№4

Махмудов Орхан
М80-405Б-18

1) Решить задачу Коши, указать область единственности и области влияния

$$\begin{cases} x u_{xx} - u_{yy} + \frac{1}{2} u_x = 0, \quad 0 < x < 1 \\ u|_{y=0} = x, \quad u_y|_{y=0} = 0 \end{cases}$$

$a = x$, $b = 0$, $c = -1$
при $x > 0$ ур-е гиперболическое

1) $\frac{dy}{dx} = \pm \frac{1}{\sqrt{x}}$

$2\sqrt{x} - y = C_1$
$2\sqrt{x} + y = C_2$

$\xi = 2\sqrt{x} - y$; $\eta = 2\sqrt{x} + y$; $u_x = \frac{u_\xi}{\sqrt{x}} + \frac{u_\eta}{\sqrt{x}}$; $u_y = u_\eta - u_\xi$

$$u_{xx} = \frac{u_{\xi\xi}}{x} + \frac{u_{\eta\eta}}{x} + \frac{2u_{\xi\eta}}{x} + \left(-\frac{1}{2x^{\frac{3}{2}}}\right)(u_\xi + u_\eta)$$

$$u_{yy} = u_{\xi\xi} + u_{\eta\eta} - 2u_{\xi\eta}$$

$$x u_{xx} - u_{yy} = 2u_{\xi\eta} + 2u_{\xi\eta} - \frac{1}{2\sqrt{x}}(u_\xi + u_\eta) = 4u_{\xi\eta} - \frac{1}{2}u_x$$

$4u_{\xi\eta} = 0$; $u_{\xi\eta} = 0$

$u = F(\xi) + g(\eta) = F(2\sqrt{x} - y) + g(2\sqrt{x} + y)$

$F(2\sqrt{x}) + g(2\sqrt{x}) = x$

$u_y = u_\eta - u_\xi = g'_\eta(\eta) - F'_\xi(\xi) = 0$: $F'_y(\sqrt{2}x) + g'_y(\sqrt{2}\cdot x) = 0$

$C_1 + F(2\sqrt{x}) = g(2\sqrt{x}) - C_1$; $u_\eta = u_\xi \Rightarrow F(\xi) = \frac{\xi^2}{8} + C_1$; $g(\xi) = \frac{\eta^2}{8} - C_1$

$$u = \frac{1}{8}(\xi^2+\eta^2) - x + \frac{y^2}{4}$$

Обл-ть зав-ти: найдём ур-я харак., проходящих через $O(0,0)$

$$\xi = 2\sqrt{x} - y \qquad \longrightarrow \qquad \gamma_1: y = 2\sqrt{x}$$
$$\eta = 2\sqrt{x} + y \qquad\qquad\qquad \gamma_2: y = -2\sqrt{x}$$

$\Rightarrow$ Обл. зав. лежит внутри параболы $y^2 = 4x$

2) Решить задачу Гурса в указ. обл-ти:

$$\begin{cases} 2u_{xx} - 2u_{yy} + u_x + u_y = 0 & y > |x| \\ u|_{y=x} = 1, \; u|_{y=-x} = (x+1)e^x \end{cases}$$

$a=2$, $b=0$, $c=-2$
ур-е гиперболическое

$\xi = x+y$, $\eta = x-y$

$u_x = u_\xi + u_\eta$; $u_y = u_\xi - u_\eta$

$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$; $u_{yy} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}$

$8u_{\xi\eta} + 2u_\xi = 0$

$4u_{\xi\eta} + u_\xi = 0$

$u_\xi = z$

$4z'_\eta + z = 0$

$$4\frac{dz}{d\eta} = -z \Rightarrow \frac{4dz}{z} = -d\eta \Rightarrow 4\ln z = -\eta + C \Rightarrow z^4 = e^{-\eta}\cdot C_1(\xi)$$

$$z = \sqrt[4]{C_1 e^{-\eta}} = \frac{\partial u}{\partial \xi} = C_1 \cdot e^{-\frac{\eta}{4}} \; ; \; u = F_1 \cdot e^{-\frac{\eta}{4}} + C_2(\eta)$$

$$u = e^{-\frac{\eta}{4}} \cdot F(\xi) + g(\eta)$$

$$u|_{y=x} = 1 \quad \Bigg| \quad \eta = 0 \quad \Bigg| \quad u|_{\eta=0} = 1 \quad \Bigg| \quad F(2x) + g(0) = 1$$

$$u|_{y=-x} = (x+1)e^{x} \quad \Bigg| \quad \xi = 0 \quad \Bigg| \quad u|_{\xi=0} = \left(\frac{\xi+\eta}{2}+1\right)e^{\frac{\xi+\eta}{2}} \quad \Bigg| \quad e^{-\frac{2x}{4}} F(0) + g(2x) = (x+1)e^{x}$$

$$\begin{cases} F(2x) = 1 - g(0) \\ g(2x) = (x+1)e^{x} - e^{-\frac{x}{2}} F(0) \end{cases} \qquad \begin{cases} g(\eta) = \left(\frac{\eta}{2}+1\right)e^{\frac{\eta}{2}} - e^{-\frac{\eta}{4}} F(0) \\ F(\xi) = 1 - g(0) \end{cases} \Rightarrow F(0) + g(0) = 1$$

$$u = \cancel{e^{-\frac{\eta}{4}}}(1 - g(0)) + \left(\frac{\eta}{2}+1\right)e^{\frac{\eta}{2}} - \cancel{e^{-\frac{\eta}{4}}} F(0) = \left(\frac{\eta}{2}+1\right)e^{\frac{\eta}{2}} = \left(\frac{x-y}{2}+1\right)e^{\frac{x-y}{2}}$$

3) Построить ф-цию Римана оператора $\ell u = xyu_{xy} + xu_x - yu_y - u$ в обл-ти $x>0, y>0$

$$\ell u = u_{xy} + \frac{u_x}{y} - \frac{u_y}{x} - \frac{u}{xy}$$

$$\ell^* u = v_{xy} - \left(\frac{v}{y}\right)_x + \left(\frac{v}{x}\right)_y - \frac{v}{xy} = v_{xy} - \frac{v_x}{y} + \frac{v_y}{x} - \frac{v}{xy}$$

$$\ell^*_{xy} R(x, y, x_0, y_0) = 0$$

$$R|_{x=x_0} = e^{\int_{y_0}^{y} \frac{dt}{t}} = \frac{y_0}{y}.$$

$$R|_{y=y_0} = \frac{x}{x_0}$$

$$R(x_0, y, x_0, y_0) = \frac{y_0}{y}\ ; \quad R(x, y_0, x_0, y_0) = \frac{x}{x_0}$$

$$R(x, y, x_0, y_0) = \frac{xy_0}{x_0 y}$$