`PHSX815_Project3`:
Role of number of dice rolls and iterations of the trials on the uncertainties of the estimated mean values.

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1 Introduction

When we try to find the information about the characteristic, like mean, of a particular population , we often take a random sample from that population. We can then calculate the corresponding sample characteristic that can be used to summarize information about the unknown population characteristic. The population characteristic of interest is called a parameter and the corresponding sample characteristic is the sample statistic or parameter estimate. Since, the statistic is a summary of information about a parameter obtained from the sample, the value of a statistic depends on the particular sample that was drawn from the population. For this project, we want to study the role of sample size in the estimation of the mean of a Gaussian distribution from a simple dice roll experiment.

Gaussian distribution has two parameters: the mean (μ) and the standard deviation (σ) and its distribution function is,

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \tag{1}$$

The likelihood for the N measurements for a Gaussian distribution is thus calculated as,

$$\prod_i^N f(x_i|\mu) = \prod_i^N \left[\frac{1}{\sqrt{2\pi\sigma^2}} exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right)\right] \tag{2}$$

Then we need to find the value of the mean that maximizes our likelihood function. Thus the best estimate for μ comes out to be,

$$\hat{\mu}_{MLE} = \frac{1}{N}\sum_i^N x_i \quad \text{where} \quad \frac{\partial}{\partial\hat{\mu}}\prod_i^N f(x_i|\mu) = 0 \tag{3}$$

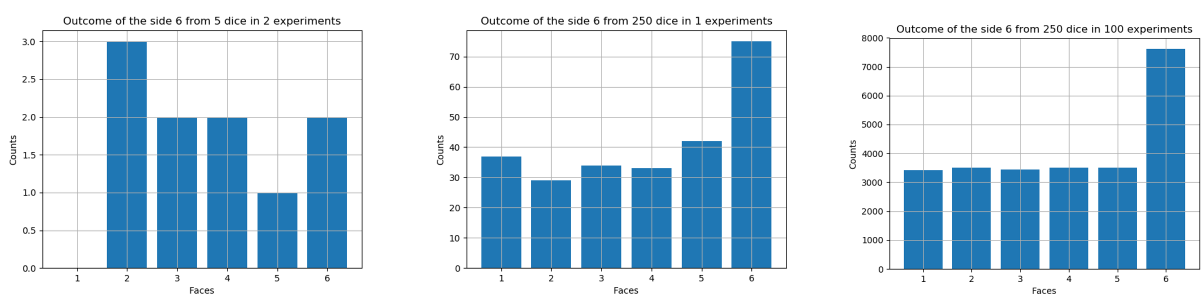
In order to find the estimate the uncertainty of these estimates we can run many iterations of the experiment for different values of means and analyze the observed means. This can be done by simply looking at the standard deviation of the gaussian distribution obtained from the slice of the Neymann construction.

2 Code and Experimental Simulation

For this project, we have simulated a dice that gives the outcome 6, 30 $\%$ of the times, we throw it. The codes have been written in such a way that the we can throw different number of dices for different number of times. Each time we throw it, we can get the probability and average number of times the outcome 6 appears for a particular number of dice thrown. Then the multiple iteration of the same experiment with same weight and number of dice thrown should give us the Gaussian distribution of the mean as predicted by the central limit theorem [1]. To find the MLE of the mean we will call the 'scipy.stats.norm' routine that will fit our histogram of the observed means. After that we have written codes to generate the Neymann Construction that will simulate the different observed means for a range of true value of means. The slice at each true value will have a normally distributed observed means. Thus by looking at the standard deviation of the particular slice, we can estimate the uncertainty of our estimated mean.

3 Analysis

The weight we have assigned for the outcome 6 is 0.30. If we look at the bar diagram of the outcomes this weight is reflected. We can immediately notice from the fig:1 that with more number of experiments,however, the weight that we have given gets more and more reflected. For limited number of dice rolls, we cannot guess, with enough confidence, the weight of the outcome with enough confidence as in the case of 2 trials of 5 dice rolls, as shown in fig: 5a.



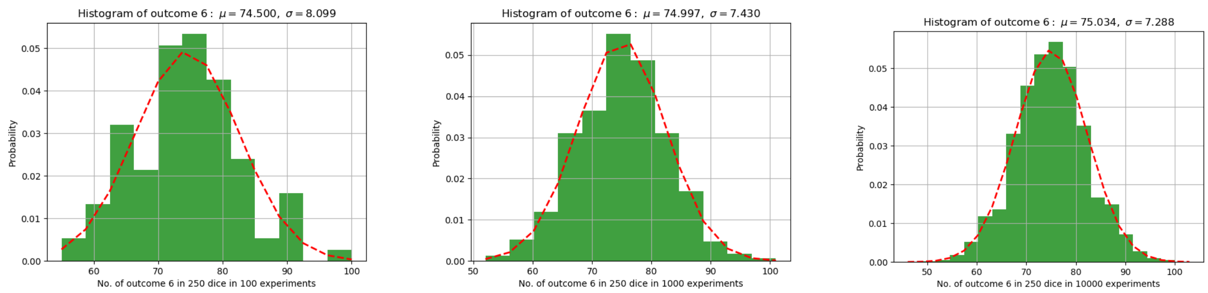
(a) 5 rolls and 2 trials (b) 250 rolls and 1 trial (c) 250 rolls and 100 trials

Figure 1: Outcome of side six for different number of dice rolls and trials

3.1 Gaussian fit using Maximum Likelihood Estimation

From the fig : 1, it is clear that 250 dice rolls should be good enough to realize the weight and average number of outcome six. So we move on to simulate the 100, 1000 and 10000 trials of the 250 dice rolls. The histogram of the average number of sixes appearing in each trial is plotted. The histogram is then fitted by using the normal distribution whose mean is the maximum likelihood estimation (MLE) given our data from the simulation.

We can see from fig : 2 that the MLE of the mean is very close to the value 75 which should be the ideal value in our case since the weight given was 0.3 which corresponds to the mean value of 75 for 250 dice rolls. Clearly, the MLE estimation becomes more and more accurate for larger number of trials and for 10000 trials, it is almost equal. The standard deviation also decreases with the increase in number of trials which gives us the hint that our estimation of the average is getting more and more reliable.

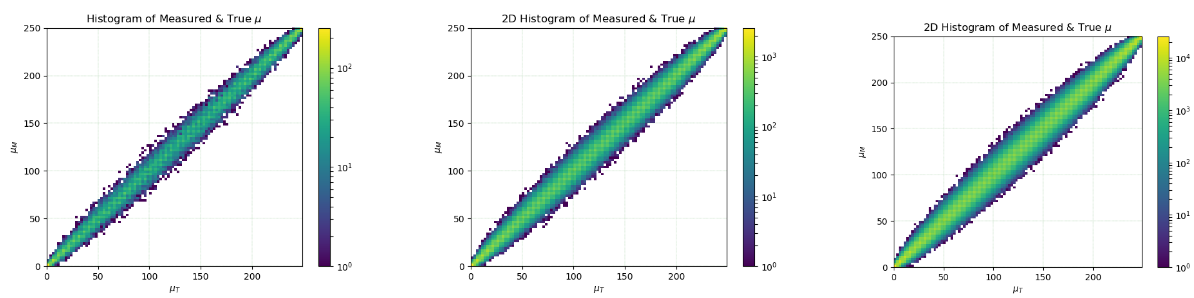


(a) 250 rolls and 100 trials (b) 250 rolls and 1000 trials (c) 250 rolls and 10000 trials

Figure 2: Gaussian fit for the average number of the outcome six from different number of trials

3.2 Neyman Construction and uncertainty of the estimation

To visualize the spread of the observed mean values for different true mean values we have plotted 2D histograms using Neyman Construction. The plots are shown in fig : 3



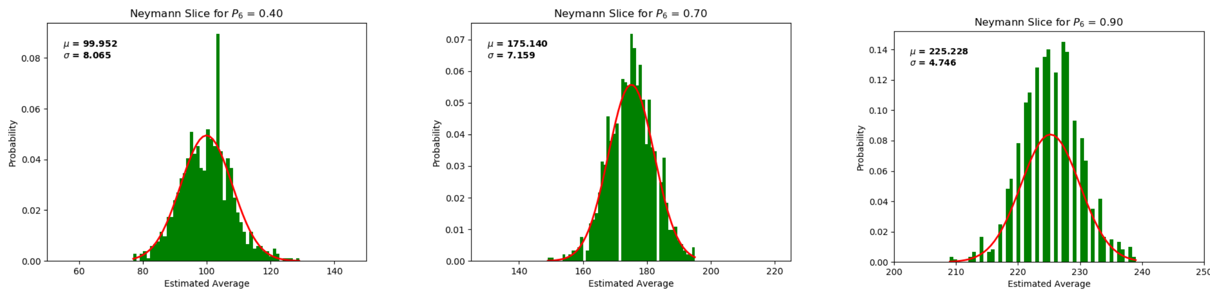
(a) 250 rolls and 100 trials (b) 250 rolls and 1000 trials (c) 250 rolls and 10000 trials

Figure 3: Neyman Construction for different number of trials of 250 dice rolls

Our Neyman construct for different number of trials shows that in our simulation most of the observed values are centered around the true value that we simulated it with. As we increase the number of trials, it seems as if the width of the spread of the observed values has also increased which is more pronounced at the centre that corresponds to the weight of the outcome six around 0.5. This is due to the fact that when we simulate the larger number of trials, it becomes more likely that we will observe wide value of means even though such outcomes are insignificantly low in comparison to the number of times we observe it near around the true value. To fully understand the spread we should take the slice of the 2D histogram for different numbers at different value of means and look at the full wave half maxima of the distribution.

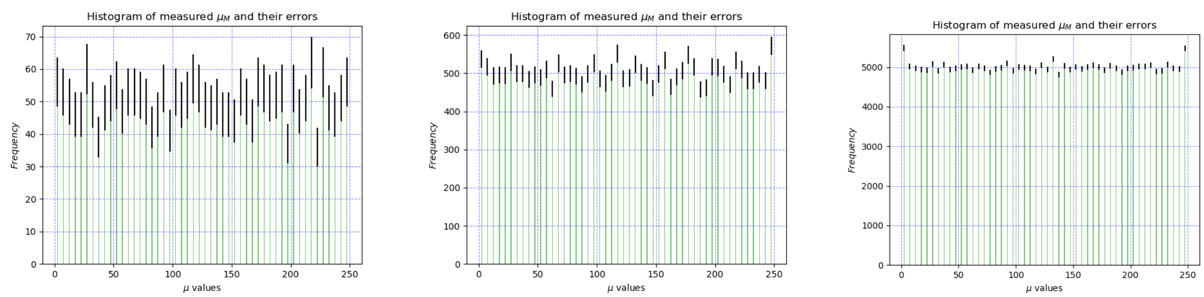
The slices shown in fig: 3 shows that for the higher value of the average the standard deviation or the spread is lesser as was predicted by the Neyman construct itself for 250 rolls and 1000 trials. We can also realize that the uncertainty of our observed parameter reduces with the increase in number of trials as shown in fig: 5 . The dark vertical lines on top of bins indicate the magnitude of the uncertainty for different values of observed means. This fact comes from the inverse relation between number of measurements and the value of stastistical uncertainty: more measurement leads to less statistical uncertainty.

The uncertainty on the measured values can also be studied by looking at the pull on the parameter being estimated. The distribution of the pull is Gaussian in nature and in an ideal case the value of



(a) 250 rolls and 1000 trials (b) 250 rolls and 1000 trials (c) 250 rolls and 1000 trials

Figure 4: Slice of Neyman Construct at different value of averages/weights of the outcome 6.



(a) 250 rolls and 10 trials (b) 250 rolls and 100 trials (c) 250 rolls and 1000 trials

Figure 5: Uncertainty on the measured parameters for different number of trials

mean should be zero and standard deviation equal to 1. For our simulation with 250 dice rolls and 1000 trials, we can see in fig: 6 that the mean is close to zero and the standard deviation of the pull is 0.85 which is close to value 1. This further confirms that our Neyman construct is reliable.

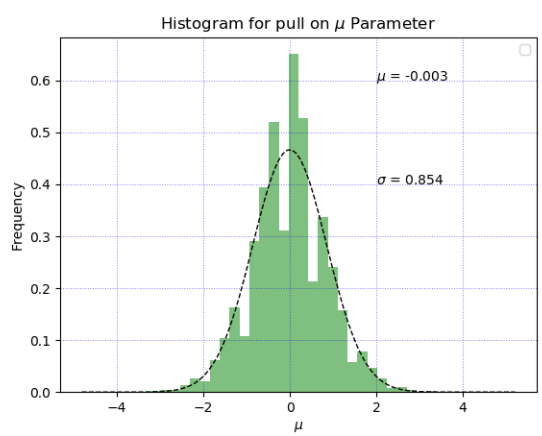


Figure 6: Pull on the parameter μ from our Neymann construct

4 Conclusion

Our Simulations and Analysis reveals that the more number of trials are always favourable if we want to estimate the parameter of the population and the uncertainty of the estimation is also reduced as our number of measurements are increased.

References

[1] Pierre-Simon Laplace. Mémoire sur les approximations des formules qui sont fonctions de très grands nombres et sur leur applications aux probabilités. *Memoires de l'Academie des Sciences de Paris*, 1810.