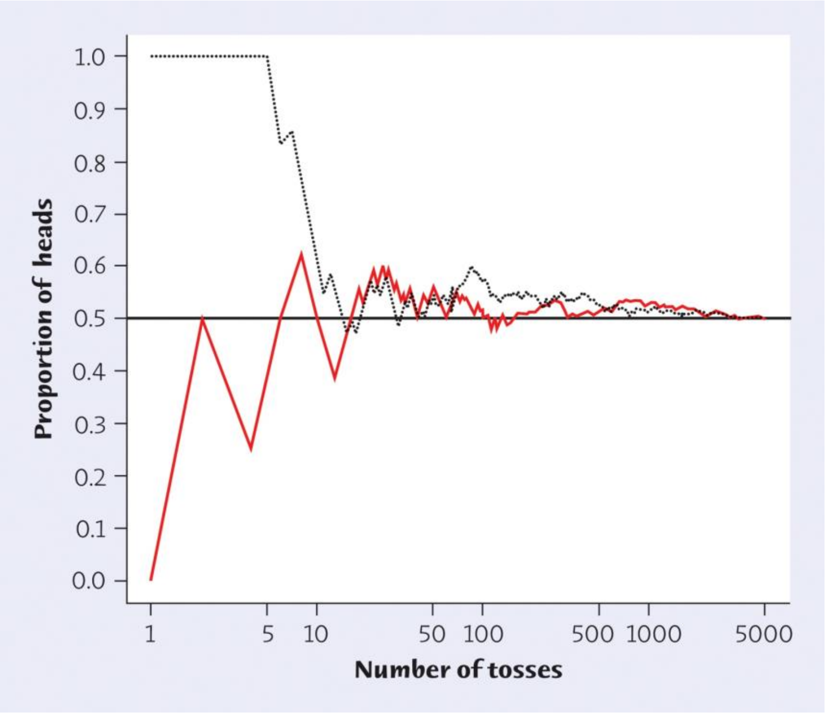
CHAPTER 3

Probability

3.1 Defining Probability

The Idea of Probability

- Coin tossing:
  - 5000 tosses: Proportion of heads
  - Red line:
    - Begins tail (proportion 0), head (proportion 0.5), tail (proportion 0.33), tail (proportion 0.25)
  - Black line:
    - Begins head, head, head, head, head (proportion 1), tail
- Make more and more tosses, the proportion of heads get close to 0.5 and stays there
- Probability 0.5 means: “occurs half the time in a very large number of trials”



However, proportion in a small number of tosses can be far from the probability.

Probability describes what happens in the long run.

The Idea of Probability

A phenomenon is *random* if individual outcomes uncertain, but there is a regular distribution of outcomes in a large number of repetitions.

Probability of an outcome is the proportion of times the outcome would occur if we observed the random process an infinite number of times.

- Probability is defined as a proportion, and it always takes values between 0 and 1. It may also be displayed as a percentage between 0% and 100%.

Law of large Numbers

Law of large numbers states that as more observations are collected, the proportion of occurrences with a particular outcome, $\hat{p}_n$, converges to the probability of that outcome, p.

Example:

When tossing a fair coin, if heads comes up on each of the first 10 tosses, what do you think the chance is that another head will come up on the next toss? 0.5, less than 0.5, or more than 0.5?

$\underline{H}\ \underline{H}\ \underline{H}\ \underline{H}\ \underline{H}\ \underline{H}\ \underline{H}\ \underline{H}\ \underline{H}\ \underline{H}\ \underline{?}$

Probability

Sample Space (S)

is the set of all possible outcomes for a random phenomenon.

Example: Toss a coin:

Roll two dice:

Event (A, B, …)

is an individual outcome or a set of outcomes of a random phenomenon

Example:

Probability of equally likely events.

- For some experiments, every outcome is equally likely to occur
- If every element is equally likely, then for some event A,

$$P(A) = \frac{Number\ of\ elements\ in\ A}{Number\ of\ elements\ in\ S}$$

Example: What is the probability of drawing a face card (Jack, Queen, King) from a standard deck of 52 cards?

Examples

A “die”, the singular dice, is a cube with six faces numbered 1, 2, 3, 4, 5, 6.

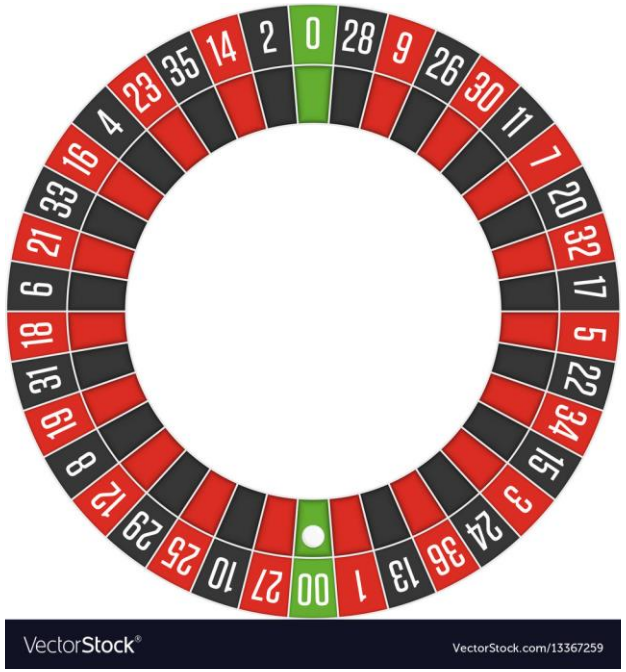
a) What is the chance of getting 1 when rolling a die?

b) What is the chance of getting either 1, 2, 3, 4, 5, 6 on the next roll?

Practice

The game of roulette involves spinning a wheel with 38 slots: 18 red, 18 black, and 2 green. A ball is spun onto the wheel and will eventually land in a slot, where each slot has an equal chance.

a) You watch a roulette wheel spin 3 consecutive times and the ball lands on a red slot each time. What is the probability that the ball will land on red slot on the next spin?



b) You watch a roulette wheel spin 300 consecutive times and the ball lands on a red slot each time. What is the probability that the ball will land on a red slot on the next spin?

c) Are you equally confident of your answers to part (a) and (b)? Why or why not?

Disjoint or Mutually Exclusive Outcomes

Two outcomes are called *disjoint* or *mutually exclusive* if they cannot both happen.

Example: roll a die, the outcomes 1 and 2 are disjoint since they cannot both occur.

Addition Rule of Disjoint outcomes

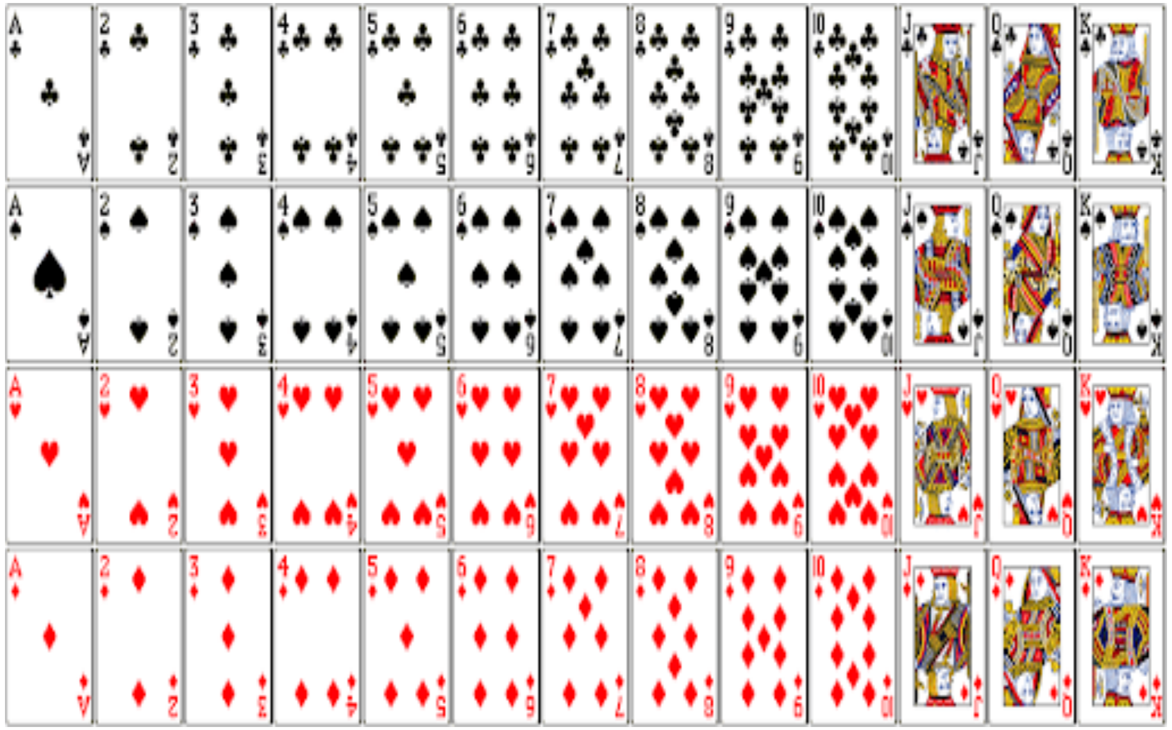
If A and B represent two disjoint outcomes, then the probability that one of them occurs is given by

$$P(A \text{ or } B) = P(A) + P(B)$$

Examples

Determine if the statements below are true of false and explain your reasoning.

a) Drawing a face card and drawing an ace from a full deck of playing cards are mutually exclusive events.



b) Drawing a face card (Jack, queen, or king) and drawing a red card from a full deck of playing cards are mutually exclusive events.

Probability When Events are Not Disjoint

General Addition Rule:

If A and B are any two events, disjoint or not, then the probability that at least one of them will occur is

$$P(A\ or\ B) = P(A) + P(B) - P(A\ and\ B)$$

Where P(A and B) is the probability that both events occur.

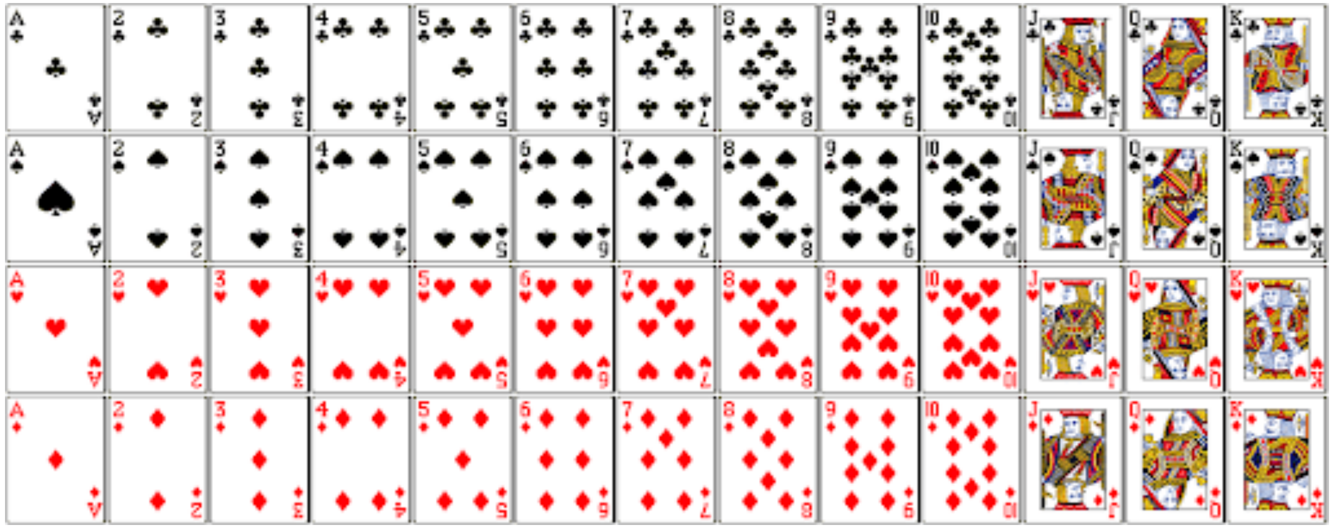
< Ex > Information from 1700 couples

What is the probability that a randomly selected couple has no children under the age of 18 or only the husband works?

	Number of Children under 18 Years Old			
Worked	**0**	**1**	**2 or More**	**Total**
Husband only	172	79	174	**425**
Wife only	94	17	15	**126**
Both spouses	522	257	370	**1149**
Total	**788**	**353**	**559**	**1700**

Example

What is the probability of drawing a jack or a red card from a well shuffled full deck?



Practice

What is the probability that a randomly sampled student thinks marijuana should be legalized <u>or</u> they agree with their parents' political views?

	Share Parents' Politics		
Legalize MJ	No	Yes	Total
No	11	40	51
Yes	36	78	114
Total	47	118	165

Probability Distribution

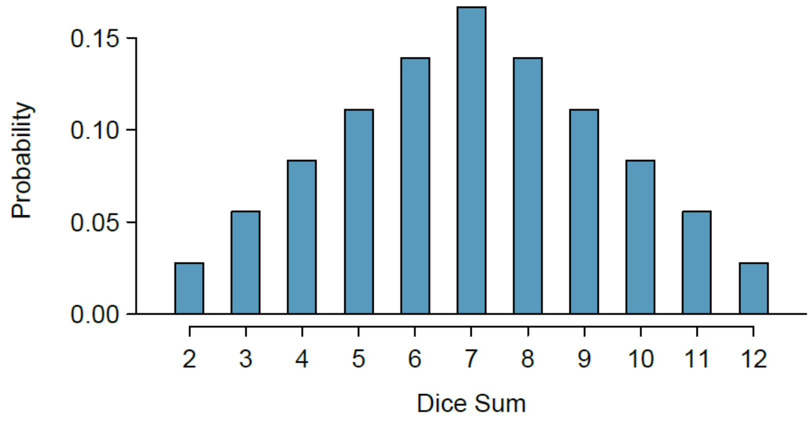
A *probability distribution* lists all possible events and the probabilities with which they occur.

Table below shows the probability distribution for the sum of two dice.

Dice sum	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Rules for probability distributions:

1. The events listed must be disjoint
2. Each probability must be between 0 and 1
3. The probabilities must total 1
4. The Probability of any event is the sum of
   the probabilities of the outcomes making up the event.



Complement of an event

The complement of event A is denoted A^c, and A^c represents all outcomes not in A. A and A^c are mathematically related:

$$P(A) + P(A^c) = 1$$

Example:

Let A represent the event where we roll two dice and their total is less than 12.

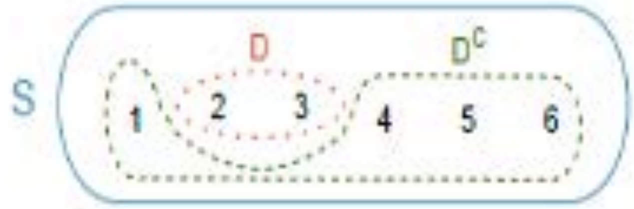
a) What does the event A^c represent?

b) Determine $P(A^c)$

c) Determine $P(A)$

Sample Space: This set of all possible outcomes is called the sample space.

< **Ex** > Rolling a die produces the sample space { 1, 2, 3, 4, 5, 6 }



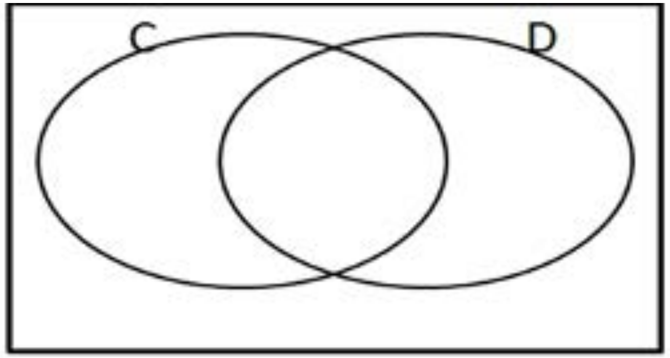
Let $D = \{2, 3\}$: event that the outcome of a die roll is 2 or 3

Then, $D^c = \{\, 1, 4, 5, 6 \,\}$

> Suppose that 40% of adults like cats (C), 45% like dogs (D), and 25% like both.
Find the probability that a randomly selected adult likes either cats or dogs.

(1) Using the Addition Rule: $P(C)=0.4, P(D)=0.45, P(C \text{ and } D)=0.25$

2)) Using the Venn Diagram:



Amy and Ben want to be partners of their law firm. Suppose Amy's probability of promotion is 0.6 and Ben's is 0.5. There is a 20% chance that both of them are promoted.
(a) Find the probability that Amy is promoted, but Ben is not.

(b) Find the probability that neither of them is promoted.

Practice

Find the following probabilities for rolling two dice:

a) The sum of the dice is not 6.

b) The sum is at least 4.

c) The sum is no more than 10.

Independence

Two processes are *independent* if knowing the outcome of one provides no useful information about the outcome of the other.

Multiplication rule for independent processes.

If A and B represent events from two different and independent processes, then the probability that both A and B occur can be calculated as the product of their separate probabilities:

$$P(A \text{ and } B) = P(A) \times P(B)$$

Example

In a multiple-choice exam, there are 5 questions and 4 choices for each question. Nancy has not studied for the exam at all and decided to randomly guess the answers. What is the probability that :

a) The first question she gets right is the 5^{th} question?

b) She gets all of the questions right?

c) She gets at least one question right?

Practice

About 9% of people are left-handed. Suppose 2 people are selected at random from the U.S. population. Because the sample size of 2 is very small relative to the population, it is reasonable to assume these two people are independent.

a) What is the probability that both are left-handed?

b) What is the probability that both are right-handed?

c) Suppose 5 people are selected at random. What is the probability that all are right –handed?

d) Suppose 5 people are selected at random. What is the probability that not all of the people are right-handed?

e) What is the probability that the first person is male and right-handed?