MA 602: Advanced Numerical Methods in Engineering

Tutorial # 4   January 13, 2020

Question 1 Plot the ball $B[0,1] = \{x \in \Re^2 : \|x\| \leq 1\}$, with respect to 1-norm, ∞ - norm and 2-norm, respectively.

Question 2 Let $X = C[-1,1]$ be the function space of all real valued continuous functions on $[-1,1]$. Find $\|f\|$ of the following functions with respect to ∞ - norm and L_2 - norm.

$$(i)\ f(x) = x^2 - \frac{1}{2}\ (ii)\ f(x) = x(x+1)(x-1)$$

$$(iii)\ f(x) = \sin \pi x\ (iv)\ f(x) = \cos \pi x$$

Question 3 Let $X = C[-1,1]$ be the vector space of all real valued continuous functions on $[-1,1]$. Let $S = \{1, x, x^2, x^3, \cdots\}$ be a set of linearly independent functions in X. Orthonormalize the set S with respect to the following two inner products on $C[-1,1]$.

$$(i)\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$$

$$(ii)\langle f, g\rangle = \int_{-1}^{1} [\frac{1}{\sqrt{1-x^2}}]f(x)g(x)dx$$

The resulting polynomials are called Legendre polynomials and Chebyshev polynomials, respectively. These sets of polynomials are very important from numerical analysis point of view and will be used in subsequent chapters.

Question 4 Let $X = C(-\infty, \infty)$ be the space of all real valued continuous functions on $(-\infty, \infty)$. Let $S = \{1, x, x^2, x^3, \cdots\}$ be a set of linearly independent functions in X. Orthonormalize the set S with respect to the following inner product.

$$\langle f, g\rangle = \int_{-\infty}^{\infty} e^{-x^2} f(x)g(x)dx$$

The resulting polynomials are called Hermite polynomials.

Question 5 Consider the following system

$$\begin{aligned} 1.0001x_1 + x_2 &= 2.0001 \\ x_1 + 1.0001x_2 &= 2.0001 \end{aligned}$$

It has $(1,1)$ as the exact solution. A bad approximation to the solution is $(2.0000, 0.0001)$. However, the residual error is small. Find it.

Question 6 Consider the following system

$$\begin{aligned} 0.913x_1 + 0.659x_2 &= 0.254 \\ 0.780x_1 + 0.563x_2 &= 0.217 \end{aligned}$$

Verify that $x = (1, -1)$ is the exact solution of this system. Consider the approximate solutions $y = (0.999, -1.0001)$ and $z = (0.341, -0.087)$. Compute the residual error r_y and r_z in both cases and find out which one is a better approximation of x. Also, find the relative errors e_y and e_z.

Question 7 Let

$$A = \begin{pmatrix} 1.0001 & 1 \\ 1 & 1.0001 \end{pmatrix}$$

be the coefficient matrix of the above system. Find its inverse. Also, find $\|A\|_1, \|A\|_\infty, \|A^{-1}\|_1, \|A^{-1}\|_\infty$.

Question 8 Find the condition number of the following matrices w.r.t. 1-norm and ∞-norm.

$$(i) \begin{pmatrix} 100 & 99 \\ 99 & 88 \end{pmatrix} \quad (ii) \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

Question 9 Find the condition number of the following matrices by using the spectral norm.

$$(i) \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix} \quad (ii) \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix} \quad (iii; \begin{pmatrix} 99 & 100 & 100 \\ 100 & 99 & 100 \\ 100 & 100 & 99 \end{pmatrix}$$

Question 10 Estimate the lower bound of the condition numbers w.r.t. (a) 1-norm and ∞- norm for the matrices given in question number 8 and (b) w.r.t. 2-norm for matrices in question number 9, by using some appropriate theorem and compare the results with the actual condition numbers.

Question 11 (MATLAB) - Wilson matrix Consider the following matrix.

$$A = \begin{pmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{pmatrix}$$

Verify that the linear system $Ax = b$ for $b = (32, 23, 33, 31)$ has $x = (1, 1, 1, 1)$ as a solution. If $\hat{b} = (32.1, 22.9, 33.1, 30.9)$ then verify that $\hat{x} = (1, 1, 1, 1) + (8.2, -13.6, 3.5, -2.1)$ is a solution. So, the system is not stable. Using MATLAB show that A^{-1} is given by

$$A^{-1} = \begin{pmatrix} 25 & -41 & 10 & -6 \\ -41 & 68 & -17 & 10 \\ 10 & -17 & 5 & -3 \\ -6 & 10 & -3 & 2 \end{pmatrix}$$

Find the condition number of the matrix.

Question 12(MATLAB) - Hilbert matrix The entries h_{ij} of the Hilbert matrix H are of the form $h_{ij} = \frac{1}{i+j-1}$. That is

$$H = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & . & . & . & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & . & . & . & \frac{1}{n+1} \\ . & . & . & . & . & . & . \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & . & . & . & \frac{1}{2n-1} \end{pmatrix}$$

Using MATLAB show that the condition number of H grows exponentially with n $(748, 2.8 \times 10^4, 9.4 \times 10^5, 2.9 \times 10^7, ...)$