(s;t)\\
\qquad\qquad -\frac{1}{\xi}(\tilde{h}_s^{t,\varepsilon,v,*})'\tilde{h}_s^{t,\varepsilon,v,*}L_s^{t,\varepsilon,v}\Big)ds+z^\varepsilon(s;t)'dW_s^*,\quad s\in[t,T],\\
l^\varepsilon(T;t)=L_T^{t,\varepsilon,v}-\mathbb{E}_t^*[L_T^{t,\varepsilon,v}]-\mu_1 x.
\end{cases}
$$

By the similar arguments as in Lemma 3.1, the expectations of $\tilde{X}_s^{t,\varepsilon,v,*}p^\varepsilon(s;t)$ and $(\tilde{X}^{t,\varepsilon,v,*})^2$ under measure $\tilde{\mathbb{Q}}^*$ are equal to the expectations of $L_s^{t,\varepsilon,v}l^\varepsilon(s;t)$ and $(L^{t,\varepsilon,v})^2$ under measure $\mathbb{Q}^*$. Thus we have

$$
\begin{aligned}
&\tilde{\mathbb{E}}_t^*\int_t^{t+\varepsilon}\big[\Lambda^\varepsilon(s;t)'\eta\big]ds\\
&=\mathbb{E}_t^*\int_t^{t+\varepsilon}\Big[\Big(\alpha_s^{t,\varepsilon,v}L_s^{t,\varepsilon,v}l^\varepsilon(s;t)-\frac{1}{\xi}(L_s^{t,\varepsilon,v})^2\tilde{h}_s^{t,\varepsilon,v,*}\Big)'\eta\Big]ds.
\end{aligned}
$$

We notice that

$$
\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\Big|\mathbb{E}_t^*\int_t^{t+\varepsilon}\Big[\Big(\alpha_s^{t,\varepsilon,v}L_s^{t,\varepsilon,v}l^\varepsilon(s;t)-\frac{1}{\xi}(L_s^{t,\varepsilon,v})^2\tilde{h}^*({}^s\alpha^{t,\varepsilon,v},s)\Big)'\eta\Big]ds\\
&\qquad -\mathbb{E}_t^*\int_t^{t+\varepsilon}\Big[\Big(\alpha_s^*X_s^*p(s;t)-\frac{1}{\xi}(X_s^*)^2\tilde{h}^*({}^s\alpha^*,s)\Big)'\eta\Big]ds\Big|\\
&\le\lim_{\varepsilon\downarrow 0}\frac{K}{\varepsilon}\mathbb{E}_t^*\int_t^{t+\varepsilon}\Big[\Big|\alpha_s^{t,\varepsilon,v}L_s^{t,\varepsilon,v}l^\varepsilon(s;t)-\frac{1}{\xi}(L_s^{t,\varepsilon,v})^2\tilde{h}^*({}^s\alpha^{t,\varepsilon,v},s)\\
&\qquad\qquad -\alpha_s^{t,\varepsilon,v}L_s^{t,\varepsilon,v}p(s;t)+\frac{1}{\xi}(L_s^{t,\varepsilon,v})^2\tilde{h}^*({}^s\alpha^*,s)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \alpha_s^{t,\varepsilon,v} L_s^{t,\varepsilon,v} p(s;t) - \frac{1}{\xi}(L_s^{t,\varepsilon,v})^2 \tilde{h}^*({}^s\alpha^*, s) \\
&\quad - \alpha_s^* X_s^* p(s;t) + \frac{1}{\xi}(X_s^*)^2 \tilde{h}^*({}^s\alpha^*, s)\Big|\Big] ds \\
&\le K \lim_{\varepsilon \downarrow 0} \Big(\sqrt{\frac{1}{\varepsilon}\mathbb{E}_t^* \int_t^{t+\varepsilon} \Big[|\alpha_s^{t,\varepsilon,v} L_s^{t,\varepsilon,v}|^2\Big] ds} \sqrt{\frac{1}{\varepsilon}\mathbb{E}_t^* \int_t^{t+\varepsilon} \Big[|l^\varepsilon(s;t) - p(s;t)|^2\Big] ds} \\
&\quad + \sqrt{\frac{1}{\varepsilon}\mathbb{E}_t^* \int_t^{t+\varepsilon} \Big[\big|\tilde{h}^*({}^s\alpha^*, s)\big|^2\Big] ds} \sqrt{\frac{1}{\varepsilon}\mathbb{E}_t^* \int_t^{t+\varepsilon} \Big[|\frac{1}{\xi}(L_s^{t,\varepsilon,v})^2 - \frac{1}{\xi}(X_s^*)^2|^2\Big] ds} \\
&\quad + \frac{1}{\varepsilon}\mathbb{E}_t^* \int_t^{t+\varepsilon} \Big[\Big|\frac{1}{\xi}(L_s^{t,\varepsilon,v})^2 \tilde{h}^*({}^s\alpha^{t,\varepsilon,v}, s) - \alpha_s^{t,\varepsilon,v} L_s^{t,\varepsilon,v} p(s;t)\Big|\Big] ds \\
&\quad + \frac{1}{\varepsilon}\mathbb{E}_t^* \int_t^{t+\varepsilon} \Big[\Big|\frac{1}{\xi}(L_s^{t,\varepsilon,v})^2 \tilde{h}^*({}^s\alpha^*, s) - \alpha_s^* X_s^* p(s;t)\Big|\Big] ds \Big).
\end{aligned}
$$

By the stability results of BSDE, (see, e.g. Yong and Zhou [38, Theorem 3.3, Chap. 7]), the first term tends to 0. The second term also converges to 0. For the last second term,

$$
\begin{aligned}
&\frac{1}{\varepsilon}\mathbb{E}_t^* \int_t^{t+\varepsilon} \Big[\Big|\frac{1}{\xi}(L_s^{t,\varepsilon,v})^2 \tilde{h}^*({}^s\alpha^{t,\varepsilon,v}, s) - \alpha_s^{t,\varepsilon,v} L_s^{t,\varepsilon,v} p(s;t)\Big|\Big] ds \\
&= \frac{1}{\varepsilon}\mathbb{E}_t^* \int_t^{t+\varepsilon} \Big[\Big|(L_s^{t,\varepsilon,v})^2 \alpha_s^{t,\varepsilon,v} \big(M({}^s\alpha^{t,\varepsilon,v}, s) - N({}^s\alpha^{t,\varepsilon,v}, s) - \Gamma({}^s\alpha^{t,\varepsilon,v}, s)\big) \\
&\qquad - \alpha_s^{t,\varepsilon,v} L_s^{t,\varepsilon,v} \big(M({}^s\alpha^*, s) X_s^* - N({}^s\alpha^*, s)\mathbb{E}_t^*[X_s^*] - \Gamma({}^s\alpha^*, s)x\big)\Big|\Big] ds \\
&\le \sqrt{\frac{1}{\varepsilon}\mathbb{E}_t^* \int_t^{t+\varepsilon} \Big[|\alpha_s^{t,\varepsilon,v} L_s^{t,\varepsilon,v}|^2\Big] ds} \\
&\quad \times \Big(\sqrt{\frac{1}{\varepsilon}\mathbb{E}_t^* \int_t^{t+\varepsilon} \Big[|M({}^s\alpha^{t,\varepsilon,v}, s) L_s^{t,\varepsilon,v} - M({}^s\alpha^*, s) X_s^*|^2\Big] ds} \\
&\qquad + \sqrt{\frac{1}{\varepsilon}\mathbb{E}_t^* \int_t^{t+\varepsilon} \Big[|N({}^s\alpha^{t,\varepsilon,v}, s) L_s^{t,\varepsilon,v} - N({}^s\alpha^*, s)\mathbb{E}_t^*[X_s^*]|^2\Big] ds} \\
&\qquad + \sqrt{\frac{1}{\varepsilon}\mathbb{E}_t^* \int_t^{t+\varepsilon} \Big[|\Gamma({}^s\alpha^{t,\varepsilon,v}, s) L_s^{t,\varepsilon,v} - \Gamma({}^s\alpha^*, s)x|^2\Big] ds} \Big) \\
&\to 0.
\end{aligned}
$$

Similarly, we can also prove that the last term tends to 0. Thus this lemma is proved. □

A.3 Proof of Lemma 4.2

Proof By definition,

$$
\begin{aligned}
&\mathbb{E}_t^* \int_t^T \left[\delta H(s, {}^s\alpha^*, X^*)\right] ds \\
&= \mathbb{E}_t^* \int_t^T \Big[\Big(\big(\theta_s + \tilde{h}^*({}^s\alpha^{t,\varepsilon,v}, s)\big)' \alpha_s^{t,\varepsilon,v} X_s^* - \big(\theta_s + \tilde{h}^*({}^s\alpha^*, s)\big)' \alpha_s^* X_s^* \Big) p(s;t) \\
&\qquad - \frac{1}{2\xi} \big(|\tilde{h}^*({}^s\alpha^{t,\varepsilon,v}, s)|^2 - |\tilde{h}^*({}^s\alpha^*, s)|^2 \big) (X_s^*)^2 + X_s^* k(s;t)' v \mathbf{1}_{s\in[t,t+\varepsilon)} \Big] ds \\
&= \mathbb{E}_t^* \int_t^T \Big[\Big(\big(\theta_s + \tilde{h}^*({}^s\alpha^*, s)\big) p(s;t) X_s^* + k(s;t) X_s^* \Big)' v \mathbf{1}_{s\in[t,t+\varepsilon)} \\
&\qquad + \big(\tilde{h}^*({}^s\alpha^{t,\varepsilon,v}, s) - \tilde{h}^*({}^s\alpha^*, s)\big)' \Big(\alpha_s^{t,\varepsilon,v} X_s^* p(s;t) \\
&\qquad\qquad - \frac{1}{2\xi} \big(\tilde{h}^*({}^s\alpha^{t,\varepsilon,v}, s) + \tilde{h}^*({}^s\alpha^*, s) \big) (X_s^*)^2 \Big) \Big] ds.
\end{aligned}
$$

The first term in the second equality can be handled straightforwardly. Now, we deal with the second term. For simplicity, we introduce the following notations,

$$
\begin{aligned}
A_s^\varepsilon &= \tilde{h}^*({}^s\alpha^{t,\varepsilon,v}, s) - \tilde{h}^*({}^s\alpha^*, s), \\
A_s &= \xi v \mathbf{1}_{s\in[t,t+\varepsilon)} \big(M({}^s\alpha^*, s) - N({}^s\alpha^*, s) - \Gamma({}^s\alpha^*, s) \big), \\
B_s^\varepsilon &= \alpha_s^{t,\varepsilon,v} X_s^* p(s;t) - \frac{1}{2\xi} \big(\tilde{h}^*({}^s\alpha^{t,\varepsilon,v}, s) + \tilde{h}^*({}^s\alpha^*, s) \big) (X_s^*)^2, \\
B_s &= \frac{1}{2} v \mathbf{1}_{s\in[t,t+\varepsilon)} \big(M({}^s\alpha^*, s) - N({}^s\alpha^*, s) - \Gamma({}^s\alpha^*, s) \big) x^2.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \Big| \mathbb{E}_t^* \int_t^T \Big[(A_s^\varepsilon)' B_s^\varepsilon - A_s' B_s \Big] ds \Big| \\
&= \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \Big| \int_t^T (A_s^\varepsilon)' \mathbb{E}_t^*[B_s^\varepsilon] - A_s' \mathbb{E}_t^*[B_s^\varepsilon] + A_s' \mathbb{E}_t^*[B_s^\varepsilon] - A_s' B_s ds \Big| \\
&\le \lim_{\varepsilon\downarrow 0} \sqrt{\frac{1}{\varepsilon} \int_t^T \big| \mathbb{E}_t^*[B_s^\varepsilon] \big|^2 ds} \sqrt{\frac{1}{\varepsilon} \int_t^T |A_s^\varepsilon - A_s|^2 ds} \\
&\qquad + \sqrt{\frac{1}{\varepsilon} \int_t^T |A_s|^2 ds} \sqrt{\frac{1}{\varepsilon} \int_t^T |\mathbb{E}_t^*[B_s^\varepsilon] - B_s|^2 ds}.
\end{aligned}
$$

Recall the expressions for $p(s;t)$, $k(s;t)$, and $\tilde{h}^*({}^s\alpha^*, s)$ in (4.1), (4.2), and (3.9). Note that $M({}^s\alpha^{t,\varepsilon,v}, s)$, $N({}^s\alpha^{t,\varepsilon,v}, s)$, $\Gamma({}^s\alpha^{t,\varepsilon,v}, s)$ uniformly converge to $M({}^s\alpha^*, s)$, $N({}^s\alpha^*, s)$, $\Gamma({}^s\alpha^*, s)$ by Lemma 3.3, we have

$$
\frac{1}{\varepsilon} \int_t^T |\mathbb{E}_t^*[B_s^\varepsilon] - B_s|^2 ds
$$

$$
\begin{aligned}
&= \frac{1}{\varepsilon}\int_t^T \Big|\alpha_s^{t,\varepsilon,v}\big(M({}^s\alpha^*,s)\mathbb{E}_t^*[(X_s^*)^2] - N({}^s\alpha^*,s)(\mathbb{E}_t^*[X_s^*])^2 - \Gamma({}^s\alpha^*,s)\mathbb{E}_t^*[X_s^*]x\big) \\
&\qquad - \frac{1}{2\xi}\Big(\xi\alpha_s^{t,\varepsilon,v}\big(M({}^s\alpha^{t,\varepsilon,v},s) - N({}^s\alpha^{t,\varepsilon,v},s) - \Gamma({}^s\alpha^{t,\varepsilon,v},s)\big) \\
&\qquad\qquad + \xi\alpha_s^*\big(M({}^s\alpha^*,s) - N({}^s\alpha^*,s) - \Gamma({}^s\alpha^*,s)\big)\Big)\mathbb{E}_t^*[(X_s^*)^2] \\
&\qquad - \frac{1}{2}v\mathbf{1}_{s\in[t,t+\varepsilon)}\big(M({}^s\alpha^*,s) - N({}^s\alpha^*,s) - \Gamma({}^s\alpha^*,s)\big)x^2\Big|^2 ds \\
&\le \frac{1}{\varepsilon}\int_t^T \Big|\alpha_s^{t,\varepsilon,v}M({}^s\alpha^*,s)\mathbb{E}_t^*[(X_s^*)^2] \\
&\qquad - \frac{1}{2}\Big(\alpha_s^{t,\varepsilon,v}M({}^s\alpha^{t,\varepsilon,v},s)\mathbb{E}_t^*[(X_s^*)^2] + \alpha_s^*M({}^s\alpha^*,s)\mathbb{E}_t^*[(X_s^*)^2] \\
&\qquad\qquad + v\mathbf{1}_{s\in[t,t+\varepsilon)}M({}^s\alpha^*,s)x^2\Big)\Big|^2 ds \\
&\quad + \frac{1}{\varepsilon}\int_t^T \Big|\alpha_s^{t,\varepsilon,v}N({}^s\alpha^*,s)(\mathbb{E}_t^*[X_s^*])^2 \\
&\qquad - \frac{1}{2}\Big(\alpha_s^{t,\varepsilon,v}N({}^s\alpha^{t,\varepsilon,v},s)\mathbb{E}_t^*[(X_s^*)^2] + \alpha_s^*N({}^s\alpha^*,s)\mathbb{E}_t^*[(X_s^*)^2] \\
&\qquad\qquad + v\mathbf{1}_{s\in[t,t+\varepsilon)}N({}^s\alpha^*,s)x^2\Big)\Big|^2 ds \\
&\quad + \frac{1}{\varepsilon}\int_t^T \Big|\alpha_s^{t,\varepsilon,v}\Gamma({}^s\alpha^*,s)\mathbb{E}_t^*[X_s^*]x \\
&\qquad - \frac{1}{2}\Big(\alpha_s^{t,\varepsilon,v}\Gamma({}^s\alpha^{t,\varepsilon,v},s)\mathbb{E}_t^*[(X_s^*)^2] + \alpha_s^*\Gamma({}^s\alpha^*,s)\mathbb{E}_t^*[(X_s^*)^2] \\
&\qquad\qquad + v\mathbf{1}_{s\in[t,t+\varepsilon)}\Gamma({}^s\alpha^*,s)x^2\Big)\Big|^2 ds \\
&\to 0.
\end{aligned}
$$

The remaining part about $|A_s^\varepsilon - A_s|^2$ can be proved similarly. Therefore,

$$
\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathbb{E}_t^*\int_t^T \Big[\Big(\tilde{h}^*({}^s\alpha^{t,\varepsilon,v},s) - \tilde{h}^*({}^s\alpha^*,s)\Big)'\Big(\alpha_s^{t,\varepsilon,v}X_s^*p(s;t) \\
&\qquad\qquad - \frac{1}{2\xi}\big(\tilde{h}^*({}^s\alpha^{t,\varepsilon,v},s) + \tilde{h}^*({}^s\alpha^*,s)\big)(X_s^*)^2\Big)\Big]ds \\
&= \frac{\xi}{2}\big(M({}^t\alpha^*,t) - N({}^t\alpha^*,t) - \Gamma({}^t\alpha^*,t)\big)^2|v|^2x^2.
\end{aligned}
$$

□

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