Fixed Income Analytics in DRIP

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v2.06, 15 July 2016

Section I: Preface

Terminology Background

Framework Glossary

1. Self-Jacobian: Self-Jacobian refers to the Jacobian of the Objective Function at any point in the variate to the Objective Function at the segment nodes, i.e., $\frac{\partial Y(t)}{\partial Y(t_K)}$.
2. Point-Measure State-Transform: Point-Measure transform refers to the one-to-one transform between a state measure at a predictor ordinate and its corresponding observation (e.g., discount factor from zero-coupon bond price observations).
3. Convolved-Measure State-Transform: Convolved-Measure transform refers to the many-to-one transform between a state metric/predictor ordinate combination to a given observation, i.e., a set of state metric/predictor ordinate pairs together imply an observation (e.g., zero rates from swap fair premium).
4. Discount-Curve Native Forward Curve: For discount curves built out of instruments dependent on forward rates, those rates and their discount curve usage ranges together constitute the discount curve's native forward curve range.

Document Layout

Section II: Jurisdictions, Instruments, Trading, and Settlement Conventions

Associations and Exchanges

Associations

1. ISDA: Many rules and standards are proposed or collected by financial associations. Chief among the, the ISDA, was founded in 1985. In particular ISDA publishes the ISDA definitions. Reference => http://www2.isda.org/
2. British Bankers' Association: The British Banker's Association (BBA) is the trade association for the UK banking and the financial services sector. Reference => http://www.bba.org.uk/
3. EURIBOR-EBF: The EURIBOR-EBF is international non-profit association founded in 1999 with the launch of the Euro. Its members are the national banking associations of the member nations of the European Union which are involved in the Euro-zone and the Euro system. Reference => http://www.euribor-ebf.eu/
4. Australian Financial Markets Association: AFMA was formed in 1986. Reference => http://www.afma.com.au
5. Danish Bankers' Association: The Danish Bankers' Association is an organization representing the banks in Denmark. It has the overall responsibility for the CIBOR indices. Reference => http://www.finansraadet.dk
6. Wholesale Markets' Brokers' Association: The WMBA is an association of the London brokers. Reference => http://www.wmba.org.uk
7. Japanese Bankers' Association: The Japanese Bankers' Association is a financial organization whose members consist of banks, bank holding companies, and bankers' associations in Japan. Reference => http://www.zenginko.or.jp/en/

Exchanges

1. Introduction: There are many exchanges where financial instruments are traded throughout the world. This section includes the main ones where the interest rate derivatives are listed. Over the years a lot of mergers and acquisitions have taken place between the exchanges. The names and the organization structures have changed, and will certainly change again.
2. Australian Securities Exchange: In the interest rate landscape, the main products are the AUD bank bill futures and their options, and AUD bond futures. Reference => http://www.asx.com.au/
3. BM&FBovespa - Brazil: BM&FBovespa was created in 2008 through the integration between Sao Paolo Stock Exchange (Bolsa de Valores de Sao Paolo) and the Brazilian Mercantile and Futures Exchange (Bolsa de Mercadorias e Futuros). Reference => http://www.bmfbovespa.com.br
4. CME Group: The CME group is a result of mergers between the Chicago Mercantile Exchange (CME), the Chicago Board of Trade (CBOT), the New York Mercantile Exchange (NYMEX), and COMEX. In the interest rate landscape, the main products are the interest rate futures (on LIBOR) and their options listed on CME, the federal funds futures listed in CBOT, and the bond futures and their options listed in CBOT. CME also runs a swap clearing business. Reference => http://www.cmegroup.com
5. EUREX: Eurex is a derivatives exchange jointly operated by the Deutsche Borse AG and the SIX Swiss Exchange. It started its derivatives trading in 1998. In the interest rate landscape, the main products are the interest rate futures (on EURIBOR) and their options, and EUR bond futures. Reference => http://www.eurexchange.com/index.html
6. Intercontinental Exchange - ICE: ICE is a relatively recent exchange active mainly in commodity, energy, and credit. It is involved in interest rate derivatives mainly through its (as of November 2013) acquisition of NYSE EuroNext. Reference => http:///www.theice.com
7. LCH.ClearNet: LCH.ClearNet group is a clearing house serving major exchanges and platforms, as well as a range of OTC markets. ClearNet is owned 77.5% by its clients and 22.5% by the exchanges. Reference => http://www.lchclearnet.com
8. MEFF - Spain: MEFF is an official secondary market regulated by the Spanish laws and under the supervision of the Spanish National Securities Market Commission. Reference => http://www.meff.com

9. Montreal Exchange: The Montreal Exchange is an electronic exchange dedicated to the development of the Canadian derivative markets. Reference => http://www.m-x.ca/
10. NASDAQ OMX: In the interest rate landscape the main products are the Nordic futures; CIBOR futures, STIBOR futures, and Swedish Bond Futures. They are also known for publishing the CIBOR and the SIOR rates. NASDAQ OMX also runs an exchange in London – the NLX (New London eXchange). References => http://www.nasdaqomx.com; http://www.nasdaqomx.com/transactions/markets/nlx
11. NYSE EuroNext: NYSE EuroNext resulted from the mergers/acquisitions between EuroNext, NYSE< LIFFE, and AMEX. The exchange was acquired by InterContinental Exchange in November 2013. In the interest rate landscape the main products are the interest rate future (on LIBOR and EURIBOR), and their listed options on LIFFE. Reference => http://www.euronext.com
12. Singapore Exchange - SGX: In the interest rate landscape, the products are Japanese and Singaporean Government Bond Futures, JPY (LIBOR and TIBOR), the Eurodollar STIR futures/options, and SGD futures. SGX also runs a swap clearing business. Reference => http://www.sgx.com
13. Tokyo Stock Exchange: In the interest rate landscape, the main products are JPY bond futures. Reference => http://www.tse.or.jp/english
14. South African Futures Exchange (SAFE): The Johannesburg Stock Exchange's Interest Rate Market offers bond futures and JIBAR 3M STIR futures. Reference => http://www.safex.co.za.

Date Conventions

Day Count Conventions

1. 1/1: The day count fraction is always 1.0; Definition 4.16 (a) in 2006 ISDA definitions (Open Gamma (2012)).
2. 30/360 Methods: Here the DCF is computed as

$$DCF = \frac{360(Y_2 - Y_1) + 30(M_2 - M_1) + (D_2 - D_1)}{30}$$

   The main differences are on how $Y_{1,2}$, $M_{1,2}$, and $D_{1,2}$ are calculated.
3. Generic 30/360: The generic 30.360 rules are:
   - If $D_1 == 31$, set $D_1 = 30$
   - If $D_2 == 31$ and $D_1 == 30/31$, set $D_2 = 30$

     This day count is also called 30/360 US, 30U/360, Bond Basis, 30/360, or 360/360. The last 3 are the ones used in the 2006 ISDA conventions. A variation of this uses an EOM Convention, which applies the following addition rule: If the EOM flag is turned on, and $D_{1,2}$ are the last days of February, the set $D_{1,2} = 30$. ISDA (as set out in 4.16(f)) does not use the EOM flag.
4. 30E/360: This is definition 4.16(g) in the 2006 ISDA definitions. The date adjustment rules are the following: If $D_1 == 31$, change it to $D_1 = 30$. Do the same for D_2 as well. This is also referred to as EUROBOND basis.
5. 30E/360 (ISDA): This is definition 4.16 (h) in ISDA 2006 definitions. The date adjustment rules are the following: a) If D_1 is the last day of the month, D_1 becomes 30. b) If D_2 is the last day of February, but not the termination date, or if $D_2 == 31$, then $D_2 = 30$.

6. 30E+/360 (ISDA): The date adjustment rules are the following: a) If $D_1 == 31$, the set $D_1 = 30$. b) If $D_2 == 31$, then change $D_2 = 1$ and $M_2 = M_2 + 1$. This convention is also called 30E+/360.
7. Act/360: This is definition 4.16 (e) in the 2006 ISDA definitions. The accrual factor is

$$DCF = \frac{d_2 - d_1}{360}$$

where $d_2 - d_1$ is the number of days between the 2 dates. This is the most widely used convention for the money market instruments (maturity below one year). This day count is also called the MONEY MARKET BASIS, Actual 360, or French Money Market Basis.
8. Act/365 Fixed: This is definition 4.16 (d) in the 2006 ISDA definitions. The accrual factor is

$$DCF = \frac{d_2 - d_1}{365}$$

where $d_2 - d_1$ is the number of days between the 2 dates. The number 365 is used even in a leap year. This convention is also called the ENGLISH MONEY MARKET BASIS.
9. Act/365L: This convention, described in ICMA Rule 251.1 (i), is seldom used. It was originally designed for the Euro-Sterling FRNs. It is used to only compute the accrual factor of a coupon. The computation of the factor requires 3 dates – the accrual start date (d_1), the accrual factor date (d_2), and the accrual end date (d_3).
10. Application of 365L: For semi-annual coupons (the type of coupons for which it was originally designed for), the accrual factor is

$$DCF = \frac{d_2 - d_1}{Days\ in\ the\ End\ Year}$$

where $Days\ in\ the\ End\ Year$ is the number of days in the year contained by d_3. This convention is extended to annual coupons by

$$DCF = \frac{d_2 - d_1}{Denominator}$$

where $Denominator$ is 366 if 29 February is between d_1 (exclusive) and d_3 (inclusive), or 365 otherwise. The convention is also called Act/365 Leap Year.

11. Act/365A: The accrual factor here is

$$DCF = \frac{d_2 - d_1}{Denominator}$$

where $Denominator$ is 366 if 29 February is between d_1 (exclusive) and d_2 (inclusive), or 365 otherwise. The convention is also called Act/365 Actual.

12. NL/365: The accrual factor is

$$DCF = \frac{Numerator}{365}$$

where $Numerator = d_2 - d_1 - 1$ if 29 February is between d_1 (exclusive) and d_2 (inclusive), or $d_2 - d_1$ otherwise. The convention is also called Act/365 No Leap Year.

13. Act/Act ISDA: This is definition 4.16(a) in 2006 ISDA definitions. The accrual factor is

$$DCF = \frac{Days\ in\ a\ Non-Leap\ Year}{365} + \frac{Days\ in\ a\ Leap\ Year}{366}$$

where the period first day is include and the period last day is excluded.

14. Act/Act ICMA – No Adjustment: This is taken from 4.16 (c) of the 2006 ISDA definitions. This convention is defined in Rule 251 of the ICMA Rule book. The accrual factor is

$$DCF = \frac{1}{Freq} \times Adjustment$$

where $Freq$ is the number of coupons per year and $Adjustment$ depends on the type of stub period. Where NO_ADJUSTMENT is set, $Adjustment = 1$, so the accrual factor becomes simply $\frac{1}{Freq}$.

15. Act/Act ICMA – Short at Start: Here the adjustment is computed as a ratio. The numerator is the number of days in the period, and the denominator is the number of days between the standardized start date computed as the coupon end date minus the number of month corresponding to the frequency (e.g., $\frac{12}{Freq}$) and the end date.
16. Act/Act ICMA – Long at Start: Two standardized start dates are computed as the coupon end date minus once and twice the number of months corresponding to the frequency. The numerator is the number of days between the start date and the denominator is the number of days between the first and the second standardized start date. The adjustment is the ratio of the numerator by the denominator plus 1.
17. Act/Act ICMA – Short at End: The adjustment is computed as a ratio. The numerator is the number of days between the start date and the standardized end date computed as the coupon start date plus the number of months corresponding to the frequency (i.e., $\frac{12}{Freq}$).
18. Act/Act ICMA – Long at End: Two standardized dates are computed as the coupon start dates plus once and twice the number of months corresponding to the frequency. The numerator is the number of months between the end date and the first standardized end date, and the denominator is the number of days between the second and the first standardized end date. The adjustment is the ratio of the numerator to the denominator plus 1.
19. Business/252: This day count is also called BUS/252. This day count is based on the business, not the calendar days. The accrual factor is

$$DCF = \frac{Business\ Days}{252}$$

where the numerator is the number of business days in a calendar year from and including the start date up to and excluding the end date. This day count convention is used in particular in the Brazilian market.

Business Day Conventions

1. <u>Following</u>: Business day convention is the convention for the adjustment of the dates when the specified date is not a good business day. The adjustment is applied with respect to a specified calendar. In the *following* convention, the adjusted day is the following business day.
2. <u>Preceding</u>: Here the adjusted day is the preceding good business day. This convention is often linked to loans and is a translation of the amount that should be paid on or before a specific date.
3. <u>Modified Following</u>: Here the adjusted date is the following good business day unless that date falls on the next calendar month, in which case it is taken to be the preceding good business day. This is the most used convention for interest rate derivatives.
4. <u>Modified Following Bi-monthly</u>: The adjusted date is the following good business day unless that adjusted date crosses mid-month (15^{th}) or the end of month, in which case the adjusted date is the preceding good business day.
5. <u>End-of-Month</u>: Where the tart of a period is on the final business day of a calendar month, the end date is also on the final business day of the end month (not necessarily the corresponding calendar date in the month).

References

- Open Gamma (2012): Interest Rate Instruments and Market Conventions Guide *Quantitative Research* **Open Gamma**.

Overnight and IBOR-like Indexes

IBOR Indexes - Introduction

1. The Indexes: IBOR-like indexes are related to interbank lending for maturities ranging from 1D to 1Y. They are usually computed as a trimmed average between rates contributed by the participating banks. The rates are the banks' estimates, but usually do not refer to the actual transactions. The most common usage of these indexes in IRD is in IRS and caps/floors.
2. IBOR Indices for the main Currencies:

Currency	Name	Maturities	Convention	Spot Lag
CHF	LIBOR	ON-12M	Act/360	2
EUR	EURIBOR	1W-12M	Act/360	2
EUR	EURIBOR	1W-12M	Act/365	2
EUR	LIBOR	ON-12M	Act/360	2
EUR	LIBOR	ON-12M	Act/360	0
GBP	LIBOR	ON-12M	Act/365	0
JPY	LIBOR	ON-12M	Act/360	2
JPY	Japan TIBOR	1W-12M	Act/365	2
JPY	EuroYen TIBOR	1W-12M	Act/360	2
USD	LIBOR	ON-12M	Act/360	2

3. IBOR Indices for the Other Currencies:

Currency	Name	Maturities	Convention	Spot Lag
AUD	BBSW	1M-6M	Act/365 F	0
CAD	CDOR	1M-12M	Act/365 F	0

CZK	PRIBOR		Act/360	2
DKK	CIBOR	1W-12M	Act/360	0
HKD	HIBOR	1M-12M	Act/365 F	0
HUF	BUBOR		Act/360	2
IDR	IDRFIX		Act/360	2
INR	MIFOR		Act/365 F	2
NOK	NIBOR		Act/360	2
NZD	BBR		Act/365	0
PLN	WIBOR		Act/365	2
RMB	SHIBOR	ON-12M	Act/360	0
SEK	STIBOR		Act/360	2
SKK	BRIBOR		Act/360	2
SGD	SIBOR		Act/365 F	2
SGD	SOR		Act/365 F	2
ZAR	JIBAR	1M-12M	Act/365 F	0

4. LIBOR: LIBOR is an acronym for London Interbank Offered Rate. It is calculated by Thomson Reuters on behalf of the British Bankers' Association. Major banks submit their cost of borrowing unsecured funds for several tenors and currencies.
5. LIBOR Administration: Up until 2012 there were 15 tenors in 10 currencies (AUD, CAD, DKK, EUR, JPY, NZD, GBP, SEK, CHF, and USD). Some have been phased out in the first half of 2013 – to 42 rates. Rates are now published for 6 currencies (EUR, EUR same day, JPY, GBP, CHF, USD) and seven tenors (ON/SN, !W, 1M, 2M, 3M, 6M, and 12M). NYSE Euronext will be in charge of administration through its subsidiary NYSE Euronext Rates Administration Limited, starting from 2014.
6. LIBOR Conventions: For all the currencies apart from EUR and GBP, the period between the fixing date and the value date will be 2 London business days after the Fixing date. However, if the day is not a business day at the corresponding financial center, it will be adjusted to the following day at both London and the location – this date will be the new value date. The business day convention is modified following, and the EOM rule applies. For all currencies except GBP, the day count convention is Act/360. References =>

http://www.bbalibor.com/technical-aspects/fixing-value-and-maturity;
http://www.nyx.com/libor

Main IBOR-Indices

1. GBP LIBOR: The fixing date and the value date are the same (0 day spot lag). The day count convention is Act/365.
2. EUR LIBOR: The value date is 2 TARGET business days after the fixing date.
3. EURIBOR: The day count convention is Act/360 and the spot lag is 2 business days. The business day convention is *modified following* and the EOM rule applies. The related calendar is TARGET. There are 43 contributor banks, and the rates are published at 11:00 AM CET. Reference => http://www.euribor-ebf.eu/euribor-org/about-euribor.html
4. JPY TIBOR: TIBOR is the acronym for Tokyo InterBank Offered Rate. It is published by the Japanese Bankers Association. There are 2 types of TIBOR. The *Japanese Yen TIBOR* rates reflect the prevailing rates on the unsecured call market. The *EuroYen TIBOR* rates are the rates from offshore Japan market. The JBA TIBOR is calculated by the JBA as a prevailing market rate based on quotes for 13 different maturities (1W, 1M-12M) provided by banks as of 11:00 AM each business day. The day count convention is Act/365 for the domestic market and Act/360 for the EuroYen market. Reference => http://www.zenginkyo.or.jp/en/tibor/the-jba-tibor/

Other IBOR-Indices

1. AUD BBSW: The rate is Bank Bill Rates (BBSW) and is published by the Australian Financial Markets Association, and the maturities are 1M-6M. The day count convention is Act/365, and the spot lag is zero days. The business day convention is *modified following bi-monthly*. The rates are published at 10:00 AM. Reference => http://www.afma.com.au/data/bbsw.html

2. CAD CDOR: CDOR is the acronym for Canadian Dealer Offered Rate. CDOR is determined daily from a survey of 9 market makers' in bank acceptances (BA). The survey is conducted at 10:00 AM each business day, with the results being quoted by 10:15 AM on the same day. The day count convention is Act/365. The fixing date and the value date are the same (0 spot lag). Reference => http://www.m-x.ca/marc_terme_bax_cdor_en.php
3. DKK CIBOR: CIBOR is the acronym for Copenhagen InterBank Offered Rate. It is the reference interest rate for liquidity offered in the inter-bank market (in Denmark) on an uncollateralized basis for maturities from 1W-12M. NASDAQ OMX publishes CIBOR on a daily basis at 11:00 AM. The Danish Bankers' Association has the overall responsibility for CIBOR. The day count convention is Act/360. References => http://www.finansraadet.dk; http://nasdaqomxnordic.com/obligationer/danmark.cibor
4. HKD HIBOR: Hungarian InterBank Offered Rate. Act/365 and *Modified Following*.
5. INR MIFOR: MIFOR is the acronym for Mumbai Interbank Forward Offered Rate. The day count convention is Act/365 and the spot lag is 2 days. It is published for 1M, 2M, 3M, 6M, and 12M tenors.
6. NOK NIBOR: NIBOR is the acronym for Norwegian InterBank Offered Rate. The day count convention is Act/360 and the business day convention is *modified following*.
7. RMB SHIBOR: SHIBOR is the acronym for SHanghai InterBank Offered Rate. The day count convention is Act/360 and the spot lag is zero days. It is published for ON, 1W, 2W, 1M, 2M, 3M, 6M, 9M, and 12M tenors. Reference => http://www.shibor.org/shibor/web/html/index_e.html
8. SEK STIBOR: STIBOR is the acronym for STockholm InterBank Offered Rate. The day count convention is Act/360. The business day convention is *modified following*.
9. SGD SIBOR: SIBOR is the acronym for Singapore InterBank Offered Rate. The day count convention is Act/365. An individual ABS SIBOR contributor bank contributes the rate at which it can borrow funds were it to do so by asking for and accepting inter-bank offers in a reasonable market size, just prior to 11:00 AM. The indexes are computed by the Association of Banks in Singapore. Reference => http://www.abs.org.sg
10. SGD SOR: SOR in the acronym for Swap Offered Rate. It is implied from the USD LIBOR and the forex forwards. The indexes are computed by the Association of Banks in Singapore.

11. ZAR JIBAR: JIBAR is the acronym for Johannesburg InterBank Agreed Rate. The rate is calculated daily by SAFEX as the average prime lending rate quoted independently by a number of different banks. The rate is available in 1M, 3M, 6M, and 12M tenors.

Overnight Index Definitions

1. Setup: Overnight indexes are related to inter-bank lending on a one-day horizon. Most indexes are for overnight loans, and some are for tomorrow/next loans. The rates are computed as a weighted average of the actual transactions.
2. Main Currencies: To note: The publication lag is the number of days between the period start and the rate publication. A lag of 0 indicates that the rate is published on the start date, 1 indicates that the rate is published on the end date, and -1 indicates that the rate is published one day earlier.

Currency	Index Name	Reference	Convention	Lag
CHF	TOIS	TN	Act/360	-1
EUR	EONIA	ON	Act/360	0
GBP	SONIA	ON	Act/365	0
JPY	TONAR	ON	Act/365	1
USD	Fed Fund	ON	Act/360	1

3. Other Common Currencies:

Currency	Index Name	Reference	Convention	Lag
AUD	RBA ON/AONIA	ON	Act/365	0
CAD	CORRA	ON	Act/365	1
DKK	DNB TN	TN	Act/360	-1
CZK	CZEONIA		Act/360	
HKD	HONIX	ON	Act/365	0

HUF	HUFONIA	ON	Act/360	
INR	MIBOR ON	ON	Act/365	0
INR	MITOR	TN	Act/365	0
NZD	NZIONA	ON	Act/365	0
PLN	POLONIA	ON	Act/365	
SEK	TN SIBOR/STIBOR	TN	Act/360	-1
SGD	SONAR	ON	Act/365	0
ZAR	SAFEX ON Deposit Rate	ON	Act/365	
ZAR	SAONIA	ON	Act/365	

Overnight Index Committees and Meeting Dates

1. Meetings: Overnight rates are strongly influenced by central banks monetary policy decisions. The meeting dates of the main central banks can be found on the following sites:
    a. http://www.federalreserve.gov/monetarypolicy/fomccalendars.htm
    b. http://www.ecb.int/events/calendar/mgcgc/html/index.en.html
    c. http://www.bankofengland.co.uk/monetarypolicy/Pages/decisions.aspx
2. TN vs. Reference Lag: TN essentially refers to an overnight lending contract between $T + 1B$ and $T + 2B$, i.e., the TN contract. Therefore, these published rates correspond to a -1 publication lag; in fact, this is indeed the case for all the TN OIS contracts (CHF, DKK, SEK, etc.)
3. CHF TOIS: The reference rate used is the TOIS rate with TN inter-bank fixing. The index is calculated by Cosmorex AG, a division of Tullet Prebon.
4. EUR EONIA: EONIA is the acronym for Euro OverNight Index Average. It is computed as a weighted average of all the overnight unsecured lending transactions undertaken in the inter-bank market, initiated within the Euro area by the contributing banks (rounded to 3 decimal places). It is calculated by the European Central Bank. The rate is published in the evening

(CET 19:00) of the period start date. Day Count convention is Act/360. Reference => http://www.euribor-ebf.eu/euribor-eonia-org/about-eonia.html.

5. EUR EURONIA: This is the weighted average of all the unsecured Euro overnight cash transactions brokered in London by WMBA member firms between midnight and 16:15 CET with all the counterparts with a minimum deal size. Reference => http://www.wmba.org.uk/pages/index.cfm?page_id=32.
6. GBP SONIA: SONIA is an acronym for Sterling OverNight Index Average. It is the weighted average of all unsecured overnight sterling cash transactions brokered in London by WMBA member firms between midnight and 16:15 CET with all the counterparts with a minimum deal size of GBP 25 million (rounded to 4 decimal places). The rate is published in the evening (around 17:00 CET) of the period start date. The day count convention is Act/365. Reference => http://www.bba.org.uk/policy/article/sterling-overnight-index-average-sonia-a-guide/benchmarks
7. JPY TONAR Uncollateralized Overnight Call Rate: TONAR is an acronym for Tokyo OverNight Average Rate. It is the weighted average of all unsecured overnight cash transactions between financial institutions. The rate is published by Bank of Japan (BOJ), and the day count is Act/365. A provisional result is published on the evening (17:15 JST, except on the last business day of the month, when it is at 18:15 JST) of the period start. The final result is published on the morning (10:00 JST) of the end date. Reference => http://www.boj.or.jp/en/statistics/market/short/mutan
8. USD Effective Federal Funds Rate: The daily effective federal funds rate is a volume weighted average of the rates on trades arranged by the major brokers. The effective rate is calculated by the Federal Reserve Bank of New York using the data provided by the brokers and is subject to revision. The rate is published in the morning (between 7:00 AM and 8:30 AM EST) of the period end date. The day count is Act/365. Reference => http://www.newyorkfed.org/markets/omo/dmm/fedfundsdata.cfm
9. AUD RBA Interbank Overnight Cash Rate Survey AONIA: The rate is computed by the Reserve Bank of Australia (RBA). It is the weighted average of the rates at which a sample of banks transact in the domestic interbank market for overnight funds. The interbank overnight cash rate calculated from the survey is published on electronic media services at the conclusion of each trading day. The rate is published on the evening of the period start

date, and the day-count is Act/365. Reference => http://www.rba.gov.au/mkt-operations/tech-notes/interbank-survey.html

10. CAD CORRA: CORRA is the average for the Canadian Overnight Repo Rate Average. It is the weighted average of overnight general (non-specific) collateral repo trades that occur through designated inter-dealer brokers between 6:00 and 16:00 EDT on the specified date as reported by the Bank of Canada. The rate is published on the morning (9:00) of the end date, and the day count convention is Act/365. Reference => http://www.bankofcanada.ca/rates/interest_rates/money-market-yields
11. DKK Danmarks Nationalbank Tomorrow/Next: The TN money market rate is calculated and published by the Danmarks Nationalbank. The TN interest rate is an uncollateralized day-to-day interest rate for money market lending. Calculation of the TN interest rate is based on the daily reports from 11 banks. Each bank reports the day-to-day uncollateralized inter-bank lending and the average rates on these loans. The report is made with a time lag of one, e.g., Monday's lending is reported on Tuesday. The day-count convention is Act/360. Reference => http://www.nationalbanken.dk/dnuk/rates.nsf/side/reference_rates!opendocument
12. NZD NZIONA: The rate used is a reference rate equal to the official cash rate for that date set by RBNZ. It is publishes as of 10:00 AM Wellington time, and the day count is Act/365.
13. SEK SIOR/TN STIBOR: STIBOR (Stockholm Interbank Offered Rate) is a reference rate that shows an average of interest rates at which a number of banks active in the Swedish money market are willing to lend to one another without collateral at different maturities. The reference rate is SIOR or TN STIBOR. The rate is published by the OMX exchange. SIOR is a reference rate equal to the daily fixing for the Swedish krona tomorrow next deposits as published at 11:00 AM Stockholm time, on the day that is one Stockholm banking day prior to the start of the payment period. Reference => http://www.swedishbankers.se
14. SGD SONAR: The SONAR rate is published by the association of banks in Singapore. The rate is published at 11:00 AM Singapore time on the period start date, and the day count convention is Act/365.
15. ZAR - SFX ZAR OND: The rate SFX ZAR OND rate is published by SAFEX JIBAR. SAFEX publishes the rate which is the average rate that is receives on its deposits with the banks, weighted by the size of the investments placed at each bank.

16. ZAR - SAONIA: The SAONIA rate is the weighted average rate paid on unsecured, interbank, overnight funding. For more details see *South African Financial Markets (G West, Financial Modeling Agency, 2009)*.

Over-The-Counter Instruments

Forward Rate Agreement

1. Description: FRA's are OTC contracts linked to an IBOR-like index. At the trade date, a *Reference Rate* (R), a start period, and a *Reference Index* are all agreed to. The end period is equal to the start period plus the index tenor (i.e., a 6M start period and a 3M tenor result in a nominal 9M end period).
2. FRA Reference Period: The start of the reference period is computed from the spot date by adding the index spot lag, and then the spot tenor (using the business day convention and the calendar of the index). The reference period's end date is computed by adding the index spot lag and then the end period. The *Fixing Date* (or the exercise date) is the spot lag before the start date.
3. FRA Accrual DCF: The accrual factor between the start date and the end date (in the index day count) is denoted by δ. Occasionally the dates (and sometimes the accrual) described above are not calculated, but simply agreed upon arbitrarily by the counter-parties (usually changing the dates by 1-2 days for convenience and/or operational reasons).
4. FRA Settlement: The *FRA Settlement Date* is the start date (and NOT the end date). For the FRA buyer, the settlement day pay amount is

$$\delta\frac{L_\theta - R}{1 + \delta L_\theta}$$

   where L_θ is the value of the reference index on the fixing date, and R is the strike. The payoff for the FRA seller is obviously the same amount with the opposite sign.
5. FRA Accounting Treatment: In some accounting schemes, since the payment is always regarded as accruing between the start and the end dates, the FRA instrument is regarded as "being alive" even if it has already fully settled.

6. <u>FRA Period Mismatch</u>: The FRA's end date can be slightly different from the end date of the theoretical deposit underlying the IBOR rate. This potential mismatch comes from a mismatch comes from a difference in adjustment of the non-good business days between the different ways to compute the period.
7. <u>IMM FRA's</u>: FRAs can also be traded as IMM FRAs, i.e., FRAs with accrual dates equal to consecutive IMM dates (just as in STIR futures). The underlying IBOR rate has a tenor the one relevant to the IMM dates' frequency (3M IBOR for the quarterly dates and 1M IBOR for monthly dates).

Interest Rate Swaps

1. <u>Interest Rate Swaps (Fixed for IBOR)</u>: Exchange of fixed payments for a floating stream of payments linked to an IBOR-like index. Typical payment/accrual periods and their generation rules are outlined below, and, since these are OTC, any variant is possible if agreed by the counterparts.
2. <u>Forward Date Generation</u>: The dates are computed from the start (or the settlement) date. The last date of the stream will be the start date plus the total length (maturity tenor) of the leg. The intermediate dates are regularly spaced, and the first one is the non-standard period. For example, a 15-month leg with a 6-month period can pay after 3 ($= 15 - 2 \times 6$), 9, and 15 months.
3. <u>IRS - Date Adjustment</u>: Dates adjustment is done by the business day convention and the EOM rule. All the dates are first computed without adjustment, and then all the dates are adjusted.
4. <u>IRS Stubs</u>: The non-standard period is referred to as the stub. It can be short (shorter than one period) or long (between one and two periods in length). The reason that the non-standard period is the first one is that once that period is finished, the instrument then has regular periods (similar to a standard one). If the stub was the last period, the swap could never become a standard one.
5. <u>Start Date</u>: The start (or the settlement) date of the swap is usually a certain lag called the spot lag after the trade date. The most common lag is 2 business days. The start date can also

be a forward date. In that case, the start date is the forward period tenor plus the spot lag. The forward period tenor is specified W/M/Y.

6. Payer (Buyer) and Receiver (Seller) Swaps: The terms *Payer* and *Receiver* refer to the fixed leg of an IRS. The swap is a payer for one party if the swap pays the fixed leg and receives the floating leg. The payer swap for one party is the receiver swap for the other. Like FRA, the swap buyer buys the floating leg by paying the fixed.

Vanilla IRS

1. Notional and Coupon: In a vanilla IRS, all the coupons have the same notional, and the coupons on the fixed leg have the same rate.
2. Payments: The payments on the fixed leg are regularly placed, most with a 6M/12M gap. The payments on the floating leg are also regularly spaced, most with the 3M/6M gap. The fixing date for the floating payment is the index spot lag before the period start date. This lag is usually part of the index, and usually the same as the swap start spot lag.
3. Floater/index mismatch: The dates of the period corresponding to the deposit underlying the IBOR index can be slightly different from that of the coupon period. The difference is created by the adjustments due to non-good business days.
4. Main Currencies:

Currency	**Spot**	**Fixed**		**Floating**		
	Lag	**Period**	**Convention**	**Index**	**Period**	**Convention**
USD (NY)	2	6M	30/360	LIBOR	3M	Act/360
USD (London)	2	1Y	Act/360	LIBOR	3M	Act/360
EUR: 1Y	2	1Y	30/360	EURIBOR	3M	Act/360
EUR: > 1Y	2	1Y	30/360	EURIBOR	6M	Act/360
GBP: 1Y	0	1Y	Act/365	LIBOR	3M	Act/365
GBP: > 1Y	0	6M	Act/365	LIBOR	6M	Act/365
JPY	2	6M	Act/365	TIBOR	3M	Act/365
JPY	2	6M	Act/365	LIBOR	6M	Act/360

CHF: 1Y	2	1Y	30/360	LIBOR	3M	Act/360
CHF: > 1Y	2	1Y	30/360	LIBOR	6M	Act/360

5. Other Currencies:

Currency	**Spot**	**Fixed**		**Floating**		
	Lag	**Period**	**Convention**	**Index**	**Period**	**Convention**
AUD: 1Y – 3Y	1	3M	Act/365	BBSW	3M	Act/365
AUD: >= 4Y	1	6M	Act/365	BBSW	6M	Act/365
DKK	2	1Y	30/360	CIBOR	6M	Act/360
INR: <= 1Y	2	1Y	Act/365	MIFOR	3M	Act/365
INR: > 1Y	2	6M	Act/365	MIFOR	6M	Act/365
HKD	0	3M	Act/365	HIBOR	3M	Act/365
NOK	2	1Y	30/360	NIBOR	6M	Act/360
HZD	0	6M	Act/365	BKBM	3M	Act/365
PLN	2	1Y	Act/Act ISDA	WIBOR	6M	Act/365
SEK	2	1Y	30/360	STIBOR	6M	Act/360
SGD	2	6M	Act/365	SIBOR	6M	Act/365
SGD	2	6M	Act/365	SOR	6M	Act/365
ZAR	0	3M	Act/365	JIBAR	3M	Act/365

6. Composition - Multi-reset Swaps: In some cases, the period between the payments is equal to the IBOR index, but a multiple thereof. The fixing rates are compounded over the sub-periods up to the payment at the end. The main currency for which this is a standard for vanilla swaps is CAD.
7. Reset Periods - Compounding: The description of the compounded coupon is as follows: The associated times are denoted $\{t_i\}_{i=0}^{n}$. The fixing for the period $[t_{i-1}, t_i]$ is denoted by $\{r_i\}_{i=1}^{n}$, and the corresponding accrual fraction in the index convention is δ_i. The fixing takes place at a date typically prior to the start of the accrual period, with the difference between the fixing date and the start date being referred to as the spot lag.

a. Geometric Compounding => The coupon pays at t_n the amount (to be multiplied by the notional)

$$\prod_{i=1}^{n}(1+\delta_i r_i)-1$$

b. Compounded Spread => In case a spread is agreed to on the compounding leg, there are 3 standard ways to deal with the compounded spread: *COMPOUNDING*, *FLAT COMPOUNDING*, and *COMPOUNDING WITH SPREAD AS A SIMPLE INTEREST*. These methods are described in the ISDA document *Alternate compounding methods for the OTC derivative transaction (2009)*.

c. CAD Swap => Multi-reset Composition:

Currency	**Spot**	**Fixed**		**Floating**		
	Lag	**Period**	**Convention**	**Index**	**Period**	**Convention**
CAD: 1Y	0	1Y	Act/365	CDOR	1Y	Act/365
CAD: >= 1Y	0	6M	Act/365	CDOR	6M	Act/365
CNY	2	3M	Act/365	CNY-Repo	3M	Act/365

8. IMM Dated Swap: Like for FRA, there exists the IMM dates IRS. These swaps pay the fixed and the floating legs at the IMM dates. The most common are the quarterly IMM dates on the floating legs based on the IBOR 3M rates. It is also common that the fixed leg payment is every 2nd (semi-annual) to 4th (quarterly).
9. In-arrears Swap: Another type of IBOR swaps is a swap with fixing in-arrears. Here the reference period for the IBOR index and the accrual period for the coupon are disjoint, i.e., the accrual period precedes the IBOR period. Thus, the start date of the IBOR period is the payment date. The fixing date for the floating payment is the index spot lag before the accrual period end date.
10. Short and Long Tenors: For some swaps, the period between the payments is not equal to the index tenor. The payment period can be shorter than the index period (the short tenor swap), or longer (long tenor swap). Typically this type of swap has a 3M pay accrual period on 6M-

12M IBOR index (short) or an annual pay accrual period on a 3M-6M IBOR index (long). The long/short tenor swap can also be of the (fixing) in-advance or the in-arrears type.

11. Step-up and step-down: The rate paid on the fixed leg coupons does not need to be the same for each coupon. The swap is then referred to as a step-up when the coupons increase and step-down when they decrease.
12. Amortized, Accruing, and Roller Coaster Swaps: The coupon notional does not need to be the same for all the coupons. In most cases, the coupons are the same for both the legs of the period. If the notional is decreasing over time, the swap is called *Amortizing Swap*, and if is increases it is called *Accruing*. If the notional first increases and then decreases up to maturity, it is called a *Roller Coaster*.

Interest Rate Swaps (Basis Swaps: IBOR for IBOR)

1. Description: In a basis swap, both legs are floating legs and depend on an IBOR index in the same currency. In most cases, the indexes have different tenors. A spread above the IBOR index is paid in one of the legs. The quote convention used quotes the spread the spread over IBOR on the shorter tenor leg, in such a way that the spread is positive.
2. EUR Basis Swap: In EUR alone, the basis swap is quoted as 2 different swaps. For e.g., the quote of EURIBOR 3M vs. EURIBOR 6M at 12 bp has the following meaning: In the first swap you receive a fixed rate and pay 3M EURIBOR. In the second swap you pay the same fixed rate plus a spread of 12 bp and receive 6m EURIBOR. Note that with this convention the spread is paid on an annual basis, like the standard fixed leg of a fixed-float IBOR swap. Even if the quote refers to the spread of a 3M vs. 6M swap, the actual spread is paid annually with a fixed leg convention.
3. Compounded Basis Swap: The multi-reset composition functionality described above is not restricted to fix-float IBOR swaps alone. Some basis swaps are also traded on a compounded basis to align the payments on both the legs. For example, a basis 1M LIBOR vs. 3M LIBOR swap can be quoted with the 1M LIBOR compounded over 3 periods and paid quarterly in line with the 3M period. Note that the exact convention for the spread convention needs to be

indicated for the trade. The multi-reset composition of the shorter leg is currently the standard in USD.

Cross-Currency Swap, IBOR for IBOR

1. Description: Here the notional is not the same in both legs as they are in different currencies. The notional in one leg is usually the notional on the other translated to the other currency through an exchange rate. The rate is often an exchange rate at the moment of trade as agreed between the parties. The notional is paid out on both the legs, at the start and at the end of the swap.
2. Non-MTM FX Cross-Currency Swap: In some cases, the FX rates used are not in line with the market rates. Often this can be abused to disguise some debts from the accounting rules. These types of cross-currency swaps at non-market exchange rates were famously used by Greece to hide some of its debts when it entered the Euro. The swaps used for construction are at-the-money exchange rates (although initially set, therefore non-MTM).
3. MTM Cross-Currency Swap: There also exist cross-currency swaps with the FX rate reset, and this is called the *Cross-Currency Mark-to-Market Swap*. They are specified in article 10 of the 2006 ISDA definitions. For each period, the FX rate that is observed at the beginning of the period is used for the following period. The notional of one of the legs is unchanged, and the other is adapted according to the new exchange rate. At each payment date and MTM amount is paid. The amount is calculated as the new notional in the adapted leg minus the previous notional. This is equivalent, up to netting, to exchanging notionals at the start and at the end of each period. This feature was introduced to reduce the credit risk induced by the movement of the FX rates.
4. Cross-Currency Swap Stream Construction: Bothe legs of a cross-currency swap are linked to an IBOR like index. In the standard cross-currency swaps, the IBOR tenor on both the legs is the same. The payments are done on the same day for both the legs to reduce the credit risk. This means that the payment calendar is the joint calendar of both the swaps involved in the swap.

5. Typical Cross Currency Swaps: The most liquid cross-currency swaps exchange 3 month payments. Even if the index of one of the currency is 6M as its most commonly used one, the cross-currency swaps may still use 3M payments. This is in particular the case with USD/JPY and USD/EUR swaps, and these use 3M payments, even if 6M EUR EURIBOR and 6M JPY LIBOR are the standard floating references for these currencies.
6. Spreaded Cross-Currency Swaps: The cross-currency swaps also pay a spread on one of the legs. On which currency leg the spread is paid out depends on the currency pairs. When one of the currencies is USD, the convention is usually USD LIBOR flat vs. the other currency plus a spread. The 2 exceptions to this rule are – USD/MXN Peso Swaps and USD/CLP Chilean Peso Swaps.

Constant Maturity Swaps

1. Description: Constant Maturity Swaps are in some sense similar to the standard IRS, in that the swap is composed of 2 legs, and each leg has its own payment type. One leg is typically a fixed leg or an IBOR leg. The other is a floating leg, the rate of which is based on a swap index.
2. Key Features: The difference to that of a standard IBOR leg is that rate of the index period can be very different from the period on which it is paid. The CMS floating leg usually pays on a quarterly or on a semi-annual basis a swap rate. The most popular swap indexes are indices based on the 2Y, the 5Y, the 10Y, the 20Y, and the 30Y swaps.
3. CMS Fixing: The details of the fixing and the payment are similar to that of the IBOR coupons. The coupon fixing can be in-advance or in-arrears. For the fixing in advance, the fixing takes place at the start of the accrual period. For the fixing in-arrears, the fixing takes place at the end of the accrual period. The lag between the reference rate and the fixing is that spot lag of the swap index.
4. EUR CMS: In EUR, the most common CMS have quarterly payments on both legs. The non-CMS leg is 3M EURIBOR.
5. Swap Rate Fixings for the Main Currencies:

Currency	Spot	Fixed		Floating			
	Lag	Period	Convention	Index	Period	Convention	Fixing Time
EUR: 1Y	2	1Y	30/360	EURIBOR	3M	Act/360	11:00 CET
EUR: > 1Y	2	1Y	30/360	EURIBOR	6M	Act/360	11:00 CET
EUR: 1Y	2	1Y	30/360	EURIBOR	3M	Act/360	12:00 CET
EUR: > 1Y	2	1Y	30/360	EURIBOR	6M	Act/360	12:00 CET
EUR: 1Y	2	1Y	30/360	LIBOR	3M	Act/360	10:00 GMT
EUR: > 1Y	2	1Y	30/360	LIBOR	6M	Act/360	10:00 GMT
EUR: 1Y	2	1Y	30/360	LIBOR	3M	Act/360	11:00 GMT
EUR: > 1Y	2	1Y	30/360	LIBOR	6M	Act/360	11:00 GMT
USD	2	6M	30/360	LIBOR	3M	Act/360	11:00 EST
USD	2	6M	30/360	LIBOR	3M	Act/360	15:00 EST
GBP: 1Y	2	1Y	Act/365	EURIBOR	3M	Act/365	12:00 CET
GBP: > 1Y	2	6M	Act/365	EURIBOR	6M	Act/365	12:00 CET
CHF: 1Y	2	1Y	30/360	LIBOR	3M	Act/360	10:00 GMT
CHF: > 1Y	2	1Y	30/360	LIBOR	6M	Act/360	10:00 GMT

Swap Indexes

1. Introduction: The most common usage of these indexes is in CMS and CMS caps/floors. Swap rates for CHF, EUR, GBP, JPY, and USD are established by ISDA in co-operation with Reuters (now Thomson Reuters) and InterCapital Brokers (ICAP). The main fixing details are shown in the table above.
2. ISDA EUR: There are 4 daily fixings – 2 for swaps vs. LIBOR, and 2 for EURIBOR. The LIBOR fixings are at 10:00 GMT and 11:00 GMT. The EURIBOR fixings are at 11:00 CET and 12:00 CET. The maturities are 1Y-10Y, 12Y, 15Y, 20Y, 25Y, and 30Y. All are 6M tenor swaps, except for the 1Y maturity which is 3M.
3. ISDA USD: There are 2 fixings – at 11:00 EST and 15:00 EST. The maturities are 1Y-10Y, 15Y, 20Y, and 30Y. All swaps are vs. 3M LIBOR.

4. ISDA GBP: There is one fixing, at 11:00 GMT. The maturities are 1Y-10Y, 12Y, 15Y, 20Y, 25Y, and 30Y. All the swap fixings are quoted for 3M , except for the 1Y which is for 6M.
5. ISDA CHF: There is one fixing at 11:00 GMT. The maturities are 1Y-10Y. All swaps are vs. 6M except for the 1Y which is vs. 3M.
6. ISDA JPY: There are 2 fixings – at 10:00 and at 15:00 Tokyo time. The maturities are 1Y-10Y, 12Y, 15Y, 20Y, 25Y, 30Y, 35Y, and 40Y. All the swaps are vs. 6M. Note that for the JPY there is also an 18M fixing.

Overnight Indexed Swaps

1. Description: The overnight indexed swaps (OIS) exchange a leg of fixed payments for a leg of floating payments. The start (or the settlement) date of the swap is a certain lag (the spot lag) after the trade date. The most common lag is 2 business days.
2. Payments: The payments on the fixed leg are regularly spaced by the given period. Most of the OIS have a single payment if the maturity is shorter than 1Y, and a 12M period for longer swaps. The payments on the floating legs are also regularly spaced, usually on the same dates as the fixed leg. The amount paid on the floating leg is computed by compounding the rates.
3. Conventions for the Main Currencies:

Currency	**Spot**	**Fixed**		**Floating**		
	Lag	**Period**	**Convention**	**Index**	**Pay Lag**	**Convention**
USD <= 1Y	2	tenor	Act/360	Fed Fund	2	Act/360
USD > 1Y	2	1Y	Act/360	Fed Fund	2	Act/360
EUR: <= 1Y	2	tenor	Act/360	EONIA	2	Act/360
EUR: > 1Y	2	1Y	Act/360	EONIA	2	Act/360
GBP: <= 1Y	0	tenor	Act/365	SONIA	1	Act/365
GBP: > 1Y	0	1Y	Act/365	SONIA	1	Act/365
JPY: <= 1Y	2	tenor	Act/365	TONAR		Act/365
JPY: > 1Y	2	1Y	Act/365	TONAR		Act/365

4. Conventions for the Other Currencies: Pay Lag => The lag in days between the last fixing publication and the payment.

Currency	Spot	Fixed		Floating		
	Lag	Period	Convention	Index	Pay Lag	Convention
AUD <= 1Y	1	tenor	Act/365	RBA ON	1	Act/365
AUD > 1Y	1	1Y	Act/365	RBA ON	1	Act/365
CAD: <= 1Y	0	tenor	Act/365	CORRA	0	Act/365
CAD: > 1Y	0	1Y	Act/365	CORRA	0	Act/365
INR: <= 1Y	1	tenor	Act/365	ON MIBOR	1	Act/365
INR: > 1Y	1	6M	Act/365	ON MIBOR	1	Act/365
SGD: <= 1Y	2	tenor	Act/365	SONAR	1	Act/365

5. Payment Calculation: Let

$$0 < t_0 < t_1 < \cdots < t_n < t_{n+1}$$

be the relevant dates (all good business days) in the composite floating leg period. Let δ_i be the accrual factor between t_i and t_{i+1} $(1 \leq i \leq n)$ and δ the accrual factor for the period $[t_i, t_{n+1}]$. The overnight rates paid between t_i and t_{i+1} are given at t_i by ${F_i}^{ON}$. The paid amount is

$$\prod_{i=1}^{n}\left(1 + \delta_i {F_i}^{ON}\right) - 1$$

multiplied by the notional. The payment is usually not done at the end of the period t_{n+1}, but at a certain lag after the last fixing publication date. The reason for the lag is that the actual amount is known only at the very end of the period; the payment lag allows for a smooth settlement.

6. USD OIS: In USD the payment is 2D after the end of the fixing period. These two days are computed from the final publication date, which is at the end of the last period, plus 2 lag days.
7. EUR OIS: In EUR, the payment is 1D after the end of the fixing period. This one day is computed as the final publication date, which is the start of the previous period and 1D before the end of the previous period, plus 2 lag days.
8. OIS Committee Meetings: A somehow popular choice of start or end dates for the OIS Swaps are the dates of the relevant committee meetings, as shown earlier in the table.
9. Federal Funds Swaps: Federal Fund Swaps are a USD peculiarity. These are swaps exchanging quarterly USD LIBOR payments for the quarterly average of the effective USD Federal Funds Rate. They are often called as the Feds or the FED Swaps.
    a. Arithmetic Compounding => The particularity is that the rates paid is an arithmetic average of the Fed Funds Rates; the rates are not compounded as in the traditional OIS. The quarterly coupon payment is not equal to a 3M OIS.
    b. Fed Fund Coupon Calculation => Let

    $$0 < t_0 < t_1 < \cdots < t_n < t_{n+1}$$

    be the relevant dates (all good business days) in the composite floating leg period. Let δ_i be the accrual factor between t_i and t_{i+1} $(1 \leq i \leq n)$ and δ the accrual factor for the period $[t_i, t_{n+1}]$. The overnight rates paid between t_i and t_{i+1} are given at t_i by ${F_i}^{ON}$. The paid amount is

    $$\sum_{i=1}^{n} \delta_i {F_i}^{ON}$$

    multiplied by the notional.
    c. Fed Funds Final Fixing => The final Fed Funds effective fixing is applied to the last 2 fixing days. In terms of the above formula, it needs to be re-cast as

$$\sum_{i=1}^{n-1} \delta_i F_i{}^{ON} + \delta_n F_{n-1}{}^{ON}$$

It is possible to trade absent the rate cut-off, but this requires the counterparty to make payment on the same day the last fixing information is published.

d. Fed Funds vs. LIBOR Swaps => Here, the swaps are quoted with a spread over the Fed Funds ON leg. A quote of s often in bp means that the swap exchanges LIBOR for ON average plus a spread of s. The spread is usually positive. The computation of the interest on the ON floating leg is additive with simple compounding, and the spread is also additive with simple compounding. The multiple compounding alternatives as in IBOR compounding are not present here. The payment is (excluding the final day repeated fixing)

$$\sum_{i=1}^{n-1} \delta_i F_i{}^{ON} + \delta s$$

In some cases, the fed fund swaps are traded against 1M LIBOR. This type of swaps is less liquid.

10. OIS Indexes: The OIS Index Rates are reference rates to the standard OIS. As an example, the EONIA index is the average rate of rates provided by the prime banks to 3 decimal places that each panel bank believes is the mid-market rate of EONIA swap quotations between prime banks. It is quoted for spot value ($T + 2B$) and on an Act/360 basis with annual payments. The fixing time is 11:00 CET. The index covers swaps from 1W to 24M. The indexes are computed by the EURIBOR-EBF association, and were launched in 2005. Reference => http://www.euribor-ebf.eu/eoniaswap-org/about-eoniaswap.html

Swaption

1. Description: A swaption is an option on a swap. It is characterized by an exercise date and an underlying swap. The exercise date is on or before the swap start date. The option gives its holder the right (but not the obligation) to enter in the underlying swap. In theory an option can be written on any underlying swap. In practice, a large majority of swaptions are written on the vanilla interest rate swap.
2. Strike: The strike of an option is the common fixed rate across all the fixed leg coupons. If the underlying swap has a different rate for each coupon (in a step-up or step-down swap, for example), the strike is ill-defined (at least as in the “common strike”).
3. Payer/Receiver Swaptions: The term payer/receiver swaption refers to the payer/receiver of the underlying swap. A swaption is a payer/receiver swaption if the party long the option has the right to enter into a payer/receiver swap. Note that the payer/receiver indicator refers to the long party. Thus, if one is short a receiver option and the swaption is exercised, he enters into a payer swap (a receiver swap for the other party which is long the option). A payer swaption for one party is the receiver swaption for the other.
4. Swaption Dates: A swaption exercise date and its underlying swap start date are computed as follows for the standard swaptions. The swaption is described by an exercise tenor and an underlying swap tenor (like, as in, 6M by 10Y). The exercise date is computed as the spot date plus an exercise tenor, using the relevant calendar and the business day convention of the underlying swap. The swap settlement date is computed as the exercise date plus the underlying swap (or the underlying swap index) spot lag.
5. Swaption Settlement Conventions:

Currency	**Method**	**Sub-method**	**Expiry**
EUR	Cash Settled	IRR	11:00 CET
GBP	Cash Settled	IRR	11:00 GMT
CHF	Cash Settled	IRR	11:00 CET
DKK	Cash Settled	IRR	11:00 CET
NOK	Cash Settled	IRR	11:00 CET
SEK	Cash Settled	IRR	11:00 CET
USD	Cash Settled	Exact Curve	11:00 EST
JPY	Physical Delivery		17:00 Tokyo

AUD Physical Delivery

6. Physical Delivery Swaptions: When the swaption is settled with physical delivery, at the exercise date the parties enter into an actual swap (the underlying swap).
7. Cash Settled Swaptions: When the swaption is cash-settled, a cash amount is paid (by the short party to the long party) at the exercise date (or, more precisely, spot lag after the exercise), and the actual swap is not entered into.
8. Yield Settled Swaptions: The cash amount to be paid to the long party is computed from a swap fixing rate using a conventional valuation formula of the theoretical underlying swap. The valuation is done using the swap fixing rate as the IRR for the swap. The cash-settled swaption can only be written on a vanilla swap with the standard convention. This is the standard convention for EUR and GBP.
    a. Yield Settlement – Computation => For a swaption with strike K and maturity M, the amount paid for a fixing s is $G(s)$ where m is the cash-annuity

$$G(s) = \sum_{i=1}^{mM} \frac{1/m}{\left(1 + \frac{1}{m}s\right)^i}$$

    where m is the number of payments per year.
9. Cash Settle Swaption – Exact Curve: The term cash-settled can also refer to another way to compute the cash amount. This approach is usually used for the USD cash-settled swaptions. The cash amount to be exchanged is explicitly calculated as the value of the underlying swap. To value the swap, a full yield curve (not just the fixing swap rate) has to be agreed to by the parties.
10. Upfront and Forward Premium: The standard for the options has been for a long time *Spot* payment. The premium relative to the option paid by the buyer to the seller was done at the spot date from the trade date. With the crisis that started in 2007, the credit risk awareness increased, and most of the dealers decided to change the standard to a forward premium. Since September 2010, in the main currencies, the premium is paid at the same date the swaption itself is settled. This is in general from the spot date from the exercise date.

Forex and Forward Swaps

1. Description: A forex swap is essentially a contract on the interest-rate differences, and therefore similar to a cross-currency swap. The conventions on these transactions are similar to the conventions on interest-rate swaps.
2. Currency Pair Order: FX rates are usually quoted for the currency pairs in the conventional order. For the main currencies, the orders are: EUR/USD, GBP/USD, JPY/USD, and GBP/EUR. The first currency in these pairs is called the *Base Currency*, and the second one is called the *Quote Currency*.
3. Conventional Currency Strength Table:
    a. Strength #1 => EUR
    b. Strength #2 => GBP
    c. Strength #3 => AUD
    d. Strength #4 => NZD
    e. Strength #5 => USD
    f. Strength #6 => CAD
    g. Strength #7 => CHF
    h. Strength #8 => JPY
    i. Strength #9 => Other
4. FX Forward: An FX/Forex forward is simply another FX transaction taking place at a forward date. The payments are in one amount in one currency vs. another amount in another currency. The amount in the other currency is the base currency multiplied by the exchange rate agreed. The rate is often quoted in 2 parts – the spot rate and the *Forward Rate*.
5. FX Swap: The FX/Forex Swap is the exchange of an FX spot and an FX forward. An FX spot rate and an FX forward rate are first agreed to. The signs of the spot and the forward amount in the same currency are opposite. For e.g., in the following EUR/USD trade, the jargon used for FX swap trade would be something like: *I buy spot and sell forward 3M EUR vs. USD for 10m with 10 (forward) points and a spot of 1.25*. This means that on the spot date I receive € 10m and pay \$ 12.5m, and at $T + 3M$ I pay € 10m and receive

$(1.25 + 0.0010) \times \$\ 10m = \$\ 12.51m$. The Spot part is called the *Near Leg* and the Forward part is called the *Far Leg*.

6. FX Building Blocks: An FX spot is a pure currency trade. The FX swap is mainly an interest rate trade, it is a trade on the differences between interest rates in the 2 currencies. As the amounts in each currency are paid and received, there is almost no currency exposure, similar to the cross-currency swap with initial and final exchange of notional. In the Forex market, the trader sees an FX forward as a net between the FX spot and the FX swap, as opposed visualizing the FX Swap as a combination of the FX spot and the FX forward. The FX forward mixes up the currency exposure and the rate exposure, and is therefore not looked at as a building block.
7. Forward Point Quotation Factors: For most of the currencies, the forward point code is the same as the currency code. This is not the case, however, for non-deliverable currencies. The Bloomberg code is built from the prefix in the table below, the maturity (1W, 1M, 2M etc.), and the postfix _CRNCY.

Base Currency	**Other Currency**	**Factor**	**BBG Code Prefix**
AUD	EUR	10,000	AUDEUR
AUD	EUR	10,000	AUD
EUR	GBP	10,000	EURGBP
EUR	JPY	100	EURJPY
EUR	USD	10,000	EUR
GBP	JPY	100	GBPJPY
GBP	USD	10,000	GBP
USD	BRL	10,000	BCN
USD	CAD	10,000	CAD
USD	CHF	10,000	CHF
USD	CNY	1	CCN
USD	EGP	10,000	EPN
USD	HUF	100	HUF
USD	INR	100	INR
USD	JPY	100	JPY

USD	KRW	1	KRW
USD	MXN	10,000	MXN
USD	PLN	100	PLN
USD	TRY	100	TRY
USD	TWD	1	TWD
USD	ZAR	10,000	ZAR

8. Forward Points: The *Forward Points* are quoted for the currency pairs in the conventional order. The mechanism of forward points is the same as that for FX forward and FX swap. The points are added to the FX spot rate to obtain the FX forward rate. For a spot rate s and points p, the forward rate is $s + p$. The points are usually quoted with a conventional factor (just as the interest rates, which are quoted in percentage). The factor is dependent upon the currency pair, as can be seen in the table above.

Exchange Traded Instruments

Introduction

1. Exchange Month Codes: The exchange traded instruments with a regular schedule (like futures) use the following code to refer to the corresponding months:
    a. January – F
    b. February – G
    c. March – H
    d. April – J
    e. May – K
    f. June – M
    g. July – N
    h. August – Q
    i. September – U
    j. October – V
    k. November – X
    l. December – Z
2. Contract Event Dates:
    a. Expiry Date.
    b. Last Trading Date.
    c. Final Settlement Date.

In addition, for contracts with *Physical Deliverables*, the following additional dates apply:

    d. First Delivery Date.
    e. Final Delivery Date.
    f. Delivery Notice Date.

Typically all the dates are computed from a single date pivot (often the expiry date pivot).

Overnight Futures

1. Overnight Index Linked Futures: The overnight index futures are linked to an average of overnight rates over a certain period (usually a calendar month). Here we look at in detail at the Fed Fund Futures, 1M EONIA Indexed Futures, and 1D Interbank Deposit Futures Contract – Brazil. All these are exchange traded.
2. Federal Funds Futures: The 30D Federal Funds Futures (called Fed Funds Futures) are based on the monthly average of the overnight Fed Funds rate for the contract month. The notional is $5m. The contract months are the first 36 calendar months. They are quoted on CBOT for USD.
    a. Pricing => Let

$$0 < t_0 < t_1 < \cdots < t_n < t_{n+1}$$

    be the relevant dates for the Fed Funds Futures, with t_1 being the first business day of the month, t_{i+1} the business day following t_i, and t_{n+1} the first business day of the following month. Let δ_i be the accrual factor between t_i and t_{i+1} $(1 \le i \le n)$ and δ the day count fraction for the full period $[t_1, t_{n+1}]$. The day count convention for USD ON rate is Act/360. If the overnight rate between t_i and t_{i+1} is given at t_i as ${F_i}^{ON}$, the futures price on the final settlement date t_{n+1} is

$$\Phi_{t_{n+1}} = 1 - \frac{1}{\delta}\left[\sum_{i=1}^{n} \delta_i {F_i}^{ON}\right]$$

    The margining is done on the price multiplied by the notional, and described by the monthly accrual fraction (i.e. $\frac{1}{12}$).
3. 1M EONIA Indexed Futures: The contract was introduced in 2008, and is traded on LIFFE. The notional is €3m and the underlying overnight rate is EONIA. The delivery month covers a single ECB Reserve Maintenance Period. The number of available delivery months will be

limited to the number of Reserve Maintenance Periods for which dates have been published by the ECB.

a. Pricing => The Exchange Delivery Settlement Price (EDSP) is one minus the EDSP Rate. The EDSP Rate is calculated as

$$\frac{1}{\delta}\left[\prod_{i=1}^{n}\left(1+\delta_i {F_i}^{ON}\right)\right]-1$$

Reference => https://globalderivatives.nyx.com/contract/content/29179/contract-specification

4. <u>1D InterBank Deposit Futures Contract – Brazil</u>: They are also called 1D Futures, and are traded in BM&FBovespa. The underlying is the daily interest rate compounded until the contracts' expiration date. The rate is the Average 1D InterBank Deposit Rate (1D) as calculated by CETIP.

a. Contract Details => The *Expiration Date* is the first business day of the contract. The *Last Trading Day* is the business day preceding the expiration date. The quotations are expressed as a rate per annum compounded daily based on a 252-day year to 3 decimal places. This the *Trading Price* is related to the *Quoted Rate* r as

$$\wp=\frac{100{,}000}{(1+r)^{\frac{n}{252}}}$$

where n is the number reserves between the trade date and the day preceding the expiration date.

b. Margining => On the trade date t, the margin is computed as (to be multiplied by the real value and the number of contracts)

$$\mathcal{M}=\mathbb{PA}_t-\mathbb{PO}$$

where $\mathbb{PA}_t$ is the contract settlement price on t, and $\mathbb{PO}$ is the initial price paid for the contract. The 1D margining increment can be computed as

$$\Delta\mathcal{M} = \mathbb{PA}_t \times (\mathbb{PA}_{t-1} \times \mathbb{FC}_t)$$

where $\mathbb{FC}_t$ is the indexation factor estimates as

$$\mathbb{FC}_t = [1 + \mathbb{DI}_{t-1}]^{1/252}$$

where $\mathbb{DI}_{t-1}$ is the 1D rate corresponding to the period $[t-1, t]$.

Short-Term Interest Rate (STIR) Futures

1. IBOR Based STIR Futures: IBOR-based STIR Futures are also called Interest Rate Futures. The settlement mechanisms are common across all currencies, but the STIR contract details differ on the notional, the underlying rate index, and the exchange on which they are quoted. He dates relevant to the futures are based on the 3rd Wednesday of the month (and adjusted to the following day if the Wednesday is not a good business day) – this date is the *Start Date* of the IBOR rate underlying the future.
2. STIR Future Contract Dates: The rate is fixed at a *Spot Lag* prior to the *Start Date* (using the currency spot lag seen earlier). The fixing, thus, usually takes place on the Monday or on the Wednesday itself. The fixing date is also the *Last Trading Date* for the future. The *End Date* of the IBOR period is usually 1-3 months after the *Start Date* depending on the type of the future (using the conventions associated with the relevant IBOR index).
3. STIR Futures – Margining: The margining process works in the following way. For a given closing price (as published by the exchange), the daily margin paid is that price minus the *Reference Price* multiplied by the notional and the accrual factor of the future. Equivalently, it is the price difference multiplied by 100 *and* by the *Point Value* (*Point Value* being the margin that results from a 1% change in the price). The reference price is the trade price on the trade date and the previous closing price on subsequent dates.

4. <u>STIR Futures - Prices/Ticks</u>: The futures price at t is denoted by Φ_t^j. On the fixing date, at the moment of publication of the underlying IBOR rate L_t^j, the future price is

$$\Phi_t^j = 1 - L_t^j$$

Before that moment, the price evolves according to the market dynamics including bid/offer. The *Tick Value* is the value of the smallest increment in price. The price usually changes in $\frac{1}{2}$ or 1 bp increments.

5. <u>STIR Futures - Designation Codes</u>: The futures are designated by character codes. The first part is dependent upon the data provider, and is usually 2-4 characters. The second part describes the month (from the month code table), and the year, with its last digit. As interest rate futures are quoted 10 years only, there is no ambiguity in using only one digit for the year. Note also that this means that when a future reaches its last trading date, a new one is created a couple of days later with the same name, but for a 10Y maturity in the future.

Currency Specific Futures

1. <u>Main Currencies</u>:

Currency	Tenor	Exchange	Underlying	Notional
CHF	3M	LIFFE	LIBOR	CHF 1m
EUR	3M	EUREX	EURIBOR	€ 1m
EUR	3M	LIFFE	EURIBOR	€ 1m
EUR	3M	NLX	EURIBOR	€ 1m
GBP	3M	LIFFE	LIBOR	£ ½ m
GBP	3M	NLX	LIBOR	£ ½ m
JPY	3M	SGX/CME	TIBOR	JPY 100 m
JPY	3M	SGX	LIBOR	JPY 100 m

USD	3M	CME	LIBOR	$ 1m
USD	1M	CME	LIBOR	$ 3m
USD	3M	SGX	LIBOR	$ 1m

2. Other Currencies:

Currency	**Tenor**	**Exchange**	**Underlying**	**Notional**
CAD	3M	MX	CDOR	CAD 1m
DKK	3M	OMX	CIBOR	CAD 1m
ZAR	3M	SAFEX	JIBAR	ZAR 0.1m

3. USD: USD interest rate futures are traded on CME and on LIFFE. For the 3M futures, the notional is $ 1m and the accrual fraction is ¼. The fixing index is LIBOR 3M. For 1M Futures, then notional is $ 3m and the accrual factor is $\frac{1}{12}$. In both cases, the notional to multiply the accrual factor is 250,000.
4. EUR: The EUR 3M interest rate futures are traded on LIFFE, EUREX, and NLX. The notional is € 1m and the accrual factor is ¼. The fixing index is LIBOR.
5. GBP: The GBP 3M interest rate futures are traded on LIFFE and NLX. The notional is £ 1m and the accrual factor is ¼. The fixing index is LIBOR.
6. JPY: The JPY 3M interest rate futures are traded on CME and on SGX for TIBOR-based futures and on SGX for LIBOR-based futures. The notional is JPY 100m and the accrual factor is ¼.
7. CHF: CHF interest rate futures are traded on LIFFE. The fixing is LIBOR 3M.The notional is CHF 1m and the accrual factor is ¼.
8. AUD: Underlying Index: AUD BBSW 3M. Margin Based on

$$\frac{Price}{1 + X\ DT}$$

9. CAD: The CAD 3M interest rate futures (3M Canadian Banker's Acceptance Futures) are traded on MX. The notional is CAD 1m and the accrual factor is ¼. The fixing index is

CDOR. The contract months are the quarterly March, June, September, and December months for up to 3 years, plus the 2 nearest non-quarterly months (serials). Reference => http://www.m-x.ca/produits_taux_int_bax_en.php

10. ZAR: The 3M ZAR interest rate futures are traded on SAFEX. The notional is ZAR 0.1 m and the accrual factor is ¼. The fixing index is JIBAR 3M, and the futures are traded 8 quarters ahead. Reference => http://www.jse.co.za/Libraries/Interest_Rate_Market_-_Products_Documentation/Jibar_FuturesContract_specifications.sf/b.ashx

Interest Rate Futures Option – Premium

1. Definition: An option on futures is described by the underlying future, an option expiration date θ, a strike K, and an option type (call/put). The expiration is before or on the futures trading date: $\theta < t_0$. The premium type options referred to here are the American type and pay premium upfront at the transaction date. The premium type is traded on CME and SGX, and there is no margining process for the option. On the CME, the options are Eurodollar futures (1M and 3M), on SGX the options are Eurodollar futures (1M), on JPY LIBOR futures, and JPY TIBOR futures.
2. Upfront Option Types: There are 3 types of options - the quarterly options, the mid-curve options, and the serial options. The quarterly options expire at the last trading day of the underlying future, i.e., $\theta = t_0$. The serial and the mid-curve options expire before the futures' last trading date. For the serial, the delay is 1-2 months (plus one weekend). For the mid-curve options, the delay is 1, 2, or 4 years. The quoted price for the option follows the same rule as the futures. For a quoted price, the amount paid is multiplied by the notional and the accrual factor of the underlying future.

Interest Rate Futures Option – Margin

1. Definition: An option on futures is described by the underlying future, the *Option Expiration Date* θ, a strike K, and an option type (call/put). The expiration is before or on the futures

trading date: $\theta < t_0$. The option on futures referred to here are the American type and have a futures-like margining process. This type of option is traded on LIFFE for the EUR, GBP, CHF, and the USD futures (3M), and on EUREX for EUR 3M.

2. Options Margining and Quoted Price: Note that there are 2 margining processes involved in the instrument – the margining process on the underlying futures, and one on the quoted option itself. The quoted price for the option follows the same rule as that for the future. For a quoted price, the daily margin is paid on the current closing price minus the reference price multiplied by the notional and the accrual factor of the underlying future. The reference price is the trade price on the trade date, and the previous closing price on the subsequent dates.
3. The Standard Contracts List:

Currency	Tenor	Exchange	Underlying	Type
USD	3M	LIFFE	LIBOR	Option on Future
USD	3M	LIFFE	LIBOR	Mid-Curve Options
EUR	3M	LIFFE	EURIBOR	Option on Future
EUR	3M	LIFFE	EURIBOR	Mid-Curve Options
EUR	3M	LIFFE	EURIBOR	2Y Mid-Curve Options
EUR	3M	EUREX	EURIBOR	Option on Future
EUR	3M	EUREX	EURIBOR	1Y Mid-Curve Options
GBP	3M	LIFFE	LIBOR	Option on Future
GBP	3M	LIFFE	LIBOR	Mid-Curve Options
GBP	3M	LIFFE	LIBOR	2Y Mid-Curve Options
GBP	3M	LIFFE	LIBOR	Option on Future

4. Margined IRF Options: Trading Dates: For standard options (not mid-curve options), the last trading date is the same as the last trading date for the underlying future. For mid-curve options, the last trading date is 1BD before the last trading date of the future in the same month. For example, the EUR mid-curve options with expiry in March 2014 (OR14) on the March 2015 Future (ER15) have a last trading date on Friday 10 March 2014, while the March 2014 Futures (ER14) and their associated standard options (ER14) trade up to 13 March 2014.

Bank Bill Futures – AUD Style

1. Definition: The AUD Bill Futures are traded on ASX. At expiry different bills can be delivered. The bills eligible for delivery are the bills having between 85 and 95 days to maturity on the settlement date. The bills issuers can be any banks in the approved banks' list (currently there are 4 banks – ANZ Banking Group, National Australia Bank Limited, Commonwealth Bank of Australia, and Westpac Banking Corporation).

Currency	Tenor	Exchange	Underlying	Notional
AUD	3M	ASX	Bank Bill	AUD 1m

2. Delivery: The party short on the futures chooses the bill it wants to deliver – for each contract, the short party can choose up to 10 different bills of AUD 0.1 m each. Thus, the short party has a delivery option – a situation very similar to bond futures in the main currencies.
3. Expiry: The expiry date θ (also called the announcement date) is the second Friday of the future month and the *Delivery Date* t_0 is the next business day (usually Monday). The futures are quoted with fixing upto 5 years.
4. Settlement: Let t_i $(1 \leq i \leq N)$ denote the possible maturity dates of the bills. At settlement, the price received for the bill will depend on the last quoted Future Index that we denote by F_θ. The yield associated with this index is

$$R_\theta = 1 - F_\theta$$

The paid price is

$$\frac{1}{1 + \delta_i R_\theta}$$

where δ_i is the accrual factor between t_0 and t_i. For the AUD bill futures δ_i is Act/365. In exchange of the price the short party delivers the bills with the notional equivalent to that of the futures (remember, in practice, there can be at most 9 possible dates taking into account weekends).

Deliverable Swap (IRS) Futures (PV Quoted)

1. Definition: These futures are traded in CBOT/CME, and the notional is $ 0.1m per contract. The margining feature is the future-daily margin on the quoted price (note that the price is quoted in percentage points and $\frac{1}{32}^{\text{nd}}$ of a point, like the bond futures contract). The underlying swap has the standard convention for a USD swap – semi-annual bond-basis vs. 3M LIBOR. The futures are quoted swaps with tenors 2Y, 5Y, 10Y, and 30Y. The underlying swap has a fixed rate as decided by the exchange on the first trading date of the contract. The rate of change is in increments of 25 bp. The rate is NOT fixed at a pre-defined value, unlike the reference coupon of bond futures.
2. CBE/CBOT Deliverable Swap Futures in USD:

Contract	Notional	Coupon
2Y	$ 0.1m	0.50% as of March 2014
5Y	$ 0.1m	1.00% as of March 2014
10Y	$ 0.1m	2.50% as of March 2014
30Y	$ 0.1m	2.75% as of March 2014

3. CME/CBOT Deliverable Swap: Delivery/Trading Dates: The *Delivery Dates* follow the quarterly cycle standard to the interest rate futures. The *Delivery Date* is the 3rd Wednesday of the quarterly month (March, June, September, and December). The *Last Trading Date* or the *Expiry Date* is 2 trading days prior to that date, usually on the Monday.
4. Deliverable Swap – Delivery and Settlement: On the expiry date, the parties agree to enter into a swap where the party long the futures receives fixed on the swap and the party short

the futures pays the fixed. The delivered swap is cleared on a CME clearing. The *Effective Date* of the Swap is also the *Delivery Date*. The swap has an upfront payment on the delivery date. The upfront payment is obtained from the futures settlement price on the last trading date, denoted by F_θ. The amount received by the long party is

$$N \times (1 - F_\theta)$$

(if the amount is negative, it is interpreted as the absolute value paid by the long party).

Bond Futures (non AUD/NZD)

1. Definition: These bond futures are exchange traded instruments. One of their peculiarities is that the underlying is not a single instrument, but a basket. For most of the instruments the short party has an option to deliver any of the instruments in the basket.
2. Basket: The basket is composed of government bonds from a unique issuer (country) with rules on initial maturity, remaining maturity, and size to be eligible. The bond futures are traded on different exchanges for different countries. In general, there are several maturity buckets for each underlying country.
3. Conversion Factor: The bonds in the basket are transformed to be comparable through a conversion factor mechanism. The factor is such that in a certain yield environment all the bonds have the same price. The reference yield acts in a way as the strike for a delivery process.
4. Main Contracts:

Country	Currency	Exchange	Number of Contracts
Canada	CAD	MSE	3
Germany	EUR	EUREX	4
Germany	EUR	NLX	3

Italy	EUR	EUREX	2
Japan	JPY	TSE	3
Japan	JPY	LIFFE	1
Japan	JPY	SGX	1
Spain	EUR	MEFF	1
United Kingdom	GBP	LIFFE	3
United Kingdom	GBP	NLX	1
United States	USD	CBOT	5
Switzerland	CHF	EUREX	1

5. Embedded Options: Some of the other embedded options for certain currencies include:
    a. Timing Option => The delivery notice can be made inside of a period and not just on a single date. This provides some American Option flavor to the futures.
    b. Wild Card Option => The underlying bonds can be selected after the price of the future has been fixed. During the delivery period, there is a daily option between the end of the future trading at 14:00 and the end of the bond trading at 18:00. After the last trading date, there can be a period of up to 7 days where the future price is fixed, but the delivery notice has not been given yet.
6. Settlement: Suppose there are n bonds in the basket. Let $AccruedInterest_i(t)$ denote the accrued interest of bond i at the delivery date t. The conversion factor associated with bond i is K_i. The bond future delivery notice takes place at some date before the actual delivery, with this lag usually being around 1-2 days. If the *Futures Price* is denoted by F, at the delivery time the short party can choose the bond he wishes to deliver and receives at the delivery date the amount

$$F \times K_i + AccruedInterest_i(t_0)$$

    a. Settle Price Clarification => The term *Price* used above is the standard in the jargon for futures, however, it should be viewed as a *Number* or as a *Traded Reference Index.* The *Future Price* is never paid in itself. It only serves as an input for the eventual computation.

Country-specific Bond Futures - USD

1. Treasury Bond Futures: The futures on United States debt are traded on CBOT. The price is quoted in percentage points and 32nd of a point. Note that the last trading day and the last delivery date are not the same for all the basket underlyings.
2. Conversion Factor: The description of the price used in the delivery (using an explicit quote from the exchange) is: *The invoice price equals a future settlement price times a conversion factor, plus accrued interest. The conversion factor is the price of the delivered bond (USD 1 par value) to yield 6%*. The conversion factor is provided by the exchange and does not need to be computed by the users. Nevertheless, there are clear rules to compute them. The values do not change through the life of the future.
3. Long Futures: The Ultra T-Bond futures, the US Treasury Bond Futures, and the 10Y US Treasury Note Futures all have the same last trading day and the last delivery date. The last trading day is *the 7th business day preceding the last business day of the delivery month. Trading in expiring contracts closes at 12:01 PM on the last trading day*. Previously the US Treasury Long Bond Futures referred to all bonds with maturities greater than 15 years. That range has recently (since March 2011) been divided into 2 futures – the Ultra T-Bond Futures, and the US Treasury Bond Futures.
    a. Ultra T-Bond Futures => The underliers of the Ultra T-Bond Futures are the *US Treasury Bonds with remaining term to maturity of not less than 25 years from the first day of the futures contract delivery month*.
    b. US Treasury Bond Futures => Formerly called the 30 years futures, the deliverable grade for the Treasury Bond Futures are *bonds with remaining maturity of at least 15 years, but less than 25 years, from the first date of the delivery month*. These are also known as *Classic Bond Futures*. The Treasury Bond Futures are less liquid than the 5Y and the 10Y futures. To match the US Treasury Naming Convention, the futures would be better called *Note Futures*.
    c. Catalog => Volume refers to the Monthly volume, as on October 2013.

Contract	Maturity	Notional	Yield	Code	Volume
Ultra T-Bond	> 25Y	$ 0.1m	6.00 %	UB/UL/LBE	1,387,996
30Y Bond	15Y - 25Y	$ 0.1m	6.00 %	ZB/US	6,193,997
10Y Bond	6½Y – 10Y	$ 0.1m	6.00 %	ZN/TY	21,265,689
5Y Note	4Y2M – 5Y3M	$ 0.1m	6.00 %	ZF/FV	10,198,247
3Y Note	2Y9M - 3Y	$ 0.2m	6.00 %	Z3N/3YR	0
2Y Note	1Y9M – 2Y	$ 0.2m	6.00 %	ZT/TU	3,132,990

4. 10Y Treasure Note Futures: *US Treasury Notes with a remaining term to maturity of at least 6½Y, but no more than 10Y, from the first date of the delivery month*.
5. 5Y Treasury Note Futures: The last trading day is *the last business day of the calendar month*. The last delivery day is *the 3rd business day following the delivery day*. The eligible bonds are *US Treasury notes with the original term to maturity if note more than 5Y3M, and a remaining term to maturity of not less than 4Y2M as of the first day of the delivery month*.
6. 3Y Treasury Note Futures: The last trading day is *the last business day of the transaction month*. The notional is $ 0.2m. The eligible bonds are *US Treasury Notes that have an original maturity of 5Y3M and a remaining maturity of not less than 2Y9M from the first day of the delivery month, but not more than 3Y from the last day of the delivery month*.
7. 2Y Treasury Note Futures: The notional is $ 0.2m. The eligible bonds are *US Treasury Notes that have an original maturity of 5Y3M and a remaining maturity of not less than 1Y9M from the first day of the delivery month, but not more than23Y from the last day of the delivery month*.

Country-specific Bond Futures - Germany

1. German € Bond Futures Catalog: Volume is as of December 2011.

Contract	Maturity	Notional	Yield	Volume
EURO-BUXL	24Y – 35Y	€ 0.1m	6.00 %	222, 821

EURO-BOND	8½Y – 10½Y	€ 0.1m	6.00 %	11,778,488
EURO-BOBL	4½Y – 5½Y	€ 0.1m	6.00 %	7,252,498
EURO-SCHATZ	1¾Y – 2¼Y	€ 0.1m	6.00 %	8.659,722

2. The Contracts: All the futures are traded on EURX and NLX, except for EURO-BUXL which is only traded on EUREX. *The delivery option arising out of a short position may only be fulfilled by the delivery of certain securities issued by the Federal Republic of Germany with a remaining term on the delivery day within the remaining term of the underlying*. To be eligible, the debt securities must have a minimum issue of € 5bn.
3. Trading/Delivery Dates: The Delivery Date is *the 10th calendar day of the respective quarterly month, if this day is an exchange day; otherwise it is the exchange day immediately succeeding that day*. The last trading day is *2 exchange days prior to the Delivery Day of the relevant maturity month*.
4. Reference Yields: Note that the reference yield for the EURO-BUXL, which the most recent among the others, is 4% (and not 6% as for the majority of the others).

Country-specific Bond Futures - Spain

1. € Bond Futures: The BONO10 Futures Contract on the Spanish 10Y Government Bond was launched on 29 May 2012 by MEFF. The volumes are currently very low (quoted below for October 2013).

Contract	**Maturity**	**Notional**	**Yield**	**Volume**
BONO 10	> 8½Y	€ 0.1m	6.00 %	253

2. Underliers: The underlying asset is a national government bond with a 6.00% annual coupon and a maturity of 10Y. The contract face value is € 0.1m. The expiration day is the 10th day of the month of maturity (if holiday, the next business day). The last trading and the registration days are 2 business days prior to the expiration date. The bonds in the basket are Spanish government bonds with a remaining life of no less than 8½Y.

3. Settlement: The settlement price at the expiration date is calculated by dividing the CTD bond market price (ex-coupon) at the end of the session by the conversion factor of the bond. The market price of the CTD bond will be the closing price determined by SENAF.

Country-specific Bond Futures - £

1. UK £ Bond Futures Catalog: Volume is monthly as of December 2010. Note – the change from 6.00 % coupon to a lower coupon took place with the December 2011 contract.

Contract	Maturity	Notional	Yield	Volume
Long GILT Futures	8Y9M – 13Y	£ 0.1m	6.00 %/4.00 %	476,025
Medium GILT Futures	4Y – 6Y3M	£ 0.1m	6.00 %/4.00%	183
Short GILT Futures	1Y6M – 3Y3M	£ 0.1m	6.00 %/ 3.00%	1,131

2. The Contracts: All the futures are traded on LIFFE, and the Long GILTs are also traded on NLX. The first notice day in 2BD prior to the first day of the delivery month. The *Last Notice Day* is the first business day after the *Last Trading Day*. The *Last Trading Day* is 2BD prior to the last business day of the delivery month. The *Delivery Day* is any business day in the delivery month (at the sellers' choice). The deliverable bonds are subject to a coupon range of 3.00% around the reference yield.

Country-specific Bond Futures - ¥

1. Japan ¥ Bond Futures: These are traded on TSE. The notional is ¥ 100m. The *Final Settlement Day* is the 20th of each contract month. The *Last Trading Day* is the 7th business day prior to each delivery date. Trading for the new contract month begins on the business day following the last trading day. There also used to be a 20Y JGB futures, but its trading was halted in December 2002 due to lack of volume.

Contract	Maturity	Notional	Reference Yield	Volume
10Y JGB Futures	7Y - 10Y	¥ 100m	6.00 %	657,356
5Y JGB Futures	4Y – 5¼Y	¥ 100m	3.00 %	

Options On Bond Futures (non AUD/NZD) - Premium

1. Description: An option on futures is described by the underlying future, and expiration date θ, the strike K, and an option type (Call/Put). The option expiration is on or before the last trading date of the futures, i.e., $\theta \leq t_0$. Premium-type options pay a premium upfront at the transaction date, and are of American type. As such, there is no margining process for them. This type is traded on CBOT for USD bond futures.
2. CBOT Options on USD Bond Futures: The contract months are the first 3 consecutive contract months (2 serial expirations and one quarterly expiration) plus the next 4 months in the March, June, September, and December quarterly cycle. The serials exercise into the first nearby quarterly futures contract. Quarterlies exercise into the futures contracts of the same delivery period. The *Last Trading Day* is the *Last Friday* which precedes by at least 2BD the last business day of the month preceding the option month. The options are quoted in $\frac{1}{64}$th of a point.
3. USD Bond Futures Options Catalog: Volumes quoted here are monthly for October 2013. The codes are for CME Globex (Electronic Platform)/Open Outcry (Trading Floor) Call-Put.

Contract	Maturity	Notional	Codes	Volume
Ultra-Bond	> 25Y	$ 0.1m	OUB/OUL	3,786
Classic Bond	15Y – 25Y	$ 0.1m	OZB/CG-PG	1,247,787
10Y Note	6½Y – 10Y	$ 0.1m	OZN/TC-TP	7,710,256
5Y Note	4Y2M – 5¼Y	$ 0.1m	OZF/FL-FP	1,752,940
2Y Note	1¾Y – 2Y	$ 0.1m	OZT/TUP-TUC	197,574

Options On Bond Futures (non AUD/NZD) - Margin

1. Description: An option on futures is described by the underlying future, and expiration date θ, the strike K, and an option type (Call/Put). The option expiration is on or before the last trading date of the futures, i.e., $\theta \leq t_0$. The margin type options are American type and have a future-style method of margining process for the option. This type is traded on EUREX for bond futures.
2. EUR - EUREX Margin Options: The contract months are the first 3 consecutive contract months (2 serial expirations and one quarterly expiration) plus the next month in the March, June, September, and December quarterly cycle. For calendar months, the maturity month of the futures contract is the quarterly month following the expiration month of the option. For quarterly months, the maturity month of the underlying futures contract and the expiration month of the option are identical.
3. Trading/Settlement Days: *Last Trading Day for the Option Series introduced from September 1, 2011 is the last Friday prior to the first calendar day of the option expiration month, followed by at least 2 exchange days prior to the first calendar day of the option expiration month. Exception => If this Friday is not an exchange day, or if this Friday is an exchange day but is followed by only one exchange day prior to the first calendar day of the option expiration month, the exchange day immediately preceding that Friday is the* ***Last Trading Day****. For the purposes of this exception, an exchange day is an exchange day at both the EUREX exchanges as well as being a Federal work day at the US*. Reference => http://www.eurexchange.com/products/INT/FIX/OGBL_en.html

AUD-NZD Bond Futures

1. Introduction: The Australian and New Zealand futures are settled in cash against a standardized bond. The standardized bond yield is computed as an average of actual bond yields for AUD, and as a linear interpolation of actual bond yields for NZD.
2. Basket Weightings: The average yield cash delivery implies that the futures behave roughly like a weighted average of the underlying. The weights are not exactly equal, but they do not

change too much with level of the rates. One single bond will never represent the future exactly, but the mixture of bonds that best represent the future does not vary too much with time and rates.

3. Characteristics: Compared to the non AUD/USD bond futures, the AUD bond futures traded in SFE have very different characteristics. The main difference is that they settle in cash vs. the average yield of the underlying bonds. The exact mechanism of the settlement (which is non-trivial) is described below.
4. Maturity Types: There exist 2 maturity types for the SFE Australian Treasury Bond Futures – the 3Y and the 10Y futures. Beyond the maturity the other characteristics of these futures types are similar. Both have a notional of 0.1m AUD per contract. The 3Y futures are more liquid than the 10Y one.
5. Settlement Yield Rules: The general scheme for choosing the yields used in settlement is that a set of randomly chosen dealer quotes is selected (after discarding extreme quotes). The selection of the underlying bonds does not appear to be captured by a very precise rule. A certain number of bonds is chosen by the exchange, often around 3 underlying bonds. The maturities are between 2Y and 4Y for the 3Y futures, and 8Y – 12Y for the 10Y futures. Reference => http://www.asx.com.au
6. Settle Yield Calculation: Suppose there are N bonds underlying the future. Since the contract settles in cash, the settlement is done against the average yield of the underlying bonds. Let $Y_{i,\theta}$ $(1 \le i \le n)$ be the yields on the fixing date for the underlying bonds. The reference yield for the settlement is

$$Y_\theta = \frac{1}{N}\sum_{i=1}^{N} Y_{i,\theta}$$

This yield is used to calculate the final future index yield and the equivalent bond price, as shown below.

7. Reference Price: The time t *Futures Price* (all the caveats regarding the price being a jargon rather than an actual economic quantity applies here) is denoted by Φ_t. All the margining payments related to SFE bond futures are done using a reference bond price R_t is computed

from the future index in the following way: Let m be the number of payments ($m = 6$ for semi-annual 3Y futures and $m = 20$ for semi-annual 10Y futures). Then

$$\mathrm{Y}_t = 1 - \Phi_t$$

$$\nu_t = \frac{1}{1 + \frac{Y_t}{2}}$$

$$\mathrm{R}_t = 0.03 \frac{1 - {\nu_t}^m}{{Y_t}/{2}} + {\nu_t}^m$$

Finally, this reference price is multiplied by notional value, which is AUD 0.1m per contract.

8. Reference Price Calculation: The expression seen above for R_t is simply a consequence of using semi-annual 3Y/10Y bond with $c = 6.00\ \%$ coupon, and semi-annual yield Y_t. It is just a special case of the expression

$$\mathrm{R}_t = \sum_{i=1}^{m} \frac{c}{2} \frac{1}{\left(1 + \frac{Y}{2}\right)^i} + \frac{1}{\left(1 + \frac{Y}{2}\right)^m} = \frac{c}{2} \frac{\vartheta - \vartheta^{m+1}}{1 - \vartheta} + \vartheta^m = \frac{c}{2} \frac{1 - \vartheta^m}{\frac{Y}{2}} + \vartheta^m$$

Section III: Treasury Futures and Options

Treasury Futures Trading and Hedging

Introduction and Contract Detail Specifications

- Exchange Traded Treasury Futures Contract: A widely used risk management instrument in debt capital markets is the government bond futures contract. This is an exchange traded standardized contract that fixes the price today at which a specified quantity and quality of a treasury bond will be delivered at a date during the expiry month of the futures contract.
- Treasury Futures vs. STIR Futures: Unlike short term interest rate futures instruments which only require cash settlement, treasury futures require the actual settlement of the bond when they are settled.
- Exchange Traded Futures - The Concept: The *futures contract* is an agreement between two counterparties that fixes the terms of the exchange that will take place between them at a future date. They are standardized agreements as opposed to OTC ones that are traded on an exchange, so they are also referred to as *exchange traded futures*.
- Contracts Traded on the LIFFE: In the UK financial futures are traded on the LIFFE – the London International Financial Futures Exchange – which opened in 1982. LIFFE is the biggest financial futures exchange in Europe in terms of the volume of the contracts traded. There are four classes of contracts traded on the LIFFE: the short-term interest rate contracts, the long-term interest rate contracts (treasury futures), the currency contracts, and the stock index contracts.
- Maturity and Settlement Prices: Most futures contracts traded on exchanges around the world trade at three month maturity intervals, with the fixed maturity dates fixed at March, June, September, and December each year. This includes contracts traded on the LIFFE. Therefore at pre-set times during a year a contract for each of these months will *expire*, and a final *settlement* price is determined for it.
- Liquidity in the Traded Contracts: The father out one goes in maturity the less liquid the trading is in that contract. It is normal to observe liquid trading only for the *front month*

contract (the current contract, so if we are trading in April 2015 the front month is the June 2015 future) and possibly one or two of the next contracts. The liquidity of the contract diminishes the farther out one trades in the maturity range.

- Futures Position Close-out Types: When a party establishes a position in a futures contract, it can either run the position out to maturity, or close out the position between the trade date and the maturity date. If the position is closed out the party will have either a loss or a profit to book.
- Treasury Futures Physical/Cash Settlement: If a position is held to maturity, the party that is long the futures will take delivery of the underlying asset (bond) at the settlement price; the party that is short the futures will deliver the underlying asset. This is referred to as *physical settlement*, or sometime confusingly, as *cash settlement* (Chaudhary (2011)).
- LCH - The London Clearing House: There is no counterparty risk associated with exchange traded futures because of the role of a *clearing house* such as the London Clearing House. This is the body through which the contracts are settled. The clearing house acts as the buyer to all contracts sold on the exchange, and as seller to all contracts that are bought. So in the London market the LCH acts as the counterparty to all transactions, so that settlement is effectively guaranteed.
- Margin Deposited by the Participants: The LCH requires all participants to deposit a *margin* with it – a cash sum that is the cost of conducting business (plus brokers' commissions). The size of the margin depends on the size of a party's net *open* position in the contracts (an open position is a position in a contract that is held overnight and not closed out).
- Margin Types - Maintenance vs. Variation: There are two types of margins – *maintenance margin* and *variation margin*. Maintenance margin is the minimum level required to be held at the clearing house; the level is set by the exchange. Variation margin is the additional amount that must be deposited to cover any trading losses, as well as when the size of the net open positions increases.
- Repo Margins vs. Exchange Margins: Margin in the repo is a safeguard against the drop in the value of the collateral supplied against a loan of cash. The margin deposited at a futures exchange clearing house acts as essentially "good faith" funds required to provide comfort to the exchange that the futures trader is able to satisfy the obligations of the futures contract.
- Sample CBOT UST Futures Contract: Source: CBOT

Unit of Trading	UST Bond with a Notional Value of $100K and a Coupon of 8%
Deliverable Grades	UST with a Minimum Maturity of 15Y from the first day of the Delivery Month
Delivery Months	March, June, September, and December
Delivery Date	Any Business Day during the Delivery Month
Last Trading Day	12:00 NOON, 7th Business Day before the last Business Day of the Delivery Month
Quotation	Percent of Par expressed as Points and Thirty-two Seconds of a Point, e.g., 108-16 is $108\frac{16}{32}$ or 108.50
Minimum Price Movement	$\frac{1}{32}$
Tick Value	$31.25
Trading Hours	07:20 – 14:00 (Pit Trading) 17:20 – 20:05 22:30 – 06:00 (Screen Trading)

- <u>Notional Coupon of the Contract</u>: The terms of the contract relate to a UST bond with a minimum maturity of 15Y and a *notional coupon* of 8%. The futures contract specifies a notional coupon to prevent delivery and liquidity problems that would arise if there was a shortage of bonds with exactly the coupon required, or f one market participant purchased a large proportion of all the bonds in issue with the required coupon.
- <u>Choice in the Deliverable Bond</u>: For exchange-traded futures the short future can deliver any bond that fits the maturity criterion specified in the contract terms. Of course a long futures would like to deliver a high-coupon bond with significant accrued interest whereas the short-futures would want to deliver a low coupon bond with low accrued interest.
- <u>Treasury Futures Invoice Amount Calculation</u>: The challenge above is accounted for by the way the *invoice amount* (the amount paid by the long futures to purchase the bond) is calculated. The invoice amount on the expiry date is given as

$$Inv_{Amount} = P_{Futures} \times CF + AI$$

where Inv_{Amount} is the invoice amount, $P_{Futures}$ is the price of the futures contract, CF is the conversion factor, and AI is the bond accrued interest.

- Bonds in the Delivery Basket: Any bond the meets the maturity specification of the futures contract is said to be in the *delivery basket* – the group of bonds that are eligible to be delivered into the futures contract. Every bond in the delivery basket will have its own *conversion factor* that is used to equalize the coupon and the accrued interest differences of all the delivery bonds.
- Announcement of the Conversion Factor: The exchange will announce the conversion factor for each bond before trading in a contract begins; the conversion factor for a bond will change over time, but remains fixed for one individual contract. That is if a bond has a conversion factor of 1.091252 this will remain fixed for the life of the contract.
- Notional Coupon Impact of the CF: If a contract specifies a bond with a notional coupon of 4% (e.g., a long GILT future on LIFFE) the conversion factor will be less than 1.0 for bonds with a coupon lower than 4% and higher than 1.0 for bonds with a coupon higher than 4%.
- Formal Definition of Conversion Factor: The conversion factor (or the price factor) gives the price of an individual cash bond such that its yield to maturity on the delivery date of the futures contract equals the notional coupon of the contract. The product of the conversion factor and the futures price is the forward price available in the futures market for that cash bond (plus the cost of funding referred to as the gross basis).
- Yield Impact on Bond's Cheapness: Although conversion factors equalize the yield on bonds, the bonds on the delivery basket trade at different yields, and for this reason they are not *equal* at the time of delivery. Certain bonds will be cheaper than the others, and one bond will be the *cheapest-to-deliver* bond.
- Determination of the Cheapest-to-Deliver: The cheapest-to-deliver bond is the one that gives the greatest return from a strategy of buying the bond and simultaneously selling the futures contract, and the closing out the positions on the expiry of the contract. This so-called *cash-and-carry trading* is actively pursued by proprietary trading desks.
- CTD vs. Futures Basis Trading: If a contract is purchased and then held to maturity the buyer will receive – via the exchange's clearing house – the cheapest-to-deliver bond. Traders

exploit these price differentials between the futures and the CTD bond – known as *basis trading*.

- <u>LIFFE Long GILT Futures Contract</u>: A sample contract specification of the long GILT futures traded on the LIFFE is shown in the table below. There is also a medium GILT contract on the LIFFE that was introduced in 1998, having been discontinued in the 1990's. This trades a notional 5Y GILT with eligible GILT's being in the 4Y to the 7Y maturity.
- <u>LIFFE Long-GILT Futures Specification</u>: Source => LIFFE

Unit of Trading	UK GILT's have a face value of 100,000 GBP, a notional coupon of 7%, and a notional maturity of 10Y – changed from the contract value of 50,000 GBP from the September 1998 contract
Deliverable Grades	UK GILT's with maturities ranging from $8\frac{3}{4}$Y to 13Y from the first day of the delivery month (changed from 10-15Y from the December 1998 contract)
Delivery Months	March, June, September, and December
Delivery Date	Any Business Day during the Delivery Month
Last Trading Day	11:00, 2BD before 1BD before the delivery month
Quotation	Percent of Par expressed as points and one-hundredth's of a point, for example 114.56
Minimum Price Movement	0.01 of One Point (One Tick)
Tick Value	10 GBP
Trading Hours	8:00 – 18:00 hours. All Trading conducted Electronically on LIFFE $CONNECT^{TM}$ Platform

- <u>Repo Trade Cost-of-Carry</u>: It is the level of the Repo rate in the market (r_{Repo}) compared to the running yield of the underlying bond (y) that sets the price of the futures contract. $r_{Repo} - y$ is the net financing cost in the arbitrage trade and is known as the *cost of carry*.
- <u>Positive vs. Negative Carry Trade</u>: If the running yield on the bond is higher than the funding cost (the repo rate) this corresponds to positive funding or *positive carry*. Negative funding (*negative carry*) is when the repo rate is higher than the running yield.

- Futures vs. Cash Forward Price: The level of $r_{Repo} - y$ determines whether the futures price above the cash forward market or below it. If the carry is positive ($r_{Repo} > y$) then the futures trades below the forward price, known as trading at a *discount*. Where $r_{Repo} > y$ (*negative carry*) the futures price will be at a premium to the forwards price. If the net funding cost was zero, such that we had neither positive nor negative carry, the futures price would equal the underlying bond price.
- Impact of the Yield Curve Slope: The cost of carry related to a bond futures contract is a function of the yield curve.
- Impact of Positive Yield Curve Slope: In a positive yield curve environment the repo rate is likely to be lower than the running yield of the bond, so the cost of carry is likely to be positive. As there is generally only a liquid market in the long bond futures out to contracts that mature up to one year from the trade date, with a positive yield curve it would be unusual to have a short term repo rate higher than the running yield on the long bond.
- Negative Yield Curve Slope Impact: So in such an environment (as above) the futures trades at a discount to the forward. If there is a negative sloping yield curve then the futures trades at a premium to the forward. It is in the circumstances of changes in the shape of the yield curve that opportunities for relative value and arbitrage trading arise, especially as the CTD bond for a given futures changes with changes in the curve.
- Integrates Clearing Houses and Exchanges: The clearing house may be owned by the exchange itself, such as the one associated with CME (the CME Clearing House), or maybe a separate entity such as the London Clearing House, which settles transactions on LIFFE. The LCH is also involved in running clearing systems for swaps and repo products in certain currencies.

References

- Chaudhary, M. (2011): Bank Asset and Liability Management: Strategy, Trading, and Analysis **Wiley**.

Identification of the CTD in Basket

Motivation for the Conversion Factor

1. Bond Coupon Price/Size Differential: The purpose of the conversion system is to put all of the deliverable bonds on more or less equal footing. All else equal, bonds with big coupons have higher prices than those with smaller coupons.
2. Normalization of the Deliverable Bond Price: The conversion factor systems equalizes the price of the deliverable bonds by pricing them to the reference yield. A deliverable bond's conversion factor tells you what the price of that bond would be per $1 of par value, at the yield corresponding to the reference yield.
3. Relative Market-To-Reference Yield: Bonds with coupons greater than the reference yield have a conversion factor greater than 1, while those with coupons smaller than the reference yield have conversion factors less than 1.
4. Lowest Normalized Invoice Price Calculation: Thus the cheapest-to-deliver bond is not necessarily the one with the lowest market price, because futures seller have to buy the bond they are going to deliver against the contract. The cheapest-to-deliver bond is the bond with the lowest price relative to the invoice price.
5. CTD Relationship to the Invoice Price: If the CTD bond costs more than the invoice price, it is closer to the invoice price than any other deliverable bond. If it costs less than the invoice price, it is farthest from the invoice price than any other bond.
6. Futures Settlement PnL for CTD: In other words the cheapest-to-deliver bond is the bond that results in the smallest loss to the greatest profit to the futures seller.

Illustration – Old vs. Active Treasury

1. 30Y Treasury CTD Futures Contract: The table below (Stanton (1999) provides all the relevant data on both the bond that is the cheapest-to-deliver as of 30 July 1999 against the futures contract (the 15 February 2015 contract), and the most recently issued 30Y Treasury Bond. The full price is what the futures seller would have to pay to acquire the bond, and the futures invoice price is what we would receive for it.
2. Profit Associated with the CTD: As can be seen from the last column, delivering the 15 February 2015 bond would cost the seller 1.32, or $1,320 per contract, while delivering the active contract would cost 9.50, or $9,500 per contract.
3. How Cheapskates figure CTD: Data as of 29 July, 1999. Source => CBOT.
    a. Full Price => Quote Price Plus Accrued Interest
    b. Futures invoice Price => Futures Price times Factor, Plus Accrued Interest
    c. Deliverer's Profit or Loss => Full Price Minus Futures invoice Price

Maturity	**Coupon**	**Quoted Price**	**Accrued Interest**	**Full Price**	**Factor**	**Futures Invoice**	**Deliverer's Profit or Loss**
15 Feb 2015	11.25%	148.56	1.41	149.97	1.28	148.65	-1.32
15 Feb 2029	5.25%	88.87	0.66	89.53	0.69	80.03	-9.50

Market Parameters Influencing CTD Calculation

1. CTD Characteristics Determining Futures Profit: The characteristics of a bond that causes it to result in the largest profit or the smallest loss for the futures seller at the delivery date are its duration and convexity (funding/repo/replication sensitivity in general).
2. Price-Rates Relationship and Impact: When interest rates fall bond price rises, and bonds with the longest durations rise the most. Conversely when interest rates rise bond prices fall, with the longest duration instruments falling the most. So the bonds that benefit the least

from the broad decline in the interest rates since 1991 are the short duration instruments – bonds with relatively short maturities and relatively large coupons.

3. Determination of the CTD Issue: At any time the cheapest-to-deliver issue will be the one with the lowest converted price where the converted price is the bond's price divided by its CBOT conversion factor.
4. Current Yield vs. Reference Yield: Therefore when interest rates are below the reference rate the short duration issue will be the cheapest to deliver, and its price would have risen the least. Conversely if rates were to rise above the reference yield, causing prices to fall, the cheapest to deliver issue would be the one whose price fell the farthest – a long-duration issue.

Impact of Yield Curve Changes

1. Base Scenario Yield Curve CTD: As demonstrated in the table below (Stanton 1999)), the first row of numbers provide the particulars on the bond that was the cheapest-to-deliver against the March 2000 contract at close on 29 July 1999. The three bonds are shown, as is their converted prices (the lowest in ***italicized bold***).
2. Yield Curve Change CTD Impact: Data as of 29 July 1999, Source: CBOT.

Maturity	**Coupon**	**Duration (Years)**	**Price**	**Price if Yield**		**Factor**	**Converted Price**		
				Shed 25 bp	**Added 25 bp**			**Shed 25 bp**	**Added 25 bp**
15 Aug 2021	8.000%	11.17	119.625	123.083	116.309	1.2398	***96.487***	99.276	93.813
15 Aug 2019	8.125%	10.38	119.844	123.113	116.698	1.2505	96.609	***99.244***	94.073
15 Aug 2022	7.250%	11.39	111.187	114.465	107.952	1.1523	96.492	99.336	***93.684***

3. Yield Curve Bumped Down CTD: The second row gives particulars on the bond that would have been the CTD at the previous close if the yields were 25 bp lower (and price were higher). As can be seen that is a shorter duration instrument whose price would have benefitted the least from the decline in the rates.
4. Yield Curve Bumped Up CTD: Finally the third row of numbers tells all about the bond that would have been the cheapest-to-deliver if the yields were 25 bp higher (and the prices lower) – a longer duration instrument whose price would have fallen the most.

References

- Stanton, M. (1999): What makes a Bond Cheapest to Deliver against the Futures Contract?

Valuation of Treasury Futures Contract

Futures Contract and Mark to Market

1. MTM and the Contract Value: Each day prior to the expiration date the long and the short positions are marked to market. The buyer gets $G(t) - G(t - 1D)$ and the seller gets $-[G(t) - G(t - 1D)]$. It costs nothing to get into or out of a futures contract ignoring transaction costs (Carpenter (2011)). Therefore in equilibrium, the futures price on any day is set to make the PV of all the contract cash flows equal to zero.
2. Basic Futures Contract Valuation Setup: In a basic futures contract without delivery options the buyer agrees to take delivery of the underlying asset from the seller at a specified expiration date T. Associated with the contract is the futures price $G(t)$ which varies in equilibrium with time and with market conditions. On the expiration date the buyer pays the seller $G(T)$ for the underlying assets.
3. Mechanics of Marking to Market: Consider buying the contract at time t and selling it at $t + 1D$. It essentially costs nothing to buy and sell the contract, so the payoff from this strategy is just the profit or loss from the MTM: $G(t + 1D) - G(t)$
4. Setting $G(t)$ from the MTM Mechanism: $G(t + 1D)$ is a random martingale, so $G(t)$ is set today to make the market value of the next day's random payoff $G(t + 1D) - G(t)$ equal to zero.
5. Risk Neutral MTM Payoff Replication: The market value of the martingale mark-to-market $G(t + 1D) - G(t)$ is the cost of replicating that payoff. We can represent that cost in its usual way as a discounted expected value under the risk-neutral probability distribution.
6. The Risk Neutral Expectation: To make this market value zero, today's futures price must be the expected value of tomorrow's futures price under the risk-neutral probability distribution

$$\mathbb{E}_t^{\mathbb{Q}_t^{t+1D}}[G(t + 1D) - G(t)] = 0$$

which implies that

$$G(t) = \mathbb{E}_t[G(t + 1D)]$$

7. Terminal Payoff Cash Convergence: Consider entering the futures contract the instant before it expires. The long position would instantly pay the futures price and receive the underlying asset. The payoff would be $V(T) - G(T)$ where $V(T)$ is the spot price of the underlying on the expiration date. In the absence of arbitrage, since it costs nothing to enter into either side of the contract, the (known) payoff must be zero:

$$G(T) = V(T)$$

8. Futures Price without Delivery Options: Consider a basic futures contract on a bond. To determine the current price $G(0)$ we start at the expiration date of the futures when the futures price is equal to the spot price of the underlying bond, then work backwards each MTM date to determine the futures price that makes the next MTM payoff worth zero.
9. Futures vs. Forward Price Match: When there are no further marks to market remaining before the expiration date of the contract, the forward and the futures prices are the same. Further, if the interest rates are uncorrelated with the value of the underlying asset, the forward and the futures price are the same. Such an assumption may be reasonable in the case of stock index futures or commodities futures.
10. Futures Price less than the Forward: When the underlying asset is an interest rate product (e.g., bond, interest rate forward, swap, etc.) the asset's price typically is negatively correlated with the interest rates. In the case the futures price becomes lower than the forward price.
11. MTM Profit and Loss Flow: The profit or loss from the forward contract is

$$V(T) - F(0) = F(T) - F(0)$$

and is received all at the end at time T, and

$$NPV[F(T) - F(0)] = 0$$

The cumulative profit or loss from the futures contract is

$$V(T) - G(0) = G(T) - G(0)$$

but this is paid out intermittently through the marks to market.

12. Reinvestment of Gains and Losses: Consider reinvesting all the gains and the losses from marking to market to the expiration date. Gains would be re-investd at low rates, losses at high rates, so to make the NPV equal to zero, the futures price must start out lower than the forward price.

Role of the Clearing Corporation

1. Clearing Corporation as the Counter Party: All buyers and sellers trade with a Clearing Corporation associated with each exchange, so there is no counterparty risk. The marking to market provision limits the credit risk faced by the clearing corporation. Commissions on the futures contract on the CBOT/CME are about $25 or less, and are fully negotiable.
2. Initial, Maintenance, and Variation Margins: Upon entering into a futures contract, the investor must post initial margin, which is interest bearing. If the balance in the margin account falls below the maintenance margin, the investor must post variation margin to restore it to its initial level.

Delivery Options for the Underlying

1. Impact of the Delivery Options: The delivery options for the futures contract listed below make the futures price very different from the forward price. In particular, these delivery options reduce the equilibrium futures price.

2. Quality and Timing Delivery Options: Using the quality option, the seller can deliver any bond with maturity in a given range using a conversion factor. With the timing option, the seller can deliver any time during the expiration month.
3. Wildcard/End-Of-Month: The futures exchange closes early in the afternoon, but the bonds keep trading. With the wildcard option, the seller can announce the delivery anytime until the bond markets close. The end-of-month option uses the fact that the futures stop trading 8 business days before the end of the month.

Implied Repo Rate for Futures

1. Motivation behind Implied Repo Rate: Prior to the delivery date, some practitioners identify the cheapest-to-deliver bond as the one with the highest *implied repo rate*.
2. Definition of the Implied Repo Rate: The implied repo rate is the hypothetical rate of return earned from buying a deliverable bond, selling the futures, and then delivering the bond on the futures contract at an assumed date (ignoring mark-to-market, treating the futures like a forward).
3. Typical Implied Repo Rate Directionality: The implied repo rate is typically below the bond market's repo rate because the seller of the futures can exploit other options (such as wildcard, end-of-month, etc.) as well.

Net Basis for Treasury Futures

1. Definition of the Net Basis: An alternate approach is go choose the bond with the minimum *net basis*. This is the hypothetical loss incurred by buying the bond, financing the purchase in the repo market, selling the futures, and delivering the bond into the futures contract on an assumed delivery date. Again this ignores marking-to-market, treating the futures like a forward.

2. Directionality of the Net Basis: The net basis is typically negative because the seller of the futures contract can exploit other delivery options (e.g., wildcard, end-of-month, etc.) as well.

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Section IV: Funding and Forward Curve Construction and Customization

Curve Builder Features

Overview

1. Smoothness Criterion Evolution: Smoothness formulation is related to the minimization of strain energy (Schwarz (1989)), and the relation to Natural cubic spline (Burden and Faires (1997)), financial cubic spline (Adams (2001)) has been explored.
2. Empirical vs. Theoretical Curve Builder Frameworks: Zangari (1997) and Lin (2002) discuss this in detail.
    - Theoretical Term Structure posit explicit term structure for a variable known as short rate of interest whose values are extracted, possibly, from a statistical analysis of market variables (Vasicek (1977), Cox, Ingersall, and Ross (1985), Rebonato (1998), Barzanti and Corradi (1998), Golub and Tilman (2000)).
    - For bonds/treasuries see Nelson and Siegel (1987), Diament (1993), Svensson (1994), Soderlind and Svensson (1997), Tanggaard (1997). Effectiveness of such treatments is examined in Christensen, Diebold, and Rudebusch (2007), and Coroneo, Nyholm, and Vidova-Koleva (2008).
    - Hybrid methods use empirically determined yield curve inside of a theoretical model (Hull and White (1990), Heath, Jarrow, and Morton (1990), Ron (2000)).
    - A fairly comprehensive (although a bit dated) description of yield curve construction is given in Andersen and Piterbarg (2010).
    - Notes on some of the standard implementations by vendors are available in Jurcaga (2010), Lipman and Mercurio (2010), White (2012a), White (2012b), Gibbs and Goyder (2012), Misys (2012).

Discount Curves

1. Exact instrument quote match: Does the builder scheme successfully construct the curve if the quotes do not pose arbitrage? Conversely, for inexact matches, does the builder algorithm converge rapidly, and minimal error (Hagan and West (2006), Hagan and West (2008))?
2. Implied Forward Rates: Taken to be typically 1M or 3M forwards – how much should it matter, and how smooth/positive/continuous are they (McCulloch (1971))?
3. Locality: How local is the interpolating builder? If an input is changed, does the interpolator change only nearby, or is there spillover to non-adjacent far-off segments?
4. Stability of the Forward Rates: How sensitive are the forward rates to change in the inputs? The Jacobian analysis below shows the results for several splining scenarios.
    a. Forward rates are chosen for the curve behavior examination because it is the most elemental entity whose continuous/smooth behavior is meaningful to the practitioner.
5. Hedge Locality: Does most of the delta risk for a given instrument get assigned to the hedging instruments that have maturities close to the given instrument?
6. Sequential vs. Tenor Delta: Does the cumulative tenor delta equal to the aggregate (i.e., parallel shifted) delta? Le Floc'h (2013) examines the importance of this.

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Curve Construction Methodology

Approach

1. Instrument Setup: Identify the calibration instruments, and set up the instrument baseline. This includes initializing the span/segments, as well as the "tuning parameter" to achieve the desired "inner" and the "outer" calibrations.
2. Span/segment stretch set up: Calibrate the segments one by one using the calibration measures/inputs.
3. Tuning Adjustment: Adjust tuners to achieve the desired "boundary" condition.

State Span Design Components

1. Base Quantification Metric Retrieval: This refers to the functionality for retrieval of the State Quantification Metric Response Value at different predictor ordinates, the relative values, and canonical (possibly categorical) representations.
2. Targeted State Metric Computation: This functionality computes state/model specific targeted state metrics (e.g., LIBOR for a discount Curve, I Spread etc.) that may be absolute or relative.
3. Sensitivity Jacobian: This functionality provides for the ability to extract sensitivity Jacobian at the following levels:
    - Cross Quantification Metric (Quantification Metric 1 to Quantification Metric 2) Sensitivity Jacobian
    - External Manifest Metric to Quantification Metric Sensitivity Jacobian
4. Calibration Input Manifest Measure Retrieval: This functionality records and retrieves the calibration input manifest measure set and other relevant calibration details.

- It needs to be remembered that the calibration input manifest measure set need not just be instrument quotes, but also "event" rates such as user specified turns meant to account for items such as year-end yield adjustments, periods of high activity etc. (Ametrano and Bianchetti (2009), Kinlay and Bai (2009)). In the case of turns, they may be modeled as discrete latent state jumps across specific pairs of dates, of a user-specified magnitude.
- Exogenously specified State Differentials => As just noted, certain state attributes maybe exogenously specified (e.g., turns, bases, etc.). These state shift differentials may be applied before or after the calibration step.

5. Scenario State Span Re-construction: This functionality re-constructs the state using adjusted, bumped, or otherwise scenario-tweaked quantification metrics and/or manifest measures.
6. Boot State Span: This functionality is used in boot state spans. Here, there needs to be the ability to set the boot values at the node knots, and the build the segment.
7. Non-linear State Span: This functionality sets up the non-linear fixed-point extraction process and the corresponding target match criterion evaluator.

Curve Calibration From Instruments/Quotes

1. Instrument Conventions: Market Conventions for all the typical calibration instruments such as deposits, futures, FRA, IRS, float-float basis swaps, OIS, cross currency swaps etc. are available in a wide variety of publications (e.g., Open Gamma (2012)).
2. Construction from Single Instrument/Quote Set: If there is only one type instrument/quote set to be calibrated from, you can simply "spline" through the constituent segments. In particular, if there are no value limitations/constraints, then spline construction may be achieved directly from the points (e.g., bond yield curve).
    - Questionable if quote interpolation is necessary for even the single instrument set, since this results in double interpolation – the first on the quote space, and the second on the span/segment canonical space.

3. Construction from Diverse/Multiple Instrument/Quote Set: Given a diverse set of instruments and/or quotes, we need canonical quote-independent/quote-transforming measure formulation that is valid across the full instrument stretch.
4. Curve Span/Segment Latent State Quantification Metric: This is the metric used to quantify the latent state represented by the curve.
   - For discount curves, this can be the discount factor/zero rate/forward rate.
   - For forward curves, this can be the absolute forward rate/forward rate basis.
   - For credit curves, this can be survival factor/cumulative hazard rate/ forward hazard rate.
   - For recovery curves, this can be the expected loss/recovery, of the forward loss/recovery.
5. Cumulative vs. Incremental Quantification Metric: The incremental segment quantification metric Φ may be extracted from an appropriate transformation of the cumulative span quantification metric Z:

$$\Phi \to \frac{\partial(ZS)}{\partial S}$$

   where S is the span variate (specifically, time in this case).
6. Relation between the Quantification Metric: More generally

$$\Phi \to \mathfrak{I}\left(Z, S, \frac{\partial Z}{\partial S}\right)$$

   where $\mathfrak{I}$ comes from the physics of the process. For the discount curve, the credit curve, and the recovery curve

$$\mathfrak{I}\left(Z, S, \frac{\partial Z}{\partial S}\right) \to \frac{\partial(ZS)}{\partial S}$$

7. Cumulative Quantification Metric from Incremental Quantification: Cumulative may be extracted from the incremental forwards using the quadrature formulation, as they are integrands over the segment dimension. For survival/discount/recovery curves

$$
\mathrm{Z} = \frac{\int_0^t \Phi(S)dS}{t}
$$

8. Structure of the Cumulative vs. Incremental Forward: Forward quantification metric is more sharp-edged/swinging than cumulative quantification metric, which, by virtue of the quadrature construct, is smoother.
   - Therefore, single instrument/quote interpolation may be able to use the forward quantification metric, and imply the cumulative quantification metric.
   - Multiple instrument/quote should use the cumulative manifest metric, and perhaps imply the forward quantification metric using the segment <-> span transformation relationship.
9. Constraints on the Forward Quantification Metric: Depends on the physics of the underlying process.
   - For survival curve

     $$\Phi \geq 0$$

     and this is a hard constraint.
   - For discount curve, there are no such constraints.
   - For recovery curve, the constraint is that

     $$\Phi \geq 0$$

10. Constraints on the Cumulative Quantification Metric: Again depends on the underlying process behind the corresponding stochastic state variate (i.e., the QM).
    - For survival curve, if Z is the cumulative survival/hazard rate, $Z \geq 0$, and it should be monotonically decreasing - this is a hard constraint.
    - For discount curve, if Z is the discount factor, then $Z \geq 0$. Beyond this there are no constraints.
11. Interpolating in the Forward Quantification Metric Space: For survival/discount, due to the exponential nature of the formulation, splining on Φ can very often cause the prior two constraints to be violated – so relatively speaking, the choice is less stable.

12. Span/Segment Quantification Metric Relationship:

- Discontinuity in the cumulative quantification metric automatically implies discontinuity in the forward quantification metric.
- Continuous, but non-differentiable cumulative quantification metric implies discontinuity in the forward quantification metric.
- Continuity in the first derivative of cumulative quantification metric implies continuous, non-differentiable forward quantification metric.
- Most generally, C^k continuity of cumulative quantification metric (represented using, e.g., C^k splines) implies continuous, C^{k-1} continuity of the forward quantification metric.
- Certain splines become problematic for highly uneven segment lengths, e.g., cubic splines will be unsatisfactory for the situation where you start with close set of nodes and move to a sparser set (Burden and Faires (1997)). This is because the curve is too convex and bulging for points far away from each other.

13. Span Quantification Metric – "Effective" Rate/Hazard Rate: This can simply be defined as

$$\zeta = -\frac{\log Z}{t}$$

where Z is either the discount factor (for the discount curve) or the survival factor (for the survival curve). This needs to be matched for 4 powers (quartic) for polynomial spline, or for three derivatives for non-polynomial (e.g., tension) splines.

Calibration Considerations

1. Exponential/Hyperbolic Tension Splines as a Natural Basis for DF representation: This is popular (Sankar (1997), Securities Industry and Financial Markets Association (2004), Andersen (2005)) because the discount factor often simply monotonically decreases in time (e.g., as in an exponential). Obviously this basis will not be suitable for forward/zero rates.
   - The Trouble with the High-Tension Tension Splines is: This causes the segment responses to be almost linear with the predictor, therefore:

- For big gaps in the predictor ordinates, “linear” can soon become a huge problem.
- NASTY, NASTY low-tenor forward’s starting near the segment edges.
- High Tension implies high local forward interest (using above).
- While Renka (1987) shows an automatic way to extract to specify the tension parameter, the resulting C^1 presents fundamentally no more of an advantage than a C^1 cubic (Le Floc’h (2013)).
- Other issues with the impact of automatic selection (see Preuss (1978)) and the corresponding implications for sensitivities remain.

2. <u>Sensitivity of the Forward Rate to the Spot Measure</u>: The forward rate/DF sensitivity to the spot quote is not just low, but also ends up producing multiple matching results.
   - In particular, the presence of root multiplicity within a single segment (as is the case for polynomial splines) reduces the calibration to a needle in a haystack search – with huge demands on intelligent heuristics placed on the searcher.
3. <u>Pay Date DF Pre-computation</u>: This method is outlined in Kinlay/Bai, and is NOT a robust method, for the following reasons:
   - It starts by estimating the DF’s parametrically (using constant forwards) between dates.
   - Fine pay date grids (owing to, say, diverse/overlapping instrument types, and diverse/overlapping quote types) means that the interpolation grid becomes highly clustered, and this produces challenges for many splining techniques.
4. <u>Non-linear DV01</u>: The DV01-type terms

$$\sum_{j=1}^{n} l_j \Delta_j D_f(t_j)$$

are non-linear on both the discount factor and the forward rate for the generic (i.e., non-telescoping) interest-rate product – this is what makes the curve calibration using the Kinlay/Bai and the Andersen schemes difficult.

   a. While relating the discount factor the LIBOR rate as

$$D_f(t) = \left\{ \prod_{i=1}^{\eta(t)-1} \left[\frac{1}{1 + L_i(t_i - t_{i-1})} \right] \right\} \frac{1}{1 + L_{\eta(t)-1}\left(t - t_{\eta(t)-1}\right)}$$

may help simplify the formulation, it still does not reduce non-linearity. Here $\eta(t) - 1$ refers to the instrument maturity that precedes the time t.

5. <u>No Arbitrage Conditions</u>:
   - Forward Rates => No Arbitrage for Forwards implies that

$$\frac{\partial}{\partial t}[t \cdot r(t)] \geq 0$$

   although this can easily seen to be violated in several legitimate instances.

   - Options => Arbitrage free Implied Volatility Surface for Call Options (Homescu (2011))

$$\frac{\partial}{\partial t}[C(t, K)] \geq 0$$

   and

$$\frac{\partial^2}{\partial t^2}[C(t, K)] \geq 0$$

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Curve Construction Formulation

Introduction

1. Cash flow PV Linearity in Discount Factor and Survival: Simply put, the PV of a single cash flow is

$$PV = C_f \times D_f$$

or more generally

$$PV = C_f \times D_f \times S_P \times X$$

where C_f is the cash flow, D_f is the discount factor, S_P is the survival probability, and X is the FX rate. The challenge is to re-cast the measure computation in a manner that retains the formulation linearity in the latent state (it is already linear in D_f, S_P, and X, so that simplifies things a bit).

- Re-casting all the product/measure calibration as a linear equation depends on the product/manifest measure combination, but many typical formulations satisfy this criterion.

2. Linearized Discount Curve Formulation Schemes:

- Single Segment Giant Spline => Use all the market observations to construct all the linearization constraints to synthesize one giant multi-basis spline.
- One Spline Segment per adjacent cash flow pair => This gives maximal control, but ends up being way too computationally involved, as their will be as many spline segments as there are cash flow pairs.

- One Spline Segment per Instrument Maturity => Here a unique spline segment will be used between adjacent calibration instrument maturities. This ordering is identical to typical instrument level bootstrapping.
- Transition Spline => This retains the spline cluster per each instrument group. This representation is valuable when you have instruments assembling in cluster (as cash/EDF/swaps etc., which is obviously a typical arrangement). Judicious choice of knots and instruments etc. reduce the chances of jumps/bumps, although can still be a challenge.

3. Nomenclature:
    - Instrument Set => $l = 1, \cdots, a$
    - Segment exclusive to instrument l spans the times $\tau_{l-1} \to \tau_l$
    - Instrument l has b cash flows indexed by $j = 0, \cdots, b - 1$
    - Segment l's spline coefficients α_{il} are determined by l's cash flows and market quotes.
    - Each Segment has $i = 0, \cdots, n - 1$, i.e., n basis function set representing the discount factor.
    - Instrument l's cash flow j has a pay date of τ_{jl}.
4. Importance of some of the Linear Algebra Operations: While most of what is used in spline systems for linearized curve building can be achieved using a robust linear system solver (e.g., Gauss Elimination, see Press, Teukolsky, Vetterling, and Flannery (1992)), robust matrix inversion algorithms are needed for Jacobian estimation.

Segment Linear Discount Curve Calibration

1. Step #1: Identify and sort instruments by their maturities.
    - In between two maturities lies a segment, and the curve start date demarcates the start of the first (exclusive) segment.
2. Step #2: For each instrument, extract the coefficient of each discount factor (which corresponds to the net cash flow at that node).
3. Step #3: Say that the market PV quote of instrument l is Q_l. This indicates

$$Q_l = \sum_{j=0}^{b-1} c_{jl} D_f(t_{jl}) = \sum_{\substack{j=0 \\ t_{jl} \le \tau_{l-1}}}^{b-1} c_{jl} D_f(t_{jl}) + \sum_{\substack{j=0 \\ t_{jl} > \tau_{l-1}}}^{b-1} c_{jl} D_f(t_{jl})$$

5. Step #4: Given that all segment l cash flows whose pay date is less than τ_{l-1} belong to the prior periods, their discount factors should be computable. Thus

$$P_l = \sum_{\substack{j=0 \\ t_{jl} \le \tau_{l-1}}}^{b-1} c_{jl} D_f(t_{jl})$$

   should be pre-computed.

6. Step #5: The segment specific constraint now becomes

$$Q_l = P_l + \sum_{\substack{j=0 \\ t_{jl} > \tau_{l-1}}}^{b-1} c_{jl} D_f(t_{jl}) \Rightarrow \sum_{\substack{j=0 \\ t_{jl} > \tau_{l-1}}}^{b-1} c_{jl} D_f(t_{jl}) = Q_l - P_l$$

7. Step #6: In terms of the segment spline coefficients α_{il} and the segment basis functions f_{il}, the constraint gets re-specified as follows:

$$D_f(t_{jl}) = \sum_{i=0}^{n-1} \alpha_{il} f_{il}(t_{jl})$$

$$Q_l - P_l = \sum_{\substack{j=0 \\ t_{jl} > \tau_{l-1}}}^{b-1} c_{jl} D_f(t_{jl}) = \sum_{\substack{j=0 \\ t_{jl} > \tau_{l-1}}}^{b-1} c_{jl} \left[\sum_{i=0}^{n-1} \alpha_{il} f_{il}(t_{jl}) \right]$$

   Notice that

$$\Omega_l = \sum_{\substack{j=0 \\ t_{jl} > \tau_{l-1}}}^{b-1} \alpha_{jl} f_{jl}(t_{jl})$$

can be pre-computed. Thus, the above becomes

$$\sum_{\substack{j=0 \\ t_{jl} > \tau_{l-1}}}^{b-1} \alpha_{jl} f_{jl}(t_{jl}) \Omega_l = Q_l - P_l$$

8. Step #7: Of course, in general Q_l need not just be the P – it just needs to be any measure linearizable in the discount factor.
9. Cash D_f Loading:
   - Given a rate calibration measure r_l

$$D_f(\tau_l) = \frac{1}{1 + r_l \tau_l}$$

10. Futures D_f Loading:
    - Given a rate calibration measure r_l

$$\frac{-D_f(\tau_{l-1})}{r_l(\tau_l - \tau_{l-1})} + D_f(\tau_l) = 0$$

    - Given a price based calibration measure P_l

$$-P_l D_f(\tau_{l-1}) + D_f(\tau_l)$$

11. Fixed Stream D_f Loading: Given a price measure P_l

$$P_l = \sum_{j=0}^{b-1} c\Delta(t_{j-1}, t_j) D_f(t_j)$$

where c is the coupon.

12. Floating Stream D_f Loading: Given a price measure P_l

$$P_l = \sum_{j=0}^{b-1} s\Delta(t_{j-1}, t_j) D_f(t_j) - [D_f(t_0) - D_f(t_m)]$$

where s is the floater spread.

13. IRS D_f Loading:

- For a par swap IRS

$$PV_{Fixed} - PV_{Floating} = 0$$

$$\sum_{j=0}^{b-1} c\Delta(t_{j-1}, t_j) D_f(t_j) - \sum_{j=0}^{b-1} s\Delta(t_{j-1}, t_j) D_f(t_j) + [D_f(t_0) - D_f(t_m)] = 0$$

- Given a price measure P_l

$$P_l = \sum_{j=0}^{b-1} c\Delta(t_{j-1}, t_j) D_f(t_j) - \sum_{j=0}^{b-1} s\Delta(t_{j-1}, t_j) D_f(t_j) + [D_f(t_0) - D_f(t_m)]$$

14. Bond D_f Loading:

- Given a dirty price measure P_l

$$P_l = \sum_{j=0}^{b-1} c\Delta(t_{j-1}, t_j) D_f(t_j) + \sum_{k=0}^{a-1} N_k(t_k) D_f(t_k)$$

- Given a yield measure, the yield can be converted to the dirty price measure P_l.
- Given a spread over TSY measure, it may also be converted to the dirty price measure P_l through the yield.

Curve Jacobian

1. Representation Jacobian: Every curve implementation needs to generate the Jacobian of the following latent state metric using its corresponding latent state quantification metric:
   - Forward Rate Jacobian to Quote Manifest Measure
   - Discount Factor Jacobian to Quote Manifest Measure
   - Zero Rate Jacobian to Quote Manifest Measure
2. Importance of the Representation Self-Jacobian: Representation Self-Jacobian computation efficiency is critical, since Jacobian of any function $F(Y)$ is going to be dependent on the self-Jacobian $\frac{\partial Y(t)}{\partial Y(t_k)}$ because of the chain rule.
3. Forward Rate - DF Jacobian:

$$F(t_A, t_B) = \frac{1}{t_B - t_A} \log \frac{\partial D_f(t_A)}{\partial D_f(t_B)}$$

$$\frac{\partial F(t_A, t_B)}{\partial D_f(t_k)} = \frac{1}{t_B - t_A} \left\{ \frac{1}{D_f(t_A)} \frac{\partial D_f(t_A)}{\partial D_f(t_k)} - \frac{1}{D_f(t_B)} \frac{\partial D_f(t_B)}{\partial D_f(t_k)} \right\}$$

   where $F(t_A, t_B)$ is the forward rate between t_A and t_B, and $D_f(t_k)$ is the discount factor at time t_k.
4. Zero Rate to Forward Rate Equivalence: This equivalence may be used to construct the Zero Rate Jacobian From the Forward Rate Jacobian. Thus the above equation may be used to extract the Zero Rate micro-Jacobian.
5. Zero Rate - DF Jacobian:

$$\frac{\partial Z(t)}{\partial D_f(t_k)} = \frac{1}{t - t_0}\left\{\frac{1}{D_f(t)}\frac{\partial D_f(t)}{\partial D_f(t_k)}\right\}$$

where $Z(t)$ is the zero rate at time t.

6. Analytical Sensitivity vs. Quote Bumped Sensitivity: In general, when dealing with the splined mechanisms for curve cooking, it may not be accurate to depend on the quote bumped sensitivity, because it may end up throwing it to a totally different curve builder scheme (Le Floc'h (2013)).
    - Also, analytical sensitivities may be estimated right during the calibration itself. However, analytical-to-quote sensitivities implies two-stage Jacobian – the Jacobian of the quote to the state representations, then the Jacobian of the state representation to the sensitivity measure.
    - In-situ Calibration Sensitivities => Measure to state sensitivities maybe generated quiet readily, depending on the calibration mode.
        - For linear calibrator, this is simply the state Jacobian inverse.
        - In some non-linear search techniques (esp. open ones like the Newton's method, but with the closed schemes as well), sensitivity Jacobians are automatically (or using light adjustment) generated as part of the calibration itself.
    - Spline coefficient sensitivity to segment/node inputs => High sensitivity of the spline coefficients to the node inputs across specific stretches indicates instability in curve (re-) construction and the corresponding deltas (i.e., spurious deltas and leakage). Le Floc'h (2013) examines this for several standard interpolating estimators in use.
7. Quote Jacobian via the Discount Factor Latent State:
    - $c \Longrightarrow 0, \cdots, d-1$ Calibration Components
    - $q_c \Longrightarrow q_0, \cdots, q_{d-1}$ Corresponding Quotes
    - Let's say the Derivative PV is

$$P = \sum_{j=1}^{m} \Upsilon_j D_f(t_j) \Longrightarrow \frac{\partial P}{\partial q_c} = \sum_{j=1}^{m} \Upsilon_j \frac{\partial D_f(t_j)}{\partial q_c}$$

Thus what is typically needed to estimate product-to-quote sensitivities via the Discount Factor latent state is $\frac{\partial D_f(t_j)}{\partial q_c}$.

8. Quote->Zero Rate Jacobian:

$$\frac{\partial Q_j(t)}{\partial Z(t_k)} = (t_k - t_0)\left\{D_f(t_k)\frac{\partial Q_j(t)}{\partial D_f(t_k)}\right\}$$

where $Z(t_k)$ is the zero-rate at time t_k.

9. PV - Quote Jacobian:

$$\frac{\partial PV_j(t)}{\partial Q_k} = \sum_{i=1}^{n}\left\{\frac{\partial PV_j(t)}{\partial D_f(t_i)} \Big/ \frac{\partial Q_j(t)}{\partial D_f(t_i)}\right\}$$

10. Cash Rate DF micro-Jacobian:

$$\frac{\partial r_j}{\partial D_f(t_k)} = \frac{1}{D_f(t_j)}\frac{1}{t_j - t_{START}}\frac{\partial D_f(t_j)}{\partial D_f(t_k)}$$

where r_j is the cash rate quote for the j^{th} Cash instrument, and $D_f(t_j)$ is the discount factor at time t_j.

11. Cash Instrument PV-DF micro-Jacobian:

$$\frac{\partial PV_{CASH,j}}{\partial D_f(t_k)} = -\frac{1}{D_f(t_{j,SETTLE})}\frac{\partial D_f(t_j)}{\partial D_f(t_k)}$$

There is practically no performance impact on construction of the PV-DF micro-Jacobian in the adjoint mode as opposed to the forward mode, due to the triviality of the adjoint.

12. Futures Quote-DF micro-Jacobian:

$$\frac{\partial Q_j}{\partial D_f(t_k)} = \frac{1}{D_f(t_{j,START})} \frac{\partial D_f(t_j)}{\partial D_f(t_k)} - \frac{D_f(t_j)}{{D_f}^2(t_{j,START})} \frac{\partial D_f(t_{j,START})}{\partial D_f(t_k)}$$

where Q_j is the Quote for the j^{th} Futures with start date of $t_{j,START}$ and maturity of t_j.

13. Futures PV-DF micro-Jacobian:

$$\frac{\partial PV_j}{\partial D_f(t_k)} = \frac{1}{D_f(t_{j,START})} \frac{\partial D_f(t_j)}{\partial D_f(t_k)} - \frac{D_f(t_j)}{{D_f}^2(t_{j,START})} \frac{\partial D_f(t_{j,START})}{\partial D_f(t_k)}$$

There is practically no performance impact on construction of the PV-DF micro-Jacobian in then adjoint mode as opposed for forward mode, due to the triviality of the adjoint.

14. Interest Rate Swap DF micro-Jacobian:

$$Q_j DV01_j = PV_{Floating,j}$$

where Q_j is the quote for the j^{th} IRS maturing at t_j, $DV01_j$ is the DV01 of the IRS, and $PV_{Floating,j}$ is the floating PV of the IRS.

$$\frac{\partial [Q_j DV01_j]}{\partial D_f(t_k)} = \frac{\partial [PV_{Floating,j}]}{\partial D_f(t_k)}$$

$$\frac{\partial [Q_j DV01_j]}{\partial D_f(t_k)} = \frac{\partial Q_j}{\partial D_f(t_k)} DV01_j + Q_j \frac{\partial DV01_j}{\partial D_f(t_k)}$$

$$\frac{\partial DV01_j}{\partial D_f(t_k)} = \sum_{i=1}^{j} N(t_i) \Delta_i \frac{\partial D_f(t_i)}{\partial D_f(t_k)}$$

$$PV_{Floating,j} = \sum_{i=1}^{j} L_i N(t_i) \Delta_i D_f(t_i)$$

$$\frac{\partial PV_{Floating,j}}{\partial D_f(t_k)} = \sum_{i=1}^{j} \frac{\partial L_i}{\partial D_f(t_k)} N(t_i) \Delta_i D_f(t_i) + \sum_{i=1}^{j} L_i N(t_i) \Delta_i \frac{\partial D_f(t_i)}{\partial D_f(t_k)}$$

15. Interest Rate Swap PV-DF micro-Jacobian: See Hull (2002) for the preliminaries.

$$\frac{\partial PV_{IRS,j}}{\partial D_f(t_k)} = \sum_{i=1}^{j} N(t_i) \Delta(t_{i-1}, t_i) \left\{ (c_j - L_i) \frac{\partial D_f(t_i)}{\partial D_f(t_k)} - D_f(t_i) \frac{\partial L_i}{\partial D_f(t_k)} \right\}$$

There is no performance impact on construction of the PV-DF micro-Jacobian in then adjoint mode as opposed for forward mode, due to the triviality of the adjoint. Either way the performance is $\mathcal{O}(n \times k)$, where n is the number of cash flows, and k is the number of curve factors.

16. Credit Default Swap DF micro-Jacobian:

$$PV_{CDS,j} = PV_{Coupon,j} - PV_{LOSS,j} + PV_{ACCRUED,j}$$

where j refers to the jth CDS Contract with a maturity t_j, c_j is its Coupon, $PV_{CDS,j}$ is the full contract PV, $PV_{Coupon,j}$ is the PV of the coupon leg of the CDS contract, $PV_{LOSS,j}$ is the PV of the loss leg of the CDS contract, and $PV_{ACCRUED,j}$ is the PV of the accrual paid on default.

$$PV_{Coupon,j} = c_j \sum_{i=1}^{j} N(t_i) \Delta_i S_P(t_i) D_f(t_i)$$

$$\frac{\partial PV_{Coupon,j}}{\partial D_f(t_k)} = c_j \sum_{i=1}^{j} N(t_i) \Delta_i S_P(t_i) \frac{\partial D_f(t_i)}{\partial D_f(t_k)} + \frac{\partial c_j}{\partial D_f(t_k)} \sum_{i=1}^{j} N(t_i) \Delta_i S_P(t_i) D_f(t_i)$$

$$PV_{LOSS,j} = \int_{0}^{t_j} N(t)[1 - R(t)]D_f(t)dS_P(t)$$

$$\frac{\partial PV_{LOSS,j}}{\partial D_f(t_k)} = \int_{0}^{t_j} N(t)[1 - R(t)]\frac{\partial D_f(t)}{\partial D_f(t_k)}dS_P(t)$$

$$PV_{ACCRUED,j} = c_j \sum_{i=1}^{j}\left[\int_{t_{j-1}}^{t_j} N(t)\Delta(t_{i-1}, t_i)D_f(t)dS_P(t)\right]$$

$$\frac{\partial PV_{ACCRUED,j}}{\partial D_f(t_k)} = \frac{\partial c_j}{\partial D_f(t_k)}\sum_{i=1}^{j}\left[\int_{t_{j-1}}^{t_j} N(t)\Delta(t_{i-1}, t_i)D_f(t)dS_P(t)\right] + c_j \sum_{i=1}^{j}\left[\int_{t_{j-1}}^{t_j} N(t)\Delta(t_{i-1}, t_i)\frac{\partial D_f(t)}{\partial D_f(t_k)}dS_P(t)\right]$$

17. Credit Default Swap DF micro-Jacobian:

$$\frac{\partial PV_{CDS,j}}{\partial D_f(t_k)} = c_j \sum_{i=1}^{j} N(t)\Delta(t_{i-1}, t_i)\frac{\partial D_f(t_i)}{\partial D_f(t_k)}S_P(t_i) + \frac{\partial c_j}{\partial D_f(t_k)}\sum_{i=1}^{j}\left[\int_{t_{j-1}}^{t_j} N(t)\Delta(t_{i-1}, t_i)D_f(t)dS_P(t)\right] + \sum_{i=1}^{j}\left[\int_{t_{j-1}}^{t_j} N(t)\{c_j\Delta(t_{i-1}, t_i) - [1 - R(t)]\}\frac{\partial D_f(t)}{\partial D_f(t_k)}dS_P(t)\right]$$

There is no performance impact on construction of the PV-DF micro-Jacobian in then adjoint mode as opposed for forward mode, due to the triviality of the adjoint. Either way the performance is $\mathcal{O}(n \times k)$, where n is the number of cash flows, and k is the number of curve factors.

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Stream-based Calibration

Latent State Formulation Metric (LSFM)

1. Case for LSFM: In addition to the quantification metric employed as described above for quantifying the latent state, we also need a "Latent State Formulation Metric". The LSFM is the metric that dictates the formulation specification for the predictor/response constraint relation for the latent state at hand. For e.g., commonly price/PV based formulation (i.e., predictor/response relation determination) is used in the discount curve construction using swap calibration instruments, whereas direct manifest measure observations (e.g., the observed FRA rate or the par forward deposit rate etc.) are used for forward curve construction (if zero coupon bond prices are available, they are form direct manifest measure maps of the discount factor quantification metric). While the quantification metric representation is chosen the same across all the constituent segments/stretches to facilitate ancillary objectives (e.g., smoothness/C^k requirements), the LSFM chosen need not be subject to such limitations. The only demand is that, using the manifest measure, the formulation metric result in a linear relation involving the LSQM's corresponding to a given segment/stretch/span.
2. Latent State QM as the Formulation Metric: In this case the relation becomes trivial, as $QM \equiv FM$, and the $QM \leftrightarrow FM$ Jacbian reduces to unity (thus producing a unit QM loading).

Stream Inference Setup

1. The Calibration Entities: The principle quantities involved in the latent state calibration are the latent state response variables, the manifest quote measure, and the formulation metric. Typical latent state calibration relations are set up so that the linearity between the latent state quantification metric and the formulation metric are maintained (there are notable exceptions,

however - e.g., $PV_{Deposit}$ $vs. FwdRate_{Deposit}$). However, the relationship between the manifest measure quote and the latent state quantification/formulation metric WILL NOT be linear, generally speaking.

2. Latent State Quantification Metric Ordinate Affixation or Predictor Tagging: Typical quantification metrics for latent states (such as collateral, funding, FX etc.) affix/tag their responses to the pay date predictor ordinate node. The forward latent state (quantified using, say, the forward rate) is an exception, as seen below.
3. Forward Rate Quantification Metric "Affixation Ordinate" Choice: While the "affixation ordinate" for the discount factor is point-wise unique (i.e., it corresponds to the pay date), similar affixation for the forward rate is unique only to within the segment range (i.e., either the start/end of the period). This would allow the choice of any of reset/start/end as a viable nominal affixation ordinate. However, the specific choice of the inference routines (e.g., a boot calibrator) may render some choices of the affixation ordinate more convenient than the others. For instance, the boot calibrator utilizes the notion of sequential segment build-out, thus a particular choice of the affixation ordinate (namely the start/end date against the reset date) may fit in very well with the marking scheme applied to attach the quote to its corresponding exclusive manifest segment.

Coupon Period-Based Calibration Specification

1. Period Latent State Loading: The period formulation metric may require the latent state response values at one/more time predictor ordinates. Recalling that the latent state loading at the appropriate predictor ordinate represents the linear calibration coefficient for the latent states, single point formulation metric requires single point state loading, and multi-point formulation metric requires multi-point state loading.
2. Single Point State Formulation Metric: Whenever the formulation metric is dependent only on a single latent state response value realization, we require just a single corresponding loading. Examples include single period single reset forward rate, period terminal discount factor, period survival probability, period pay FX rate, etc.
3. Single Point State Loading: In the case of PV formulation metric, the PV for period j is

$$PV_j = \Delta(t_{j-1}, t_j) \times \mathcal{L}_{Compounded,X}(t_{j-1}, t_j) \times S_P(t_j) \times D_f(t_j) \times FX(t_j) \\ \times Convexity(t, t_{j-1}, t_j)$$

a. Credit Loading =>

$$\Delta(t_{j-1}, t_j) \times \mathcal{L}_{Compounded,X}(t_{j-1}, t_j) \times D_f(t_j) \times FX(t_j) \times Convexity(t, t_{j-1}, t_j)$$

b. Funding Loading =>

$$\Delta(t_{j-1}, t_j) \times \mathcal{L}_{Compounded,X}(t_{j-1}, t_j) \times S_P(t_j) \times FX(t_j) \times Convexity(t, t_{j-1}, t_j)$$

c. Forward Loading =>

$$\Delta(t_{j-1}, t_j) \times S_P(t_j) \times D_f(t_j) \times FX(t_j) \times Convexity(t, t_{j-1}, t_j)$$

d. FX Loading =>

$$\Delta(t_{j-1}, t_j) \times \mathcal{L}_{Compounded,X}(t_{j-1}, t_j) \times S_P(t_j) \times D_f(t_j) \times Convexity(t, t_{j-1}, t_j)$$

e. Predictor Ordinate Anchoring => As will be seen later, it is common to anchor the credit loading and the forward loading to period end date predictor ordinate, while funding and FX loading are anchored to period pay date predictor ordinate.

4. Multi-Point Loading - Multi-Reset: This corresponds to the joint cases of a) multi-reset periods per coupon period, and b) the reset periods get compounded arithmetically. In this situation a compounding adjustment identical to the typical non-merged state forward/funding convexity adjustment is applied to each reset period k as

$$ForwardLoading_k = \Delta(t_{j,k-1}, t_{j,k}) \times S_P(t_j) \times D_f(t_j) \times FX(t_j) \times Convexity(t, t_{j,k-1}, t_{j,k}) \quad (10.6)$$

5. Multi-Point Loading - Quadrature: Quadrature-based multi-point loading results from the state response realizations being evaluated using a quadrature routine, e.g., loss quadrature grid – and therefore is continuous. While this is the primary distinction between multi-point reset and multi-point quadrature loadings, in practice the quadrature loadings also tend to be discretized – although a finer granularities.

Stream-Based Calibration Specification

1. Calibration State Loadings and Stream Sensitivities Sought: We consider the case of curve construction for the discount rates and the forward rate latent state as a concrete example. In this case we seek:
    a. Distinct Discount State Segment-local Quantification Metric Loading
    b. Distinct Discount State Segment-local Quantification Metric Jacobian Loading
    c. Distinct Forward State Segment-local Quantification Metric Loading
    d. Distinct Forward State Segment-local Quantification Metric Jacobian Loading
    e. Merged Discount/Forward State Segment-local Quantification Metric Loading
    f. Merged Discount/Forward State Segment-local Quantification Metric Jacobian Loading
2. Boot Stretch Calibration: In addition to the above, given that we are going to be focused on a boot stretch with manifest measure exclusivity $[a_+ \rightarrow b]$, we also seek to determine the leading formulation metric contribution from the leading segments/regimes of the stretch $[0 \rightarrow a]$. Given the boot framework, we presume no contributions arising out of the trailing segments.
3. Fixed and Floating Streams: We treat each stream as the calibration unit, since the potential state merging and telescoping occur at this level. The fixed and the floating streams are:

$$\mathfrak{I}_x(a,b) = c\sum_{i=1}^{a} \Delta(t_{i-1}, t_i)\, D_f(t_i) + c\sum_{i=b+1}^{b} \Delta(t_{i-1}, t_i)\, D_f(t_i)$$

$$\mathfrak{I}_l(a,b) = \sum_{i=1}^{a} [\mathcal{L}(t_{i-1}, t_i) + \beta]\Delta(t_{i-1}, t_i)\, D_f(t_i) + \sum_{i=a+1}^{b} \mathcal{L}(t_{i-1}, t_i)\Delta(t_{i-1}, t_i)\, D_f(t_i) + \beta\sum_{i=a+1}^{b} \Delta(t_{i-1}, t_i)\, D_f(t_i)$$

β is the floating stream basis, and we've partitioned the manifest measure exclusive segment $[a_+ \to b]$ into the floater and the basis parts.

4. Leading Stream Contribution:
    a. Merged/non-merged Fix =>

$$\mathfrak{I}_x(a,b) = c\sum_{i=1}^{a} \Delta(t_{i-1}, t_i)\, D_f(t_i)$$

    b. Merged/non-merged Floater =>

$$\mathfrak{I}_l(a,b) = \sum_{i=1}^{a} [\mathcal{L}(t_{i-1}, t_i) + \beta]\Delta(t_{i-1}, t_i)\, D_f(t_i)$$

    c. The Merged Floater reduces to

$$\mathfrak{I}_l(a,b) = D_f(t_0) - D_f(t_a) + \beta\sum_{i=1}^{a} \Delta(t_{i-1}, t_i)\, D_f(t_i)$$

5. Fixed Stream D_f Loading:

$$\frac{\partial \mathfrak{I}_x(a,b)}{\partial D_f(t_k)} = \begin{cases} c\Delta(t_{k-1},t_k) & k \in [a_+ \to b] \\ 0 & k \notin [a_+ \to b] \end{cases}$$

6. Floating Stream D_f Loading:

$$\frac{\partial \mathfrak{I}_l(a,b)}{\partial D_f(t_k)} = \begin{cases} [\mathcal{L}(t_{k-1},t_k) + \beta]\Delta(t_{k-1},t_k) & k \in [a_+ \to b] \\ 0 & k \notin [a_+ \to b] \end{cases}$$

7. Distinct Forward Rate: Given that the fixed stream relies on no floating stream payments, the forward latent quantification metric loadings will be NULL. However, for the floating stream

$$\frac{\partial \mathfrak{I}_l(a,b)}{\partial F_f(t_k)} = \begin{cases} \Delta(t_{k-1},t_k)D_f(t_k) & k \in [a_+ \to b] \\ 0 & k \notin [a_+ \to b] \end{cases}$$

8. Loadings vs. Constraint for the Stream: While a given stream may not have explicit dependence on the specified latent state, it may still participate in the constraint generation process. For instance, in the case of the fix-float swap seen above, the fixed stream does not have dependence on the floating rate, but will still contribute to the net PV (in this case through the PV formulation metric).

Calibration of Multi-Stream Component

1. Loadings Consolidation as Linear Overlays: In the case of linear formulation metric, the loadings and the constraints of the individual streams are simply overlaid across onto the component level. Thus, for the component index $j = 1, \ldots, m$, the consolidated constraint becomes

$$\sum_{j=1}^{m}\sum_{i=1}^{n} \alpha_{ij}\mathcal{L}_{ij} = \sum_{j=1}^{m} \mathbb{C}_j$$

where α_{ij} corresponds to the loading for node i and component i, and $\mathbb{C}_j$ is the corresponding constraint wo within the containing segment . Clearly this generalizes well to the case of more than 2 components, with the only limitation being that there can only be one outstanding quantification metric to be inferred (of course, in the case of merged latent states, as single quantification metric suffices to uniquely quantify multiple latent states).

2. Formulation Metric Consistency across the Streams and the Component: As is obvious, the inference within a single segment (for all streams contained within the component's manifest measure exclusive segment) NEED to share the same formulation metric. While the unitary loadings generation entity is still the stream, the latent state sequence builder interacts (via the specified manifest measure quotes) only with the component. This implies that the component needs to maintain an intimate awareness of the layout/metric of the corresponding constituent streams, and may "create/translate/introduce" stream-specific manifest measures during the calibration run.
3. Fixed Income Product Aggregations: Just as cash flows get aggregated into streams, streams get aggregated onto components, and components onto products. Also cash flows inside a stream get telescoped out to simplify valuation/loadings generation. Likewise, entire streams may get telescoped off inside constituent product components – in particular this feature is utilized in "package calibrations" – as in CCBS discount/forward, and USD OIS using LIBOR-OIS and LIBOR-fixed swaps.

Spanning Spline

Formulation and Set up

1. Spline vs. Boot Span: For the purposes of this discussion, the main difference between spline span and boot span is that, in boot span, the segment boundaries HAVE to line up with the instrument maturity edges. In spline spans, however, additional criterion-based knots may be used to determine the boundaries (e.g., parametric knot insertion in line with regression spline approaches).
2. Basic Setup: All instruments and quotes fall into one set of constraints as

$$\sum_{j=0}^{b-1} c_{jl} D_f(t_{jl}) = Q_l$$

   where $l = 1, \cdots, a$.

   - In general, $a < b$, so you have $b - a$ degrees of freedom.
3. Local Ordinate Re-formulation: The spline extends from $t_{START} \to t_{b-1}$. Setting

$$x_i = \frac{t_i - t_{START}}{t_{b-1} - t_{START}}$$

$$\sum_{j=0}^{b-1} c_{jl} D_f(t_{jl}) \Rightarrow \sum_{j=0}^{b-1} c_{jl} D_f(x_{jl})$$

   Further

$$D_f(t_{START}) = D_f(x = 0) = 1$$

4. Basis Formulation: Setting

$$D_f(x) = \sum_{i=0}^{n-1} \alpha_i f_i(x)$$

$$\sum_{j=0}^{b-1} c_{jl} \sum_{i=0}^{n-1} \alpha_i f_i(t_j) = Q_l \Rightarrow \sum_{i=0}^{n-1} \alpha_i \left\{ \sum_{j=0}^{b-1} c_{jl} f_i(t_j) \right\} = Q_l$$

Thus, if

$$n = a$$

there now are a equations and a unknowns.

5. Monotonicity Preservation in Spanning Splines: The heterogeneity of the calibration instruments demands special techniques for monotonicity maintenance (Hagan West (2006) described in detail earlier was a sample).
   - Stringent monotonic constraints introduced by Hyman (1983) was relaxed by Dougherty, Edelman, and Hyman (1989), and this was works well in practice in its ability to maintain monotonicity (Ametrano and Bianchetti (2009a), Le Floc'h (2013), also implemented in Quantlib (2009)).
   - Intermediate filter constraints introduced by Steffen (1990) and their variants treated in some detail by Huynh (1993) – all suffer from the same unnatural dips or cook bumps.
6. Pros: As always, the degrees of freedom may be expanded beyond a to allow for optimizing spline construction (covered in the spline builder section).
7. Cons: With many basis functions (esp. for polynomials), the inevitable Runge's phenomenon takes over.

Challenges with the Spanning Spline Approach

1. Problems with Cubic Polynomial Spline: Too well known to documented – spurious inflection, too much concavity/convexity at widely separated predictor nodes (esp. in long end), and no guarantee of positivity where desired.
    - As noted in Le Floc'h (2013), monotone variants (including Hagan and West (2006), Wolberg and Alfy (1999), Hyman (1983)) of the standard cubic spline have differing degrees of problems since they are attempt to model the entire span with a single representation.
2. Problems with Quartic Spline: While this makes the interpolation very smooth (Adams and van Deventer (1994), van Deventer and Inai (1997), Adams (2001), Lim and Xiao (2002), Quant Financial Research (2003)), the stiffness needed for shape-preservation is completely lost. Other troubles as with cubic splines (spurious inflection, too much concavity/convexity at widely separated predictor nodes (esp. in long end), and no guarantee of positivity where desired) as well Runge's swings are also present.

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Monotone Decreasing Splines

Motivation

1. These are spline basis functions that monotonically decrease over the given interval. Valuable for representing discount factors.
2. Why represent discount factors? Because the pay-offs are linearizable in them, so working with them implies working with the linear rates space representation, and all the advantages that come with that.

Exponential Rational Basis Spline

1. Basis Function Set:

$$\left\{1, \frac{1}{1+t}, e^{-t}, \frac{e^{-t}}{1+t}\right\}$$

2. Monotone Decreasing Nature: Each of the above basis functions is decreasing. If

$$\beta_i > 0 \;\forall\; i$$

then, conservatively speaking, the functional form is monotonically decreasing.

- Alternatively, we may also require that no inflection exist within the given segment, but that is hard to enforce with this set.

Exponential Mixture Basis Set

1. Motivation: Since the discounting function goes as e^{-t}, an exponential mixture basis such as $e^{-\lambda_i t}$ may be a good choice, as they are both intuitively monotone, and linear combinations of them produce convexity/concavity.
2. Basis Function Set:

$$\{e^{-\lambda_i t}\}; i \in [0, \cdots, n-1]$$

   - Choosing λ_i: Since for C^2 continuity we require 4 basis functions, we choose $\lambda_{Floor} = 0$, λ_{Low}, λ_{Medium}, and λ_{High}. $\lambda_{Floor} = 0$ accounts for adjusting jumps.
   - Typical values can be: $\lambda_{Low} = 0.01\%$, $\lambda_{Medium} = 5\%$, and $\lambda_{High} = 25\%$.
   - Parallel with Tension Splines => λ_i are comparable to tension splines.
   - With this choice, C^k may be maintained for $k \geq 2$, thereby making the forwards continuous, preserving locality, imparting segment convexity/concavity. Thus all the smoothing schemes may be maintained.
3. Similarity with exponential/hyperbolic tension splines: Very similar in formulation. However, given that with exponential/hyperbolic basis set spline at one of basis functions has a non-negative exponential argument, that basis function becomes monotonically increasing.
   - Further, while estimation of the exponential tension needs to be done extraneously (Renka (1987)), here we appeal to the intuitive physics, as shown.

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Hagan West (2006) Smoothness Preserving Spanning Spline

Monotone/Convexity Preserving Estimator

1. Premise: This is primarily focused on a quadratic interpolant, but it also contains heterogeneously inserted sub-segment knots in effect to achieve the desired monotonicity, convexity, and positivity effect.
2. Philosophy:
   - This is mainly meant for forward rates inside finance, although bit more general outside of it.
   - The observation set $\{z_i\}_{i=1}^{n}$ is simply a quantity conserved on a per-segment basis, e.g., the segment mean of the state variate response, i.e.,

$$z_i = \frac{1}{\tau_i - \tau_{i-1}} \int_{\tau_{i-1}}^{\tau_i} y(t)dt$$

   - $y(t)$ is positive and piece-wise quadratic inside of $\{\tau_{i-1}, \tau_i\}$.
   - The node response value y_i at the predicate ordinate τ_i is linearly interpolated from the observations at z_i and z_{i+1} (obviously edges will be treated slightly differently).
   - Based on the specified monotonicity maintenance and convexity preservation criteria, the algorithm identifies and inserts knots. Zero or more knots may need to be inserted.
   - The quadratic interpolant is essentially a Bessel C^1 Hermite interpolant.
   - Finally, similarity response value may be applied for positivity, and range-bounded-ness.
3. Steps:
   - Infer the response node value y_i at the predicate ordinate τ_i is linearly interpolated from the observations at z_i and z_{i+1} as:

$$y_i = \frac{\tau_i - \tau_{i-1}}{\tau_{i+1} - \tau_{i-1}} z_{i+1} + \frac{\tau_{i+1} - \tau_i}{\tau_{i+1} - \tau_{i-1}} z_i, i \neq 0, n$$

$$y_0 = z_1 - \frac{1}{2}[y_1 - z_1]$$

$$y_n = z_n - \frac{1}{2}[y_{n-1} - z_n]$$

- Work out the "Z-score" metric within $\{\tau_{i-1}, \tau_i\}$:

$$g_{i-1} = y(\tau_{i-1}) - z_i = y_{i-1} - z_i$$

$$g_i = y(\tau_i) - z_i = y_i - z_i$$

Further, we work in the local predictor ordinate space x, where

$$x = \frac{\tau - \tau_{i-1}}{\tau_i - \tau_{i-1}}$$

- Apply the appropriate adjustments for the monotonicity/convexity enforcement at the appropriate zones:
    - Case I => This case arises when either

$$g_{i-1} > 0 \ AND \ \frac{1}{2} g_{i-1} \geq g_i \geq -2g_{i-1}$$

or

$$g_{i-1} < 0 \ AND \ -\frac{1}{2} g_{i-1} \leq g_i \leq -2g_{i-1}$$

is true. In this case the function

$$g(\tau) = g_{i-1}(1 - 4x + 3x^2) + g_i(-2x + 3x^2)$$

can be used unchanged, as the original construct is already monotone and convex.

- Case II => This case arises when either

$$g_{i-1} > 0 \; AND \; g_i \geq -2g_{i-1}$$

or

$$g_{i-1} < 0 \; AND \; g_i \leq -2g_{i-1}$$

is true. Here, insert a knot at

$$\eta = \frac{g_i + 2g_{i-1}}{g_i - g_{i-1}}$$

The segment univariate becomes

$$g(\tau) = g_{i-1}; 0 \leq x \leq \eta$$

and

$$g(\tau) = g_{i-1} + (g_i - g_{i-1})\left[\frac{x - \eta}{1 - \eta}\right]^2; \eta < x \leq 1$$

- Case III => This case arises when either

$$g_{i-1} > 0 \; AND \; 0 > g_i > -\frac{1}{2}g_{i-1}$$

or

$$g_{i-1} < 0 \; AND \; 0 < g_i \leq -\frac{1}{2} g_{i-1}$$

is true. Here, insert a knot at

$$\eta = \frac{3g_{i-1}}{g_i - g_{i-1}}$$

The segment univariate becomes

$$g(\tau) = g_{i-1} + (g_i - g_{i-1}) \left[\frac{\eta - x}{\eta}\right]^2 ; 0 \leq x < \eta$$

and

$$g(\tau) = g_i ; \eta \leq x \leq 1$$

- Case IV => This case arises when either

$$g_{i-1} \geq 0 \; AND \; g_i \geq 0$$

or

$$g_{i-1} \leq 0 \; AND \; g_i \leq 0$$

is true. Here, insert a knot at

$$\eta = \frac{g_i}{g_i + g_{i-1}}$$

Setting

$$A = -\frac{g_i g_{i-1}}{g_i + g_{i-1}}$$

the segment univariate becomes

$$g(\tau) = A + (g_i - A)\left[\frac{\eta - x}{\eta}\right]^2 ; 0 \leq x < \eta$$

and

$$g(\tau) = A + (g_i - A)\left[\frac{x - \eta}{1 - \eta}\right]^2 ; \eta \leq x \leq 1$$

Positivity Preserving Estimator

1. Positivity of the interpolant: Hagan and West (2006) guarantee this by setting the following bounds:

$$y_0 = bound[0, y_0, 2z_1]$$

$$y_n = bound[0, y_n, 2z_n]$$

$$y_i = bound[0, y_i, 2 \times min(z_i, z_{i+1})] ; i \neq 0, n$$

Ameliorating Estimator

1. Amelioration (i.e., Smoothing) of the Interpolant - Steps:

- #1: Expand the Range at the edges => Add an interval at the beginning and at the end. Define

$$\tau_{-1} = \tau_0 - (\tau_1 - \tau_0)$$

$$z_0 = z_1 - \frac{\tau_1 - \tau_0}{\tau_2 - \tau_0}(z_2 - z_1)$$

$$\tau_{n+1} = \tau_n + (\tau_n - \tau_{n-1})$$

$$z_{n+1} = z_n - \frac{\tau_n - \tau_{n-1}}{\tau_n - \tau_{n-1}}(z_n - z_{n-1})$$

  Complete the linear interpolation of the response variate across all the intervals as before.

- #2: Set the Extraneous Bounds Parametrically/Empirically => Assume that the left and the right mini-max bounds are set extraneously for each segment, i.e., $y_{i,LeftMin}$, $y_{i,LeftMax}$, $y_{i,RightMin}$, and $y_{i,RightMax}$ are extraneously set. They may be set either point-by-point, or using another parametrization. This ensures locality, at expense of C^k, however.
  - Check if the given response value is inside of the specified range, i.e.,

$$\min\left(y_{i,LeftMax}, y_{i,RightMax}\right) \geq y_i \geq \max\left(y_{i,LeftMax}, y_{i,RightMax}\right)$$

    set as follows:
    - If $y_i < \min\left(y_{i,LeftMax}, y_{i,RightMax}\right)$ $y_i = \min\left(y_{i,LeftMax}, y_{i,RightMax}\right)$.
    - If $y_i > \max\left(y_{i,LeftMin}, y_{i,RightMin}\right)$ $y_i = \max\left(y_{i,LeftMin}, y_{i,RightMin}\right)$.
  - Otherwise:
    - If $y_i < \min\left(y_{i,LeftMax}, y_{i,RightMax}\right)$ $y_i = \min\left(y_{i,LeftMax}, y_{i,RightMax}\right)$.
    - If $y_i > \max\left(y_{i,LeftMin}, y_{i,RightMin}\right)$ $y_i = \max\left(y_{i,LeftMin}, y_{i,RightMin}\right)$.
- #3: Re-work the edges =>

- If $|y_0 - z_0| > \frac{1}{2}|y_1 - z_0|$ then $y_0 = z_1 - \frac{1}{2}|y_1 - z_0|$.
- If $|y_n - z_n| > \frac{1}{2}|y_{n-1} - z_n|$ then $y_n = z_n + \frac{1}{2}|y_{n-1} - z_n|$.
- If y_0 is already explicitly specified (as the zero-day rate in some markets) use that instead.
- Finally, if needed re-apply the positivity enforcement across all the segments as before.

Harmonic Spline Extension to the Framework above

1. Harmonic Splines and Continuous Limiters extension: Le Floc'h (2013) applies the harmonic splines originally introduced by Fritsch and Butland (1984), and extends the monotonicity preserving limiters of Van Leer (1974) and Huynh (1993) by using rational functions.
2. Harmonic Forwards in Hagan-West: Couple of interesting items to note: Given

$$m_{i+1} = \frac{y_{i+1} - y_i}{t_{i+1} - t_i}$$

on substituting

$$y_i = -z_i t_i$$

you get

$$z_{i+1} = -m_i$$

and

$$-s_i = +f_i$$

3. Estimation of the node forwards using Harmonic mean: Apply the above now to get

$$\frac{1}{f_i} = \frac{t_i - t_{i-1} + 2(t_{i+1} - t_i)}{3(t_{i+1} - t_{i-1})}\frac{1}{z_i} + \frac{t_{i+1} - t_i + 2(t_i - t_{i-1})}{3(t_{i+1} - t_{i-1})}\frac{1}{z_{i+1}}$$

if $z_i z_{i+1} > 0$, and $f_i = 0$ otherwise. After this, the regular Hagan-West may be applied without the need to enforce monotonic or convexity constraints, as it now is monotonic/convex by construction.

Minimal Quadratic Estimator

1. Design Philosophy: The algorithm extracts the spline coefficients keeping in mind the following:
    - Formulate using a 2nd degree quadratic polynomial for each segment
    - Maintain the Conserved Quantities
    - Maintain the Segment Edge Continuities
    - Optimize for the linear combination of two penalties:
        - Jump of the inter-segment discontinuities on the first derivatives
        - Curvature of the second derivative
2. Step #1: Preservation of the Conserved Quantity Set: This results in the following equation:

$$z_i = a_i + \frac{1}{2}b_i h_i + \frac{1}{3}c_i {h_i}^2$$

3. Step #2: Edge Continuity Constraint:

$$a_{i+1} = a_i + b_i h_i + c_i {h_i}^2$$

4. Step #3: Minimize the Penalty:
    - Jump of the inter-segment discontinuities on the first derivatives

$$J_{1i} = [b_i + 2c_i h_i - b_{i+1}]^2 = [(b_i - b_{i+1}) + 2c_i h_i]^2$$

$$= (b_i - b_{i+1})^2 + 4c_i h_i(b_i - b_{i+1}) + 4{c_i}^2{h_i}^2$$

- Curvature of the second derivative

$$J_{2i} = [h_i \cdot 2c_i]^2 = 4{c_i}^2{h_i}^2$$

- Complete Penalty Formulation =>

$$P(w) = wJ_{2i} + (1-w)J_{1i} = 4wc_i h_i(b_i - b_{i+1}) + 4{c_i}^2{h_i}^2$$

- Minimizing $P(w)$ =>

$$\frac{\partial P(w)}{\partial c_i} = 0 \Longrightarrow 4wc_i h_i(b_i - b_{i+1}) + 8c_i{h_i}^2 = 0$$

$$\frac{\partial^2 P(w)}{\partial {c_i}^2} = 8{h_i}^2 > 0$$

so minimum exists.

5. <u>Equation Set and Unknowns Analysis</u>:
   - $z_i = a_i + \frac{1}{2}b_i h_i + \frac{1}{3}c_i{h_i}^2$ => One per segment => $n-1$ Equations
   - $a_{i+1} = a_i + b_i h_i + c_i{h_i}^2$ => One per common edge => $n-2$ Equations
   - $4wc_i h_i(b_i - b_{i+1}) + 8c_i{h_i}^2 = 0$ => One each for all c_i up to c_{n-2} => $n-2$ Equations
   - Total number of linear equations => $3n-5$
   - Total number of unknowns => $3n-3$
   - As always, the final 2 conditions from natural, financial, or the not-a-knot clamped boundary conditions.

References

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Extrapolation in Curve Construction

1. Latent State Choice for the Extrapolator: The quantification metric used to extrapolate the latent state may be completely different from that used to infer within the span.
    - This clearly indicates that the span spans the extrapolated range as well. Further, the extrapolator should be a property of the Span, not any stretch.
2. Extrapolator Construction: At the span edges, the C^k continuity constraints may be passed onto the extrapolator as well. These may take the form of the stretch boundary conditions (natural/financial etc.).
3. State Space Extrapolation using Synthetic Observations: This is really what it is. In particular, to get the desired left/right boundary behavior, you may insert synthetic observations at either end to produce the desired custom behavior (this may also be used in lieu of the explicit boundary condition specification).

Multi-Pass Curve Construction

Motivation

1. Introduction: This is composed of one shape preserving pass on the inferable state quantification metric, followed by one or more "smoothing passes".
2. Shape Preserving Pass: The shape preservation pass occurs on the "native designate" measure, preferably one that is linearly inferred from the manifest measure. The primary objective of the shape preservation pass is to maintain the monotonicity, the convexity, the locality, and possibly the positivity of the quantification metric.
   - The output of the shape-preserving pass is a span on the quantification metric that is "well-behaved", and one that contains a new set of "truthness" nodes on which the eventual smoothing can be done.
3. Shape Preservation Variants:
   - Linear in the Discount Factor Quantification Metric => They are obviously the best shape preserver (owing to the perfection in the match and zero curvature penalty), but they no inherent convexity/concavity in them, so it gets harder for the smoothing stage.
   - Constant forward rate bootstrapping may also be used.
4. Smoothing Pass: Here you smooth on the appropriate quantification metric that is deemed to be a better hidden-state characterizer.
5. Advantages of the Shape-Preserving Pass:
   - Separation between Shape-preservation and smoothing.
   - Choice of convenient, yet potentially different metrics across shape-preserving and smoothing.
   - The final state representation quantification metric need not be linear on the manifest measure.
   - The granularity/precision of fit of the curve automatically adjusts with information (i.e., cash flow event dates such as pay dates), thereby making it inherently more precise.

- PCHIP techniques may be applied more conveniently on the smoothing pass.
- Other closeness of fit techniques (such as least squares methodologies etc.) become much more relevant on the smoothing pass.

6. Disadvantages of the Shape-Preserving Pass:
   - Calculation overhead penalty associated with the dual pass (although, by choosing linearity between manifest measure/quantification metric and the quantification metric/ quantification metric combinations this adverse impact maybe reduced).
   - Artifacts produced during shape-preservation (again, there will be artifacts associated with just about any basis representation).

Bear Sterns Multi-Pass Curve Building Techniques

1. DENSE Methodology: This method is outlined in Nahum (2004).
   - Cash/Forwards => Piece-wise constant forwards. Turn Spreads imposed as needed.
   - Swaps => Shape Preserving uniform tension splines.
   - RAW Swaps Inputs => Quarterly swap rates are now re-implied from the curve constructed in the earlier stage.
   - From these new swap quotes, a new curve is constructed using quarterly constant forward rates (constant forward rates methodology is called RAW).
2. DUAL DENSE Methodology: Again, this method is outlined in Nahum (2004).
   - Short end (Cash/Futures) => Daily forwards (i.e., constant daily forwards or cdf) latent state implied.
   - Long End => Same methodology as DENSE, except for the non-uniform tension that is applied across quarterly swap contracts.

References

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Transition Spline (Or Stitching Spline)

Motivation

1. Spline per Instrument Grouping: Another possibility is to use transition spline to bridge across different instrument groups – this simply needs to adjust to the smoothness/truthness constraints of each of the instrument groups.
   - Essentially, transition splines connect spline families across instrument group (each instrument essentially belongs to its own spline cluster).
2. Design:
   - May use discontinuous Hermite splines in the transition area, or higher order basis (say, with an appropriate C^k constraint), or even an optimizing transition spline.
   - Instrument choice is critical if we are to avoid steep transition slopes (esp. tight group gaps, and steep measure drops). These are challenges in any mechanism, but possibly a lot more here.
   - Construct single instrument spanning spline curves, then demarcate/spec out the instrument range, finally bridge in the transition splines.
   - Transition splines may also be used to stitch in arbitrary instruments together, each belonging to its own separate group, although it is hard to find a practical need for such a construct.
   - In general, instrument group boundaries need not strictly coincide with the instrument termination nodes (esp. in case of stitch-in splines). Boundaries may be inserted using any of the appropriate knot insertion techniques.
3. Advantages:
   - These preserve the curve character embedded in each instrument grouping, which can be a sub-set of a vaster instrument set.

- By retaining the localization to the corresponding instrument grouping, the hedges produced by the transition spline may, in principle, be better than those produced by the typical ones.

4. Disadvantages:
   - Of course, by construction, they do not allow for overlapping instrument groups (which, however, may not be a problem in the practical world). This forces a decision on the instrument set choices and boundaries.
   - Technically, the single “natural spline boundary condition” is not applicable across all the unprocessed instrument groups – this is really what is compromised.
     - How much the effectiveness is compromised due to the above may be estimated using targeted metrics, say the span DPE.
5. Transition Segment in the Transition Spline: This needs at least $2k + 2$ basis functions for representation, as it needs to “mate out” the left stretch and the right stretch ($k + k$ for each of the C^k continuity spec - plus 2 more, one at each end to match up the point node).
6. Using Transition Splines for Calibration Instrument Selection: As shown in Figures 2 and 3 below, the transition stretch represented in figure 2 is narrower, and therefore more abrupt/jumpy (with corresponding implications for the forward rates) than that in Figure 3. A criteria based approach is necessary to develop this.

Stretch Modeling Using Transition Splines

1. Information Propagation across Stretches: All the truthness/smoothness information of the predecessor stretch is captured by the stretch’s calibrated span parameters. Any state inference for predictors in a given domain needs to be deferred to the domain’s span stretch.
   - The corollary to the above is that trailing stretches will typically need information from the leading stretches for state inference/estimation (leading/trailing here are set in regards to the inference flow (or information flow)). Applied to discount curve cooking, the leading stretch that uses cash instruments is essentially self-calibrating, whereas the trailing stretch of swap instruments is going to rely on information that comes out of the cash calibration. Going into swap segments, the information will propagated in the form

of RVC's, so they will need to be handled right from the left-most segment of each stretch.

- Regular Stretches vs. Finance Curve Stretches => For typical stretch construction, all you need is the transmission of the segment-to-segment continuity constraints through C^k. For segment curve builders, however,

$$Constraint\ (Segment_N) = f(Segment_i, \cdots, Segment_{N-1})$$

i.e., more construction information in addition to just the C^k is required (mostly via explicit evaluation of arbitrary points in earlier segments' stretches).

2. Response Stretches: Markov response state variables may follow distinct behavior in different predictor stretches. For example, the discount factor/zero rate/swap rate may be characterized using one set of representations for the cash stretch, whereas the swap stretch may use a different set.
3. Why Response Stretches exist: Is it simply because of the instrument choice (cash for the front end, swap for the back end, etc.), or is there a more fundamental driver? Can't say one way or the other, but the fact is we empirically attempt to match point-by-point in a left to right manner (we do this today) without compromising the empirical characteristics of each instrument group. We call each of these groups manifest groups, since they could be result of specific product manifest measures).
4. Manifest Group Contribution to the Response Signal Strength: Say that a signal strength contribution to a specific response signal is proportional to its liquidity (to improve accuracy, you may make it sided liquidity). As you move from left to right in the predictor space, by working in terms of the liquidity-fade of the left stretch to the liquidity-explode of the right stretch, you may be able to characterize the response space more naturally (with less dependence on explicit stitching splines, or on artificially inserted knots).
5. Liquidity-Fade and Liquidity-Explosion in practice: In practice the actual predictor ordinates across the manifest stretches will be too discrete for tracking the liquidity-fade and liquidity-explosion. Thus, it may be more appropriate to operate on predictor windows. If convenient and admissible, the predictor window boundaries may also coincide with the segment boundaries.

Stretch Partition/Isolation in Transition Splines

1. Definition: A given calibratable predictor ordinate/response realization space is called a span. The span is partitioned into stretches. Stretches can be either core stretches or transition stretches. Both the core stretches and the transition stretches are built from segments (within which the response values may be represented using basis splines). Core stretch are inferred to truthness and the smoothness signals, and the transition stretches provide the explicit bridge between the core stretches that may not be possible using the plain core stretch representations.
2. Information Patterns: With a higher unit, information propagation is associated with each sub-unit entities below. Across peer units, information exchange is materially similar in nature. Across higher units, information exchange may be more parsimonious (although it may still happen between lower entities belonging to the higher units).
3. Information Localization and Transmission: Intra-segment information propagation occurs through smoothness constraints such as C^k.
4. Stretch-Level Information Localization: In the spline case, this happens though boundary-condition delimitation/isolation (i.e., natural/financial/clamped boundary conditions based isolation is applicable to within a single stretch).
5. Stretch-Stretch Transmission: These are not bound by the equivalent isolation constraints, therefore the connecting/transition splines need to have a qualitatively different nature.
6. Transition/Connecting Splines: By definition, since they are the bridge between the stretches, they need to have greater degrees of freedom for a complete bridge.

Knot Insertion vs. Transition Splines

1. Equivalence: In some sense, they are equivalent in that inserting knots also attempts to complete the bridge. However, transition splines are more customizable, since the splines that flank the knots are assumed in the literature to be variants of the others.

2. Advantages on Knot Insertion: Remember that transition splines need $2k + 2$ basis function. Thus, for high k, you are stuck with higher-order polynomials (for e.g.), along with all the Runge's oscillations/instabilities that it brings. Suitable choice of knots may minimize this.
3. Advantage of Transition Spline: Knots are stretch response altering (via their C^k criteria), whereas transition splines enable each stretch to retain their character.

Overlapping Stretches

1. Premise: By definition, stretch fade-out and stretch explode axiomatizations imply predictor ordinate overlapping stretches.
2. Stretch Boundaries: Each stretch constituting an overlapping stretch needs to have its boundaries identified. What ***do*** not necessarily overlap are the smoothness constraints.
3. Overlapping Stretch – Problem Statement:
   - Predictor Ordinate Stretches overlap.
   - Stretches (and by implication, their predicate ranges) are contained/telescoped.
   - Smoothness constraints may not overlap, in which case they are posited to be distinct in each of the constituent stretches.
   - Truthness should be strictly telescopically contained/localized, i.e., there is a ***manifest measurement exclusivity*** to each stretch.
   - A consequence of this is that the inferred state response variable will be propagated, but not (necessarily) the smoothness criterion.

Penalizing Exact/Closeness of Fit and Curvature Penalty

Motivation

1. Least Squares Exact Fit vs. Best Fit: Unlike in functional analysis/financial curve construction, in machine learning "exact fit" is treated as a rarity in machine learning, as there is presumed to be an irreducible manifest measure generation error. Here we assume that there are processes the result in "zero manifest measure uncertainties" – in other words, these are "quotes" that are explicitly honored.
2. Basic Setup: As described in the companion Spline Library Documentation, the regularized regression loss/penalizer may be decomposed and worked out as

$$Gross\ Penalizer = Fitness\ Match\ Penalizer + Curvature\ Penalizer$$

$$\mathfrak{R} = \frac{1}{q}\mathfrak{R}_F + \lambda\mathfrak{R}_C$$

$$\mathfrak{R}_F = \sum_{p=0}^{q-1} W_p\left(y_p - Y_p\right)^2$$

$$\mathfrak{R}_C = \int_{x_l}^{x_{l+1}} \left(\frac{\partial^m y}{\partial x^m}\right)^2 dx$$

3. Estimation of λ: While the segment spline coefficients are computed by minimizing $\mathfrak{R}$, λ is often extraneously supplied as a tuner that trades the prefect high degree of fit to the curvature. Tanggaard (1997) suggests using a few methods to estimate λ:
   - Using the GCV criterion as demonstrated by Craven and Wahba (1979) and Wahba (1990).

- From the smoothing spline viewpoint, set the number of basis functions, then search for the corresponding λ using the technique listed in Tanggaard (1997).

4. Measurement Filtering vs. Best Fit Weighted Response: These approaches are very similar, in that the Best Fit Weighted Response "steers" the calibrated spline basis and their coefficients to accommodate the measurements in the uncertain sense (potentially by incorporating measurement uncertainty).
    a. If the measurement uncertainty/variance is explicitly known, the Andersen (2005), the Tanggaard (1997), and/or the GCV techniques may be used to extract better estimate for λ - through Andersen RMS γ^2 estimator, Craven/Wahba's GCV, or Tanggaard's trace-based λ estimator.
    b. Differences => However, it needs to be remembered that, for current curve construction methodologies, a key requirement is the γ^2 matches (i.e., exactly reproducing state estimations) – which is not the typical case for the filtered state estimations.
5. Effectiveness of State Representation Quantification Metric: The combination of curvature penalty, the length penalty, and the closeness of fit penalty must be taken together to gauge the effectiveness of the chosen Quantification Metric/Smoothing spline scheme set. Alternatively, full simulations of the manifest metric (with induced noise terms as explained in for e.g., Fisher, Nychka, and Zervos (1994)) and their corresponding evaluations are also appropriate, although they tend to be time consuming (and possibly overkill).

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Index/Tenor Basis Swaps

Component Layout and Motivation

1. Basis Swap Market: Although Basis Swaps did exist even earlier (Tuckman and Porfirio (2003), Morini (2008)), post-crisis segmentation (attributable, among other things, to the preference towards receiving higher frequency payments) intensified these differentials (Mercurio (2009)).
2. Origins of Basis Swap Existence: In principle, these are expected to represent embedded duration counter-party credit risk. The "good" model should couple embedded credit risk with the sided flow dynamics (i.e., the credit quality of the counter-party that enters into the long/short side of the greater frequency leg, etc.)
3. Float-Float Swap as a Combination of Two Fix-Float Swaps: The proxy of the float-float as two fix-floats would be perfect if both the fixed legs had the same frequency and day count conventions. In terms of the tenor basis swap conventions, pre-2008 the convention was to quote the float-float basis directly – post-2008, however, it was quoted as a combination listed above.
4. The Discounting Curve: Challenges regarding the uniqueness in relation to the instrument choice for building the discount curve have been identified by Henrard (2007). The issues stem primarily from the uncollateralized nature of deposits and forwards, therefore, these are typically replaced by OIS/EONIA and Futures (Madigan (2008)).
    - Interest Rate Swap continues to be used for the discount curve calibration, as it possesses the following characteristics:
        - Par IRS'es are collateralized at inception.
        - Collateral margining may be applied over time.
        - IRS is the only liquidly available fix-float swap, and as such effectively implies just a single forward curve.

- Convexity adjustment for extracting the rate from future/forward price => Since futures/forwards act effectively as a zero coupon bond, the transformation of price to the latent zero/forward rate requires a dynamical volatility based curve evolution model. Sophisticated, comprehensive approaches are available in literature (see for e.g., Kirikos and Novak (1997), Jackel and Kawai (2005), Brigo and Mercurio (2006), Piterbarg and Renedo (2006)); common practitioner approaches, however, employ simpler approaches such as the Hull-White one-factor short-rate model (Hull and White (1990)).

5. <u>Multi Curve vs. Forward Smoothness</u>: Given that the discount curve and the forward curve are essentially distinct in the multi-curve latent state, the stringent demands that all forwards stay smooth (as in the single discount curve that covers all the basis curve scenarios) may be relaxed.
   - Forwards Implied in the Discount Curve => Since the forwards are used only for the "core" tenor pillars in the discount curve, only those forwards need to be subject to the extra discounting constraints (e.g., 6M forwards). By discount curve construction this will typically be the case, as the forwards period will always straddle/span fully a single reset pillar.
6. <u>Point- vs. Convolved-Measure State Transform</u>:
   - Point-Measure transform refers to the one-to-one transform between a state measure at a predictor ordinate and its corresponding observation (e.g., discount factor from zero-coupon bond price observations). Since these may be expressed as straightforward transformations, the observation-state non-linearity may be easily accommodated.
   - Convolved measure-state transforms introduce what are effectively observation constraints across predictor ordinate/state response combinations. Non-linearity introduces complications, therefore usage of spline-based linearization constraints are highly effective.
7. <u>Reset-Date Forward-Rate Pair Constraint in Discount Curve Building</u>: The yM tenor (e.g., $yM \Longrightarrow 6M$) may be extracted only at the reset start/end date (depending on the reset rate-time axis label) from the discount curve, i.e., only the pair $[yM, Forward_{yM}]$ makes sense. In other words, this is the only set of dates for which the information on forward rates is available. Splining is an obvious option at the other dates.
   - yM Tenor/DF Relationship =>

$$PV_{yM} = \sum_{j=1}^{m} \Delta(j-1, j) F_{yM}(t_j) D_f(t_j)$$

For PV_{yM} to be telescoped away into

$$PV_{yM} = D_f(t_0) - D_f(t_i)$$

the requirements are: Period Accrual End Date == Period Reset End Date == Period Pay Date. This is the main reason why the period dates are adjusted before the cash flows are rolled out.

8. Alternative View: Discount Curve IS the yM Forward Curve: To automatically ensure uniqueness and consistency of the latent state space, it may also be more restrictively imposed that the native yM Forward Curve be implied entirely off of the discount curve. Thus, the native yM Forward Curve may now be implied at all nodes, not just at the reset nodes as postulated earlier. This automatically eliminates the state basis between these measures; further, this is still not too restrictive in terms of the native yM Forward Curve smoothness for same reasons as before.
9. Basis between the yM Forward Curve and the Discount Curve: Given that basis constraints are of paramount consideration in other markets, why not look at the basis between discount curve and its native forward curve? This is because neither the latent state underpinning the forward curve or that underpinning the discount curve is entirely observable (unlike, say basis between a bond and the issuer's underlying CDS). Thus an extraneous observation model is necessary. By convention, the current practice achieves this by construction – the formulation mandates that the discount curve and the "discounting-native" forward curve be alternate quantification metrics of the same latent state.

Formulation

1. Float-Float Swap Setup: The phenomenology and flow details laid out in Figure 5 are based off of descriptions and details provided by ISDA (2006), Ametrano and Bianchetti (2009a), Bianchetti (2012)). The two swap legs are:
   - The "known" or the "Reference" leg. Forwards of this leg come from the discount curve's IRS contracts, and 6M LIBOR/EURIBOR is the most common such tenor. We generalize this with a basis spread, i.e., the "effective" forward is $F_{6M} + S_{6M}$, where F_{6M} and S_{6M} stand for the corresponding forward and the spread.
   - The "unknown" or the "Derived" leg with a tenor of xM. Forwards of this leg are computed from the corresponding basis market quotes. We generalize this with a basis spread, i.e., the "effective" forward is $F_{xM} + S_{xM}$, where F_{xM} and S_{xM} stand for the corresponding forward and the spread.
2. Basic Formulation Setup:

$$PV_{xM} = \sum_{j=1}^{m} \Delta(j-1, j)\left[F_{xM}(t_j) + S_{xM}\right]D_f(t_j)$$

$$PV_{6M} = \sum_{a=1}^{b} \Delta(a-1, a)[F_{6M}(t_a) + S_{6M}]D_f(t_a)$$

   - Equivalence of S_{xM} and S_{6M} => Since both S_{xM} and S_{6M} are additive, we work in a space that is essentially an adjusted forward rate space, with

$$F_{6M,Adj} \rightarrow F_{6M} + S_{6M}$$

   and

$$F_{xM,Adj} \rightarrow F_{xM} + S_{xM}$$

   While this is straightforward to accommodate in the case of $6M$, from a calibration point-of-view, we can work off of a basis $6M$ space, and re-adjust back after splining.

3. Basis Swap Calibration Formulation:

$$PV_{xM} = PV_{6M}$$

implies that

$$\sum_{j>m_l}^{m} \Delta(j-1,j) F_{xM,Adj}(t_j) D_f(t_j) = \sum_{a=1}^{b} \Delta(a-1,a) F_{6M,Adj}(t_a) D_f(t_a) - \sum_{j=1}^{m_l} \Delta(j-1,j) F_{xM,Adj}(t_j) D_f(t_j)$$

For all but the left most basis swap, $m_l > 0$.

4. Basis Swap Calibration Constraint Specification: Set

$$\aleph_m = \sum_{a=1}^{b} \Delta(a-1,a) F_{6M,Adj}(t_a) D_f(t_a) - \sum_{j=1}^{m_l} \Delta(j-1,j) F_{xM,Adj}(t_j) D_f(t_j)$$

Notice that $\aleph_m$ maybe fully computed from before. Recognize that

$$F_{xM,Adj}(t) = \sum_{i=1}^{n} \beta_i f_i(t)$$

1. Combine above to get the calibration constraint

$$\aleph_m = \sum_{i=1}^{n} \beta_i \left\{ \sum_{j>m_l}^{m} \Delta(j-1,j) f_i(t_j) D_f(t_j) \right\}$$

5. Reference/Derived Par Spread Relations: For parity,

$$PV_{Reference} + DV01_{Reference}S_{Reference} + PV_{Derived} + DV01_{Derived}S_{Derived} = 0$$

Setting

$$S_{Derived} = 0$$

$$S_{Reference} = -\frac{PV_{Reference} + PV_{Derived}}{DV01_{Reference}}$$

Likewise

$$S_{Derived} = -\frac{PV_{Reference} + PV_{Derived}}{DV01_{Derived}}$$

Remember that both $S_{Reference}$ and $S_{Derived}$ can be negative.

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Multi-Stretch Merged Curve Construction

Motivation

1. Discount Curve composed of Forward Rate Stretches: The discount curve span may be viewed as being composed of overlapping/non-overlapping forward rate stretches, i.e., adjacent or otherwise 3M Tenor forward stretch, 6M Tenor forward stretch, etc. This visualization is a consequence of the representation of the “single discount curve latent state”, whose alternate/parallel quantification metrics are composed off of those stretches of forward rates that share the latent state space with the global discount curve.
2. Out-of-Native Stretch Arbitrage: If one seeks a forward rate outside these stretches for the given tenor/index combination, there can be no expectations of no-arbitrage, i.e., there will be a basis between the forward implied by this latent space quantification metric and the forward rate under consideration.
    - Likewise, inside the stretch there should be no implied basis, since the diver latent state is identical/fully correlated.
3. Merging/de-merging of the Latent State along the Predictor Ordinates: If you imagine the rates state space being characterized by a set of latent states (which may be highly correlated), each state may ideally be characterized by a quantification metric that is native to the state physical view. Thus, the unification of the sub-states in a stretch may be viewed as state-merging (i.e., one quantification metric may be inferred from another within a merged space via a trivial transformation).
4. Probit-based Latent State Merger Analysis: Given that the discount/forward latent states merge/de-merge, it might it particularly amenable to a common-factor probit (or even a logistic) analysis of the merger driver dynamics. The challenge would then be to link the driver dynamics to the maturity based predictor ordinate.

Merge Stretch Calibration

1. Cross-Stretch Calibration: Clearly the latent state span characterized by multiple stretches will in turn be composed of latent state merge sub-stretches. The merged stretch may be followed by de-merged stretch, etc.
2. Calibration Challenges:
    a. What would be most optimal cross-representation inside the merge sub-stretch (i.e., the state representation needs to be smooth for both the discount factor latent state as well as the forward curve latent state)?
    b. On the other hand in the solitary segment sub-stretch, you may have more representation freedom, but may still need to carry over the smoothness constraints from the merged sub-stretch. How can this be done? Can the transition spline treatment above be effectively employed here? In other words, what would be appropriate transition zone applicable to the sub-stretch?

Latent State Manifest Measure Sensitivity

Introduction

1. $\frac{\partial PV_{Swap}}{\partial q_c}$: Remember that the floater leg PV goes as $D_f(t_0) - D_f(t_n)$. Thus, these terms tend to dominate both the PV and the manifest measure sensitivity calculations. The fixed annuity per-coupon-date cash-flow is smaller comparatively, and that is reflected in the Jacobians.
2. Latent State Sensitivities to the Product Segments:
   - Cash/Deposit => Here the sensitivities are to within a single segment, since it is spot starting.
   - Future/FRA => Here the manifest sensitivities are to the two straddling segments, since it is forward starting.
   - IRS => Sensitivities through multiple segment/preceding segments, but concentrated a more on the edges for the reasons seen above.
3. Latent State Sensitivities Signs: The far end is always negative, since the PV decreases with the increase in the manifest measure sensitivity – this is valid across all products. The near end is positive for Future/FRA as well as IRS, as that corresponds to the shorted side. There is no shorted side for cash.
4. Latent State Segment Manifest Measure Sensitivity: It may be appropriate to imagine that, for a given segment, the latent state sensitivity is contributed to only from the current and the prior segment manifest quotes.
   - Justification for the above => As may be observed from Figure 6, sensitivity has to be zero at the start of the current and at the end of the next. "Current" is completely determined through the constraints and the matches corresponding to intra-segment observations, but the "next" dependence propagation is exclusively via C^k transmission constraints, and devoid of targeted segment-specific contributions.
5. Design of Manifest Measure Sensitivity Segment Tail: Since the tail is, technically, a strict fade, any monotonically decreasing-to-zero function will work – the smoother the better.

6. Preceding Manifest Measure Sensitivity Basis Function: Given the shape in figure 6, a choice of sinusoidal function would serve as appropriate set of basis. Alternately, the same basis as manifest measure sensitivity (which should, strictly speaking, be the same basis used for quantifying the latent state response) may also be used – the head/tail C^k may provide additional customization.
7. Preceding Manifest Measure Sensitivity Fade off/Retain: The two possibilities of the transmission of the preceding manifest measure stem from the differing nature of the current manifest metric. If the current manifest measure is of the "retain" type, the preceding manifest measure sensitivity is retained as is (i.e., uses a flat through transmission of the right edge value). If the current manifest measure demands that the preceding manifest measure sensitivity fade, then the preceding manifest measure sensitivity is faded off/decayed, as seen above.
8. Fade off/retention sequence: Further, if the current is "retain", all the segment manifest measure sensitivities of the earlier segments since the last fade-off are transmitted, and replicated.
9. Preceding Manifest Measure Sensitivity Customization: The fade-off/retain preceding manifest measure sensitivity customization should be applied on a stretch-by-stretch basis:
   - Cash/Deposit Stretch => Use Fade-off
   - Futures/FRA Stretch => Use Retain
   - IRS Stretch => Use Fade-off

Float-Float Manifest Measure Sensitivities

1. Float-Float Reference Leg Sensitivity to the Derived Leg Basis:

$$V_{Ref} = \sum_{i=1}^{n} D_f(t_i)\left[L_{Ref}(t_{i-1}, t_i) + b_{Ref}\right]\Delta_{Ref}(t_{i-1}, t_i) \Longrightarrow \frac{V_{Ref}}{b_{Der}} = 0$$

2. Float-Float Derived Leg Sensitivity to the Derived Leg Basis:

$$\sum_{j=1}^{n} D_f(t_j)[L_{Der}(t_{j-1},t_j) + b_{Der}]\Delta_{Der}(t_{j-1},t_j)$$

$$= \sum_{j=1}^{k} D_f(t_j)[L_{Der}(t_{j-1},t_j) + b_{Der}]\Delta_{Der}(t_{j-1},t_j)$$

$$+ \sum_{j=k+1}^{n} D_f(t_j)[L_{Der}(t_{j-1},t_j) + b_{Der}]\Delta_{Der}(t_{j-1},t_j)$$

where $(k+1,\cdots,n)$ cash flow instances belong to the manifest measure exclusive segment c.

- Closed Form => The non-exclusive (i.e., the earlier segments and stretches) do NOT contribute to the current manifest measure sensitivity. Thus, the sensitivity becomes

$$\sum_{j=k+1}^{n} D_f(t_j)\frac{\partial L_{Der}(t_{j-1},t_j)}{\partial b_{Der}}\Delta_{Der}(t_{j-1},t_j) = -\sum_{j=k+1}^{n} D_f(t_j)\Delta_{Der}(t_{j-1},t_j)$$

where the right hand side is the manifest measure exclusive segment incremental derived floating leg annuity.

3. Float-Float Derived Basis Sensitivity Transmission Rule: Given that the floating leg sensitivity could potentially be OVERLAPPING, the preceding manifest measure sensitivity choice will be FADE ON, not RETAIN.
4. Tenor Basis Swap Sensitivity:

$$\frac{\partial L_{Der,j}}{\partial b_{Ref}} = -DV01_{Ref}(0,\cdots,m-1)$$

$$\frac{\partial L_{Der,j}}{\partial b_{Der}} = -DV01_{Der}(0,\cdots,m-1)$$

Their magnitudes must be similar, save for the annuity flow differences.

6. Multi Leg Basis Sensitivity Points: For each of the constituent legs and their corresponding manifest measure, the symbolic sensitivities need to be computed/splined in. The sensitivity nodes will be the payment dates, along with an additional cross-leg “final upfront’. The current manifest measures are the derived leg basis, the reference leg basis, and the interest rate sensitivity.
7. Multi-Metric Latent State Calibration: As long as the latent state is linearizable among the multiple metrics, such a calibration is possible. Further, a chain sweep multi-metric sensitivity Jacobian is estimable on the calibration pass. Of course, the preceding quote sensitivity control must be customizable on a per-manifest measure basis.

Multi-Reset Floating Period

1. The Setup: This small sections concerns itself with the case where the reset tenor that is different from the floating period tenor. In this situation, there are 2 specific impacts to be considered:
    a. The compounding rule to accumulate the reset periods onto the floating period, and
    b. The associated convexity correction mismatch since the terminal measure numeraire for the floater period pay is different from that of the reset period terminal measure.
2. Convexity Correction vs. Quanto Adjustment: Remember that the forward and the discount latent states form part of the same shared latent state in the case of convexity correction, and the convexity adjustment stems purely from the terminal measure mismatch, as just observed. Quanto adjustment, however, is applied across multiple distinct latent states that are non-merged, e.g., funding vs. collateral vs. forward vs. FX latent states.
3. Origin of the Convexity Correction: In practical settings, the convexity correction occurs only when a) the floater periods encapsulates the multiple reset periods, AND b) these reset periods DO NOT compound geometrically (e.g., the compounding is arithmetic). As examples, these reset periods include overnight fixings applied via a corresponding index, 3M reset vs. 6M floater etc. The latter is the case for certain standard sovereign IRS’es (CAD).

4. Merged/non-merged Latent State Convexity/Quanto Estimation: Each constituent segment/stretch/regime is still expressed using distinct stochastic (e.g., Brownian) component partitions, in practice the merged state convexity adjustment ends up looking very similar to that of quanto adjustment. Thus, generalizing from the above, the PV of the forward rate $\mathcal{L}(t \to \ t + \tau)$ between t and $t + \tau$ paid at time T in a different currency X looks identical (save for the specific volatilities involved) in both cases (please note the integration time limit differences):

$$E_0^{Q_D^T}\left[\mathcal{L}(t \to \ t + \tau) D_f(T) X(T)\right] =$$

$$\mathcal{L}(0; t \to \ t + \tau) D_f(0; T) X(0; T) \times$$

$$e^{\left\{\int_0^t \rho_{D\mathcal{L}}(s)\sigma_D(s)\sigma_{\mathcal{L}}(s)ds + \int_0^T \rho_{DX}(s)\sigma_D(s)\sigma_X(s)ds + \int_0^t \rho_{\mathcal{L}X}(s)\sigma_{\mathcal{L}}(s)\sigma_X(s)ds\right\}}$$

OIS Valuation and Curve Construction

Base Framework and Environment Setup

1. <u>The OIS Model</u>: Given the compounded overnight rate $R_{ON}(T_{i-1}, T_i)$, the par OIS Rate $R_{ON}^{OIS}(t, T, S)$ is given as

$$R_{ON}^{OIS}(t, T, S) = \frac{\sum_{i=1}^{n} D_f(t, T_i) R_{ON}(T_{i-1}, T_i) \tau_{ON}(T_{i-1}, T_i)}{Fixed\ Leg\ DV01}$$

   As expected, given that this corresponds to the par OIS, this telescopes to

$$R_{ON}^{OIS}(t, T, S) = \frac{D_f(t, T_0) - D_f(t, T_n)}{Fixed\ Leg\ DV01}$$

   The compounded rate $R_{ON}(T_{i-1}, T_i)$ is computed from the daily overnight fixes as (Mercurio (2011))

$$R_{ON}(T_{i-1}, T_i) = \frac{1}{\tau_{ON}(T_{i-1}, T_i)} \prod_{k=1}^{n_i} [\{1 + R_{ON}(T_{i,k-1}, T_{i,k}) \tau_{ON}(T_{i,k-1}, T_{i,k})\} - 1]$$

2. <u>Stringency on the OIS Spline Construction</u>: Since the OIS has shown itself to dip into the negative territory (Whitall (2010), Cameron (2011), Atkins and Jones (2012), Carver (2012), Lipman (2012)), the corresponding demands on the shape preserving splines need to be accommodative.

OIS Valuation Extensions and Approximations

1. <u>OIS Extensions Using Fed Fund Basis Quotes</u>: For some jurisdictions (say, USD), the OIS quotes are not widely available beyond the 10Y tenor. Therefore the OIS discount curve is constructed using the USD LIBOR – Fed Fund Basis Swap Quotes that trade till the 30Y tenor. Since both the OIS and the Fed Funds Basis Swap Quotes are projected from the forwards of the Fed Funds Effective Rate FEDL01, no arbitrage arguments may be used to extract the OIS Curve (Bloomberg (2011a), Bloomberg (2012), Bloomberg (2013)).
2. <u>USD OIS Curve Construction</u>: As discussed partly in White (2012a), the USD OIS curve construction can occur in the same manner as that done for building discount curves from CCBS and IRS quotes – i.e., the OIS long end OIS curve can be constructed from fed funds-LIBOR basis swap (a float-float swap) and IRS.
3. <u>Fed Fund OIS Basis Swap</u>:

$$PV_{Fed\ Fund\ Leg} = \sum_{i=0}^{n_{FF}-1} \Delta_{FF}(t_i, t_{i+1})\,[\mathcal{L}_{FF}(t_i, t_{i+1}) + b_{FF}]D_f(t_{i+1})$$

$$PV_{LIBOR\ Leg} = \sum_{j=0}^{n_L-1} \Delta_L(t_j, t_{j+1})\,[\mathcal{L}_L(t_j, t_{j+1}) + b_L]D_f(t_{j+1})$$

$$\begin{aligned} PV_{Fed\ Fund\ LIBOR} &= PV_{Fed\ Fund\ Leg} + S_L PV_{LIBOR\ Leg} \\ &= \sum_{i=0}^{n_{FF}-1} \Delta_{FF}(t_i, t_{i+1})\,[\mathcal{L}_{FF}(t_i, t_{i+1}) + b_{FF}]D_f(t_{i+1}) \\ &+ S_L \sum_{j=0}^{n_L-1} \Delta_L(t_j, t_{j+1})\,[\mathcal{L}_L(t_j, t_{j+1}) + b_L]D_f(t_{j+1}) \end{aligned}$$

   a. <u>Basis Free LIBOR Leg PV</u>: Setting

$$PV_{Fed\ Fund\ LIBOR} = 0$$

we get

$$\sum_{j=0}^{n_L-1} \Delta_L(t_j, t_{j+1}) \left[\mathcal{L}_L(t_j, t_{j+1})\right] D_f(t_{j+1})$$

$$= -\frac{1}{S_L}\left\{\sum_{i=0}^{n_{FF}-1} \Delta_{FF}(t_i, t_{i+1}) \left[\mathcal{L}_{FF}(t_i, t_{i+1}) + b_{FF}\right] D_f(t_{i+1})\right\}$$

$$- b_L \sum_{j=0}^{n_L-1} \Delta_L(t_j, t_{j+1})\, D_f(t_{j+1})$$

Working from an IRS point-of-view, by matching the fixed and the floating legs we get

$$\sum_{j=0}^{n_L-1} \Delta_L(t_j, t_{j+1}) \left[\mathcal{L}_L(t_j, t_{j+1})\right] D_f(t_{j+1}) = -\frac{s}{S_L} \sum_{k=0}^{m-1} \Delta(t_k, t_{k+1})\, D_f(t_{k+1})$$

Here S_L stands for the notional sign (typical convention is to assume -1 for the floater side), and s stands for the swap rate.

4. Consolidated Discount Curve Builder Relation:

$$\sum_{i=0}^{n_{FF}-1} \Delta_{FF}(t_i, t_{i+1}) \left[\mathcal{L}_{FF}(t_i, t_{i+1}) + b_{FF}\right] D_f(t_{i+1}) + S_L b_L \sum_{j=0}^{n_L-1} \Delta_L(t_j, t_{j+1})\, D_f(t_{j+1})$$

$$= s \sum_{k=0}^{m-1} \Delta(t_k, t_{k+1})\, D_f(t_{k+1})$$

5. Curve/Quote Dependence for the Cross DC Builder: Remember

$$\sum_{i=0}^{n_{FF}-1} \Delta_{FF}(t_i, t_{i+1})\, \mathcal{L}_{FF}(t_i, t_{i+1}) D_f(t_{i+1}) = D_f(t_0) - D_f(t_{i+1})$$

This clearly shows that there is no explicit market curve dependence for building the OIS curve – the only quote dependences are on b_{FF}/b_L (one of them, typically b_L, is zero) and s. What the equation provides is to create a sequence of linear constraints over $\{D_f(t_i)\}$, $\{D_f(t_j)\}$, and $\{D_f(t_k)\}$.

6. Bloomberg (2013) Approximation for the OIS Curve Rate, given LIBOR Level: If, say, the 10Y LIBOR is known, and so is the FF-LIBOR basis, Bloomberg (2013) approximates the OIS rate as

$$\hat{s}_N{}^{OIS} = \left[1 + \frac{r_Q - b_N{}^{FF}}{4}\right]^4 - 1$$

where $b_N{}^{FF}$ is the Fed Funds OIS Basis, $\hat{s}_N{}^{OIS}$ is the corresponding 10Y LIBOR, and

$$r_Q = \left\{\left(1 + \frac{s_N{}^{LIBOR} \times \frac{360}{365}}{2}\right)^{\frac{2}{4}} - 1\right\} \times 4$$

7. Bloomberg (2013) Enhanced Approximation: The approximation above is in place because the daily discrete compounding applied over specific holidays, weekends etc. becomes very expensive to compute. Therefore Bloomberg (2013) introduces an additional accuracy enhancement to compensate for the daily compounding of the FF using a flat curve to get $\hat{s}_N{}^{OIS,ENH}$ as:

$$\hat{s}_N{}^{OIS,ENH} = \hat{s}_N{}^{OIS} + \left[\left(1 + \frac{\hat{s}_N{}^{OIS}}{360}\right)^{90} - 1 - \frac{\hat{s}_N{}^{OIS}}{4}\right] \times 4 = \left[\left(1 + \frac{\hat{s}_N{}^{OIS}}{360}\right)^{90} - 1\right] \times 4$$

OIS FX-Basis Swap Valuation and Approximations

1. OIS-FX-Basis Swap Definition: Consider 2 OIS floating streams of the corresponding currencies CCY1 and CCY2 respectively. This package of both the legs together is called the OIS FX-Basis Cross-Currency Swap. Further, assume that this is collateralized in currency #1.
2. OIS-FX-Basis Swap Valuation: This valuation is straightforward.

$$PV_{1,N} = \sum_{i=1}^{N} f_{1,i}{}^{OIS}\Delta(t_{i-1}, t_i)D_{1,f}{}^{OIS}(t_i) + D_{1,f}{}^{OIS}(t_N)$$

   and

$$PV_{1,N} = \gamma(0)\left[\sum_{j=1}^{M}\{f_{2,j}{}^{OIS}\Delta\left(t_{j-1}, t_j\right) + b_{2,N}{}^{OIS,FX}\}D_{2,f}{}^{OIS}\left(t_j\right) + D_{2,f}{}^{OIS}(t_M)\right]$$

   where $\gamma(0)$ is the appropriate starting FX Rate.
3. LIBOR FX-Basis Swap: This is identical to the OIS-FX-Basis Swap, except that the floating leg now pays LIBOR floating plus a basis. The valuation is done precisely as in the case of the OIS-FX-Basis Swap, with the OIS part replaced by the LIBOR part.
4. Approximating the OIS FX Basis using the LIBOR FX Basis: Since LIBOR FX-Basis is more widely traded than OIS-FX Basis, Bloomberg (2011b) claim to have developed an approximation for the OIS-FX-Basis Spread in terms of the LIBOR-FX-Basis Spread that is intuitive, simple, easy to use, and very accurate (they demonstrate this using comparative reconciliations):

$$b_{2,N}{}^{OIS,FX} = b_{2,N}{}^{LIBOR,FX} - \left[s_{1,N}{}^{LIBOR} - s_{1,N}{}^{OIS}\right] + \left[s_{2,N}{}^{LIBOR} - s_{2,N}{}^{OIS}\right]$$

Arithmetic Accrual Convexity Correction

1. <u>One Floater Unit Paid out at the non-terminal Time</u>: As shown in Figure 7, this corresponds to the classic change of measure paradigm. The payout time is T, and the forward period is $(t, t+\tau)$. The PV of the accrual unit becomes

$$PV = \langle L_\tau(t) D_f(T) \rangle = E_0^{Q_D^T}\left[L_\tau(t) D_f(T)\right]$$

2. <u>Equivalent Martingale Forward Valuation</u>:

$$\langle L_\tau(t) D_f(T) \rangle = D_f(0,T) E_0^{Q_D^T}\left[L_\tau{}'(t) D_f{}'(T)\right]$$

where $D_f{}'(T)$ is simply the de-drifted martingale devoid of the drift $D_f(0,T)$, and where we assume (by the fundamental theorem) that $L_\tau(t)$ is a martingale itself. Setting up the dynamics for $L_\tau(t)$ and $D_f{}'(T)$ as

$$\Delta L_\tau = \sigma_L L_\tau \Delta W_L$$

and

$$\Delta D_f' = \sigma_D D_f' \Delta W_D$$

along with their joint moves

$$\Delta W_D \Delta W_L = \rho_{DL} \sigma_D \sigma_L \Delta t$$

we get

$$\langle L_\tau(t) D_f(T) \rangle = L_\tau(0,t) D_f(0,T) e^{\int_0^t \rho_{DL}(s) \sigma_D(s) \sigma_L(s) ds}$$

where $L_\tau(0,t)$ and $D_f(0,T)$ stand for today's expectations of $L_\tau(t)$ and $D_f(T)$ respectively.

3. Convexity Adjusted Accrual PV: The model above for $L_\tau(t)$ and $D_f(T)$ is pretty generic – but deterministic (i.e., non-local). This may now be applied to assess the convexity adjustment to be used for each of the daily payments on the overnight fund index for the period j as

$$PV_j = D_f(T_j) \sum_i L_{ON}(t_{j,i}) \Delta(t_{j,i}, t_{j,i+1})\, e^{\int_0^{t_{j,i}} \rho_{ON,D}(t)\sigma_D(t)\sigma_{ON}(s)dt}$$

Composed Period Latent State Loadings

1. Composed Period Compounding – Arithmetic: In literature, the notional of arithmetic/geometric accruals is often spelt out in terms of rates averaging, i.e., arithmetic vs. geometric averaging over the composable rates periods. The arithmetic accrual over the composable periods (with the index i running over the composable periods) is given as

$$\left[\sum_{i=1}^{n} L_i(t_i - t_{i-1})\, \vartheta_i(\tau_n)\right] D_f(\tau_n) S_P(\tau_n) X(\tau_n)$$

   thereby clearly inducing the convexity adjustment $\vartheta_i(\tau_n)$ across each composable period.
2. Composed Period Compounding - Geometric: Here the period PV becomes

$$\left\{\prod_{i=1}^{n} [1 + \ L_i(t_i - t_{i-1})] - 1\right\} D_f(\tau_n) S_P(\tau_n) X(\tau_n)$$

   Using the fact that

$$L_i(t_i - t_{i-1}) = \frac{\widetilde{D}_f(t_{i-1})}{\widetilde{D}_f(t_i)} - 1$$

where $\widetilde{D}_f$ refers to the discount factor quantification metric of the corresponding forward rate L, we reduce the above to

$$\prod_{i=1}^{n}[1+\ L_i(t_i-t_{i-1})]=\frac{\widetilde{D}_f(t_{i-1})}{\widetilde{D}_f(t_i)}$$

Thus, this induces a single convexity correction, at the end of the composite period.

3. Composite Period - Accruals: The difference between the above analysis and the one for accruals is that, in the case of accruals, all the unit periods' rates preceding the accrual valuation date have been realized. This makes the analysis more straightforward. Given that the fixings have been realized, we have ONLY ONE convexity adjustment across all the realized periods for both arithmetic and geometric compounding – the one at the period terminal anchor.
4. Merged Forward/Funding State Loading Under Arithmetic Compounding: Using the discount factor quantification metric above for the case of arithmetic compounding above, we get

$$\left[\sum_{i=1}^{n}L_i(t_i-t_{i-1})\,\vartheta_i(\tau_n)\right]D_f(\tau_n)S_P(\tau_n)X(\tau_n)$$
$$=\left[\sum_{i=1}^{n}\frac{\widetilde{D}_f(t_{i-1})-\widetilde{D}_f(t_i)}{\widetilde{D}_f(t_i)}\vartheta_i(\tau_n)\right]D_f(\tau_n)S_P(\tau_n)X(\tau_n)$$

This demonstrates that, since there will be a mismatch between the terminal measures at t_i and τ_n, the telescoping will not occur in this case. Thus the merged forward/funding calibration trial becomes a non-linear exercise, thereby not making itself amenable to much of the linear scenarios seen above.
5. Separated Forward and Funding State Loadings under Arithmetic Compounding: Since the expression

$$\left[\sum_{i=1}^{n} L_i(t_i - t_{i-1})\,\vartheta_i(\tau_n)\right] D_f(\tau_n) S_P(\tau_n) X(\tau_n)$$

is linear separately in both L_i and D_f (as well as S_P and X), linear state loadings should absolutely be generatable for those states, the basis spline representations described above are applicable.

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Spline-Based Credit Curve Calibration

1. Overview: Andersen (2003) has made an initial effort in this regard.

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Section V: Multi-Curve Construction and Product Valuation

Correlated Multi-Curve Build-out

Introduction

1. Regime Segmentation: Indicators of regime changes in the interest rate markets, e.g., divergence between XIBOR-based deposits vs. OIS/EONIA, FRA's vs. forwards implied by consecutive deposits etc. are discussed in Ametrano and Bianchetti (2009b), Goldman Sachs (2009), and Bianchetti (2012) among others.
2. Pre-Crisis Segmentation: Segmentation was already present and well-understood pre-2008, but ignore since the effects were small (Fruchard, Zammouri, and Willems (1995), Tuckman and Porfirio (2003)).
3. Prior Multi-Curve Frameworks: The cross-currency swap multi-curve framework was proposed by Boenkost and Schmidt (2005), and was extended to the 3-curve case (i.e., the discount curve, the LIBOR curve, and the bond rates instance) by Kijima, Tanaka, and Wong (2008).
    - Other Two-Curve Extensions =>
        - Morini (2008) and Morini (2009) approach this from the point-of-view of counter-party risk
        - Mercurio (2009) approaches this in terms of an extended LMM
        - Henrard (2009) approaches this using a more foundational axiomatization framework setup

Standard FRA Setup

1. Standard FRA Setup Basis:

$$F_x(t;\ T_1, T_2) = \frac{1}{\tau_x(T_1, T_2)}\left[\frac{1}{P_x(t;\ T_1, T_2)} - 1\right]$$

for tenor x.

$$P_x(t;\ T_1, T_2) = \frac{P_x(t;\ T_2)}{P_x(t;\ T_1)}$$

is the discount factor for tenor x. From Brigo and Mercurio (2006),

$$FRA_x(T_2;\ T_1, T_2, K, N) = N\tau_x(T_1, T_2)[L_x(T_1, T_2) - K]$$

where

$$L_x(T_1, T_2) = \frac{1 - P_x(T_1, T_2)}{\tau_x(T_1, T_2)P_x(T_1, T_2)}$$

i.e., $L_x(T_1, T_2)$ is the T_1-spot LIBOR maturing at

$$T_2 = T_1 + xM$$

2. <u>LIBOR-Standard FRA Specification</u>:

$$\begin{aligned} &FRA_x(t;\ T_1, T_2, K, N) \\ &= P_x(t, T_2)\mathbb{E}_t^{Q_x^{T_2}}[FRA_x(T_2;\ T_1, T_2, K, N)] \\ &= N\tau_x(T_1, T_2)P_x(t, T_2)\mathbb{E}_t^{Q_x^{T_2}}[L_x(T_1, T_2) - K] \\ &= N\tau_x(T_1, T_2)P_x(t, T_2)\mathbb{E}_t^{Q_x^{T_2}}[F_x(t;\ T_1, T_2) - K] \end{aligned}$$

where

$$F_x(t;\ T_1,T_2) = \mathbb{E}_t^{Q_x^{T_2}}[L_x(T_1,T_2)]$$

is the corresponding Standard FRA rate, $\mathbb{E}_t^{Q_x^{T_2}}$ denotes the expectation taken at time t with respect to the measure Q_t (within the filtration $\mathcal{F}_t$ that encodes the market information available at t) over the $\mathbb{E}_t^{Q_x^{T_2}}$-forward measure corresponding to the numeraire $P_x(t,T_2)$.

3. Multiplicative Standard FRA Basis:

$$\mathcal{N}(t;\ T_1,T_2) = \frac{F_f(t;\ T_1,T_2)}{F_d(t;\ T_1,T_2)}\frac{\tau_f(T_1,T_2)}{\tau_d(T_1,T_2)} = \frac{P_f(T_1,T_2)}{P_d(T_1,T_2)}\left[\frac{P_f(t,T_2)-P_f(t,T_1)}{P_d(t,T_2)-P_d(t,T_1)}\right]$$

which simply results in

$$P_f(t;\ T_1,T_2) = \frac{1}{1+F_d(t;\ T_1,T_2)\mathcal{N}(t;\ T_1,T_2)\tau_d(T_1,T_2)}$$

4. Additive Standard FRA Basis:

$$P_f(t;\ T_1,T_2) = \frac{1}{1+[F_d(t;\ T_1,T_2)+\mathcal{N}'(t;\ T_1,T_2)]\tau_d(T_1,T_2)}$$

implies

$$\mathcal{N}'(t;\ T_1,T_2) = F_d(t;\ T_1,T_2)[\mathcal{N}(t;\ T_1,T_2)-1]$$

5. Forward Basis Bootstrapping Relations: These come from Bianchetti (2012)

$$P_{d,i} = \frac{P_{f,i}\mathcal{N}_i}{P_{f,i-1}-P_{f,i}+P_{f,i}\mathcal{N}_i}P_{d,i-1} = \frac{P_{f,i}\mathcal{N}_i}{P_{f,i-1}-P_{f,i}\mathcal{N}_i'\tau_{d,i}}P_{d,i-1}$$

$$P_{f,i} = \frac{P_{d,i}}{P_{d,i}+\left(P_{d,i-1}+P_{d,i}\right)\mathcal{N}_i}P_{f,i-1} = \frac{P_{d,i}\mathcal{N}_i}{P_{d,i-1}+P_{d,i}\mathcal{N}_i'\tau_{d,i}}P_{f,i-1}$$

6. <u>Quanto-Adjusted Standard FRA Evolution Dynamics</u>: The forward rates are martingales in their own measure ${Q_t}^f$:

$$\frac{\Delta F(t;\ T_1, T_2)}{F(t;\ T_1, T_2)} = \sigma_F(t)\Delta {W_F}^{T_2} : t \leq T_1$$

Likewise, we employ an analogy with the quanto-based approaches used in the FX world to derive the dynamics of a multiplicative quanto-adjustment X_{fd} as

$$\frac{\Delta X_{fd}(t;\ T_1, T_2)}{X_{fd}(t;\ T_1, T_2)} = \sigma_F(t)\Delta {W_X}^{T_2} : t \leq T_2$$

with

$$\Delta {W_F}^{T_2}\Delta {W_X}^{T_2} = \rho_{FX}\Delta t$$

- FX Quanto Analogy Application => The Standard FRA Payoff at T_2 is $F(t;\ T_1, T_2)$. The payoff over the "local/domestic" currency using the domestic numeraire is

$$\frac{F(t;\ T_1, T_2)}{X_{fd}(t;\ T_1, T_2)}$$

  This quantity, by using the FX quanto analogy, is a martingale in the domestic/discounting measure.
- Quanto Drift Adjustment => This sets us up for the application of the change of numeraire that produces an additional drift, i.e.,

$$\mathbb{E}_t^{Q_d^{T_2}}\left[\frac{F(t;\ T_1, T_2)}{X_{fd}(t;\ T_1, T_2)}\right] \approx -\rho_{FX}\sigma_F\sigma_X$$

i.e.,

$$\mathbb{E}_t^{Q_d^{T_2}}\left[\frac{F_f(t;\ T_1,T_2)}{X_{fd}(t;\ T_1,T_2)}\right] = F_f(t;\ T_1,T_2)QA_{fd}(t;\ T_1,\sigma_F,\sigma_X,\rho_{FX}) = e^{-\int_t^{T_1}\rho_{FX}(u)\sigma_F(u)\sigma_X(u)du}$$

where $QA_{fd}(t;\ T_1,\sigma_F,\sigma_X,\rho_{FX})$ is the multiplicative quanto adjustment (Jamshidian (1989), Geman, El Karoui, and Rochet (1995), Brigo and Mercurio (2006)).

- Additive Quanto Adjustment => Define this as Bianchetti (2012) does:

$$\begin{aligned} QA'_{fd}(t;\ T_1,\sigma_F,\sigma_X,\rho_{FX}) &= \mathbb{E}_t^{Q_d^{T_2}}\left[F_f(T_1;\ T_1,T_2)\right] - F_f(t;\ T_1,T_2) \\ &= F_f(t;\ T_1,T_2)\left[QA_{fd}(t;\ T_1,\sigma_F,\sigma_X,\rho_{FX}) - 1\right] \end{aligned}$$

- Additive/Multiplicative Basis Adjustment =>
  - Multiplicative Basis Adjustment $BA_{fd}(t;\ T_1,T_2)$ is given as:

$$BA_{fd}(t;\ T_1,T_2)QA_{fd}(t;\ T_1,\sigma_F,\sigma_X,\rho_{FX}) = \frac{\tau_f(T_1,T_2)}{\tau_d(T_1,T_2)}\frac{\mathbb{E}_t^{Q_d^{T_2}}\left[L_f(T_1,T_2)\right]}{\mathbb{E}_t^{Q_d^{T_2}}\left[L_d(T_1,T_2)\right]}$$

  - Additive Basis Adjustment $BA'_{fd}(t;\ T_1,T_2)$ is given as:

$$\begin{aligned} & BA'_{fd}(t;\ T_1,T_2) + QA'_{fd}(t;\ T_1,\sigma_F,\sigma_X,\rho_{FX}) \\ &\qquad = \tau_f(T_1,T_2)\mathbb{E}_t^{Q_d^{T_2}}\left[L_f(T_1,T_2)\right] - \tau_d(T_1,T_2)\mathbb{E}_t^{Q_d^{T_2}}\left[L_d(T_1,T_2)\right] \end{aligned}$$

- Mean-reverting Deterministic Volatility Form => This is outlined in Andersen and Piterbarg (2010), and it possesses distinctive properties that enable it to capture certain kinds of physics:

$$\sigma(t,T) = \frac{\alpha(t)}{a}\left[1 - e^{-a(T-t)}\right]$$

In his collateral choice option calculation, Piterbarg (2012) uses $\alpha(t) = 0.50\%$ and $a = 40\%$.

7. Standard FRA Price:

$$FRA_d(t;\ T_1, T_2, K, N) = NP_d(t, T_2)\tau_f(T_1, T_2)\left\{\mathbb{E}_t^{Q_d^{T_2}}\left[F_f(T_1;\ T_1, T_2)\right] - K\right\}$$

$$FRA_d(t;\ T_1, T_2, K, N) = NP_d(t, T_2)\tau_f(T_1, T_2)\{QA_{fd}(t;\ T_1, \sigma_F, \sigma_X, \rho_{FX}) - K\}$$

8. Standard FRA Quanto-Adjusted Par Swap Rate:

$$S_f(T, S) = \frac{\sum_{i=1}^{n} P_d(t, T_i)\tau_f(T_{i-1}, T_i)F_f(t;\ T_{i-1}, T_i)QA_{fd}\left(t;\ T_{i-1}, \sigma_{F,i}, \sigma_{X,i}, \rho_{FX,i}\right)}{\sum_{j=1}^{m} P_d\left(t, S_j\right)\tau_d\left(S_{j-1}, S_j\right)}$$

9. Applicability of the Quanto Adjustment Formulas: It is important to remember that these quanto adjustments above are primarily for textbook/standard FRAs. Further, for these FRA's, the observed prices/rates can be worked out explicitly in the discounting measure in itself anyway, thus rendering the quanto correlation effects irrelevant. Obviously, both the discounting and the native FRA measure converge at the FRA exercise date due to the fixing.

Standard FRA Options

1. Caplet/Floorlet Options: Caplet/Floorlet Options on a T_1-spot rate exercised date with the payoff maturity at T_2 is given by (Bianchetti (2012), Bianchetti and Carlicchi (2012))

$$cf(T_2;\ T_1, T_2, K, w, N) = N\tau_f(T_1, T_2)\max\left\{w\left(L_f(T_1, T_2) - K\right)\right\}$$

Thus,

$$cf(T_2;\ T_1,T_2,K,w,N) = N\tau_f(T_1,T_2)\mathbb{E}_t^{Q_d^{T_2}}\left[\max\left\{w\left(L_f(T_1,T_2)-K\right)\right\}\right]$$

- Closed Form Expression =>

$$cf(T_2;\ T_1,T_2,K,w,N)$$
$$= N\tau_f(T_1,T_2)Black\left(F_f(t;\ T_1,T_2)QA_{fd}(t;\ T_1,\sigma_F,\sigma_X,\rho_{FX}),K,\mu_f,\sigma_f,w\right)$$

where

$$Black\left(F,K,\mu_f,\sigma_f,w\right) = w[F\Phi(wd_+) - K\Phi(wd_-)]$$

with

$$d_+ = \frac{\log\frac{F}{K} + \mu(t,T) \pm \frac{1}{2}\sigma^2(t,T)}{\sigma(t,T)};w = \begin{cases} +1 & Caplet \\ -1 & Floorlet \end{cases}$$

$$\mu(t,T) = \int_t^T \mu(u)du = -\int_t^T \sigma_f(u)\sigma_X(u)\rho_{fX}(u)du$$

$$\sigma^2(t,T) = \int_t^T \sigma^2(u)du = \int_t^T {\sigma_f}^2(u)du$$

- Cap/Floor Option Prices =>

$$CF(t;\ T,K,w,N) = \sum_{i=1}^{n} cf(t;\ T_{i-1},T_i,K_i,w_i,N_i)$$

Plug in the earlier developed relation for the cap/floor prices.

- SABR/LIBOR Cap Volatility Functional Form => The claim is that the industry uses the following humped function for capturing the cap volatility:

$$\sigma(t) = [a + b(T - t)]e^{-c(T-t)} + d$$

Rebonato, McKay, and White (2009) assign physical meanings to a, b, c, and d, as well as how to calibrate this model using caplet volatilities.

No Arbitrage and Counter-party Risk Based Standard FRA Formulation

1. Setup: Following Mercurio (2009), we set

$$P_f(t, T) = P_d(t, T) R\left(t;\ t, T, R_f\right)$$

where

$$R\left(t;\ t, T, R_f\right) = R_f + \left(1 - R_f\right) \mathbb{E}_t^{Q_d}[q_d(T_1, T_2)]$$

given R_f is the recovery rate, and

$$q_d(T_1, T_2) = \mathbb{E}_t^{Q_d}\left[\mathrm{I}_{\tau(t) > T}\right]$$

is the counter-party default probability associated with the default time $\tau(t)$.

2. Counter-party risk based Risky XIBOR and Standard Forward:

$$L_f(T_1, T_2) = \frac{1}{\tau_f(T_1, T_2)}\left[\frac{1}{P_f(T_1, T_2)} - 1\right] = \frac{1}{\tau_f(T_1, T_2)}\left[\frac{1}{P_d(T_1, T_2)} \frac{1}{R\left(T_1;\ T_1, T_2, R_f\right)} - 1\right]$$

$$F_f(t;\ T_1,T_2) = \frac{1}{\tau_f(T_1,T_2)}\left[\frac{P_f(t,T_1)}{P_f(t,T_2)} - 1\right] = \frac{1}{\tau_f(T_1,T_2)}\left[\frac{P_d(t,T_1)}{P_d(t,T_2)}\frac{R\left(t;\ t,T_1,R_f\right)}{R\left(t;\ t,T_2,R_f\right)} - 1\right]$$

The corresponding Standard FRA price is

$$F_f(t;\ T_1,T_2) = \frac{P_d(t,T_1)}{R\left(t;\ T_1,T_2,R_f\right)} - P_d(t,T_2)\left[1 + K\tau_f(T_1,T_2)\right]$$

3. Counter-party risk Quanto Adjustment:

$$QA_{fd}(t;\ T_1,T_2) = \frac{P_d(t,T_1)\dfrac{1}{R\left(t;\ T_1,T_2,R_f\right)} - P_d(t,T_2)}{P_d(t,T_1)\dfrac{R\left(t;\ t,T_1,R_f\right)}{R\left(t;\ t,T_2,R_f\right)} - P_d(t,T_2)}$$

$$QA'_{fd}(t;\ T_1,T_2) = \frac{1}{\tau_f(T_1,T_2)}\frac{P_d(t,T_1)}{P_d(t,T_2)}\left[\frac{1}{R\left(t;\ T_1,T_2,R_f\right)} - \frac{R\left(t;\ t,T_1,R_f\right)}{R\left(t;\ t,T_2,R_f\right)}\right]$$

Morini (2009) expresses the counter party risk spot exchange rate in terms of the credit variables.

4. Counter-party risk Basis Adjustment:

$$BA_{fd}(t;\ T_1,T_2) = \frac{P_d(t,T_1)R\left(t;\ t,T_1,R_f\right) - P_d(t,T_2)R\left(t;\ t,T_2,R_f\right)}{\left[P_d(t,T_1) - P_d(t,T_2)\right]R\left(t;\ t,T_2,R_f\right)}$$

$$BA'_{fd}(t;\ T_1,T_2) = \frac{1}{\tau_d(T_1,T_2)}\frac{P_d(t,T_1)}{P_d(t,T_2)}\left[\frac{R\left(t;\ t,T_1,R_f\right)}{R\left(t;\ t,T_2,R_f\right)} - 1\right]$$

Market FRA Setup

1. Standard FRA vs. Market FRA: Using the time T payoff, the standard FRA value is

$$FRA_{STD}\left(t, T, L_{x,i}, K\right) = \tau_L(T_{i-1}, T_i) P_C(t, T_i)\left[F_{x,i}(t) - K\right]$$

The payoff at time T for the market FRA is

$$FRA_{MKT}(T_{i-1}, T_i, K) = \frac{[L_x(T_{i-1}, T_i) - K]\tau_L(T_{i-1}, T_i)}{1 + L_x(T_{i-1}, T_i)\tau_L(T_{i-1}, T_i)}$$

2. Pricing of the Market FRA: We employ the formulation presented in Mercurio (2010), simplifying the notation a little. Setting

$$\tau_L(T_{i-1}, T_i) \to \tau_{1,2}$$

and

$$L(T_{i-1}, T_i) \to L(T_1, T_2)$$

we get

$$FRA_{MKT}(T_1;\ T_1, T_2, K) = \frac{[L(T_1, T_2) - K]\tau_{1,2}}{1 + L(T_1, T_2)\tau_{1,2}}$$

    - Reduced discounting measure representation for Market FRA => Unfortunately, given the above payoff definition for the market FRA's, further discounting measure representations are not possible.

3. The Par FRA: Since FRA's are fully collateralized, we work assuming that overnight rate is the collateral rate. The par FRA rate at time t is the K above that results in the net value of zero, i.e.,

$$\mathbb{E}_t^{Q_C^{T_1}}\left[\frac{[L(T_1,T_2)-FRA_{PAR}(t;\ T_1,T_2)]\tau_{1,2}}{1+L(T_1,T_2)\tau_{1,2}}|\mathcal{F}_t\right]=0$$

or

$$\mathbb{E}_t^{Q_C^{T_1}}\left[1-\frac{FRA_{PAR}(t;\ T_1,T_2)\tau_{1,2}}{1+L(T_1,T_2)\tau_{1,2}}|\mathcal{F}_t\right]=0$$

Thus

$$FRA_{PAR}(t;\ T_1,T_2)=\frac{1}{\tau_{1,2}\mathbb{E}_t^{Q_C^{T_1}}\left[\frac{1}{1+L(T_1,T_2)\tau_{1,2}}|\mathcal{F}_t\right]}-\frac{1}{\tau_{1,2}}$$

4. <u>Measure Change from $\mathbb{E}_t^{Q_C^{T_1}}$ to $\mathbb{E}_t^{Q_C^{T_2}}$</u>: If the terminal payoff only depended on T_1, the above evaluation can be made in the terminal measure $\mathbb{E}_t^{Q_C^{T_1}}$. However, since $L(T_1,T_2)$ also depends on the collateral account's numeraire evolution from T_1 to T_2, we need to apply the appropriate risk neutral measure $P_C(T_1,T_2)$ and change the measure to $\mathbb{E}_t^{Q_C^{T_2}}$.
    - T_2-Forward Measure Numeraire Changes => Every cash flow needs to be discounted at its terminal/payout date, a consequence of the basic Arrow replication principle. Thus, the "inner" contingent claims need to be evaluated using their own terminal measure. This necessitates a measure change, which, using the corresponding forward numeraire change, results in the deterministic discount factor being pulled out (this numeraire corresponds to the discount factor spanning the inner start and the end dates).
5. <u>Expectation Under $\mathbb{E}_t^{Q_C^{T_2}}$</u>:

$$\mathbb{E}_t^{Q_C^{T_1}}\left[\frac{1}{1+L(T_1,T_2)\tau_{1,2}}|\mathcal{F}_t\right]=P_C(T_1,T_2)\mathbb{E}_t^{Q_C^{T_2}}\left[\frac{1}{P_C(T_1,T_2)}\frac{1}{1+L(T_1,T_2)\tau_{1,2}}|\mathcal{F}_t\right]$$
$$=\frac{P_C(t,T_2)}{P_C(t,T_1)}\mathbb{E}_t^{Q_C^{T_2}}\left[\frac{1+L_C(T_1,T_2)\tau^C{}_{1,2}}{1+L(T_1,T_2)\tau_{1,2}}|\mathcal{F}_t\right]$$

where

$$L_C(T_1, T_2) = \frac{1}{\tau^C{}_{1,2}} \left[\frac{P_C(t, T_2)}{P_C(t, T_1)} - 1\right]$$

6. <u>FRA Par Rate Expression</u>: Thus, the par FRA Rate becomes

$$FRA_{PAR}(t;\ T_1, T_2) = \frac{1}{\tau_{1,2} \frac{P_C(t, T_2)}{P_C(t, T_1)} \mathbb{E}_t^{Q_C^{T_2}} \left[\frac{1 + L_C(T_1, T_2)\tau^C{}_{1,2}}{1 + L(T_1, T_2)\tau_{1,2}} | \mathcal{F}_t\right]} - \frac{1}{\tau_{1,2}}$$

7. <u>FRA Par Rate in Terms of the Collateral Forward Rate</u>: Remembering that

$$F_C(t;\ T_1, T_2) = \frac{1}{\tau^C{}_{1,2}} \left[\frac{P_C(t, T_2)}{P_C(t, T_1)} - 1\right]$$

we get

$$FRA_{PAR}(t;\ T_1, T_2) = \frac{1 + F_C(t;\ T_1, T_2)\tau^C{}_{1,2}}{\tau_{1,2} \mathbb{E}_t^{Q_C^{T_2}} \left[\frac{1 + L_C(T_1, T_2)\tau^C{}_{1,2}}{1 + L(T_1, T_2)\tau_{1,2}} | \mathcal{F}_t\right]} - \frac{1}{\tau_{1,2}}$$

It is easy to see that under the single curve case

$$FRA_{PAR}(t;\ T_1, T_2) = F_C(t;\ T_1, T_2)$$

8. <u>Modeling the Dynamics of $F_C(t;\ T_1, T_2)$ and $FRA(t;\ T_1, T_2)$</u>: We choose the convenient shifted log normal evolution form for $F_C(t;\ T_1, T_2)$ and $FRA(t;\ T_1, T_2)$:

$$\Delta F_C(t;\ T_1, T_2) = \sigma_{1,2}^C \left[\frac{1}{\tau_{1,2}^C} + F_C(t;\ T_1, T_2)\right] = \Delta Z_2^C$$

and

$$\Delta F_{FRA}(t;\ T_1, T_2) = \sigma_{1,2}\left[\frac{1}{\tau_{1,2}} + F_{FRA}(t;\ T_1, T_2)\right] = \Delta Z_2$$

where $\sigma_{1,2}$ and $\sigma_{1,2}^{C}$ are the constant instantaneous volatilities, and ΔZ_2 and ΔZ_2^{C} are the $\mathcal{F}_t$-adapted $Q_C^{T_2}$ Brownians with instantaneous correlations $\rho_{1,2}$.

9. Evaluating the Closed-Form for the Dynamics:

$$F_C(T_1;\ T_1, T_2) = F_C(t;\ T_1, T_2)e^{-\frac{1}{2}{\sigma_{1,2}^{C}}^{2}(T_1-t)+\sigma_{1,2}^{C}[Z_2^{C}(T_1)-Z_2^{C}(t)]}$$

and

$$FRA(T_1;\ T_1, T_2) = FRA(t;\ T_1, T_2)e^{-\frac{1}{2}{\sigma_{1,2}}^{2}(T_1-t)+\sigma_{1,2}[Z_2(T_1)-Z_2(t)]}$$

The evolution stops at T_1, since both F_C and FRA cease their evolutions beyond their fixing time, i.e., T_1.

10. Connection to Fixings: Remember that $L_C(T_1, T_2)$ and $L(T_1, T_2)$ are linked to their corresponding market FRA via

$$L_C(T_1, T_2) = F_C(T_1;\ T_1, T_2)$$

and

$$L(T_1, T_2) = FRA(T_1;\ T_1, T_2)$$

Also, since this caters only to the evolution of the forward rates, they are still in their stochastic forms, with the expectations taken to within the T_1 measure only at the final stage.

11. Evaluation of the Expectation:

$$\mathbb{E}_t^{Q_C^{T_2}}\left[\frac{1 + L_C(T_1, T_2){\tau^C}_{1,2}}{1 + L(T_1, T_2)\tau_{1,2}} | \mathcal{F}_t\right]$$

$$= \mathbb{E}_t^{Q_C^{T_2}}\left[\frac{1 + F_C(t;\ T_1, T_2){\tau^C}_{1,2}}{1 + FRA(t;\ T_1, T_2)\tau_{1,2}} | \mathcal{F}_t\right] e^{-\frac{1}{2}\left({\sigma_{1,2}^C}^2 - {\sigma_{1,2}}^2\right)(T_1 - t) + \sigma_{1,2}^C[Z_2^C(T_1) - Z_2^C(t)] - \sigma_{1,2}[Z_2(T_1) - Z_2(t)]}$$

Evaluation of this stochastic integral leads to

$$\mathbb{E}_t^{Q_C^{T_2}}\left[\frac{1 + L_C(T_1, T_2){\tau^C}_{1,2}}{1 + L(T_1, T_2)\tau_{1,2}} | \mathcal{F}_t\right] = \frac{1 + L_C(T_1, T_2){\tau^C}_{1,2}}{1 + L(T_1, T_2)\tau_{1,2}} e^{-\left({\sigma_{1,2}^C}^2 - \rho_{1,2}\sigma_{1,2}\sigma_{1,2}^C\right)(T_1 - t)}$$

12. <u>Closed Form Par FRA and Convexity Correction</u>:

$$FRA_{PAR}(t;\ T_1, T_2) = \frac{1}{\tau_{1,2}}\left[1 + F_C(t;\ T_1, T_2){\tau^C}_{1,2}\right]\left(e^{-\left({\sigma_{1,2}}^2 - \rho_{1,2}\sigma_{1,2}\sigma_{1,2}^C\right)(T_1 - t)} - 1\right)$$

Thus the convexity correction is computed as

$$FRA_{PAR}(t;\ T_1, T_2) - FRA_{STD}(t;\ T_1, T_2)$$
$$= \frac{1}{\tau_{1,2}}\left[1 + F_C(t;\ T_1, T_2){\tau^C}_{1,2}\right]\left(e^{-\left({\sigma_{1,2}}^2 - \rho_{1,2}\sigma_{1,2}\sigma_{1,2}^C\right)(T_1 - t)} - 1\right)$$

13. <u>Constant FRA-Collateral Forwards Basis</u>: Mercurio (2010) shows that the dual log-normal formulation above results in corrections of the order of one bp in most cases. Thus, a case is made for analyzing the impact of using a small/constant FRA-forward collateral basis.
14. <u>Constant FRA-Collateral Forwards Basis Formulation</u>:

$$FRA_{PAR}(t;\ T_1, T_2) - FRA_{STD}(t;\ T_1, T_2) = S_{1,2}$$

a small positive constant. The $FRA_{PAR}(t;\ T_1, T_2)$ then becomes

$$FRA_{PAR}(t;\ T_1,T_2)=\frac{1+\left[FRA(t;\ T_1,T_2)-S_{1,2}\right]{\tau^C}_{1,2}}{\tau_{1,2}\mathbb{E}_t^{Q_C^{T_2}}\left[\frac{1+\{L_C(T_1,T_2)-S_{1,2}\}{\tau^C}_{1,2}}{1+L(T_1,T_2)\tau_{1,2}}|\mathcal{F}_t\right]}-\frac{1}{\tau_{1,2}}$$

$$=\frac{1+\left[FRA(t;\ T_1,T_2)-S_{1,2}\right]{\tau^C}_{1,2}}{{\tau^C}_{1,2}-\left({\tau^C}_{1,2}-\tau_{1,2}+\tau_{1,2}{\tau^C}_{1,2}S_{1,2}\right)\mathbb{E}_t^{Q_C^{T_2}}\left[\frac{1}{1+L(T_1,T_2)\tau_{1,2}}|\mathcal{F}_t\right]}-\frac{1}{\tau_{1,2}}$$

15. <u>Shifted Log Normal Dynamics for $L(T_1,T_2)$</u>: Assuming shifted log normal dynamics for $L(T_1,T_2)$, and assuming

$${\tau^C}_{1,2}=\tau_{1,2}$$

- an equivalent simplifying basis assumption would correspond to

$$\tau_{1,2}FRA_{PAR}(t;\ T_1,T_2)-{\tau^C}_{1,2}F_C(t;\ T_1,T_2)=S_{1,2}$$

- we get

$$FRA_{PAR}(t;\ T_1,T_2)=\frac{1+\left[FRA(t;\ T_1,T_2)-S_{1,2}\right]\tau_{1,2}}{\tau_{1,2}-\left({\tau_{1,2}}^2S_{1,2}\right)\frac{1}{1+FRA(t;\ T_1,T_2)\tau_{1,2}}e^{{\sigma_{1,2}}^2T_1}}-\frac{1}{\tau_{1,2}}$$

16. <u>Convexity Correction</u>: The corresponding convexity correction is

$$FRA_{PAR}(t;\ T_1,T_2)-FRA(t;\ T_1,T_2)$$

$$=S_{1,2}\left[e^{{\sigma_{1,2}}^2T_1}-1\right]\frac{\tau_{1,2}{S_{1,2}}^2e^{{\sigma_{1,2}}^2T_1}\left[e^{{\sigma_{1,2}}^2T_1}-1\right]}{1+FRA(t;\ T_1,T_2)\tau_{1,2}-\tau_{1,2}{S_{1,2}}^2e^{{\sigma_{1,2}}^2T_1}}$$

$$\approx S_{1,2}\left[e^{{\sigma_{1,2}}^2T_1}-1\right]$$

to the leading order in $S_{1,2}$. Since the convexity adjustment goes as $e^{{\sigma_{1,2}}^2T_1}$, it may appear that the straightforward dependence on linear, unadjusted T_1 may cause the correction to blow up at sufficient maturities. However, the shifted log-normal volatility is at least one order of magnitude smaller than the corresponding log normal volatility, i.e.,

$$\sigma_{1,2} \approx \frac{\tau_{1,2}{\sigma_{1,2}}^{LN} FRA(t;\ T_1, T_2)}{1 + FRA(t;\ T_1, T_2)}$$

where ${\sigma_{1,2}}^{LN}$ is the log normal volatility.

Futures

1. Futures Terminal Price/Payoff: Futures Payoff is

$$1 - L(T_{k-1} - T_k)$$

Thus

$$\mathbb{E}_t^{Q_C^{T_{k-1}}}[1 - L(T_{k-1} - T_k)] = 1 - \mathbb{E}_t^{Q_C^{T_{k-1}}}[L(T_{k-1} - T_k)]$$

Here the measure $Q_C^{T_{k-1}}$ will be treated as the collateralized discounting measure, as the futures are collateralized transactions (the collateralized discounting measure can be a discretely re-balanced bank account numeraire – called the spot-T measure).

2. Treatments of Convexity Adjustment: Several treatments of the convexity adjustment exist in the literature:
   - Both Kirikos and Novak (1997) and Henrard (2005) use the 1-factor Hull and White (1990) model
   - Piterbarg and Renedo (2006) use the stochastic volatility model
   - Mercurio (2009) and Mercurio (2010) use the multi-curve extended Market Model – that is what we consider here.

3. Terminology for the Extended Multi-Curve Market Model: Borrowing from Mercurio (2009) and Mercurio (2010) we get:
   - $Z^d = \{{Z_1}^d, \cdots, {Z_M}^d\} \Longrightarrow$ The M-dimensional discretely balanced bank-account numeraire measure Q_C^T, and its Brownian components
   - $$F_h^C(t) = F^C(t;\ T_{k-1}, T_k) = \frac{1}{\tau_k^c}\left[\frac{P_C(t, T_{k-1})}{P_C(t, T_k)} - 1\right]$$

     is the discounting curve forward rate, and τ_k^c is the time interval.
   - σ_k and σ_h^c are respective deterministic volatilities of L_k and F_h^c (instantaneous)
   - $\rho_{k,h}^{L,F}$ is the instantaneous correlation between L_k and F_h^c
4. Change of Numeraire to get to $(T_{k-1}, T_k]$: Applying the change of measure on successive segments to get to $(T_{k-1}, T_k]$, the extended market model predicts that

$$\Delta L_k(t) = \sigma_k L_k(t) \sum_{h=\beta(t)}^{k} \frac{\rho_{k,h}^{L,F}\tau_k^c\sigma_h^c F_h^C(t)}{1 + \tau_k^c F_h^C(t)}\Delta t + \sigma_k L_k(t)\Delta Z_k^d(t)$$

   where

$$\beta(t) = m \quad T_{m-2} < t \leq T_{m-1}; m \geq 1$$
$$\beta(0) = 0 \quad t \in \left(T_{\beta(t)-2}, T_{\beta(t)-1}\right]$$

5. Full Drift Freeze: For computational convenience, we freeze the drift evolution at its time 0 value as

$$\mu_k = \sigma_k \sum_{h=0}^{k} \frac{\rho_{k,h}^{L,F}\tau_k^c\sigma_h^c F_h^C(0)}{1 + \tau_k^c F_h^C(0)}$$

   Now we can evolve $L_k(t)$ as

$$\Delta L_k(t) = \mu_k L_k(t)\Delta t + \sigma_k L_k(t)\Delta Z_k^d(t)$$

- Future Price => The Futures Price Valuation now becomes straightforward, as

$$V_t = 1 - \mathbb{E}_t^{Q_C^{T_{k-1}}}[L(T_{k-1}, T_k)|\mathcal{F}_t] \approx 1 - L_k(t)e^{\mu_k(T_{k-1}-t)}$$

$e^{\mu_k(T_{k-1}-t)}$ then is the convexity adjustment, and given that $L_k(0)$ is a market observable, V_0 may be computed from

$$L_k(0) = (1 - V_0)e^{\mu_k(T_{k-1}-t)}$$

6. <u>Drift Freeze Adjustment #2</u>: Here, only the values of the forward rates at frozen at time 0, not $\beta(t)$, and

$$\mu_k = \sigma_k \sum_{h=\beta(t)}^{k} \frac{\rho_{k,h}^{L,F}\tau_k^c\sigma_h^c F_h^C(0)}{1 + \tau_k^c F_h^C(0)}$$

Thus

$$V_t = 1 - \mathbb{E}_t^{Q_C^{T_{k-1}}}[L(T_{k-1}, T_k)|\mathcal{F}_t] \approx 1 - L_k(t)e^{\int_t^{T_{k-1}}\mu_k(u)du}$$

7. <u>Price/Convexity Adjustment</u>:

$$\mathbb{E}_t^{Q_C^{T_{k-1}}}[L(T_{k-1}, T_k)|\mathcal{F}_t] = L_k(0)e^{\sigma_k \sum_{h=0}^{k}\left[\int_{T_{h-1}}^{T_h}\sum_{j=h+1}^{k}\frac{\rho_{k,h}^{L,F}\tau_h^c\sigma_h^c F_h^C(0)}{1+\tau_k^c F_h^C(0)}du\right]}$$

$$= L_k(0)e^{\sigma_k \sum_{h=0}^{k}\sum_{j=h+1}^{k}\frac{\rho_{k,h}^{L,F}\tau_h^c\sigma_h^c F_h^C(0)}{1+\tau_k^c F_h^C(0)}(T_h - T_{h-1})} = L_k(0)e^{\sigma_k \sum_{j=1}^{k}\frac{\rho_{k,j}^{L,F}\tau_j^c\sigma_j^c F_j^C(0)}{1+\tau_j^c F_j^C(0)}T_{j-1}}$$

Thus

$$L_k(0) = (1 - V_0)e^{\sigma_k \sum_{j=1}^{k} \frac{\rho_{k,j}^{L,F} \tau_j^C \sigma_j^C F_j^C(0)}{1+\tau_j^C F_j^C(0)} T_{j-1}}$$

Multi-Curve Swap Valuation

1. Interest Rate Futures: Just like the standard FRA, interest rate futures are key for the discovery of the forward swap rates and volatilities, and therefore for their calibration. These do trade, and appear to referred to literature under different names – STIR (short-term interest rate) future (Henrard (2013)), or FSIRS (forward starting IRS) (Bianchetti (2012)).
2. Swap Annuities in the Discount/Forward Measure:

$$A_d(t, S) = \sum_{i=1}^{m} \tau_d(S_{i-1}, S_i) P_d(t, S_i)$$

and

$$A_f(t, S) = \sum_{j=1}^{m} \tau_d(T_{j-1}, T_j) P_f(t, T_j)$$

The swap rate in the forward measure would be

$$S_f(t;\ T, S) = \frac{\sum_{i=1}^{m} \tau_f(S_{i-1}, S_i) F_f(S_{i-1}, S_i) P_f(t, S_i)}{A_d(t, S)}$$

and similar expressions may be computed for $S_d(t;\ T, S)$.

3. Dynamics of $S_f(t;\ T, S)$: In its own measure, $S_f(t;\ T, S)$ is a martingale, i.e.,

$$\frac{\Delta S_f(t;\ T, S)}{S_f(t;\ T, S)} = \nu_f(t; T, S) \Delta W_F^{T,S}$$

with $t \geq T_0$, T_0 being the starting/effective date of the swap contract.

4. Swap Annuity Exchange Rate: Given that the swap annuity is the numeraire, we may introduce the swap annuity exchange rate quanto adjustment by resorting to FX-type quanto adjustment we have seen before, i.e.,

$$Y_{fd}(t,S) = \frac{A_f(t,S)}{A_d(t,S)}$$

Further, we consider the dynamics of $Y_{fd}(t,S)$

$$\frac{\Delta Y_{fd}(t,S)}{Y_{fd}(t,S)} = \nu_Y(t,S)\Delta W_Y^S$$

along with

$$\Delta W_F^{T,S}\Delta W_Y^S = \rho_{FY}(t;T,S)\Delta t$$

for

$$t \geq T_0$$

5. Swap Annuity Quanto Adjustment:

$$\mathbb{E}_t^{Q_d^{T,S}}[SwapPV] = A_d(t,S)\mathbb{E}_t^{Q_d^{T,S}}\left[\frac{S_f(t;T,\ S)}{Y_{fd}(t,S)}\right]$$

Given that, by construction, $\mathbb{E}_t^{Q_d^{T,S}}\left[\frac{S_f(t;T,S)}{Y_{fd}(t,S)}\right]$ is a martingale in the $Q_d^{T,S}$ measure, this produces the drift adjustment

$$QA_{fd}(t;T,S,\nu_F,\nu_Y,\rho_{FY}) = e^{-\lambda_f} = e^{-\int_t^{T_0}\rho_{FX}(u,T,S)\nu_Y(u,S)\nu_F(u,T,S)du}$$

6. Quanto Adjusted Par Swap Rate: Setting

$$\mathbb{E}_t^{Q_d^{T,S}}[SwapPV] = A_d(t,S)\mathbb{E}_t^{Q_d^{T,S}}\left[S_f(T_0;T,S)\right]$$

we get

$$\mathbb{E}_t^{Q_d^{T,S}}\left[S_f(t;T,S)\right] = S_f(T_0;T,S)QA_{fd}(t;T,S,\nu_F,\nu_Y,\rho_{FY})$$

and

$$QA'_{fd}(t;T,S,\nu_F,\nu_Y,\rho_{FY}) = S_f(t;T,S)\left[QA_{fd}(t;T,S,\nu_F,\nu_Y,\rho_{FY}) - 1\right]$$

where $QA_{fd}(t;T,S,\nu_F,\nu_Y,\rho_{FY})$ is the multiplicative swap quanto adjustment, and $QA'_{fd}(t;T,S,\nu_F,\nu_Y,\rho_{FY})$ is the additive swap quanto adjustment.

7. Swaption Pricing: The T_0-Spot swap rate with an exercise date at T_0 is given by

$$SwaptionPrice(T_0;T,S,K,w,N) = NA_d(T_0,S)\max\left(w\left[S_f(t;T,S) - K\right]\right)$$

The corresponding price at $t < T_0$ is

$$SwaptionPrice(t;T,S,K,w,N) \\ = NA_d(t,S)Black\left(wS_f(t;T,S)QA_{fd}(t;T,S,\nu_F,\nu_Y,\rho_{FY}),w,\lambda_f,\nu_f,\rho_{fY}\right)$$

where λ_f is the drift seen before.

8. Multiple Underlying Interest Rates: When two or more underlying interest rate are present, the pricing expressions can become considerably more complicated (e.g., spread options – see Brigo and Mercurio (2006)).

9. Generalization for Joint Multi Factor Numeraire: Say that the Brownian dynamics of evolution of n latent states are

$$\frac{\Delta A_i}{A_i} = \mu_i \Delta t + \sigma_i \Delta W_i;\ \Delta W_i \Delta W_i = \rho_{ij} \Delta t$$

for $i, j = 1, \ldots, n$. The evolution for $\prod_{i=1}^{n} A_i$ is guided by

$$\frac{\Delta[\prod_{i=1}^{n} A_i]}{[\prod_{i=1}^{n} A_i]} = \left[\sum_{i=1}^{n} \mu_i(t) + \sum_{i=1}^{n} \sum_{j>i}^{n} \rho_{ij}(t)\sigma_i(t)\sigma_j(t)\right] \Delta t$$

From this it is easy to derive the joint numeraire

$$\prod_{i=1}^{n} A_i = e^{\int_{t_0}^{t} \left\{\sum_{i=1}^{n} \mu_i(s) + \sum_{i=1}^{n} \sum_{j>i}^{n} \rho_{ij}(s)\sigma_i(s)\sigma_j(s)\right\} ds}$$

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Cross Currency Basis Swap

Product Details and Valuation

1. Background: From Fujii, Shimada, and Takahashi (2010a, 2010b, 2010c, 2010d), for a USDJPY CCS that uses a USD discounting, the set of constitutive equations are

$$CCS_{USD} = \sum_{n=1}^{N} \delta_{n,USD} \mathbb{E}_{t,USD}^{T_n}\left[L_{a,USD}(T_{n-1}, T_n)\right] P_{USD}(t, T_n) + \sum_{m=1}^{M} \delta_{m,USD} \mathbb{E}_{t,USD}^{T_m}\left[L_{b,USD}(T_{m-1}, T_m)\right] P_{USD}(t, T_m)$$

   i.e., the USD Leg in itself is a full basis swap. Likewise, the JPY leg is also a full basis swap. Thus,

$$PV_{JPY,USD}(t) = CCS_{JPY} + X_{JPY,USD} CCS_{USD}$$

   - The CCS basis swap is quoted typically on the non-funding leg. In the above case that would be the JPY leg.

2. Valuation: Generalizing from the approach of Fujii, Shimada, and Takahashi (2010), we use the following nomenclature:
   - a, b are the corresponding LIBOR legs.
   - $i = 1, \cdots, N$ for leg a, and $j = 1, \cdots, M$ for leg b, where i and j are the cash flow indices.
   - V_{ab} is the PV of the USD segment of the CCS, i.e.,

$$V_{ab} = \sum_{i=1}^{N} \Delta_{i,USD} \mathbb{E}_{t,USD}^{T_i} [L_{a,USD}(T_{i-1}, T_i)] P_{USD}(t, T_i) + \sum_{j=1}^{M} \Delta_{j,USD} \mathbb{E}_{t,USD}^{T_j} [L_{b,USD}(T_{j-1}, T_j)] P_{USD}(t, T_j)$$

This need not be zero.

3. CCS Valuation Base Setup: From the USDJPY CCS market information we get for the JPY basis swap

$$\sum_{i=1}^{N} \Delta_{i,JPY} [L_{a,JPY}(T_{i-1}, T_i) + l_{aC}] P_{JPY}(t, T_i) + \sum_{j=1}^{M} \Delta_{j,JPY} [L_{b,JPY}(T_{j-1}, T_j) + l_{bC}] P_{JPY}(t, T_j) = V_{ab}$$

From the JPY tenor basis swap market information we get

$$\sum_{i=1}^{N} \Delta_{i,JPY} [L_{a,JPY}(T_{i-1}, T_i) + l_{aT}] P_{JPY}(t, T_i) + \sum_{j=1}^{M} \Delta_{j,JPY} [L_{b,JPY}(T_{j-1}, T_j) + l_{bT}] P_{JPY}(t, T_j) = V_{ab}$$

Here we've removed all expectation operators $\mathbb{E}_{t,JPY}^{T_i}$ etc. l_{aC}, l_{bC} are the basis quotes in the cross currency markets, and l_{aT}, l_{bT} are the tenor basis market quotes.

4. Basis Quotes Inputs: Notice that both swaps need to refer to the same tenor/maturity set a, b and T_N, T_M - that is the only restriction. If the quote for the tenor swap is not available, they will need to be implied off of others – this provides a very strong motivation to mark the forward curve off of the forwards basis space. This may also require extrapolation – we'll get to that messy challenge later.

Building the CCS Discount Curve

1. Setup: Subtracting the basis quote legs from the CCS legs, we get

$$[l_{aC} - l_{aT}] \sum_{i=1}^{N} \Delta_{i,JPY} P_{JPY}(t, T_i) + [l_{bC} - l_{bT}] \sum_{j=1}^{M} \Delta_{j,JPY} P_{JPY}(t, T_j) = V_{ab}$$

Actually,

$$l_{aC} \equiv l_{aC}(T_N)$$

$$l_{bC} \equiv l_{bC}(T_M)$$

$$l_{aT} \equiv l_{aT}(T_N)$$

and

$$l_{bT} \equiv l_{bT}(T_M)$$

to explicitly spell out the bootstrapping dependence. In particular, if

$$l_{bC} = 0$$

and

$$l_{bT} = 0$$

then we have the really simple following bootstrapping relationship:

$$\sum_{i=1}^{N} \Delta_{i,JPY} P_{JPY}(t, T_i) = \frac{V_{ab}}{l_{aC}(T_N) - l_{aT}(T_N)}$$

2. Symmetry in the Discount Curve Calibration from CCS: Notice that V_{ab}, when taken to the other side, becomes $-\frac{V_{ab}}{X_{JPY,USD}}$. Thus from the same market quotes we may calibrate the USD discount curve using the JPY discount curve as well.
3. Parallel between Forward and Discount Curve Construction:
   - Relevant leg for the Forward Curve Extraction => Derived Component Derived Leg (which corresponds to a Floating stream, as the derived component is a float-float swap).
   - Relevant leg for the Discount Curve Construction => Again derived component Derived Leg, this now corresponds to a Fixed Stream, as the derived component is a fix-float (IRS) component.
   - Remember that the basis is always placed on the reference component – either the reference leg or the derived leg.
4. CCBS Cross Bases Computation: Say $\kappa_{\alpha,\beta}$ is the cross-currency basis.

$$\kappa_{\alpha,\beta} = \frac{PV_{DC} + XPV_{RC}}{DV01_{\alpha,\beta}}$$

where α indicates the component side (i.e., reference or derived), and β indicates the component stream under consideration (i.e., the reference stream or the derived stream). $DV01_{\alpha,\beta}$ corresponds to the DV01 of stream β inside of component α. X is the FX of the derived/reference cross, i.e., units of the derived currency in terms of the reference currency.

Custom CCBS Based Curve Construction SKU

1. Curve Construction from CCBS: For each currency leg, there are 3 potential latent states – the forward latent states 1 & 2, and the discounting latent state. Thus there are 6 of them in all. This contributes to the cognitive confusion – the number of latent states.

2. Latent State Determination using CCBS: Typically we are given a one set of currency parameters, and made to determine the other. The practical use case would be for:
    a. Computing the funding curve parts – this includes the merge-stretched discount curve and its corresponding forward curve
    b. Computing the non-funding forward curve – in this case we need to have either the merge-stretch discount and forward, or the distinct discount/forward curves.
3. Separated "Derived Product" Latent States: Since we infer the latent states of the derived sides from the inputs, in we general we are required to extract:
    a. The derived forward state
    b. The reference forward latent state
    c. The discounting latent state

    Thus the state composites are built strictly on the availability of the additional contingent inputs (beyond the CCBS quotes).
4. CCBS Derived Forward vs. Discount Curve Construction: If the derived product discount curve and the derived product reference leg forward curves are available, the derived product derived leg products then become spline constructible using the CCBS quotes alone. Otherwise, additional external inputs and/or simplifying assumptions are needed (these assumptions help set the merge states).
5. Joint Latent State Estimations: If there are non-linear couplings between 2/more latent states above, the linear splined state extractions become infeasible without some kind of kernel transformation. This is among the 2 troubles with Fujii, Takahashi, and Shimada (2010a, 2010b, 2010c, 2010d):
    a. The instrument set pair of CCBS/IRS used for calibration purposes needs to be completely paired up
    b. Linear reductions (either directly or by using kernel transformations) need to be possible by the elimination of the correspondingly paired CCBS/IRS streams
6. CCBS Deep-Drill Cognitive Challenge: CCBS is possibly the biggest challenge so far in terms of back-and-forth cognitive switch-in/switch-out, owing to the following:
    a. The variety of the latent states
    b. The manner in which these latent states interleave

c. Estimation of each of them at each segment level using the corresponding "deep-drill" instrument quotes
d. Cross influence via the C^k transmission criterion across the latent states
e. Interleaving of the sensitivities as well

Mark-To-Market Cross-Currency Swap Valuation

1. The "MTM" in MTM Cross Currency Swap: In this context the term "MTM" is a misnomer in that it is simply an MTM in the FX dimension, i.e., it is an FX-MTM cross-currency swap. The floating nature ensures "MTM" in the floater dimension, however, it does not address MTM in the FX/floating and the forward cross-currency dimensions.
2. MTM is not Collateralization: For the reasons above, the MTM cross-currency may not be viewed as a collateralized transaction – not even in the non-continuous collateralization limit. Collateralization impact is essentially undetermined in account of the reasons above – even in the absence of a stochastic cross-currency basis, under-collateralization/over-collateralization is determined entirely by the sign of the FX/floater correlation.
3. MTM for Dual Stream Instruments: Instruments in this category include the fix-float and the float-float swaps. As noted above, while explicit MTM'ing using the forward construction is not necessary in this case, the instruments are still exposed to collateral/forward, collateral/funding, and forward/funding volatilities/correlations, which are not easy to hedge for – thereby making a case for "MTM" type agreements (esp. for the basis).

Mark-To-Market Cross-Currency Swap – Valuation Formulation

1. Setup:

$$PV_{RC} = PV_{RL} + PV_{DL} \rightarrow X_0 PV_{RC} = X_0 PV_{RL} + X_0 PV_{DL}$$

$$X_0 PV_{RC} = X_0 \left[\sum_i \mathcal{L}_{RL}(t_{i-1}, t_i) DCF_{RL}(t_{i-1}, t_i) D_f(t_i) \right] + X_0 \left[\sum_j \mathcal{L}_{DL}(t_{j-1}, t_j) DCF_{DL}(t_{j-1}, t_j) D_f(t_j) \right] + X_0 \kappa_{\alpha,\beta} DV01_{\alpha,\beta}$$

- The last term simply indicates that the basis can be either on the reference leg or on the derived leg.
- We will assume that $\kappa_{\alpha,\beta}$ is indepdendent of the FX process or the discount factor process.

2. <u>Reference Component PV in Derived Currency</u>: Consider a forward starting CCBS, maturing at time t_0. The PV of the reference component in the derived currency at time t can be expressed as:

$$\begin{aligned} X_0 PV_{RC}(t, t_0) &= \left[\sum_i \mathcal{L}_{RL}(t_{i-1}, t_i) DCF_{RL}(t_{i-1}, t_i) D_f(t, t_i) \right] X(t, t_0) \\ &+ \left[\sum_j \mathcal{L}_{DL}(t_{j-1}, t_j) DCF_{DL}(t_{j-1}, t_j) D_f(t, t_j) \right] X(t, t_0) \\ &+ X(t, t_0) \kappa_{\alpha,\beta} DV01_{\alpha,\beta}(t) \\ &= X(t, t_0) D_f(t, t_0) \left[\sum_i \mathcal{L}_{RL}(t_{i-1}, t_i) DCF_{RL}(t_{i-1}, t_i) D_f(t_0, t_i) \right. \\ &\left. + \sum_j \mathcal{L}_{DL}(t_{j-1}, t_j) DCF_{DL}(t_{j-1}, t_j) D_f(t_0, t_j) + \kappa_{\alpha,\beta} DV01_{\alpha,\beta}(t_0) \right] \\ &= X(t, t_0) D_f(t, t_0) PV_{RC}(t_0, t_0) \end{aligned}$$

3. <u>t_m-Forward MTM CCBS PV</u>: From

$$PV_{CCBS,m} = PV_{DC,m} + XPV_{RC,m}$$

we get

$$E_t^{Q_D^{tm}}\left[PV_{CCBS,m}\right] = E_t^{Q_D^{tm}}\left[PV_{DC,m}\right] + E_t^{Q_D^{tm}}\left[XPV_{RC,m}\right]$$

which implies

$$E_t^{Q_D^{tm}}\left[PV_{CCBS,m}\right] = E_t^{Q_D^{tm}}\left[PV_{DC,m}\right] + E_t^{Q_D^{tm}}\left[PV_{RC,m}\right]E_t^{Q_D^{tm}}\left[X(t,t_m)D_f(t,t_m)\right]$$

4. <u>Joint X, D_f Evolution</u>:

$$\Delta X = \mu_X X \Delta t + \sigma_X X \Delta W_X$$

$$\Delta D_f = \mu_D D_f \Delta t + \sigma_D D_f \Delta W_D$$

$$\Delta X \Delta D_f = \rho_{DX}\sigma_X\sigma_D X D_f \Delta t$$

$$\Delta\left(XD_f\right) = XD_f\{(\mu_X + \mu_D + \rho_{XD}\sigma_X\sigma_D)\Delta t + \sigma_X\Delta W_X + \sigma_D\Delta W_D\}$$

$$E_t^{Q_D^{tm}}\left[\Delta\left(XD_f\right)\right] = E_t^{Q_D^{tm}}\left[XD_f\right]\{(\mu_X + \mu_D + \rho_{XD}\sigma_X\sigma_D)\Delta t\}$$

$$E_t^{Q_D^{tm}}\left[XD_f\right] = e^{\int_t^{tm}[\mu_X(s)+\mu_D(s)+\rho_{XD}(s)\sigma_X(s)\sigma_D(s)]ds}$$

5. <u>MTM Reference Component Adjustment</u>:

$$E_t^{Q_D^{tm}}\left[X(t,t_m)D_f(t,t_m)\right] = E_t^{Q_D^{tm}}\left[X(t,t_m)\right]E_t^{Q_D^{tm}}\left[D_f(t,t_m)\right]e^{\int_t^{tm}[\rho_{XD}(s)\sigma_X(s)\sigma_D(s)]ds}$$
$$= X(t,t_m)D_f(t,t_m)M_A(t,t_m)$$

Here

$$X(t, t_m) = E_t^{Q_D^{t_m}}[X(t, t_m)] = e^{\int_t^{t_m} \mu_X(s) ds}$$

$$D_f(t, t_m) = E_t^{Q_D^{t_m}}\left[D_f(t, t_m)\right] = e^{\int_t^{t_m} \mu_D(s) ds}$$

The MTM Adjuster $M_A(t, t_m)$ is given from

$$M_A(t, t_m) = e^{\int_t^{t_m} [\rho_{XD}(s) \sigma_X(s) \sigma_D(s)] ds}$$

6. Aggregated MTM CCBS PV:

$$PV_{CCBS,MTM}(t) = \sum_m PV_{CCBS,m}(t) = \sum_m PV_{DC,m}(t) + \sum_m X(t, t_m) PV_{RC,m}(t)$$

$$PV_{CCBS,MTM}(t) = \sum_m D_f(t, t_m) PV_{DC,m}(t_m) + \sum_m D_f(t, t_m) X(t, t_m) M_A(t, t_m) PV_{RC,m}(t_m)$$

$$PV_{CCBS,MTM}(t) = \sum_m D_f(t, t_m) PV_{DC,m}(t_m) + \sum_m D_f(t, t_m) X(t, t_m) PV_{RC,m}(t_m) + \sum_m D_f(t, t_m) X(t, t_m) [M_A(t, t_m) - 1] PV_{RC,m}(t_m)$$

$$PV_{CCBS,MTM}(t) = PV_{CCBS,non-MTM}(t) + \sum_m D_f(t, t_m) X(t, t_m) [M_A(t, t_m) - 1] PV_{RC,m}(t_m)$$

7. MTM Adjustment to MTM CCBS PV: From above

$$PV_{CCBS,MTM}(t) = PV_{CCBS,non-MTM}(t) + MTM_{Adjustment}(t)$$

where

$$MTM_{Adjustment}(t) = \sum_m D_f(t, t_m) X(t, t_m) [M_A(t, t_m) - 1] PV_{RC,m}(t_m)$$

The $MTM_{Adjustment}(t)$ correction term vanishes as $\rho_{DX} \rightarrow 0$ as one would expect. Further owing to the presence of the exponentials, when $\rho_{DX} > 0$ the correction term dominates as $\rho_{XD}, \sigma_X, and\ \sigma_D$ tend higher. If negatively correlated (i.e. $\rho_{DX} < 0$) the correction essentially reduces towards $-\sum_m D_f(t, t_m) X(t, t_m) PV_{RC,m}(t_m)$, and the contribution to $PV_{CCBS,MTM}(t)$ from $PV_{RC,m}(t)$ diminishes.

8. <u>Absence of the explicit Cross Currency Basis in the MTM Correction</u>: Since the basis is confined exclusively to the reference component, it is automatically incorporated into $PV_{RC,m}$ and its derivative terms above. Throughout the only assumption made about the cross currency basis is that it evolve independently of the discount factor and the FX rate.

Absolute/Relative MTM Application

1. <u>Component Pair Relative MTM Generalization</u>:
    a. The "MTM"able component pair consists of 2 components – the reference and the derived.
    b. "MTM"ing occurs at discrete MTM dates – which is most typically set to the coupon dates of either of the streams.
    c. The MTM process can be absolute or relative, i.e., in the relative MTM'ing the reference is MTM'ed w.r.t. the derived.
    d. The Reference Component is decomposed into forward components – each forward being built out from the forward MTM dates.
    e. Valuation of each of the stripped forward component may be customized to correspond to employ its own model/assumptions/market data/evolution dynamics.
2. <u>Component Pair Absolute MTM Generalization</u>: Situation here is identical to the "Relative" Case, except for items c) (the reference and the derived are MTM'ed independent of each

other, not relative to one another) and e) (of course, the forward evolution/valuation is still going to be based off of one set market parameters and their realization).

3. Generalized Absolute/Relative Valuation Adjustment Market Data: Models that drive the valuation in this case would rely on the following external market data:
    a. Funding/FX Volatility/Correlation
    b. Funding/Forward Volatility/Correlation
    c. Collateral/Funding Volatility/Correlation
    d. Collateral/FX Volatility/Correlation
    e. Collateral/Forward Volatility/Correlation
    f. Joint modes implied from the combinations above
4. No Convexity Adjustment for non-MTM Contracts: Given that the convexity adjustment is applicable only to dynamic (i.e., MTM) jointly-evolved state-specification in the contract, no joint state convexity corrections with respect to the given state will be applied to non-MTM counterparts of a specific numeraire.

Per-trade Risk Isolation Components

1. Underlier Security Price Market Risk
2. Discount Factor Risk
3. Forward Rate Risk
4. Currency/FX Risk
5. Basis Risk (on any Risk Factor)
6. Funding Risk
7. Collateral Risk
8. Counter-party Risk

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Section VI: Collateralized Valuation and XVA Metric Generation

Collateralized Agreements and Derivatives Valuation

Background

1. Background: While economies without risk-free rates have been considered in the past (Black (1972)), typical derivatives pricing treatments have assumed the existence of such rates as a matter of course (e.g., Duffie (2001)).
2. Holy Grail of Curve Construction: Combining multiple curves, partial collateralization involving multiple currencies, with liquidity, counter party risk, funding, and credit risk factored in into a dynamic approach is treated in a variety of papers (Pallavicini and Tarenghi (2010), Fujii, Shimada, and Takahashi (2010a), Fujii, Shimada, and Takahashi (2010b), Fujii, Shimada, and Takahashi (2010c), Fujii and Takahashi (2011a, 2011b), Castagna (2012), Henrard (2013)).
3. Treatments of CVA/DVA: Partial collateralization results in non-zero counter-party risk, and these cases are covered in Burgard and Kjaer (2011a, 2011b), Brigo, Pallavicini, Buescu, and Liu (2012), Crpey (2012a), Crpey (2012b). Considerations regarding the risk of an "average" counter-party are treated in Morini (2009).

Introduction and Motivation

1. Counter party Credit Risk Free Asset: Closest to a counterparty credit-risk free asset is an asset that is fully collateralized on a continuous basis (ISDA (2009), ISDA (2011), Sawyer (2011), Piterbarg (2012)), i.e., the collateralized asset produces cash flows that are continuous with changes in both the derivative MTM and the collateral coupon. Macey (2011) and Piterbarg (2012) illustrate how to retain the traditional risk-neutral valuation in a collateralized context.

2. Collateralized Asset Process: At the inception of a fully collateralized trade, there is no cash exchange, i.e., the upfront payment amount is returned back as collateral. Further, in exchange for the continuous pay streams above, the trade can be cancelled at any time with zero net value for either side.
3. Price of a Collateralized Asset: The price of a collateralized asset is effectively the outstanding level of the collateral account, i.e., a collateralized transaction is an asset with a zero-drift price process and with the given cumulative dividend flows (Duffie (2001)).
4. Collateral Cash Flows: $V(t)$ is the asset price paid by A to B, and B posts this amount back as collateral. A now pays the contractual collateral coupon flow $c(t)$ back to B. In time unit Δt, the cash flow that is exchanged (i.e., paid to A) is $V(t + \Delta t) - V(t) - c(t)V(t)\Delta t$, i.e.,

$$\Delta\chi(t) = \Delta V(t) - c(t)V(t)\Delta t$$

   Once this is exchanged, the transaction can terminate, and A can keep the collateral.

Two Collateralized Assets

1. Setup: Assume that each of the assets follows its corresponding real-world measures, but are exposed to the same risk factor ΔW, i.e.,

$$\Delta V_i(t) = \mu_i(t)V_i(t)\Delta t + \sigma_i(t)V_i(t)\Delta W$$

   for

$$i = 1, 2$$

2. Hedge Portfolio: Say that the corresponding collateralized account for each of these assets has the dynamics

$$\Delta\chi_i(t) = \Delta V_i(t) - c(t)V_i(t)\Delta t$$

Construct a hedge portfolio using $-\sigma_1(t)V_1(t)$ of asset 2 and $+\sigma_2(t)V_2(t)$ of asset 1. The net change in the real world collateralized portfolio of these two assets is:

$$\Delta\chi_{12}(t) = \sigma_2(t)V_2(t)[\Delta V_1(t) - c(t)V_1(t)\Delta t] - \sigma_1(t)V_1(t)[\Delta V_2(t) - c(t)V_2(t)\Delta t]$$

$$\Delta\chi_{12}(t) = V_1(t)V_2(t)[\sigma_2(t)\{\mu_1(t) - c(t)\} - \sigma_1(t)\{\mu_2(t) - c(t)\}]\Delta t$$

3. <u>Application of the Collateral Rules</u>: The above amount is known at time t, and maybe exchanged at $t + \Delta t$, at zero additional cost to either party. Thus, the only way both can enter into this transaction is if the net cash flow is zero (this is the collateralized version of no arbitrage). This produces

$$\frac{\mu_1(t) - c(t)}{\sigma_1(t)} = \frac{\mu_2(t) - c(t)}{\sigma_2(t)}$$

4. <u>Differences with Traditional Risk Neutral Pricing</u>: The main difference is: in the traditional risk-neutral pricing, the hedged portfolio grows at the "risk-free" rate. In collateralized pricing, the COLLATERALIZED + HEDGED portfolio grows at ZERO rate (i.e., does not grow at all) after incremental netting! Therefore the "risk-free" rate does not enter into this setting at all.
5. <u>Measure Change</u>: Create a new measure t where

$$\Delta W_Q = \Delta W + \frac{\mu_i(t) - c(t)}{\sigma_i(t)}$$

In this new measure, the individual assets grow as

$$\Delta V_i(t) = c(t)V_i(t)\Delta t + \sigma_i(t)V_i(t)\Delta W_Q$$

using which we estimate $V_i(t)$ as

$$V_i(t) = \mathbb{E}_t^Q\left[e^{-\int_t^T c(s)ds}V_i(T)\right]$$

As may be observed, measure Q looks like the traditional risk neutral measure.

6. Different Collateral Rates: The collateral rates $c_i(t)$ can be asset-specific within changing any of our principal conclusions, and $V_i(t)$ now becomes

$$V_i(t) = \mathbb{E}_t^Q\left[e^{-\int_t^T c_i(s)ds}V_i(T)\right]$$

Examples would be, say, a stock collateralized at its repo rate (or other funding rate), while the derivative would be collateralized at it collateral rate (e.g., Piterbarg (2010)).

7. Other Variants: Other collateralization variants include varying collateral processes, different counter-parties etc. Typically all these only end up varying the drift, thus you get

$$\frac{P_1(t,T)}{P_2(t,T)} = \frac{\mathbb{E}_t^{Q_1}\left[e^{-\int_t^T c_1(s)ds}V(T)\right]}{\mathbb{E}_t^{Q_2}\left[e^{-\int_t^T c_2(s)ds}V(T)\right]}$$

Of course, the collateralization drift can also be stochastic. This measure change from collateralization scheme #1 to collateralization scheme #2 induces a drift to the scheme #2 as

$$\mathbb{E}_t^Q\left[e^{-\int_t^T [c_2(s)-c_1(s)]ds}V(T)\right]$$

8. Many Collateralized Assets: Will quickly flip through this, as Piterbarg (2012) spells out the details. N-dimensional asset $\vec{V}$ possesses the real-world dynamics

$$\Delta\vec{V}(t) = \vec{\mu}^T(t)\vec{V}(t)\Delta t + \vec{\sigma}^T(t)\vec{V}(t)\Delta\vec{W}$$

A linearly combined weight set $\vec{w}$ of the hedge portfolio satisfies the constraint

$$\vec{w}^T \vec{\sigma} = 0$$

Using the collateral cash flow matching arguments presented above, we get

$$\vec{w}^T \left[\vec{\mu}^T \vec{V} - \vec{c}^T \vec{V} \right] = 0$$

Measure Change => As before, there exists a measure Q with the drift vector $\vec{c}$, one for each asset, such that an adjustment $\vec{\lambda}$ can be made to the real world measure making it

$$\Delta \vec{V}(t) = \vec{c}^T(t) \vec{V}(t) \Delta t + \vec{\sigma}^T(t) \vec{V}(t) \left[\Delta \vec{W} + \vec{\lambda} \Delta t \right]$$

such that $\Delta \vec{W} + \vec{\lambda} \Delta t$ can become drift-less. Once again, in this new measure, the individual assets follow

$$V_i(t) = \mathbb{E}_t^Q \left[e^{-\int_t^T c_i(s) ds} V_i(T) \right]$$

Setup of the Collateral Curve Dynamics

1. Short-Rate Collateral Curve: Piterbarg (2010) considers the risk-free curve for lending, a curve that corresponds to the safest available collateral (cash). The corresponding short rate is denoted by $r_C(t)$ – where C stands for CSA, since the assumption is that this is the agreed upon overnight rate paid among collateral dealers under the CSA.
2. HJM Parametrization of the Collateral Curve: It is convenient to parametrize term curves in terms of discount factors $P_C(t, T), 0 \leq t \leq T < \infty$. Standard HJM theory applies with the following dynamics for the yield curve:

$$\frac{\Delta P_C(t, T)}{P_C(t, T)} = r_C(t) \Delta t - \sigma_C(t, T)^T \Delta W_C(t)$$

where W_C is a d-dimensional Brownina motion under the risk-neutral measure P and σ_C is a vector-valued, d-dimensional stochastic process.

3. Asset-specific Funding/Repo Rate: Piterbarg (2010) considers derivative contracts on a particular asset where the price process is denoted $S(t); t \geq 0$. The short-rate on funding secured by this asset is r_R (R for repo).
4. Rates for Unsecured Funding: Finally the short-rate for unsecured funding is denoted $r_C(t), t \geq 0$. As a rule, it would be expected that

$$r_C(t) \leq r_R(t) \leq r_F(t)$$

5. Funding Spread as a Default Premium: The existence of non-zero short rate spreads between short-rates of different collateral can be cast in the language of credit risk, by introducing joint defaults between the bank and the various assets used as collateral for funding.
6. Default Intensity of the Bank: In particular, the funding spread

$$s_F(t) \triangleq r_F(t) - r_C(t)$$

can be thought of as the stochastic intensity of default of the bank. The dynamics of the intensity is pursued in Gregory (2009) and Burgard and Kjaer (2009), while Piterbarg (2009) postulates the dynamics of the funding curves directly instead.

Collateralized Black-Scholes Formulation

1. Dynamics of the Derivative Value: This section examines how the regular Black-Scholes pricing methodology changes in the presence of CSA. Let $S(t)$ be an asset that follows in real-world the dynamics

$$\frac{\Delta S(t)}{S(t)} = \mu_S(t)\Delta t + \sigma_S(t)\Delta W_S(t)$$

2. Full Change in the Derivative Value: Let $V(t, S)$ be a derivative on the asset. By Ito's lemma it follows that

$$\Delta V(t) = \mathcal{L}\big(V(t)\big)\Delta t + \mathcal{X}(t)\Delta V(t)$$

where $\mathcal{L}$ is the standard pricing operator

$$\mathcal{L} = \frac{\partial}{\partial t} + \frac{1}{2}\sigma_S{}^2(t)S^2\frac{\partial^2}{\partial S^2}$$

and $\mathcal{X}(t)$ is the option's delta

$$\mathcal{X}(t) = \frac{\partial V(t)}{\partial S}$$

3. Full/Partial Collateral Cash Account: Let $C(t)$ be the collateral, i.e., the cash held in the collateral account, at time t against the derivative. For flexibility this amount may be different from $V(t)$.
4. Replicating Portfolio for the Derivative Payoff: To replicate the derivative at time t we hold $\mathcal{X}(t)$ units of the asset and $\gamma(t)$ units of cash. The value of the replication portfolio, which we denote by $\Pi(t)$ is equal to

$$V(t) = \Pi(t) = \mathcal{X}(t)S(t) + \gamma(t)$$

where

$$\gamma(t) = C(t) + [V(t) - C(t)] - \mathcal{X}(t)S(t)$$

5. Decomposition of the Cash Account: The cash amount $\gamma(t)$ is split among a number of accounts;
   a. Amount $C(t)$ is in collateral
   b. Amount $V(t) - C(t)$ needs to be borrowed/lent from the treasury desk

c. Amount $\mathcal{X}(t)S(t)$ is borrowed to finance the purchase of $\mathcal{X}(t)$ assets. It is secured by the assets purchased.

d. The assets pay dividend at the rate $r_D(t)$

6. Full Growth of the Cash Account: The growth of all the cash accounts is given by

$$g(t)\Delta t = [r_C(t)C(t) + r_F(t)\{V(t) - C(t)\} - r_R(t)\mathcal{X}(t)S(t) + r_D(t)\mathcal{X}(t)S(t)]\Delta t$$

7. Applying the Self-Financing Criterion: On the other hand, from

$$V(t) = \mathcal{X}(t)S(t) + \gamma(t)$$

using the self-financing criterion one gets

$$g(t)\Delta t = \Delta V(t) - \mathcal{X}(t)\Delta S(t)$$

which becomes, by Ito's lemma

$$\Delta V(t) - \mathcal{X}(t)\Delta S(t) = \mathcal{L}\big(V(t)\big)\Delta t = \left[\frac{\partial V(t)}{\partial t} + \frac{1}{2}{\sigma_S}^2(t)S^2\frac{\partial^2 V(t)}{\partial S^2}\right]\Delta t$$

8. Funding/Collateral Derivative Valuation PDE: Thus one obtains

$$\left[\frac{\partial V(t)}{\partial t} + \frac{1}{2}{\sigma_S}^2(t)S^2\frac{\partial^2 V(t)}{\partial S^2}\right]$$
$$= r_C(t)C(t) + r_F(t)\{V(t) - C(t)\} - r_R(t)\mathcal{X}(t)S(t) + r_D(t)S(t)\frac{\partial V(t)}{\partial S}$$

which after re-arrangement results in

$$\frac{\partial V(t)}{\partial t} + [r_R(t) - r_D(t)]S(t)\frac{\partial V(t)}{\partial S} + \frac{1}{2}{\sigma_S}^2(t)S^2\frac{\partial^2 V(t)}{\partial S^2}$$
$$= r_F(t)V(t) - C(t)[r_F(t) - r_C(t)]$$

9. Solution Using Feynman-Kac Integral: The solution may be obtained by essentially following the steps that lead to the Feynman-Kac formula (Karatzas and Shreve (1997)) and is given by

$$V(t) = \mathbb{E}_t\left[e^{-\int_t^T r_F(u)du} V(T) + \int_t^T e^{-\int_t^u r_F(v)dv}\{r_F(u) - r_C(u)\}C(u)du\right]$$

in the measure in which the asset grows at the rate $r_R(t) - r_D(t)$, that is

$$\frac{\Delta S(t)}{S(t)} = [r_R(t) - r_D(t)]\Delta t + \sigma_S(t)\Delta W_S(t)$$

10. The Right "Risk-Free" Rate: Note that if the probability space is rich enough, it can be taken to be the same risk-neutral measure P in

$$\frac{\Delta P_C(t,T)}{P_C(t,T)} = r_C(t)\Delta t - \sigma_C(t,T)^T \Delta W_C(t)$$

Thus this derivation validates the view of Barden (2009) (and Hull (2006)) that the repo rate $r_R(t)$ is the right "risk-free" rate to use when valuing derivatives on $S(t)$.

Collateralization and Funding Derivative Valuation

1. Incremental Change in the Derivative Value: By re-arranging the Feynman-Kac expression above for $V(t)$ one obtains another useful expression for the valuation of the derivative:

$$V(t) = \mathbb{E}_t\left[e^{-\int_t^T r_C(u)du} V(T) - \int_t^T e^{-\int_t^u r_C(v)dv}\{r_F(u) - r_C(u)\}\{V(u) - C(u)\}du\right]$$

It can be seen that

$$\mathbb{E}_t[\Delta V(T)] = [r_F(t)V(t) - \{r_F(t) - r_C(t)\}C(t)]\Delta t = [r_F(t)V(t) - s_F(t)C(t)]\Delta t$$

2. Derivative Value Under Full Collateralization: Thus the rate of growth in the derivative security is the funding spread $s_F(t)$ applied to the collateral. In particular, if the collateral is equal to the value $V(t)$ then

$$\mathbb{E}_t[\Delta V(T)] = r_C(t)C(t)\Delta t$$

and therefore

$$V(t) = \mathbb{E}_t\left[e^{-\int_t^T r_C(u)du}V(T)\right]$$

and the derivative value grows at the risk free (i.e., collateral) rate. The final value is the only payment that appears in the discounted expression as the other payments net out given the assumption of full collateralization. This is consistent with the drift in

$$\frac{\Delta P_C(t,T)}{P_C(t,T)} = r_C(t)\Delta t - \sigma_C(t,T)^T\Delta W_C(t)$$

as $P_C(t,T)$ corresponds to the deposits secured by cash collateral.

3. Derived Value Under Unsecured Trading: On the other hand if the collateral is zero then

$$\mathbb{E}_t[\Delta V(T)] = r_F(t)V(t)\Delta t$$

and the rate of growth is equal to the bank's unsecured funding rate, or, using the credit risk language, adjusted for the probability of the bank default.

4. Collateral and Funding Measure Numeraires: Therefore the case

$$C = V$$

could be handled using the measure that corresponds to the risk free bond

$$P_C(t) = \mathbb{E}_t\left[e^{-\int_t^T r_C(u)du}\right]$$

as a numeraire, and likewise, the case

$$C = 0$$

corresponds to the risky bond

$$P_F(t) = \mathbb{E}_t\left[e^{-\int_t^T r_F(u)du}\right]$$

as a numeraire.

5. Portfolio Effects of the Collateral Position: When two dealers are trading with each other the collateral is applied to the overall value of the portfolio between them with the positive exposures on some trades offsetting the negative exposures on the other trades (so-called netting). Hence the valuation of individual trades should take into account the collateral position of the whole portfolio.
6. Simplification of Full Collateralization/Trading: Fortunately in the simples case of the collateral being a linear function of the exact value of the portfolio - the case that includes both the no-collateral case

$$C = 0$$

as well as the full collateral case

$$C = V$$

- the value of the portfolio is just the sum of the values of the individual trades (with the collateral attributed to the trades by the same linear function). This easily follows from the pricing formula linearity of C and V in

$$V(t) = \mathbb{E}_t\left[e^{-\int_t^T r_C(u)du}V(T) - \int_t^T e^{-\int_t^u r_C(v)dv}\{r_F(u) - r_C(u)\}\{V(u) - C(u)\}du\right]$$

Collateral PDE Formulation

1. <u>PDE Collateralization Treatments</u>: Bjork (2009), Piterbarg (2010), Castagna (2011), Fujii and Takahashi (2011a, 2011b), Henrard (2012), Piterbarg (2012), Ametrano and Bianchetti (2013), and Han, He, and Zhang (2013) extend the no-arbitrage to the collateralization case.
2. <u>Review of Derivative PDE Using Replication</u>: The derivative that is replicated using n assets and a bond via

$$V = nS + B$$

undergoes the evolution through the self-financing formulation

$$\Delta V = n\Delta S + \Delta B$$

This is matched to the derivative change

$$\Delta V = \left[\frac{\partial V}{\partial t} + \frac{1}{2}{\sigma_S}^2 S^2 \frac{\partial^2 V}{\partial S^2}\right]\Delta t + \frac{\partial V}{\partial S}\Delta S$$

Equating the two, setting

$$\frac{\partial V}{\partial S} = n$$

to eliminate stochasticity, and noticing that

$$\Delta B = rB\Delta t$$

we get

$$\left[\frac{\partial V}{\partial t} + \frac{1}{2}{\sigma_S}^2 S^2 \frac{\partial^2 V}{\partial S^2}\right]\frac{1}{r} = B$$

Using the expression for V, this may be re-composed as the Black-Scholes PDE from

$$\frac{\partial V}{\partial t} + \frac{1}{2}{\sigma_S}^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} = rV$$

(Harrison and Kreps (1979), Harrison and Pliska (1981), Harrison and Pliska (1983)).

3. Derivative Replication with Collateral Account: The replication strategy now involves the assets, the bank funding account, and the collateral account.

$$V = aS + B + C$$

where a and B are the number of assets and the bank funding notional account, respectively, and C is the collateral account. Under perfect collateralization

$$C \equiv V$$

Further

$$\Delta C = r_C V \Delta t$$

$$\Delta B = r_f B \Delta t$$

and

$$\Delta S = \mu_S S \Delta t + \sigma_S S \Delta W$$

4. Derivative Value Change: Applying the self-financing condition

$$\Delta V = a\Delta S + \Delta B + \Delta C$$

Using the perfect collateral condition we get

$$V = aS + B + V$$

which implies

$$B = -aS$$

Thus

$$\Delta V = a\Delta S + r_f B \Delta t + r_C V \Delta t$$

results in

$$\Delta V = a\Delta S - a r_f S \Delta t + r_C V \Delta t$$

We refer to the quantity

$$\Gamma(r_C, r_f) = -a r_f S \Delta t + r_C V \Delta t$$

as the cash account.

5. The Collateralization PDE:

$$\Delta V = \left[\frac{\partial V}{\partial t} + \frac{1}{2}{\sigma_S}^2 S^2 \frac{\partial^2 V}{\partial S^2}\right]\Delta t + \frac{\partial V}{\partial S}\Delta S = a\Delta S - a r_f S \Delta t + r_C V \Delta t$$

Setting

$$a = \frac{\partial V}{\partial S}$$

we get

$$\frac{\partial V}{\partial t} + \frac{1}{2}{\sigma_S}^2 S^2 \frac{\partial^2 V}{\partial S^2} + r_f S \frac{\partial V}{\partial S} = r_C V$$

Re-casting using the appropriate measure terminology, we get

$$V(S, t) = \mathbb{E}_t^{Q^f}[D_C(S, T) V(S, T)]$$

where

$$D_C(t, T) = e^{-\int_t^T r_C(u) du}$$

Forward Contract Valuation

1. Repo’d Zero Strike Call Option:
    a. Asset Delivery at Future Time => Possibly the simplest derivative contract on an asset is the promise to deliver this asset at a given future time T. The contract should be seen as a zero strike call option with expiry T. In the standard theory, of course, the value of the derivative is the same as the value of the asset itself (in the absence of dividends).
    b. Forwards Value and Derivative Value => The payout of the derivative is given by

$$V(T) = S(T)$$

and the value at time t, assuming no CSA, is given by

$$V_{ZSC}(t) = \mathbb{E}_t\left[e^{-\int_t^T r_F(u)du} S(T)\right]$$

On the other hand, if

$$r_D(t) \neq 0$$

then

$$S(t) = \mathbb{E}_t\left[e^{-\int_t^T r_R(u)du} S(T)\right]$$

as follows from

$$\frac{\Delta S(t)}{S(t)} = [r_R(t) - r_D(t)]\Delta t + \sigma_S(t)\Delta W_S(t)$$

and clearly

$$V_{ZSC}(t) \neq S(t)$$

c. Repo Impact on the Value => The difference in the value between the derivative and the asset are now easily understood as the zero-strike call-option carries the credit risk of the bank, while the asset $S(t)$ does not. Or, in the language of funding, the asset $S(t)$ can be used to secure the funding – which is reflected in the corresponding repo rate applied – while $V_{ZSC}(t)$ cannot be used for such a purpose.

2. <u>No-CSA Forwards Valuation</u>:

a. Forwards Contract without CSA => This section considers a forward contract on $S(t)$ where at a time t the bank agrees to deliver the asset at time T against a cash payment at time T.

b. No-CSA Forward Contract Definition => A no-CSA forward contract could be seen as a derivative with payout $S(T) - F_{NoCSA}(t, T)$ at a time T where $F_{NoCSA}(t, T)$ is the forward price at a time t for a delivery at T. As the forward contract is cost free, we have by

$$V(t) = \mathbb{E}_t\left[e^{-\int_t^T r_C(u)du}V(T) - \int_t^T e^{-\int_t^u r_C(v)dv}\{r_F(u) - r_C(u)\}\{V(u) - C(u)\}du\right]$$

that

$$0 = \mathbb{E}_t\left[e^{-\int_t^T r_F(u)du}\{S(T) - F_{NoCSA}(t, T)\}\right]$$

so we get

$$F_{NoCSA}(t, T) = \frac{\mathbb{E}_t\left[e^{-\int_t^T r_F(u)du}S(T)\right]}{\mathbb{E}_t\left[e^{-\int_t^T r_F(u)du}\right]}$$

c. Valuation of the No-CSA Forward => From the above expression for $F_{NoCSA}(t, T)$ define

$$P_F(t, T) \triangleq \mathbb{E}_t\left[e^{-\int_t^T r_F(u)du}\right]$$

Note that this is essentially a credit-risky bond issued by the bank. Thus the expression for $F_{NoCSA}(t, T)$ can be re-written as

$$F_{NoCSA}(t, T) = \widetilde{\mathbb{E}}_t^T[S(T)]$$

where the measure $\tilde{P}_T$ is defined by the numeraire $P_F(t, T)$ as

$$e^{-\int_0^t r_F(u)du} P_F(t, T) = \mathbb{E}_t \left[e^{-\int_0^T r_F(u)du} \right]$$

is a P-martingale. Thereby $F_{NoCSA}(t, T)$ is a $\tilde{P}$-martingale.

d. No-CSA Forward Probability Measure => Note that the value of the asset under no-CSA at time t is given by

$$\mathbb{E}_t[\Delta V(T)] = r_F(t) V(t) \Delta t$$

to be

$$V(t) = \mathbb{E}_t \left[e^{-\int_0^T r_F(u)du} V(T) \right] = P_F(t, T) \widetilde{\mathbb{E}}_t^T [V(T)]$$

so it could be calculated simply by taking the expected value of the payout in the risky T-forward measure.

3. <u>Forwards Contract with CSA</u>:
    a. Full CSA Forward Contract Definition => Now let us consider a forward contract covered by a CSA where we assume that the collateral posted C is always equal to the value of the contract.
    b. Valuation of the CSA-Based Forward => Let the forward price $F_{CSA}(t, T)$ be fixed at t; then the value from

$$V(t) = \mathbb{E}_t \left[e^{-\int_t^T r_C(u)du} V(T) - \int_t^T e^{-\int_t^u r_C(v)dv} \{r_F(u) - r_C(u)\} \{V(u) - C(u)\} du \right]$$

is given by

$$0 = V(t) = \mathbb{E}_t\left[e^{-\int_t^T r_C(u)du} V(T)\right] = \mathbb{E}_t\left[e^{-\int_t^T r_C(u)du}\{S(T) - F_{CSA}(t,T)\}\right]$$

so we get

$$F_{CSA}(t,T) = \frac{\mathbb{E}_t\left[e^{-\int_t^T r_C(u)du} S(T)\right]}{\mathbb{E}_t\left[e^{-\int_t^T r_C(u)du}\right]}$$

Comparing this with

$$F_{NoCSA}(t,T) = \frac{\mathbb{E}_t\left[e^{-\int_t^T r_F(u)du} S(T)\right]}{\mathbb{E}_t\left[e^{-\int_t^T r_F(u)du}\right]}$$

we see that

$$F_{NoCSA}(t,T) \neq F_{CSA}(t,T)$$

By the arguments similar to the no-CSA case we obtain

$$F_{CSA}(t,T) = \mathbb{E}_t^T[S(T)]$$

where P^T is the standard T-forward measure – that is a measure defined by

$$P_C(t,T) \triangleq \mathbb{E}_t\left[e^{-\int_t^T r_C(u)du}\right]$$

as its numeraire.

c. CSA Based Forward Probability Measure => Note that the value of an asset under CSA at a time t with a payout $V(t)$ is given by

$$V(t) = \mathbb{E}_t\left[e^{-\int_t^T r_C(u)du} V(T)\right] = P_C(t,T) = \mathbb{E}_t^T[V(T)]$$

so it could be simply calculated by taking the expected value of the payout in the risk-free T-forward measure.

4. Calculating CSA Convexity Adjustment:
   a. The Funding Basis Spread Numeraire => This section focusses on the difference between the CSA and the non-CSA forward prices. It can be seen that

$$F_{NoCSA}(t,T) = \widetilde{\mathbb{E}}_t^T[S(T)] = \frac{\mathbb{E}_t\left[e^{-\int_t^T r_F(u)du} S(T)\right]}{P_F(t,T)}$$

$$= \frac{\mathbb{E}_t\left[e^{-\int_t^T r_C(u)du} e^{-\int_t^T \{r_F(u) - r_C(u)\}du} S(T)\right]}{P_F(t,T)}$$

$$= \frac{P_C(t,T)}{P_F(t,T)} \mathbb{E}_t^T\left[e^{-\int_t^T s_F(u)du} S(T)\right] = \mathbb{E}_t^T\left[\frac{M(T,T)}{M(t,T)} S(T)\right]$$

   where

$$M(t,T) \triangleq \frac{P_F(t,T)}{P_C(t,T)} e^{-\int_t^T s_F(u)du}$$

   is a P^T-martingale, as

$$M(t,T) = \mathbb{E}_t^T\left[e^{-\int_t^T s_F(u)du}\right]$$

   b. CSA vs. no-CSA Convexity => It can be noted trivially that

$$\mathbb{E}_t^T\left[\frac{M(T,T)}{M(t,T)}\right] = 1$$

   so

$$F_{NoCSA}(t,T) - F_{CSA}(t,T) = \mathbb{E}_t^T\left[\left\{\frac{M(T,T)}{M(t,T)} - \mathbb{E}_t^T\left[\frac{M(T,T)}{M(t,T)}\right]\right\}\{S(t,T) - F_{CSA}(t,T)\}\right]$$

$$= \frac{1}{M(t,T)} Covariance_t^T[M(T,T), F_{CSA}(T,T)]$$

c. Funding Spread Dynamical Model => To obtain the actual value of the adjustment the joint dynamics of $s_F(u)$ and $S(u), u \geq t$ needs to be postulated. A simple model presented later shows the results of these corrections.

5. Futures vs. CSA Forward Contracts:
    a. Futures vs. CSA Forward – Similarity => At first sight the forward contract under CSA looks like a futures contract on the asset. With the futures contract, the daily price difference gets credited/debited to the margin account. In the same way, as the forward prices move, a CSA forward contract also specifies that money exchanges hands.
    b. Futures vs. CSA Forward – Differences => There is, however, an important difference. Consider the value of the forward contract at

$$t' > t$$

a contract that was entered at time t, so

$$V(t) = 0$$

Then

$$V(t') = \mathbb{E}_{t'}\left[e^{-\int_{t'}^{T} r_C(u)du}\{S(T) - F_{CSA}(t,T)\}\right]$$

$$= \mathbb{E}_{t'}\left[e^{-\int_{t'}^{T} r_C(u)du} S(T)\right] - \mathbb{E}_{t'}\left[e^{-\int_{t'}^{T} r_C(u)du}\right] F_{CSA}(t,T)$$

From

$$F_{CSA}(t,T) = \frac{\mathbb{E}_t\left[e^{-\int_t^T r_C(u)du} S(T)\right]}{\mathbb{E}_t\left[e^{-\int_t^T r_C(u)du}\right]}$$

one gets

$$V(t') - V(t) = \mathbb{E}_{t'}\left[e^{-\int_{t'}^T r_C(u)du}\right]\{F_{CSA}(t',T) - F_{CSA}(t,T)\}$$

so the difference between the contract values on t and t' that exchanges hands on t' is equal to the discounted T difference in the forward prices. For a futures contract the difference will not be discounted.

c. Futures vs. CSA Forward Convexity => Therefore the types of convexity seen in the futures contract are different from those seen in the CSA vs. non-CSA forward contracts, a conclusion different from those reached by Johannes and Sundaresan (2007).

European Style Options

1. CSA vs. non-CSA Pricing:
    a. Basic European Option Pricing Setup => Consider a European style option on $S(T)$ with a strike K. Depending on the presence of absence of CSA we get two prices:

$$V_{CSA}(t) = \mathbb{E}_t\left[e^{-\int_t^T r_C(u)du}\{S(T) - K\}^+\right]$$

and

$$V_{NoCSA}(t) = \mathbb{E}_t\left[e^{-\int_t^T r_F(u)du}\{S(T) - K\}^+\right]$$

where for the CSA case we assume that the collateral posted C is always equal to the option value V_{CSA}.

b. CSA vs. non-CSA Numeraires => By the same measure change arguments as in the previous sections we get

$$V_{CSA}(t) = P_C(t,T)\mathbb{E}_t^T[\{S(T) - K\}^+]$$

and

$$V_{NoCSA}(t) = P_F(t,T)\widetilde{\mathbb{E}}_t^T[\{S(T) - K\}^+]$$

c. CSA vs. non-CSA Raw Moments => The difference between the measures $\widetilde{P}_t^T$ and P_t^T not only manifests itself in the mean of $S(T)$ – as already established – but also reveals itself in the characteristics of the distribution of $S(\cdot)$ such as its variance and higher moments.

2. <u>Distribution Impact of Convexity Adjustment</u>:
    a. No-CSA European Option Price => To see ow a change of measure affects the distribution of $S(\cdot)$, using

$$F_{NoCSA}(t,T) = \mathbb{E}_t^T\left[\frac{M(T,T)}{M(t,T)}S(t,T)\right]$$

one has

$$V_{NoCSA}(t) = \mathbb{E}_t^T\left[\frac{M(T,T)}{M(t,T)}\{S(T) - K\}^+\right]$$

where $M(t,T)$ is defined as

$$M(t,T) \triangleq \frac{P_F(t,T)}{P_C(t,T)}e^{-\int_t^T s_F(u)du}$$

b. Conditional on $S(T)$ Option Price => From this, by conditioning on $S(T)$ one obtains

$$V_{NoCSA}(t) = P_F(t,T)\mathbb{E}_t^T\left[\alpha\big(t,T,S(T)\big)\{S(T)-K\}^+\right]$$

where the deterministic function $\alpha(t,T,x)$ is given by

$$\alpha(t,T,x) = \mathbb{E}_t^T\left[\frac{M(T,T)}{M(t,T)}|S(T)=x\right]$$

c. Linearization of the Conditional Funding Basis => Using the approach of Antonov and Arneguy (2009), Piterbarg (2010) approximates the function $\alpha(t,T,x)$ by a function that is linear in x;

$$\alpha(t,T,x) = \alpha_0(t,T) + \alpha_1(t,T)x$$

and obtains α_0 and α_1 by minimizing the squared differences while using the fact that

$$F_{CSA}(t,T) = \mathbb{E}_t^T[S(T)]$$

and

$$\mathbb{E}_t^T\left[\frac{M(T,T)}{M(t,T)}\right] = 1$$

as

$$\alpha_1(t,T) = \frac{\mathbb{E}_t^T\left[\frac{M(T,T)}{M(t,T)}S(t,T)\right] - F_{CSA}(t,T)}{Variance_t^T[S(T)]}$$

and

$$\alpha_0(t, T) = 1 - \alpha_1(t, T) F_{CSA}(t, T)$$

d. Slop of the Conditional Funding Basis Distribution => Recognizing the term $\mathbb{E}_t^T \left[\frac{M(T,T)}{M(t,T)} S(t, T) \right] - F_{CSA}(t, T)$ as the convexity adjustment of the forward between the n-CSA and the CSA versions of $F(t, T)$ one can write

$$\alpha_1(t, T) = \frac{F_{NoCSA}(t, T) - F_{CSA}(t, T)}{Variance_t^T[S(T)]}$$

e. Collateral vs. Funding Measure Relation => Differentiating

$$V_{NoCSA}(t) = P_F(t, T) \mathbb{E}_t^T \left[\alpha\big(t, T, S(T)\big) \{S(T) - K\}^+ \right]$$

twice with respect to K one obtains the probability density functions (PDFs) of $S(T)$ under the two measures as

$$\tilde{P}_t^T(S(T) \in [K, K + \Delta K]) = [\alpha_0(t, T) + \alpha_1(t, T) K] P_t^T(S(T) \in [K, K + \Delta K])$$

So the PDF of $S(T)$ under the no-CSA measure is obtained by the density of $S(T)$ under the CSA measure by multiplying it with a linear function. It is not hard to see that the main impact of such a transformation is on the slope of the volatility smile of $S(\cdot)$.

3. Stochastic Funding Model: This section considers a simple stochastic funding model to estimate the impact of collateral rules on forwards and options. Consider an asset that follows a log-normal process

$$\frac{\Delta S(t)}{S(t)} \cong \mathcal{O}(\Delta t) + \sigma_S \Delta W_S(t)$$

and a funding spread that follows a simple one-factor Gaussian model of interest rates

$$\Delta s_F(t) = -\aleph_F[\theta - s_F(t)]\Delta t + \sigma_F \Delta W_F(t)$$

with

$$\langle \Delta W_F(t) \Delta W_S(t) \rangle = \rho \Delta t$$

where ρ is the correlation between the asset price and the funding spread.

a. Evolution of F_{CSA} under P => It is also assumed for simplicity that $r_C(t)$ and $r_R(t)$ are deterministic, and

$$r_D(t) = 0$$

Then

$$F_{CSA}(t, T) = \mathbb{E}_t^T[S(T)]$$

and

$$\frac{\Delta F_{CSA}(t, T)}{F_{CSA}(t, T)} = \sigma_S \Delta W_S(t)$$

with $W_S(t)$ being a Brownian motion under the risk-neutral measure P.

b. Evolution of the Credit-Risky Numeraire => On the other hand

$$\frac{\Delta P_F(t)}{P_F(t)} = \mathcal{O}(\Delta t) - \sigma_F b(T - t) \Delta W_F(t)$$

where

$$b(T-t) = \frac{1 - e^{-\aleph_F (T-t)}}{\aleph_F}$$

c. Evolution of the Funding Spread => As $M(t,T)$ is a martingale under P (since $r_C(t)$ is deterministic, the measures P and P_T coincide) one has from

$$M(t,T) \triangleq \frac{P_F(t,T)}{P_C(t,T)} e^{-\int_t^T s_F(u)du}$$

that

$$\frac{\Delta M(t,T)}{M(t,T)} = -\sigma_F b(T-t) \Delta W_F(t)$$

d. Evolution of the Convexity Adjustment => Both $M(t,T)$ and $F_{CSA}(t,T)$ are martingales under P so it follows that

$$\frac{\Delta [M(t,T) F_{CSA}(t,T)]}{M(t,T) F_{CSA}(t,T)} = \mathcal{O}(\Delta t) + \rho \sigma_S \sigma_F b(T-t) \Delta W_F(t)$$

Using the fact that

$$F_{NoCSA}(0,T) - F_{CSA}(0,T) = \mathbb{E}\left[\frac{M(T,T)}{M(0,T)} \{F_{CSA}(T,T) - F_{CSA}(0,T)\}\right]$$

one gets

$$F_{NoCSA}(0,T) = F_{CSA}(0,T) e^{-\int_0^T \rho \sigma_S \sigma_F b(T-t) dt} = F_{CSA}(0,T) e^{-\rho \sigma_S \sigma_F \frac{T - b(t)}{\aleph_F}}$$

and in the case

$$\aleph_F = 0$$

$$F_{NoCSA}(0,T) - F_{CSA}(0,T) = F_{CSA}(0,T)\left[e^{-\frac{1}{2}\rho\sigma_S\sigma_F T^2} - 1\right]$$

e. Tenor Dependence of Convexity Correction => Note that the adjustment grows roughly as T^2. A similar formula was obtained by Barden (2009) using a model in which the funding spread was functionally linked to the value of the asset.

Cross Currency Model

1. LCH.ClearNet Collateral Rules: Single currency trades (currently mostly swaps) are collateralized in their own currencies, but multi-currency trades (e.g., cross currency swaps) are typically collateralized in USD.
2. Building Blocks: The building blocks typically are a) Domestic-Currency Collateralized Domestic Zero Coupon Bonds, b) Foreign-Currency Collateralized Foreign Zero Coupon Bonds, c) Collateralized FX Contracts. In practice, the former (the collateralized zeros) may not trade, whereas collateralized FX contracts typically do.
3. Foreign Bonds Collateralized in Domestic Currency: Consider a foreign zero-coupon bond collateralized with domestic collateral. The price of this zero coupon bond in foreign currency is $P_{f,d}(t)$. If $X(t)$ is the forex rate (i.e., the number of domestic units per foreign unit), the collateral account cash flow growth is

$$\Delta\chi_i(t) = \Delta\left[P_{f,d}(t)X(t)\right] - c_d(t)\left[P_{f,d}(t)X(t)\right]\Delta t$$

where $c_d(t)$ is the domestic collateral rate.

4. Collateralization of the FX: If $r_{d,f}(t)$ is the rate agreed on a domestic loan collateralized by foreign collateral, then the FX collateral account cash flow growth is

$$\Delta\chi_X(t) = \Delta X(t) - X(t)\Delta t$$

The contention by Piterbarg (2012) is that there is no relation between the collateralization rates $r_{d,f}(t)$, $c_d(t)$, and $c_f(t)$.

5. Collateralization Using Domestic Collateral:

$$\begin{bmatrix} \frac{\Delta X(t)}{X(t)} \\ \frac{\Delta P_{d,d}(t)}{P_{d,d}(t)} \\ \frac{\Delta P_{f,d}(t)}{P_{f,d}(t)} \end{bmatrix} = \begin{bmatrix} r_{f,d}(t) \\ c_d(t) \\ c_f(t) \end{bmatrix} \Delta t + \sigma_d \Delta W_d$$

Thus, under the domestic collateralization risk-neutral measure Q_d we have the following:

$$X(t) = \mathbb{E}_t^{Q_d}\left[e^{-\int_t^T r_{d,f}(s)ds} X(T)\right]$$

$$P_{d,d}(t,T) = \mathbb{E}_t^{Q_d}\left[e^{-\int_t^T c_d(s)ds}\right]$$

$$P_{f,d}(t,T) = \frac{1}{X(t)} \mathbb{E}_t^{Q_d}\left[e^{-\int_t^T c_d(s)ds} X(T)\right]$$

6. Collateralization Using Foreign Collateral:

$$\begin{bmatrix} \frac{\Delta\left[\frac{1}{X(t)}\right]}{\left[\frac{1}{X(t)}\right]} \\ \frac{\Delta P_{f,f}(t)}{P_{f,f}(t)} \\ \frac{\Delta P_{d,f}(t)}{P_{d,f}(t)} \end{bmatrix} = \begin{bmatrix} -r_{d,f}(t) \\ c_f(t) \\ c_d(t) \end{bmatrix} \Delta t + \sigma_f \Delta W_f$$

Thus, under the foreign collateralization risk-neutral measure Q_f we have the following:

$$\frac{1}{X(t)} = \mathbb{E}_t^{Q_f}\left[e^{\int_t^T r_{d,f}(s)ds}\frac{1}{X(T)}\right]$$

$$P_{f,f}(t,T) = \mathbb{E}_t^{Q_f}\left[e^{-\int_t^T c_f(s)ds}\right]$$

$$P_{d,f}(t,T) = X(t)\mathbb{E}_t^{Q_d}\left[e^{-\int_t^T c_d(s)ds}\frac{1}{X(T)}\right]$$

7. $\underline{P_{d,f}(t,T)}$ and $\underline{P_{f,d}(t,T)}$ Numeraires: These are effectively the cross-currency, oppositely collateralized numeraires, i.e., one unit of domestic/foreign currency collateralized using the corresponding foreign/domestic collateral. Thus these numeraires, as such, can form the basis for cross-currency discount curves employed in cross-currency swaps. Further, while these building blocks are primarily only discounting oriented – securities with forward/floater leg may also require a quanto adjustment to be applied.
8. Cross Currency Model Parameters: All the model parameters and the process dynamical parameters in the set of equations above can be independently observed.
9. "Implied" Cross Currency Risk Free Rate: The measure change from Q_d to Q_f under the Q_d measure is captured by the Q_d martingale

$$M(t) = \frac{\partial Q_f}{\partial Q_d} = e^{-\int_t^T r_{d,f}(s)ds}\frac{X(t)}{X(0)}$$

Thus, the corresponding growth rate $r_{d,f}(t)$ also helps clarify the references to the "cross-currency risk-free Rates" (e.g., Fujii and Takahashi (2011a, 2011b)) – viz., they are instantaneous FX collateralization rate using the foreign collateral.

10. Forward Forex Contract Collateralized with Domestic Collateral: This contract pays

$$X(T) - K$$

in the domestic currency, and is collateralized using the domestic collateral. Thus

$$\mathbb{E}_t^{Q_{f,d}}[X(T) - K] = X(t)P_{f,d}(t,T) - KP_{d,d}(t,T)$$

Therefore, the par strike K for this contract is

$$K = \frac{X(t)P_{f,d}(t,T)}{P_{d,d}(t,T)}$$

11. Forward Forex Contract Collateralized with Foreign Collateral: This contract pays $1 - \frac{K}{X(T)}$ in the foreign currency, and is collateralized using the foreign collateral. Thus

$$\mathbb{E}_t^{Q_f}\left[1 - \frac{K}{X(T)}\right] = P_{f,f}(t,T) - K\frac{P_{d,f}(t,T)}{X(T)}$$

Therefore, the par strike K for this contract is

$$K = \frac{X(T)P_{f,f}(t,T)}{P_{d,f}(t,T)}$$

12. Same Currency Collateralization:

$$V_{d,d}(t,T) = \mathbb{E}_t^{Q_{d,d}{}^T}[X(T) - K] = P_{d,d}(t,T)[X(T) - K]$$

and

$$V_{f,f}(t,T) = \mathbb{E}_t^{Q_{f,f}{}^T}\left[1 - \frac{K}{X(T)}\right] = \left[1 - \frac{K}{X(T)}\right]P_{f,f}(t,T)$$

No rocket science, really, with simple forwards. Question, however, is that whether $X(T)$ would ever be domestically collateralized, and that whether $\frac{1}{X(T)}$ would ever be collateralized in foreign currency. Same currency collateralization is uncommon presumably for these reasons.

13. Market Quotes for Collateralized Forex Forwards: Strictly speaking, all Forex Forwards should always be collateralized using either foreign or domestic collateral. Thus, the Forward Prices should be different depending on the collateralization currency. However, this DOES NOT appear to be the market practice, as the quotes are independent of collateral.

Collateral Choice Model

1. Setup: Here, an American style path-dependent collateral is chosen at every incremental step by opting for the collateral among the choices available that maximizes the incremental collateral cash flow.
2. Motivation: Collateralization at the domestic collateral accrual rate is c_d. On switching over to the foreign collateral, the rate becomes $c_f + r_{d,f}$. Thus at each time step we want to maximize the incremental collateral cash flow

$$\max(c_d, c_f + r_{d,f}) = c_d + \max(c_f + r_{d,f} - c_d, 0)$$

We begin by setting

$$q_{d,f} = c_f + r_{d,f} - c_d$$

3. Dynamics of $Q_{d,f}$: Consider the dynamics of

$$Q_{d,f} = \frac{P_{d,f}}{P_{f,f}}$$

This entity has a drift

$$q_{d,f} = c_f + r_{d,f} - c_d$$

First of all, the dynamics of $c_d(t)$, $r_{d,f}(t)$, and $c_f(t)$ may be worked out using one of several typically accepted practices – e.g., the HJM-type dynamics, or an even more simplified Hull-White type dynamics.

- Using

$$Q_{d,f} = \frac{P_{d,f}}{P_{f,f}}$$

  it is fairly straightforward to show that

$$\frac{\Delta Q_{d,f}}{Q_{d,f}} = q_{d,f}(t)\Delta t + \sigma_q(t)\Delta W_q$$

  where

$$\sigma_q(t)\Delta W_q = \sigma_f(t)\Delta W_f + \sigma_X(t)\Delta W_X - \sigma_d(t)\Delta W_d$$

4. <u>Piterbarg (2012) Expression for $q_{d,f}(t)$</u>: Piterbarg (2012) employs a combination of HJM machinery as listed above and additional techniques outlined in Andersen and Piterbarg (2010) to obtain $q_{d,f}(t)$.
5. <u>Collateral Choice - Deterministic $q_{d,f}(t)$</u>: If $q_{d,f}(t)$ is deterministic, there will be no optionality involved; however, depending upon the sign of $q_{d,f}(t)$, there will be a collateral switch at each time increment. Piterbarg (2012) demonstrates this in his framework by turning the volatility explicitly down to zero.
6. <u>Deterministic and Incremental Curve Decay Collateral</u>: If the collateral discounting path choice can be proxied using a “curve roll up” phenomenon, the collateral choice discount factor becomes

$$P_{d,CC}(t_0,t_n) = \prod_{i=1}^{n} \min\left(\{P_{d,j}(t_{i-1},t_i)\}_{j=1}^{r}\right)$$

where

$$j = 1, \cdots, r$$

are the r possible collateral choices, $j = 0$ is the domestic collateral curve, $P_{d,j}(t_{i-1},t_i)$ is the discount factor between t_{i-1} and t_i for one unit of domestic currency collateralized using the foreign collateral j, and $P_{d,CC}(t_{i-1},t_i)$ is the collateral choice discount factor between t_{i-1} and t_i for one unit of domestic currency collateralized using the most appropriate incremental collateral. Note that this discount curve is artificial and deterministic.

- Advantages of using deterministic collateral choices => All the advantages stem from the computational simplicity. They are:
  - More than one collateral currency may be used, thus optimizing over the multiple collateral choices (USD, GBP, EUR, JPY, etc.)
  - Empirical Curve Representations using splining techniques may be usable

7. Valuing the Collateral Choice Option: The value we seek is of the form

$$P_{d,d}(0,T)\mathbb{E}_t^{Q_d^T}\left[e^{-\int_t^T \max(q_{d,f}(s),0)ds}V(T)\right]$$

where $V(T)$ is the terminal payoff at the time instant T. It may be a fixed amount (i.e., the fixed swap rate) or a variable amount (the floating swap coupon).

- Closed Form => Typically

$$\mathbb{E}_t^{Q_d^T}\left[e^{-\int_t^T \max(q_{d,f}(s),0)ds}\right]$$

has to be computed using Monte-Carlo or a PDE, therefore we seek an alternative fast analytic approximation. By Jensen's inequality, Piterbarg (2012) noticed that

$$\mathbb{E}_t^{Q_d^T}\left[e^{-\int_t^T \max(q_{d,f}(s),0)ds}\right] \geq e^{-\int_t^T \max(q_{d,f}(s),0)ds}$$

This approximation may be used to compute the fixed leg value for the swap above. For the floater leg, the term $V(T)$ may be pushed outside to a separated expectation to get

$$P_{d,d}(0,T)\mathbb{E}_t^{Q_d^T}[V(T)]\mathbb{E}_t^{Q_d^T}\left[e^{-\int_t^T \max(q_{d,f}(s),0)ds}\right]$$

Piterbarg (2012) performs a full set of comparison to demonstrate that these approximations behave favorably with the Monte-Carlo under several situations.

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CVA And Funding Adjustments PDE

Counterparty Risk and Funding Costs

1. PDE Derivation of Adjustments – Approach: Burgard and Kjaer (2012a) derive the partial differential equation (PDE) representation for the value of financial derivatives with bilateral counterparty risks and funding costs. The model is very general in that the funding rate for lending and borrowing and the MTM value at default can be exogenously specified.
2. Buy-Back of Own Bonds: The buying back of a party's own bonds is a key part of the delta hedging strategy; they discuss how the cash account of the replication strategy provides sufficient funds for this.
3. Full Value as Default Payout: First they consider the case where the mark-to-market at default is given by the derivative, which includes the counterparty risk. They find that the resulting pricing PDE becomes non-linear, except in those special cases where the non-linear terms vanish and the Feynman-Kac representation of the total value can be obtained. In these cases the total value of the derivative can be decompose into a default-free part and a bilateral credit valuation and funding adjustment part.
4. Fair Derivative Value at Payout: Next they assume that the MTM value at default is given by the fair economic value of the derivative. This time the resulting PDE is linear and the corresponding Feynman-Kac representation is used to decompose the total value of the derivative into a default-free value plus bilateral credit valuation and funding cost adjustments.

Motivation, Literature Scan, and Approach

1. Counterparty Credit Risk Definition: Counterparty credit risk implicitly embedded in derivative contracts has become increasingly relevant in recent market conditions. This risk

represents the possibility that the counterparty defaults while owing money under the terms of a derivative contract, or more precisely, if the mark-market value of the derivative is positive to the seller at the time of the default of a counterparty.

2. OTC Counterparty Risk Mitigation: While for exchange traded products the counterparty risk is mitigated by the presence of the exchange as an intermediary, this is not the case for OTC products. For these a number of different techniques are being used to mitigate the counterparty risk, most commonly by means of netting agreements and collateral mechanisms. The details of these agreements are, for example, published by the International Swaps and Derivatives (ISDA) 2002 Master Agreement.
3. Bilateral Counterparty Credit Risk: However the counterparty faces a similar risk of the seller defaulting when the mark-to-market value is positive to the counterparty. Taking into account the credit risk of both the parties is commonly referred to as considering the bilateral counterparty risk. When doing so the value of the derivative to its seller is influenced by its own credit quality.
4. Counterparty Credit Risk Coverage: Papers, books, and book chapters that develop techniques for the valuation of the derivatives and derivative portfolio under counterparty risk include, but are not limited to, Jarrow and Turnbull (1995), Jarrow and Yu (2001), Brigo and Mercurio (2007), Li and Tang (2007), Pykhtin and Zhu (2007), Alavian, Ding, Whitehead, and Laiddicina (2008), Gregory (2009), and Cesari, Aquilina, Charpillon, Filipovic, Lee, and Manda (2009).
5. Counterparty Risk With Funding Costs: There are other areas where the credit of the seller is relevant, in particular in terms of the MTM accounting of its own debt, as well as the effect it has on its funding costs. Piterbarg (2010) discusses the funding costs on derivative valuations when collateral has to be posted. Thus Burgard and Kjaer (2012a) combine the effects of the seller's credit on its own funding costs with that on the bilateral counterparty risk into a unified framework.
6. Black-Scholes PDE Formulation Extension: Further they use the hedging argument to derive extensions to the Black-Scholes PDE in the presence of bilateral counterparty risk in the presence of bilateral jump-to-default model and include funding considerations in the financing of the hedge positions. In addition they consider two scenarios for the determination of the derivative MTM at default – namely that recovery is on the total risky

value or that it is on the counterparty riskless value. The latter corresponds to the most common approach taken in the literature.

7. The ISDA 2002 Master Agreement: The total value of the derivative will then depend upon which of the 2 MTM values is used at default. For contracts following the ISDA 2002 Master Agreement the value of the derivatives upon the default of one of the counterparties is determined by a dealer poll. There is no reference to the counterparties, and one could reasonably expect the derivative value to be the counterparty riskless value, i.e., the second case considered.
8. Risky Derivative Value at Payout: In the case where the default-risky derivative value is used as the mark-to-market, Burgard and Kjaer (2012a) derive a pricing PDA that is in general non-linear and demonstrate that the unknown risky price can be found by solving a non-linear integral equation. Under certain conditions on the payoff the non-linear terms vanish and the Feynman-Kac representation of the resulting linear PDE is examined.
9. Fair Derivative Value at Payout: In the case where the counterparty derivative price is used as the mark-to-market, the resulting pricing PDE is linear. As in the first case the Feynman-Kac representation can be used to decompose the risky derivative value into a counterparty risk-free part, a funding part, and a bilateral credit valuation adjustment (CVA) part.
10. Granular Hedging Accounts and Strategies: By using a fine-grained hedging strategy to derive their results Burgard and Kjaer (2012a) ensure that the hedging costs of all considered risk factors are included in their derivative price such that the decomposition of the risky price is a generalization of the result commonly found in literature. Further they get explicit expressions for hedges, which is important for risk management.
11. Own Credit Hedging Risk Caveats: There have been discussions about how a seller can hedge out its own credit risk – Cesari, Aquilina, Charpillon, Filipovic, Lee, and Manda (2009) contain a summary. The strategy described by Burgard and Kjaer (2012a) includes the repurchase by the seller of its own bonds to hedge out its own credit risk. On the face of it this may seem like a futile approach since of the bond purchase were funded by issuing more debt (i.e., more bonds), the seller would have in effect achieved nothing in terms of hedging its own credit risk.
12. Differentiated Buy-Back Funding Strategy: However the replication strategy presented shows how the funding for the purchase of the seller's own bond is achieved through the cash

account of the hedging strategy. The hedging strategy (including the premium of the derivative) generates the cash needed to repurchase the sellers' own bond.

13. Multiple Assets and Netting Extensions: Although all the results presented in Burgard and Kjaer (2012a) are for one derivative on one underlying asset following the specified dynamics, extension to the situation of a netted derivatives portfolio on several underlying following generalized diffusion dynamics is straightforward.

Notation, Symbology, and Key PDEs

1. **1.** The Nature of the Derivative: Consider a derivative contract $\hat{V}$ between a seller B and a counterparty C that may both default. The asset S is not affected by the default of either B or C and is assumed to follow a Markov process with a generator $\mathcal{A}_t$. Similarly we let V denote the same derivative between parties that cannot default.
2. **2.** Payout on a Default: At the default of either the counterparty or the seller the value of the derivative to the seller $\hat{V}$ is determine by using an MTM rule M which may equal $\hat{V}$ or V. Throughout the convention used is that the positive derivative values correspond to seller assets and counterparty liabilities.
3. **3.** Notations, Parameter Definitions, and Caveats:
    - **a.** r => Risk-free rate
    - **b.** r_B => Yield on a Recovery-less Bond of Seller B
    - **c.** r_C => Yield on a Recovery-less Bond of Counterparty C
    - **d.** λ_B =>

$$\lambda_B = r_B - r$$

    - **e.** λ_C =>

$$\lambda_C = r_C - r$$

f. r_F => Seller funding rate for the borrowed cash on the seller's derivative replication cash account.

$$r_F = r$$

if the derivative can be used as a collateral.

$$r_F = r + (1 - R_B)\lambda_B$$

if the derivative cannot be used as a collateral.

g. s_F =>

$$s_F = r_F - r$$

h. R_B => Recovery on the derivative MTM value in case the seller B defaults

i. R_C => Recovery on the derivative MTM value in case the counterparty C defaults

4. <u>PDE for $\hat{V}$ when $M = \hat{V}$</u>: When the MTM at default is given by

$$M = \hat{V}$$

then $\hat{V}$ satisfies the PDE

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = (1 - R_B)\lambda_B \hat{V}^- + (1 - R_C)\lambda_C \hat{V}^+ + s_F \hat{V}^+$$

5. <u>PDE for $\hat{V}$ when $M = V$</u>: When the MTM at default is given by

$$M = V$$

then $\hat{V}$ satisfies the PDE

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C)\hat{V} = -(R_B \lambda_B + \lambda_C)\hat{V}^- - (\lambda_B + R_C \lambda_C)\lambda_C \hat{V}^+ + s_F \hat{V}^+$$

6. Credit Funding Adjustment when $M = V$: Let

$$M = V$$

and

$$r_F = r + s_F$$

Then

$$\hat{V} = V + U$$

and the credit funding adjustment U is given by

$$U(t,S) = -(1 - R_B)\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V^-\left(u, S(u)\right)\right]du$$

$$- (1 - R_C)\int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V^-\left(u, S(u)\right)\right]du$$

$$- \int_t^T s_F(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V^+\left(u, S(u)\right)\right]du$$

where

$$D_k(t,u) = e^{-\int_t^u k(v)dv}$$

is the discount factor between times t and u using the rate k. If

$$s_F = 0$$

then U is identical to the regular bilateral CVA derived in many of the papers.

7. Justification of the Buy-Back Strategy: Another important result of Burgard and Kjaer (2012a) is the justification on which the seller's own-credit risk can be taken into account. In the hedging strategy considered the risk is hedged out by the seller buying back its own bonds. It is shown that the cash needed for doing so is generated through the replication strategy.

Model Setup and the Derivation of the Bilateral Risky PDE

1. Underlying Traded Assets and Securities: Consider an economy with the following four traded assets:
   a. P_R => Default-risk free zero-coupon bond
   b. P_B => Default-risky, zero-recovery, zero-coupon bond of party B
   c. P_C => Default-risky, zero-recovery, zero-coupon bond of counterparty C
   d. S => The Spot Asset with no Default Risk
2. Zero-Recovery Bonds Building Blocks: Both the risky bonds P_B and P_C pay $1 at some future date T if the issuing counterparty has not defaulted, and 0 otherwise. These simplistic bonds are useful for modeling and can be used as building blocks for more complex bonds, including the ones with zero recovery.
3. Dynamics of the Underlying Assets: The processes for he assets P_B, P_C, P_R, and S under their corresponding probability measures are specified by

$$\frac{\Delta P_B}{P_B} = r_B(t)\Delta t - \Delta J_B$$

$$\frac{\Delta P_C}{P_C} = r_C(t)\Delta t - \Delta J_C$$

$$\frac{\Delta P_R}{P_R} = r(t)\Delta t$$

$$\frac{\Delta S}{S} = \mu(t)\Delta t - \sigma(t)\Delta W(t)$$

where $W(t)$ is a Weiner process,

$$r(t) > 0$$

$$r_B(t) > 0$$

$$r_C(t) > 0$$

and

$$\sigma(t) > 0$$

are deterministic functions of t and J_B and J_C are two point processes that jump from 0 to 1 on the default of B and C respectively.

4. Asset Independence of the Seller/Counterparty: The assumptions above indicate that the hedging can be done by P_B and P_C alone, and later this assumption is relaxed. Further the spot asset price S is assumed to be unaffected by a default of either B or C. Finally B is taken to be the *seller* and C the *counterparty* respectively.
5. Terminal Derivative Payout at Maturity: It is assumed that the parties B and C enter a derivative on the spot asset S that pays the seller B the amount

$$H(S) \in \mathbb{R}$$

at maturity T. Thus in this convention the payout scenario

$$H(S) \geq 0$$

means that the seller receives cash or asset from the counterparty.

6. Credit Risky Derivative Present Value: The value of the derivative to the seller at time t is denoted $\hat{V}(t, S, J_B, J_C)$ and depends on the spot asset S of the underlying and the default states J_B and J_C respectively of the seller B and the counterparty C. Analogously $V(t, S)$ denotes the value to the seller of the same derivative as if it were a transaction between two default-free counterparties.
7. Default Derivative Close-out Claim: When party B or C default in general the mark-to-market on the derivative determines the close-out or the claim on the position. However the precise nature of this depends on the contractual details and the mechanism by which the mark-market is determined.
8. 2002 ISDA Master Agreement Specification: The 2002 ISDA Master Agreement specifies that the derivative contract will return to the surviving party the recovery value of its positive mark-to-market value (from the point of view of the surviving party) just prior to default, whereas the full mark-to-market has to be paid to the defaulting party if the mark-to-market value is negative (from the view of the surviving party). The master agreement specifies a dealer poll mechanism to establish the mark-to-market to the seller at default $M(t, S)$ without referring to the names of the counterparties involved in the derivative transaction.
9. Handling Imprecise ISDA Close-outs: From the above one would expect $M(t, S)$ to be close to $V(t, S)$ even though it is unclear if the dealers in the poll may or may not include their funding costs in the derivative prices. In case the ISDA Master Agreement is not followed there may be other mechanisms involved. Therefore Burgard and Kjaer (2012a) derive the PDE for the general case $M(t, S)$ and consider the two special cases

$$M(t, S) = \hat{V}(t, S, 0, 0)$$

and

$$M(t, S) = V(t, S)$$

10. <u>Default Close-out Boundary Conditions</u>: Let

$$R_B \in [0, 1]$$

and

$$R_C \in [0, 1]$$

denote the recovery rates on the derivatives positions for parties B and C respectively – for now they are taken to be deterministic. The above discussions result in the following boundary conditions:

$$\hat{V}(t, S, 1, 0) = M^+(t, S) + R_B M^-(t, S)$$

when the seller defaults first and

$$\hat{V}(t, S, 0, 1) = R_C M^+(t, S) + M^-(t, S)$$

when the counterparty defaults first. Li and Tang (2009), Gregory (2009), and the vast majority of the papers on the valuation of the counterparty risk use

$$M(t, S) = V(t, S)$$

11. <u>The Black Scholes Replicating Portfolio</u>: As in the usual Black-Scholes framework the derivative s hedged with a self-financing portfolio that covers all the risk factors of the model. In this case the portfolio Π the seller sets up consists of $\alpha_S(t)$ units of S, $\alpha_B(t)$ units of P_B, $\alpha_C(t)$ units of P_C, and $\beta(t)$ units of cash such that the portfolio value at time t replicates the value of the derivative to the seller, i.e.,

$$\hat{V}(t) + \Pi(t) = 0$$

Thus

$$-\hat{V}(t) = \Pi(t) = \alpha_S(t)S(t) + \alpha_B(t)P_B(t) + \alpha_C(t)P_C(t) + \beta(t)$$

12. Counterparty Credit Hedge Ratio: Before proceeding note that when

$$\hat{V} \geq 0$$

the seller will incur a loss at the counterparty default. To hedge this loss P_C needs to be shorted, so we expect that

$$\alpha_C \leq 0$$

Assuming that the seller can borrow the bond P_C at a rate close to the risk-free rate r through a repurchase agreement, the spread λ_C between the rate r_C on the bond and the cost of financing the hedge position in C can be approximated to

$$\lambda_C = r_C - r$$

Since P_C is a bond with zero recovery this spread corresponds to the default intensity of C.

13. Own Credit Risk Hedge Ratio: On the other hand, if

$$\hat{V} < 0$$

the seller will gain at own default, which can be hedged by buying back P_B bonds, so one expects that

$$\alpha_C \geq 0$$

in this case. For this to work, enough cash must be generated, and any remaining cash after the purchase of P_B bonds is invested in such a way that it does not generate additional credit risk to the seller, i.e., any remaining positive cash generates a yield at the risk free rate of r.

14. Growth in the Cash Account: Imposing that the portfolio $\Pi(t)$ is self-financing implies that

$$-\Delta\hat{V}(t) = \alpha_S(t)\Delta S(t) + \alpha_B(t)\Delta P_B(t) + \alpha_C(t)\Delta P_C(t) + \Delta\bar{\beta}(t)$$

where the growth in the cash $\Delta\bar{\beta}(t)$ may be decomposed into

$$\Delta\bar{\beta}(t) = \Delta\bar{\beta}_S(t) + \Delta\bar{\beta}_F(t) + \Delta\bar{\beta}_C(t)$$

i.e., it is composed of three parts – the asset cash growth, the growth from the counterparty cash flow, and the growth from a unified (bond + cash) account.

15. Instantaneous Growth vs. Portfolio Rebalancing: The growth above is the growth in the cash account before re-balancing of the portfolio. The self-financing condition ensures that after Δt the rebalancing can happen at zero overall cost. This distinction is clarified in detail by Brigo, Buescu, Pallavicini, and Liu (2012).

16. Growth From the Asset Position $\Delta\bar{\beta}_S(t)$: The asset position provides a dividend income of $\alpha_S(t)\gamma_S(t)S(t)$ at a financing cost of $-\alpha_S(t)q_S(t)S(t)$ so

$$\Delta\bar{\beta}_S(t) = \alpha_S(t)[\gamma_S(t) - q_S(t)]S(t)\Delta t$$

The value of $q_S(t)$ may depend on the risk-free rate $r(t)$ and the repo rate of $S(t)$.

17. Growth From the Counterparty Position $\Delta\bar{\beta}_C(t)$: Using the arguments above, the seller will short the counterparty bonds using a repurchase agreement and incur financing costs of

$$\Delta\bar{\beta}_C(t) = -\alpha_C(t)r(t)P_C(t)\Delta t$$

Note the use of $r(t)$ instead of $r_C(t)$.

18. Positive/Negative Cash Account Asymmetry on $\Delta\bar{\beta}_F(t)$: From the above analysis any surplus cash held by the user after the own-bonds have been purchased must earn the risk-free rate

$r(t)$ in order to not introduce any further credit risk to the seller. If borrowing money the seller needs to pay the rate $r_F(t)$. Thus the own bonds/cash account uses a funding/hedging scheme that is not symmetric.

19. Unsecured Funding vs. Derivative as Collateral: For this funding rate, the following two cases are distinct; where the derivative itself can be used as a collateral for the required funding, on no haircut $r_F(t)$ is set to $r(t)$. If however the derivative cannot be used as a collateral, the funding rate is set to the yield of the unsecured seller bond with recovery R_B, i.e.,

$$r_F(t) = r(t) + (1 - R_B)\lambda_B$$

20. Own Bond Cash Position Growth: In practice the second instance above is the more realistic case. Keeping $r_F(t)$ general for now,

$$\Delta\bar{\beta}_F(t) = \left[r(t)\left(-\hat{V} - \alpha_B P_B\right)^+ + r_F(t)\left(-\hat{V} - \alpha_B P_B\right)^-\right]\Delta t$$

$$= r(t)\left(-\hat{V} - \alpha_B P_B\right)\Delta t + s_F(t)\left(-\hat{V} - \alpha_B P_B\right)^-\Delta t$$

where the funding spread

$$s_F = r_F - r$$

i.e.,

$$s_F = 0$$

if the derivative cannot be used as a collateral, and

$$s_F = (1 - R_B)\lambda_B$$

if it cannot.

21. Cumulative Cross Cash Growth Account: From the above analysis it follows that the total change in the cash account (here t is dropped from the notation where applicable to improve clarity) is given by

$$\Delta\bar{\beta} = (\gamma_S - q_S)\alpha_S S\Delta t + \left[r\left(-\hat{V} - \alpha_B P_B\right) + s_F\left(-\hat{V} - \alpha_B P_B\right)^{-}\right]\Delta t - r\alpha_C P_C \Delta t$$

22. Change of the Replication Portfolio Value: Using the $\Delta\bar{\beta}$ computed above in

$$-\Delta\hat{V} = \alpha_S \Delta S + \alpha_B \Delta P_B + \alpha_C \Delta P_C + \Delta\bar{\beta}$$

one gets

$$\begin{aligned}
-\Delta\hat{V} = \alpha_S \Delta S + \alpha_B P_B(r_B\Delta t - \Delta J_B) + \alpha_C P_C(r_C\Delta t - \Delta J_C) + \Delta\bar{\beta} \\
+ \left[(\gamma_S - q_S)\alpha_S S + r\left(-\hat{V} - \alpha_B P_B\right) + s_F\left(-\hat{V} - \alpha_B P_B\right)^{-} - r\alpha_C P_C\right]\Delta t \\
= \left[-r\hat{V} + s_F\left(-\hat{V} - \alpha_B P_B\right)^{-} + (\gamma_S - q_S)\alpha_S S + (r_B - r)\alpha_B P_B \right. \\
\left. + (r_C - r)\alpha_C P_C\right]\Delta t - \alpha_B P_B \Delta J_B - \alpha_C P_C \Delta J_C + \alpha_S \Delta S
\end{aligned}$$

23. Ito's Lemma for Jump Diffusion: On the other hand, by Ito's lemma for jump diffusion and the assumption that a simultaneous jump of both the seller and the counterparty is a zero-probability event, the derivative value moves by

$$\Delta\hat{V} = \frac{\partial\hat{V}}{\partial t}\Delta t + \frac{\partial\hat{V}}{\partial S}\Delta S + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2\hat{V}}{\partial S^2}\Delta t + \mathcal{D}\hat{V}_B \Delta J_B + \mathcal{D}\hat{V}_C \Delta J_C$$

where

$$\mathcal{D}\hat{V}_B = \hat{V}(t, S, 1, 0) - \hat{V}(t, S, 0, 0)$$

and

$$\mathcal{D}\hat{V}_C = \hat{V}(t, S, 0, 1) - \hat{V}(t, S, 0, 0)$$

can be computed from

$$\hat{V}(t,S,1,0) = M^+(t,S) + R_B M^-(t,S)$$

and

$$\hat{V}(t,S,0,1) = R_C M^+(t,S) + M^-(t,S)$$

24. Asset and Bond Hedge Ratios: Equating the $\Delta\hat{V}$ from the cash growth and the Ito's lemmas, the following choice of α_S, α_B, and α_C eliminates all risks in the portfolio:

$$\alpha_S = \frac{\partial \hat{V}}{\partial S}$$

$$\alpha_B = \frac{\mathcal{D}\hat{V}_B}{P_B} = -\frac{\hat{V} - (M^+ + R_B M^-)}{P_B}$$

$$\alpha_C = \frac{\mathcal{D}\hat{V}_C}{P_C} = -\frac{\hat{V} - (M^- + R_C M^+)}{P_C}$$

25. Cash Account Evolution Expression: Hence the cash account evolution

$$\Delta\bar{\beta}_F(t) = r(t)\left(-\hat{V} - \alpha_B P_B\right)\Delta t + s_F(t)\left(-\hat{V} - \alpha_B P_B\right)^- \Delta t$$

can be rewritten as

$$\Delta\bar{\beta}_F(t) = [-r(t)R_B M^- - r_F(t)M^+]\Delta t$$

so the amount of cash deposited by the seller at the risk-free rate equals $-rR_B M^-$ and the amount borrowed at the funding rate r_F equals $-M^+$

26. Derivative as Collateral Black Scholes: Introducing the parabolic differential operator $\mathcal{A}_t$ as

$$\mathcal{A}_t\hat{V} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \hat{V}}{\partial S^2} + (\gamma_S - q_S) S \frac{\partial \hat{V}}{\partial S}$$

it follows that $\hat{V}$ is the solution to the PDE

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - r\hat{V} = s_F\left(\hat{V} + \mathcal{D}\hat{V}_B\right)^+ - \lambda_B \mathcal{D}\hat{V}_B - \lambda_C \mathcal{D}\hat{V}_C$$

$$\hat{V}(T, S) = H(S)$$

where

$$\lambda_B = r_B - r$$

and

$$\lambda_C = r_C - r$$

27. Incorporation of the Boundary Condition: Inserting

$$\mathcal{D}\hat{V}_B = \hat{V}(t, S, 1, 0) - \hat{V}(t, S, 0, 0)$$

and

$$\mathcal{D}\hat{V}_C = \hat{V}(t, S, 0, 1) - \hat{V}(t, S, 0, 0)$$

with the boundary conditions

$$\hat{V}(t, S, 1, 0) = M^+(t, S) + R_B M^-(t, S)$$

and

$$\hat{V}(t, S, 0, 1) = R_C M^+(t, S) + M^-(t, S)$$

into

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = s_F\left(\hat{V} + \mathcal{D}\hat{V}_B\right)^+ - \lambda_B \mathcal{D}\hat{V}_B - \lambda_C \mathcal{D}\hat{V}_C$$

finally gives

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = (\lambda_B + \lambda_C)\hat{V} + s_F M^+ - \lambda_B(R_B M^- + M^+) - \lambda_C(R_C M^+ + M^-)$$

$$\hat{V}(T, S) = H(S)$$

where the relation

$$\left(\hat{V} + \mathcal{D}\hat{V}_B\right)^+ = (R_B M^- + M^+)^+ = M^+$$

has been used.

28. Fair Derivative Black Scholes Value: In contrast the risk-free value V satisfies the regular Black-Scholes PDE

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = 0$$

$$V(T, S) = H(S)$$

Thus if one interprets λ_B and λ_C as effective default rates then the difference between $\hat{V}$ and V may be interpreted as follows.

29. Self/Counterparty Default Impact: The term $(\lambda_B + \lambda_C)\hat{V}$ is the additional growth rate the seller B requires on the risky asset $\hat{V}$ to compensate for the risk that default of either the seller the seller or the counterparty will terminate the derivative contract.
30. Receivables Funding Impact Hedge Strategy: The term $s_F M^+$ is the additional funding cost for negative values of the cash account for the hedging strategy.
31. Own Default Close-out: The term $-\lambda_B(R_B M^- + M^+)$ is the adjustment in the growth rate that the seller can accept because of the cash flows occurring at own default.
32. Counterparty Default Close-out: The term $-\lambda_C(R_C M^+ + M^-)$ is the adjustment in the growth rate that the seller can accept because of the cash flow occurring at the counterparty default.
33. Modeling of the Extinguisher Trade: The terms $(\lambda_B + \lambda_C)\hat{V}$, $-\lambda_B(R_B M^- + M^+)$, and $-\lambda_C(R_C M^+ + M^-)$ are related to counterparty risk whereas the term $s_F M^+$ represents the funding cost. From this interpretation it follows that the PDE for a so-called *extinguisher trade*, whereby it is agreed that no party gets anything at default, is obtained by removing the terms $-\lambda_B(R_B M^- + M^+)$ and $-\lambda_C(R_C M^+ + M^-)$ from

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - r\hat{V} = (\lambda_B + \lambda_C)\hat{V} + s_F M^+ - \lambda_B(R_B M^- + M^+) - \lambda_C(R_C M^+ + M^-)$$

Using $\hat{V}(T, S)$ As Mark-to-Market at Default

1. Conceptually Simple Payout Condition: This section considers the case where the payments in default are based on $\hat{V}(T, S)$ so that

$$M(T, S) = \hat{V}(T, S)$$

in the boundary condition

$$\hat{V}(t, S, 1, 0) = M^+(t, S) + R_B M^-(t, S)$$

and

$$\hat{V}(t, S, 0, 1) = R_C M^+(t, S) + M^-(t, S)$$

Conceptually this is the simpler case since if the defaulting party is in the money with respect to a derivative contract, then there is no additional impact on the profit and the loss at the point of default.

2. <u>Simplification of the PDE for $\hat{V}$</u>: Similarly if the surviving party is in the money with respect to the derivative contract, then its loss is simply $(1 - R)\hat{V}$. In this case

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = (\lambda_B + \lambda_C)\hat{V} + s_F M^+ - \lambda_B(R_B M^- + M^+) - \lambda_C(R_C M^+ + M^-)$$

reduces to

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = s_F \hat{V}^+ - (1 - R_B)\lambda_B \hat{V}^- - (1 - R_C)\lambda_C \hat{V}^+$$

$$\hat{V}(T, S) = H(S)$$

where

$$s_F = 0$$

if the derivative can be posted as collateral and

$$s_F = (1 - R_B)\lambda_B$$

if it cannot.

3. <u>Own Credit And Counterparty Hedges</u>: Moreover the hedge ratios α_B and α_C are given by

$$\alpha_B = -\frac{(1 - R_B)\hat{V}^-}{P_B}$$

and

$$\alpha_C = -\frac{(1 - R_C)\hat{V}^+}{P_C}$$

so that

$$\alpha_B \geq 0$$

and

$$\alpha_C \geq 0$$

and the replication strategy generates enough cash $-\hat{V}^-$ for the seller to purchase back its own bonds.

4. Interpretation of the Own Credit Hedge: The cash available to the seller is $-\hat{V}^-$, of which the fraction $1 - R_B$ is invested in buying back the recovery-less bond B and the fraction R_B is invested risk-free. This is equivalent to investing a total amount of $-\hat{V}^-$ into purchasing a seller bond $\bar{B}$ with recovery R_B.
5. Credit Valuation Adjustment (CVA) Formulation: In the counterparty risk literature it is customary to write

$$\hat{V} = V + U$$

where U is called the credit valuation adjustment or the CVA. Inserting this representation into

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = s_F \hat{V}^+ + (1 - R_B)\lambda_B \hat{V}^- + (1 - R_C)\lambda_C \hat{V}^+$$

$$\hat{V}(T, S) = H(S)$$

$$\frac{\partial V}{\partial t} + \mathcal{A}_t V - rV = 0$$

$$V(T, S) = H(S)$$

yields

$$\frac{\partial U}{\partial t} + \mathcal{A}_t U - rU = s_F (V + U)^+ + (1 - R_B)\lambda_B (V + U)^- + (1 - R_C)\lambda_C (V + U)^+$$

$$U(T, S) = 0$$

where V is known and acts as the source term.

6. Funding + CVA Feynman-Kac Integral: Furthermore one may apply the Feynman-Kac integral to the PDE for U, which with the additional assumption of deterministic interest rates produces the following non-linear integral equation:

$$U(t, S) = -(1 - R_B) \int_t^T \lambda_B(u) D_r(t, u) \mathbb{E}_t\left[\left\{V\big(u, S(u)\big) + U\big(u, S(u)\big)\right\}^-\right] du$$
$$- (1 - R_C) \int_t^T \lambda_C(u) D_r(t, u) \mathbb{E}_t\left[\left\{V\big(u, S(u)\big) + U\big(u, S(u)\big)\right\}^+\right] du$$
$$- \int_t^T s_F(u) D_r(t, u) \mathbb{E}_t\left[\left\{V\big(u, S(u)\big) + U\big(u, S(u)\big)\right\}^+\right] du$$

It follows that one can compute U first by computing V and then solving either the non-linear PDE above or the integral equation.

7. Receivable/Payable + Funding Numeraire: Before proceeding with the study of the two cases

$$s_F = 0$$

and

$$s_F = (1 - R_B)\lambda_B$$

it is worthwhile to examine a few instances where $\hat{V}$ corresponds to 1 UNIT seller receivable/payable, where those bonds may be with and without recovery.

8. The Seller sells 1 Unit Payable to the Counterparty:
    a. Zero-recovery Payable Numeraire => The first case considered is one unit sold by the seller to the counterparty C. In this situation

    $$\hat{V} = \hat{V}^- = -1$$

    and

    $$R_B = 0$$

    Since only deterministic rates and spreads are considered, there is no risk with respect to the underlying market factors, and the term $\mathcal{A}_t\hat{V}$ vanishes, so

    $$\frac{\partial\hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - r\hat{V} = s_F\hat{V}^+ + (1 - R_B)\lambda_B\hat{V}^- + (1 - R_C)\lambda_C\hat{V}^+$$

    and

    $$\hat{V}(T, S) = H(S)$$

    become

$$\frac{\partial \hat{V}}{\partial t} = (r + \lambda_B)\hat{V} = r_B \hat{V}$$

and

$$\hat{V}(T, S) = -1$$

with the solution

$$\hat{V}(t) = -e^{-\int_t^T r_B(s) ds}$$

as expected for

$$\hat{V}(T, S) = -1$$

b. Non-zero Recovery Payable Numeraire => If on the other hand the recovery

$$R_B \neq 0$$

then

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = s_F \hat{V}^+ + (1 - R_B)\lambda_B \hat{V}^- + (1 - R_C)\lambda_C \hat{V}^+$$

becomes

$$\frac{\partial \hat{V}}{\partial t} = \{r + (1 - R_B)\lambda_B\}\hat{V}$$

and

$$\hat{V}(T, S) = -1$$

with the solution

$$\hat{V}(t) = -e^{-\int_t^T \{r(s) + (1 - R_B)\lambda_B(s)\} ds}$$

As expected the rate $r + (1 - R_B)\lambda_B$ payable on a bond with recovery is equal to the unsecured funding rate r_F that the seller has to pay on negative cash balances when the derivative cannot be posted as a collateral.

9. <u>The Seller Buys 1 Unit Receivable From C</u>:
    a. Zero-Recovery Receivable Funding Numeraire => If $\hat{V}$ describes the purchase of 1 Unit receivable by the seller from the counterparty (i.e., the seller lends to the counterparty without recovery) then

$$\hat{V} = \hat{V}^- = +1$$

and

$$R_C = 0$$

and

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = s_F \hat{V}^+ + (1 - R_B)\lambda_B \hat{V}^- + (1 - R_C)\lambda_C \hat{V}^+$$

and

$$\hat{V}(T, S) = H(S)$$

become

$$\frac{\partial \hat{V}}{\partial t} = (r_F + \lambda_C)\hat{V} = [r_F + (r_C - r)]\hat{V}$$

and

$$\hat{V}(T, S) = -1$$

b. Zero-Recovery Receivable Collateralized Numeraire => In this case, if the seller can use the derivative (i.e., the loan asset) as collateral for the funding of its short position within its replication strategy, then (neglecting haircuts)

$$r_F = r$$

the risk-free rate. Then the net result in this case is then

$$\frac{\partial \hat{V}}{\partial t} = r_C \hat{V}$$

so

$$\hat{V}(t) = -e^{-\int_t^T r_C(s) ds}$$

as expected for

$$\hat{V}_C(t) = P_C(t)$$

c. Non-zero Recovery Numeraire => If on the other hand

$$R_C \neq 0$$

then

$$\hat{V}(t) = -e^{-\int_t^T \{r(s)+(1-R_C)\lambda_C(s)\}ds}$$

as expected.

10. <u>The Case $r_F = r$</u>:
    a. PDE With Derivatives as Collateral => If the derivative can be posted as collateral the PDE

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - r\hat{V} = s_F\hat{V}^+ + (1 - R_B)\lambda_B\hat{V}^- + (1 - R_C)\lambda_C\hat{V}^+$$

and

$$\hat{V}(T, S) = H(S)$$

become

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - r\hat{V} = (1 - R_B)\lambda_B\hat{V}^- + (1 - R_C)\lambda_C\hat{V}^+$$

and

$$\hat{V}(T, S) = H(S)$$

which is a non-linear PDE that needs to be solved numerically unless

$$\hat{V} \geq 0$$

or

$$\hat{V} \leq 0$$

b. Feynman-Kac Integral for $\hat{V}(t, S) \leq 0$ => Assuming that

$$\hat{V} \leq 0$$

i.e., the seller sold an option to the counterparty so

$$H(S) \leq 0$$

and that all rates are deterministic the Feynman-Kac representation of $\hat{V}$ is given by

$$\hat{V}(t, S) = \mathbb{E}_t\left[D_{r+(1-R_B)\lambda_B}(t, T)H\big(S(T)\big)\right]$$

where

$$D_k(t, T) = -e^{-\int_t^T k(s)ds}$$

is the discount factor over $[t, T]$ given that rate k.

c. Own-Credit Adjustment for Payables => Alternatively if for

$$\hat{V} \leq 0$$

we insert the ansatz

$$\hat{V} = V + U_0$$

where the zero subscript in U_0 indicates that the CVA U_0 is computed at

$$s_F = 0$$

into

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = s_F \hat{V}^+ + (1 - R_B)\lambda_B \hat{V}^- + (1 - R_C)\lambda_C \hat{V}^+$$

and

$$\hat{V}(T, S) = H(S)$$

and apply the Feynman-Kac theorem, and finally use that

$$V(t, S) = D_r(t, u)\mathbb{E}_t\big[V\big(u, S(u)\big)\big]$$

one then gets

$$U_0(t, S) = -V(t, S)\left[\int_t^T (1 - R_B)\lambda_B(u) D_{(1-R_B)\lambda_B(u)}(t, u) du\right]$$

d. Counterparty Adjustment for Receivables => When

$$\hat{V} \geq 0$$

i.e., the “seller” bought an option, symmetry yields that

$$U_0(t, S) = -V(t, S)\left[\int_t^T (1 - R_C)\lambda_C(u) D_{(1-R_C)\lambda_C(u)}(t, u) du\right]$$

Thus one concludes that if

$$\hat{V} \leq 0$$

then U_0 depends only on the credit of the seller, whereas if

$$\hat{V} \geq 0$$

it depends only on the counterparty credit.

11. <u>The Case $r_F = r + (1 - R_B)\lambda_B$</u>:
    a. The Derivative Cannot Serve as Collateral => If the derivative cannot be posted as collateral then

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = s_F \hat{V}^+ + (1 - R_B)\lambda_B \hat{V}^- + (1 - R_C)\lambda_C \hat{V}^+$$

and

$$\hat{V}(T, S) = H(S)$$

become

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = (1 - R_B)\lambda_B \hat{V}^- + \{(1 - R_B)\lambda_B + (1 - R_C)\lambda_C\} \hat{V}^+$$

and

$$\hat{V}(T, S) = H(S)$$

which again is a non-linear PDE.

    b. Feynman-Kac Integral for $\hat{V}(t, S) \leq 0$ => For

$$\hat{V} \leq 0$$

one writes

$$\hat{V} = V + U$$

and it is easy to see that

$$U = U_0$$

given in

$$U_0(t, S) = -V(t, S)\left[\int_t^T (1 - R_B)\lambda_B(u) D_{(1-R_B)\lambda_B(u)}(t, u) du\right]$$

so $\hat{V}$ is given by

$$\hat{V}(t, S) = \mathbb{E}_t\left[D_{r+(1-R_B)\lambda_B}(t, T) H\big(S(T)\big)\right]$$

c. Feynman-Kac Integral for $\hat{V}(t, S) \geq 0$ => If

$$\hat{V}(t, S) \geq 0$$

then

$$\hat{V}(t, S) = \mathbb{E}_t\left[D_{r+k}(t, T) H\big(S(T)\big)\right]$$

where

$$k = (1 - R_B)\lambda_B + (1 - R_C)\lambda_C$$

d. Own/Counterparty Credit Adjustment => Analogous to the case

$$r_F = r$$

one may make the ansatz

$$\hat{V} = V + U$$

and show that

$$U(t, S) = -V(t, S) \int_t^T k(u) D_k(t, u) du$$

Comparing this $U(t, S)$ with

$$U_0(t, S) = -V(t, S) \int_t^T (1 - R_C) \lambda_C(u) D_{(1 - R_C)\lambda_C}(t, u) du$$

shows that when the "seller" buys an option from the counterparty it encounters an additional funding spread

$$s_F = (1 - R_B) \lambda_B$$

Using $V(T, S)$ As Mark-to-Market at Default

1. <u>$M(t, S) = V(t, S)$ Case for the PDE for $\hat{V}$</u>: This section considers the scenario where the payments in the case of default are based on V and hence use

$$M(t, S) = V(t, S)$$

in the boundary conditions

$$\hat{V}(t, S, 1, 0) = M^{+}(t, S) + R_B M^{-}(t, S)$$

and

$$\hat{V}(t, S, 0, 1) = R_C M^{+}(t, S) + M^{-}(t, S)$$

The PDE for $\hat{V}$

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = (\lambda_B + \lambda_C)\hat{V} + s_F M^{+} - \lambda_B (R_B M^{-} + M^{+}) - \lambda_C (R_C M^{+} + M^{-})$$

then becomes

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C)\hat{V} = -(R_B \lambda_B + \lambda_C)V^{-} - (R_C \lambda_C + \lambda_B)V^{+} + s_F V^{+}$$

$$\hat{V}(T, S) = H(S)$$

2. <u>Own-Credit and Counterparty Hedges</u>: The above PDE is linear and has a source term on the right hand side. On writing

$$\hat{V} = V + U$$

the hedge ratios become

$$\alpha_B = \frac{U + (1 - R_B)V^{-}}{P_B}$$

and

$$\alpha_C = \frac{U + (1 - R_C)V^+}{P_C}$$

Comparing α_B with

$$\alpha_B = -\frac{(1 - R_B)\hat{V}^-}{P_B}$$

shows that in the current case default triggers a windfall cash flow of U that needs to be taken into account in the hedging strategy.

3. Valuation Adjustment Feynman Kac Integrals: Writing

$$\hat{V} = V + U$$

also gives the following PDE for U:

$$\frac{\partial U}{\partial t} + \mathcal{A}_t U - (r + \lambda_B + \lambda_C)U = s_F V^+ + (1 - R_B)\lambda_B V^- + (1 - R_C)\lambda_C V^+$$

$$U(T, S) = 0$$

Thus application of the Feynman-Kac theorem yields

$$U(t, S) = -(1 - R_B)\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t, u)\mathbb{E}_t\left[V^-\left(u, S(u)\right)\right]du$$

$$-(1 - R_C)\int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t, u)\mathbb{E}_t\left[V^+\left(u, S(u)\right)\right]du$$

$$+\int_t^T s_F(u) D_{r+\lambda_B+\lambda_C}(t, u)\mathbb{E}_t\left[V^+\left(u, S(u)\right)\right]du$$

The CVA U can be calculated using $V(t, S)$ as a known source term when solving the PDE in the integral or in the differential form.

4. Derivatives as Cash Account Collateral: In the case where we can use the derivative as a collateral for the funding of our cash account, i.e.,

$$s_F = 0$$

the last term in the expression for $U(t, S)$ above vanishes and the equation reduces to the regular bilateral CVA derived in many papers and books cited earlier, for example Gregory (2009).

5. Zero Funding Cost Credit Valuation Adjustment: The bilateral benefits in this case does not come from any own default, but from being able to use the cash generated from the hedging strategy and buy back own bonds, thus generating an excess return of $(1 - R_B)\lambda_B$. This CVA that corresponds to the case

$$M = V$$

and

$$s_F = 0$$

is denoted U_0.

6. Unsecured Funding Rate Valuation Adjustment: In practice, however, the derivative cannot normally be used as a collateral and

$$U(t,S) = -(1-R_B)\int_t^T \lambda_B(u)D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V^-\big(u,S(u)\big)\right]du$$
$$-(1-R_C)\int_t^T \lambda_C(u)D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V^+\big(u,S(u)\big)\right]du$$
$$+\int_t^T s_F(u)D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V^+\big(u,S(u)\big)\right]du$$

gives a consistent adjustment of the derivative prices for bilateral counterparty risk and funding costs. In the specific case where the funding spread corresponds to that of the unsecured B bond (with recovery R_B), i.e.,

$$s_F = (1-R_B)\lambda_B$$

the first and the third terms may be merged and U may be re-written as

$$U(t,S) = -(1-R_B)\int_t^T \lambda_B(u)D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V\big(u,S(u)\big)\right]du$$
$$-(1-R_C)\int_t^T \lambda_C(u)D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V^+\big(u,S(u)\big)\right]du$$

7. <u>Payable and Receivable Funding Impact</u>: The first and the last term of the previous equation now not only contain the bilateral asset described above, but also the funding liability arising from the fact that the higher rate

$$r_F = r + (1-R_B)\lambda_B$$

is paid when borrowing for the hedging strategy's cash account.

Funding and Default Payoff Examples

1. Funding and Default Payoff Scenarios: In this section the total derivative value $\hat{V}$ for a call option bought by the "seller" is computed in the following 4 cases:
    a. Case I =>

$$M = \hat{V}$$

    and

$$s_F = 0$$

    b. Case II =>

$$M = \hat{V}$$

    and

$$s_F = (1 - R_B)\lambda_B$$

    c. Case III =>

$$M = V$$

    and

$$s_F = 0$$

    d. Case IV =>

$$M = V$$

and

$$s_F = (1 - R_B)\lambda_B$$

2. <u>One-Sided "Bought" Call CVA</u>: A bought call is a one-sided trade that satisfies

$$V \geq 0$$

and

$$\hat{V} \geq 0$$

and furthermore if the rates are constant the CVAs U/U_0 for the four cases above simplifies to

a. Case I =>

$$U_0 = -V(t,S)\left[1 - e^{-(1-R_C)\lambda_C(T-t)}\right]$$

b. Case II =>

$$U = -V(t,S)\left[1 - e^{-\{(1-R_B)\lambda_B+(1-R_C)\lambda_C\}(T-t)}\right]$$

c. Case III =>

$$U_0 = -V(t,S)\frac{(1-R_C)\lambda_C}{\lambda_B+\lambda_C}\left[1 - e^{-(\lambda_B+\lambda_C)(T-t)}\right]$$

d. Case IV =>

$$U = -V(t, S)\frac{(1 - R_B)\lambda_B + (1 - R_C)\lambda_C}{\lambda_B + \lambda_C}\left[1 - e^{-(\lambda_B + \lambda_C)(T - t)}\right]$$

3. CVA and Funding Impact Analysis: Thus all the four CVA's are linear in $V(t, S)$. All are negative since the seller faces counterparty risks and funding costs when

$$s_F = 0$$

but does not have any bilateral asset because of the one-sidedness of the option payoff. Further the effect of the funding cost is significantly larger than choosing

$$M = \hat{V}$$

or

$$M = V$$

for a bought option. For a sold option the impact of the funding cost does not have any effect.

Counterparty Funding and PDE Extensions

1. Liquid Markets CVA/Funding Impact: The valuation adjustments presented here are particularly relevant when pricing interest rate swaps and vanilla options since these markets are very liquid and having an analytical model that does not take fully all costs into account may consume all profits from a deal depending upon the funding and the credit spreads.
2. Derivatives with more General Payouts: Though the examinations above were conducted on a simple one-asset, one-derivative Black Scholes framework, the results can be immediately extended to derivatives with more general payments than $H\big(S(T)\big)$. These could be Asian options or interest rate swaps.

3. Netted Portfolios with Multiple Trades: In this case the vales V and $\hat{V}$ represent the net derivatives portfolio value rather than the value of a single derivative.
4. Generalized Multi-asset Diffusion Dynamics for Multiple Underlyings: The only restriction is that the asset price SDE's satisfy technical conditions such that the option pricing PDE (now multi-dimensional) admits a unique solution given by the Feynman-Kac representation. Note that if the number of assets exceeds two or three it is computationally more efficient to compute the CVA using Monte-Carlo simulation combined with numerical integration rather than solving the high-dimensional PDE.
5. Stochastic Interest Rates: This is essential for interest rate derivatives, and the effect would be that the discounting in the CVA expression would happen inside the expectation operator.
6. Stochastic Hazard Rates: One way of introducing default time dependence and right/wrong way risk would be to make λ_B and λ_C stochastic and correlate them with each other and with other market factors. This would, again, simply imply that we do not move the discount factors outside of the expectation operator in

$$U(t,S) = -(1-R_B)\int_t^T \lambda_B(u)D_r(t,u)\mathbb{E}_t\left[\left\{V\left(u,S(u)\right)+U\left(u,S(u)\right)\right\}^-\right]du$$
$$-(1-R_C)\int_t^T \lambda_C(u)D_r(t,u)\mathbb{E}_t\left[\left\{V\left(u,S(u)\right)+U\left(u,S(u)\right)\right\}^+\right]du$$
$$-\int_t^T s_F(u)D_r(t,u)\mathbb{E}_t\left[\left\{V\left(u,S(u)\right)+U\left(u,S(u)\right)\right\}^+\right]du$$

or in

$$U(t,S) = -(1-R_B)\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V^-\left(u,S(u)\right)\right]du$$
$$-(1-R_C)\int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V^+\left(u,S(u)\right)\right]du$$
$$+\int_t^T s_F(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V^+\left(u,S(u)\right)\right]du$$

Also the generator $\mathcal{A}_t$ would incorporate terms corresponding to the new stochastic state variables.

7. Explicitly Specified Default Time Dependence: Another way of specifying default time dependence is by specifying simultaneous defaults. This could be done by setting J_0, J_1, and J_2 be independent point processes and then setting

$$J_B = J_0 + J_1$$

and

$$J_C = J_0 + J_2$$

This approach is the well-known Marshall-Olkin copula and would require some basket default instrument for perfect replication. The hazard rates λ_0, λ_1, and λ_2 of J_0, J_1, and J_2 could be made stochastic in which case the right and the wrong way risk can be modeled as well.

Balance Sheet and Funding Cost Management

1. Funding Position Dependent Cost Adjustment: In both Piterbarg (2010) and Burgard and Kjaer (2012a) the size of the funding cost adjustment is dependent on the specific way the

funding is achieved and thus gives rise to prices that are dependent on the funding position of the issuer. The counterparty would clear the price with the best funding position.

2. Funding Cost as Seller Windfall: Burgard and Kjaer (2012b) show that the funding cost term is related to the windfall to the issuer's bondholders upon default of the issuer. This leads them to examine the impact of the derivative asset and the funding positions on the balance sheet from within the confines of a simple balance sheet model.
3. Funding Strategy Balance Sheet Impact: They demonstrate that this impact on the balance sheet and the overall funding position of the issuer reduces the effective marginal funding spread for new positions to zero.
4. Strategies for Balance Sheet Impact Neutralization: Burgard and Kjaer (2012b) discuss two strategies for how the balance sheet impact can directly be neutralized, mitigating the need for a funding cost adjustment to a derivatives price. If such strategies can be put into practice, they lead back to a state where the symmetric prices between the issuer and the counterparty can be achieved.

Unified Framework for Bilateral Counterparty Risk and Funding Adjustments

1. CVA, DVA, and FVA Expressions: Burgard and Kjaer (2012b) list the explicit integral representations for the CVA, the DVA, and the FCA.

$$U(t, S) = CVA + DVA + FCA$$

$$CVA = -(1 - R_C) \int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t\left[V^+\left(u, S(u)\right)\right] du$$

$$DVA = -(1 - R_B) \int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t\left[V^-\left(u, S(u)\right)\right] du$$

$$FVA = -\int_t^T s_F(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t\left[V^+\left(u, S(u)\right)\right] du$$

2. Analysis of the XVA Expressions: It is clear from the above expression that while both the CVA and the FVA are related to the credit position of the issuer, they do not double-count the issuer's credit, but capture the exposures of the mark-to-market value of the derivatives of the opposite sign and as such are opposite sides of the same coin. The DVA term itself can be seen as a funding benefit as it arises from the issuer using a positive cash account to buy back its own bonds, earning the spread on it while at the same time hedging out own credit risk on the derivatives position.
3. Own Credit Hedge Using Bonds: It should be noted that the hedging strategy leading to the set of expressions above involves the issuer re-purchasing its own bonds. It does not involve any dealings in the CDS. The term λ_B is the spread of a zero-recovery bond over the risk-free rate. It is not the hazard rate derived from the CDS market. Thus if there is a basis between the bonds and the CDS for the issuer B, it is the bond market that counts in determining the λ_B used in the DVA and the FVA terms.
4. Contribution of U to the Hedges: The contributions of U to the hedge ratios

$$\alpha_B = \frac{U + (1 - R_B)V^-}{P_B}$$

and

$$\alpha_C = \frac{U + (1 - R_C)V^+}{P_C}$$

on the other hand comes from the fact that, upon default the close-out amount of V differs from the risky value of the derivative just prior to the default by the amount U because the credit and the funding adjustments for the trade disappear on default of the counterparty or the issuer. Thus a credit risk on the full amount of U needs to be hedged out and this is achieved by means of taking positions in zero-recovery bonds B and C corresponding to the full value of U.

5. Symmetry between the CVA and the DVA: Without the FVA term the risky value $\hat{V}$ would be symmetric in B and C, i.e., the counterparty and the issuer, following the same methodology, would agree on the same price. If however the funding costs for the replication strategy are included, and the funding spread is non-zero, then the two parties hedging out their risks and pricing in their funding costs would not agree on the price. A counterparty that wants to buy this derivative would buy it from the seller with the smallest FVA value, i.e., the lowest funding cost.
6. Balance Sheet Funding Costs Mitigation: Burgard and Kjaer (2012b) also clarify the origins of the FVA term within the ambit of their framework in more detail, and outline different ways of how the funding costs can be mitigated. Doing so successfully can reduce the FVA cost to zero and produce prices that are independent of the funding costs of the issuer, and therefore symmetric.

Simple Model for the Impact of Derivative Asset on Balance Sheet and Funding

1. Balance Sheet and Funding Model: As discussed earlier, derivatives and their funding positions contribute to the issuer asset and liability positions on the issuer default. Therefore they themselves should impact the funding costs of then issuer. Burgard and Kjaer (2012b) quantify this feedback effect using a simple balance sheet and funding model.
2. Pre-trade Asset and Liability: Assuming that, as in a reduced form credit model, the default of the issuer is driven by an instantaneous default process with default intensity λ. Prior to entering into the derivative contract, let A_0 be the expected assets on default of the issuer and L_0 be the liabilities, so that the expected recovery on default is

$$R_0 = \frac{A_0}{L_0}$$

3. Pre-trade Instantaneous Incremental Funding Cost: Within this simple setup, the funding spread s_F of the issuer over the risk-free rate that compensates for the expected loss upon its default is

$$s_F = (1 - R_0)\lambda$$

Thus the instantaneous funding cost f_0 over the time Δt for a total liability L_0 is

$$f_0 \Delta t = [r + (1 - R_0)\lambda] L_0 \Delta t$$

4. Addition of a Derivative Transaction: Let the seller now add a derivative with a positive value d as an asset, resulting in total assets of

$$A_1 = A_0 + d$$

The positive value d corresponds to $-V^-$ in the previous analysis. The corresponding negative cash is funded by adding a corresponding liability giving a total new liability of

$$L_1 = L_0 + d$$

Thus the new expected recovery is now

$$R_1 = \frac{A_1}{L_1} = \frac{A_0 + d}{L_0 + d}$$

5. Post-Trade Incremental Funding Cost: Assuming that the seller has hedged the market and the counterparty risk, the addition of the derivative asset does not change the default intensity of the seller. Thus the instantaneous funding cost after adding the derivative is

$$f_1\Delta t = [r + (1 - R_1)\lambda]L_1\Delta t = r(L_0 + d)\Delta t + (L_1 - A_1)\lambda\Delta t$$
$$= rL_0\Delta t + rd\Delta t + (L_0 - A_0)\lambda\Delta t = rd\Delta t + rL_0\Delta t + (1 - R_0)\lambda L_0\Delta t$$
$$= rd\Delta t + f_0\Delta t$$

6. Effective Incremental Post-Trade Funding Cost: Thus the effective funding cost for the additional liability d is $rd\Delta t$. While the new liability d draws a new funding spread $(1 - R_1)\lambda$ the change on the recovery and its effect on the funding of the total liabilities results in an effective funding rate for d that is the risk-free rate. Thus within this balance sheet model the spread s_F is zero.
7. Balance Sheet Funding Cost Mitigation: While this balance sheet model is somewhat simplistic, it shows that the proper accounting for the effects of the derivative assets on the balance sheet can mitigate the funding costs and bring the FVA terms down to zero. With a vanishing FVA term, the equation

$$U = CVA + DVA$$

yields an adjustment U and a risky value $\hat{V}$ that are symmetric between the issuer B and the counterparty C.
8. Operational Challenges with the above Approach: Practically, however, the challenge is an operational one in that the benefit of the balance sheet impact is difficult to pin down at the moment of trading and hedging the derivative contract and therefore difficult to allocate as a benefit to the derivatives trading desk.

Balance Sheet Management to Mitigate Funding Costs

1. Issuer Windfall and Liability Matchup: Another way to shield the balance sheet from the impact of both the derivative asset and the funding liability is to actively manage the balance sheet in such a way that the windfall from the derivative asset and the funding position upon default of the issuer is balanced out by a corresponding liability.

2. Multiple Classes of Issuer Bonds: The above can be achieved if the issuer can freely trade two of its own bonds P_1 and P_2 with different recovery rates R_1 and R_2, i.e., different seniority.
3. Dynamics of the Bonds and the Assets: The setup is changed from before in that there are now 4 hedging instruments - P_1, P_2, P_C, and S. All the positive and negative cash in the cash account is invested/raised by buying back and/or issuing P_1/P_2 bonds. The assets follow the dynamics

$$\frac{\Delta P_1}{P_1} = r_1 \Delta t - (1 - R_1) \Delta J_B$$

$$\frac{\Delta P_2}{P_2} = r_2 \Delta t - (1 - R_2) \Delta J_B$$

$$\frac{\Delta P_C}{P_C} = r_C \Delta t - \Delta J_C$$

$$\frac{\Delta S}{S} = \mu \Delta t - \sigma \Delta W$$

where

$$R_1 \in [0, 1)$$

and

$$R_2 \in [0, 1)$$

and

$$R_1 < R_2$$

Neither of the recoveries R_1 or R_2 need equal the derivative recovery rate R_B.

4. Payoff Replication Hedge Portfolio: As before the replicating hedge portfolio Π is setup and given by

$$\Pi = \alpha_1 P_1 + \alpha_2 P_2 + \alpha_C P_C + \alpha_S S + \beta_S + \beta_C$$

where

$$\beta_S = -\alpha_S S$$

is the funding account for the asset position and

$$\beta_C = -\alpha_C P_C$$

is the funding position for the P_C bonds.

5. Application of the Self-Financing Criterion: The fact that Π is meant to be a replicating self-financing hedge portfolio implies that

$$\Pi = \alpha_1 P_1 + \alpha_2 P_2 = -\hat{V}$$

$$\Delta\Pi = -\Delta\hat{V}$$

so repeating the delta-hedging arguments of Burgard and Kjaer (2012a) and defining

$$s_1 = r_1 - r$$

and

$$s_2 = r_2 - r$$

yields

$$\alpha_S = -\frac{\partial \hat{V}}{\partial S}$$

$$\alpha_1(1 - R_1)P_1 + \alpha_2(1 - R_2)P_2 = \mathcal{D}\hat{V}_B$$

$$\alpha_C P_C = \mathcal{D}\hat{V}_C$$

$$\alpha_1 s_1 P_1 + \alpha_2 s_2 P_2 + \alpha_C \lambda_C P_C = -\frac{\partial \hat{V}}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \hat{V}}{\partial t^2} + r\hat{V} - (q_S - \gamma_S) S \frac{\partial \hat{V}}{\partial S}$$

where

$$\mathcal{D}\hat{V}_B = \hat{V}(t, S, 1, 0) - \hat{V}(t, S, 0, 0)$$

and

$$\mathcal{D}\hat{V}_C = \hat{V}(t, S, 0, 1) - \hat{V}(t, S, 0, 0)$$

with $\hat{V}(t, S, 1, 0)$ and $\hat{V}(t, S, 0, 1)$ given by

$$\hat{V}(t, S, 1, 0) = M^+(t, S) + R_B M^-(t, S)$$

and

$$\hat{V}(t, S, 0, 1) = M^-(t, S) + R_C M^+(t, S)$$

with

$$M = V$$

6. Hedge Ratios and Consolidated PDEs: From the above equations one can determine α_1 and α_2 to be

$$\alpha_1 = -\frac{R_2\hat{V} - V^+ - R_B V^-}{(R_2 - R_1)P_1}$$

$$\alpha_2 = -\frac{-R_1\hat{V} + V^+ + R_B V^-}{(R_2 - R_1)P_2}$$

which implies the following pricing PDE:

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - r\hat{V} = s_1\frac{R_2\hat{V} - V^+ - R_B V^-}{R_2 - R_1} + s_2\frac{-R_1\hat{V} + V^+ + R_B V^-}{R_2 - R_1} - \lambda_C\left(V^- + R_C V^+ - \hat{V}\right)$$

7. Zero Bond Funding Spread Basis: If furthermore one assumes zero differential between the bond basis and the funding spread, i.e.,

$$s_1 = (1 - R_1)\lambda_B$$

and

$$s_2 = (1 - R_2)\lambda_B$$

then the previous equation simplifies to

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - (r + \lambda_B + \lambda_C)\hat{V} = -(R_B\lambda_B + \lambda_C)V^- - (\lambda_B + R_C\lambda_C)V^+$$

8. Issue Senior and Repurchase Junior: Comparing this PDE with

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - (r + \lambda_B + \lambda_C)\hat{V} = -(R_B\lambda_B + \lambda_C)V^- - (\lambda_B + R_C\lambda_C)V^+ + s_F V^+$$

implies financing of the negative cash account at vanishing spread

$$s_F = 0$$

In practice the strategy involves issuing the senior P_2 bonds and using some of the proceeds to re-purchase the junior and hence higher-yields P_1 bonds.

9. Excess Return Balancing the Funding Costs: The excess return generated by this strategy exactly offsets the funding costs so the net financing rate becomes r. At the same time the combined positions of the P_1 and the P_2 bonds ensures there is no windfall to the bondholders in the case of default of the issuer while V is positive (and the cash account negative).
10. Junior/Senior Positions at Default: These are shown in the table below which summarizes the total bond positions at the issuers own default and which partially offsets the value of the derivative defined in

$$\hat{V}(t, S, 1, 0) = M^{+}(t, S) + R_B M^{-}(t, S)$$

and

$$\hat{V}(t, S, 0, 1) = M^{-}(t, S) + R_C M^{+}(t, S)$$

P_1 Position Value	$\frac{R_1 R_2 \hat{V} - R_1 V^{+} - R_1 R_B V^{-}}{R_2 - R_1}$
P_2 Position Value	$-\frac{-R_2 R_1 \hat{V} + R_2 V^{+} + R_2 R_B V^{-}}{R_2 - R_1}$
Total Position Value	$-(V^{+} + R_B V^{-})$

11. Default Windfall Balancing out Funding Costs: Thus if the issuer is able to offset the impact of the derivative and its funding on the balance with combination of going long senior bonds

and shot junior bonds, then the windfall is effectively monetized while the issuer is alive, and by doing so the funding cost term is reduced to zero.

Funding Strategies and Costs Impact

1. Abstract: The economic values of derivatives depends on their funding costs, because they can result in windfalls or shortfalls to bondholders on their firm's default. But this depends on not just who is funding them, but how – so the resulting adjustments depend on the funding strategy deployed. It is another layer of complexity to derivatives pricing, as argued by Burgard and Kjaer (2013).
2. Derivative ITM vs OTM Funding: Assuming that the issuer can hedge its own default when the derivative position is in the money and so provides funding is more realistic than assuming that the issuer can freely dynamically trade spread positions on its own bonds. When the derivative is out of the money and the issuer requires funding, a post-default windfall to the issuer's estate is generated. In that case a funding cost adjustment (FCA) is added in to compensate. There have been a flurry of papers prosing alternative approaches from different authors, including Morini and Prampolini (2012), Brigo, Pallavicini, and Perini (2012), Crepey (2013a, 2013b), and perhaps most famously Hull and White (2012a) that use risk-neutral valuation principles to examine the question.
3. Classical Price with Bilateral CVA: Such an approach, discounting all expected cash flows at the risk free rate results in the classical price with the bilateral CVA. But this disregards the preferences of the different stake holders regarding the value of the pre- and post- own-default cash flows. This is justified if all the risks – including own default – are hedgeable so that net post-default cash flows are zero. But, as mentioned, they are not.
4. Impact on the Shareholders: Shareholders are primarily interested in the pre-default cash flows of derivatives and their hedges, but post-default cash flows matter for bondholders, as they contribute to the recovery realized. Shareholders only care about the latter through the balance sheet effects that are in practice hard to realize and account for.
5. Selective Disregarding of Default Cash flows: Some authors have considered cases where the post-default cash flows on the funding leg are disregarded, but not the ones of the derivative.

But it is not clear why some post-default cash flows should be disregarded but not others, and without specifying the funding strategies, the resulting recursive relations cannot be solved.

6. Crepey's Generalization Approach: Crepey's generalization of the original Markovian framework to non-Markovian processes using backward stochastic equations is elegant, but difficult to solve explicitly.
7. Selective Hedging of Default Cash Flows: Burgard and Kjaer (2013) look at funding strategies in terms of holding or issuing own bonds. The strategies hedge out some but not all cash flows at own default. The economic value of a derivative to the shareholders is then given by assuming that they disregard any remaining post-default cash flows and pre-default balance sheet effects.
8. Implementation of the Custom Funding Strategies: The funding cost adjustment is then given by the discounted expected value of the post-default cash flows. The strategies previously considered are then special cases of those considered by Burgard and Kjaer (2013), thus generalizing the previous work. Dealers can consider their own strategies and decide which adjustments represent the economic funding costs they expect to change while in business.
9. CSA Variants and Set-offs: Burgard and Kjaer (2013) also consider boundary conditions covering practical cases such as one-way or two-way credit support annexes (CSAs) governing collateral agreements and the so called set-offs. Set-offs are particularly interesting as they mitigate the need for funding cost adjustments. Their explicit calculations and numerical results show that different funding strategies can yield quiet different funding adjustments and asymmetries in the absence of set-of provisions.

Generalized Semi-Replication and Pricing PDE

1. Imperfect Own Credit Hedge: Consider a derivative contract, possibly collateralized, between an issuer B and a counterparty C with an economic value $\hat{V}$ that incorporates the risk of the counterparty and the issuer and any net funding costs the issuer may encounter prior to own default. This section describes a general semi-replication strategy that the issuer can deploy to perfectly hedge out any market factors and counterparty default, but which may provide a perfect hedge in the event of the issuer's "own" default.

2. Portfolio of Tradeable Instruments: The tradeable instruments used in this strategy are a counterparty zero-coupon zero-recovery bond P_C, two issuer "own" bonds P_1 and P_2 of different seniorities, i.e., different recoveries R_1 and R_2 respectively, and a market instrument S that can be used to hedge out the market factor for the derivative contract (e.g., stock).
3. Underlying Assets and Market Factors Dynamics: The setup can be easily extended to many market factors. The following standard dynamics for these instruments are assumed

$$i = 1,2$$

$$\Delta S = \mu S \Delta t + \sigma S \Delta W$$

$$\Delta P_C = r_C {P_C}^{-} \Delta t - {P_C}^{-} \Delta J_C$$

$$\Delta P_i = r_i {P_i}^{-} \Delta t - (1 - R_i) {P_i}^{-} \Delta J_B$$

where J_B and J_C are the default indicators for B and C respectively, and

$${P_{i/C}}^{-} = P_{i/C}(t^{-})$$

are the pre-default bond prices.
4. Risk-Neutral Bond-Funding Spread: Without loss of generality, P_1 is the junior bond, i.e.,

$$R_1 < R_2$$

and

$$r_1 > r_2$$

In case of zero basis between bonds of different seniority, it is trivial to show that

$$r_i - r = (1 - R_B)\lambda_B$$

where r is the risk-free rate, λ_B corresponds to the spread of a potentially hypothetical zero-recovery zero-coupon bond of the issuer.

5. Generalized Boundary Conditions at Default: Let $\hat{V}(t, S, J_B, J_C)$ be the total economic value of the derivative to the issuer. Using Kjaer (2011) the general boundary conditions at default of the issuer or the counterparty are given by

$$\hat{V}(t, S, 1, 0) = g_B(M_B, X)$$

if B defaults first, and

$$\hat{V}(t, S, 0, 1) = g_C(M_C, X)$$

if C defaults first with general close-out amounts M_B and M_C, and collateral X.

6. Collateral Extensions to Boundary Conditions: If

$$M_B = M_C = V$$

these boundary conditions are called “regular”, where V is the classic Black-Scholes price of the derivative, i.e., without counterparty and own-default risks and no funding costs. For example the regular bilateral boundary conditions with collateral are defined as

$$g_B = (V - X)^+ + R_B(V - X)^- + X$$

and

$$g_C = R_C(V - X)^+ + (V - X)^- + X$$

7. Extensions to Alternate Close-outs: Burgard and Kjaer (2012a) consider alternate close-out cases for

$$M_B = M_C$$

and Brigo and Morini (2011) extend this to the cases where

$$M_B \neq M_C$$

and include the cost of funding in the close-out amounts. Separately Burgard and Kjaer (2012c) apply the present framework to such funding aware close-outs.

8. Regular Bilateral Uncollateralized Close-outs: An example for the close-out functions are the regular bilateral close-outs with collateral, which are described by

$$g_B = V^+ + R_B V^-$$

and

$$g_C = R_C V^+ + V^-$$

Later other examples such as one-way CSA and set-offs are examined.

Semi-Replication

1. The Semi-replication Hedge Portfolio: For the semi-replication the hedge portfolio Π is set up as

$$\Pi(t) = \alpha_S(t)S(t) + \alpha_1(t)P_1(t) + \alpha_2(t)P_2(t) + \alpha_C(t)P_C(t) + \beta_S(t) + \beta_C(t) - X(t)$$

with $\alpha_S(t)$ units of $S(t)$, $\alpha_{1/2}(t)$, and $\alpha_C(t)$ units of own and counterparty bonds respectively, cash accounts $\beta_S(t)$ and $\beta_C(t)$, and a collateral account $X(t)$.

2. The Asset Financing Cash Accounts: The cash accounts β_S and β_C are used to finance the S and the P_C positions, i.e.,

$$\alpha_C P_C + \beta_C = 0$$

and

$$\alpha_S S + \beta_S = 0$$

and assumed to pay net rates of $(q_S - \gamma_S)$ and q_C respectively, where γ_S may be the dividend income. The hedge positions may be collateralized or repo'ed, so q_S and q_C maybe the collateral or the repos rates respectively.

3. The Replication Portfolio Funding Constraint: The derivative collateral balances X are assumed to be fully re-hypothecable and pay the amount r_X and

$$X > 0$$

corresponds to the counterparty having posted the amount X with the issuer. The strategy shall be designed such that

$$\hat{V} + \Pi = 0$$

except possibly at the issuer default. The issuer bond positions $\alpha_1 P_1$ and $\alpha_2 P_2$ are used to finance/invest any remaining cash that is not funded via the collateral, which yields the following *funding constraint*

$$\hat{V} - \Pi + \alpha_1 P_1 + \alpha_2 P_2 = 0$$

4. Evolution of the Derivative Hedge Portfolio: The evolution of the hedge portfolio Π defined in

$$\Pi = \alpha_S S + \alpha_1 P_1 + \alpha_2 P_2 + \alpha_C P_C + \beta_S + \beta_C - X$$

is given by

$$\Delta\bar{\Pi} = \alpha_S \Delta S + \alpha_1 \Delta P_1 + \alpha_2 \Delta P_2 + \alpha_C \Delta P_C + \Delta\bar{\beta}_S + \Delta\bar{\beta}_C - \Delta\bar{X}$$

where ΔS, ΔP_1, ΔP_2, and ΔP_C are given earlier and $\Delta\bar{\beta}_S$, $\Delta\bar{\beta}_C$, and $\Delta\bar{X}$ are changes to the cash and the collateral accounts, excluding rebalancing.

5. Funding Costs for Cash Accounts: As in Burgard and Kjaer (2012a) the hedge account β_S is collateralized with the financing rate q_S and income via dividend γ_S.
6. Growth in the Cash Accounts: Likewise the counterparty bond position is assumed to be setup via a repo transaction costing a repo rate q_C. The derivative collateral account is assumed to cost a collateral rate r_X. Excluding rebalancing these yield the following increments in the accounts:

$$\Delta\bar{\beta}_S = \alpha_S S(\gamma_S - q_S)\Delta t$$

$$\Delta\bar{\beta}_C = -\alpha_C q_C P_C \Delta t$$

$$\Delta\bar{X} = -r_X X \Delta t$$

7. Incremental Change in the Derivative Portfolio: With the pre- and the post-default values of the issuer bond position given by

$$P = \alpha_1 P_1 + \alpha_2 P_2$$

and

$$P_D = \bar{R}_1 \alpha_1 P_1 + \bar{R}_2 \alpha_2 P_2$$

respectively, inserting

$$\Delta S = \mu S \Delta t + \sigma S \Delta W$$

$$\Delta P_C = r_C P_C{}^- \Delta t - P_C{}^- \Delta J_C$$

and

$$\Delta P_i = r_i P_i{}^- \Delta t - (1 - R_i) P_i{}^- \Delta J_B$$

and the expressions for $\Delta\bar{\beta}_S$, $\Delta\bar{\beta}_C$, and $\Delta\bar{X}$ into

$$\Delta\bar{\Pi} = \alpha_S \Delta S + \alpha_1 \Delta P_1 + \alpha_2 \Delta P_2 + \alpha_C \Delta P_C + \Delta\bar{\beta}_S + \Delta\bar{\beta}_C - \Delta\bar{X}$$

results in

$$\Delta\bar{\Pi} = [r_1 \alpha_1 P_1 + r_2 \alpha_2 P_2 + \lambda_C \alpha_C P_C + \alpha_S S(\gamma_S - q_S) - r_X X]\Delta t + (P_D - P)\Delta J_B - \alpha_C P_C \Delta J_C + \alpha_S \Delta S$$

where

$$\lambda_C \equiv r_C - q_C$$

is the spread of the zero-coupon bond price P_C yield over its repo rate, i.e., the financing rate of the counterparty default position.

8. Ito's Lemma for the Derivative Contract: The evolution of the derivative $\Delta\hat{V}$ on the other hand is given by Ito's lemma for jump diffusions as

$$\Delta\hat{V} = \frac{\partial\hat{V}}{\partial t}\Delta t + \frac{\partial\hat{V}}{\partial S}\Delta S + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2\hat{V}}{\partial t^2}\Delta t + \mathcal{D}\hat{V}_B \Delta J_B + \mathcal{D}\hat{V}_C \Delta J_C$$

with

$$\mathcal{D}\hat{V}_B = \hat{V}(t, S, 1, 0) - \hat{V}(t, S, 0, 0) = g_B - \hat{V}$$

and

$$\mathcal{D}\hat{V}_C = \hat{V}(t, S, 0, 1) - \hat{V}(t, S, 0, 0) = g_C - \hat{V}$$

9. <u>The Consolidated Derivative Portfolio Increment</u>: Combining the evolution of the derivative in

$$\Delta\hat{V} = \frac{\partial\hat{V}}{\partial t}\Delta t + \frac{\partial\hat{V}}{\partial S}\Delta S + \frac{1}{2}\sigma^2 S^2\frac{\partial^2\hat{V}}{\partial t^2}\Delta t + \mathcal{D}\hat{V}_B\Delta J_B + \mathcal{D}\hat{V}_C\Delta J_C$$

with the hedge portfolio

$$\Delta\overline{\Pi} = [r_1\alpha_1P_1 + r_2\alpha_2P_2 + \lambda_C\alpha_CP_C + \alpha_SS(\gamma_S - q_S) - r_XX]\Delta t + (P_D - P)\Delta J_B - \alpha_CP_C\Delta J_C + \alpha_S\Delta S$$

gives

$$\Delta\hat{V} + \Delta\overline{\Pi} = \left[\frac{\partial\hat{V}}{\partial t} + \frac{1}{2}\sigma^2S^2\frac{\partial^2\hat{V}}{\partial t^2} + r_1\alpha_1P_1 + r_2\alpha_2P_2 + \lambda_C\alpha_CP_C + \alpha_SS\frac{\partial\hat{V}}{\partial S}(\gamma_S - q_S) - r_XX\right]\Delta t + (g_B + P_D - X)\Delta J_B + \left(\mathcal{D}\hat{V}_C - \alpha_CP_C\right)\Delta J_C + \left(\frac{\partial\hat{V}}{\partial S} + \alpha_S\right)\Delta S$$

where the term in front of ΔJ_B follows from the fact that

$$\hat{V} - P + X = 0$$

from the funding constraint

$$\hat{V} - X + \alpha_1P_1 + \alpha_2P_2 = 0$$

10. <u>Asset and Counterparty Hedges</u>: From the derivatives portfolio increment we can eliminate the stock price and the counterparty risks by choosing

$$\mathcal{D}\hat{V}_C = \alpha_C P_C$$

and

$$\alpha_S = -\frac{\partial \hat{V}}{\partial S}$$

which yields

$$\Delta\hat{V} + \Delta\overline{\Pi} = \left[\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t\hat{V} + r_1\alpha_1 P_1 + r_2\alpha_2 P_2 + \lambda_C\mathcal{D}\hat{V}_C - r_X X\right]\Delta t + (g_B + P_D - X)\Delta J_B$$

where

$$\mathcal{A}_t\hat{V} = \frac{1}{2}\sigma^2 S^2\frac{\partial^2 \hat{V}}{\partial t^2} + \alpha_S S\frac{\partial \hat{V}}{\partial S}(\gamma_S - q_S)$$

11. <u>Incorporating the Hedge Ratios</u>: On using the zero bond basis relation

$$r_i - r = (1 - R_B)\lambda_B$$

$$\hat{V} - X + \alpha_1 P_1 + \alpha_2 P_2 = 0$$

and

$$\mathcal{D}\hat{V}_C = \alpha_C - \hat{V}$$

the above becomes

$$\Delta\hat{V} + \Delta\overline{\Pi} = \left[\frac{\partial\hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - (r + \lambda_B + \lambda_C)\hat{V} - s_X X + \lambda_B g_B + \lambda_C g_C - \epsilon_h\lambda_B\right]\Delta t + \epsilon_h\Delta J_B$$

where

$$s_X = r_X - r$$

and

$$\epsilon_h = g_B + P_D - X$$

12. Own-Credit Default Hedging Error: From the jump term above it follows that upon issuer default there is a hedge error of size ϵ_h. While alive, on the other hand, the issuer correspondingly incurs a cost/gain of size $-\epsilon_h\lambda_B$ per unit time.
13. Hedge Error Windfall and Shortfall: It can thus be seen that the combination of the derivative $\hat{V}$ and the hedge portfolio Π is risk free as long as the issuer is alive. At issuer default the jump term $\epsilon_h\Delta J_B$ gives rise to a hedge error of size ϵ_h. The hedge error can be a windfall or a shortfall and its size depends on the post-default value of the own bond portfolio, and thus the funding strategy employed.
14. Windfall/Shortfall Accrual Gain/Bleed: While alive, on the other hand, the issuer correspondingly incurs a cost/gain of size $-\epsilon_h\lambda_B$ per unit time. Tis can be seen as the running spread to pay for the potential windfall/shortfall upon issuer default.
15. Collateral/Hedge Error Derivative PDE: Since the issuer wants the strategy to evolve in a self-financing fashion while he is alive the total drift term above should become zero. This produces the following PDE for the risky economic value $\hat{V}$ for the derivative:

$$\frac{\partial\hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - (r + \lambda_B + \lambda_C)\hat{V} = s_X X - \lambda_B g_B - \lambda_C g_C + \epsilon_h\lambda_B$$

$$\hat{V}(T, S) = H(S)$$

where $H(S)$ is the payout of the derivative at maturity.

16. PDE for the Valuation Adjustment: To estimate the correction

$$U = \hat{V} - V$$

to the risk-free Black Scholes price V the Black-Scholes PDE for V is used to get the PDE for U to be

$$\frac{\partial U}{\partial t} + \mathcal{A}_t U - (r + \lambda_B + \lambda_C)U = s_X X - \lambda_B(g_B - V) - \lambda_C(g_C - V) + \epsilon_h \lambda_B$$

$$U(T, S) = H(S)$$

17. Decomposition of U into Components: Applying the Feynman-Kac theorem to this PDE gives

$$U = CVA + DVA + FVA + COLVA$$

with

$$CVA = -\int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t\left[V(u) - g_C\left(V(u), X(u)\right)\right] du$$

$$DVA = -\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t\left[V(u) - g_B\left(V(u), X(u)\right)\right] du$$

$$FCA = -\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t[\epsilon_h(u)] du$$

$$COLVA = -\int_t^T s_X(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t[X(u)] du$$

$$D_k(t,u) = e^{-\int_t^u k(v)dv}$$

is the discount factor between t and u for a rate k. The measure of the expectations in these equations is such that S drifts at the rate $q_S - \gamma_S$. The sum of DVA and FCA is sometimes referred to as FVA.

18. Symmetry of CVA, DVA, and $COLVA$: Here the sum of the CVA, the DVA, and the $COLVA$ is symmetric in that is identical – with sign flipped – when computed by the issuer and the counterparty, respectively.
19. Lack of Symmetry in FCA: The $FCA - \int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t[\epsilon_h(u)] du$ on the other hand is the discounted survival probability weighted expected value of the hedge error ϵ_h implied by the semi-replication strategy chosen. Because the hedge error on own default is different for the issuer and for the counterparty, the FCA is not symmetric. This is a generalization of the result presented in Burgard and Kjaer (2012b) for regular bilateral close-outs and a particular choice of bonds that states that the FCA is the cost of generating a windfall to the issuer bondholders in the case of a default. If the issuer wants to break-even while being alive, the cost has to be included in the derivative price charged to the counterparty.
20. Asset Addition Funding Rate Impact: The analysis above assumes that the funding rates r_1 and r_2 remain unaffected by the addition of the derivative asset and the funding positions. Burgard and Kjaer (2012b) have noted that the presence of potential windfall has a positive balance sheet effect; it improves the recovery rate to the bondholders and therefore the funding spread of the issuer should go down. Hull and White (2012a) have used a similar argument.
21. Balance Sheet Funding Cost Mitigation: For a simple balance sheet model with floating funding costs Burgard and Kjaer (2012b) have demonstrated that this effect can result in an effective marginal funding rate that corresponds to the risk-free rate.

22. Practical Challenges with the Mitigation: However as discussed there, in practice the balance sheet effect on the funding costs is rather indirect, fraught with accounting issues, and in general only feed through over time. Therefore the current treatment assumes that the issuer disregards this effect.

Examples of Different Bond Portfolios

1. Strategies Generating Different Hedge Errors: Using the general framework developed above Burgard and Kjaer (2013) provide three different examples of semi-replication strategies that generate different hedge errors ϵ_h and therefore different valuation adjustments.
2. Zero FCA and Windfall Only: The first strategy, if employed, allows for perfect replication and generates zero FCA. The second one is equivalent to the setup used in Burgard and Kjaer (2012a) for the bilateral close-outs and ensures that there is never a shortfall at issuer default and only a windfall (potentially).
3. Windfall or Shortfall Generation Strategy: The third strategy assumes hedging with a single issuer bond. It generates both potential windfall and potential loses post-default and is an extension of the model derived in Piterbarg (2010).
4. Strategy Economic Value and Adjustments: The different strategies generate different economic values – and therefore different adjustments – to the issuer while he is alive. They demonstrate the assumptions implicitly being made when using different adjustment formulas in practice.
5. Bilateral and Funding Curve Close-outs: Throughout it is assumed that the close-out value is V, i.e., that

$$M_B = M_C = V$$

Funding aware close-outs are discussed separately in Burgard and Kjaer (2012c).

Perfect Replication – The FCA Vanishes

1. Hedging Windfalls and Shortfalls: The first case considered is the one where the issuer is able to perfectly hedge out the windfall/shortfall at own default. This corresponds to the case discussed in the Section *Balance-Sheet Management to Mitigate Funding Costs* in Burgard and Kjaer (2012b) and also covers the risk-neutral approach outlined in Hull and White (2012a).
2. PDE Corresponding to the Perfect Hedge: Perfect hedge is equivalent to the hedge error ϵ_h being zero, i.e.,

$$g_B + P_D - X = 0$$

The valuation PDE

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C)\hat{V} = s_X X - \lambda_B g_B - \lambda_C g_C + \lambda_B \epsilon_h$$

becomes

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C)\hat{V} = s_X X - \lambda_B g_B - \lambda_C g_C$$

$$\hat{V}(T, S) = H(S)$$

Correspondingly the FCA given by

$$FCA = \int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t[\epsilon_h(u)] du$$

vanishes.

3. Corresponding Own Portfolio Hedge Ratios: The hedge ratios α_1 and α_2 that achieve this perfect replication are determined by the no-windfall condition

$$g_B + P_D - X = 0$$

and the funding constraint

$$\hat{V} - X + \alpha_1 P_1 + \alpha_2 P_2 = 0$$

These two conditions provide two equations that can be solved to find

$$\alpha_1 = \frac{R_2 \hat{V} - g_B + (1 - R_2) X}{(R_1 - R_2) P_1}$$

$$\alpha_2 = \frac{R_1 \hat{V} - g_B + (1 - R_1) X}{(R_2 - R_1) P_2}$$

4. Valuation Adjustment Feynman-Kac Integrals: As an example, for the regular bilateral conditions

$$g_B = (V - X)^+ + R_B (V - X)^- + X$$

and

$$g_C = R_C (V - X)^+ + (V - X)^- + X$$

these adjustments specialize to

$$CVA = -(1 - R_C) \int_t^T \lambda_C(u) D_{r + \lambda_B + \lambda_C}(t, u) \mathbb{E}_t[\{V(u) - X(u)\}^+] du$$

$$DVA = -(1 - R_B) \int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t[\{V(u) - X(u)\}^-] du$$

with

$$FCA = 0$$

and

$$COLVA = 0$$

if

$$r_X = r$$

5. Case of Classical Bilateral CVA: If the derivatives are not collateralized, i.e.,

$$X = 0$$

these adjustments correspond to the classical bilateral CVA. Thus the classical bilateral CVA can be achieved when perfect replication is possible. As mentioned, in practice such a dynamic balance sheet management via actively traded spread options between junior and senior bonds is in general not a viable option.

Semi-Replication with No Shortfall at Own-Default

1. Collateralized without Shortfall at Default: Burgard and Kjaer (2013) demonstrate a bond portfolio and hedging strategy that constitute an equivalent to the one presented in Burgard

and Kjaer (2012b) extended to more general conditions, including the possibility of collateral.

2. Dynamic Trading of Two Bonds: It still involves dynamic trading of two bonds, but is more conservative than the dynamic trading strategy of the previous section as it does not aim at monetizing the potential windfall upon own default by entering into an offsetting position between the two bonds.
3. Bilateral CVA Plus a DVA: While this generates potential windfalls at own default, it does not generate shortfalls, and has the additional advantage that for regular bilateral close-outs without collateral it results in the usual bilateral CVA adjustment plus a funding cost adjustment, so presents a simple extension to the existing framework, where the derivatives dealer does not think he can monetize the windfall.
4. P_1/P_2 Buy-Sell Strategy: The strategy involves a zero recovery bond P_1 with

$$R_1 = 0$$

and a recovery bond

$$R_2 = R_B$$

The issuer runs the following bond positions:

   a. Invest of fund the difference between $\hat{V}$ and V by buying or issuing P_1 bonds.
   b. Hold the number of P_2 bonds given the following funding constraint:

$$\hat{V} - X + \alpha_1 P_1 + \alpha_2 P_2 = 0$$

5. Corresponding Own-Portfolio Hedge Ratios: The strategy is this defined by the following values of α_1 and α_2:

$$\alpha_1 P_1 = -\left(\hat{V} - V\right) = U$$

$$\alpha_2 P_2 = -\alpha_1 P_1 - \hat{V} + X = -(V - X)$$

6. Collateral Adjusted Junior Bond Hedge: The strategy is thus symmetric between positive and negative funding, and the risk-free value V not covered by the collateral X is funded/invested via own unsecured bonds with recovery R_B. Only the adjustment U, which falls away on own-default, is funded/invested via a zero-recovery bond. As such this strategy looks more palatable from a regulatory and accounting perspective than the perfect replication strategy which attempts to actively extract the funding spread from the balance sheet by issuing own senior bonds to buy back own junior bonds.
7. Regular Bilateral Close-out Case: This bond portfolio is equivalent to the one described in Burgard and Kjaer (2012a, 2012b) for the bilateral close-outs considered there. This section analyzes the setup in more detail for the case of regular bilateral close-out given in

$$g_B = (V - X)^+ + R_B(V - X)^- + X$$

and

$$g_C = R_C(V - X)^+ + (V - X)^- + X$$

For this case the hedge error ϵ_h specializes to

$$\epsilon_h = (1 - R_B)(V - X)^+$$

which is always a windfall – possibly zero – to the bondholders of the issuer.
8. Zero-Recovery Own Bond Off-setter: Therefore this strategy is characterized by the ability of the issuer to perfectly hedge out the difference U between the risky value of the derivative before own default and the close-out amount after own default by means of trading own bonds at zero-recovery.
9. R_B Recovery Own Bond Balance: The remainder, i.e., the difference between the close-out and the collateral is invested/funded using own bonds with recovery R_B.This part generates the windfall ϵ_h to the issuer's bondholders when $V - X$ is in the money and the issuer defaults.

10. <u>CVA, DVA, FCA, and $COLVA$</u>: The CVA, the DVA, the FCA, and the $COLVA$ specialize to

$$CVA = -(1 - R_C) \int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t[\{V(u) - X(u)\}^+] du$$

$$DVA = -(1 - R_B) \int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t[\{V(u) - X(u)\}^-] du$$

$$FCA = -(1 - R_B) \int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t[\{V(u) - X(u)\}^+] du$$

and

$$COLVA = -\int_t^T s_X(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t[X(u)] du$$

11. <u>Consolidation of the DVA and the FCA</u>: It is possible to combine the DVA (a funding benefit) and FCA (a funding cost) into a funding value FVA as

$$FVA = DVA + FCA = -(1 - R_B) \int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t[V(u) - X(u)] du$$

Note that Hull and White (2012a, 2012b) refer to this FCA as FVA. Burgard and Kjaer (2013) use the notation that is consistent with their previous papers and with Gregory (2012).

12. <u>Uncollateralized Classical Bilateral CVA/DVA</u>: For uncollateralized derivatives

$$X = 0$$

and the $COLVA$ term vanishes. The CVA and the DVA then correspond to the classical bilateral CVA. The FCA term provides the funding cost adjustment on top.

13. Gold-Plated Two-Way CSA: For gold-plated two-way CSAs where

$$X = V$$

the CVA, the DVA, and the FVA terms are all zero. If the collateral rate is the risk-free rate, then the $COLVA$ – which represents the spread earned by the issuer – also vanishes. In this case the risky price $\hat{V}$ of the derivative becomes the risk-free price V as well. The intuition is that the collateral cash is exactly what is needed to fund the hedge and eliminate all counterparty risk and as a consequence

$$\alpha_1 = \alpha_2 = 0$$

This corresponds to the result for fully collateralized trades of Piterbarg (2010).

14. One-Way CSA Issuer Posting: Another special case worth considering is that of a one-way CSA whereby the issuer only posts collateral when the risk-free value of the trade is out-of-the-money, i.e.,

$$X = V^-$$

One-way CSAs are common when the issuer trades with sovereign counterparties or with sovereign-like public entities.

15. One-Way CSA Valuation Adjustments: In this case the adjustments specialize to

$$CVA = -(1 - R_C) \int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t[V(u)^+] du$$

$$FCA = -(1 - R_B) \int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t[V(u)^+] du$$

and

$$COLVA = -\int_t^T s_X(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t[V(u)^-] du$$

Unsurprisingly the introduction of one-way CSA makes the DVA vanish while leaving the CVA and the FCA terms unchanged.

16. Implication of One-Way CSA: Unlike the uncollateralized case any cash available when the derivative is out-of-the-money must be handed over as collateral and thus cannot be used to generate a funding benefit. And unlike two-way CSA there is no influx of collateral cash that can be used to fund the hedge when the derivative is in-the-money. The issuer is thus faced with the uncollateralized and the 2-way cases, and need to charge a higher price to the counterparty to compensate for that in order to break even.

Set-offs

1. Definition of the Set-off Mechanism: It is also instructive to study the case of the so-called set-offs. A set-off is a legal agreement that allows the surviving party to settle the outstanding derivative claims of the defaulting party by means of supplying the bonds of the defaulting party at nominal value rather than cash.
2. Boundary Conditions for Set-offs: Since post-default these bonds trade at their recoveries, this type of settlement is valuable to the surviving party. Explicitly for regular bilateral set-offs without collateral, the boundary conditions are given by

$$g_B = R_B V$$

and

$$g_C = R_C V$$

Inserting this into

$$\alpha_1 P_1 = -U$$

and

$$\alpha_2 P_2 = -(V - X)$$

implies that the hedge error ϵ_h disappears.

3. Set-offs CVA and DVA: The CVA and the DVA adjustments in this case are given by

$$CVA = -(1 - R_C) \int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t[V(u)] du$$

$$DVA = -(1 - R_B) \int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t[V(u)] du$$

4. Issuer and Counterparty Price Symmetry: Significantly with

$$\epsilon_h = 0$$

the FCA term vanishes resulting in symmetric prices between the issuer and the counterparty. Thus adoption of close-outs would mitigate the need for economic funding for adjustments.

Semi-Replication with a Single Bond

1. Strategy P_1/P_2 Hedge Ratios: This section considers a very simple strategy where the issuer uses a single own bond with recovery R_F, i.e.,

$$\alpha_1 P_1 = 0$$

and

$$\alpha_2 P_2 = -(\hat{V} - X) = -(V + U - X)$$

where the second line follows from the funding constraint

$$\hat{V} - X + \alpha_1 P_1 + \alpha_2 P_2 = 0$$

For aesthetic reasons the remaining bond P_F and its yield are relabeled

$$r_F = r + s_F$$

The hedge ratios imply that the issuer raises all necessary net cash by issuing P_F-bonds and invests any surplus net cash by repurchasing the same bonds.

2. Insufficient Hedge Degrees of Freedom: With a single bond, once the funding constraint is fulfilled, there are no degrees of freedom left for the issuer to hedge out his own default. This is in contrast to the previous setup of strategy I where the issuer is able to hedge out its own default risk when the trade is out-of-the-money at least.
3. Hedge Error Produced by the Strategy: For the own bond portfolios with the hedge ratios above the hedge error ϵ_h amount to

$$\epsilon_h = g_B + P_D - X = g_B - R_F \hat{V} - (1 - R_F) X$$

where the default value of the own bond portfolio given by

$$P_D = -R_F(\hat{V} - X)$$

has been used.

4. Recursive Setup Formulation for the FCA: Using this hedge error in the general adjustment expression

$$FCA = -(1 - R_B) \int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t[\epsilon_h(u)] du$$

yields a recursive relationship. This is because

$$\hat{V} = V + U$$

appearing on the RHS of the expression for ϵ_h above includes the contribution from U and thus the FCA itself.

5. Elimination of Recursion ith the Original PDE: The best way to deal with this situation is by going back to the PDE

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C) \hat{V} = s_X X - \lambda_B g_B - \lambda_C g_C + \lambda_B \epsilon_h$$

and insert the hedge error above to obtain

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r_F + r_C) \hat{V} = -(r_F - r_C) X - \lambda_C g_C$$

$$\hat{V}(T, S) = H(S)$$

6. Uncollateralized Discounting-with- Funding Approach: The boundary condition g_B does not enter – this is because in this strategy there is no attempt to hedge own default. It is worth noting that for uncollateralized trades, i.e., trades with

$$X = 0$$

and zero counterparty risk –

$$\lambda_C = 0$$

– the PDE specializes to a simple funding-with-discounting approach as in Piterbarg (2010).

7. Special Case – Discounting-with-Funding: Thus discounting-with-funding is a special case of this strategy and assumes that the issuer deals with any funding requirement or surplus by using a single funding instrument and is happy to generate a windfall or shortfall upon own default.
8. PDE for Gross Valuation Adjustment: Similarly inserting the hedging error

$$\epsilon_h = g_B + P_D - X = g_B - R_F\hat{V} - (1 - R_F)X$$

into the PDE

$$\frac{\partial U}{\partial t} + \mathcal{A}_t U - (r + \lambda_B + \lambda_C)U = s_X X - \lambda_B(g_B - V) - \lambda_C(g_C - V) + \lambda_B \epsilon_h$$

$$U(T, S) = 0$$

gives the adjustment U for this strategy as

$$\frac{\partial U}{\partial t} + \mathcal{A}_t U - (r + \lambda_C)U = -\lambda_C(g_C - V) + s_F(V - X) + s_X X$$

$$U(T, S) = 0$$

9. Collateralized Regular Bilateral Close-outs: This step carries out an analysis of the regular bilateral close-outs with collateral as given in

$$g_B = (V - X)^+ + R_B(V - X)^- + X$$

and

$$g_C = R_C(V - X)^+ + (V - X)^- + X$$

For these

$$g_C - V = -(1 - R_C)(V - X)^+$$

Applying the Feynman-Kac theorem we obtain

$$U = CVA_F + DVA_F + FCA_F + COLVA_F$$

with

$$CVA_F = -(1 - R_C)\int_t^T \lambda_C(u) D_{r_F + \lambda_C}(t, u)\mathbb{E}_t[\{V(u) - X(u)\}^+]du$$

$$DVA_F = -\int_t^T s_F(u) D_{r_F + \lambda_C}(t, u)\mathbb{E}_t[\{V(u) - X(u)\}^-]du$$

$$FCA_F = -\int_t^T s_F(u) D_{r_F + \lambda_C}(t, u)\mathbb{E}_t[\{V(u) - X(u)\}^+]du$$

$$COLVA_F = -\int_t^T s_X(u) D_{r_F + \lambda_C}(t, u)\mathbb{E}_t[X(u)]du$$

10. $\underline{DVA_F \text{ and } FCA_F \text{ as } FVA_F}$: Combining DVA_F and FCA_F into FVA_F results in

$$FVA_F = DVA_F + FCA_F = -\int_t^T s_F(u) D_{r_F+\lambda_C}(t,u) \mathbb{E}_t[V(u) - X(u)] du$$

11. <u>Comparison with the No Shortfall Case</u>: These adjustments are very similar to the ones in the strategy described in *semi-replication with no shortfall at own default* except that the discounting used is $D_{r_F+\lambda_C}(t,u)$ rather than $D_{r_F+\lambda_B+\lambda_C}(t,u)$. There is no reference to λ_B, only to the funding rate r_F of the P_F bond used in the own-bond portfolio of the semi-replication strategy.
12. <u>Inapplicability as Generalized Valuation Adjustment</u>: It should be noted that the adjustments CVA_F, DVA_F, FCA_F, and $COLVA_F$ are not direct specializations of the general adjustments defined in

$$CVA = -\int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t\left[V(u) - g_C\left(V(u), X(u)\right)\right] du$$

$$DVA = -\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t\left[V(u) - g_B\left(V(u), X(u)\right)\right] du$$

$$FCA = -\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t[\epsilon_h(u)] du$$

$$COLVA = -\int_t^T s_X(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t[X(u)] du$$

In particular FCA_F does not correspond anymore to the discounted expectation of the hedge error ϵ_h upon default.

13. <u>FCA As Collateralized Adjustment Difference</u>: Obviously the expectation above for FCA_F still equals the difference between $CVA_F + FVA_F + COLVA_F$ and the classical bilateral CVA with collateral. The FCA of

$$FCA = -\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t[\epsilon_h(u)] du$$

can thus be calculated as

$$FCA = FCA_F + (CVA_F - CVA) + (FVA_F - FVA) + (COLVA_F - COLVA)$$

The adjustments of the special cases of uncollateralized, gold-plated 2-way CSA, and 1-way CSA can then be derived easily equivalently to those in the strategy *semi-replication with no shortfall at own default*.

14. <u>Simplicity and Relevance of the Strategy</u>: The strategy specified above is thus very simple to understand and implement. It is also of practical relevance not least because dealers who simply discount by the funding rate assume this strategy implicitly (for zero counterparty risk), including potential windfalls and shortfalls to their estate upon own default.

Burgard and Kjaer (2013) Case Study

1. <u>Strategy Specific Valuation Asymmetry Estimation</u>: Burgard and Kjaer (2013) provide an illustrative case study for the generalized bilateral CVA – the sum of CVA from $CVA = -\int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t\big[V(u) - g_C\big(V(u), X(u)\big)\big] du$ DVA from $DVA = -\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t\big[V(u) - g_B\big(V(u), X(u)\big)\big] du$ and the generalized FCA from $FCA = -\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t[\epsilon_h(u)] du$ and the valuation asymmetries for strategies I and II computed from the perspective of the issuer and the counterparty (i.e., with counterparty's FCA) respectively together with the issuer's hedge error $\epsilon_h(t_0)$ in case of

immediate default of the issuer for three sample trades. For strategy II the FCA is computed as per $FCA = FCA_F + (CVA_F - CVA) + (FVA_F - FVA) + (COLVA_F - COLVA)$

2. ITM, ATM, and OTM Swaps: The sample trades are 10Y swaps with $100m notional where the issuer pays fixed and receives 6M LIBOR floating. Burgard and Kjaer (2013) consider different close-out provisions for 3 fixed rates – 3.093% (OTM), 2.693% (ATM), and 2.293% (ITM).
3. Dealer Specific Funding Value Adjustments: The adjustments computed from the perspective of the counterparty show that if both sides include their funding costs their economic values may be far apart and the two parties may not agree on the deal. If the counterparty does not include the funding costs in general it will deal with the issuer with the lowest funding costs.
4. Differences between Strategies I and II: As shown in Burgard and Kjaer (2013) the differences between the adjustments of the strategies I and II are in general not particularly big (they increase with funding rate), but as expected strategy II has potentially significant shortfalls upon issuer default, whereas strategy I only generates windfalls.
5. Impact of using Setoffs: As discussed earlier, when using set-offs the impact of the funding cost is mitigated. When following the strategy I the FCA vanishes completely and symmetric prices are obtained. Even when implementing strategy II the FCA prices are pretty small and the prices are close to being symmetric. Setoff close-outs are an attractive way of mitigating the need for funding cost adjustments.
6. Asymmetric Valuation and Hedge Error: Burgard and Kjaer (2013) show how asymmetric valuation and hedging error can vary across funding strategies I and II outlined above, and how they interact with CSA's and set-offs. Each in turn prices an out-of-the money (OTM), at-the-money (ATM), and in-the-money (ITM) 10Y $100m swap with the OTM and the ITM swaps 40 bp either side of the ATM level.
7. Impact of the Issuer Bond Spread: Issuer bond spreads are considered at 100 bp and 500 bp respectively, while the counterparty spread is set constant at 300 bp. Total adjustments are calculated for the issuer and the counterparty as well as the bilateral CVA and the hedging error. Uncollateralized one-way CSAs in the counterparty's favor, as well as cases including a set-off are also considered.
8. Adjustment Impact between the Sides: The adjustments differ between the two counterparties creating an asymmetry in the derivative's valuation. The degree of this and the resulting

hedging error depends on the funding and the collateralization strategy. In the 100 bp case the uncollateralized swap has a valuation asymmetry of 50 bp of the notional with the magnitude decreasing from OTM through ATM to ITM.

9. One-Way CSA Adjustment Impact: The introduction of the one-way CSA increases the size of the issuer's adjustment but reduces the asymmetry to 20-30 bp with the amount of reduction skewed from the opposite direction from ITM to OTM. This is because under the one-way CSA the issuer has to post more collateral for an OTC swap, thus reducing the funding benefit.
10. XVA Metrics for Strategy I: Strategy I has zero hedge error for OTM and ITM, but produces roughly 2% notional hedging error in the ITM case regardless of the existence of the CSA. In the presence of the set-off the FCA is eliminated, and so the total adjustment is equal to the bilateral CVA.
11. XVA Metrics for Strategy II: Strategy II is more complex, but the valuation asymmetry is dramatically reduced – by roughly a factor of 10 in the case study. The hedging error is also reduced with the biggest reduction coming for the ITM case. When the issuer is ITM and defaults, the setoff implies that the counterparty can pay back the full present value of the trade using the issuer bond notional rather than cash, which reduces the post-default bond-holder windfall.
12. Directional Dependence on the Issuer Spread: The interesting point is that the quantitative findings of the study are independent of the issuer's bond spread – only the magnitudes change with a greater proportional reduction in the valuation asymmetry and the hedging error of the strategy II in the presence of a set-off, for instance – as can be seen from the results for the 500 bp case.

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Accounting for OTC Derivatives: Funding Adjustments and Re-hypothecation Option

Status of Current FCA/FBA Accounting

1. Motivation: Banks hold and routinely exercise the option of freely re-hypothecating variation margin across counter-parties and trades.
2. FCA/FBA Accounting Standards: However, the emerging FCA/FBA cost accounting metrics for funding costs are mostly formulated in terms of netting set specific metrics that fail to properly account for re-hypothecation benefits to common Equity Tier 1 Capital (CET1).
3. Double Counting in FCA/FBA: Additionally, the FCA/FBA standard introduces a double counting issue between the funding benefits and the DVA which leads ultimately to the violation of the fundamental accounting tenet of asset-liability symmetry.
4. FVA/FDA Accounting Objectives: Albanese and Andersen (2014) propose an alternative accounting framework meant to rectify some of the problems in existing standards. This new accounting method, which they call FVA/FDA, explicitly incorporates the re-hypothecation option into its definition of funding costs, and maintains consistency with the Modigliani-Miller Theorem, with fair-value and asset-liability symmetry principles, and with Basel III rules for DVA and equity capital.
5. Pricing at the CET1 Indifference Level: They argue that derivative pricing necessitates an incremental assessment of capital structure impact on new trades and propose that the entry prices should be struck at the indifference level for CET1.
6. Departure From FCA/FBA Accounting: Unlike the FCA/FBA method, the FVA/FDA accounting does not result in outright net-income write-offs due to funding costs.

Comparison Between FCA/FBA and FDA/FVA

1. Where are they similar: FCA/FBA accounting and FVA/FDA accounting lead to very similar and quantitatively close conclusions in the particular case of a portfolio consisting of a single netting set and a single trade.
2. Portfolio Sets and Portfolio Sizing: However material differences arise in the case of large portfolios. Albanese and Andersen (2014) discuss a case study with a representative portfolio whereby the CET1 adjustments for funding are 3 times as large as the ones required in FVA/FDA accounting.
3. Impact on Prices and VAs: After the portfolio effects are accounted for, incremental entry prices for individual trades differ materially between the FCA/FBA and the FVA/FDA methods, with the FCA/FBA accounting often displaying sizeable and risky pricing biases between derivative payables and receivables.
4. Abbreviations and Expansions:

Abbreviation	Expansion
A	Asset Account
BCBS	Basel Committee on Banking Supervision
CA	Contra-Asset
CCP	Central Counter-party (i.e., a Clearing House)
CDS	Credit Default Swap
CET1	Common Equity Tier 1 Capital
CL	Contra-Liability
CFD	Central Funding Desk
CSA	ISDA Credit Support Annex Agreement
CVA	Credit Valuation Adjustment, same as FTDCVA
CVA_{CL}	Contra-Liability entry for Credit Valuation Adjustment
DVA	Debt Valuation Adjustment

DVA2	Funding Debt Adjustment (same as FDA)
EE	Expected Exposure
ENE	Expected Negative Exposure
EPE	Expected Positive Exposure
FBA	Funding Benefit Adjustment
FCA	Funding Cost Adjustment
FDA	Funding Liability Adjustment (same as DVA2)
FTDCVA	First-to-default CVA, same as CVA
FTP	Funding Transfer Pricing
FVA	Funding Valuation Adjustment
PFV	Portfolio Fair Valuation
KVA	Capital Valuation Adjustment
L	Liability Account
OIS Rate	Overnight Index Swap Rate
OTC	Over-the-Counter
RE	Retained Earnings
REPO	Re-purchase Agreement
RHO	Re-hypothecation Option Benefit
SFVA	Symmetric Funding Value Adjustment
UCVA	Unilateral CVA
VM	Variation Margin

XVA	"X" Valuation Adjustment (short-hand for all valuation adjustments, such as CVA, DVA, FVA, etc.)

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Funding and Re-Hypothecation Adjustment - Motivation

OTC vs. Repo Markets

1. Repo Market Infrastructure: One key indicator of trade liquidity is the existence of an efficient REPO market infrastructure in support of market-making activities. In a liquid markets, such as those trading government bonds, security acquisitions may be financed by reverse REPO trades.
2. Funding in OTC Derivatives Market: The OTC derivatives market, on the other hand, is not directly supported by a REPO infrastructure, and uncollateralized derivatives must instead be financed by other means.
3. OTC Hedges Variation Margin Funding: As derivatives receivables are an inefficient form of collateral, the rates banks face when funding variation margin (VM) for their hedges for uncollateralized derivatives are typically close to those for unsecured funding.

Modus Operandi of Funding Desks

1. Funding the Derivatives Hedging Cost: Without an efficient REPO market infrastructure, managers cannot prevent or hedge the wealth transfer with unsecured derivatives trading. Instead they must seek to recoup the loss to shareholders by passing on the cost to the clients.
2. The Funding Cost Transfer Chain: At a more granular level, this cost transfer commonly goes through a chain that starts when a bank treasury issues unsecured debt to raise funds needed for uncollateralized derivatives trading activities.
3. The CFD/CVA Desks: In the standard bank setup, the running spread cost of the bank issuance is subsequently passed by the treasury to a central funding desk (CFD) which consolidates the management of the funding costs on behalf of the bank's trading functions. In many cases the CFD is merged with the CVA desk in a unified trading operation.

4. FTP Policy and Client Pricing: The CFD, in turn, transfers costs of funding to business line desks on an upfront basis by implementing a funding transfer pricing policy. Finally business line desks charge the costs to clients by embedding them into deal structures.

MMT And Asset-Liability Symmetry

1. Modigliani-Miller Theorem and Indifference Pricing: According to the famous Modigliani-Miller Theorem (MMT – Modigliani and Miller (1958)) the indifference price of new trade should not depend the rate on which the bank funds the VM on the hedges.
2. Wealth Transfer Across the Capital Structure: However, inefficient funding strategies may still give rise to wealth transfer across the capital structure of the bank, directed from the shareholders to the senior creditors (Albanese and Iabichino (2013)).
3. Funding Charges in Accounting Statements: While estimates for funding costs have been used and incorporated into funding costs informally for decades, it is only post-crisis that the banks have attempted to recognize these costs in official accounting statements by adding funding related valuation adjustments (FVA) to the existing CVA and DVA credit risk adjustments.
4. FVAs and Asset Write-downs: Despite unresolved controversies in the literature, in the last quarter of 2013 funding costs were reflected in the accounts at various banks including JP Morgan, Deutsche Bank, Nomura, and others. Not only did these give rise to very material adjustments to CET1, they also led to, due to asymmetries in the fair-value adjustments, asset write-downs that at least in one case exceeded $1 billion (JP Morgan Press Release (2013)).

Rigorous Framework For Funding Costs

1. Impact of Basel Committee Recommendations: The question of how and whether to include funding costs in accounts is complicated by the recent decision of the national regulators to accept the recommendation by the Basel Committee (Basel Committee On Banking

Supervision (2012)) and mandate the exclusion of the DVA and other own-credit benefits from CET1.

2. Impact of DVA Capital Exclusion: The decision is relevant in the context of funding strategies as popular accounting methods for funding costs are inter-twined with the definition (or sometimes re-definition) of the DVA. In a realistic case-study example, Albanese and Andersen (2014) show that the impact of the DVA capital exclusion is to triple the CET1 deductions for funding whenever the FCA/FBA accounting method is used.
3. Motivation for the Rigorous Funding Valuation: Further, Albanese and Andersen (2014) show that such adverse impacts are to a large extent due to logical faults in the FCA/FBA accounting, and are not justified in a more rigorous framework for funding costs.

Funding Set VM RHO Computation

1. Rationale Behind the VM RHO Computation: The calculation of the CET1 deductions and the FTP amounts within lines of businesses are essentially always model based and depend strongly on a variety of assumptions and approximations.
2. VM Re-hypothecation Across Netting Sets: Of particular relevance here is how the re-hypothecation option (RHO) for the VM is treated. As banks are allowed to re-hypothecate across netting sets in the same business line portfolio, the RHO is valuable and routinely exercised through the shifting of the cash collateral hedges from receivable hedges to payable hedges.
3. Funding Set Portfolio RHO Valuation: As a consequence, rigorously computed funding cost adjustment is an aggregate portfolio level amount that cannot be linearly decomposed across netting sets. Specifically re-hypothecation dictates that funding costs be calculated through a portfolio level simulation with scenarios that are shared at the *finding set* level, whereby a funding set is defined as a portfolio of unsecured or partly collateralized trades and their corresponding hedges among which the variation margin can be re-hypothecated.

Shortcomings of Traditional CVA Systems

1. Funding Cost Valuation Implementation Challenges: Modeling challenges here get intertwined with technology implementation difficulties. Funding cost calculations that require aggregation of trade and collateral values across netting are rarely a good fit for traditional CVA systems optimized for individual netting sets.
2. CVA Systems Retrofit for RHO: In particular modeling of re-hypothecation using common distributed computing setups is often awkward compared to an in-memory architecture where all counter-party credits are simulated dynamically and all scenarios are shared.
3. Approximations to Funding Cost Calculations: To bypass technical implementation difficulties and to re-use grid-based CVA systems, many market participants have implemented an approximate accounting method for funding costs based on metrics that are additive over netting sets.
4. FCA/FBA Netting Set Additivity: The popular FCA/FBA accounting method was designed with linear aggregation in mind. However the netting set additivity inherent in the FCA/FBA accounting typically overlaps between the funding benefits (as captured by the FBA) and the DVA on the derivative payables.
5. Adjusting for DVA Double Counting: The resulting double-counting is handled normally by an outright replacement of DVA with FBA (Castagna (2011), Caccia (2013)). This replacement inevitably intertwines the DVA on payables (a CET1 deduction) with the RHO (which should instead *add* to CET1).

Addressing the Shortcomings of FCA/FBA Accounting

1. Rigorous Modeling of the RHO: To address the shortcomings of the FCA/FBA accounting, Albanese and Andersen (2014) examine an alternate accounting method – denoted FVA/FDA – on which the RHO is modeled rigorously and care is taken to make entries of financial statements as meaningful as possible.
2. "Going Concern" Viewpoint of Accounting: Reconciling financial statements is often a delicate task as the accrual principles and the "going concern" viewpoint of financial

accounting inevitably clashes with the notion of fair market value, DVA, and balance-sheet wealth transfers.

3. Accounting Theory Consistency of the FVA/FDA: Unlike the FCA/FBA Accounting method, the FVA/FDA accounting method is simultaneously consistent with the MMT, the risk-neutral pricing, and the general accounting principles such as asset-liability symmetry. Further in FVA/FDA accounting there is no double-counting of the DVA, and the funding cost adjustment to CET1 has the correct directional dependence with respect to the bank's own credit spread.
4. Impact on Income, CET1, and Price: Consistently with the MMT, the FVA/FDA funding value adjustments do not impact income, but do affect both CET1 and the entry price levels. This should provide sufficient incentives for the trading personnel to manage their funding costs properly, something that cannot be said for classical accounting principles that ignore funding costs entirely.

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Albanese and Andersen (2014) Results Summary

Valuation Adjustment Estimation Framework Setup

1. Book of Uncollateralized OTC Trades: Albanese and Andersen (2014) consider an OTC book containing trades with multiple unsecured counterparties, alongside back-to-back hedges with dealers or clearing houses. The unsecured counterparties do not post VM in full while hedges are fully collateralized.
2. Rate on the VM Collateral: In case the unsecured book is a net receivable, the hedge book is net payable, and the bank needs to procure VM to hedge counterparties. In this situation the bank receives a rate of OIS on the collateral posted as VM.
3. Definition of Funding Value Adjustment: The funding value adjustment (FVA) is defined as the PV of the carry cost of funding VM, net of the OIS receipts on the posted collateral.

OTC Books Funding Set Decomposition

1. Definition of the Derivative Funding Sets: OTC books are decomposed into *funding sets*, defined as trade sets for which VM for hedges can be re-hypothecated across all trades. Funding sets may span a large number (possibly thousands) of netting sets.
2. FVA Additivity over Funding Sets: The FVA is additive over funding sets, but not over netting sets. Therefore the valuation is difficult to carry out with standard CVA systems, and the industry is currently focused on using simpler alternatives which are linear over netting sets.
3. Funding Costs over Netting Sets: Funding costs would be additive over netting sets if either one of the following two mutually exclusive hypothesis hold:
    a. HY1 => Re-hypothecation is possible only between hedges to trades in individual netting sets;

b. HY2 => The collateral received from hedges to each payable netting set can be fully re-hypothecated as VM for hedges against other receivable netting sets.

4. Complete Cross netting Set Re-hypothecation: This leads to a symmetric FVA (SFVA) which recognizes a re-hypothecation benefit to all VM received, whereas HY1 leads to the FCA metric that aggregates funding costs linearly over all netting sets. The metric

$$FBA = FCA - SFVA$$

measures the difference between the funding costs under assumptions HY1 and HY2.

5. Shortcomings of FCA and SFVA: Note that in the above expression the FCA generally overstates the funding costs because it neglects the RHO for hedges to unsecured trades in different netting sets. The SFVA on the other hand has errors of opposite sign since it overvalues the RHO.

Inconsistent Booking Under the FCA/FBA

1. Overlap of FBA and DVA: In the interpretation of the FBA, it is often noted that the FBA overlaps with the DVA on payables – in their case study, Albanese and Andersen (2014) note that the FBA is about 20% larger than the standard DVA on payables.
2. DVA CL Replacement with the FBA: As a result advocates of FCA/FBA accounting remove the regular DVA contra-liability (CL) entry on the financial accounting statement and effectively replace it with the FBA number. In fact to better comply with the accounting laws, the CL entry may be broken into two pieces – the DVA, plus a new "funding" term equal to $FBA - DVA$. The net CL entry, however, is still FBA.
3. New Accounting Rules on DVA: DVA on payables is always recognized as a gain on the income statement and until 2012 could theoretically be considered a contribution to CET1. Under FCA/FBA accounting, this would ultimately lead to a net CET1 reduction relating to funding equal to the SFVA.
4. Payables DVA Contribution to CET1: In 2012, the DVA on payables was de-recognized as contributing to CET1 (Basel Committee On Banking Supervision (2012)). As the FBA is

basically re-classified as DVA, this effectively prevents the FBA from contributing to CET1 and sets the overall CET1 deduction for funding equal to the FCA.

5. DVA Cross Netting Set Re-hypothecation Benefit: As a consequence the re-hypothecation benefits across netting sets are ignored altogether in CET1. As discussed in Albanese and Andersen (2014), this has material impact on both accounting, management, and trading decisions.
6. FCA/FBA Accounting Compromise Solution: The compromise solution in FCA/FBA accounting includes the following:
    a. Enter FCA and unilateral CVA (UCVA) as CET1 deductions.
    b. Eliminate DVA on payables from accounts, replace it with FBA, and enter this amount as a contra-liability (CL) adjustment not contributing to CET1.
    c. Transfer UCVA and SFVA to clients.
    d. As we explain later in the paper, the end result is that the FTP's are struck at the indifference levels to the income.
7. Breakage of the Asset Liability Symmetry: Besides the issues that have already been mentioned, it is clear from the booking rules above that the FCA/FBA accounting implies a loss of asset-liability symmetry, since the DVA on payables is eliminated on favor of the FBA even though CVA is supplemented with (rather eliminated in favor of) FCA. The lack of symmetry is problematic from an accounting standpoint and contradicts FASB 159 adopted in 2007.
8. SFVA as a CET1 Deduction: A possible way around this asymmetry involves deducting SFVA (rather than FCA) from the equity capital. In this case the DVA double counting issue manifests itself by the fact that the SFVA inherits from the DVA a wrong-way sensitivity with respect to the own-credit of the bank, i.e., it may decrease (causing the CET1 to increase) whenever the bank credit deteriorates.
9. FTP Policies in FCA/FBA Accounting: As mentioned above, the funding related FTP policies in FCA/FBA accounting normally pass through the SFVA amount, i.e., include FBA benefits. Prior to 2012 rules, this could be argued to be reasonable from a share-holder perspective (as proxied by CET1, at least). Yet, since FBA has been currently demoted to the status of contra-liability that is not recognized in equity capital considerations, the FCA/FBA FTP policy induces deal-flow volatility to CET1.

10. Interpretation of the FCA/FBA FTP Policy: One way to interpret the effect above is that the FCA/FBA FTP policies are based on indifference pricing to the overall firm (including senior creditors), rather than just share-holders as is normally desired.
11. Inaccuracies of the FCA/FBA Accounting: In addition to the above-mentioned undesirable side-effects, numerical experiments show that the net FTP amounts are often too large in absolute value, despite the inclusion of the FBA benefits. This effect is due to large inaccuracies in modeling VM re-hypothecation and leads to incorrect firm-level hedge ratios for market risk.

Improvements Offered by the FVA/FDA Accounting

1. Proposals to Overcome the FCA/FBA Drawbacks: To overcome the shortcomings of the FCA/FBA accounting, Albanese and Andersen (2014) propose an accounting methodology that:
    a. Reflects and justifies Basel III regulatory requirements regarding counterparty credit risk;
    b. Is consistent with generally accepted accounting principles;
    c. Is consistent with the tenets of classical finance theory, such as the MMT and risk neutral valuation.

    Within this framework they then consistently value cash flow streams for VM funding and re-hypothecation strategies.
2. CET1 as a Shareholder Proxy: Their proposal fundamentally uses CET1 as a proxy for shareholder value and defines FVA as a discounted expectation of future funding costs occurring whenever there is an overall deficit of the VM at the book level. Future scenarios where there is a net excess of OTC collateral do *not* contribute to the FVA.
3. Corporate Finance Interpretation of FVA/FDA: A corporate finance interpretation of this FVA metric equates it to the present value of the wealth transfer from the bank shareholders to the bank senior creditors as a result of the bank entering into OTC trades with unsecured funding.

4. Validity of Asset Liability Symmetry: Importantly the FVA definition is such the funding adjustments are entirely divorced from the DVA on payables, wherefore the asset-liability symmetry still holds and the own-credit metric of the FVA has the correct sign.
5. Embedding of RHO in FVA: Since the RHO is embedded in the valuation of the FVA, the FVA amount is much smaller than the FVA amount in the case of portfolios of realistic size – about one third as large in the case study portfolio of Albanese and Andersen (2014).
6. Consistency of the FVA/FDA with MMT: In FVA accounting, the MMT is satisfied, as the FVA is accompanied by an offsetting CL adjustment which does not overlap with the DVA on payables. This contra-liability is named DVA2 by Hull and White (2014) and is referred to as FDA by Albanese and Andersen (2014). In the case of FCA/FBA accounting the term DVA2 is not meaningful because the approximations involved break the MMT and compromise a rigorous capital structure interpretation.
7. CVA Fair Valuation in FVA/FDA: Within the FVA/FDA framework, the fair valuation of CVA is most naturally a bilateral one (sometimes known as first-to-default CVA, or FTDCVA). As this fair value contains a DVA-like element of self-default benefit, guidelines in the Basel Committee On Banking Supervision (2012) suggest that FTDCVA cannot be directly deducted from CET1.
8. UCVA, FTDCVA, and CVA-CL Metrics: From the FTDCVA number a unilateral CVA (UCVA) number can be split out and it may be recorded as a contra-asset (CA) adjustment that is subtracted from CET1. The remaining "self-CVA" term is listed as a contra-liability (CL) and is to be excluded from CET1.
9. Elements of FVA/FDA Accounting - Summary: In summary the FVA/FDA accounting with rigorous RHO modeling includes the following elements:
    a. The UCVA and the FVA are both entered as CA adjustments and CET1 deductions recognizing the full benefit of the RHO to CET1;
    b. The DVA and the FDA are both entered as CL adjustments, as is the part of the FTDCVA that involves benefits from the bank defaults. None of the CL adjustments are to be counted for the CET1 purposes.
    c. The FTP is designed to immunize the CET1 from deal-flow volatility and to transfer the incremental costs of FVA and UCVA capital deductions to clients.

10. Income, CET1, and MMT Impact: Note that a) and b) preserve the standard CVA-DVA accounting at the net income level, as the FVA and the FDA adjustments cancel against each other. CET1, however, is affected by the funding costs.
11. Conservative Computation of the FTP: The FTP rule in the FVA/FDA accounting aims at preserving the CET1 capital, a principal that is fundamentally more conservative than the one followed in FCA/FBA accounting, where one only insists that new trades not have a negative impact on the income.
12. CET1 Impact from Deal Flow Volatility: In FVA/FDA accounting, deal flow still engenders volatility of the contra-liabilities (such as the DVA) as well as the fair value of the bank itself. However, due to our alignment of CET1 and the shareholder value, mitigating this volatility is irrelevant from the viewpoint of the shareholder.
13. FCA/FBA vs. FVA/FDA FTP Comparison: Notwithstanding the stronger FTP requirement, the fact that the RHO is properly modeled means that the FTP amounts obtained in the FVA/FDA accounting are generally quiet reasonable and often materially smaller than those in the FCA/FBA methodology.
14. FCA/FBA vs. FVA/FDA Derivatives Valuation: Relative to FVA/FDA, FCA/FBA accounting is observed to systematically undervalue the derivatives payables and over-value derivatives receivables, thus potentially giving rise to biased sub-optimal positioning of the OTC book.
15. Non-linear RHO Funding Set Contribution: It should be emphasized that under the FVA/FDA accounting, the notion of an individual unsecured trade price loses its meaning because all trades within the same funding set contribute non linearly to the RHO of the funding set. To a lesser extent, individual unsecured trade price loses its meaning in FCA/FBA accounting as well, since here the smallest possible additive unit is a netting set.
16. Valuation Across Entire Funding Sets: It is therefore possible to conclude that in order to correctly account for funding adjustments, one needs to value derivatives in the context of *entire* funding sets.
17. Funding Set Level Scenario Simulation: In FVA/FDA accounting, in order to account for collateral thresholds and to model re-hypothecation benefits correctly, whenever w new possible trade is priced, one needs to evolve dynamically the full portfolio valuation along with all the CDS curves for all counterparties and compute book level incremental statistics.

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CET1 Capital Deductions in Basel III and Capital Structure Considerations

CET1 Deductions

1. DVA as Full CET1 Deduction: The BCBS recommended in 2012 that the DVA be fully deducted from CET1.
2. BCBS Statement on DVA Impact: The relevant wording from the Basel Committee on Banking Supervision (2012) is:

   Therefore, after considering all the views, the Basel Committee is of the view that all DVA's for derivatives should be fully deducted in the calculation of CET1. The deduction of the DVA's is to occur at each reporting date and requires deducting the spread premium over the risk-free rate for derivative liabilities. In effect, this would require banks to value their derivatives for CET1 purposes as if they (but not their counterparties) were risk free, and to deduct unrealized gains both at the inception of the derivative trade and afterwards, when the creditworthiness of the bank deteriorates.
3. Credit Quality Impact On CET1: The BCBS rule ensures that a bank cannot claim increases in CET1 solely due to the deterioration in its own credit quality, consistently with the spirit of Basel III accord (Basel Committee On Banking Supervision (2011)).
4. Impact on CVA and FVA: While Basel Committee On Banking Supervision (2012) nominally only deals with the derivatives liability and the DVA, it is generally understood that the disallowance of the CET1 increases from deteriorating bank credit is a universal principal that extends to CVA and FVA as well.
5. BCBS Language Relevant to CVA: For instance, for CVA some relevant language is (from Federal Register (2014)):

   CVA equals the credit valuation adjustment that the bank has recognized in its balance sheet valuation of any OTC derivatives contract in its netting set. For the purposes of this *paragraph, CVA does not include any adjustments to CET1 attributable to changes in its own credit risk since the inception of the transaction with the counterparty.*

6. Unilateral CVA vs. Bilateral FTDCVA: From a modeling standpoint, adherence to this particular language can be achieved by deducting from the capital a unilateral UCVA metric as opposed to a smaller bilateral FTDCVA.

"Going Concern" or Defaultable Banks?

1. Regulatory Intent Behind the Deductions: The intent of the regulator regarding both the DVA and the CVA capital deductions is to exclude from CET1 the present value of the cash flows that benefit the bank after the default.
2. Regulatory Notion of "Going Concern": Care must be taken to not extend the regulatory exclusion to a fair value setting. For instance, the regulatory notion of valuation from the viewpoint of "going concern", in which the bank is assumed to be unable to default, is clearly at odds with both the reality and with the objective of consistent market pricing.
3. Unintended Consequence of "Going Concern": If taken literally and applied out the intended context, this no-default assumption has the unwanted and undesirable side effect of increasing the prices on the bank-issued debt and significantly lowering the bank's funding costs.
4. Funding Spread as a Liquidity Spread: We note in particular that if one were to consistently assume that the bank cannot default, then the spread separating the bank funding rates from the OIS rates would need to be interpreted as a liquidity spread. This is one of the possible financial interpretation behind the FCA/FBA accounting rules, along with an alternate approach based on modeling debt buy-back strategies.
5. Trouble with the Liquidity Spread Approach: The liquidity spread assumption is hardly defensible; typical funding spreads are in the range of 50-400 bp while typical liquidity spreads are below 5 bp.
6. Funding Spread without Credit Risk: It would be very difficult to construct a financial interpretation to funding spreads which does not involve the credit risk of the bank. The debt buy-back argument is more subtle and is discussed later.

7. No-default View on Funding Considerations: Besides making fair-value considerations awkward, the no bank default view is not reasonable for funding considerations either. Specifically the FVA is a fair valuation of a cash flow stream resulting from a funding strategy that the bank clearly cannot implement past its own default; once the bank is in a state of default, its funding spread ins infinite, and the bank is unable to borrow funds on an unsecured basis or to conduct most other trading activities. As such any correct measure for funding costs must inescapably reference default by the bank (and its counterparties, for that matter).
8. Wealth Transfer across the Capital Structure: From yet another angle, FVA admits the financial interpretation as an internal wealth transfer across the capital structure of the bank resulting from the implementation of a funding strategy and is not the price of an asset sold to the counterparty on which the bank can default. However, wealth transfers can stop once the default occurs and the equity holders are wiped out.

Categorization of Cash-flow Streams

1. Accounting Merger/Unification Viewpoint Framework: Since a straight "going concern" assumption is inadequate for evolving a comprehensive accounting framework, Albanese and Andersen (2014) put together an alternate framework which reproduces and justifies the regulator mandated CET1 deductions for CVA and DVA, but which is also meaningful from the viewpoints of funding costs, classical Finance Theory, and the generally accepted accounting principles.
2. Fundamental Cash flow Steam Types: For this purpose, Albanese and Andersen (2014) propose that the cash-flow streams be fundamentally classified into the following 5 types:
    a. CF1 => Contractually promised cash flow streams excluding all bank and counterparty credit risk events;
    b. CF2 => Trade-related cash flows resulting from counter party defaults, but excluding bank default events;
    c. CF3 => Trade related cash flows resulting from the bank default;

d. CF4 => Cash flows streams derived from dynamic trading strategies (such as funding strategies) implemented by the banks and taking place prior to the bank default;
e. CF5 => Cash flow steams deriving from dynamic trading strategies (such as funding strategies) implemented by the bank and taking place at or after the default of the bank.

3. Derivative Contractual Agreement Cash Flows: Any derivatives contract can always be split into separate contractual agreements generating cash flows to types CF1, CF2, and CF3.
4. Dynamic Trading Strategy Cash Flows: Similarly cash flows arising from any dynamic trading strategy can be modeled as a split between types CF4 and CF5.
5. Cash Flows Contributing to CET1: We assume that the splits have been carried out such that each unit of account referring to either a counter party contract or a trading strategy is matched to the relevant cash flow stream. We then designate the units of account whose underlying cash flow streams are of the types CF1, CF2, and CF4 as contributing to CET1, while units of account whose cash flow streams are of type CF3 and CF5 do not contribute to CET1.
6. Counterparty Contract vs. Trading Strategy: A key difference between a counterparty contract and a trading strategy is that the former is settled with a counterparty at the time of the bank default while the latter simply terminates at that point in time. The reason why the 2 cases are treated differently is that a contractually promised cash flow reflects an obligation by the bank that extends to the last maturity, independently of whether a bank defaults or remains a going concern until then.
7. Trading Flows at Bank Default: Cash flows deriving from trading strategies cannot be implemented past the time of the default of the bank, at which time the bank goes into receivership and is unable to carry out normal trading activities. Nevertheless the implementation of the trading strategies prior to default has consequences on a post-default basis which have an impact on the default claim held by senior creditors. These are cash flows of type CF4.
8. Cash Flow Streams:
    a. Collateralized Transactions => Collateralized transactions involve the cash flow streams of type CF1 which are immune from counterparty credit risk.

b. UCVA => The UCVA refers to a cash flow stream of type CF2 and is, effectively, the price of a CVA protection contract promised by the bank, excluding the effects of bank default.
c. DVA => The DVA refers to cash flow streams of the type CF3 as it represents the benefit the senior creditors of the bank obtain from the default of the bank on derivative liabilities.
d. CVA CL => The CVA contra-liability is the DVA component of the CVA; like the ordinary DVA on liabilities it is a cash flow of type CF3.
e. FVA => The FVA can be interpreted as the price of the strategy of borrowing VM collateral up to the time the defaults (after which it becomes impossible to borrow any further). This is a cash flow stream of type CF4.
f. FDA => The FDA is the post-default benefit to the senior creditors deriving from owning a title to the portfolio of derivative receivables whose hedges were funded on an unsecured basis. The FDA corresponds to a cash flow steam of type CF5.

9. Corporate Finance Interpretation of CET1: The cash flow rules above are designed to allow for a Corporate Finance Interpretation of CET1 as a proxy for the value of the bank assets to the shareholders – or at least the contributions to CET1 from derivatives trading activities.
10. Cash Flows after Bank Default: In particular, since the shareholders are indifferent to the cash flows occurring at or after the bank default, such cash flow streams should not contribute to CET1 – excluding here the feedback effects of the type discussed in Burgard and Kjaer (2011). Also note that by ensuring that the cash flow types of CF3 have no impact on CET1, we achieve the stated regulatory objective of deducting both FVA and UCVA from CET1.
11. RHO Exercise Impact on CET1: By the same token, funding costs are assessed at the market level, and the benefits resulting from the exercise of the RHO at times prior to the bank default contributes positively to CET1.
12. Self-Default Benefit - Unilateral CVA: One particular rationale for the regulatory mandate to deduct full UCVA from CET1 was to avoid the scenario where an increase in the bank's own credit spread could lead to a higher CET1.
13. Bank Default Continuation Funding Spread: One may ask whether a similar principle applies to FVA, necessitating the definition of a unilateral FVA. However such a remedy would be

difficult to justify within a consistent modeling framework, since we would have to introduce the notion of continuation funding spread for the bank at and after its own default.

14. Peer Proxy Continuation Funding Spread: While such a notion could potentially be based on average or minimum peer spreads (post bank default), satisfying the Basel III principles does not depend on it as the FVA defined here normally increases as a function of the bank's own credit spread.
15. Credit Spread Impact on FVA: The FVA is impacted by rising bank credit spreads in 2 different directions; rising spreads tend to decrease FVA because funding costs are cut short by a bank default; yet rising spreads also tend to increase the FVA as the spread paid on unsecured lending increases. Albanese and Andersen (2014) show that in normal circumstances the latter effect dominates.
16. Own-Credit Sensitivity in FCA/FBA: Albanese and Andersen (2014) also show that this *not* the case for SFVA, which is the metric for funding costs that are transferred to the clients in the form of FTP in FCA/FBA accounting. Since the SFVA can be shown to have own-credit sensitivities of the wrong sign when applied to portfolios containing mostly payables, its use as a CET1 deduction would be problematic from a regulatory standpoint. For this reason alone, the CET1 deduction in FCA/FBA accounting must be the full FCA.

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Accounting Principles, Units of Accounts, and Valuation Adjustment Metrics

Accounting Rules

1. Accounting Principles for Derivative Portfolios: We list here the key accounting principles for derivatives portfolios. KPMG (2011) and PWC (2011) contain a discussion on them.
2. Units of Account:
    a. Trading Securities (AP0) => Derivatives contracts are categorized as trading securities and must be listed on the balance sheet on their fair market value. Changes to the fair market value are registered as income or loss on the income statement.
    b. Units of Account and Linearity (AP1) => Financial portfolios are decomposed into elementary *units of account* satisfying a linearity property, in the sense that the fair valuation of the portfolio is the sum of the fair valuations of the units of account.
    c. Fair Value as an Exit Price (AP2) => The *fair value* of a unit of account is defined in IFRS13 as the price that would be received if one were to sell the unit of account in an orderly trade between the market participants at the measurement date. This is called the *exit price* of the unit of account. In the determination of the exit prices neither transaction costs (such as broker commissions) nor entity-specific production costs are considered as part of the fair valuation.
    d. Model Based Valuation (AP3) => When a unit of account is not tradeable in the active market, its fair value may be determined using a model based valuation technique. In this case an income based approach is allowed, where the cash flows of the trades present valued. This includes cash flows associated with the default of one of the counterparties.
    e. Symmetry Principle (AP4) => When (as is often the case) there is no active market for the liability transfer, IASB accounting standards require that the fair value of the liability be determined from the perspective of the asset investor.

f. Non-Performance Risk (AP5) => According to FASB 157, the benefit to senior creditors on the non-performance of the liabilities should be captured. Non-performance risk includes the effects of credit risk, as well as any other factors the influence the likelihood of fulfilling contractual obligations.

3. Application to Derivative Security Accounting: To consider the application of these principles for derivatives securities accounting, Albanese and Andersen (2014) consider the situation where a bank B transacts in partially collateralized OTC derivatives with a set of n credit-risky counterparties.
4. Collateral Posting and CSA Agreements: Each counterparty holds a portfolio of derivatives under an over-arching CSA agreement involving a netting clause, and possibly, but not necessarily, collateral posting obligations for the variation margin (VM).
5. ISDA 2009 Closeout Amount Protocols: In the absence of full collateralization, closeout protocols in place to govern settlements whenever a bank or a counterparty defaults need to be explicitly considered; standard protocols are the ISDA Market Quotation Close-out Convention of 1992 (most common) and the ISDA Close-out Amount Protocol of 2009. Note that closeout protocols are relevant for the fair valuation because of the accounting principles AP3 and AP5 above.
6. Decomposition of the Bank OTC Portfolio: Before discussing any valuation metrics, it needs to be clarified how the bank OTC portfolio is linearly decomposed into units of account.
7. Decomposition of the Unsecured Derivative Contract: The above question is non-trivial as an unsecured derivative contract in isolation generally cannot be considered a proper unit of account= because netting clauses cause valuations to depend on the netting set to which the trade belongs.
8. Funding Sets vs. Netting Sets: The RHO for VM causes valuation to depend on funding sets defined as the largest set of trades among which re-hypothecation is possible. Notice that funding sets can cut across netting sets as it is entirely possible that the VM is re-hypothecated separately across distinct business lines contributing to the same netting sets.
9. Fully Collateralized Counterparty Derivative Trades: Fully collateralized trades done with traders or with CCP's generally have negligible credit risk, and shall be considered here to be default free; fir valuation of such trades is additive at the trade level. The valuation of the

unsecured derivative assumed to be fully collateralized in cash is referred to as its *default-free valuation*.

10. Bank Counterparty Credit Valuation Adjustments: The impact of the counterparty and the bank credit risk on valuations is then captured by other units of accounts called *adjustments*. Adjustments make reference to sub-portfolios as opposed to individual trades and can be interpreted as the valuation of derivatives referring to sub-portfolios as underlying.
11. The Asset Account (A): Albanese and Andersen (2014) construct a reference to stream-lined accounting framework of OTC derivatives based on six balance sheet accounts. The Asset Account (A) refers to the receivable units of account referring to cash-flow streams of the type CF1.
12. The Liabilities Account (L): This balance sheet account includes payable units of account referring to cash flow streams of type CF1.
13. The Contra Asset Account (CA): This balance sheet account includes payables adjustments with underlying cash-flow streams of types CF2 and CF4; these are deducted from CET1.
14. The Contra Liabilities Account (CL): This balance sheet account includes receivable adjustments referring to the cash flow streams of the type CF3 and CF5; these do not contribute to CET1.
15. The Retained Earnings Account (RE): This balance sheet account includes provisions that are set aside to meet future obligations such as funding costs and credit default losses. These entries do contribute to CET1.
16. The Equity Account (PFV): This balance sheet account is defined in such a way that the basic accounting equation

$$A + RE - CA = PFV + L - CL$$

is satisfied. The Equity Account has the meaning of Portfolio Fair Valuation and the variation of the PFV over an accounting period is called the *Income*.

17. The CET1 Capital Measure Components: The CET1 is a capital measure that requires the exclusion of the value of units of accounts referencing cash-flows of the type CF3 and CF5 taking place at or after the time of default of the bank B. As those types of cash flows are captured in the CL account, we exclude this account from the common equity and write

$$CET1 = PFV - CL = A - L - CA + RE$$

18. CET1 as Bank Shareholder Value: Within their framework, Albanese and Andersen (2014) interpret CET1 as the value of the bank to the shareholders, while PFV is the combined value to the shareholders and the senior creditors. Here the term "senior creditors" is used to refer to a class of creditors which is different from collateral lenders, and which either have priority or are at the same level of seniority as the collateral lenders.
19. Computation of the CET1 Deduction: Upon entering into a derivative transaction, the CET1 is subjected to an incremental deduction denoted by ΔCA and is augmented through earnings by the FTP amount received from the clients over and above the default-free valuation. Attributing the FTP to the Retained Earnings (RE) account the CET1 variation is given by

$$\Delta CET1 = -\Delta CA + \Delta RE$$

20. Computation of the Income Increment: A nee trade also affects the CL adjustments by an incremental amount ΔCL, an amount tied to the benefits of the bank default and therefore only having an impact on the senior creditors' wealth. The ΔCL term is excluded from CET1, but affects the income and the fair valuation of the bank.

$$\Delta PFV = \Delta Income = \Delta CET1 + \Delta CL = -\Delta CA + \Delta CL + \Delta RE$$

Contra-Asset and Contra-Liability Accounting for Credit Risk

1. CVA for the i^{th} Netting Set: To examine the CA and the CL accounts more closely, consider first the fair valuation of the credit risk for the i^{th}netting set, denoted by CVA_i. The quantity can be interpreted as the value of the default protection contract implicitly sold by the bank to the counterparty i, with the notional set to the netting set value at the time of the counterparty default.

2. Decomposition of the CVA Components: As shown in Albanese and Andersen (2014), the precise valuation methodology to be used depends on the closeout rules. The total CVA is computed by summing the CVA_i across n netting sets and is commonly split into 2 components:

$$CVA = FTDCVA = UCVA - CVA_{CL}$$

3. The Unilateral CVA Component – UCVA: The UCVA component is booked as a CA adjustment and is a unilateral CVA metric independent of the closeout rules, i.e., it is the present value of all the counterparty credit risk losses resulting from the default of the counterparty computed under the assumption that the bank does not default.
4. The CL CVA Component CVA_{CL}: The CVA_{CL} is loosely speaking the DVA component of the CVA, i.e., it is the benefit the bank senior creditors receive at the time of the bank default by, in effect, no longer accepting to sustain future counterparty credit losses. It is booked as a CL adjustment. The magnitude of the CVA_{CL} is closely linked to the closeout specifications; Albanese and Andersen (2014) give the relevant equation for CVA_{CL} using the ISDA 1992 closeout rules.
5. Non-Performance Risk Accounting Principle: According to the accounting principle AP5 on non-performance risk, cash flows taking place at or after the default of the bank should be present valued and accounted for.
6. Symmetry Principle Applied towards DVA: By virtue of principle AP4, the value of the default protection sold implicitly by the unsecured counterparty to the bank should be valued as the CVA assessed by the counterparty against the bank. This amount is the DVA for payables, the reporting of which was mandated by FASB 159 in 2007.
7. DVA Impact on CET1 Numbers: DVA enters accounts as a CL adjustment, and as is references cash flows ensuing a bank default, it is excluded from CET1. A split such as the one seen for CVA is not meaningful for DVA, as this quantity only involves post-default cash flows (with no direct relevance to equity holders and to the capital).
8. "Going Concern" Impact on the CVA: It should be noted here in passing that not all banks account for the CVA_{CL} term, as they effectively equate the CVA with the UCVA term. While this "going concern" definition is convenient in a number of ways (e.g., regulatory and

accounting definitions of CVA are better aligned), it is hard to argue that it is correct or consistent with DVA accounting.

Contra-Asset and Contra-Liability Accounting for Funding

1. CA and CL Credit Risk: Accounting for the credit risk through the UCVA (contra-asset entry) and the $DVA + CVA_{CL}$ term (the contra-liability term) as seen before is a fairly well-established practice even if there are minor differences in the way banks sometimes define CVA and DVA.
2. Supplementing the CVA and the DVA Metrics: As discussed before, recently the CVA and the DVA risk metrics have been supplemented by the quantities meant to account for funding cash flows to which the banks are subject to through their postings of the VM on the hedges.
3. Nature of the Posted VM: VM is normally paid in cash (or to a lesser extent) in highly liquid short term government debt and maybe re-hypothecated across trades within the same funding set. The bank posts VM on collateralized derivative liabilities and receives VM on collateralized derivative assets.
4. Funding Adjustment Unit of Account: For funding costs, we work with a unit of account at the funding set level and define FVA as the discounted value of the book level funding costs arising whenever the funding set has a net collateral deficit. Future states of the world whereby the funding set is a net receiver of the VM are modeled as *not* contributing to funding assets.
5. Handling the Different VM States: On the one hand the excess cash deposited as VM can earn a riskless rate of OIS. But on the other hand, derivative counterparties pledging VM are also entitled to interest rate payments at a matching OIS rate. In total, the states of the world where the bank enjoys accumulation of excess VM collateral are modeled as having zero funding costs.
6. Alternate VM Cash Management Strategies: As discussed in Albanese and Andersen (2014), the zero benefit assumption around excess collateral maybe considered a conservative assumption, and some researchers have considered strategies where the bank buys back long-term debt with excess VM (Burgard and Kjaer (2011a, 2011b, 2011c, 2013)).

7. VM Strategies Capital Structure Impact: However these strategies are not very difficult to implement in practice because the VM is very volatile, they also have zero impact on income if one insists on MMT consistency. As a consequence, if the deleveraging transactions occur at fair valuation, wealth is not transferred between share-holders and the senior creditors, i.e., the benefit is not truly there.
8. Origin of the Funding Cost Impact: Neglecting basis spreads, funding costs for the VM procurement are non-zero because the collateral lenders receive only partial recovery upon bank default. If we denote the recovery rate to collateral lenders with ${R_B}^C$, then the case where there is a perfectly functioning REPO market for derivatives would have

$${R_B}^C = 1$$

9. FVA Definition and CET1 Impact: In reality

$${R_B}^C < 1$$

   i.e., recovery is only partial because of market inefficiencies and the spreads for collateralized borrowing are very close the spreads for unsecured borrowing. The FVA is defined as the discounted expectation of the funding costs up until the time of bank default. Hence the FVA is booked as a CA adjustment and a CET1 deduction.
10. Origin of the Funding Benefit Impact: When short term debt is issued to fund the VM collateral, the lenders providing the funds are exposed to the bank default risk. The flip-side of the risk is the DVA-like benefit held by the bank. To account for it, we introduce an FDA entry as the present value of the depreciation of the collateral debt at the time of the bank default, due to incomplete recovery.
11. FDA Definitions and the CET1 Impact: That is the FDA is the present value of $1 - {R_B}^C$ times the notional borrowed for VM funding purposes, received at the time of the bank default. Since the FDA makes reference to cash flows happening at or after the bank default, the FDA is booked as a CL adjustment and excluded from the CET1.
12. Shareholder Debt Holder Transfers: For the MMT to hold, the FVA must equal the FDA. In this case, the funding strategies, and in particular the value of ${R_B}^C$ do not affect the fair

valuation of the bank. However if ${R_B}^C$ is strictly less than 100% then the deal flow induces a wealth transfer from the shareholders to the senior creditors.

13. FVA/FDA Cash Flow Transfer View: In accounting terms if we decrease the value of ${R_B}^C$ from 1 down to 0, a portion of CET1 is gradually demoted to the status of contra-liability, reflecting the wealth transfer from the shareholders to the senior creditors. The FVA may therefore be considered the wealth transfer lost from the shareholders, while the FDA is the amount earned by the senior creditors.

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Accounting Cash Flows

Accounting Cash Flow Setup Framework

1. XVA Metrics Valuation Formulas Setup: Having so far been limited to a largely qualitative accounting discussion, the next step is to follow the treatment set out in Albanese and Andersen (2014) and proceed to provide concrete valuation formulas for the XVA metrics. For this the precise funding and credit related flows taking place in OTC derivatives trading need to be considered, both before and after the default of the counterparties and the bank itself.
2. Modeling Details in the XVA Computation: Since elaborate cash flow details can obscure the main concepts, this section omits certain less essential minutiae, such as flows associated with ratings dependent CSA thresholds and credit-risk sensitive closeout conditions. This is later followed by a more complete cash flow representation, much of which is implemented in the case study carried out by Albanese and Andersen (2014).
3. Transmitting XVA Charges to Clients: Whenever the bank enters into an unsecured trade, the XVA adjustments are accompanied by charges to clients which can be structured in various ways. In their treatment Albanese and Andersen (2014) assume that the XVA related FTP payments are simply upfront and due at the inception of the trade in question.
4. Nature and Type of FTP: As described earlier, business line trading desks, in fact, typically pay FTP charges to the CVA/CFD desks on an upfront basis. However, assuming upfront structure for the client charges is a stylized approximation, since often costs are paid in the instalments embedded in the unsecured derivative structure itself.
5. Default Free Counterparty Portfolio Value: Working within the modeling framework briefly described before, let us introduce the notation

$$V_i^U(t) \; \forall \; i = 1, \cdots, n$$

for the default valuation at the time t of the portfolio held with the counterparty

$$i = 1, \cdots, n$$

computed by neglecting all funding and credit risks including that of the bank itself.

6. Fully Collateralized Netting Set Value: That is, ${V_i}^U(t)$ represents the value of the netting set i in a world where all trades are collateralized in full. ${V_i}^U(t)$ is set to 0 in case the i^{th} counterparty is in a state of default at time t.
7. Counterparty Value Netting Set Additivity: Assuming no closeout risk and no initial margin, the fair valuation of fully collateralized books is additive over individual trades and therefore also additive over netting sets, i.e., we can meaningfully define a total default-free portfolio value of

$$V^U(t) = \sum_i {V_i}^U(t)$$

8. Hedging of the Unsecured Trade: For simplicity's sake for now we assume that the unsecured trades are hedged on a precise back-to-back basis with the hedge trades having a valuation ${V_i}^H(t)$ equal to exactly $-{V_i}^U(t)$. Indeed, one way to interpret the role of the CFD and the CVA desks is that, after the appropriate compensation, they allow the lines of business desks to hedge risk as if there were not funding or credit risk.
9. Value of the Hedge Trades: The common value of the default-free unsecured trades and hedges is denoted as follows:

$$V_i(t) = -{V_i}^H(t) = {V_i}^U(t)$$

It is also assumed for now that neither the bank nor the counterparty post any VM to each other; later these hypotheses are relaxed and extensions including partial VM postings with CSA collateral thresholds are considered.

10. VM Re-hypothecation Across Hedge Trades: Banks typically have a separate funding set for each business line and jurisdiction. Within each such dedicated book, banks always exercise

the RHO for the VM across hedges (albeit not necessarily optimally – collateral systems of many banks tend to be rudimentary).

11. Net VM Across Hedge Trades: The net VM posted at any given time is given by the sum

$$C_{VM}(t) = \sum_{i} C_{VM,i}(t)$$

where

$$C_{VM,i}(t) = V_i(t)\mathbb{I}_{t<\tau_B}\mathbb{I}_{t<\tau_i}$$

(this is a simplification, as there is typically a material VM from the CVA hedges – this complication is discussed down later).

12. Impact of the Bank/Counterparty Default: Notice the presence of the indicator functions $\mathbb{I}_{t<\tau_i}$ and $\mathbb{I}_{t<\tau_B}$ above, reflecting the fact that the default of either the bank or the counterparty results in an immediate settlement of collateralized hedges. The sign convention is that if $C_{VM}(t)$ is positive (negative) then the bank is a net poster (receiver) of the collateral on the hedges.
13. Assumption of the REPO Rate Value: For simplicity's sake it is assumed that the difference between the OIS rate and the REPO rate for general collateral is quantitatively immaterial and denote both rates with $r_{OIS}(t)$.
14. Recovery Rates Across Debt Classes: Another assumption is that the bank has at least two classes of debt. One is unsecured senior debt with recovery rate R_B used for regulatory costs, initial margin, and administrative costs. The other is debt used to finance VM collateral imperfectly secured by derivative receivables. This second class of debt is modeled as having a recovery rate ${R_B}^C$, a funding rate equal to $r_B(t)$ and a spread over OIS equal to

$$s_B(t) = r_B(t) - r_{OIS}(t) \geq 0$$

15. Estimation of the Funding Spread: If there existed an efficient REPO market for unsecured OTC derivatives, collateral lenders would be guaranteed a full recovery on the VM, i.e., we would have

$$R_B{}^C = 1$$

and the funding spread would be 0. In general

$$R_B{}^C \le 1$$

and the risk neutral valuation of the overnight funding spread is given by

$$s_B(t) = \left(1 - R_B{}^C\right)\lambda_B(t)$$

where $\lambda_B(t)$ is the probability of the rate of default at the time t (Lando (1998)). In practice an additional liquidity spread may apply, but this basis is ignored for the present discussion.

    a. FVA Spread vs. FDA Spread => As pointed out by Morini and Prampolini (2011), FVA/FDA indifference assumes that the CDS-bond basis is zero. This is not the case in practice, since FVA is calculated from bond yield spread whereas FDA should, in theory, be calculated from the CDS spread.

16. Hybrid Debt Class Type Assumptions: It needs to be stressed that the assumption that the debt is divided into 2 classes is only formal and does not restrict the generality of the argument. In the general case one can still assume that the traded debt securities are hybrids between the theoretical bonds of the two types considered.

Cash Flows Related to VM Funding

1. Ignore CVA-related Funding Costs: We start by considering the funding flows on the interval $[t, t + \Delta t]$. To simplify the exposition the funding costs for any VM arising from the default hedges that the CVA desk may have entered into are ignored.

2. Borrowing Cost for the VM Funding: Suppose first that the net VM collateral $C_{VM}(t)$ posted by the bank is *positive*, whereby the bank needs to borrow to fund its overall VM position. In this scenario the bank treasury is assumed to issue short-term unsecured debt into the market to raise the necessary funds.
3. Funding Rate on the Collateral: More specifically, if in the time interval $[t, t + \Delta t]$ the bank has to fund a net collateral shortage

$$C_{VM}(t) > 0$$

the treasury issues $C_{VM}(t)$ worth of short-term debt for this purpose, either unsecured or backed by derivative receivables. Then interest charge on the unsecured debt in the time interval $[t, t + \Delta t]$ at the CFD funding rate $r_B(t)$ is $C_{VM}(t)r_B(t)\Delta t$.
4. CFD Role in the Bank Setup: The required VM collateral is then routed through the inter-dealer positions where an interest-rate amount of $C_{VM}(t)r_{OIS}(t)\Delta t$ is received back. As illustrated pictorially in Albanese and Andersen (2014), when

$$C_{VM}(t) > 0$$

the CFD experiences a net negative cash flow in the amount of $-C_{VM}(t)s_B(t)\Delta t$. FVA is defined as the present value of this negative carry over the lifetime of the funding set or until the time of bank default, whichever comes first.
5. Interest Rate on VM Receivables: In case the total collateral requirement for the VM is *negative*, i.e.

$$C_{VM}(t) < 0$$

the treasury OTC funding program is not called upon. In this case we assume that the bank treasury would invest the excess collateral in short-term securities, yielding on average (nearly) OIS levels. As the bank is liable to paying OIS on hedge counterparties on the net collateral received, the bank is therefore assumed to receive no benefits and face no costs due to VM posting obligations when

$$C_{VM}(t) < 0$$

6. Handling the Net Excess Collateral: In principle one can imagine that the net excess collateral would be passed by the CFD to the treasury, which would in turn use the funds to retire outstanding long-term deb. If this were the case on would imagine that the CFD receives from the treasury a benefit based on the interest-savings on the long-term debt, resulting in positive carry.
7. Rapid Fluctuation of the VM Levels: The above assumption is an aggressive one as VM varies greatly in short time scales and generally constitutes an unstable base from which to retire debt. In practice it is far easier to let debt mature than to retire long-dated bonds on the secondary market.
8. Art of Treasury Liquidity Management: Normally collateral is over-provisioned allowing for buffers as the treasury needs to manage liquidity prudently. There are additional costs to collateral over-provisioning and risk management of the liquidity buffers, all of which are ignored here.
9. Funding Rates and Strategies - Assumption: Besides the above "neutral" assumptions about investment returns, the definition of FDA used here makes additional assumptions about treasury funding strategies and funding rates. The specified ones are listed below.
    a. Worst-case Funding Rate Scenario => The worst scenario regarding funding rates is assumed, i.e., that derivative receivables cannot be passed as collateral and that VM borrowing is entirely unsecured.
    b. Best Case Over-provisioning Scenario => The best-case scenario regrading over-provisioning is also assumed, that is the bank procures on an overnight basis only the cash collateral that is strictly needed for VM borrowing, not more.
10. FVA and Funding Strategy Link: Both of the above assumptions are reasonable for accounting purposes, although they admittedly represent idealizations of the actual funding behavior (which no doubt varies from bank to bank). In this view, it is important to note that FVA calculations are non-unique and tied to the *actual* funding strategy to which a firm commits, a fundamental difference from the ideas underlying fair value pricing of derivatives.

11. Derivative Contract Fair Value Theory: According to classical finance theory, if there exists a replication strategy, then the cost of replication/hedging equals the fair value. Hence if one replication strategy is theoretically possible, then the cost of implementing it ought to equal the fair value *whether or not* the strategy is actually implemented. If more than one replication strategy can potentially be implemented, absence of arbitrage indicates that the cost of implementing each one equals the fair value.
12. MMT on Funding Cost Impact: For funding cost valuation, the MMT tells us that the fair value of funding strategies for trades entered at fair value is zero, as FVA does not represent the fair value of an asset, but instead the wealth transfer amount from the shareholders to the senior creditors.

Cash Flows at Counterparty Default

1. Counterparty Triggered Default Flows: If τ_i is the default time of the counterparty i, let $D_i(\tau_i)$ be the default cash flow received from the bank from the counterparty i as part of the default triggered closeout of the i^{th} portfolio. $D_i(\tau_i)$ is governed by the ISDA agreement between the bank and the i^{th} counterparty.
2. ISDA Standard Closeout Assumption: For simplicity's sake it is assumed that the closeout procedure follows the ISDA Market Quotation Protocol of 1992, and extensions to the other protocols are covered later.
3. Applying the ISDA 1992 Protocol: In the prevailing interpretation of the ISDA 1992 closeout protocol, if either the counterparty or the bank itself default prior to the maturity of the trade portfolio, then the portfolio is settled at default free levels, i.e., XVA adjustments are excluded from the calculation of the settlement amount.
4. Counterparty Default Cash Flow: In other words, the default cash flows at time τ_i received the bank from counterparty i is

$$D_i(\tau_i) = \mathbb{I}_{\tau_i < \tau_B}\left[\mathbb{I}_{V_i(\tau_i) \geq 0} R_i V_i(\tau_i) + \mathbb{I}_{V_i(\tau_i) < 0} V_i(\tau_i)\right]$$

where τ_B is the default time of the bank and

$$R_i \in [0, 1]$$

is the recovery rate received from the counterparty i. Observe the presence of the default indicator $\mathbb{I}_{\tau_i < \tau_B}$ in this expression, a reflection of the fact that if the bank defaults prior to the counterparty i, then no cash flow takes place at time τ_i since the unsecured derivative portfolio held with counterparty i is assumed to have been unwound at τ_B.

5. Sign of the Counterparty Default Cash Flow: $D_i(\tau_i)$ may be positive or negative, and can be considered an inherent part of the portfolio flows of counterparty i. Notice that if

$$V_i(\tau_i) \geq 0$$

and

$$\tau_B > \tau_i$$

the above equation represents a loss to bank B. The present value of this loss is the CVA_i for portfolio i.

6. Hedging the Counterparty Risk: As indicated earlier, the cash flow formula above assumes that the counterparty credit risk above is left unhedged. In reality the CVA desk seeks to layoff the counterparty credit risk through the purchase of credit hedges such as single names and index CDS.
7. CVA Hedge Collateral VM Pool: This setup does not alter our conclusions about how to account for CVA, but it does mean that CVA hedges are typically entered with dealers/exchanges on a fully collateralized basis, which in itself produces VM postings that can be introduced into the overall VM "pool" generated by the client trades.

Cash Flows at Bank Default

1. Collateral Impact on Bank Default: At the bank default time τ_B, any posted collateral amount stays with the holder, and all the inter-dealer positions are torn up, with no economic impact to any dealer counterparty. Similarly the default of a dealer used as a counterparty for hedging purposes has no impact as all credit risk is covered by collateral.
2. Explicit Cash Flows: The positions with n counterparties are typically settled at ISDA terms, leading to a loss for those counterparties that have a previous exposure to the bank. Under the standard ISDA 1992 terms, the effective default cash flow at time τ_B received by the bank from counterparty i may be written as

$$D_i(\tau_B) = \mathbb{I}_{\tau_i > \tau_B}\left[\mathbb{I}_{V_i(\tau_B) \geq 0} V_i(\tau_B) + \mathbb{I}_{V_i(\tau_B) < 0} R_B V_i(\tau_B)\right]$$

   where

$$R_B \in [0, 1]$$

   is the recovery rate for the bank B.
3. Credit Losses to Counterparty: Notice that the loss to counterparty is therefore $-(1 - R_B) V_i(\tau_B)^-$. Senior bank creditors, who in aggregate have a claim on derivative receivables of value $V_i(\tau_B)^-$ for the i^{th} counterparty, would similarly recover a fraction $R_B V_i(\tau_B)^-$ of their claim.
4. DVA Gain to the Bank: Again for the sake of simplicity it is assumed that the recovery rate on the senior debt and on the derivatives portfolio are identical. Notice that if

$$V_i(\tau_B) < 0$$

   then the equation for $D_i(\tau_B)$ represents a gain for the bank relative to a fully collateralized payout of $V_i(\tau_B)$. The present value of this gain is the DVA_i associated with counterparty i.
5. Non additivity of the VM Collateral: Finally note that if is the recovery rate for the VM collateral lenders, upon defaulting the bank does not return (i.e., effectively receives) the following amount in VM cash:

$$D_i(\tau_B) = \left(1 - R_B{}^C\right)\left[\sum_i V_i(\tau_B)^-\right]$$

Since the total amount of VM collateral borrowed is reduced by exercising the RHO, the amounts lent and recovered from the collateral lenders cannot be represented as a linear sum over netting sets.

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Credit and Funding Valuation Adjustments

Introduction

1. Cash Flow Impact On Accounting: Having outlined all relevant cash flows above, the next step is to work out how these cash flows are valued, and how they affect the capital structure of the OTC book. In particular, this section looks at in detail the relevant XVA metrics needed for the accounting treatment outlined earlier.

CVA and DVA

1. Bilateral FTD CVA Formulation: The cash flows triggered by the counterparty defaults were under ISDA 1992 were discussed earlier. The loss in value due to counterparty i is measured by the present value of the protection against the value loss inherent in

$$D_i(\tau_i) = \mathbb{I}_{\tau_i < \tau_B}\left[\mathbb{I}_{V_i(\tau_i) \geq 0} R_i V_i(\tau_i) + \mathbb{I}_{V_i(\tau_i) < 0} V_i(\tau_i)\right]$$

2. Expression for CVA_i or $FTDCVA_i$:

$$CVA_i = FTDCVA_i = \mathbb{E}\left[e^{-\int_0^{\tau_i} r_{OIS}(u)du}\mathbb{I}_{\tau_i<\tau_B}(1-R_i)V_i(\tau_i)^+\right]$$

$$= \int_0^\infty \mathbb{E}\left[e^{-\int_0^{t} r_{OIS}(u)du}\mathbb{I}_{t<\tau_B}(1-R_i)V_i(t)^+\right]\mathbb{Q}(\tau_i \in [t, t+dt])$$

$$= \int_0^\infty \mathbb{E}\left[e^{-\int_0^{t}\{r_{OIS}(u)+\lambda_i(u)\}du}\lambda_i(t)\mathbb{I}_{t<\tau_B}(1-R_i)V_i(t)^+\right]dt$$

$$= \int_0^\infty \mathbb{E}\left[e^{-\int_0^{t}\{r_{OIS}(u)+\lambda_i(u)+\lambda_B(u)\}du}\lambda_i(t)(1-R_i)V_i(t)^+\right]dt$$

3. <u>Applying a Stochastic Default Intensity</u>: Here $\mathbb{E}[\cdot]$ denotes the expectation in the risk neutral measure $\mathbb{Q}$ and $\lambda_i(t)$ is the (possibly stochastic) default intensity for counterparty i. The concept of default intensity was introduced by Lando (1998) in the context of reduced for models based on Cox processes, but has become meaningful (and quite useful) also for many structural processes.
4. <u>Unilateral CVA Contra-Asset Adjustment</u>: As discussed in the previous section, the CVA entry to be booked as a contra-asset is the unilateral UCVA metric given by the expression

$$UCVA_i = \int_0^\infty \mathbb{E}\left[e^{-\int_0^{t}\{r_{OIS}(u)+\lambda_i(u)\}du}\lambda_i(t)(1-R_i)V_i(t)^+\right]dt$$

without the indicator function $\mathbb{I}_{t<\tau_B}$.

5. <u>CVA Contra-Liability Adjustment Expression</u>: This entity is accompanied by a CL adjustment defined as follows:

$$CVA_{CL,i} = UCVA_i - FTDCVA_i = \int_0^\infty \mathbb{E}\left[e^{-\int_0^{t}\{r_{OIS}(u)+\lambda_i(u)\}du}\mathbb{I}_{t\geq\tau_B}\lambda_i(t)(1-R_i)V_i(t)^+\right]dt$$

6. <u>DVA Contra-Liability Adjustment Formulation</u>: The $CVA_{CL,i}$ can be interpreted as the "DVA of the CVA", i.e., the benefit the senior creditors have on the default of the bank on the default protection contract implicitly sold to the counterparties. The more substantial benefit

associated with the option of defaulting on the underlying derivatives is instead captured by a DVA term defined as

$$DVA_i = \int_0^{\infty} \mathbb{E}\left[e^{-\int_0^t \{r_{OIS}(u)+\lambda_B(u)\} du} \mathbb{I}_{t<\tau_i} \lambda_B(t)(1-R_B)V_i(t)^-\right] dt$$

7. Valuation Across Netting Sets: CVA and DVA numbers maybe added across counterparties giving rise to

$$FTDCVA = CVA = \sum_i CVA_i$$

$$UCVA = \sum_i UCVA_i$$

$$CVA_{CL} = \sum_i CVA_{CL,i}$$

$$DVA = \sum_i DVA_i$$

8. Accounting of CA/CL Adjustments: The booking of these quantities as CA and CL adjustments was covered earlier and is tabulated below.

FVA and FDA

1. Funding Payments Cash Flow Stream: As explained earlier, to find fair valuation expressions for the FVAs, one needs to value a continuous cash flow stream for funding costs at the rate given by the spread in

$$s_B(t) = r_B(t) - r_{OIS}(t) \geq 0$$

It was argued earlier that the contribution to this cost in the time period $[t, t + \Delta t]$ is $C_{VM}(t)^+ s_B(t) \Delta t$.

2. Funding Cost Valuation Expression: The discounted present value of the future funding costs is defined as follows:

$$FVA = \mathbb{E}\left[\int_0^\infty e^{-\int_0^t r_{OIS}(u)du} C_{VM}(t)^+ s_B(t) dt\right]$$

$$= \mathbb{E}\left[\int_0^\infty e^{-\int_0^t \{r_{OIS}(u) + \lambda_B(u)\}du} \left\{\sum_i V_i(t) \mathbb{I}_{t < \tau_i}\right\}^+ s_B(t) dt\right]$$

where the second equality follows form the definition of the net collateral $C_{VM}(t)$ in

$$C_{VM}(t) = \sum_i V_i(t) \mathbb{I}_{t < \tau_i} \mathbb{I}_{t < \tau_B}$$

As explained earlier we include the cash flows only up to the time of the default of the bank.

3. Accommodating the Timing Element of Default: The are a plethora of competing definitions of FVA, some of which deliberately avoid the complication of a default timing element. Carver (2013) contains details on the "dark art" of FVA definitions.
4. Funding Spreads for VM Receivables: Due to the lack of infrastructure to guarantee water-tight collateralization mechanics with unsecured derivatives receivables as underlying, funding spreads for VM collateral are observed to be near unsecured levels even in the rare cases where derivatives receivables are nominally mentioned as collateral.
5. PV of the Funding Benefits: The present value of the gain that senior creditors of the bank gain due to the inability of the collateral lenders to recover in full on default is referred to as FDA here. To compute this quantity, observe that at the time of a bank default, senior creditors will gain a benefit equal to the fraction $1 - {R_B}^C$ of the pool of derivative receivables, while a fraction ${R_B}^C$ goes to the collateral lenders.

6. FDA Valuation Adjustment Formulation: From the above we have that

$$FDA = \mathbb{E}\left[\int_0^{\tau_B} e^{-\int_0^t r_{OIS}(u)du}\left(1 - R_B{}^C\right)\lambda_B(t)\left\{\sum_i V_i(t)\mathbb{I}_{t<\tau_i}\right\}^+ dt\right]$$

$$= \mathbb{E}\left[\int_0^{\infty} e^{-\int_0^t \{r_{OIS}(u)+\lambda_B(u)\}du}\left(1 - R_B{}^C\right)\lambda_B(t)\left\{\sum_i V_i(t)\mathbb{I}_{t<\tau_i}\right\}^+ dt\right]$$

Thanks to the risk neutral valuation relation

$$s_B(t) = \left(1 - R_B{}^C\right)\lambda_B(t)$$

for the bank credit spreads, we find

$$FVA = FDA$$

consistent with the discussion in the earlier section.

7. MMT Consistency of FVA/FDA Accounting: The FVA and the FDA contribute with opposite signs to the bank fair valuation and cancel each other on the bank balance sheet, in agreement with the MMT. In other words, the fair valuation of the bank assets is indifferent to the funding strategy employed, and therefore to the FVA and the FDA. This relation was first noticed in Hull and White (2012), where the FVA is called DVA2 and the equation

$$FVA = FDA$$

was stated as a direct consequence of the MMT.

FCA and FBA

1. Motivation Behind the FCA/FBA Accounting: FCA/FBA accounting is an approximation to the funding value adjustment motivated by a desire to base the methodology upon netting set specific metrics that are computable by means of traditional CVA systems. This methodology is in a sense an extension to large books of the work in Piterbarg (2010) and Burgard and Kjaer (2011) which focused on the case of individual trades treated in isolation and not in a portfolio context.
2. FCA/FBA Methodology - SFVA Computation: To better understand the logic behind FCA/FBA accounting, we note that if hypothetically the bank was never a net receiver of VM collateral and there were no collateral thresholds, then the funding cost of a cash flow $X(T)$ would be represented as

$$SFVA_X = V_X(0) - {V_X}^*(0)$$

where

$$V_X(t) = \mathbb{E}_t\left[e^{-\int_t^T r_{OIS}(u)du} X(T)\right]$$

and

$${V_X}^*(t) = \mathbb{E}_t\left[e^{-\int_t^T r_B(u)du} X(T)\right]$$

where $\mathbb{E}_t$ represents the time t expectation in the risk-neutral measure. One interpretation of this formula is that the CFD desk has access to borrowing and lending lines at a funding rate of $r_B(t)$ and are not subject to any credit risk.
3. Application of the Feynman-Kac Theorem: By the Feynman-Kac Theorem we have that

$$\mathbb{E}_0\left[e^{-\int_0^T r_B(u)du} X(T)\right] = \mathbb{E}_0\left[e^{-\int_0^T r_{OIS}(u)du} X(T) - \int_0^T e^{-\int_0^t r_{OIS}(u)du}\{r_B(t) - r_{OIS}(t)\}{V_X}^*(t)dt\right]$$

and also by symmetry

$$\mathbb{E}_0\left[e^{-\int_0^T r_{OIS}(u)du}X(T)\right] = \mathbb{E}_0\left[e^{-\int_0^T r_B(u)du}X(T) - \int_0^T e^{-\int_0^t r_B(u)du}\{r_{OIS}(t) - r_B(t)\}V_X(t)dt\right]$$

4. Simplification of the SFVA Calculation: It follows that

$$SFVA_X = V_X(0) - {V_X}^*(0)$$

may be rewritten as

$$SFVA_X = \mathbb{E}_0\left[\int_0^T e^{-\int_0^t r_{OIS}(u)du}s_B(t){V_X}^*(t)dt\right] = \mathbb{E}_0\left[\int_0^T e^{-\int_0^t r_B(u)du}s_B(t)V_X(t)dt\right]$$

In practice, the dependence on ${V_X}^*(t)$ in the exposure integral is inconvenient, so the second equality is probably the most useful in applications.

5. Applying SFVA to the Netting Set: Extending the result above to the more general portfolio case with collateralized hedges, we write the SFVA for counterparty i as

$$SFVA_i = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t r_B(u)du}s_B(t)V_i(t)dt\right]$$

Let us remark that assuming there are no collateral thresholds, the $SFVA_i$ decomposes further into a sum over the SFVA of trades contained in the i^{th} netting set. Collateral thresholds break this property and make the SFVA a netting set specific amount.

6. Decomposition into FCA/FBA Metrics: The netting set specific SFVA can be decomposed into cost and benefit components as follows.

$$SFVA_i = FCA_i - FBA_i$$

where

$$FCA_i = \mathbb{E}_0\left[\int_0^{\infty} e^{-\int_0^t r_B(u)du} s_B(t) V_i(t)^+ dt\right]$$

and

$$FBA_i = \mathbb{E}_0\left[\int_0^{\infty} e^{-\int_0^t r_B(u)du} s_B(t) V_i(t)^- dt\right]$$

We also introduce the aggregate amounts

$$SFVA = \sum_i SFVA_i$$

$$FCA = \sum_i FCA_i$$

$$FBA = \sum_i FBA_i$$

CA and CL Adjustments

1. Lack of Consistency in FCA/FBA: While, as described earlier, FVA/FDA accounting cleanly splits the CA and CL adjustments, the lack of consistency in the FCA/FBA accounting requires some effort to strike a reasonable compromise between conflicting assumptions. For this purpose, notice that the FBA in

$$FBA_i = \mathbb{E}_0\left[\int_0^{\infty} e^{-\int_0^t r_B(u)du} s_B(t) V_i(t)^- dt\right]$$

is quiet similar to the DVA in

$$DVA_i = \int_0^{\infty} \mathbb{E}\left[e^{-\int_0^t \{r_{OIS}(u)+\lambda_B(u)\}du} \mathbb{I}_{t<\tau_i} \lambda_B(t)(1-R_B)V_i(t)^-\right] dt$$

yet it differs from it in two ways.

2. Effective Discount Rate in FCA/FBA: First the FBA expression does not contain the indicator function $\mathbb{I}_{t<\tau_i}$. Second, the FBA losses are discounted at the rate

$$r_B(t) = r_{OIS}(t) + \lambda_B(t)(1-R_B)$$

rather than at (effectively) the rate $r_{OIS}(t) + \lambda_B(t)$. Only in the unlikely case of

$$R_B = 0$$

does the discounting rule in the two expressions match. In general we have that

$$FBA > DVA$$

3. FBA as a CL Adjustment: The FBA partially accounts for the RHO, but always contains a large overlap with the standard DVA on payables. Since the two components of FBA cannot be disentangled from each other, the full FBA entry is normally configured as a CL adjustment, similar to the way DVA is treated. On the other hand the FCA is clearly a CA deduction from CET1 that adds to the usual CVA deduction.
4. Comparison of FCA/FDA and FCA/FBA:

	CA Adjustment	**CL Adjustment**

FVA/FDA	$UCVA + FVA$	$CVA_{CL} + DVA + FDA$
FCA/FBA	$UCVA + FCA$	$CVA_{CL} + FBA$

5. FCA/FBA vs. FDA/FVA CA Adjustments: To comment on the relative magnitudes of the CA and CL adjustments between the two methods, notice that since the FCA/FBA approximation effectively amounts to recognizing the re-hypothecation benefits only within the individual netting sets, one expects that the FCA is much larger than the FVA, i.e., the FCA/FBA accounting produces much larger CA deductions than the FVA/FDA accounting.
6. FVA/FDA vs FCA/FBA CL Adjustments: As for the CL adjustments, note that the CL adjustment under the FCA/FBA accounting effectively drops the DVA term in order to avoid double-counting. The case study conducted by Albanese and Andersen (2014) confirms these conclusions. They also quantify the RHO amount, the degree of symmetry breaking in the FCA/FBA accounting, and the difference between FBA and DVA.

Own Credit Sensitivities

1. Expected Positive Exposure Valuation Formulation: As discussed earlier CET1 deduction should generally not decrease when the bank spreads increase. To investigate the sensitivity of the FVA to the bank spread, it is necessary to estimate the discounted Expected Positive Exposure (EPE) up to time t as

$$EPE(t) = \mathbb{E}\left[\int_0^T e^{-\int_0^t r_{OIS}(u)du}\left\{\sum_i V_i(t)\mathbb{I}_{t<\tau_i}\right\}^+ dt\right]$$

where T (the upper integration limit) is the longest maturity on any trade in the funding set in question.
2. FVA Funding Spread Impact Theorem: Assume that $s_B(t)$ is independent of $r_{OIS}(t)$ and $\left\{\sum_i V_i(t)\mathbb{I}_{t<\tau_i}\right\}^+$ and let $s_B(t)$ be subject to a perturbation of the type

$$s_B(t) \to s_B(t) + \epsilon h(t)$$

where ϵ is a scalar and

$$h(t) \geq 0$$

is a bounded deterministic function. Also define the Gateaux derivative

$$\mathcal{D}_h FVA = \left.\frac{\partial FVA}{\partial \epsilon}\right|_{\epsilon=0}$$

Then

$$\mathcal{D}_h FVA \geq 0$$

if and only if

$$\int_0^T EPE(t)\mathbb{E}\left[e^{-\int_0^t \lambda_B(s)ds}\left\{h(t) - \frac{1}{1 - {R_B}^C} s_B(t) \int_0^t h(s)\, ds\right\}\right] dt \geq 0$$

3. <u>FVA Funding Spread Impact - Proof</u>: Defining

$$q_B = \frac{1}{1 - {R_B}^C}$$

and re-casting FVA as a function of ϵ in

$$s_B(t) \to s_B(t) + \epsilon h(t)$$

$$FVA = \mathbb{E}\left[\int_0^T e^{-\int_0^t \{r_{OIS}(s)+q_B[s_B(s)+\epsilon h(s)]\}ds}\{s_B(t)+\epsilon h(t)\}\left\{\sum_i V_i(t)\mathbb{I}_{t<\tau_i}\right\}^+ dt\right]$$

where we have used

$$s_B(t) = \left(1 - {R_B}^C\right)\lambda_B(t)$$

Straightforward calculus shows that

$$\mathcal{D}_h FVA = \frac{\partial FVA}{\partial \epsilon}\bigg|_{\epsilon=0}$$

$$= \mathbb{E}\left[\int_0^T e^{-\int_0^t \{r_{OIS}(s)+\lambda_B(s)\}ds}\left\{h(t)\right.\right.$$

$$\left.\left.- q_B s_B(t)\int_0^t h(s)\,ds\right\}\left\{\sum_i V_i(t)\mathbb{I}_{t<\tau_i}\right\}^+ dt\right]$$

$$= \int_0^T EPE(t)\mathbb{E}\left[e^{-\int_0^t \lambda_B(s)ds}\left\{h(t) - q_B s_B(t)\int_0^t h(s)\,ds\right\}\right]dt$$

where the second equality follows from the independence assumption. The criterion

$$\int_0^T EPE(t)\mathbb{E}\left[e^{-\int_0^t \lambda_B(s)ds}\left\{h(t) - q_B s_B(t)\int_0^t h(s)\,ds\right\}\right]dt \geq 0$$

then follows.

4. Gateaux Function Lower Bound Corollary: In the setting of the previous theorem, if

$$h(t) \geq \frac{1}{1 - {R_B}^C}\frac{\mathbb{E}\left[s_B(t)e^{-\int_0^t \lambda_B(s)ds}\right]}{\mathbb{E}\left[e^{-\int_0^t \lambda_B(s)ds}\right]}\int_0^t h(s)\,ds \geq 0\ \forall\ t \in [0, T]$$

then

$$\mathcal{D}_h FVA \geq 0$$

5. Gateaux Function Lower Bound Proof: This corollary follows directly from

$$\int_0^T EPE(t)\mathbb{E}\left[e^{-\int_0^t \lambda_B(s)ds}\left\{h(t) - \frac{1}{1 - {R_B}^C} s_B(t) \int_0^t h(s)\, ds\right\}\right] dt \geq 0$$

and from the fact that

$$EPE(t) \geq 0$$

Further it turns out that it is a sufficient but conservative condition that ensures

$$\mathcal{D}_h FVA \geq 0$$

6. Right-way Regulatory Sensitivity Criterion: Suppose for instance that h and s_B are positive constants, i.e., we have a flat credit curve that is being shifted up in a parallel fashion. "Right-way" regulatory sensitivity is guaranteed by

$$h(t) \geq \frac{1}{1 - {R_B}^C} \frac{\mathbb{E}\left[s_B(t) e^{-\int_0^t \lambda_B(s)ds}\right]}{\mathbb{E}\left[e^{-\int_0^t \lambda_B(s)ds}\right]} \int_0^t h(s)\, ds \geq 0 \ \forall\ t \in [0, T]$$

if

$$h \geq \frac{s_B h T}{1 - {R_B}^C}$$

or

$$s_B \leq \frac{1 - {R_B}^C}{T}$$

7. Right-way Sensitivity Typical Behavior: Some typical values for the constants in $\frac{1-{R_B}^C}{T}$ are

$${R_B}^C = 40\%$$

and

$$T = 10$$

which yields

$$s_B \leq 6\%$$

a condition that is rarely violated (common values for s_B are $1 - 2\%$).

8. Alternate Right-way Regulatory Sensitivity: Alternatively, if we assume

$$EPE(t) \approx EPE$$

is approximately constant in

$$\int_0^T EPE(t)\mathbb{E}\left[e^{-\int_0^t \lambda_B(s)ds}\left\{h(t) - \frac{1}{1 - {R_B}^C} s_B(t) \int_0^t h(s)\, ds\right\}\right] dt \geq 0$$

then it can be written (again for constant h and s_B)

$$\mathcal{D}_h FVA \approx EPE \cdot \int_0^T e^{-\lambda_B t}\left(h - \frac{1}{1 - {R_B}^C} s_B t\right) dt \approx EPE \cdot hT\left[1 - \frac{1}{2} q_B s_B T\right]$$

which leads to a bound that is twice as loose as before

$$s_B \le \frac{2\left(1 - {R_B}^C\right)}{T}$$

9. Normal Situation FVA Regulatory Sensitivity: In most situations the $EPE(t)$ terms in

$$\mathcal{D}_h FVA = \int_0^T EPE(t) \mathbb{E}\left[e^{-\int_0^t \lambda_B(s) ds}\left\{h(t) - q_B s_B(t) \int_0^t h(s)\, ds\right\}\right] dt$$

typically peak long before T, which would widen the bound even further. All in all it is safe to say that in normal conditions FVA increases when s_B is decreased.

10. SFVA Funding Spread Perturbation Impact: If we repeat the arguments behind the FVA funding spread impact theorem for the SFVA defined in

$$SFVA_i = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t r_B(u) du} s_B(t) V_i(t) dt\right]$$

the result is easily seen to be

$$\mathcal{D}_h SFVA = \mathbb{E}\left[\int_0^T e^{-\int_0^t \{r_{OIS}(u) + s_B(u)\} du}\left\{h(t) - s_B(t) \int_0^t h(s)\, ds\right\}\left\{\sum_{i=1}^n V_i(t)\right\} dt\right]$$

11. Normal Conditions SFVA Regulatory Sensitivity: In case $h(t)$ satisfies the conditions above that ensure the positivity of $\mathcal{D}_h FVA$ and in situations where the funding sets are prevailingly

net payables, the SFVA has wrong sign sensitivities and is unsuitable as a CET1 deduction. This issue is expanded later on.

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Triggers and Close-out Adjustments

Introduction

1. Motivation: This section provides a greater detail and a realistic analysis of the cash flows and the close out protocols.
2. Default-Free Fair Valuation: Let

$$V_i^U(t) \; \forall \; i = 1, \cdots, n$$

be the default free valuation at time t of the portfolio held with counterparty i, i.e., the valuation computed by neglecting the upfront XVA payments, and also neglecting credit risks and funding costs. The default free value of the unsecured book is additive over trades, in particular

$$V^U(t) = \sum_i V_i^U(t)$$

3. FTP Transfer via Super-Replication: Hedges are typically struck at par and engender cash flows across time that super-replicate the cash flows of the unsecured trades. Strict super-replication is required in order to ensure a positive return on equity.
4. Value Adjustment on Super-Replication: In the case of swaps adjustments are usually embedded as a fixed spread in addition to either the fixed leg or the floating leg or both. The cash flows are such that typically the default free valuation of the unsecured portfolio is positive at inception.
5. CFD Routing of the Hedge Flows: The hedge typically consists of one or a portfolio of collateralized swaps entered in the inter-dealer market, initially struck at par and whose cash

flows super replicate those of the unsecured trade. The excess cash flows are then routed to the CVA desk and the CFD desk at the time when they are received.

6. Reducing of the VM Posting Impact: In the case of swaptions and FX options premia are typically paid at maturity and struck at the level for which the present value of the option are zero. This structure lessens the amount of VM to be exchanged. Unsecured trade often pay an upfront premium which is added to the book cash account. By transforming a portion of the upfront payment into a payment at the option maturity on achieves super-replication in this case as well. Excess cash flows are given by the XVA adjustments.
7. Hedge Portfolios across Netting Sets: In general hedges are not specific to individual netting sets. Even if hedges are entered initially on a deal-specific basis, the compression cycles of the collateralized swap portfolio reduce the number of hedge trades and obfuscates the attribution of the individual netting sets.
8. Super Replicated Hedge Portfolio Value: Nevertheless by means of a hedge attribution analysis, it is possible to arrive at the concept of the value of the hedge book for a single portfolio i – we call it ${V_i}^H(t)$. Super-replication is achieved when

$${V_i}^H(t) \leq {V_i}^U(t)$$

   The present value of the difference ${V_i}^U(t) - {V_i}^H(t)$ is the FTP.
9. Value of the Hedge Book: The value of the hedge book is denoted as

$${V_i}^H(t) = \sum_i {V_i}^H(t)$$

10. Gap Risk on Collateralized Hedges: As the inter-dealer market and the exchanges require the posting of the collateral in full, the hedge trades are affected by the bank/counterparty credit risk only because of the gap risk exposure. For simplicity's sake the gap risk on collateralized hedges is neglected here since the analysis and the notations become very heavy. However a professional system implementation should account for it as the effect can be quite material.

Collateral Triggers and Close-outs

1. CSA Based Collateral Trigger Levels: Typically CSA collateral agreements include time-dependent collateral trigger levels $\Gamma_i(t)$ that dependent on the counterparty rating level and is such that, if the exposure of the counterparty surpasses $\Gamma_i(t)$ then the counterparty is obliged to post collateral above that threshold. The bank has a similar trigger level $\Gamma_{B,i}(t)$ for each counterparty i. Initial margin on unsecured trade is not a common practice as they would be ineffective unless thresholds are struck at virtually zero level.
2. Collateral Trigger Impact on VM: The equation for the variation margin accounts can be refined for ratings dependent collateral thresholds and also for the fact that variation margins are received on the CVA hedges. The more general expression is

$$C_{VM,i}(t) = \mathbb{I}_{\tau_B>t}\mathbb{I}_{\tau_i>t}\left[-V_i^{\,H}(t) + \left\{V_i^{\,U}(t) - \Gamma_i(t)\right\}^+ - \left\{V_i^{\,U}(t) - \Gamma_{B,i}(t)\right\}^- - CVA_{i,CA}(t)\right]$$
$$\approx \mathbb{I}_{\tau_B>t}\mathbb{I}_{\tau_i>t}\left[-V_i^{\,H}(t) + \left\{V_i^{\,U}(t) - \Gamma_i(t)\right\}^+ - \left\{V_i^{\,U}(t) - \Gamma_{B,i}(t)\right\}^-\right]$$

   where again the approximation is implemented in the Albanese and Andersen (2014) case study.
3. Generalized Hedge VM Funding Rate: The equation

$$s_B(t) = r_B(t) - r_{OIS}(t)$$

   for the spread $s_B(t)$ can also be extended. In general $s_B(t)$ should defined as the spread between the funding rate of the bank on the short-term debt instruments used for the purpose of financing variation margin and the rate received on the VM posted. In general the interest rate received on the VM posted may differ from OIS only by a small spread.
4. Trigger Based FVA/FDA Impact: The portfolio FVA is given by (note the VM related alteration)

$$FVA = \mathbb{E}\left[\int_0^{\infty} e^{-\int_0^t\{r_{OIS}(u)+\lambda_B(u)\}du}\left\{\sum_i C_{VM,i}(t)\right\}^+ s_B(t)dt\right]$$

The FDA is sensitive to the value of bank receivables at the time of the bank impact and is given by

$$FDA = \mathbb{E}\left[\int_0^{\tau_B} e^{-\int_0^t \{r_{OIS}(u)+\lambda_B(u)\}du}\left(1 - R_B{}^C\right)\lambda_B(t)\left\{\sum_i \max\left(\min\left(V_i{}^U(t), \Gamma_i(t)\right), -\Gamma_{B,i}(t)\right)\right\}^+ dt\right]$$

5. Composite Trigger-Based FVA Formulation: On the above basis, the expression for asymmetric FVA from the bank perspective in

$$FVA = \mathbb{E}\left[\int_0^{\infty} e^{-\int_0^t \{r_{OIS}(u)+\lambda_B(u)\}du}\left\{\sum_i C_{VM,i}(t)\right\}^+ s_B(t)dt\right]$$

needs to be amended to

$$FVA = \mathbb{E}\left[\int_0^{\infty} e^{-\int_0^t \{r_{OIS}(u)+\lambda_B(u)\}du}\left\{\sum_i \left[-V_i{}^H(t)\mathbb{I}_{\tau_i>t} + \left\{V_i{}^U(t)\mathbb{I}_{\tau_i>t} - \Gamma_i(t)\right\}^+ - \left\{V_i{}^U(t)\mathbb{I}_{\tau_i>t} - \Gamma_{B,i}(t)\right\}^-\right]\right\}^+ s_B(t)dt\right]$$

This equation accounts for the VM to be posted to or received from unsecured derivative counterparties depending on the thresholds.

6. Asset-Liability Super Replication Mismatch: Notice that FVA and FBA are not precisely equal but they are very close. Because of the inequality

$$V_i{}^H(t) \le V_i{}^U(t)$$

it turns out

$$FVA \leq FDA$$

This discrepancy is due to the fact that the FDA payments are embedded in the deal structure, and at the time of the bank default, the senior creditors still hold a claim to future FTP payments for funding without the obligation (or even the capability) to continue hedging.

7. Super Replication Impact on MMT: All in all, since banks enact slightly super-replicating strategies (as opposed to precise replication), in order to have a positive return on equity, the FDA tends to be slightly larger than the FVA.
8. Wealth Transfer From Derivative Counterparties: The above can be viewed as a wealth transfer from the derivatives counterparties to the senior creditors. The MMT is still not invalid as the game is still zero sum, as long as one includes in the analysis not only shareholders and senior creditors but also the derivatives counterparties.
9. Trigger Based FCA/FBA Impact: The expression for $C_{VM,i}(t)$ above can be used to extend

$$FCA_i = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t r_B(u)du} s_B(t) V_i(t)^+ dt\right]$$

as

$$FCA_i = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t r_B(u)du} s_B(t) C_{VM,i}(t)^+ dt\right]$$

while

$$FBA_i = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t r_B(u)du} s_B(t) V_i(t)^- dt\right]$$

becomes

$$FBA_i = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t r_B(u)du} s_B(t) C_{VM,i}(t)^- dt\right]$$

Incorporating ISDA 1992 Close-outs

1. Impact of the ISDA Closeouts: The representation of the cash flows at the time of counterparty defaults under the ISDA 1992 closeout protocol in

$$D_i(\tau_i) = \mathbb{I}_{\tau_i<\tau_B}\left[\mathbb{I}_{V_i{}^U(\tau_i)\geq 0} R_i V_i{}^U(\tau_i) + \mathbb{I}_{V_i{}^U(\tau_i)<0} V_i{}^U(\tau_i)\right]$$

is modified as follows to include collateral thresholds:

$$D_i(\tau_i) = \mathbb{I}_{\tau_i<\tau_B}\left[\mathbb{I}_{V_i{}^U(\tau_i)\geq 0} R_i \min\left(V_i{}^U(\tau_i), \Gamma_i(\tau_i)\right) + \mathbb{I}_{V_i{}^U(\tau_i)<0} V_i{}^U(\tau_i)\right]$$

Similarly

$$D_i(\tau_B) = \mathbb{I}_{\tau_i>\tau_B}\left[\mathbb{I}_{V_i{}^U(\tau_B)\geq 0} V_i{}^U(\tau_B) + \mathbb{I}_{V_i{}^U(\tau_B)<0} R_B V_i{}^U(\tau_B)\right]$$

becomes

$$D_i(\tau_B) = \mathbb{I}_{\tau_i>\tau_B}\left[\mathbb{I}_{V_i{}^U(\tau_B)\geq 0} V_i{}^U(\tau_B) + \mathbb{I}_{V_i{}^U(\tau_B)<0} R_B \max\left(V_i{}^U(\tau_B), -\Gamma_B(\tau_B)\right)\right]$$

2. Exit Price of the Trade: Under the 2009 ISDA Close-out rules, cash flows differ in that $V_i{}^U(\tau_i)$ is replaced by the exit price of the trade, including the residual DVA_i at the time of the default. On the other hand, in case the bank defaults first, then the i^{th} counterparty has a right to recover its own DVA, which is the CVA. This implies that when valuing CVA from the bank viewpoint, these default losses happen after the default of the bank itself.

3. <u>Closeout Payments after the Default</u>:

$$D_i(\tau_i) = \mathbb{I}_{\tau_i<\tau_B}\left[\mathbb{I}_{V_i^{(+)}(\tau_i)\geq 0}R_i \min\left(V_i^{(+)}(\tau_i), \Gamma_i(\tau_i)\right) + \mathbb{I}_{V_i^{(+)}(\tau_i)<0}V_i^{(+)}(\tau_i)\right]$$

$$D_i(\tau_B) = \mathbb{I}_{\tau_i>\tau_B}\left[\mathbb{I}_{V_i^{(-)}(\tau_B)\geq 0}V_i^{(-)}(\tau_B) + \mathbb{I}_{V_i^{(-)}(\tau_B)<0}R_B V_i^{(-)}(\tau_B)\right]$$

where

$$V_i^{(+)}(\tau_i) = V_i^U(\tau_i) + DVA_i(\tau_i) \approx V_i^U(\tau_i)$$

and

$$V_i^{(-)}(\tau_B) = V_i^U(\tau_B) - CVA_i(\tau_B) \approx V_i^U(\tau_B)$$

4. <u>SFVA Estimation Under Collateral Trigger</u>: The definition of SFVA in

$$SFVA_i = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t r_B(u)du} s_B(t) V_i(t) dt\right]$$

is extended as

$$SFVA_i = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t r_B(u)du} s_B(t) \max\left(\min\left(V_i^U(\tau_i), \Gamma_i(\tau_i)\right), -\Gamma_B(\tau_B)\right) dt\right]$$

5. <u>Trigger Based UCVA and DVA</u>: The valuation formula for the UCVA under ISDA 2009 including CSA thresholds is given by

$$UCVA_i = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t \{r_{OIS}(u)+\lambda_i(u)\}du} \lambda_i(t)(1-R_i) \min\left(\left\{V_i^U(t)\right\}^+, \Gamma_i(t)\right) dt\right]$$

The DVA is given by the following extension

$$DVA_i = \mathbb{E}_0\left[\int_0^{\infty} e^{-\int_0^t \{r_{OIS}(u)+\lambda_B(u)\}du}\lambda_B(t)(1-R_B)\left\{-\min\left(\left\{V_i{}^U(t)\right\}^-,\Gamma_B(t)\right)\right\}dt\right]$$

6. <u>MTM ISDA Closeout Impact</u>: Under ISDA 2009, at the time of the bank default, the bank needs to mark-to-market the derivatives by also including the CVA discount. Symmetrically, whenever the counterparty defaults the bank is also entitled to recover a DVA benefit.
7. <u>Modified Bilateral FTD CVA Adjustment</u>: The DVA benefit entitlement, however, typically gives rise to a negligible correction that is currently ignored in order to avoid the need for nested simulations for the calculation of the CVA itself. Hence the fair valuation formula for the CVA is

$$\begin{aligned} FTDCVA_i = \int_0^{\infty} \mathbb{E}\Big[e^{-\int_0^t \{r_{OIS}(u)+\lambda_i(u)\}du}\mathbb{I}_{t<\tau_B}\lambda_i(t)(1 \\ - R_i)\min\left(\left\{V_i{}^U(t)+DVA_i(t)\right\}^+,\Gamma_i(t)\right)\Big]dt \\ \approx \int_0^{\infty} \mathbb{E}\left[e^{-\int_0^t \{r_{OIS}(u)+\lambda_i(u)\}du}\mathbb{I}_{t<\tau_B}\lambda_i(t)(1-R_i)\min\left(\left\{V_i{}^U(t)\right\}^+,\Gamma_i(t)\right)\right]dt \end{aligned}$$

8. <u>Modified CVA CL and CA</u>: In this case we also have an alteration to the contra-liability given by the present value of the default losses occurring after the bank default, i.e.,

$$\begin{aligned} CVA_{CL,i} = -\int_0^{\infty} \mathbb{E}\Big[e^{-\int_0^t \{r_{OIS}(u)+\lambda_i(u)\}du}\mathbb{I}_{t\geq\tau_B}\lambda_i(t)(1 \\ - R_i)\min\left(\left\{V_i{}^U(t)+DVA_i(t)\right\}^+,\Gamma_i(t)\right)\Big]dt \\ \approx -\int_0^{\infty} \mathbb{E}\left[e^{-\int_0^t \{r_{OIS}(u)+\lambda_i(u)\}du}\mathbb{I}_{t\geq\tau_B}\lambda_i(t)(1-R_i)\min\left(\left\{V_i{}^U(t)\right\}^+,\Gamma_i(t)\right)\right]dt \end{aligned}$$

The sum

$$FTDCVA_i = UCVA_i + CVA_{CL,i}$$

is the first-to-default unilateral CVA and represents the fair valuation of the counterparty credit.

VM Rehypothecability Across Funding Sets

1. Restrictions on VM Rehypothecation: In their case study, Albanese and Andersen (2014) model rehypothecation by assuming that the VM received can be posted back to meet posting requirements by the hedges in the same funding set. This is a simplifying assumption that may have to be refined in practical implementations, For instance, there may be restrictions in the CSA preventing banks from being able to rehypothecate the VM.
2. Impact of the Rehypothecation Ban: Rehypothecation bans are sometimes triggered by a degradation in the bank credit quality. Whenever this happens, the cost of funding is effectively increased and the bank has a greater interest in "flattening out" its book.
3. Funding Costs Under Stress Conditions: A careful simulation of the funding costs should account for the resulting increase in funding costs that occur whenever such stress conditions manifest themselves.

References

- Albanese, C., and L. Andersen (2014): Accounting for OTC Derivatives: Funding Adjustments and the Re-Hypothecation Option **eSSRN**.

Entry Prices, Exit Prices, and Trade FTP

Trade and Portfolio FTP Estimation

1. The Breakeven Trade FTP: We recall that the FTP policy defines the costs or benefits that must at a minimum be passed to the clients in excess of the default-free trade values (FTP can be negative if the trade reduces the credit risk of the funding costs for the bank). In other words, the FTP defines what constitutes a "break-even" transaction and thereby determines the bank's *entry price*, i.e., the price that the bank would bid to acquire a trade or possibly a collection of trades.
2. Marginal FTP Impact on Portfolios: As mentioned earlier XVA-aware accounting systems do not aggregate portfolios linearly over trades, so FTP policies cannot operate only at the trade level but must take into consideration the effect that a new trade has on the risk profiles of existing trade aggregations (at the netting set or at the funding set levels). For this reason it is reasonable to set out FTP policies in terms of *marginal* increments to portfolio level metrics.
3. FCA/FBA vs FVA/FDA Magnitude Differences: As the choice of the accounting methods have material implications on the choice of various accounting quantities, the FTP policies are necessarily different in the FCA/FBA and the FVA/FDA frameworks. This is discussed in the next two sections. The topic of *exit prices* and fair value accounting are subsequently covered.
4. Transaction Cost Incorporation in Pricing: By convention entry prices include transaction costs and exit prices exclude them. For the purposes of this discussion the assumption throughout here is that the transaction costs are zero.

FTP For FCA/FBA Accounting

1. Netting Set Granularity For FTP: In the case of FCA/FBA accounting, the relevant units of account refer either to default-free trades or (for CVA, DVA, FCA, and FBA) to the netting sets. There are no funding set level units of account in the FCA/FBA method.
2. Credit and Funding Cost Components: In the FCA/FBA accounting, the standard FTP policy is to transfer to the clients at a minimum the incremental amount

$$FTP = \Delta UCVA + \Delta SFVA$$

   $\Delta UCVA$ and $\Delta SFVA$ cover the counterparty credit risk and the funding charges respectively, and are paid by the business line trading desk to the CVA and CFD desks.
3. FTP Impact on Equity Capital: From

$$\Delta CET1 = -\Delta CA + \Delta RE$$

   we see that the incremental impact of a new trade on the equity capital is

$$\Delta CET1 = -\Delta CA + FTP = -\Delta UCVA - \Delta FCA + \Delta UCVA + \Delta SFVA = -\Delta FBA$$

   while the change in liabilities is

$$\Delta CL = \Delta FBA$$

   By design the FTP policy in FCA/FBA accounting ensures that the impact on income from adding a new trade is zero, i.e.,

$$\Delta PFV = \Delta CET1 + \Delta CL = 0$$

4. FTP Alignment with the Shareholders' Wealth: By tailoring the FTP in such a way that new trades have no impact on income, the deals flows induces volatility in CET1, as is evidenced from

$$\Delta CET1 = -\Delta FBA$$

As CET1 is a proxy for shareholders wealth, it may be argued that

$$FTP = \Delta UCVA + \Delta SFVA$$

does not fully align executive incentives with shareholders' interests. An FTP policy designed for CET1 indifference would, however, not be viable for FCA/FBA accounting due to the sheer size of the full FCA, as is demonstrated in the case study by Albanese and Andersen (2014).

FTP For FVA/FDA Accounting

1. FTP as a CET1 Enhancer: In FVA/FDA accounting, FVA and UCVA are CET1 deductions and DVA, FVA, and CVA_{CL} are CL adjustments. The upfront charge for a given trade that a given business line desk needs to pay the CFD desk in order to ensure that CET1 stays constant is

$$FTP = \Delta UCVA + \Delta FVA$$

With this choice

$$\Delta CET1 = 0$$

2. Netting and Funding Set Granularity: The calculation of FTP involves assessing the marginal impact on two separate units of account:
    a. The netting set associated with the counterparty to the trade in question.
    b. The overall funding set under which the VM may be re-hypothecated. This calculation in particular is not always trivial – this is treated later.
3. FTP Shareholder Centric View: By focusing on CET1

$$FTP = \Delta UCVA + \Delta FVA$$

effectively expresses a share-holder centric view in the computation of the FTP, where the proxy for "true" deal in the FTP is only the shareholder part of the overall value that a trade brings to the firm.

4. FTP Impact on Net Income: For trades that involve bond holder benefit post bank default, any such benefits are ignored when passing costs to the client. When it comes to net income the FTP policy therefore has an impact given by

$$\Delta PFV = \Delta CL = \Delta DVA + \Delta FDA + \Delta CVA_{CL}$$

Again this is a marginal calculation at the level of both the netting set (DVA, CVA_{CL}) and the funding set (FVA, FDA).

5. Alignment Across Debt and Equity: It needs to be emphasized that while the FTP policy in FVA/FDA accounting is based on principles more conservative than those for FCA/FBA, the inclusion of RHO into FVA causes the funding cost component of the CA account to be materially smaller in the absolute value than FCA/FBA accounting (by about a factor of two in the case study examples of Albanese and Andersen (2014)). As a consequence CET1 immunization is practical in the case of FVA/FDA method and results in an FTP principle that aligns the interests of the bank managers and the shareholders.

Exit Prices and Fair Valuation

1. Deciding on the Unit of Account: Exit prices are important from an accounting point of view as they provide a model-independent method to access fair valuation. As a general approach, the first step is to decide what is to be considered a unit of account.
2. Auctioning for an Account Unit: If a unit of account can be transferred in a market with sufficient liquidity and without altering any of the expected cash flows an auction process may be run and a best bidder chosen.

3. Model Pricing for the Account Unit: Failing the execution of the auction process, a model-based valuation method is used with a model calibrated consistently to all available market pricing information for the relevant risk factors.
4. Consistency with the Accounting Principles: According to accounting principles AP0 and AP1 above, the exit prices discovered through the steps outlined in the general approach above are what should be recorded as fair value for derivative assets and liabilities on the balance sheet.
5. Granularity of a Single Trade: Unless trades are fully collateralized, the notion that a single trade can ever be considered a unit of account is, as discussed before, an incorrect one. So while it may be tempting to ask for the market price of a single trade, the question becomes truly meaningful only when asked about entire netting set portfolios – and sometimes even that may be too granular.
6. Granularity of a Full Portfolio: Further when discussing portfolio prices, not only must the netting set portfolios be held fixed, but so must also the credit quality of the bank and its counterparty – otherwise cash flows induced by the default are not the same. For instance, trying to assign a portfolio from a low-rated bank to a high-rated bank automatically changes the cash flow characteristics of a bilateral portfolio and violates the apples-to-apples provision in the auctioning setup above.
7. Compatibility with the Bidding Bank: Even if the bidding bank has exactly the right credit spread, the bidding bank very likely has trades with the counterparty in question. The bid is therefore not for the original portfolio in question, but for a combined portfolio of old and new trades.
8. Need to invoke Model Pricing: Due to the above effects it is exceedingly rare that an auction of the type discussed is ever practical; instead the model based pricing approach is virtually always called upon.

FVA/FDA Accounting

1. Exit Price Trade Level Granularity: As we have seen, exit price at the level of a netting set operates at too narrow a unit account, as funding costs due to re-hypothecation must still be

modeled at the funding set level. In the FVA/FDA method this certainly is required for the FTP and entry price calculations as just seen above, but is not required for fair value calculations.

2. CA and CL Adjustment Cancellation: The reason for this, of course, is that cash flows associated with the funding operations are modeled as having zero net value for the firm as a whole, consistent with the MMT. This manifests itself in the cancellation of the CA and the CL adjustments for funding, such that funding costs never make it to the level of net income and net fair asset/liability values on the balance sheet.
3. Price Impact on Net Income: DVA and CVA, however, generally do not cancel out, and their difference shows up in net fair values. Specifically for the FVA/FDA accounting, the Portfolio Fair Value (PFV) of the derivatives is, according to the table seen earlier,

$$PFV = A - L - CA + CL = A - L - FTDCVA + DVA$$

an expression that contains in $A - L$ the present values of promised cash flows, and in $-FTDCVA + DVA$, the present values of the lost cash-flows due to default.
4. Trade Level Cash Flow Focus: The above flows are focused on cash flows generated by the trades in isolation and does not reference how the trades are funded. As a consequence, as mentioned above, $A - L - FTDCVA + DVA$ is additive over netting sets. However, this is not the case for FTP expressions in FVA/FDA accounting which are aligned CET1 (a non-additive metric over netting sets).

FCA/FBA Accounting

1. Price Impact on Net Income: In FCA/FBA accounting funding costs are allowed to hit net income as the PFV is given by

$$PFV = A - L - UCVA - FCA + FBA = A - L - UCVA - SFVA$$

This measure is a mixture of partial and incomplete cash flow valuations with a measure of funding costs thrown in. As in the expression for PFV in FVA/FDA accounting, the expression is additive over netting sets.

2. Caveat - Fair Valuation of CVA: UCVA is generally not the fair valuation of the CVA, as it ignores self-survival. Similarly the expression for PFV in FCA/FBA accounting does not contain the DVA and the approximation

$$FBA \approx DVA$$

is a poor one.

3. Violation of the MMT Principles: As mentioned earlier it is clear that

$$PFV = A - L - UCVA - SFVA$$

above violates both the MMT and the principle of asset-liability symmetry. One should also ask the question whether the presence of SFVA in

$$PFV = A - L - UCVA - SFVA$$

amounts to an entity-specific cost adjustment decoupled from trade cash flows, something that violated the principles of exit pricing and normally not allowed in fair value accounting.

4. Popular Justification for the FCA/FBA Accounting: Proponents of the FCA/FBA accounting often attempt to argue away the entity specific nature of the PFV expression by suggesting that their bank's credit spreads are "representative", wherefore the SFVA represents a market average that can be used for exit pricing purposes. This is a questionable line of reasoning for several reasons.

5. Choice of Appropriate Funding Spread: First, even if the bank's funding spread is close to the market average at some point, this may cease to be the case in the future, especially if the bank itself approaches default. And second it is unclear if the market average spread is a metric that can be used for exit pricing and PFV computations – perhaps the "best" spread

(i.e., high spreads for receivables and low spreads for payables) could be argued to be more appropriate.

6. Industry Standard Proxy for Funding: A related line of thought suggest that the SFVA term in

$$PFV = A - L - UCVA - SFVA$$

is not to be computed at the bank's own spread, but at a separately marked industry spread. This removes the entity specific nature of the FCA/FBA PFV, but also decouples FBA entirely from DVA, whereby FCA/FBA accounting would violate one of the accounting principles.

7. Impact of Exogenously Marked Spread: One also wonders where the industry spread curve is supposed to come from, especially since such curves are very hard to detect at the level of individual trades (since FTP entry prices operate on netting or funding set metrics only). The idea of an exogenously marked spread is treated in detail in the next section.

References

- Albanese, C., and L. Andersen (2014): Accounting for OTC Derivatives: Funding Adjustments and the Re-Hypothecation Option **eSSRN**.

Liquidity Spreads, Asset Liability Symmetry, and Alternative Allocations for Excess Collateral

Motivation

1. Assumptions Underlying the FCA/FBA Scheme: This section reviews the FCA/FBA accounting framework by examining a variety of assumptions that have been put forward to justify at least some of the elements of the FCA/FBA accounting ideas. These assumptions are quiet strong, and not necessarily realistic.
2. Working Capital Management for Derivatives: Derivatives are normally funded on a short-term basis, as the funding needs associated with derivatives trades typically exhibits considerable variation through time. The inherent variability in turn would make it unlikely that the bank's treasury department would commit to systematically using excess collateral from derivative trading to retire general term debt from the bank's liabilities.
3. FCA/FBA Working Capital Assumptions: Yet, as seen earlier, this assumption is essentially one that is required to make sense of some of the FCA/FBA accounting ideas.
4. Returns on the Excess Collateral: Earlier it was suggested that one use a more conservative and reasonable assumption that the excess collateral is simply invested in short-term risk-free investments, earning a rolling rate of $r_{OIS}(t)$. This way no shareholder gain is generated out of variable excess collateral.

Working Capital Management and Operations

1. Shareholder/Creditor Income Share: The sum of the values of the bank to the shareholders and the senior creditors add up in income statements and define the PFV of the bank. The value of the debt from collateral lenders is excluded from this calculation.

2. Handling of Excess Working Capital: Whenever the bank finds itself in the situation where it has excess cash the bank managers have an option to deleverage by buying back the senior debt. According to MMT neither the decision to leverage up with collateral lenders or de-leveraging by debt buy-back have a net impact on income. However the two decision differ in terms of wealth transfer between the shareholders and the senior creditors.
3. Fair Value Bond Buy-back: The interest paid to collateral lenders triggers a wealth transfer between the shareholders and the senior creditors of a bank and this is quantified by the FVA and the FDA. Whenever bonds are bought back/sold to from the senior creditors at their fair value, the wealth of both the shareholders and the senior creditors is not affected by the decision to de-leverage.

Equity Gain and Debt Gain

1. Equity/Debt Gain Overview and Definition: An alternative to the FVA/FDA assumption would be to assume that the short-term collateral excesses can be invested in strategies that lead to time 0 shareholder and debt-holder increases of EG ("Equity Gain") and DG ("Debt Gain") respectively.
2. MMT Consistency Across the Gains: It would generally be a stretch to build into accounting statements any "sure thing" on firm-wide profitably of investment strategies, it seems that one should at least require

$$EG + DG = 0$$

   This is of course the condition required to satisfy MMT.
3. CA and CL Impact of Gains: EG and DG are specific to whatever investment strategy that the treasury commits to, but should one somehow be able to project their values, the FVA/FDA accounting method could be adapted as follows:

$$CA\ Entries := FVA + UCVA - EG$$

$$CL\ Entries := CVA_{CL} + FDA + DVA + DG$$

4. Funding Definitions Impact on Accounting: This is a good time once again to point out that the funding cost definitions are not immutable, but depend strongly on the assumptions made on how the finds are raised and invested. Baking any such assumptions into the accounting numbers obliges the bank to actually follow strategies on which it based its accounting numbers.
5. Maintenance of the Asset-Liability Symmetry: As

$$EG + DG = 0$$

implies

$$EG = -DG$$

it is clear that introducing EG and DG into the FVA/FDA accounting would preserve the asset-liability symmetry and would not have any effects on the Net Income. However if $EG \neq 0$ new terms would arise at the balance sheet accounting level and would potentially affect CET1.
6. Positive EG and Debt Covenants: As bank managers should not engage in trading strategies where

$$EG < 0$$

the introduction of the EG and the DG terms would realistically only involve the cases where

$$EG > 0$$

and therefore

$$DG < 0$$

i.e., it is possible to only consider collateral investment strategies that prevent wealth transfers from shareholders to senior creditors. Such strategies are possible in principle but are non-trivial to setup since bond covenant put serious restrictions on any activities that enrich the shareholders solely at the expense of senior creditors.

7. Counterparty Credit Risk Transfer: Transferring counterparty credit risk to third party investors may prevent wealth transfers from shareholders to senior creditors. However unless such strategies are properly quantified and executed, theoretical equity gains should not be reflected in accounts just because they are possible. As a consequence the FVA/FDA assumption of

$$EG = DG = 0$$

is a more reasonable and a rigorous one.

8. Shareholder Equity Gain Formulation: Pursuing the extensions above a bit further, supposing that the investment benefits are assumed to accrue at the rate of $s_G(t)$ where $s_G(t)$ is interpreted as the spread over $r_{OIS}(t)$ returns. In this case we would write, long the same lines as

$$FVA(t) = \mathbb{E}\left[\int_0^\infty e^{-\int_0^t \{r_{OIS}(s)+\lambda_B(s)\}ds} s_B(t) \left\{\sum_i V_i\,(t)\mathbb{I}_{t<\tau_i}\right\}^+ dt\right]$$

$$EG = EG(s_G) = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t \{r_{OIS}(s)+\lambda_B(s)\}ds} s_G(t) \left\{\sum_i V_i\,(t)\mathbb{I}_{t<\tau_i}\right\}^- dt\right]$$

9. Equity Gain as Return Rate: For the special case where

$$s_G = s_B$$

we can observe that

$$FVA - EG(s_B) = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t \{r_{OIS}(s)+\lambda_B(s)\}ds} s_B(t) \left\{\sum_i V_i\,(t)\mathbb{I}_{t<\tau_i}\right\} dt\right]$$

where now the max operator has disappeared from the expectation. Approximating the recovery rate of the bank as zero, i.e., setting

$$R_B = 0$$

we find that

$$FVA - EG(s_B) \approx FCA - FBA = SFVA$$

10. Debt Retirement using Equity Gain: Of course setting

$$s_G = s_B$$

in

$$EG = EG(s_G) = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t \{r_{OIS}(s)+\lambda_B(s)\}ds} s_G(t) \left\{\sum_i V_i\,(t)\mathbb{I}_{t<\tau_i}\right\}^- dt\right]$$

basically amounts to debt retirement – a strategy that was questioned, so this cannot be endorsed. Nevertheless

$$FVA - EG(s_B) \approx SFVA$$

does help understand better the underpinning of the FCA, the FBA, and the SFVA terms.

11. Gain Accounting vs FCA/FBA Accounting: It needs to be emphasized, however, that setting

$$s_G = s_B$$

and following the CA and the CL entries above does not reproduce the FCA/FBA method, not even approximately. For instance, note that the FCA/FBA method sets the CA account to $CVA + FCA$ whereas

$$FVA - EG(s_B) \approx SFVA$$

results in a CA account of $CVA + SFVA$

$$CA\ Entries := FVA + UCVA - EG$$

and

$$CL\ Entries := CVA_{CL} + FDA + DVA + DG$$

constitutes an accounting method with asset-liability symmetry, whereas FCA/FBA method does not.

Liquidity Based Analysis and Treatment

1. Asset-Liability Symmetry without MMT: Let us note that it is possible to preserve asset-liability symmetry without satisfying the MMT. Suppose that the entire market decides that a liquidity spread of s_L - unrelated to the compensation of the default risk – applies to all the discounting operations on unsecured derivatives. Incorporation of this spread would only mean a universal redefinition of the default-free security value V_i – something that would result in asset and earnings re-statements across firms, but would not break the asset-liability symmetry.
2. Liquidity Value Adjustment Metric Formulation: More concretely we define a Liquidity Value Adjustment (LVA) as

$$LVA(s_L) = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t \{r_{OIS}(s)+\lambda_B(s)\}ds} s_L(t)\left\{\sum_i V_i\,(t)\right\}dt\right]$$

$$= \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t \{r_{OIS}(s)+\lambda_B(s)\}ds} s_L(t)\left\{\sum_i V_i(t)^+\right\}dt\right]$$

$$+ \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t \{r_{OIS}(s)+\lambda_B(s)\}ds} s_L(t)\left\{\sum_i V_i(t)^-\right\}dt\right] \triangleq LVA_A(s_L) - LVA_L(s_L)$$

3. <u>CA/CL Components of LVA</u>: One obvious way of accounting for liquidity spreads would be to let $LVA_A(s_L)$ and $LVA_L(s_L)$ to be entered as contra-asset and contra-liability, respectively. It is clear here that $LVA_L(s_L)$ - unlike all other terms in

$$CL\ Entries := CVA_{CL} + FDA + DVA + DG$$

– is not associated with self-default or wealth transfers, and therefore should *not* be excluded from CET1. Note also that introducing the LVA would result in an earnings impact equal to the LVA.

4. <u>LVA in FCA/FBA Accounting</u>: A radical idea is to assume that the funding spread s_L is unrelated to default and in actuality is just a friction-type liquidity spread. In that case one may interpret the SFVA as an LVA by noting the identity

$$SFVA = LVA(s_L)$$

as well as

$$FCA = LVA_A(s_L)$$

and

$$FBA = LVA_L(s_L)$$

5. Symmetric Asset-Liability Accounting Rule: In this interpretation we would use the following accounting rule:

$$CA\ Entries := UCVA + LVA$$

$$CL\ Entries := CVA_{CL} + DVA$$

where effectively only $CVA_{CL} + DVA$ (but not FBA) would need exclusion from the regulatory capital.
6. DVA Double Counting and Resolution: Besides stretching the imagination by assuming that the liquidity spreads are not as large as credit spreads, this rule effectively double counts the DVA. Note that this does not preserve the MMT (and results in a funding impact on the earnings), but does preserve the asset-liability symmetry. If as in FCA/FBA accounting, DVA is removed from the CL entries to avoid double counting, the accounting symmetry is broken.

Problems with the Gain Accounting

1. SFVA Wrong-Way Sensitivity Impact: The approach above is problematic not only from an accounting angle, but also due to the regulatory viewpoint due to the wrong-way sensitivity of the SFVA on the bank's own credit.
2. Misplaced Incentives from FCA/FBA Accounting: In case there were a generalized acceptance of the SFVA being interpreted as an LVA and qualified for a CET1 deduction, the FTP's computed using the SFVA would incentivize traders to buy payables and sell receivables, since they overstate both the funding costs for the latter and the funding benefits for the former.
3. Worsening Bank Credit CET1 Impact: Hence portfolios of banks with material credit spreads would drift towards being net payables. One can imagine that the SFVA for the worst funders would become negative and develop a wrong-sign sensitivity with respect to own credit. A

vicious cycle may ensue since the worsening of the credit of the bank would ipso facto increase equity capital.

4. Sudden Accounting Regime Change Impact: If a blow-up occurs and the accounting standards need to be changed for all the market participants to enforce an FCA or an FVA deduction, the system-wide impact on the regulatory capital would be pervasive.

References

- Albanese, C., and L. Andersen (2014): Accounting for OTC Derivatives: Funding Adjustments and the Re-Hypothecation Option **eSSRN**.

Albanese and Andersen (2014) Case Study

Case Study Setting and Purpose

1. Errors Associated with the FCA/FBA Accounting: While market participants are generally aware of the fact that the FCA/FBA accounting is approximate, little is known about the magnitude of the errors involved. This is particularly true of the RHO due to the computational challenges involved, the proper accounting of which involves simulation of the entire funding set with multiple netting sets.
2. FCA/FBA vs. FDA/FVA Accounting Comparison: In their section on case study findings, Albanese and Andersen (2014) attempt to shed some light on the materiality of the errors by using a realistic test-bed to compare the FCA/FBA and the FVA/FDA accounting results. They obtained results using $Global\ Valuation\ Esther^{TM}$, an in memory risk analytics system designed for the simulation of massive OTC portfolios.
3. Global Valuation Esther and Athena: Esther uses an mathematical framework based on operator algebras as discussed in Albanese, Bellaj, Gimonet, and Pietronero (2011) and references therein. Model calibrations were taken from $Global\ Valuation\ Athena^{TM}$ data service and refer to 7 July 2014.
4. The Fixed Income OTC Portfolio: As their test case, they use a realistic portfolio of fixed income derivatives of about 100,000 trades, 1,600 counterparties, and a variety of collateral agreements, some involving thresholds. The portfolio contains trades in 8 different currencies, including swaps, cross-currency swaps, swaptions, FX forwards, and options. The simulation entails 100 time steps at each of which they find scenarios for the default-free valuation of all the netting sets.
5. Types of Netting Sets Considered: Most netting sets in their tests are at least partially unsecured, with the collateral thresholds being either positive finite or infinite. Some large netting sets are of the unilateral "government" type, i.e., the threshold for the bank is zero and the threshold for the counterparty is infinite. Caveat – if all netting sets were of the

government type, then the FCA would equal the FVA, as the government type netting sets do not contribute to the RHO.

6. Fully Collateralized Netting Set Construction: The portfolio contains also a few fully collateralized netting sets with both thresholds at zero, but they do not contribute to funding and contribute to CVA only mildly through the closeout gap risk – which Albanese and Andersen (2014) do capture in their tests, even though they do not discuss in their paper.
7. Custom Scenario Bank CDS Curves: Since funding metrics depend crucially on the bank funding rates, Albanese and Andersen (2014) carry out the calculation for two different funding curves. One corresponds to the 5Y CDS spread of 106 bp and the other to a 5Y CDS spread of 274 bp. Funding costs are simulated dynamically and consistently with the funding curves.
8. Full Market Risk Factors Simulation: CDS curves for all counterparties are simulated dynamically, as needed for ratings dependent collateral policies (of which the test portfolios had a few) and the analysis of credit-correlation effects related to loss-distributions and stress testing. Albanese and Andersen (2014) also simulate all relevant market risk factors and derivative security prices. The various XVA metrics are evaluated by computing the XVA expressions after incorporating thresholds and closeouts.

Scenario Estimation of the XVA Metrics

1. Book Level Incremental and Cumulative Errors: Albanese and Andersen (2014) found that in their simulation trials the funding set calculations were well-behaved. As expected, they found that the incremental FVA is noisier than the book level FVA. In order to the book-level FVA errors to below the 0.5% mark and the incremental metrics within the 2% mark, it was necessary to run about 100,000 scenarios.
2. Netting Set Granularity XVA Caching: Albanese and Andersen (2014) retained the scenario information in memory aggregated at the netting set granularity. By having the scenarios cached in memory the calculation of the incremental XVA metrics was quite efficient. Incremental metrics of interest include the FVA, the symmetric FVA, as well as the UCVA, the FTDCVA, and the DVA.

3. <u>Funding Curve Scenario XVA Metrics</u>: Albanese and Andersen (2014) demonstrate the XVA portfolio metrics for the two funding curve scenarios. Accounting metrics are computed using the accounting rules listed earlier. Accounting entries are computed separately for the following cases:
    a. Only counterparty credit risk is accounted for
    b. Funding is accounted for using FCA/FBA accounting
    c. Funding is accounted for using FVA/FDA accounting
4. <u>CET1 and MMT Accounting Impact</u>: Albanese and Andersen (2014) demonstrate that there are material difference between the FVA/FDA accounting and the FCA/FBA accounting. In particular the CET1 charges for funding are about triple in the FCA/FBA accounting as opposed to the FVA/FDA accounting. Since the FVA/FDA accounting is consistent with the MMT, the write-off one faces by adding funding entries on top of credit adjustments in NILL.
5. <u>FVA/FDA vs. FCA/FBA CL Impact</u>: Contra Liabilities in FVA/FDA accounting are slightly larger than in FCA/FBA accounting. This happens because although the FDA is substantially smaller than FBA, the DVA is preserved as is required by the asset-liability symmetry, and there is no overlap.
6. <u>EE, EPE, and ENE Estimates</u>: For FVA/FDA calculations the key statistics in measuring RHO are the Expected Exposure (EE), the Expected Positive Exposure (EPE), and the Expected Negative Exposure (ENE) of the portfolio as a whole. Using 100,000 simulation trials, Albanese and Andersen (2014) report these metrics for a range of time horizons. The book under consideration starts off as a net receivable and remains as such on average.
7. <u>Cross Netting RHO Impact</u>: The impact of modeling rehypothecation between hedges belonging to different netting sets is sizeable and receives contributions also from the states of the world where the book is a net payable, because even in such situations a fraction of the netting sets are receivables and the corresponding hedges receive collateral.
8. <u>Receivable vs Payable Book Swing</u>: Furthermore the book valuation swings to a net payable status as one can see from the fairly substantial size of the ENE. A large ENE (in this case) is an indication that the FVA is a materially non-linear metric which cannot be represented as a simple sum over the netting sets. In other words the incremental FVA for a netting set or a

trade can be computed accurately only by knowing and accounting for the positions in the entire book.

Product and Scenario Threshold Type Scenarios

1. Impact of the Swap Types Traded: In their analysis, Albanese and Andersen (2014) add their portfolio three swaps – one initially at par, one a payable, and the third initially a receivable. They then compute the XVA metrics under the following 3 scenarios.
2. Thresholded Scenario XVA Impact:
    a. The swaps are added to a netting set with a finite collateral thresholds that already contains other trades
    b. The swaps are added to a netting set with a threshold at infinity
    c. The swaps are added to a netting set which is initially empty, thus neglecting the benefits of netting
3. Netting and Collateral Threshold Impact: The FTPs differ substantially between all the cases above, and are sensitive to both collateral thresholds and netting. Bothe collateral thresholds and netting materially decrease the FTP.
4. FCA/FBA vs. FVA/FDA FTP Estimates: There are also material differences between the FTP's obtained under the FCA/FBA accounting and those using the rules of FVA/FDA accounting. In the latter case the FTPs are smaller by a factor of $1.5 - 2.0$ in absolute value. This is also remarkable from the viewpoint that the FVA/FDA method does not recognize a DVA benefit to clients. This de-recognition is however more than compensated by a correct modeling of the RHO.
5. Accounting Scheme Objective of the FTP: The incremental FTPs in the two methods achieve a different objective. In the case of the FCA/FBA method the incremental fair value of the OTC portfolio ΔPFV is NILL while the Net Equity Capital is systematically depleted, especially in the case of payables. In FVA/FDA accounting instead the Equity Capital is stable while the Bank fair value appreciates systematically.

XVA Metric Errors and Incrementals

1. Error Rates from Simulation Runs: Albanese and Andersen (2014) display the standard errors on the simulation runs – their simulation entails 100,000 scenarios and shows that the FTP for the FVA/FDA method is around 1.2-1.7%, an error of which type would be acceptable in most circumstances. Using 10,000 scenarios would imply relative errors as large as 10% and of the order of 1 bp per annum, which we would consider as being unacceptable large.
2. Tail Loss Distribution Contribution Scenarios: Further 100,000 scenarios allow carrying out of the reverse stress testing analysis by identifying extreme scenarios which either contribute to the tail of the loss distribution or invalidate a collateral procurement strategy.
3. FVA and SFVA Trade Incrementals: Albanese and Andersen (2014) demonstrate the size of the incrementals after adding one trade of the forward FVA and SFVA as follows: $FVA(t) = \mathbb{E}\left[e^{-\int_0^t \{r_{OIS}(s)+\lambda_B(s)\}ds} s_B(t)\left\{\sum_i V_i(t)\mathbb{I}_{t<\tau_i}\right\}\right]$ and $SFVA(t) = \mathbb{E}\left[e^{-\int_0^t r_B(s)ds} s_B(t)\left\{\sum_i V_i(t)\mathbb{I}_{t<\tau_i}\right\}\right]$
4. Incremental SFVA vs. Incremental FVA: They observed that incremental SFVA is systematically above the incremental FVA in the case of receivables and systematically below the FVA in the case of payables. We conclude that the approximations which are intrinsic to the FCA/FBA accounting over-estimates funding costs for receivables and also over-estimates funding benefits for payables.

Estimation of the FCA/FBA – FVA/FDA Mismatch

1. Payables Derivatives Bias FCA/FBA: The bias in favor of payables in FCA/FBA accounting is a worrisome feature of this method as it induces banks to skew their exposure in favor of the payables. If the traders are incentivized with an FTP computed by this method, they are inclined to sell out of the money options for premium upfront, effectively raising collateral and subjugating themselves to the treasury, thus incurring inefficiencies.

2. Origins of the Payables Bias: The above happens because the FCA/FBA method recognizes a benefit to the excess variation margin at a rate equal to the funding spread of the bank while the FVA/FDA method does not recognize any such benefit).
3. Threshold Impact of FCA/FBA Accounting: The SFVA is a transactional amount only in cases where there are no thresholds or the thresholds are very remote and attained only with a negligible probability. For instance the two SFVAs for the case where the trade is added to an empty netting set is full but the thresholds are neglected are very close. This happens although the FCA and the FBA are very different.
4. FCA/FBA Volatility on Bank PFV: Within FCA/FBA accounting a new trade does not cause volatility in the fair valuation of the bank as a whole given by the PFV. In this case, however, each trade causes a transfer of wealth from CET1 to CL, i.e., from shareholders to senior creditors.
5. Shareholder to Senior Creditor Transfer: On average this transfer is a net loss to CET1. It MMT held this would happen for a zero FTP. The fact that it happens for a non-zero FTP is yet again a signal of the internal inconsistency of the FCA/FBA accounting.
6. Income and CET1 FVA/FDA Impact: In the case of FVA/FDA accounting instead we propose to structure the FTP in seeking to keep CET1 constant. We could also have computed the FTP in such a way to keep the income constant and this would have given rise to zero FTP's.
7. FTP in FCA/FBA vs. FVA/FDA: The FTP in FVA/FDA accounting are lower than those in FCA/FBA accounting for a net receivable book. This signals again the internal inconsistency of the FCA/FBA accounting as the FTP policy should ensure on average that the CET1 is not depleted while it instead appears as though it is.
8. Single Swap on Single Counterparty: It is also useful to consider the case of a single swap transaction with a single counterparty. Albanese and Andersen (2014) demonstrate the CET1 changes which measure the economic value of the transaction are very close between the FCA/FBA and FDA/FVA accounting schemes. In this case the FVA/FDA methodology can be regarded as an extension of the FCA/FBA method as long as the latter is restricted to portfolios consisting of a single netting set.
9. Two Counterparties with Opposite Swaps: The second interesting particular case in Albanese and Andersen (2014) is the one described whereby there is a portfolio with two

counterparties, each with a single swap position. The swaps have identical terms except that one is a payable while the other is a receivable.

10. Collateral Offset from the Swaps: What happens in this case is that the collateral received on the hedge to one swap is nearly always the exact amount the bank is required to post on the hedge to the other swap. Discrepancies arise only in scenarios where one counterparty defaults while the other is still ongoing, in which case one of the hedge positions is closed upon default.
11. Magnitude of the Asymmetric FVA: Assuming that the CDS spread curves of the counterparties are identical, and that they reach 200 bp in 5Y, the resulting asymmetric FVA is small.
12. FCA/FBA vs. FVA/FDA CET1 Deductions: Instead the FCA for each of the two counterparties is large. As a consequence the FCA/FBA accounting method gives rise to capital deductions that are 9 times larger than the deductions resulting from the FCA/FBA method.

References

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- Albanese, C., and L. Andersen (2014): Accounting for OTC Derivatives: Funding Adjustments and the Re-Hypothecation Option **eSSRN**.

Conclusions with Funding Adjustments with RHO

Traditional Challenges with Derivative Accounting

1. Indifference Pricing of OTC Derivatives: Although funding costs may feel all too real for the derivatives trader that sees his unsecured positions bleed negative carry, well-established finance principles nevertheless insist that fair values of assets are independent of how they are funded.
2. "Going Concern" Accounting Principles Reconciliation: Reconciling the funding carry vs. the Corporate Finance Theory within the confines of the financial accounting statements is not an easy exercise, especially since traditional "going concern" accounting principles were not designed for credit-risky securities.
3. CET1 Equity Capital Regulatory Principles: Complicated matters further are the newly established regulatory principles for the CET1 equity capital that require particular care in the accounting of DVA and DVA-like adjustments.

Problems with FCA/FBA Accounting

1. Not Accounting for the RHO Handling: While some banks have put forward – and into action – the FCA/FBA method for funding cost accounting, Albanese and Andersen (2014) demonstrate that this method is not satisfactory. First FCA/FBA does not properly reflect the re-hypothecation options embedded in variation margin financing and as a result over-estimated funding related deductions from the equity capital.
2. Violation of the Asset-Liability Symmetry: Second the FCA/FBA violates the asset-liability symmetry principles of generally accepted accounting standards while breaking the Modigliani-Miller Theorem dear to financial economists.

3. Wrong-Way Bank Credit Sensitivity: Another popular variation of the FCA/FBA accounting – which involves deducting SFVA from equity capital rather than FCA – is not viable from a regulatory standpoint as the deduction has a wrong-way sensitivity with respect to the bank credit spread for portfolios which are net payables.

FVA/FDA as FCA/FBA Enhancement

1. CET1, Income, and Fair Value: Albanese and Andersen (2014) proposal for funding cost accounting aims to establish some coherence and to clear up a number of holes in the FCA/FBA accounting. In their tests this new method differs significantly from the FCA/FBA method on key accounting numbers (such as CET1, Net Income, and Fair Asset Value), yet the FVA/FDA accounting should resonate with most relevant parties.
2. MMT and Risk-Neutral Pricing: Firstly financial economists and asset pricing experts will appreciate that the accounting rules will satisfy the Modigliani-Miller Theorem and lean heavily on classic risk-neutral pricing principles.
3. Exclusion of Self-Credit Benefits: Secondly regulators will also appreciate that all self-credit benefits are collected cleanly in a Contra-Liability account that can be easily excluded from common equity for capital purposes.
4. Adherence to Accepted Accounting Principles: Thirdly accountants will appreciate the adherence to the accounting principles, and in particular to the asset-liability symmetry principle.

Trading Staff Point of View

1. Explicit Trade-Level Valuation Adjustment: For trading personnel the picture gets more complicated. On the one hand the FVA/FDA adjustment does not trigger the funding related adjustments to income and asset valuations that many prefer to see.

2. Equity Capital Based Incentive Schemes: On the other hand Albanese and Andersen (2014) describe why traders and managers drafting incentive schemes should care about the changes in CET1 than about the Net income.
3. CET1 Based FTP Estimation Schemes: In particular they highlight the link between the shareholder value and CET1 and demonstrate how a rational FTP scheme can be designed around the principle of book-level CET1 indifference pricing.

Challenges with the XVA Metric Estimation

1. Simulation Across Multiple Netting Sets: On the topic of FTP calculation there is no doubt that FVA/FDA method requires a fairly sophisticated calculation engine to support the necessary incremental FVA calculation. Being a book-level quantity the FVA (and therefore the FTP) computation involves simulating through entire books across time, involving a large number of netting sets.
2. Enhancements to Existing CVA Systems: This, in turn, requires modifications to standard CVA calculation engines that normally can only aggregate trades at the level of individual netting sets.
3. Challenges from a Computational Finance Perspective: Given the complexities involved in computing FTP's against the backdrop of an entire book position, there are challenging computational finance questions to be addressed.

Shortfalls of the FVA/FDA Scheme

1. Derivatives Focus of the FVA/FDA Accounting: While FVA/FDA leans on a number of ideas from corporate finance, Albanese and Andersen (2014) make it clear that the method is pragmatic and derivatives focused.
2. Targeted Scoping of the FVA/FDA Accounting: They do not attempt a rigorous, full-blown analysis of the balance sheet that takes into consideration many other assets and operation of a typical bank. Neither do they consider the effects of taxes, bond covenants, dividend

policies, and subtle feedback effects from investment decisions on firm-wide recovery rates (which they assume to be constant) and default probabilities.

3. Inclusive, Wide Ranging Accounting Treatment: It remains an interesting question for future whether a more rigorous and large (an necessarily complex) analysis can provide any insights that can be turned into concrete accounting rules that improve upon what they propose.

Alternate Specialized Value Adjustment Metrics

1. Capital Charge Value Adjustment: While FVA is the most prominent newcomer to the XVA alphabet soup, there are other adjustments just waiting around the block. For instance, it has been suggested that the cost of capital charges should be reflected in the deal pricing through a "capital value adjustment" or KVA (Green, Kenyon, and Dennis (2014)).
2. Trade Scenarios Requiring Initial Margin: Similarly one may consider Margin Valuation Adjustment (MVA) due to the funding cost of the initial margin (IM) for the netting sets that require IM posting. Initial Margin is required when trading with the CCPs, and due to regulatory requirement also becomes far more prevalent in the future for non-cleared products (Basel Committee on Banking Supervision (2013)).
3. MVA and KVA Estimation Complexities: The accounting for – and the associated impact of – MVA, KVA, and other metrics that come up, are topics for future research, as is their practical computation. Albanese and Andersen (2014) note that both the capital and the initial margins are complex quantities that are more involved to calculate dynamically on the path than just portfolio values (as needed for the FVA). Regression-based methods or nested simulations are likely needed here.

References

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Section VII: Assorted Calibration, Hedging, and Valuation Considerations

Convexity Corrections Associated with Margining

1. Origin of Convexity Corrections in Margining: Certain exchanges (esp. CME, on the futures) expect posting of the collateral (full or the maintenance amount) on moving out of the money. However, when in the money, you get nothing. This results in a returns mismatch asymmetry between in-the-money/out-of-the-money time snaps of the trade, i.e., this happens because you need to fund your margin gaps. This is also sometimes referred to as a one-way CSA.
2. Literature Confusion on Margin Convexity: The above asymmetry, of course, is a drag on the position value, and needs to be accounted for – and it is also referred to as convexity correction for some reason, and requires a dynamic rates model to value.
3. Modern CSA's and CCP's: CSA's essentially provide for symmetric collateral cash flow payments, therefore the situation listed above does not arise explicitly. Further some of the newer margining rules in CCP's treat in/out symmetrically (through a concept referred to as PAI – price alignment interest rate – which essentially appears like a collateral rate), so again in these term contracts these issues vanish.

Hedging Considerations

1. Curve Construction vs. Product Hedging Instrument Manifest Measure Choices: It may be preferable to incorporate a vaster universe of input instruments and their manifest measures (the manifest measures maybe both exact matches as well as the imperfect best-fit matches) in the latent state calibration. Hedging, however, may use only the most liquid set of products and their manifest measures.
2. Hedge Manifest Measure Moves: As a better approximation, of you can work out the β's of the manifest measure moves for the "non-hedging" product observation set to the manifest measure moves of the "hedging" manifest measure set, you may achieve a better and more complete PnL explain.

Product Curve Effect Attribution

Market Value Change Explain Components

1. Linear Daily Market Value Change Components: The 3 main linear principal components to the market value change are:
    - Coupon Accruals
    - Time Value Market Parameters Intrinsic
    - Market Parameters Extrinsic.

    Time is an implicit factor entity across all the three components, simply because PnL explains are conducted across distinct time entity snapshots.

Coupon Accrual Intrinsic

1. Motivation: This is applicable only to coupon/dividend bearing securities – here the coupon payout is part of the security value, and therefore causes a security value jump at the payout dates.
2. Modern Accrual Intrinsic: This should include funding flows, re-investment flows, as well as collateral flows. Collateral flows should include initial, maintenance, and valuation margins. Switches on the collateral choice numeraire generate their own flows, depending on the corresponding rolling numeraires (analogous to the traditional CTD's). Accrual flows only relate to realized cash flows, and are therefore deterministic.

Market Parameters Intrinsic

1. Motivation: This refers to the "riding the market" effects. The baseline level corresponds to the world where all the market levels stay frozen at the current instant levels. This is closes to what is referred to as θ (the intrinsic time value change).
2. Computation: Effectively this calculation addresses the question "How does the value of the derivative change as the market parameters stay frozen over the incremental period under consideration at the initial levels". STAY FROZEN is NOT the same as riding the curve. Simply put, this principal component quantifies the incremental period market curve set effects, owing to the component's maturity shrinkage by the corresponding time horizon.
3. Related Market Parameter Intrinsic Computation: The principal component measured precisely as above is referred to as the "maturity roll down" principal component. Related to this are the other ones:
   a. Maturity Roll Up => Here the derivative is valued by rolling/riding up the market curve
   b. Time Roll => Here the derivative values differences are estimated as the difference between the values at 2 distinct time snaps of the same latent state projected at the respective instants.
4. Modern Time Value Intrinsic: Roll up, roll down, and time roll are all computed on the instantaneous valuation market parameter set that determine the security value. Thus, there should be one time value intrinsic corresponding to each latent state.
5. Shape Sensitive Explain Component: The market parameter intrinsic component happens to be the most shape sensitive explain component (this includes roll down, roll up, and time roll). As a consequence, this component ends up being the most sensitive to the splined latent state representation scheme.

Market Parameters Extrinsic

1. Motivation: This principal component aims to capture the first order market move impact on the security value. This leading linear order is referred to as the "curve shift" effect, i.e., impact of the change in quote that intuitively corresponds closest to the product's extrinsic market move impact (if a single such manifest measure quote uniquely exists). Subsequent

orders (such as twist/tilt, farther quote manifest measure re-calibration impact etc.) cause higher order change impacts (e.g., convexity, butterflies, etc.)

2. Modern Market Parameters Extrinsic: Since the extrinsic market parameters simply correspond in reality to the full variety of the calibrated latent states, each of these latent state metric change triggers the corresponding linear principal component shift, and thereby a non-zero corresponding explain component. The additional "modern" latent states contributing to these factors are the collateral curves, collateral switch curves, funding curves, and re-investment curves.

Market Value Change Effects Formulation

1. The Linear Explain Components:

$$\Delta V(t_2, t_1) = V(t_2) - V(t_1)$$

$$= \sum_{t_i = t_1}^{t_i < t_2} C_f(t_i) + (t_2 - t_1) \sum_j \mathcal{R}_j(t_1)$$

$$+ \sum_j \left[\mathbb{C}_j(t_2, t_1) \left(\mathcal{L}_j(t_2) - \mathcal{L}_j(t_1) \right) \right]$$

$$\sum_{t_i = t_1}^{t_i < t_2} C_f(t_i)$$

is the cumulative carry.

$$\mathcal{R}_j(t_1) = \left\{ \frac{\partial V}{\partial \mathcal{L}_j} \frac{\partial \mathcal{L}_j}{\partial t} | \mathfrak{I}_{t_1} \right\}$$

is the per-market parameter specific roll-down.

$$\mathbb{C}_j(t_2, t_1) = \left\{\frac{\partial V}{\partial \mathcal{L}_j} | \mathfrak{I}_{t_2}\right\}$$

is the per-market parameter specific "curve shift". $\mathcal{L}_j$ refers to the latent state designated by the market value, and may need to be computed as

$$\frac{\partial V}{\partial \mathcal{L}_j} = \sum_k \frac{\partial V}{\partial q_k} \frac{\partial q_k}{\partial t}$$

where $\{q_k\}$ corresponds to the quote set required for the calibration, and $\frac{\partial V}{\partial q_k}$ is the corresponding Jacobian.

2. <u>The Linear Explain Using Value Quote Jacobian</u>: In practice what we want is

$$\Delta V(t_2, t_1) = V(t_2) - V(t_1)$$
$$= \sum_{t_i = t_1}^{t_i < t_2} C_f(t_i) + (t_2 - t_1) \sum_j \mathcal{R}_j(t_1) + \sum_j \left[\widetilde{\mathbb{Q}}_j(t_2, t_1)\left(\mathbb{Q}_j(t_2) - \mathcal{L}_j(t_1)\right)\right]$$

where

$$\widetilde{\mathbb{Q}}_j(t_2, t_1) = \left\{\frac{\partial V}{\partial \mathbb{Q}_j} | \mathfrak{I}_{t_2}\right\}$$

Section VIII: Statistical Learning in Curve Construction

Inference-Based Curve Construction

Curve Smoothing in Finance

1. Unconstrained Curve Smoothing:
    - Applicable primarily for rates/semi-liquid FX curves. Smoothing can be done here without constraints.
    - Smoothing may also be applicable to the quotes for a given instrument across several days.
    - Smoothing may also be applied over a single day curve – particularly to model the switch over from instrument to instrument (e.g., between EDF and Swaps).
2. Constrained Curve Smoothing: Applicable, for e.g., to the case of a hazard curve. The smoothing basis functions/weights combination must guarantee, from a formulation PoV, that the implied hazard rate is always greater than zero.
3. Latent State Inference as a Deep Learning Exercise: Multi-stretch, multi-pass latent state inference/representation (esp. in the financial curve construction context) can be essentially construed as a shallow version of the deep neural network.
4. Liquidity Based Weighted Signal Smoothing:
    - Fidelity at the "liquid bonds" / benchmark bond nodes
    - Lower fidelity penalty, but higher smoothness penalty for the less liquid bonds
    - Penalty measure is calculated off of the relative liquidity ranking measure (for e.g., TRACE)
5. Non Bayesian Liquidity Based Smoothing:
    - Liquidity indicator serves as a roughness/fidelity magnifier/dampener
    - Also need to penalize for over-parameterized fits using AIC/BIC (also CV/GCV – given that this is essentially a frequentist case).
    - These can be applied not just for bonds, but also CDS, rates, FX – even less liquid ones.

6. Bayesian Extension to the above: Any parametrically specified distribution needs to evolve using a hyper-prior, and the Wahba parametric Bayesian priors need evolving too.
7. Nodal Jacobian/Sensitivity Impact: As always study the impact on the locality of the perturbation, as well as the ease of Jacobian estimation – esp. if the calibration needs to occur through MCMC, non-linear optimization etc.
8. Mixtures of splines and smoothness penalties: As always estimate the impact on monotonicity, convexity, shape preservation etc. - the category item checks in Goodman's paper.
9. Knot Selection Tips: Need some tips in both situations – frequentist and Bayesian.
10. Suggestion on the locally adaptive Parametric Form: Examine the knot-to-knot smoothness and penalty by using additional locally adaptive microstructure parameters and their implications.
11. Goodman and Eilers/Marx Talking Point Issues: Criterion check for these specific "goodness" checks.

Bayesian Curve Calibration

1. Bayesian based past knowledge incorporation of survival probabilities: Given that the prior's, the posterior's and the likelihood's are all probabilities, perhaps the best starting point is for applying it to the problem of updating the survival probabilities and recovery rates based on price observations.
2. Curve Updating techniques: Need grand new formulation techniques that are based on AD and Bayesian methodologies as part of the curve updating strategies based upon individual incoming observations and their strength signals.
3. Curve Construction off of hard/soft signals: Hard Signals are typically the truthness signals. Typically reduce to one calibration parameter per hard observation, and they include the following:
    - Actual observations => Weight independent true truthness signals
    - Weights => Potentially indicative of the truthness hard signal strength

Soft signals are essentially signals extracted from inference schemes. Again, typically reduce to one calibration parameter per soft inference unit, and they include the following:

- Smoothness signals => Continuity, first, second, and higher-order derivatives match – one parameter per match.
- Bayesian update metrics => Inferred using Bayesian methodologies such as maximum likelihood estimates, variance minimization, and error minimization techniques.

4. No-arbitrage hard signals: Simply indicates that ***the*** given hard observation is out of bounds and irreconcilable (i.e., no solution can be found) within the axiomatic inference space dictated by:
    - The parameter sequence implied by the other set of hard signals.
    - The model axiom schemes.
    - The inference rules.

- Directionality "bias" is inherent in calibration (e.g., left to right, ordered sequence set, etc.) – this simplifies the problem space significantly. Therefore, the same directional bias also exists in the calibration nodal sequence.

5. Parameter Space Explosion: Generally not a problem as long as it is segment-localized (in matrix parlance, as long the transition matrix is tri-diagonal, or close to it), i.e., local information discovery does not affect far away nodes/segments.

- Also maybe able to use optimization techniques to trim them.

6. Live Calibrated Parameter Updating: Use automatic differentiation to:

- Estimate parametric Jacobians (or sub-coefficient micro-Jacobians) to the observed product measures.
- Re-adjust the shifts using the hard-signal strength.
- Update the parameters from the calculated shifts.
- Re-construct the curve ever so periodically (for a full re-build, as opposed to the incremental).
- Remember that AD based parametric updates break smoothness (including continuity as Bayesian MLE's) – so use a tolerance in the shift if this is acceptable.

7. Causality Bayesian Network DAG For Credit Curve Building: See Figure 1.

- DAG searches are not really needed, since here they maybe formulated conceptually/axiomatically, as opposed to being established through a search mechanism.

8. Financial First Principles SKU: Following concepts are the core components that can be used to create the curve construction SKU:
   - Time Value of Money.
   - Latent Default Indicator.
   - Recovery on Default.
   - Imbalance premium/discount (for FX, Basis Swaps, etc.)
9. Financial Signal Analysis: Need special analysis techniques to pick out "event trends" from "concept jumps", even for highly liquid instruments.
   - Liquidity-based Signal Extraction =>
     - Identify a liquidity metric
     - Imply the "perfect liquidity" – the point at which there is no premium
     - Compute the liquidity metric for each security
     - Regress (or conceptually determine, or fit) the bid-ask spread to inverse liquidity (remember that even benchmarks only have finite liquidity, not infinite) for each security
     - Try to slap in a secular "event premium" across all the instruments, over and above liquidity
10. Systemic Finance Variables Evolution: Given that every measurement is uncertain to within bounds, it stands to reason that every distribution is also a true distribution (to within the tolerance provided by the corresponding sufficient statistics, and over a finite observation window) of the technical state of the world (i.e., technical = fundamental + a bias).
11. Technical to Fundamental Bias Estimation: This should result from the flow of the information. Non-technical/Fundamental may possibly be estimated using a bias correction applied to the technical signal – Bayesian/frequentist techniques may be of value here.
    - Proxy for non-technical behavior => Identify the non-market proxies for the fundamental drivers, and estimate market drivers as possibly lagging indices.
12. Bayesian Decomposition of Technical Signals: In general, the signal core drivers are limited (like systemic/idiosyncratic factors – alternatively, the latent state quantification metric), but the product specific manifest measures are more varied. Bayesian frameworks well suited for these.

13. Financial Stretch Identification: Bayesian classification techniques can be readily adapted for these purposes – in fact, with abundance of data, these techniques are very appropriate now.

Sequential Curve Estimation

1. Calibration Framework Drivers: Calibration is considered to occur FOR a hidden state $\vec{S}$, which is quantified using the quantification metric $\vec{X}$. $\vec{X}$ is estimated from the manifest measure $\vec{X}$.
2. Product-Measure Point-of-View: From the Dempster-Shaefer/Kalman Filter/Linear Quadratic Estimator point-of-view, the Kalman $\vec{H}$ matrix probabilistically transforms the hidden state quantification metric to an observation measure, e.g., the latent forward rate manifests itself through the swap rate.
3. Segment/Span Nomenclature vs. Curve Calibrator Nomenclature: Call the Curve Calibrator the Dempster-Shaefer Calibrator. Under this:
    - LSQM (Latent State Quantification Metric) => Elastic Variate
    - State Dimensions (Tenor Axis, X/Y Axis of predictors) => Inelastic Variates
    - Thus, the predictors are inelastic, and the responses are elastic.
4. Linearization of LSQM over the predictor axes: The Kalman $\vec{H}$ observation transformer should just linearize $\vec{M}$ onto the space of $\vec{X}$ over the predictor dimensions. Non-linearity of $\vec{X}$ over the predictors is handled through basis splines.
5. Hidden State Evolution vs. Hidden State Representation: The Kalman $\vec{H}$ matrix is more of a state modeling and state representation matrix (i.e., the update part that is fully local to the current time slice) that already brings in the manifest measure ⇔ LSQM transformation model.
6. The Curve Builder $\vec{H}$ matrix: Due to the above, the curve builder $\vec{H}$ matrix needs to accommodate the 2 possible uncertainties:
    - Uncertainty in the manifest measure

- Uncertainty in the manifest measure ⇔ LSQM transformation model. If this transformation is non-parametric, then treat it as certain/deterministic. If it is parametric, then use MLE/MAP to the handle the parameter estimation.

7. UKF Techniques applied to evolve the Curve Builder $\vec{H}$ matrix: Potential non-linearity in the curve builder $\vec{H}$ may be handled using the Jacobian EKF and/or the sigma-point UKF schemes.
8. The Curve Builder $\vec{F}$ matrix: The Curve builder $\vec{F}$ Matrix dictates the evolution from t_i to t_{i+1} as $LSQM_{i+1} = \vec{F} \times LSQM_i$. This should be explicitly posited/formulated. Again, use splining to linearize.
9. Financial Noise Covariance Estimation: May be able to extraneously determine these covariance independent of the state evolution model (if not, we may have to rely on techniques such as ALS (Rajamani (2007), Rajamani and Rawlings (2009)).

References

- Rajamani, M. R. (2007): *Data-based Techniques to improve State Estimation in Model Predictive Control* PhD Thesis **University of Wisconsin-Madison**.
- Rajamani, M. R., and J. B. Rawlings (2009): Estimation of the Disturbance Structure from Data using Semi-definite Programming and Optimal Weighting **45** 142-148.

Section IX: Bond Relative Value Metrics Generation

Credit Analytics Bond RV Calculation Methodology

Introduction

This document outlines the methodology used in Credit Analytics (release 1.4 and above) for the calculation of the bond curve-based relative value measures.

The Bond RV Measure Set

Classification of a given bond measure as an “RV” measure is somewhat arbitrary. In general, it is used (here) to refer to any of the measure that is in use for spotting relative value across bonds for a given issuer (or any similar category), and which is usually determined straight from a bond market measure (price/yield/spread to treasury). Specifically, it excludes such bond measures as DV01, loss PV, principal PV etc.

Following is the list of the RV measures - refer to the section below for a precise definition of these terms.

- Bond Basis
- Convexity
- Credit Basis
- Discount Margin
- Duration
- DV01
- G Spread (Spread to the Government/Treasury Discount Curve)
- I Spread (Interpolated Spread to the Discount Curve)

- Option Adjusted Spread
- Macaulay Duration
- Modified Duration
- Par Asset Swap Spread
- Par Equivalent CDS Spread (PECS)
- Price
- Spread over Treasury (TSY) benchmark
- Yield
- Yield Basis
- Yield Spread
- Yield01
- Zero Discount Margin (ZDM)
- Zero (Z) Spread

Asset Swap Spread

Asset swap is an estimate of the spread over a matching swap maturing at the bond's maturity. For a non-par swap, an additional spread is added by dividing the departure from par by the swap annuity.

Bond Basis

Bond Basis to Exercise (B_E) is a bond RV metric capturing the basis in the yield space. It is defined as the difference between the yield to exercise computed from the market price and the yield to exercise computed from the theoretical price off of the risk-free discount curve.

Convexity

Convexity to Exercise (C_E) measures the rate of change of duration with yield. It is defined as the change in market duration on 1 basis point increase in yield.

Credit Basis

Credit Basis to Exercise (Φ_E) captures the adjustment needed to the input credit curve to account for the bond market price. It is defined as the parallel shift needed to be applied across the input credit curves quotes to make create the credit curve that produces the market price.

Credit Basis can be negative; given that the credit curve does not typically calibrate for negative hazard rates, the credit basis may not be calculable for market prices above a certain range.

Discount Margin

Discount Margin to Exercise (Δ_E) measures that spread earned above the reference rate. For fixed coupon bonds, it is computed as the difference between market yield and the initial implied discount rate to the bond's frequency. For floaters, it is computed as the difference between market yield and the initial reference index rate.

Duration

Duration to Exercise (D_E) captures the relative rate of change of bond price with yield. It is defined as the fractional change of price as the market yield increases by 1 basis point.

DV01

DV01 to exercise captures the cash flow present-value weighted pay date durations.

G Spread

G Spread to Exercise (G_E) accounts for the Spread over the Government/Treasury Discount Curve. It is defined as the difference between the market yield to exercise of the bond and the rate calculated to the exercise date, implied from the specified discount curve constructed from the government debt instruments.

I Spread

I-Spread to Exercise (I_E) measures the spread over the specified Discount Curve interpolated to the exercise date. It is defined as the difference between the market yield to exercise of the bond and the rate interpolated to the exercise date, implied from the specified discount curve.

Macaulay Duration

Macaulay duration to exercise captures the cash flow present-value weighted pay date durations.

Modified Duration

Modified duration to Exercise (D_E) captures the relative rate of change of bond price with yield. It is defined as the fractional change of price as the market yield increases by 1 basis point.

Option Adjusted Spread

Option adjusted to Exercise (O_E) spread captures the value of the option embedded in the bond. It is calculated identical to the Z-Spread (see Z-Spread for details), although it may be based off of a different discount curve.

Par Asset Swap Spread

Par asset swap spread to Exercise (P_E) estimates the spread implied by the price that a par floater would be expected to pay. It is defined as the difference between the market price and the theoretical price computed using the discount curve, computed in units of the bond PV01 (duration times price).

Par Spread

Par spread to Exercise (Ω_E) estimates the fair fixed coupon implied by the market price that an equivalent fixed coupon bond trading at par would pay. It is defined as the difference between the market price and par, computed in units of the bond PV01 (duration times price).

Par Equivalent CDS Spread (PECS)

The PECS to Exercise (Θ_E) measures the flat credit spread premium implied by the bond price. It is computed as the implied flat spread of the fictitious CDS needed to recover the market price of the bond.

Price

The theoretical exercise price of the bond can be computed from the bond cash flows, the discount curve and/or the credit curve and recovery using the methodology described below.

Spread Over Treasury Benchmark

Treasury Spread to Exercise (S_{TSY}) accounts for the returns over the given benchmark bond. It is defined as the difference between the market yield to exercise of the bond and the yield to maturity of the specified benchmark treasury bond.

Yield

The yield to exercise (y_E) implied from the bond market price is calculated according to the equations shown below.

Yield Basis

Yield basis to Exercise is defined identically as the bond basis. See Bond Basis for details.

Yield Spread

Yield spread is defined identically as the bond basis. See Bond Basis for details.

Yield01

Yield01 (also called YV01) to Exercise (D_E) captures the relative rate of change of bond price with yield. It is defined as the fractional change of price as the market yield increases by 1 basis point.

Zero Discount Margin (ZDM)

Zero Discount Margin to Exercise (Ψ_E) estimates the excess spread over the reference index curve. It is a measure valid only for floaters; it is defined as the extra coupon spread to be applied to the reference index rate curve so as to be able to recover the market price.

Zero (Z) Spread

Z Spread to Exercise (Z_S) captures the excess spread over the discount curve. The details of implying the zero-curve and the corresponding calculation of the Z Spread are described below.

Symbol	Description
B_E	Bond Basis to Exercise
C_E	Convexity to Exercise
Φ_E	Credit Basis to Exercise
Δ_E	Discount Margin to Exercise
D_E	Duration to Exercise
G_E	G-Spread to Exercise
I_E	I Spread to Exercise
O_E	Option Adjusted Spread to Exercise
P_E	Par Asset Swap Spread to Exercise

Ω_E	Par Spread to Exercise
Θ_E	Par Equivalent CDS Spread to Exercise
Ψ_E	Zero Discount Margin to Exercise
ϵ_i	The Full Period Coupon Rate between t_{i-1} and t_i
φ_E	Government Curve implied Rate to Exercise
$\Gamma_C(i-1,i)$	Coupon Day Count Fraction between t_{i-1} and t_i
$\Gamma_Y(i-1,i)$	Yield Quote Day Count Fraction between t_{i-1} and t_i
δ_{IR}	Spread applied to the Interest Rate Curve
d_c	Coupon Day Count Convention
d_{yc}	Yield Quote Day Count Convention
f_c	Coupon Frequency
f_y	Yield Quote Frequency
t_i	Time at coupon flow #i
t_E	Exercise Date Time
y_E	Yield To Exercise
$C_f(t_i)$	Coupon Flow at Date Time t_i
$D_f(t_i)$	Discount Curve based Discount Factor at Date Time t_i
$D_f(\delta, t_i)$	δ Bumped Discount Curve based Discount Factor at Date Time t_i
$D_f(y_E, f_y, d_{yc}, t_i)$	Discount Factor at Date Time t_i given Yield To Exercise y_E, Quote Frequency f_y, Quote Day Count Convention d_{yc}
$D_f(z_S, f_y, d_{yc}, t_i)$	Discount Factor at date time t_i given the Z Spread z_S, the quote frequency f_y, Quote Day Count Convention d_{yc}
N_E	Notional at Exercise
N_j	Outstanding Notional at Date Time N_j
ΔN_j	Principal Notional Payout at Date Time N_j
$P_{Dirty}(IR_{Theo})$	Theoretical Dirty Price calculated from the input IR Curve
$P_{CR,Dirty}(IR_{Theo}, CR_{Theo})$	Theoretical Dirty Price calculated from the input IR and Credit Curves
$P_{Dirty}(\delta, IR_{Theo})$	Theoretical Dirty Price calculated from the input IR Curve with a spread adjustment

$P_{CR,Dirty}(\lambda_{CR}, IR_{Theo}, CR_{Theo})$	Theoretical Dirty Price calculated from the input IR Curve and Credit Curve, where the Credit Curve is created off of a flat spread λ_{CR}
$P_{CR,Dirty}(\delta_{CR}, IR_{Theo}, CR_{Theo})$	Theoretical Dirty Price calculated from the input IR Curve and Credit Curve, with a spread adjustment δ_{CR} applied to the Credit Curve
R_E	Discount Curve implied Rate to Exercise
$S_P(t)$	Survival Probability to time t
S_{TSY}	Treasury Benchmark Spread to Exercise (done)
y_{BMK}	Yield of the Specified Treasury Benchmark
y_E	Yield to Exercise
$y_E(IR_{Theo}, CR_{Theo})$	Theoretical Yield to exercise
$\{z_i\}$	Collection of the ordered nodes $\{z_i\}$ that constitute the Zero Curve
z_i	Zero Rate to the Date Time t_i
z_S	Z Spread

Relative Value Cross-Metric Grid

Measure	Anchor	Work-out Dependence	Non-linear Root Extraction	Manifest Measure Extraction	Cash Flow Valuation
Asset Swap Spread	PRICE	TRUE	FALSE	TRUE	FALSE
Bond Basis	YIELD	TRUE	TRUE	FALSE	TRUE
Credit Basis	PRICE	TRUE	TRUE	FALSE	TRUE
Discount Margin	YIELD	FALSE	FALSE	FALSE	FALSE
G Spread	YIELD	TRUE	FALSE	TRUE	FALSE
I Spread	YIELD	TRUE	FALSE	TRUE	FALSE
Option Adjusted Spread	PRICE	TRUE	FALSE	FALSE	TRUE

PECS	PRICE	TRUE	TRUE	FALSE	TRUE
Price	PRICE	FALSE	FALSE	FALSE	FALSE
Treasury Spread	YIELD	TRUE	FALSE	FALSE	FALSE
Yield	YIELD	FALSE	FALSE	FALSE	FALSE
Yield Spread	YIELD	TRUE	TRUE	FALSE	TRUE
Z Spread	PRICE	TRUE	TRUE	FALSE	TRUE
Convexity	PRICE	TRUE	TRUE	FALSE	TRUE
Duration	PRICE	TRUE	TRUE	FALSE	TRUE
Modified Duration	PRICE	TRUE	TRUE	FALSE	TRUE
Macaulay Duration	YIELD	TRUE	TRUE	FALSE	TRUE
Yield01	YIELD	TRUE	TRUE	FALSE	TRUE

Basic Measures

The Coupon Cash Flow of the bond at coupon date time t_i is given as

$$C_f(t_i) = \epsilon_i \Gamma_c(i-1, i) d_c$$

The Discount Factor at date time t given the yield to exercise y_E, the quote frequency f_y, and the annualized quote day count based time fraction $\Gamma_y(i-1, i)$ is given as

$$D_f(y_E, f_y, d_{yc}, t_i) = \left[\frac{1}{1 + \frac{y_E}{f_y}}\right]^{f_y \Gamma(0,t)}$$

The Zero Rate z_i to a date time t_i is determined by the solution to z_i that computes the discount factor $D_f(t_i)$ given the quote frequency f_y, and the annualized quote day count based time fraction $\Gamma_y(i-1, i)$ is given as

$$D_f(t_i) = \left[\frac{1}{1 + \frac{z_i}{f_y}}\right]^{f_y \Gamma(0,t)}$$

The Discount Factor at date time t_i given the zero rate z_i, the Z Spread z_S, the quote frequency f_y, and the annualized quote day count based time fraction $\Gamma_y(i-1, i)$ is given as

$$D_f(z_S, f_y, d_{yc}, t_i) = \left[\frac{1}{1 + \frac{z_i + z_S}{f_y}}\right]^{f_y \Gamma(0,t_i)}$$

The Principal redeemed, amortized, or capitalized at time t_j is given as

$$\Delta N_j = N_j - N_{j-1}$$

The Dirty Price of the bond at exercise given an exercise yield y_E is given as

$$P_{Dirty}(y_E) = \sum_i C_f(t_i) D_f(y_E, f_y, d_{yc}, t_i) + \Delta \sum_j N_j D_f(y_E, f_y, d_{yc}, t_j) + N_E D_f(y_E, f_y, d_{yc}, t_E)$$

The Dirty Price of the bond at exercise given a Z spread (z_S) is given as

$$P_{Dirty}(z_S) = \sum_i C_f(t_i) D_f(z_S, f_y, d_{yc}, t_i) + \sum_j \Delta N_j D_f(z_S, f_y, d_{yc}, t_j) + N_E D_f(z_S, f_y, d_{yc}, t_E)$$

The Theoretical IR implied Dirty Price $P_{Dirty}(IR_{Theo})$ of the bond at exercise calculated using the discount factors from the input discount curve is given as

$$P_{Dirty}(IR_{Theo}) = \sum_i C_f(t_i) D_f(t_i) + \sum_j \Delta N_j D_f(t_j) + N_E D_f(t_E)$$

The IR implied Dirty Price $P_{Dirty}(\delta_{IR}, IR_{Theo})$ of the bond at exercise calculated using the discount factors from the input discount curve bumped by a rate δ_{IR} is given as

$$P_{Dirty}(\delta_{IR}, IR_{Theo}) = \sum_i C_f(t_i) D_f(\delta_{IR}, t_i) + \sum_j \Delta N_j D_f\left(\delta_{IR}, t_j\right) + N_E D_f(\delta_{IR}, t_E)$$

The Theoretical Credit implied Dirty Price $P_{CR,Dirty}(IR_{Theo}, CR_{Theo})$ of the bond at exercise calculated using the discount factors and the survival probabilities from the input discount curve and the credit curve respectively is given as

$$P_{CR,Dirty}(IR_{Theo}, CR_{Theo}) = \sum_i C_f(t_i) D_f(t_i) S_P(t_i) + \sum_j \Delta N_j D_f\left(t_j\right) S_P\left(t_j\right) + N_E D_f(t_E) S_P(t_E)$$

The Theoretical Credit implied Dirty Price $P_{CR,Dirty}(\delta_{CR}, IR_{Theo}, CR_{Theo})$ of the bond at exercise calculated using the discount factors and the survival probabilities from the input discount curve and the credit curve respectively, where the credit curve is bumped by a rate δ_{CR} is given as

$$\begin{aligned} P_{CR,Dirty}(\delta_{CR}, IR_{Theo}, CR_{Theo}) \\ &= \sum_i C_f(t_i) D_f(t_i) S_P(\delta_{CR}, t_i) + \sum_j \Delta N_j D_f\left(t_j\right) S_P\left(\delta_{CR}, t_j\right) \\ &+ N_E D_f(t_E) S_P(\delta_{CR}, t_E) \end{aligned}$$

The Credit Basis to Exercise Φ_E of the bond given the market price (P_{MKT}) is given as the solution of δ_{CR} in (30.11):

$$\begin{aligned} P_{CR,Dirty}(\Phi_E, IR_{Theo}, CR_{Theo}) \\ &= \sum_i C_f(t_i) D_f(t_i) S_P(\Phi_E, t_i) + \sum_j \Delta N_j D_f\left(t_j\right) S_P\left(\Phi_E, t_j\right) + N_E D_f(t_E) S_P(\Phi_E, t_E) \end{aligned}$$

The Theoretical Credit implied Dirty Price of the bond at exercise $P_{CR,Dirty}(\lambda_{CR}, IR_{Theo}, CR_{Theo})$ is calculated using the discount factors and the survival probabilities from the input discount curve and the credit curve respectively, where the credit curve is created off of a flat spread λ_{CR}, is given as

$$P_{CR,Dirty}(\lambda_{CR}, IR_{Theo}, CR_{Theo}) = \sum_i C_f(t_i) D_f(t_i) S_P(\lambda_{CR}, t_i) + \sum_j \Delta N_j D_f(t_j) S_P(\lambda_{CR}, t_j) + N_E D_f(t_E) S_P(\lambda_{CR}, t_E)$$

The Par Equivalent CDS Spread to Exercise of the bond given the market price (P_{MKT}) is given as the solution of δ_{CR} in (30.13):

$$P_{CR,Dirty}(\delta_{CR}, IR_{Theo}, CR_{Theo}) = \sum_i C_f(t_i) D_f(t_i) S_P(\delta_{CR}, t_i) + \sum_j \Delta N_j D_f(t_j) S_P(\delta_{CR}, t_j) + N_E D_f(t_E) S_P(\delta_{CR}, t_E)$$

The Bond Spread to Treasury Benchmark at exercise S_{TSY} is computed from the Bond Yield to Exercise y_E and the given Treasury Benchmark Yield y_{BMK} as

$$S_{TSY} = y_E - y_{BMK}$$

The Bond I Spread to exercise I_E is computed from the Bond Yield to Exercise y_E and the Discount rate to Exercise implied from the input Interest Rate Curve R_E as

$$I_E = y_E - R_E$$

The Bond G Spread to exercise G_E is computed from the Bond Yield to Exercise y_E and the Discount rate to Exercise implied from the input Government Rate Curve φ_E as

$$G_E = y_E - \varphi_E$$

The Theoretical Yield to exercise $y_E(IR_{Theo})$ of the bond at exercise calculated using the discount factors from the input discount curve is given as the solution of y_E in (30.6), where the dirty price P_{Dirty} is substituted by $P_{Dirty}(IR_{Theo})$ of (30.8).

The Bond Basis at exercise B_E (also referred to as yield basis or as yield spread) is computed from the Bond Yield to Exercise y_E and the Bond Yield to Exercise y_E as

$$B_E = y_E - y_E(IR_{Theo})$$

The Bond Duration to exercise D_E is computed as the fractional change in bond market price (P_{MKT}) to the change in the market yield (Y_{MKT}) as

$$D_E = \frac{1}{P_{MKT}} \frac{\Delta P_{MKT}}{\Delta Y_{MKT}}$$

The Bond Convexity to exercise C_E is computed as the change in bond market duration to exercise (D_E) to the change in the market yield (Y_{MKT}) as

$$C_E = \frac{\Delta D_E}{\Delta Y_{MKT}}$$

The Discount Margin to Exercise Δ_E of the bond given the market yield to exercise (y_E) is given as:

$$\Delta_E = y_E - R_E$$

The Par Asset Swap Spread to Exercise (P_E) of the bond given the market price (P_{MKT}) is given as:

$$P_E = \frac{1}{P_{MKT}} \frac{P_{Dirty}(IR_{Theo}) - P_{MKT}}{D_E}$$

The Option Adjusted Spread to Exercise O_E is calculated identical to Z Spread, as a solution to z_S in (30.7).

Some Trivial Closed-Form Analytical Bond Math Results

1. Price when Yield Equals Coupon: Given the annualized coupon r, payment frequency f, period yield y, period coupon payment $c = \frac{r}{f}$, number of coupon periods to maturity n, the PV is computed from

$$PV = \sum_{m=1}^{n} \frac{\frac{r}{f}}{(1+y)^m} + \frac{1}{(1+y)^n} = \sum_{m=1}^{n} \frac{c}{(1+y)^m} + \frac{1}{(1+y)^n} = cd\frac{1-d^n}{1-d} + d^n$$

where

$$d = \frac{1}{1+y}$$

If you are just past a coupon pay so that

$$PV_{Clean} = PV_{Dirty}$$

and if

$$PV = 1$$

then we get

$$1 = cd\frac{1-d^n}{1-d} + d^n$$

which implies that

$$d = \frac{1}{1 + c}$$

and thus

$$y = c$$

2. Par Yield Dirty Price at a non-coupon Date: If ξ is the accrual fraction corresponding to the accruing period, then

$$PV_{Dirty} = \frac{\xi c}{(1 + y)^{\xi}} + \frac{c}{(1 + y)^{\xi + 1}} + \cdots + \frac{c}{(1 + y)^{\xi + n}} + \frac{1}{(1 + y)^{\xi + n}}$$

which reduces to

$$PV_{Dirty} = \frac{\xi c}{(1 + y)^{\xi}} + \frac{1}{(1 + y)^{\xi}}\left[\sum_{m=1}^{n} \frac{c}{(1 + y)^{m}} + \frac{1}{(1 + y)^{n}}\right] = \frac{1 + \xi c}{(1 + y)^{\xi}}$$

Section X: Stochastic Evolution and Option Pricing

Stochastic Calculus

Single Factor Stochastic Calculus

1. The Principal Brownian Stochastic Differential Equation: Given

$$\Delta S = \mu_S(S, t)\Delta t + \sigma_S(S, t)\Delta W_S$$

the Brownian SDE that accounts for the evolution of

$$F \equiv F(S, t)$$

is

$$\Delta F(S, t) = \left[\frac{\partial F(S, t)}{\partial t} + \mu_S(S, t)S\frac{\partial F(S, t)}{\partial S} + \frac{1}{2}\sigma_S{}^2(S, t)S^2\frac{\partial^2 F(S, t)}{\partial S^2}\right]\Delta t + \sigma_S(S, t)S\Delta W_S$$

ONLY in situations where $F(S, t)$ is second order or higher in S does the Brownian SDE have a non-trivial contribution (arising from the $\frac{\partial^2 F(S,t)}{\partial S^2}$ term).

2. Incorporation of the Weiner Process in the Brownian SDE: This is incorporated exclusively at the point

$$\underset{\Delta t \to 0}{Limit} \langle \Delta W_S{}^2 \rangle \cong \Delta t$$

Notice that in the SDE above $\langle \Delta W_S{}^2 \rangle$ shows up only in conjunction with $\frac{\partial^2 F(S,t)}{\partial S^2}$! Consistent with the limit above, one may often find similar meanings behind statements such as

$$\underset{\Delta t \to 0}{Limit} \langle \Delta W_i \Delta W_j \rangle \cong \rho_{ij} \Delta t$$

3. Non-Brownian Evolution SDE's: In general,

$$\Delta F(S, t) = \frac{\partial F(S, t)}{\partial t} \Delta t + \frac{\partial F(S, t)}{\partial S} \Delta S + \frac{1}{2} \frac{\partial^2 F(S, t)}{\partial S^2} (\Delta S)^2 + \frac{1}{2} \frac{\partial^3 F(S, t)}{\partial S^3} (\Delta S)^3 + \cdots$$

As we saw earlier, for Brownian we observed that $\underset{\Delta t \to 0}{Limit} \langle \Delta W_S{}^2 \rangle \cong \Delta t$. Of course, there could be other SDE's where the higher order terms may have specific stochastic expectations applied in the limit.

4. General Purpose Validity of the Stochastic Differential: The power behind the validity of $\Delta S = \mu_S(S, t)\Delta t + \sigma_S(S, t)\Delta W_S$ is as follows: Since both μ_S and σ_S can, in general, be functions of S and t, the incremental ΔS still **ABSOLUTELY** follows the stochastic dynamics dictated by the driver process ΔW_S.
   - Further, given the stochastic shock for ΔS may go in the opposite direction from ΔW_S, σ_S can be absolutely negative as well.
5. Convenience of the log form: By setting

$$\Delta F(S, t) = \log S$$

the driver equation now becomes

$$\Delta(\log S) = \left[\mu_S(S, t) - \frac{1}{2} \sigma_S{}^2(S, t)\right] \Delta t + \sigma_S(S, t)\Delta W_S$$

The advantages to this are:

- $S > 0$ always (by demands of the log normal form), and
- The RHS may be independent of S, and dependent only on t, which enables explicit evaluation of the dynamics.

The reason for the above advantages is that, in

$$\Delta F(S,t) = \left[\frac{\partial F(S,t)}{\partial t} + \mu_S(S,t)S\frac{\partial F(S,t)}{\partial S} + \frac{1}{2}{\sigma_S}^2(S,t)S^2\frac{\partial^2 F(S,t)}{\partial S^2}\right]\Delta t + \sigma_S(S,t)S\Delta W_S$$

the only explicitly S-dependent terms are $S\frac{\partial F(S,t)}{\partial S}$ and $S^2\frac{\partial^2 F(S,t)}{\partial S^2}$. The $\log S$ form above reduces these terms to

$$S\frac{\partial F(S,t)}{\partial S} = 1$$

and

$$S^2\frac{\partial^2 F(S,t)}{\partial S^2} = -1$$

In the case where

$$\mu_S(S,t) \equiv \mu_S(t)$$

and

$$\sigma_S(S,t) \equiv \sigma_S(t)$$

(i.e., μ_S and σ_S depend only on t), an explicit expression for S may be worked out as

$$S = S_0 e^{\int_{t_0}^{t}\left[\mu_S(t) - \frac{1}{2}{\sigma_S}^2(t)\right]dt} e^{\int_{t_0}^{t}\sigma_S(t)dW_S}$$

6. <u>Universality of the results above</u>: Despite a "particular" choice for the transformation of S for the solution seen above, the evolution dynamics specified above are ABSOLUTELY VALID UNIVERSALLY for the latent state dynamics of S. The only axiomatic stipulation is that W be Brownian with variance $\sqrt{\Delta t}$.

Multi-Factor Stochastic Calculus

1. Multi-variate non-Stochastic Evolution: Consider 2 driver deterministic processes

$$\frac{\Delta B}{B} = \mu_B \Delta t$$

and

$$\frac{\Delta F}{F} = \mu_F \Delta t$$

From above

$$\Delta(BF) = \Delta B.F + \Delta F.B = \mu_B BF\Delta t + \mu_F BF\Delta t$$

results in

$$\frac{\Delta(BF)}{BF} = \mu_B \Delta t + \mu_F \Delta t = \frac{\Delta B}{B} + \frac{\Delta F}{F}$$

2. Multi-variate Stochastic Evolution: The 2 driver processes now become

$$\frac{\Delta B}{B} = \mu_B \Delta t + \sigma_B \Delta W_B$$

and

$$\frac{\Delta F}{F} = \mu_F \Delta t + \sigma_F \Delta W_F$$

From above

$$\Delta(BF) = \Delta B.F + \Delta F.B + \Delta B.\Delta F = BF\frac{\Delta B}{B} + BF\frac{\Delta F}{F} + \Delta B.\Delta F$$

results in

$$\frac{\Delta(BF)}{BF} = \frac{\Delta B}{B} + \frac{\Delta F}{F} + \frac{\Delta B \Delta F}{BF}$$

The final component $\frac{\Delta B \Delta F}{BF}$ is attributable to the stochastic correlative cross product.

3. $\frac{\Delta B \Delta F}{BF}$ Simplification: Consider the situation where ΔW_B and ΔW_F are both Brownian such that

$$\Delta W_B \Delta W_F = \rho_{BF} \Delta t$$

Then

$$\Delta B \Delta F = \rho_{BF} \sigma_B \sigma_F BF \Delta t$$

Thus

$$\frac{\Delta B \Delta F}{BF} = \rho_{BF} \sigma_B \sigma_F \Delta t$$

which is the pure stochastic drift term. Of course, if ΔW_B and ΔW_F are related in other covariance forms (i.e., non-Brownian forms), other terms will enter the formulation.

4. Numeraire: "Numeraire" refers to the multiplicative unit that has the impact of "localizing" the corresponding stochastic factor. Implicit in our usage of the numeraire here is its multiplicative nature, thereby requiring the log-normal dynamics in the formulation as in above.

- "Change of numeraire" => This has a more formal and comprehensive treatment later, but the term here refers to the impact of the cross-numeraire correlative component which simply contributes to the drift, i.e.,

$$\frac{\Delta(BF)}{BF} = \frac{\Delta B}{B} + \frac{\Delta F}{F} + \frac{\Delta B \Delta F}{BF} = \frac{\Delta B}{B} + \frac{\Delta F}{F} + \rho_{BF}\sigma_B\sigma_F\Delta t$$

$\rho_{BF}\sigma_B\sigma_F$ is the incremental cross numeraire drift. The corresponding expression for "divided" numeraire formulation that is common in stochastic finance is

$$\frac{\Delta\left({}^{B}/_{F}\right)}{\left({}^{B}/_{F}\right)} = \frac{\Delta B}{B} - \frac{\Delta F}{F} - \frac{\Delta B \Delta F}{BF} = \frac{\Delta B}{B} - \frac{\Delta F}{F} - \rho_{BF}\sigma_B\sigma_F\Delta t$$

5. <u>Challenges with the log numeraire formulation</u>: It needs to be remembered that working in the log numeraire format space does not result in the cross-correlation term coming out explicitly. We'll see that below.
6. <u>Evolution Formulation</u>: The driver processes are

$$\frac{\Delta B}{B} = \mu_B\Delta t + \sigma_B\Delta W_B$$

and

$$\frac{\Delta F}{F} = \mu_F\Delta t + \sigma_F\Delta W_F$$

We will examine the behavior of

$$\phi \equiv \phi(B, F, t)$$

given

$$\Delta W_B \Delta W_F = \rho_{BF} \Delta t$$

- SDE => Remembering that

$$\mu_B = \mu_B(F, t)$$

$$\mu_F = \mu_F(F, t)$$

$$\sigma_B = \sigma_B(F, t)$$

and

$$\sigma_F = \sigma_F(F, t)$$

$$\Delta\phi = \left[\frac{\partial\phi}{\partial t} + \mu_B B \frac{\partial\phi}{\partial B} + \frac{1}{2}{\sigma_B}^2 B^2 \frac{\partial^2\phi}{\partial B^2} + \mu_F F \frac{\partial\phi}{\partial F} + \frac{1}{2}{\sigma_F}^2 F^2 \frac{\partial^2\phi}{\partial F^2} + \rho_{BF}\sigma_B\sigma_F BF \frac{\partial^2\phi}{\partial B \partial F}\right]\Delta t + \sigma_B B \Delta W_B + \sigma_F F \Delta W_F$$

7. $\underline{\phi = BF}$: Setting

$$\phi = BF$$

we get the following:

$$\frac{\partial\phi}{\partial B} = F$$

$$\frac{\partial^2\phi}{\partial B^2} = 0$$

$$\frac{\partial \phi}{\partial F} = B$$

$$\frac{\partial^2 \phi}{\partial F^2} = 0$$

$$\frac{\partial^2 \phi}{\partial B \partial F} = 1$$

The SDE now becomes

$$\Delta\phi = [\mu_B + \mu_F + \rho_{BF}\sigma_B\sigma_F]\Delta t + \sigma_B B \Delta W_B + \sigma_F F \Delta W_F$$

Applying the original Ito's Lemma on $\log \phi$, we now get

$$\Delta(\log \phi) = [\mu_B + \mu_F]\Delta t + \sigma_B B \Delta W_B + \sigma_F F \Delta W_F$$

Notice that the correlation cross product vanishes.

Likewise, if

$$\phi = \frac{B}{F}$$

(the typical divided numeraire), and working through the partials, the SDE becomes

$$\Delta\phi = [\mu_B - \mu_F - \rho_{BF}\sigma_B\sigma_F]\Delta t + \sigma_B B \Delta W_B - \sigma_F F \Delta W_F$$

Applying the original Ito's Lemma on this $\log \phi$, we get

$$\Delta(\log \phi) = [\mu_B - \mu_F]\Delta t + \sigma_B B \Delta W_B - \sigma_F F \Delta W_F$$

As before, the correlation cross product vanishes.

Risk-Neutral Pricing Framework

1. Probability Measure: This is typically specified in stochastic Brownian terms, using the drift μ and volatility σ. As an example, for a given stock asset, the dynamics are specified using the real-world μ and σ (i.e., potentially the realized μ and σ).
2. Risk-Neutral Probability Measure: This has its own μ and σ, the only difference being that

$$\mu \equiv r$$

   the risk free rate of return, and σ is the implied future volatility, i.e., the volatility extracted by calibrating to the market prices.

   - Real-world <-> Risk-Neutral Transforms => The risk-neutral μ/σ maybe mapped over to the real-world μ/σ using the standard measure transforms such as the continuous Girsanov and Radon-Nikodym transformations (Bjork (2004)).
   - The Drift in Risk-Neutral => In reality, the risk-neutral drift is not entirely unique. If the derivatives are collateralized, the drift should correspond to the collateral numeraire drift. Under dynamic replication/hedging, however, the drift should correspond to the appropriate numeraire that accounts for the funding cost (e.g., treasury account numeraire, etc.)

3. Replication Principle: This simply states that the derivative price is computed using the risk-free discounting of the terminal payoff sequence. The terminal payoff may be a function of the asset price, but it **MAY NOT** imply a specific asset path, or a particular payoff asset distribution.
4. Replication of the Terminal Payoff: Given that we typically attempt to replicate a known pay-off, what we are attempting to estimate (and, therefore, taken to be unknown) is today's asset price. In particular, this is true for zero-coupon bond prices ($P(t, T)$ is the unknown, and $P(T, T)$ is taken to be 1); for FX ($X(t)$ is the unknown, and $X(T)$ is known), etc. NB: P/X are derivatives, and not primary securities. Primary securities evolve on their own (real-

world or risk-neutral, as the usage may dictate), and their time value is not estimated using replication arguments.

5. Asset Price as a Martingale in its own Risk-Neutral Measure: This is because the risk-neutral asset growth rate occurs at the risk-free rate due to the multi-period self-financing requirement of the Arrow prices owing to dynamic replication (Taleb (1997), Gisiger (2010).
    - State Price Density => This refers to the distribution of the security's prices on its own measure. Unique risk-neutral state measure (i.e., the measure under which the state prices are martingales) implies that the state price density (i.e., distribution) is unique under the same measure.
    - Uniqueness of the Risk-Neutral State Measure => Why will the risk-neutral state measure be unique? This is because, as seen above, the measure's drift and volatility will be unique – the drift being unique because the asset prices are martingales, as this corresponds to **ZERO** price-of-market-risk premium (thereby making the portfolio return the risk-free rate). This is also applicable to non-Brownian scenarios, but with the stipulation that the drift still correspond to the risk-free rate. As in the case of Brownian motion, the measure parameters may be inferred/calibrated using the terminal payoff prices calibrated to the market (using the given model).
6. Unique Risk Neutral Asset Price: As can be seen above, this is a martingale. The no-arbitrage state resulting from dynamic replication causes a) the risk-free drift, and b) the unique state price.
7. Zero Coupon Bond as the Perfect Replicator: The scenario specific cash flow may be generated only by the corresponding zero-coupon bond – further, it generates ONLY the specific cash flow, and nothing else. Thus, this is the ideal Arrow replicator.
8. Measure Change: If the underlying asset's stochastic drivers and the risk-free numeraire's stochastic drivers are identical, the net volatility is simply the difference, and the asset continues to be a martingale in the numeraire's measure. Likewise, if the drivers between the asset and the risk-free numeraire's processes are orthogonal, they should be able to grow consistently independent of each other (with the asset continuing being a martingale in its own measure). If these drivers are correlated, however, the measure change amount needs to be applied as an adjustment to the asset's risk-neutral drift (this may be easily verified using straightforward application of Ito's lemma).

9. Equivalent Martingale Formulation: Let's say

$$y_1 = \alpha_1 \Delta t + \beta \Delta W$$

and

$$y_2 = \alpha_2 \Delta t + \beta \Delta W$$

Then, on setting

$$\widetilde{W} = W + \frac{\alpha_2 - \alpha_1}{\beta} t$$

you get

$$y_1 = \alpha_1 \Delta t + \beta \Delta W = \alpha_2 \Delta t + \beta \Delta \widetilde{W}$$

Thus the drift is now the same for y_1 and y_2, but under different measures (W and $\widetilde{W}$).

References

- Bjork, T. (2004): *Arbitrage Theory in Continuous Time* **Oxford University Press** Oxford.
- Gisiger, N. (2010): Risk-Neutral Probabilities Explained, eSSRN.
- Taleb, N. N. (1997): *Managing Vanilla and Exotic Options* **Wiley** New York.

Black-Scholes Methodology

Overview and Base Derivation

1. Components of the Black Scholes Pricing Framework:
    - Terminal Payoff Replication (this may show itself up as a boundary condition, really)
    - Instantaneously non-stochastic replication
    - Self-financing, indicating that the portfolio grows at the risk-free (i.e., financing/unique "risk free") rate
2. Black-Scholes Portfolio: Let V be the value of the option, S the value of the underlying, N the number of short units of the underlying, and Π the Options Portfolio. Then

$$\Pi = V - nS$$

results in

$$\Delta\Pi = \Delta V - n\Delta S$$

$$\Delta V = \frac{\partial V}{\partial S}\Delta S + \left[\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial S^2}\right]\Delta t$$

3. Delta-Hedged Portfolio:

$$n = \frac{\partial V}{\partial S}$$

The time change for Π is the same as that for V, i.e.,

$$\frac{\mathrm{dV}}{dt} = \frac{\mathrm{d\Pi}}{dt} = \frac{\partial \mathrm{V}}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial S^2} = r\Pi = r[V - nS]$$

Finally, since the dependence ΔΠ on ΔS is zero, the dependence ΔV on ΔS is non-zero! Note that

$$\frac{\partial \mathrm{V}}{\partial \mu_S} = 0$$

as there is no explicit dependence.

4. Relation between Delta, Gamma, Theta, and Option Value: A simple linear relation exists between these:

$$\frac{\partial \mathrm{V}}{\partial t} + rS\frac{\partial \mathrm{V}}{\partial S} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial S^2} - rV = 0$$

where

$$\frac{\partial \mathrm{V}}{\partial t} = \theta$$

$$\frac{\partial \mathrm{V}}{\partial S} = \Delta$$

and

$$\frac{\partial^2 V}{\partial S^2} = \Gamma$$

5. Base Equation Interpretation: The quantity $\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial S^2}$ represents the ***Option Total Time Value Change***, and $r\left[V - S\frac{\partial V}{\partial S}\right]$ represents the ***Hedged Portfolio Value Return***. Thus

The Replication Technology

1. Replication Technique:
    - Blind expectation performed over the distribution of the stochastic underlier (in this case S) will not work, since the investors' expectation of the premium (i.e., drift) over the base drift of the underlier can vary dependent on many factor (e.g., the investors' risk tolerance profile).
    - Pick the Arrow securities – the set of securities that replicate the pay-off of the asset you are pricing – at least synthesize these securities if they are not real.
    - Replication done using one per each risk factor (Stock Option needs to be replicated using one stock and one bond).
        - More strongly, it is the hedged portfolio that guarantees replication.
        - For a delta-hedged replicator, need $\frac{\partial V}{\partial S}$ short units of the underlier for one unit of the derivative.
        - Remember that, by definition, the S in Black-Scholes refers to $S(0)$ - today's value for S, so all prices, deltas, and hedge-ratios are for today.
        - Finally, remember that the actual terminal pay-off in itself could simply become a boundary condition. At exercise this pay-off can be non-differentiable (even discontinuous), but differentiable earlier.
2. Cause for the drift Elimination in the lack Scholes Portfolio: Anything that eliminates the explicit $\Delta \mathrm{W}$ coefficient (i.e., makes the portfolio non-stochastic) also eliminates the explicit $\Delta \mathrm{t}$ coefficient. Of course in the case of Brownian motion, the $(\Delta \mathrm{W})^2$ term contributes to the time dependence.
3. Pricing/Payoff Replication Portfolio:

$$\Pi = V - nS - mB$$

implies that

$$\Delta\Pi = \Delta V - n\Delta S - m\Delta B$$

with n and m fixed. Note that the instantaneous non-stochastic dynamics constraint results in a value for n and m. This is called the self-financing portfolio, in that

$$\Delta\Pi \to 0$$

as it is hedged across both S and B.

4. Pricing vs. Hedging Portfolio: Notice that the pricing/replication portfolio is instantaneous, in that it is valid on for the specific n

$$n = -\frac{\partial V}{\partial t}$$

Hedging is done using individual securities, therefore does not constitute a portfolio in that sense; further it is CERTAINLY not self-financing it the portfolio is not re-balanced.

5. Pricing as a Hedging Portfolio: Remember that the stochastic factor hedging here is only strictly instantaneous. Therefore, an incremental time instant later this same portfolio will not be hedged, that is, it will not be stochastically invariant, and will therefore need to be re-composed.
6. PRICE is only based off of terminal payoff replication and instantaneous stochasticity elimination: Other approaches (e.g., pre-Black Scholes strategies) may say that the price of a derivative is sum total of all the cash-flows that form a part of the derivative product life-cycle valued individually, and in a non-risk neutral manner, i.e., derivative product cash flows, hedging cash-flows, collateralization cash-flows, funding cash-flows, etc., i.e., cash-flows associated with a given strategy/set of strategies in the future through to maturity. Not so in the case of Black-Scholes, where only instantaneous risk-neutrality and terminal payoff replication for a given cash-flow (stochastic/deterministic) are considered.
7. "Rates" in the Black Scholes Portfolio: This corresponds to:

- The re-investment/investment returns rate if the Portfolio Value is positive.
- The funding rate if the Portfolio Value is negative.
- The funding Rate if the activity is for hedging and/or futures replication.
- The collateral rate, if the rate refers to the cash flow associated with collateralization.

8. Interpreting the Replicating Portfolio from the BS Call Formula:

$$V(S, K, T) = S\Phi(d_1) - KB\Phi(d_2)$$

where

$$B = e^{-r\tau}$$

$$d_1 = \frac{\log\frac{S}{K} + \tau\left(r + \frac{1}{2}\sigma^2\right)}{\sigma\sqrt{\tau}}$$

$$d_2 = d_1 - \sqrt{\sigma}$$

and

$$\tau = T - t$$

From this we can say that:

- $\Phi(d_1)$ is the number of replicating stocks (therefore the hedge ratio)
- $\Phi(d_2)$ is the probability of the call expiring in the money (also the bond numeraire units scaled by the strike).

9. Put-Call Parity:

$$C(K) = S\Phi(d_1) - Ke^{-\int_t^T r(t)dt}\Phi(d_2)$$

Buying a call while simultaneously selling a put with the same strike is equivalent to buying a stock and simultaneously borrowing $Ke^{-\int_t^T r(t)dt}$. Thus

$$C - P = S - Ke^{-\int_t^T r(t)dt}$$

thereby resulting in

$$P(K) = Ke^{-\int_t^T r(t)dt}\Phi(-d_2) - S\Phi(-d_1)$$

Capital Asset Pricing Model

1. Definition: The Capital Asset Pricing Model (CAPM) stipulates that the expected return of the security i in excess of the risk-free rate is

$$\mathbb{E}[r_i] - r = \beta_i\{\mathbb{E}[r_M] - r\}$$

where r_i is the return on the asset, r_M is the return on the market, and

$$\beta_i = \frac{Covariance\langle r_i, r_M\rangle}{Variance\langle r_M\rangle}$$

is the security's β to the market.

2. CAPM for the Assets: In a time increment Δt, the expected asset return is

$$\mathbb{E}[r_S\Delta t] = \mathbb{E}[\Delta S]$$

where S follows

$$\Delta S = rS\Delta t + \sigma S\Delta W$$

Using the CAPM setup, we see that

$$\mathbb{E}\left[\frac{\Delta S}{S}\right] = r\Delta t + \beta_S\{\mathbb{E}[r_M] - r\}\Delta t$$

for the asset. Likewise, for the derivative

$$\mathbb{E}\left[\frac{\Delta V}{V}\right] = r\Delta t + \beta_V\{\mathbb{E}[r_M] - r\}\Delta t$$

- Starting from

$$\Delta V = \frac{\partial V}{\partial t}\Delta t + \frac{\partial V}{\partial S}\Delta S + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(\Delta S)^2$$

substituting for $(\Delta S)^2$ and dividing throughout by V, we get

$$\frac{\Delta V}{V} = \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right]\Delta t + \frac{\partial V}{\partial S}\frac{\Delta S}{S}\frac{S}{V}$$

Applying the CAPM rule, we get

$$r_V \Delta t = \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right]\Delta t + \frac{\partial V}{\partial S}\frac{S}{V} r_S \Delta t$$

Taking covariance on both sides, and dropping the $\left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right]\Delta t$ term (since it is non-stochastic), we get

$$Covariance\langle r_V, r_M\rangle = \left(\frac{\partial V}{\partial S}\frac{S}{V}\right) Covariance\langle r_S, r_M\rangle$$

which implies

$$\beta_V = \left(\frac{\partial V}{\partial S}\frac{S}{V}\right)\beta_S$$

(Rouah (2010a)).

- Using the β_V in terms of β_S: From the expression for $\mathbb{E}\left[\frac{\Delta V}{V}\right]$, multiplying by V we get

$$\mathbb{E}[\Delta V] = rV\Delta t + \beta_V V\{\mathbb{E}[r_M] - r\}\Delta t = rV\Delta t + \frac{\partial V}{\partial S}\beta_S S\{\mathbb{E}[r_M] - r\}\Delta t$$

- Comparing this with

$$\mathbb{E}[\Delta V] = \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right]\Delta t + \mathbb{E}\left[\frac{\partial V}{\partial S}\Delta S\right]$$

using the CAPM expression for ΔS in terms of β_S, we get the Black Scholes equation

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} = rV$$

3. Black Scholes PDE from the Binomial Model: Cox, Ross, and Rubinstein (1979) derive the Black-Scholes PDE as a limit of the binomial tree model in the limit of the discretized time interval evolution.

Multi-numeraire Formulation

1. Setup: Say that the Brownian dynamics of evolution of n latent states are

$$\frac{\Delta A_i}{A_i} = \mu_i\Delta t + \sigma_i\Delta W_i$$

$$\Delta W_i \Delta W_i = \rho_{ij} \Delta t$$

for $i, j = 1, \dots, n$. The evolution for $\prod_{i=1}^{n} A_i$ is guided by

$$\frac{\Delta[\prod_{i=1}^{n} A_i]}{[\prod_{i=1}^{n} A_i]} = \left[\sum_{i=1}^{n} \mu_i(t) + \sum_{i=1}^{n} \sum_{j>i}^{n} \rho_{ij}(t)\sigma_i(t)\sigma_j(t)\right] \Delta t$$

From this it is easy to derive the joint numeraire

$$\prod_{i=1}^{n} A_i = e^{\int_{t_0}^{t} \left\{\sum_{i=1}^{n} \mu_i(s) + \sum_{i=1}^{n} \sum_{j>i}^{n} \rho_{ij}(s)\sigma_i(s)\sigma_j(s)\right\} ds}$$

References

- Cox, J. C., S. A. Ross, and M. Rubinstein (1979): Option Pricing: A Simplified Approach *Journal of Financial Economics* **7** 229-263.

Log-Normal Black Scholes Greeks

First Order Greeks

1. Notation: In all the treatments below we use ϕ and Φ to represent the point-wise Gaussian distribution density and cumulative Gaussian respectively. Further we assume that there exists no dividends discrete/continuous.
2. Vega:

$$\nu = \frac{\partial V}{\partial \sigma}$$

Sometimes the symbol κ is used instead. Products such as straddles are extremely sensitive to changes in volatility.

$$\nu_{Call} = \nu_{Put} = S\phi(d_1)\sqrt{\tau} = Ke^{-r\tau}\phi(d_2)\sqrt{\tau}$$

3. Theta:

$$\theta = -\frac{\partial V}{\partial \tau}$$

is the sensitivity with respect to time (time decay). Except for deep out-of-the money puts, most options have negative θ. θ is composed of the intrinsic (which is always positive) and the time value (which is negative).

$$\theta_{Call} = -\frac{S\sigma\phi(d_1)}{2\sqrt{\tau}} - rKe^{-r\tau}\Phi(d_2)$$

$$\theta_{Put} = -\frac{S\sigma\phi(d_1)}{2\sqrt{\tau}} - rKe^{-r\tau}\Phi(-d_2)$$

4. Rho:

$$\rho = \frac{\partial V}{\partial r}$$

is the sensitivity with respect to the interest rate.

$$\rho_{Call} = Ke^{-r\tau}\Phi(d_2)$$

$$\rho_{Put} = -Ke^{-r\tau}\Phi(-d_2)$$

Second Order Greeks

1. Gamma:

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 \mathrm{V}}{\partial S^2}$$

is always positive for long options.

$$\Gamma_{Call} = \Gamma_{Put} = \frac{\phi(d_1)}{S\sigma\sqrt{\tau}}$$

2. Vanna:

$$\mathrm{Vanna} = \frac{\partial \Delta}{\partial \sigma} = \frac{\partial \mathrm{V}}{\partial S} = \frac{\partial^2 \mathrm{V}}{\partial \sigma \partial S}$$

where the equality strictly holds if the partial second derivative exists.

$$\text{Vanna}_{Call} = \text{Vanna}_{Put} = -\frac{\phi(d_1)d_2}{\sigma} = \frac{\nu}{S}\left[1 - \frac{d_1}{\sigma\sqrt{\tau}}\right]$$

3. Vomma and Charm:
   a. Vomma/Volga/Vega Convexity is the 2nd order sensitivity to the volatility.

$$\text{Vomma} = \frac{\partial \text{v}}{\partial \sigma} = \frac{\partial^2 \text{v}}{\partial S^2} = S\phi(d_1)\sqrt{\tau}\frac{d_1 d_2}{\sigma} = \text{v}\frac{d_1 d_2}{\sigma}$$

   b. Charm is the rate of decay of delta, i.e., delta decay.

$$\text{Charm} = -\frac{\partial \Delta}{\partial \tau} = \frac{\partial \theta}{\partial S} = -\frac{\partial^2 \text{V}}{\partial S \partial \tau}$$

$$Charm_{Call} = Charm_{Put} = -\phi(d_1)\frac{2r\tau - \sigma d_2\sqrt{\tau}}{2\tau\sigma\sqrt{\tau}}$$

4. Veta and Vera:
   a. Veta => Vega Decay, given from

$$Vega_{Decay} = \frac{\partial \text{v}}{\partial \tau} = \frac{\partial^2 \text{V}}{\partial \sigma \partial \tau} = S\phi(d_1)\sqrt{\tau}\left[\frac{rd_1}{\sigma\sqrt{\tau}} - \frac{1 + d_1 d_2}{2\tau}\right]$$

   b. Vera is also called rhova, and is the rate of change of ρ with respect to volatility.

$$\text{Vera} = \frac{\partial \rho}{\partial \sigma} = \frac{\partial^2 \text{V}}{\partial \sigma \partial r}$$

Third Order Greeks

1. <u>Color</u>: Measures the time decay of gamma.

$$Color = \frac{\partial \Gamma}{\partial \tau} = \frac{\partial^3 \mathrm{V}}{\partial^2 S \partial \tau} = -\frac{\phi(d_1)}{2S\tau\sigma\sqrt{\tau}}\left[1 + d_1 \frac{2r\tau - \sigma d_2\sqrt{\tau}}{\sigma\sqrt{\tau}}\right]$$

2. <u>Speed</u>: Speed is the delta of the gamma.

$$Speed = \frac{\partial^3 \mathrm{V}}{\partial S^3} = \frac{\partial^2 \Delta}{\partial S^2} = \frac{\partial \Gamma}{\partial S} = -\frac{\phi(d_1)}{S^2\sigma\sqrt{\tau}}\left[\frac{d_1}{\sigma\sqrt{\tau}} + 1\right] = -\frac{\Gamma}{S}\left[\frac{d_1}{\sigma\sqrt{\tau}} + 1\right]$$

3. <u>Ultima</u>:

$$Ultima = \frac{\partial^3 \mathrm{V}}{\partial \sigma^3} = -\frac{\nu}{\sigma^2}\left[d_1 d_2(1 - d_1 d_2) + {d_1}^2 + {d_2}^2\right]$$

4. <u>Zomma</u>:

$$Zomma = \frac{\partial \Gamma}{\partial \sigma} = \frac{\partial \mathrm{Vanna}}{\partial S} = \frac{\partial^3 \mathrm{V}}{\partial^2 S \partial \sigma} = \phi(d_1)\frac{d_1 d_2 - 1}{\sigma^2 S\sqrt{\tau}} = \Gamma\left(\frac{d_1 d_2 - 1}{\sigma}\right)$$

Black-Scholes Extensions

Time-Dependent Black Scholes

1. Pricing Expression: If r and σ in the Black-Scholes expression are strictly time-dependent (i.e., not asset value dependent), the adjustment to the Black-Scholes expression is trivial:

$$d_1 = \frac{\log\frac{S}{K} + \int_0^T r(t)dt + \frac{1}{2}\int_0^T \sigma^2(t)dt}{\sqrt{\int_0^T \sigma^2(t)dt}}$$

$$d_2 = \frac{\log\frac{S}{K} + \int_0^T r(t)dt - \frac{1}{2}\int_0^T \sigma^2(t)dt}{\sqrt{\int_0^T \sigma^2(t)dt}}$$

   Effectively the original Black-Scholes equation may be used by applying

$$\overline{r} = \int_0^T r(t)dt$$

   and

$$\overline{\sigma} = \sqrt{\int_0^T \sigma^2(t)dt}$$

   $\overline{\sigma}$ is referred to as the root-mean squared volatility (Rebonato (2004)).

2. Time-Dependent Volatility from Implied Volatility: From

$$\sigma_{IMP}(T) = \sqrt{\int_0^T \sigma^2(t)dt}$$

you get

$$\int_0^T \sigma^2(t)dt = T\sigma_{IMP}{}^2(T)$$

which implies

$$\sigma^2(T) = \frac{\partial[T\sigma_{IMP}{}^2(T)]}{\partial T} = \sigma_{IMP}{}^2(T) + 2T\sigma_{IMP}(T)\frac{\partial\sigma_{IMP}(T)}{\partial T}$$

Thus if $\sigma_{IMP}(T)$ is known at selected time grid nodes, you may use a spline form to fit $\sigma_{IMP}(T)$ vs. T, and extract the time-dependent volatility $\sigma(T)$ from there.

3. Shortcoming of time-dependent volatility: Time-dependent Black Scholes formulation can be calibrated to the ATM implied volatility term structure $\sigma_{IMP}(T)$, but cannot reproduce as-is the complete implied volatility surface $\sigma_{IMP}(T, K)$.

Local Volatility Models

1. Local Volatility Model Definition: Here, under the risk-neutral measure

$$\Delta S = r(t)S\Delta t + \sigma(S, t)\Delta W_S$$

Thus, while $r(t)$ can be deterministic, $\sigma(S, t)$ is stochastic. Implied volatilities may be fitted using local Volatility models.

2. Risk neutral Distribution of the Asset Price at T: As shown in popular, the call-price may be computed from the risk-neutral probability density function $\sigma(S_T, T)$ as

$$C(K,T) = e^{-rT} \int_K^{\infty} (S_T - K)\varphi(S_T, T) dS_T$$

results in

$$\frac{\partial C(K,T)}{\partial K} = -e^{-rT} \int_K^{\infty} \varphi(S_T, T) dS_T$$

and

$$\frac{\partial^2 C(K,T)}{\partial K^2} = e^{-rT}\varphi(S_T, T)$$

   1. Implication of the T derivation => This shows that the risk-neutral distribution of the asset price at T can be entirely determined from the market quotes of European options.

3. Extracting Local Volatility from Market Prices: If you have an option price surface, you typically have $\frac{\partial C(K,T)}{\partial T}$, $\frac{\partial C(K,T)}{\partial K}$, and $\frac{\partial^2 C(K,T)}{\partial K^2}$. The analysis above demonstrated how to extract $\varphi(S_T, T)$ from the option price surface, now we demonstrate how to extract $\sigma(K, T)$ from the same.

$$\frac{\partial C(K,T)}{\partial T} = -rC(K,T) + e^{-rT} \int_K^{\infty} (S_T - K) \frac{\partial \varphi(S_T, T)}{\partial T} dS_T$$

$$\frac{\partial C(K,T)}{\partial T} = -rC(K,T) + e^{-rT} \int_K^{\infty} (S_T - K) \left\{ \frac{1}{2} \frac{\partial^2}{\partial {S_T}^2} [{S_T}^2 \sigma^2 \varphi] - \frac{1}{2} \frac{\partial}{\partial S_T} [r S_T \varphi] \right\} dS_T$$

where we have made use of the Fokker-Planck relation in the second step. Notice the dependence on $\frac{\partial}{\partial S_T}$ and $\frac{\partial^2}{\partial {S_T}^2}$in the integral above, so integrating the above by parts twice yields

$$\frac{\partial C}{\partial T} = -rC + \frac{1}{2} e^{-rT} \sigma^2 K^2 \varphi + re^{-rT} \int_K^\infty S_T \varphi dS_T$$

2. Using the expressions for $\frac{\partial C(K,T)}{\partial K}$ and $\frac{\partial^2 C(K,T)}{\partial K^2}$ from the previous point, you can eliminate all dependence on φ to get

$$\frac{\partial C}{\partial T} = -rK \frac{\partial C}{\partial K} + \frac{1}{2} \sigma^2 K^2 \frac{\partial^2 C}{\partial K^2}$$

which results in

$$\sigma_{LOCAL} = \sqrt{\frac{\frac{\partial C}{\partial T} + rK \frac{\partial C}{\partial K}}{\frac{1}{2} K^2 \frac{\partial^2 C}{\partial K^2}}}$$

Thus, once a 2D spline surface representing

$$C \equiv C(K, T)$$

is constructed, the corresponding σ_{LOCAL} is readily determined.

Black Normal Model Specification and Dynamics

1. <u>Setup</u>: Here

$$\Delta F(t;\ T,S) = \sigma_N \Delta W$$

This leads to

$$F(T;\ T,S) = L(T,S) \approx \mathcal{N}(F(t;\ T,S), {\sigma_N}^2 T)$$

and the corresponding probability density function is

$$\varphi(F) = \frac{1}{\sqrt{2\pi {\sigma_N}^2 T}} e^{-\frac{(F-F_0)^2}{2\pi {\sigma_N}^2 T}}$$

2. <u>Call/Put Prices</u>:

$$C(S,K) = P(0,S)\mathbb{E}_0[(F-K)^+] = P(0,S)\sigma_N\sqrt{T}\left[\frac{1}{\sqrt{2\pi}}e^{-\frac{d^2}{2}} + d\Phi(d)\right]$$

where

$$d = \frac{F_0 - K}{\sigma_N\sqrt{T}}$$

and

$$\Phi(d) = \int_K^{+\infty} \varphi(F)dF$$

Setting

$$\phi(x) = \mathcal{N}'(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

we get

$$C(S,K) = P(0,S)\sigma_N\sqrt{T}[\phi(d) + d\mathcal{N}(d)]$$

and

$$P(S,K) = P(0,S)\sigma_N\sqrt{T}[\phi(d) - d\mathcal{N}(-d)]$$

Options on Forward

Theoretical Framework and Background

1. Forwards as a Martingale: Forward is defined as that entity whose price/value at a forward time T is under consideration at a time $t < T$. The forward is treated as an entity whose expectation is time invariant under its own measure (as opposed to "spot" whose expectation grows typically with time even under its own native measure). This makes the forward a martingale. Projection of the forward from the spot occurs via the risk-neutral discounting.

- The forward is a martingale simply because the payoff has to be a martingale in its own measure. Of course the payoff may be correlated with the discounting measure, in which case the correlation adjustment needs to be applied.

2. Replication of the forward from the spot: This needs a SPOT and a funding account, thus bringing the funding measure in. Of source the funding measure may be correlated with the spot measure, in which case an adjustment needs to be applied again along the path (not on the terminal payoff distribution).

Valuation

1. Black76 Model: This is the most straightforward extension to what we saw before. Given

$$\Delta S = r(t)S\Delta t + \sigma_S(S,t)\Delta W_S$$

define

$$F(t,T) = S(t)e^{-\int_t^T r(t)dt}$$

It is straightforward to see that $F(t,T)$ is a martingale with the same volatility as that of $S(t)$. Thus, a call option on $F(t,T)$ with the strike K becomes

$$C(K) = e^{\int_t^T r(t)dt}\{F(t,T)\Phi(d_1) - K\Phi(d_2)\}$$

2. Forward Evolution: The forward evolves according to

$$\frac{\Delta F}{F} = \mu_F \Delta t + \sigma_F \Delta W_F$$

Call it the Q^f measure – all you know are μ_F and σ_F. As noticed earlier we assume

$$\mu_F = 0$$

so that martingale property of the forward is maintained.

3. Numeraire Evolution: The numeraire evolves according to

$$\frac{\Delta B}{B} = \mu_B \Delta t + \sigma_B \Delta W_B$$

- call it the Q^b measure. Q^b and Q^f are correlated via

$$\langle \Delta W_B \Delta W_F \rangle = \rho_{BF} \Delta t$$

Remember that if B refers to the discrete bond numeraire, there will be several ΔW_i's.

4. What do we seek: We seek $B_0 \int d\left(\frac{F}{B}\right)$, i.e. the PV of the Arrow security that pays F units replicated using B, where B and F follows the evolution dynamics above. In essence, we seek the evolution of $\Delta\left(\frac{F}{B}\right)$.

5. $\Delta\left(\frac{F}{B}\right)$ Evolution:

$$\Delta\left(\frac{F}{B}\right) = \left(\frac{\mathrm{B}\Delta F - \mathrm{F}\Delta B - \Delta\mathrm{B}\Delta F}{B^2}\right)$$

implies that

$$\frac{\Delta\left(\frac{F}{B}\right)}{\left(\frac{F}{B}\right)} = \frac{\Delta F}{F} - \frac{\Delta B}{B} - \frac{\Delta B}{B}\frac{\Delta F}{F}$$

which results in

$$\frac{\Delta\left(\frac{F}{B}\right)}{\left(\frac{F}{B}\right)} = (\mu_F - \mu_B - \rho_{BF}\sigma_B\sigma_F)\Delta t + \sigma_F\Delta W_F - \sigma_B\Delta W_B$$

On integrating the above, you get

$$\Delta\left(\frac{F}{B}\right) = \frac{F_0}{B_0}\Phi\left[(\mu_F - \mu_B)\Delta t - Variance\left(\frac{\Delta B}{B}, \frac{\Delta F}{F}\right) \cdot Covariance\left(\frac{\Delta B}{B}, \frac{\Delta F}{F}\right)\right]$$

6. Change of Measure: Denoted as $\frac{\partial Q^f}{\partial Q^b}$, in cases above ends up picking on the drift term $-Covariance\left(\frac{\Delta B}{B}, \frac{\Delta F}{F}\right)$. Thus,

$$\mathbb{E}^{Q^b}\left[\frac{\partial Q^f}{\partial Q^b}\right] = e^{-\int(\mu_F - \mu_B)\mathrm{d}t - Covariance\left(\frac{\mathrm{d}B}{B}, \frac{\mathrm{d}F}{F}\right)}$$

Notice the terminology/language/notation.

Stochastic Volatility Models: The Heston Model

Model Specification and Dynamics

1. Stochastic Volatility Model: Examples of stochastic volatility models are the Heston (1993) and SABR models. These are NOT local volatility models in the Dupire sense, as the volatilities here do not have an explicit dependence on the asset levels, strike, or the moneyness. Therefore they are widely applicable in markets where the common quotes are at-the money, e.g., in caps/floors/IR Options markets.
2. Stochastic Volatility Dynamics: The Heston model (Heston 1993)) assumes that the underlying S follows a Brownian stochastic process with variance $\upsilon(t)$ that follows a Cox-Ingersoll-Ross stochastic volatility dynamics:

$$\Delta S = rS\Delta t + \sqrt{\upsilon} S \Delta W_S$$

$$\Delta \upsilon = \kappa(\theta - \upsilon) + \sqrt{\upsilon}\sigma\Delta W_\upsilon$$

$$\Delta W_\upsilon \Delta W_S = \rho_{\upsilon S}\Delta t$$

3. The Heston Portfolio: The Heston portfolio consists of the first option V, the second option U, and the underlier S (Gatheral (2006)):

$$\Pi = V + \delta S + \phi U$$

From the self-financing criterion, we have

$$\Delta\Pi = \Delta V + \delta\Delta S + \phi\Delta U$$

4. Portfolio Dynamics: Applying Ito's lemma to ΔV, ΔU, and ΔS, we get

$$\Delta V = \frac{\partial V}{\partial t}\Delta t + \frac{\partial V}{\partial S}\Delta S + \frac{\partial V}{\partial \upsilon}\Delta \upsilon + \frac{1}{2}\upsilon S^2\frac{\partial^2 V}{\partial S^2}\Delta t + \frac{1}{2}\upsilon\sigma^2\frac{\partial^2 V}{\partial \upsilon^2}\Delta t + \rho\sigma\upsilon S\frac{\partial^2 V}{\partial \upsilon \partial S}\Delta t$$

$$\Delta U = \frac{\partial U}{\partial t}\Delta t + \frac{\partial U}{\partial S}\Delta S + \frac{\partial U}{\partial \upsilon}\Delta \upsilon + \frac{1}{2}\upsilon S^2\frac{\partial^2 U}{\partial S^2}\Delta t + \frac{1}{2}\upsilon\sigma^2\frac{\partial^2 U}{\partial \upsilon^2}\Delta t + \rho\sigma\upsilon S\frac{\partial^2 U}{\partial \upsilon \partial S}\Delta t$$

and for the portfolio increment

$$\begin{aligned}
& \left\{\frac{\partial V}{\partial t} + \frac{1}{2}\upsilon S^2\frac{\partial^2 V}{\partial S^2} + \rho\sigma\upsilon S\frac{\partial^2 V}{\partial \upsilon \partial S} + \frac{1}{2}\upsilon\sigma^2\frac{\partial^2 V}{\partial \upsilon^2}\right\}\Delta t \\
\Delta\Pi = \phi & \left\{\frac{\partial U}{\partial t} + \frac{1}{2}\upsilon S^2\frac{\partial^2 U}{\partial S^2} + \rho\sigma\upsilon S\frac{\partial^2 U}{\partial \upsilon \partial S} + \frac{1}{2}\upsilon\sigma^2\frac{\partial^2 U}{\partial \upsilon^2}\right\}\Delta t \\
& \left\{\frac{\partial V}{\partial S} + \phi\frac{\partial U}{\partial S} + \delta\right\}\Delta S + \left\{\frac{\partial V}{\partial \upsilon} + \phi\frac{\partial U}{\partial \upsilon} + \delta\right\}\Delta \upsilon
\end{aligned}$$

5. Risk-less Portfolio: To eliminate the dependence on ΔS and $\Delta \upsilon$, we require

$$\phi = -\frac{\frac{\partial V}{\partial \upsilon}}{\frac{\partial U}{\partial \upsilon}}$$

and

$$\delta = -\frac{\partial V}{\partial S} - \phi\frac{\partial U}{\partial S}$$

As is typical in these treatments, the risk-less portfolio should then evolve according to

$$\Delta\Pi = r\Pi\Delta \mathrm{t}$$

6. Risk-less Portfolio Dynamics: Set

$$\Delta\Pi = (A + \phi B)\Delta \mathrm{t}$$

where

$$A = \frac{\partial V}{\partial t} + \frac{1}{2}\upsilon S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma\upsilon S \frac{\partial^2 V}{\partial \upsilon \partial S} + \frac{1}{2}\upsilon\sigma^2 \frac{\partial^2 V}{\partial \upsilon^2}$$

and

$$B = \frac{\partial U}{\partial t} + \frac{1}{2}\upsilon S^2 \frac{\partial^2 U}{\partial S^2} + \rho\sigma\upsilon S \frac{\partial^2 U}{\partial \upsilon \partial S} + \frac{1}{2}\upsilon\sigma^2 \frac{\partial^2 U}{\partial \upsilon^2}$$

Putting this back into $\Delta\Pi = (A + \phi B)\Delta \mathrm{t}$ we get

$$\mathrm{r}\Pi\Delta \mathrm{t} = r(V + \delta S + \phi U)\Delta \mathrm{t}$$

7. Dynamics of the Market Price of Volatility Risk: Plug in for ϕ and δ from the portfolio hedging relations above, and re-arrange them to produce

$$\frac{A - rV + rS\frac{\partial V}{\partial S}}{\frac{\partial V}{\partial \upsilon}} = \frac{B - rU + rS\frac{\partial U}{\partial S}}{\frac{\partial U}{\partial \upsilon}} = -\kappa(\theta - \upsilon) + \lambda(S, \upsilon, t)$$

where $\lambda(S, \upsilon, t)$ is the market price of the volatility risk. Heston (1993) sets

$$\lambda(S, \upsilon, t) = \lambda\upsilon$$

i.e., the market price of volatility risk to be linear in the volatility.

8. Sources of Market Price of Risk in the Heston Model: There are two sources of market risk – one for ΔS and one for $\Delta\upsilon$. Instantaneous hedge across ΔS produces a drift for ΔS of r - the risk-free rate – essentially

$$\mathrm{r} = \lambda_S - \mu_S$$

Likewise, for $\Delta \upsilon$ it is

$$\frac{\lambda(S, \upsilon, t) - \kappa(\theta - \upsilon)}{\upsilon}$$

which is what is seen above.

- In the final option price PDE, the risk-adjusted drifts always appear as the coefficients of the corresponding option price <-> stochastic entity derivatives. Thus, $\frac{\partial U}{\partial S}$'s coefficient is rS; and $\frac{\partial U}{\partial \upsilon}$'s coefficient is $\lambda(S, \upsilon, t) - \kappa(\theta - \upsilon)$.
- One consequence of the above formulation is the need to calibrate multiple groups of parameters – the real-world ones (the mean reversion intensity κ and the steady state mean reverted level θ), and the market price of risk $\lambda(S, \upsilon, t)$.

9. Heston Option Price PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\upsilon S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma\upsilon S \frac{\partial^2 V}{\partial \upsilon \partial S} + \frac{1}{2}\upsilon\sigma^2 \frac{\partial^2 V}{\partial \upsilon^2} - rV + rS\frac{\partial V}{\partial S} + [\kappa(\theta - \upsilon) - \lambda(S, \upsilon, t)]\frac{\partial V}{\partial \upsilon} = 0$$

Expressing the above in terms of

$$x = \log S$$

we get

$$\frac{\partial V}{\partial t} + \frac{1}{2}\upsilon \frac{\partial^2 V}{\partial x^2} + \left[r - \frac{1}{2}\upsilon\right]\frac{\partial V}{\partial x} + \rho\sigma\upsilon \frac{\partial^2 V}{\partial \upsilon \partial x} + \frac{1}{2}\upsilon\sigma^2 \frac{\partial^2 V}{\partial \upsilon^2} - rV + [\kappa(\theta - \upsilon) - \lambda(S, \upsilon, t)]\frac{\partial V}{\partial \upsilon} = 0$$

As noted above Heston (1993) set $\lambda(S, \upsilon, t) = \lambda\upsilon$, simplifying the above slightly.

- Drift Terms for ΔS and $\Delta \upsilon$ => As expected the drift terms for ΔS and $\Delta \upsilon$ do not show up explicitly, since we are dealing with instantaneously hedged portfolios. However, for the reasons seen above, Heston appears to have explicitly accommodated the elements of υ dynamics via the market price of risk approach, essentially making the formulation "non-local" in υ.

Price Estimation Through Characteristic Functions

1. Call Price re-cast of the Derivative Price above: The Call Price

$$C_T(K) = e^{-r\tau}\mathbb{E}[(S_T - K)^+] = e^x K P_1(x, \upsilon, \tau) - e^{-r\tau} K P_2(x, \upsilon, \tau)$$

where

$$x = \log S$$

and

$$\tau = T - t$$

Using

$$\frac{\partial V}{\partial t} = -\frac{\partial V}{\partial \tau}$$

the pricing equation above becomes

$$-\frac{\partial V}{\partial \tau} + \frac{1}{2}\upsilon\frac{\partial^2 V}{\partial x^2} + \left[r - \frac{1}{2}\upsilon\right]\frac{\partial V}{\partial x} + \rho\sigma\upsilon\frac{\partial^2 V}{\partial \upsilon \partial x} + \frac{1}{2}\upsilon\sigma^2\frac{\partial^2 V}{\partial \upsilon^2} - rV + [\kappa(\theta - \upsilon) - \lambda(S, \upsilon, \tau)]\frac{\partial V}{\partial \upsilon} = 0$$

2. Derivative Value in terms of $P_j(x, \upsilon, \tau)$: Substituting for $C_T(K)$ using $P_1(x, \upsilon, \tau)$ and $P_2(x, \upsilon, \tau)$, Heston (1993) derived

$$e^x f\left(P_1(x, \upsilon, \tau)\right) - e^{-r\tau} f\left(P_2(x, \upsilon, \tau)\right) = 0$$

where

$$f\left(P_j\right) = -\frac{\partial P_j}{\partial \tau} + \rho\sigma\upsilon \frac{\partial^2 P_j}{\partial \upsilon \partial x} + \frac{1}{2}\upsilon \frac{\partial^2 P_j}{\partial x^2} + \frac{1}{2}\upsilon\sigma^2 \frac{\partial^2 P_j}{\partial \upsilon^2} + \left[r + u_j\upsilon\right] \frac{\partial P_j}{\partial x} - rV + \left[a + b_j\upsilon\right] \frac{\partial P_j}{\partial \upsilon}$$

Here

$$u_1 = \frac{1}{2}$$

$$u_2 = -\frac{1}{2}$$

$$a = \kappa\theta$$

$$b_1 = \kappa + \lambda - \rho\sigma$$

and

$$b_2 = \kappa + \lambda$$

Given that P_1 and P_2 are state evolution probabilities, their evolution is governed by the Fokker-Planck equation

$$-\frac{\partial P_j}{\partial \tau} + \rho\sigma\upsilon \frac{\partial^2 P_j}{\partial \upsilon \partial x} + \frac{1}{2}\upsilon \frac{\partial^2 P_j}{\partial x^2} + \frac{1}{2}\upsilon\sigma^2 \frac{\partial^2 P_j}{\partial \upsilon^2} + \left[r + u_j\upsilon\right] \frac{\partial P_j}{\partial x} - rV + \left[a + b_j\upsilon\right] \frac{\partial P_j}{\partial \upsilon} = 0$$

This results in

$$f(P_1) = 0$$

and

$$f(P_2) = 0$$

so we seek a solution for them.

3. Functional Basis Form for $P_j(x, v, \tau, \alpha)$: Use the form

$$P_j(x, v, \tau, \alpha) = e^{C_j(\tau,\alpha)+vD_j(\tau,\alpha)+i\alpha x}$$

for

$$j = 1, 2$$

This choice is attractive for several reasons:

- The dependence of $P_j(x, v, \tau, \alpha)$ on x and α is exponentially partitioned.
- Given that $P_j(x, v, \tau, \alpha)$ shows up linearly on the pricing PDE, it gets factored out under such a representation, leaving behind only the coefficients.
- As will be seen below, this results in expressing the equation set as

$$pv + q = 0$$

across all v, so that

$$p = 0$$

and

$$q = 0$$

can be solved separately.

- Finally

$$p = 0$$

may be solved entirely for D_j alone, and

$$q = 0$$

may be solved for C_j using a trivial dependence on D_j.

4. Insight into the $e^{i\alpha x}$ choice: Remember that

$$x = \log S$$

and this translates into

$$e^{i\alpha x} = i\alpha S$$

Further by imposing

$$D(0, \alpha) = 0$$

and

$$C(0, \alpha) = 0$$

as boundary conditions, Heston (1993) ensures that the time T price is either directly proportional to S or 0, as you would want.

- In effect, this choice of basis function that involves α is simply a Fourier transform (plus function/field partition) over the asset "frequency" range α, so we need a final Inverse Fourier Transform step that reverts this.

5. Exponential Basis Expansion for $P_j(x, v, \tau)$: You get

$$\left[-\frac{\partial D_j}{\partial \tau} + i\rho\sigma\alpha D_j - \frac{1}{2}\alpha^2 + \frac{1}{2}\sigma^2 {D_j}^2 + iu_j\alpha + b_j D_j\right] v + \left[-\frac{\partial C_j}{\partial \tau} + ir\alpha + aD_j\right] = 0$$

This is, of course, of the form

$$pv + q = 0$$

above, thus

$$p = 0$$

and

$$q = 0$$

may be solved separately. This results in

$$-\frac{\partial D_j}{\partial \tau} + i\rho\sigma\alpha D_j - \frac{1}{2}\alpha^2 + \frac{1}{2}\sigma^2 {D_j}^2 + iu_j\alpha + b_j D_j = 0$$

and

$$-\frac{\partial C_j}{\partial \tau} + ir\alpha + aD_j = 0$$

The first is a second degree ODE depending exclusively on D_j (this is referred to as a Riccatti's equation); the second is a first order ODE on C_j that may be solved using a straightforward integration of D_j.

6. Solution for D_j: The Riccatti equation is of the form

$$\frac{\partial D_j}{\partial \tau} = M_j + N_j D_j + R_j {D_j}^2$$

Heston (1993) and Rouah (2010b) demonstrate that the solution that incorporates the boundary condition

$$D_j(0, \alpha) = 0$$

is

$$D_j = \frac{b_j - i\rho\sigma\alpha + d_j}{\sigma^2}\left[\frac{1 - e^{d_j \tau}}{1 - g_j e^{d_j \tau}}\right]$$

where

$$d_j = \sqrt{\left(i\rho\sigma\alpha - b_j\right)^2 - \sigma^2\left(2i\alpha u_j - \alpha^2\right)^2}$$

and

$$g_j = \frac{b_j - i\rho\sigma\alpha + d_j}{b_j - i\rho\sigma\alpha - d_j}$$

7. Solution for C_j:

$$C_j = \int_0^\tau [ir\alpha + aD_j]d\tau$$

Applying the boundary condition

$$C_j(0, \alpha) = 0$$

we get

$$C_j = ir\alpha\tau + \frac{a}{\sigma^2}\left[(b_j - i\rho\sigma\alpha + d_j)\tau - 2\log\left|\frac{1 - g_j e^{d_j \tau}}{1 - g_j}\right|\right]$$

8. Enhancement by Albrecher, Mayer, Schoutens, and Tistaert (2007): They propose the following tweaks to improve performance of the Heston model (a similar proposal was also made by Gatheral (2006)). Reset

$$g_j{}' = \frac{1}{g_j} = \frac{b_j - i\rho\sigma\alpha - d_j}{b_j - i\rho\sigma\alpha + d_j}$$

Apparently this modest formulation adjustment makes a significant impact on the numerical stability. This results in

$$C_j{}' = ir\alpha\tau + \frac{a}{\sigma^2}\left[(b_j - i\rho\sigma\alpha - d_j)\tau - 2\log\left|\frac{1 - g_j{}' e^{-d_j \tau}}{1 - g_j{}'}\right|\right]$$

and

$$D_j{}' = \frac{b_j - i\rho\sigma\alpha - d_j}{\sigma^2}\left[\frac{1 - e^{-d_j \tau}}{1 - g_j{}' e^{-d_j \tau}}\right]$$

All other formulation components remain the same.

9. Solution for $P_j(x, v, \tau)$:

$$P_j(x, v, \tau) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Real \left[\frac{e^{-i\alpha x} P_j(x, v, \tau, \alpha)}{i\alpha} \right] d\alpha$$

Fourier Inversion in Characteristic Function

1. Motivation for the Fourier Inversion Form in Heston (1993): The formulation uses the result that the Fourier transform of the Heaviside function is composed simply of a Dirac component and a hyperbolic component:

$$\int e^{-2\pi i\alpha x} h(x) dx = \frac{1}{2} \delta(x) + \frac{1}{2\pi i\alpha}$$

2. Problem #1 Multi Valued Complex Log: The Fourier transform of the multi-valued complex logarithm may end up getting switched away from the principal branch. One suggestion to remedy this is to carefully keep track of the branch (Schobel and Zhu (1999), Mikhailov and Nogel (2003), Sepp (2004), Lee (2005)) along the discretized path integral of

$$f_j(x, v, \tau, \alpha) = \frac{e^{-i\pi\alpha} P_j(x, v, \tau, \alpha)}{i\alpha}$$

as α goes from 0 to ∞.

3. Problem #2 Branch Switching of the Complex Power Function: Related to the above, but the distinction is highlighted in Kahl and Jackel (2009). By setting the complex variable in the Fourier transform

$$z = re^{i\phi}$$

we get

$$z\varpi = re^{i\varpi\phi}$$

As the phase of z changes from $-\pi$ to $+\pi$, the phase of z^{ϖ} changes from $-\varpi\pi$ to $+\varpi\pi$. If $\varpi \in Z$, this is a clearly not a problem, but if $\varpi \notin Z$, the branch switching can occur.

4. Phase Correction: Kahl and Jackel (2009) narrow down on the $\frac{1-g_j e^{d_j \tau}}{1-g_j}$ term, since that is the logarithm operand, and apply the phase rotation corrections separately to the numerator and the denominator for each α evaluation – this eliminates the need to track the phase (and its jumps/discontinuities) across all subsequent α evaluations.
   - The algorithm => Phase adjustment for $g_j - 1$ is $2\pi n$, where

     $$n = int\left[\frac{Phase(g_j) + \pi}{2\pi}\right]$$

     This states that if

     $$Phase(g_j) > 0$$

     then

     $$n = 1$$

     otherwise

     $$n = 0$$

     Thus, the adjustment amount is entirely determined by $Phase(g_j)$.
   - The full correction => Phase adjustment for $g_j e^{d_j \tau} - 1$ is $2\pi m$. The full correction, therefore, becomes $\frac{1-g_j e^{d_j \tau}}{1-g_j}$, which gives

$$\log\left|\frac{1-g_j e^{d_j\tau}}{1-g_j}\right| = \log\left|\frac{\left\|1-g_j e^{d_j\tau}\right\|}{\left\|1-g_j\right\|}\right| + Phase\left(1-g_j e^{d_j\tau}\right) - Phase\left(1-g_j\right) + 2\pi(n-m)$$

5. Applicability of the Phase Corrections: Although discussed in a specific context here, these techniques may be used for all inverse Fourier transforms across all fields – in particular among the option pricing models that use the Fourier inversion integral approach (Carr and Madan (1999), Carr (2003)) based on the log-characteristic function.
6. Fourier Integration Quadrature Schemes: Adaptive quadrature schemes are required for Fourier Inversion Integrands. Since the integration limit for α goes from 0 to ∞, we seek to transform the limits to 0 to 1, which works well for adaptive schemes such as the Gauss-Labatto algorithm (Gander and Gautschi (2000)). Thus

$$\int_0^\infty f_j(x,\upsilon,\tau,\alpha)d\alpha = \int_0^1 \frac{f_j\left(x,\upsilon,\tau,\alpha(\xi)\right)}{\xi C_\infty} d\xi$$

where

$$\alpha(\xi) = -\frac{\log\xi}{C_\infty}$$

resulting in

$$\xi = e^{-\alpha(\xi)C_\infty}$$

- Determination of C_∞ => C_∞ above is estimated in the limit of $\alpha \to \infty$ for all the coefficients in the Heston formulation, giving

$$C_\infty = \frac{\sqrt{1-\rho^2}}{\sigma}(v_0 + \kappa\theta\lambda)$$

- Kahl and Jackel (2009) work out the limiting expressions for $f_1(x, v, \tau, \alpha)$ and $f_2(x, v, \tau, \alpha)$ as $\alpha \to 0$ and $\alpha \to \infty$, but unfortunately do not explicitly spell out the appropriate lower/upper bounds for α (and therefore ξ).

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Dynamical Latent State Calibration

Fokker-Planck Equations

1. Introduction: Consider a random variable x that follows

$$\Delta x = \mu(x, t)\Delta t + \sigma(x, t)\Delta W$$

with

$$x(t_0) = x_0$$

The transition probability of reaching x at $t > t_0$ is $f(x, t : x_0, t_0)$, and is given using the Fokker-Planck version of the Kolmogorov equation (see, e.g., Wang (2010a)) as

$$\frac{\partial}{\partial t} f(x, t) = \frac{1}{2}\frac{\partial^2}{\partial x^2}[\sigma^2(x, t) f(x, t)] - \frac{1}{2}\frac{\partial}{\partial x}[\sigma(x, t) f(x, t)]$$

2. Extension to Options: There is explicit requirement that x follow Brownian motion. For options, if $\varphi(S_T, T)$ is the probability of reaching S_T at T given

$$S(t_0) = S_0$$

we get

$$\frac{\partial}{\partial t}\varphi(S_T, T) = \frac{1}{2}\frac{\partial^2}{\partial {S_T}^2}\left[{S_T}^2\sigma^2(x, t)\varphi(S_T, T)\right] - \frac{1}{2}\frac{\partial}{\partial S_T}[S_T\sigma(x, t)\varphi(S_T, T)]$$

3. Feynman-Kac Relation: This is a generalization of the Fokker-Planck equation (see, for e.g., Karatzas and Shreve (1997)): Give the boundary value problem

$$\left[\frac{\partial}{\partial t} + \mu(t, X)\frac{\partial}{\partial X} + \frac{1}{2}\sigma^2(t, X)\frac{\partial^2}{\partial X^2}\right] f(t, X) = A(t, X) f(t, X) + B(t, X)$$

and the boundary condition

$$f(t, X) = \psi(X)$$

we can solve for $f(t, X)$ as

$$f(t, X) = \mathbb{E}_t{}^Q \left[D(t, T, X)\psi(X) - \int_t^T D(u, T, X) B(u, X) du\right]$$

with

$$D(t, T, X) := e^{-\int_t^T A(u,X) du}$$

where, under the measure Q, X now follows

$$\Delta X = \mu(t, X) X(t) \Delta t + \sigma(t, X) X(t) \Delta W^Q(t)$$

with

$$S(0) = S_0$$

Volatility Observations vs. Calibrations

1. Latent State Quantification Metric (LSQM) Calibration vs. Manifest Measure (MM) Observation Quotes in the Volatility Space: Both prices and certain implied volatilities are manifest measure quotes, so there is less empirical significance in interpolating their "intermediate" nodes. Calibrated deterministic/local volatility surfaces are the corresponding latent state quantification metrics, so their splined latent state representations are of greater significance.
2. LSQM Extraction from MM: Both the deterministic volatility term structures as well as Dupire's calibrated local volatility surface depend on the derivatives of the manifest measure quotes. Thus, this is significantly distinct from the equivalent treatments of the splined discount/forward/credit/basis curve latent states. Correspondingly, the C^k continuity criterion for the volatility term structure/surface requires $k + r + 1$ basis functions to denote the manifest measure representation.
    - For local volatility surfaces, in addition, multi-dimensional splines are required. Wire mesh 2D spline proxies may work in some cases (esp., for extracting term structures at the strike/term nodal anchors), but are poor alternatives for full surface splines (e.g., bi-cubic splines).
3. Deterministic Volatility – No Arbitrage Criterion: Given that

$$\sigma_{Det}(T) = \sqrt{\sigma_{IMP}{}^2(T) + 2T\sigma_{IMP}(T)\frac{\partial \sigma_{IMP}(T)}{\partial T}}$$

$\sigma_{Det}(T)$ stays valid as long as

$$\sigma_{IMP}(T) > 2T\frac{\partial \sigma_{IMP}(T)}{\partial T}$$

This can get violated for either a steeply upward sloping curve where

$$\frac{\partial \sigma_{IMP}(T)}{\partial T} > 0$$

or a humped curve. Thus this forms the basis behind no arbitrage detection in construction of deterministic volatility term structures, and of the corresponding market quotes (prices OR implied volatilities).

4. Volatility Surface Bootstrapping: Over any incremental time, the probability of moving into the money (and therefore, the corresponding option call/put payoff) increases monotonically. This monotonic in-the-money probability can be mathematically related to the incremental σ, and hence can form the basis for volatility bootstrapping.
5. State Dynamical Parameters Bootstrapping: The local volatility surface enhanced by the generalized Fokker-Planck formulation (through the Feynman-Kac formalism) is the basis behind the volatility dynamics forward diffusion. This formulation is ABSOLUTELY general and POWERFUL in its validity, with the stationary latent-state inferences falling out as a consequence of applying steady-state treatment to this behavior.
6. Risk-Neutral Forward Measures as a Volatility Surface Boot-strapper Unit: Since this measure captures the dynamics of the forward risk-neutral numeraire, the boot-strapper can serve as a suitable incremental dynamics parameters inferrer. The corresponding no-arbitrage drifts may be splined at every time snapshot, or an even better approach may be incorporated.
7. Market Volatility Quote Transform: Remember that cap/floor volatilities etc. are log-normal level quotes (often ATM). Convexity correction using shifted log-normal volatilities (i.e., volatility of $1 + \tau F$) needs to use the corresponding shifted log-normal volatility, which scaled by $\sim\tau F$, the log-normal one. Since $\tau F \sim \frac{1}{100}$ or less is common, this leads to extremely small convexity corrections.
8. Option Price Manifest Measure Quotes: In general, the option manifest measure quotes are with respect to the contract description. Thus, price would be the contract PV. In this case there is no ambiguity – thus, the price manifest measure quote maybe used to "uniquely" calibrate the latent state.
9. Option Implied Volatility Manifest Measure Quote: This quote can be a severe problem. Risk neutrality apart, implied volatility of what – the forward? The terminal payoff metric? The latter is the most sensible interpretation (i.e., implied volatility of the terminal forward payoff).
10. Latent State Dynamics Estimation: Using the combination of the "Current" Latent State and the specific suite of option manifest measures, you may infer the latent state dynamical

parameters. All that is available from the “current” latent state metric would be the central forward metrics (such as par ATM levels, etc.).

11. Discount Curve Latent State Quantification Metric as Forward Rate: This provides an additional motivation towards representing the discounting latent state using the forward rate quantification metric, as the ATM levels automatically fall out of the latent state representation.
12. Options Manifest Measure Quote: Ultimately, like the “steady state” case, the option valuation has to get formulated and eventually represented along a spline formalism (a la local volatility model) no matter what the dynamical state latent state quantification metric parameter set is. Thus, a generalized forward linear formulation would be of great value.

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Section XI: Interest Rate Dynamics and Option Pricing

HJM Model

Introduction

1. Background: Heath, Jarrow, and Morton (1992) discovered that the no-arbitrage condition for zero-coupon bond prices under the risk-neutral measure Q, whose numeraire is the bank account numeraire, implies the existence of a simple constraint between the instantaneous volatility and the instantaneous drift of the instantaneous forward rate.
2. Instantaneous Forward Rate: It is assumed that, for a *fixed Maturity* T, the instantaneous forward rate $f(t,T)$ under Q follows

$$\Delta f(t,T) = \alpha(t,T)\Delta t + \vec{\sigma}(t,T) \cdot \Delta\vec{W}(t)$$

with the initial condition

$$f(0,T) = f_M(0,T)$$

where $\vec{\sigma}(t,T) \cdot \Delta\vec{W}(t)$ is the inner product of the two vectors

$$\vec{\sigma}(t,T) = [\sigma_1(t,T), \dots, \sigma_N(t,T)]^T$$

and

$$\Delta\vec{W}(t) = [W_1(t), \dots, W_N(t)]^T$$

(Wang (2010a)).

3. Gaussian HJM: If, in addition, $\vec{\sigma}(t,T)$ does not depend on $f(t,T)$, then it is known as the Gaussian HJM, where the instantaneous forward rates are normally distributed.

Formulation

1. Instantaneous Forward Rate and Bond Price: By definition

$$f(t,T) = -\frac{\partial \log P(t,T)}{\partial T}$$

which implies that

$$P(t,T) = e^{-\int_t^T f(t,u)du} = e^{Q(t,T)}$$

where

$$Q(t,T) = -\int_t^T f(t,u)du$$

2. Derivative on $Q(t,T)$:

$$\Delta Q(t,T) = f(t,t)\Delta t - \int_t^T [\Delta f(t,u)]du = r(t)\Delta t - \int_t^T \left[\alpha(t,u)\Delta t + \vec{\sigma}(t,u)\cdot\Delta\vec{W}(t)\right]du$$

$$= r(t)\Delta t - \left[\int_t^T \alpha(t,u)du\right]\Delta t - \left[\int_t^T \vec{\sigma}(t,u)du\right]\cdot\Delta\vec{W}(t)$$

3. Price Relationship to $Q(t,T)$: Setting

$$\alpha^*(t,T) = \int_t^T \alpha(t,u)du$$

and

$$\vec{\sigma}^*(t,T) = \int_t^T \vec{\sigma}(t,u)du$$

$$\Delta Q(t,T) = r(t)\Delta t - \alpha^*(t,T)\Delta t - \vec{\sigma}^*(t,T) \cdot \Delta\vec{W}(t)$$

From this we get

$$\Delta P(t,T) = P(t,T)\left\{\left[r(t) - \alpha^*(t,T) + \frac{1}{2}\vec{\sigma}^*(t,T) \cdot \vec{\sigma}^*(t,T)\right]\Delta t - \vec{\sigma}^*(t,T) \cdot \Delta\vec{W}(t)\right\}$$

4. Discounted Zero-Coupon Bond Price Process: Since the zero-coupon bond, as a tradeable asset, must have its discounted price $\frac{P(t,T)}{B(t)}$ to be a martingale under the risk-neutral measure, the drift term of $P(t,T)$ should be $r(t)P(t,T)$, therefore

$$\alpha^*(t,T) = \frac{1}{2}\vec{\sigma}^*(t,T) \cdot \vec{\sigma}^*(t,T)$$

Taking derivative with respect to T produces

$$\alpha(t,T) = \vec{\sigma}(t,T) \cdot \int_t^T \vec{\sigma}(t,u)du$$

In other words, the drift term of the instantaneous forward rate is completely determined by the volatility term. This is the main result of Heath, Jarrow, and Morton (1992).

5. Differential Form of the Instantaneous Forward Rate: Therefore, under the risk-neutral world, HJM model says that

$$\Delta f(t,T) = \left\{ \vec{\sigma}(t,T) \cdot \int_t^T \vec{\sigma}(t,u)du \right\} \Delta t + \vec{\sigma}(t,T) \cdot \Delta \vec{W}(t)$$

with

$$f(0,T) = f_M(0,T)$$

where $f_M(0,T)$ are the exogenous inputs that ensure that these models are automatically consistent with discount bond prices at

$$t = 0$$

6. <u>HJM Integral Forms</u>:

$$f(0,T) = f_M(0,T) + \int_0^t \left\{ \vec{\sigma}(u,T) \cdot \int_u^T \vec{\sigma}(u,s)ds \right\} du + \int_0^t \vec{\sigma}(s,T) \cdot d\vec{W}(s)$$

$$\frac{\Delta P(t,T)}{P(t,T)} = r(t)\Delta t - \left\{ \int_t^T \vec{\sigma}(t,s)ds \right\} \cdot \Delta \vec{W}(t) = r(t)\Delta t - \vec{\sigma}^*(t,T) \cdot \Delta \vec{W}(t)$$

$$P(t,T) = P(0,T) + \int_0^t P(s,T)r(s)ds - \int_0^t P(s,T)\vec{\sigma}(s,T) \cdot d\vec{W}(s)$$

$$r(t) = f(t,t) = f(0,t) + \int_0^t \left\{ \vec{\sigma}(u,t) \cdot \int_u^t \vec{\sigma}(u,s)ds \right\} dt + \int_0^t \vec{\sigma}(s,t) \cdot d\vec{W}(s)$$

From the last equation on $r(t)$, it follows that the short rate process $r(t)$ is not Markovian in general.

Hull-White From HJM

1. HW Case: Hull-White is a special case of the one-factor HJM:

$$f(0,T) = f_M(0,T) + \int_0^t \left\{\vec{\sigma}(u,T) \cdot \int_u^T \vec{\sigma}(u,s)ds\right\} dt + \int_0^t \vec{\sigma}(s,T) \cdot d\vec{W}(s)$$

with the additional assumption for the volatility as a time-homogenous of the form

$$\vec{\sigma}(t,T) = \vec{\sigma}_{HW} e^{-a(T-t)}$$

2. HJM HW Short Rate Formulation: Then

$$r(t) = f(0,t) + \int_0^t \left\{\vec{\sigma}(u,t) \cdot \int_u^t \vec{\sigma}(u,s)ds\right\} du + \int_0^t \vec{\sigma}(s,t) \cdot d\vec{W}(s)$$

$$= f(0,t) + \int_0^t \left\{\vec{\sigma}_{HW} e^{-a(t-u)} \cdot \int_u^t \vec{\sigma}_{HW} e^{-a(s-u)} ds\right\} du$$

$$+ \int_0^t \vec{\sigma}_{HW} e^{-a(t-s)} \cdot d\vec{W}(s)$$

$$= f(0,t) + \frac{\left\|\vec{\sigma}_{HW}{}^2\right\|}{2a^2}[1 - e^{-at}]^2 + \vec{\sigma}_{HW} \cdot \int_0^t e^{-a(t-s)} d\vec{W}(s)$$

$$= r(0)e^{-at} + f(0,t) + \frac{\left\|\vec{\sigma}_{HW}{}^2\right\|}{2a^2}[1 - e^{-at}]^2 - f(0,0)e^{-at} + \vec{\sigma}_{HW}$$

$$\cdot \int_0^t e^{-a(t-s)} d\vec{W}(s)$$

3. Comparison with the Standard HW Model: Comparison with the regular HW model, with the observation that

$$f(0,0) = r(0)$$

shows that

$$\Delta r(t) = [\theta(t) - ar(t)]\Delta t + \vec{\sigma}(t,T) \cdot \Delta\overrightarrow{W}(t)$$

where

$$\theta(t) = \frac{\partial f(0,t)}{\partial T} + af(0,t) + \frac{\left\|\vec{\sigma}_{HW}{}^{2}\right\|}{2a}\left[1 - e^{-2at}\right]$$

G2++ - A 2-Factor HJM Model

1. Setup: G2++ is a special case of HJM where there are 2 factors. It becomes

$$\Delta f(t,T) = \alpha(t,T)\Delta t + \vec{\sigma}_1(t,T) \cdot \Delta\overrightarrow{W_1}(t) + \vec{\sigma}_2(t,T) \cdot \Delta\overrightarrow{W_2}(t)$$

and

$$\Delta\overrightarrow{W_1}(t) \cdot \Delta\overrightarrow{W_2}(t) = \rho\Delta t$$

The two factors may be determined in practice using PCA – e.g., $\Delta\overrightarrow{W_1}(t)$ stands for the change of slope, while $\Delta\overrightarrow{W_2}(t)$ can stand for the change of curvature.

2. Formulation: Assume time-homogenous exponential form

$$\vec{\sigma}(t,T) \cdot \Delta\overrightarrow{W}(t) = \vec{\sigma}e^{-a(T-t)} \cdot \Delta\overrightarrow{W_1}(t) + \vec{\eta}e^{-b(T-t)} \cdot \Delta\overrightarrow{W_2}(t)$$

Using the same approach as for the 1-factor HJM we get

$$r(t) = f(0,t) + r(0)e^{-at} + \left\{\frac{\|\vec{\sigma}^2\|}{2a^2}[1 - e^{-at}]^2 - f(0,0)e^{-at}\right\} + \vec{\sigma} \cdot \int_0^t e^{-a(t-s)} d\overrightarrow{W_1}(s)$$
$$+ r(0)e^{-bt} + \left\{\frac{\|\vec{\eta}^2\|}{2b^2}[1 - e^{-bt}]^2 - f(0,0)e^{-bt}\right\} + \vec{\eta} \cdot \int_0^t e^{-a(t-s)} d\overrightarrow{W_2}(s)$$

3. <u>G2++ Model Short Rate</u>: Let

$$\varphi(t) = f(0,t) + r(0)e^{-at} + \left\{\frac{\|\vec{\sigma}^2\|}{2a^2}[1 - e^{-at}]^2 - f(0,0)e^{-at}\right\} + r(0)e^{-bt}$$
$$+ \left\{\frac{\|\vec{\eta}^2\|}{2b^2}[1 - e^{-bt}]^2 - f(0,0)e^{-bt}\right\}$$

Setting

$$x(t) = \vec{\sigma} \cdot \int_0^t e^{-a(t-s)} d\overrightarrow{W_1}(s)$$

and

$$y(t) = \vec{\eta} \cdot \int_0^t e^{-a(t-s)} d\overrightarrow{W_2}(s)$$

we get

$$\Delta x(t) = -ax(t)\Delta t + \vec{\sigma} \cdot \Delta\overrightarrow{W_1}(t); x(0) = 0$$

$$\Delta y(t) = -by(t)\Delta t + \vec{\eta} \cdot \Delta \overrightarrow{W_2}(t); y(0) = 0$$

and

$$\Delta \overrightarrow{W_1}(t) \cdot \Delta \overrightarrow{W_2}(t) = \rho \Delta t$$

and we arrive at the G2++ model under the risk-neutral measure as

$$r(t) = x(t) + y(t) + \varphi(t)$$

HJM to LMM

1. The Forward Rate: Consider the forward rate

$$F(t; S, T) = \frac{1}{\tau(S, T)}\left[\frac{P(t, S)}{P(t, T)} - 1\right]$$

Applying the Ito lemma, we get

$$\Delta F(t; S, T) = \frac{1}{\tau(S, T)}\left[\frac{1}{P(t, T)}\Delta P(t, S) + P(t, S)\Delta\left\{\frac{1}{P(t, T)}\right\} + \Delta P(t, S)\Delta\left\{\frac{1}{P(t, T)}\right\}\right]$$

2. Application of the HJM to the Forward Rate: From the above, we get

$$\Delta P(t, T) = P(t, T)\left\{r(t)\Delta t - \vec{\sigma}^*(t, T) \cdot \Delta \vec{W}(t)\right\}$$

which implies that

$$\Delta\left\{\frac{1}{P(t,T)}\right\} = -\frac{1}{P(t,T)^2}\Delta P(t,T) + \frac{1}{P(t,T)^3}\Delta\langle P(t,T),\Delta P(t,T)\rangle$$

$$= \frac{1}{P(t,T)}\left\{[-r(t) + \|\vec{\sigma}(t,T)^*\|^2]\Delta t + \vec{\sigma}^*(t,T)\cdot\Delta\vec{W}(t)\right\}$$

so that $\Delta F(t;S,T)$ can be re-written as

$$\Delta F(t;S,T) = \frac{1}{\tau(S,T)}\left[\frac{1}{P(t,T)}\Delta P(t,S) + P(t,S)\Delta\left\{\frac{1}{P(t,T)}\right\} + \Delta P(t,S)\Delta\left\{\frac{1}{P(t,T)}\right\}\right]$$

$$= \frac{1}{\tau(S,T)}\frac{P(t,S)}{P(t,T)}\{[\|\vec{\sigma}(t,T)^*\|^2 - \vec{\sigma}^*(t,S)\cdot\vec{\sigma}^*(t,T)]\Delta t$$

$$+ [\vec{\sigma}^*(t,T) - \vec{\sigma}^*(t,S)]\cdot\Delta\vec{W}(t)\}$$

3. Shifted LIBOR Forward Rate: From the above, we see that

$$\Delta F(t;S,T) = \left[F(t;S,T) + \frac{1}{\tau(S,T)}\right][\vec{\sigma}^*(t,T) - \vec{\sigma}^*(t,S)]\cdot\left[\vec{\sigma}^*(t,T)\Delta t + \Delta\vec{W}(t)\right]$$

Now define the shifted LIBOR forward rate as

$$F^*(t;S,T) = F(t;S,T) + \frac{1}{\tau(S,T)}$$

This leads to the following log-normal process

$$\Delta F^*(t;S,T) = F^*(t;S,T)\left[\int_S^T \vec{\sigma}(t,u)du\right]\cdot\left[\vec{\sigma}^*(t,T)\Delta t + \Delta\vec{W}(t)\right]$$

4. T-Forward Measure: We know that under the T-forward measure the shifted LIBOR forward rate is a martingale. To change the measure from risk-neutral to the T-forward measure, we use Girsanov theorem (Girsanov Theorem (Wiki)) as follows:

$$X(t) = -\int_0^t \left[\int_s^T \vec{\sigma}(s,u)du\right] \cdot d\vec{W}(s) = -\int_s^T \vec{\sigma}^*(s,u) \cdot d\vec{W}(s)$$

implies that

$$\Delta X(t) = -\left[\int_s^T \vec{\sigma}(s,u)du\right] \cdot \Delta\vec{W}(s) = -\vec{\sigma}^*(s,u) \cdot \Delta\vec{W}(s)$$

resulting in

$$\vec{W}_T(t) = \vec{W}(t) - X(t)$$

Thus

$$\Delta\vec{W}_T(t) = \Delta\vec{W}(t) - \left[\Delta\vec{W}(t), \Delta X(t)\right] = \Delta\vec{W}(t) + \vec{\sigma}^*(t,T)\Delta t$$

So, under the T-forward measure

$$\Delta F^*(t;S,T) = F^*(t;S,T)\left[\int_S^T \vec{\sigma}(t,u)du\right] \cdot \Delta\vec{W}_T(t)$$

5. Instantaneous Forward Rate under T-forward Measure: Using the fact that

$$\Delta f(t,T) = \vec{\sigma}(t,T) \cdot \vec{\sigma}^*(t,T)\Delta t + \vec{\sigma}(t,T) \cdot \Delta\vec{W}(t)$$
$$= \vec{\sigma}(t,T) \cdot \vec{\sigma}^*(t,T)\Delta t + \vec{\sigma}(t,T) \cdot \left[\Delta\vec{W}_T(t) - \vec{\sigma}^*(t,T)\Delta t\right] = \vec{\sigma}(t,T) \cdot \Delta\vec{W}_T(t)$$

we can see that, under the T-forward measure the instantaneous forward rate is a martingale, or

$$\Delta f(t,T) = \vec{\sigma}(t,T) \cdot \Delta \overrightarrow{W}_T(t)$$

Thus, from the T-forward measure for $F^*(t;S,T)$ and $f(t,T)$ we can see that the expectation hypothesis holds under the T-forward measure, i.e.,

$$\mathbb{E}_t^{Q^T}[r(t)|\mathcal{F}_t] = f(t,T)$$

6. <u>Caplet/Floorlet on $F(t;S,T)$</u>: The caplet/Floorlet on rate $F(S;S,T)$ with strike K can be re-defined on $F^*(S;S,T)$ with a strike $K + \frac{1}{\tau(S,T)}$. The Black volatility seen above becomes $\int_0^S \left\| \int_S^T \vec{\sigma}(t,u)du \right\|^2 dt$.
7. <u>Blowup of the T-forward Instantaneous Forward Rate</u>: Anderson and Piterbarg (2010a, 2010b) pointed out that the log-normal proportional volatility HJM process

$$\sigma\big(t,T,f(t,T)\big) = \sigma(t,T)f(t,T)$$

leads to log-normally distributed instantaneous forward rates under the T-forward measure, i.e.,

$$\Delta f(t,T) = \vec{\sigma}(t,T) \cdot \Delta \overrightarrow{W}_T(t)$$

but the forward rates can explode in finite time to ∞ with non-zero probability. The LIBOR market model addresses this drawback.
8. <u>Generalization of LMM</u>: The typical LIBOR market model can be generalized as follows. Suppose there exists a deterministic function $\hat{\sigma}(t,T)$ such that

$$\int_S^T \sigma(s,u)du = \frac{F(t;S,T)}{F(t;S,T) + \frac{1}{\tau(S,T)}} \hat{\sigma}(t,T)$$

Then

$$\Delta F(t; S, T) = \hat{\sigma}(t, T) F(t; S, T) \Delta W_T(t)$$

HJM PCA

1. Introduction: In this section we introduce the historical estimation/calibration for HJM. We then follow it up by applying PCA for the calibration of HJM.
2. Time Homogeneity: Considering the Gaussian HJM model, time homogeneity means that volatility $\vec{\sigma}(t, T)$ depends on the remaining time to maturity, or

   $$\vec{\sigma}(t, T) = \vec{\sigma}(T - t)$$

   so that the Gaussian HJM may be written as

   $$\Delta f(t, T) = \mu(t, T) \Delta t + \vec{\sigma}(T - t) \cdot \Delta \vec{W}(t)$$

   This enables us to estimate $\vec{\sigma}$ (and hence the drift) from the historical data, e.g., to estimate the 3M volatility, we retain the time-to-maturity at 3M across observations.
3. HJM Estimation on the Spot Rate: The historical estimation maybe performed on the instantaneous forward rate, or on the more observable continuous zero rate. Recall that the continuously compounded spot rate is defined by

   $$R(t, T) = -\frac{\log P(t, T)}{\tau(t, T)}$$

   Applying the Ito lemma, we get

$$\Delta R(t,T) = -\Delta\left[\frac{1}{\tau(t,T)}\right]\log P(t,T)$$
$$-\frac{1}{\tau(t,T)}\left\{\frac{1}{P(t,T)}\Delta P(t,T) - \frac{1}{2P(t,T)^2}\Delta\langle P(t,T), P(t,T)\rangle\right\}$$
$$= -\frac{\log P(t,T)}{(T-t)^2}\Delta t - \frac{1}{\tau(t,T)}\left\{\left[r(t) - \frac{1}{2}\|\vec{\sigma}^*(t,T)\|^2\right]\Delta t - \vec{\sigma}^*(t,T)\cdot\Delta\vec{W}(t)\right\}$$
$$= \frac{1}{\tau(t,T)}\left\{\left[R(t,T) - r(t) + \frac{1}{2}\|\vec{\sigma}^*(t,T)\|^2\right]\Delta t + \vec{\sigma}^*(t,T)\cdot\Delta\vec{W}(t)\right\}$$

which yields

$$\Delta R(t,T) = \frac{1}{\tau(t,T)}\left\{\left[R(t,T) - r(t) + \frac{1}{2}\|\vec{\sigma}^*(t,T)\|^2\right]\Delta t + \vec{\sigma}^*(t,T)\cdot\Delta\vec{W}(t)\right\}$$

4. Variance/Covariance of the Instantaneous Forward Rate: Here we consider general processes of the form

$$\Delta X_i(t) = \mu_i \Delta t + \sum_{j=1}^{N}\sigma_{ij}\Delta W_j(t)$$

where X can be the instantaneous forward rate or the continuously compounded spot rate. Further the time $T - t$ is replaced by the subscript i to indicate different tenors (3M/6M etc.) Work in the correlation space eventually to determine the constituent components.
5. Dimensionality Reduction via PCA: Usually the first 3 PCA's a repacked, and they represent the 3 main yield curve movements – parallel shifts, twists (steepener/flattener), and curvature (butterfly).

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Hull-White Model

Short Rate Formulation

1. Basic Relation: This section adapted from Brigo and Mercurio (2006) and Wang (2010). As an extension of the Vasicek model, the Hull-White model assumes that the short rate follows the mean-reverting SDE

$$\Delta r(t) = [\theta(t) - ar(t)]\Delta t + \sigma W(t)$$

where σ and a are positive constants, and $\theta(t)$ is a time-dependent function that will be used to fit the current zero curve.

2. Solution to the Hull-White SDE: To solve this SDE, first apply the Ito lemma to re^{at};

$$\Delta(re^{at}) = e^{at}\Delta r + are^{at}\Delta t = (\theta\Delta t + \sigma\Delta W)e^{at}$$

Then integrate both sides over (s, t);

$$r(t)e^{at} - r(s)e^{as} = \int_s^t \theta(u)e^{au}du + \sigma\int_s^t e^{au}dW(u)$$

resulting in

$$r(t) = r(s)e^{-a(t-s)} + \int_s^t \theta(u)e^{-a(t-u)}du + \sigma\int_s^t e^{-a(t-u)}dW(u)$$

3. Fitting to the Initial Term Structure: In order to fit to the initial term structure of interest rates the time-dependent $\theta(t)$ must satisfy

$$\theta(t) = \frac{\partial f_M(0,t)}{\partial t} + a f_M(0,t) + \frac{\sigma^2}{2a}[1 - e^{-2at}]$$

where $f_M(0,t)$ is the market observed instantaneous forward rate at time 0 for maturity t.

4. Re-cast of $r(t)$: From the above, $r(t)$ can be written as

$$r(t) = r(s)e^{-a(t-s)} + \alpha(t) - \alpha(s)e^{-a(t-s)} + \sigma \int_s^t e^{-a(t-u)} dW(u)$$

where

$$\alpha(t) = f_M(0,t) + \frac{\sigma^2}{2a^2}[1 - e^{-at}]^2$$

5. Mean and Variance of $r(t)$: Therefore $r(t)$ conditional on $\mathcal{F}_s$ is normally distributed with mean and variance given respectively by

$$\mathbb{E}[r(t)|\mathcal{F}_s] = r(s)e^{-a(t-s)} + \alpha(t) - \alpha(s)e^{-a(t-s)}$$

and

$$Var[r(t)|\mathcal{F}_s] = \sigma \int_s^t e^{-2a(t-u)} du = \frac{\sigma^2}{2a}[1 - e^{-2at}]$$

6. Hull-White as an Affine Term Structure Model: Hull-White model is an *affine term structure model* where the continuously compounded spot rate is an affine function on the short rate, i.e.,

$$R(t,T) = \alpha'(t,T) + \beta(t,T)r(t)$$

7. <u>Hull-White Based Product Valuation</u>: The zero-coupon price is given by

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}$$

where

$$A(t,T) = \frac{P_M(0,T)}{P_M(0,t)} e^{B(t,T)f_M(0,t) - \frac{\sigma^2}{4a}[1-e^{-2at}]B(t,T)^2}$$

and

$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a}$$

We can also find closed-form formulas for zero-coupon bond options, caps/floors, and swaptions (Brigo and Mercurio (2006)).

Hull-White Trinomial Tree

1. <u>Decomposition</u>: To construct the Hull-White tree, it is useful to decompose the short rate into the following format:

$$r(t) = x(t) + \alpha(t)$$

where

$$\alpha(t) = f_M(0,t) + \frac{\sigma^2}{2a^2}[1 - e^{-at}]^2$$

$$\Delta x(t) = -ax(t)\Delta t + \sigma \Delta W(t); x(0) = 0$$

$$x(t) = x(s) + e^{-a(t-s)} + \sigma \int_s^t e^{-a(t-u)} dW(u)$$

2. Trinomial Tree Construction Steps: With this decomposition in hand, the tree construction can be achieved in 2 steps. In the first, one constructs a trinomial tree for $x(t)$. Then in the next, one shifts the tree by $\alpha(t)$ to being it in line with the initial term structure.

Construction of the Symmetric Trinomial Tree

1. Setting up the Tree Nodes: Denote the tree nodes by (i, j) where the time index i ranges from 0 to N and the space index j ranges from some $\underline{j}_i$ to some $\overline{j}_i$. Recall that the expectation and the variance of $x(t)$ are

$$\mathbb{E}[x(t)|\mathcal{F}_s] = x(s)e^{-a(t-s)}$$

and

$$Var[r(t)|\mathcal{F}_s] = \frac{\sigma^2}{2a}\left[1 - e^{-2a(t-s)}\right]$$

respectively. Using these in our discretized nodes (Brigo and Mercurio (2006)), we get

$$\mathbb{E}\left[x(t_{i+1})|x(t_i) = x_{i,j}\right] = x_{i,j}e^{-a\Delta t_i} \doteqdot M_{i,j}$$

and

$$Var\left[x(t_{i+1})|x(t_i) = x_{i,j}\right] = \frac{\sigma^2}{2a}\left[1 - e^{-2a\Delta t_i}\right] \doteqdot {V_i}^2$$

where

$$\Delta t_i = t_{i+1} - t_i$$

2. Incorporating the Transition Probabilities: Now, given the node $x_{i,j}$, we need to locate its subsequent nodes $x_{i+1,k+1}$, $x_{i+1,k}$, and $x_{i+1,k-1}$, associated with their transition probabilities p_u, p_m, and p_d. First we find the spatial displacement as

$$\Delta x_{i+1} = V_i\sqrt{3} = \sigma\sqrt{\frac{3}{2a}[1 - e^{-2a\Delta t_i}]}$$

Then locate the node k using

$$k = round\left(\frac{M_{i,j}}{\Delta x_{i+1}}\right)$$

where $round(x)$ indicates the integer closest to the real number x. We then set the following:

$$x_{i+1,k+1} = (k+1)\Delta x_{i+1}$$

$$x_{i+1,k} = k\Delta x_{i+1}$$

$$x_{i+1,k-1} = (k-1)\Delta x_{i+1}$$

3. Choice of Transition Probabilities: Finally the transition probabilities are chosen in such a way as to match the conditional mean and the variance:

$$p_u = \frac{1}{6} + \frac{{\eta_{j,k}}^2}{6{V_i}^2} + \frac{\eta_{j,k}}{2\sqrt{3}V_i}$$

$$p_m = \frac{2}{3} - \frac{{\eta_{j,k}}^2}{3{V_i}^2}$$

$$p_d = \frac{1}{6} + \frac{{\eta_{j,k}}^2}{6{V_i}^2} - \frac{\eta_{j,k}}{2\sqrt{3}V_i}$$

Here

$$\eta_{j,k} = M_{i,j} - x_{i+1,k} = M_{i,j} - k\Delta x_{i+1}$$

Displacing the Nodes of the Trinomial Tree

1. Calculation of the displacement α: An easy way to do this is by noticing that

$$\alpha(t) = f_M(0,t) + \frac{\sigma^2}{2a^2}[1 - e^{-at}]^2$$

and approximating the instantaneous forward rate by

$$f_M(0,t) \approx F_M(0;\ t, t + 0.5\ bp)$$

(Wang (2010)). This approach has to approximate the continuously compounded rate $R(0,t)$ by the short rate $r(0)$, therefore doesn't fit the zero curve exactly.

2. Using Arrow-Debreu Prices: The alternate approach is to use the Arrow-Debreu prices. Denote α_i as the displacement at t_i, and let Q_{ij} be the Arrow-Debreu price at node (i,j). A *State-Price Security* or the *Arrow-Debreu Security* is defined as the contract that pays $1 at a particular date and a particular time, and pays $0 in all other states. The corresponding price (i.e., the NPV) is referred to as the Arrow-Debreu Price.
3. α_i/Q_{ij} Calculation Step #1: Initialize

$$Q_{ij} = 0$$

4. α_i/Q_{ij} Calculation Step #2: Find

$$\alpha_0 = -\frac{\log P_M(0, t_1)}{t_1}$$

5. α_i/Q_{ij} Calculation Step #3: With the given α_i (where $i = 0$ at the start), calculate

$$Q_{i+1,j} = \sum_h Q_{i,h} q(h,j) e^{-(\alpha_i + j\Delta x_i)\Delta t_i}$$

where $q(h,j)$ is the probability of migrating from node (i,h) to node $(i+1,j)$.

6. α_i/Q_{ij} Calculation Step #4: With $Q_{i,j}$ in hand, find α_i by solving

$$P(0, t_{i+1}) = \sum_{j=\underline{j}_i}^{\bar{j}_i} Q_{i,j} e^{-(\alpha_i + j\Delta x_i)\Delta t_i}$$

which leads to

$$\alpha_i = \frac{1}{\Delta t_i} \log \frac{\sum_{j=\underline{j}_i}^{\bar{j}_i} Q_{i,j} e^{-j\Delta x_i \Delta t_i}}{P(0, t_{i+1})}$$

7. α_i/Q_{ij} Calculation Step #5: Loop through steps 3 and 4 to discover α_i and $Q_{i,j}$ in the eventual steps. The short rate at each node is

$$r_{i,j} = x_{i,j} + \alpha_i$$

In general, remember that if σ and/or a is a function of f, r, the tree will be non-recombining.

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Market Model of Interest Rate Dynamics

Problems with Conventional Market Practice

1. Typical Swap Derivatives Book: In most markets, caps and floors form the largest component of the average swap derivatives book, caps/floors being considered here are strips of caplets/floorlets, each of which is a call/put on the forward rate.
2. Market Pricing Practice: Conventional market practice has been to price the option assuming that the underlying forward rate is distributed log-normally with zero drift. Thus the option price is given by the Black's formula, discounted from the settlement date.
3. Trouble with the Market Practice: In an arbitrage-free setting, forward rates over consecutive time intervals are related to one another and cannot all be lognormal under one arbitrage-free measure.
4. The BGM Approach: Brace, Gatarek, and Musiela (1997) show that the market practice above can be made consistent with an arbitrage-free term structure model. Consecutive quarterly or semi-annual rates can all be lognormal while the model remains arbitrage-free. This is possible because each forward rate is lognormal under the corresponding forward (to the settlement date) arbitrage-free measure rather than under a single spot arbitrage-free measure. Lognormality under the appropriate forward and not the spot arbitrage-free measure is required to justify the use of the Black futures formula with discount for the caplet pricing.

Nomenclature and Notation

1. Origins of the Term Structure Parametrization: The term structure parametrizations considered here were originally proposed by Musiela (1993), and developed further later by Musiela and Sondermann (1993), Brace and Musiela (1994), Goldys, Musiela, and Sondermann (1994), and Musiela (1994).

2. <u>The Continuously Compounded Forward Rate</u>: We denote by $r(t, x)$ the continuously compounded forward rate prevailing at time t over the interval $[t + x, t + x + \Delta t]$. There is an obvious relationship between the Heath, Jarrow, and Morton (1992) forward rates $f(t, T)$ and $r(t, x)$, namely

$$r(t, x) = f(t, t + x)$$

For all $T > 0$ the process

$$P(t, T) = e^{-\int_0^{T-t} r(t,u)du} = e^{-\int_t^T f(t,u)du}$$

for

$$0 \leq t \leq T$$

describes the price evolution of a zero-coupon bond maturing at T.

3. <u>Evolution of $r(t, x)$</u>: We make the usual mathematical assumptions, i.e., all processes are defined on the probability space $[\Omega, \{\mathcal{F}_t \geq 0\}, \mathbb{P}]$, where the filtration $\{\mathcal{F}_t \geq 0\}$ is the $\mathbb{P}$-augmentation of the natural filtration generated by a d-dimensional Brownian motion

$$\vec{W} = \{\vec{W}(t);\ t \geq 0\}$$

We then assume that the process

$$\{r(t, x);\ x, t \geq 0\}$$

satisfies

$$\Delta r(t, x) = \frac{\partial}{\partial x}\left[\left\{r(t, x) + \frac{1}{2}|\vec{\sigma}(t, x)|^2\right\}\Delta t + \vec{\sigma}(t, x) \cdot \Delta\vec{W}(t)\right]$$

where for all

$$x \geq 0$$

the volatility process

$$\{\vec{\sigma}(t,x);\ t \geq 0\}$$

is $\mathcal{F}_t$-adapted with values in $\mathbb{R}^d$, while $|\cdot|$ and $\cdot$ stand for the usual norm and the inner product in $\mathbb{R}^d$, respectively. We also assume that the function

$$x \longmapsto \vec{\sigma}(t,x)$$

is absolutely continuous, and the derivative

$$\vec{\tau}(t,x) = \frac{\partial}{\partial x}\vec{\sigma}(t,x)$$

is bounded on $\mathbb{R}_+^2 \times \Omega$.

4. <u>The Discount Function</u>: The time evolution of the discount function

$$D(t,x) = P(t,t+x) = e^{-\int_0^x r(t,u)du}$$

is described by the discount process

$$\{D(t,x);\ x,t \geq 0\}$$

From the dynamics of $r(t,x)$, it easily follows that

$$\Delta D(t,x) = D(t,x)\big\{[r(t,0) - r(t,x)]\Delta t - \vec{\sigma}(t,x) \cdot \Delta\overrightarrow{W}(t)\big\}$$

and hence $\vec{\sigma}(t, x)$ can be interpreted as price volatility. Obviously we have

$$\vec{\sigma}(t, 0) = 0$$

5. <u>Non-Markovian Nature of the Spot Rate Process</u>: The spot rate process

$$\{r(t, 0);\ t \geq 0\}$$

satisfies

$$\Delta r(t, 0) = \frac{\partial}{\partial x} r(t, x)\bigg|_{x=0} \Delta t + \frac{\partial}{\partial x} \vec{\sigma}(t, x)\bigg|_{x=0} \cdot \Delta \overrightarrow{W}(t)$$

and hence is, in general, not Markovian.

6. <u>Spot Rate Savings Account</u>: The process

$$\beta(t) = e^{\int_0^t r(u,0)du}\ \forall\ t \geq 0$$

represents the amount generated at time

$$t \geq 0$$

by continuously re-investing $1 at the spot rate

$$r(s, 0)\ \forall\ \ 0 \leq s \leq t$$

7. <u>Martingale Nature of $P(t, T)$ under $\beta(t)$</u>: It is well-known that if for all

$$T > 0$$

the process

$$\left\{\frac{P(t,T)}{\beta(t)}; 0 \leq t \leq T\right\}$$

is a martingale under $\mathbb{P}$, then there is no arbitrage possible between the zero coupon bonds $P(t,T)$ of all maturities $T > 0$ and the savings account $\beta(t)$. Thus, from the $r(t,x)$ evolution equation, one can write that

$$\frac{P(t,T)}{\beta(t)} = P(0,T)\, e^{-\int_0^t \vec{\sigma}(s,T-s)\cdot d\vec{W}(s) - \frac{1}{2}\int_0^t |\vec{\sigma}(s,T-s)|^2 ds}$$

where the RHS is a martingale under $\mathbb{P}$. It also follows that

$$\Delta P(t,T) = P(t,T)\left\{r(t,0)\Delta t - \vec{\sigma}(t,T-t)\cdot\Delta\vec{W}(t)\right\}$$

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The BGM Model

LIBOR Rate Dynamics

1. Lognormal LIBOR Rate: To specify the model, or equivalently, to define the volatility process $\vec{\sigma}(t,x)$ in

$$\Delta r(t,x) = \frac{\partial}{\partial x}\left[\left\{r(t,x) + \frac{1}{2}|\vec{\sigma}(t,x)|^2\right\}\Delta t + \vec{\sigma}(t,x)\cdot\Delta\vec{W}(t)\right]$$

we fix

$$\delta > 0$$

and assume that for each

$$x \geq 0$$

the LIBOR rate process

$$\{L(t,x);\ t,x \geq 0\}$$

defined by

$$1 + \delta L(t,x) = e^{\int_x^{x+\delta} r(t,u)du}$$

has a lognormal volatility structure, i.e.,

$$\Delta L(t,x) = \cdots \Delta t + L(t,x)\vec{\gamma}(t,x) \cdot \Delta\vec{W}(t)$$

where the deterministic function

$$\vec{\gamma}: \mathbb{R}_+^2 \longrightarrow \mathbb{R}^d$$

is bounded and piecewise continuous.

2. <u>Formulation of the LIBOR Rate Dynamics</u>: From Ito's formula and the equation for $\Delta r(t,x)$ dynamics above, we get

$$\Delta L(t,x) = \frac{1}{\delta}\Delta\left[e^{\int_x^{x+\delta} r(t,u)du}\right]$$

$$= \frac{1}{\delta}e^{\int_x^{x+\delta} r(t,u)du}\Delta\left[e^{\int_x^{x+\delta} r(t,u)du}\right] + \frac{1}{\delta}\frac{1}{2}\left[e^{\int_x^{x+\delta} r(t,u)du}\right]|\vec{\sigma}(t,x+\delta) - \vec{\sigma}(t,x)|^2\Delta t$$

$$= \frac{1}{\delta}\left[e^{\int_x^{x+\delta} r(t,u)du}\right]\left\{\left[r(t,x+\delta) - r(t,x) + \frac{1}{2}|\vec{\sigma}(t,x+\delta)|^2 - \frac{1}{2}|\vec{\sigma}(t,x)|^2\right]\Delta t\right.$$

$$\left. + [\vec{\sigma}(t,x+\delta) - \vec{\sigma}(t,x)] \cdot \Delta\vec{W}(t)\right\}$$

$$+ \frac{1}{\delta}\frac{1}{2}\left[e^{\int_x^{x+\delta} r(t,u)du}\right]|\vec{\sigma}(t,x+\delta) - \vec{\sigma}(t,x)|^2\Delta t$$

$$= \left\{\frac{\partial}{\partial x}L(t,x) + \frac{1}{\delta}[1 + \delta L(t,x)]\vec{\sigma}(t,x+\delta) \cdot [\vec{\sigma}(t,x+\delta) - \vec{\sigma}(t,x)]\right\}\Delta t$$

$$+ \frac{1}{\delta}[1 + \delta L(t,x)][\vec{\sigma}(t,x+\delta) - \vec{\sigma}(t,x)] \cdot \Delta\vec{W}(t)$$

3. <u>The LIBOR Recurrence</u>: For the above LIBOR dynamics to be lognormal for all

$$x \geq 0$$

we need

$$\vec{\sigma}(t,x+\delta) - \vec{\sigma}(t,x) = \frac{\delta L(t,x)}{1 + \delta L(t,x)}\vec{\gamma}(t,x)$$

This recurrence defines the HJM volatility process $\vec{\sigma}(t,x)$ for all

$$x \geq \delta$$

provided $\vec{\sigma}(t,x)$ is defined on the interval

$$0 \leq x < \delta$$

We set

$$\vec{\sigma}(t,x) = 0$$

for all

$$0 \leq x < \delta$$

and the recursive solution for $\vec{\sigma}(t,x)$ for $x \geq \delta$ is

$$\vec{\sigma}(t,x) = \sum_{k=1}^{\left\lfloor \frac{x}{\delta} \right\rfloor} \frac{\delta L(t,x-k\delta)}{1+\delta L(t,x-k\delta)} \vec{\gamma}(t,x-k\delta)$$

4. <u>Evolution of $L(t,x)$</u>: From the recursive lognormal constraint, the equation for $L(t,x)$ becomes

$$\Delta L(t,x) = \left[\frac{\partial}{\partial x} L(t,x) + L(t,x)\vec{\gamma}(t,x) \cdot \vec{\sigma}(t,x+\delta)\right] \Delta t + L(t,x)\vec{\gamma}(t,x) \cdot \Delta\overrightarrow{W}(t)$$

5. <u>Recursion Dynamics of $L(t,x)$</u>: Using the recursion solution for $\vec{\sigma}(t,x)$, the process $\{L(t,x);\ t,x \geq 0\}$ satisfies

$$\Delta L(t,x) = \left[\frac{\partial}{\partial x} L(t,x) + L(t,x)\vec{\gamma}(t,x) \cdot \vec{\sigma}(t,x) + \frac{\delta L^2(t,x)}{1 + \delta L^2(t,x)} |\vec{\gamma}(t,x)|^2\right] \Delta t + L(t,x)\vec{\gamma}(t,x) \cdot \Delta\overrightarrow{W}(t)$$

Relation to the HJM Dynamics

1. <u>Introduction</u>: The BGM approach to the term structure modeling is quite different from the ones based on instantaneous continuously compounded spot or forward rates, and therefore its motivations/origins are worth examining.
2. <u>Instantaneous Effective Annual Rates</u>: The change of focus from the instantaneously compounded forward rates to the instantaneous compounded effective annual rates was first proposed by Sandmann and Sondermann (1993) in response to the impossibility of being able to price a Eurodollar futures contract with a lognormal mode of the continuously compounded spot rate.
3. <u>Lognormal Effective Annual Rate</u>: An HJM-type model based on instantaneous effective annual rates was introduced by Goldys, Musiela, and Sondermann (1994). A lognormal volatility structure was assumed on the effective annual rate $j(t,x)$ which is related to the instantaneous continuously compounded forward rate $r(t,x)$ via the formula

$$1 + j(t,x) = e^{r(t,x)}$$

4. <u>Nominal Annual Rates</u>: The case of the nominal annual rates $q(t,x)$ corresponding to $r(t,x)$, i.e.,

$$[1 + \delta j(t,x)]^{\frac{1}{\delta}} = e^{r(t,x)}$$

   was studied by Musiela (1994). It turns out that the HJM volatility process takes the form

$$\vec{\sigma}(t,x) = \int_0^x \frac{1}{\delta}\left[1 - e^{-\delta r(t,u)}\right]\vec{\gamma}(t,u)du$$

Obviously for

$$\delta = 1$$

we get the Goldys, Musiela, and Sondermann (1994) model, and for

$$\delta = 0$$

we get

$$\vec{\sigma}(t,x) = \int_0^x r(t,u)\vec{\gamma}(t,u)du$$

and hence the HJM lognormal model, which is known to explode – for

$$\delta > 0$$

no explosion occurs.

5. Option Pricing Challenges with the above Representation: Unfortunately, these formulas do not give the closed form pricing formulas for the options. In order to price a caplet, for example, one would have to use some numerically intensive algorithms. This would not be practical for model calibration, where an iterative procedure would be needed to identify the volatility $\vec{\gamma}(t,x)$ which returns the market prices for a large number of caps and swaptions.
6. Discrete Annual Rates: A key piece of the term structure puzzle was found by Miltersen, Sandmann, and Sondermann (1994). First, attention was shifted from the instantaneous forward rates $q(t,x)$ to the nominal annual rates $f(t,x,\delta)$ defined by

$$[1 + f(t, x, \delta)]^{\delta} = e^{\int_{x}^{x+\delta} r(t,u)du}$$

More importantly, however, it was shown for $\delta = 1$ the model prices a yearly caplet according to the market standard.

7. Effective Discrete Rates: Unfortunately, the volatility $\vec{\sigma}(t, x)$ was not completely identified above, leaving open the question of model selection for maturities different from

$$x = i\delta$$

as well as the solution to $\Delta r(t, x)$. These problems were partially addressed by Miltersen, Sandmann, and Sondermann (1995), where a model based on the effective rates $f(t, T, \delta)$ defined by

$$1 + \delta f(t, T, \delta) = e^{\int_{T-t}^{T+\delta-t} r(t,u)du}$$

was analyzed.

8. The BGM Approach: As indicated earlier, the BGM approach assumes a log-normal volatility structure on the LIBOR rate $L(t, x)$ defined by

$$1 + \delta L(t, x) = e^{\int_{x}^{x+\delta} r(t,u)du}$$

for all

$$x \geq 0$$

and a fixed

$$\delta > 0$$

This leads to the $\vec{\sigma}(t, x)$ for $L(t, x)$ given by

$$\vec{\sigma}(t,x) = \sum_{k=1}^{\left\lfloor \frac{x}{\delta} \right\rfloor} \frac{\delta L(t, x - k\delta)}{1 + \delta L(t, x - k\delta)} \vec{\gamma}(t, x - k\delta)$$

and

$$\Delta L(t,x) = \left[\frac{\partial}{\partial x} L(t,x) + L(t,x)\vec{\gamma}(t,x) \cdot \vec{\sigma}(t,x) + \frac{\delta L^2(t,x)}{1 + \delta L^2(t,x)} |\vec{\gamma}(t,x)|^2\right] \Delta t$$
$$+ L(t,x)\vec{\gamma}(t,x) \cdot \Delta \vec{W}(t)$$

Existence, Uniqueness, and Regularity of the LIBOR Dynamics Solution

1. Uniqueness Statement: For all

$$x \geq 0$$

let

$$\left\{\vec{\xi}(t,x);\ t \geq 0\right\}$$

be an adapted, bounded stochastic process with values in $\mathbb{R}^d$,

$$\vec{a}(\cdot, x): \mathbb{R}_+ \to \mathbb{R}^d$$

be a deterministic and piece-wise continuous function, and let

$$M(t,x) = \int_0^t \vec{a}(s,x) \cdot d\vec{W}(s)$$

For all

$$x \geq 0$$

the equation

$$\Delta y(t,x) = y(t,x)\vec{a}(t,x) \cdot \left\{ \left[\frac{\delta y(t,x)}{1 + \delta y(t,x)} \vec{a}(t,x) + \vec{\xi}(t,x) \right] \Delta t + \Delta \overrightarrow{W}(t) \right\} \; \forall \; y(0,x) > 0$$

where

$$\delta > 0$$

is a constant, has a unique and strictly positive solution on $\mathbb{R}_+$. Moreover, if for some

$$k \in \{0,1,2,\cdots\}$$

$$y(0,x) \in \mathbb{C}^k(\mathbb{R}_+)$$

for all

$$t \geq 0$$

$$\vec{a}(t,x), M(t,x), \vec{\xi}(t,x) \in \mathbb{C}^k(\mathbb{R}_+)$$

then

$$y(t,x) \in \mathbb{C}^k(\mathbb{R}_+) \; \forall \; t \geq 0$$

2. Proof:

a. Uniqueness => Since the RHS of the equation for $\Delta y(t, x)$ is locally Lipschitz-continuous with respect to y on $\mathbb{R} - \frac{1}{\delta}$ and Lipschitz-continuous on $\mathbb{R}_{+}$, there exists a unique (possibly exploding) strictly positive solution to $\Delta y(t, x)$.

b. Applying Ito Integral => By the Ito formula

$$y(t, x) = y(0, x) e^{\int_0^t \vec{a}(s,x) \cdot d\vec{W}(s) + \int_0^t \vec{a}(s,x) \cdot \left[\frac{\delta y(s,x)}{1+\delta y(s,x)} \vec{a}(s,x) + \vec{\xi}(s,x) - \frac{1}{2} \vec{a}(s,x)\right] ds}$$

for all

$$t < \tau = \inf\{t : y(t, x) = \infty, or\ y(t, x) = 0\}$$

But if

$$y(t, x) = 0$$

for some

$$t < \infty$$

then

$$y(s, x) = 0$$

for all

$$s \geq t$$

and hence

$$\tau = \inf\{t : y(t, x) = \infty\}$$

Moreover, because

$$\int_0^t |\vec{a}(s,x)|^2 ds < \infty$$

for all

$$t < \infty$$

we deduce that

$$\tau = \infty$$

c. Ito Integral to the Volterra Integral Form => Thus the Ito integral form for $y(t,x)$ above is equivalent to the following Volterra-type integral equation for

$$l(t,x) = \log y(t,x)$$

$$l(t,x) = l(0,x) + \int_0^t \vec{a}(s,x) \cdot d\vec{W}(s)$$
$$+ \int_0^t \vec{a}(s,s) \cdot \left[\frac{\delta e^{l(s,x)}}{1 + \delta e^{l(s,x)}} \vec{a}(s,x) + \vec{\xi}(s,x) - \frac{1}{2}\vec{a}(s,x)\right] ds$$

d. Lipschitz-continuity of the RHS => Because the RHS in the Volterra-type integral equation is globally Lipschitz continuous with respect to l, we deduce using the standard fixed-point arguments that exists a unique path-wise solution to the Volterra-type integral equation above. Moreover, for any

$$t \geq 0$$

$$l(t,x) \in \mathbb{C}^k(\mathbb{R}_+)$$

provided

$$l(0,x), \vec{a}(t,x), \vec{\xi}(t,x) \in \mathbb{C}^k(\mathbb{R}_+) \ \forall \ t \geq 0$$

3. Smoothness of the Solution - Statement: Let

$$\vec{\gamma}: \mathbb{R}_+{}^2 \to \mathbb{R}^d$$

be a deterministic, bounded, and piecewise continuous function

$$\delta > 0$$

be a constant, and let

$$M(t,x) = \int_0^t \vec{\gamma}(s, x+t-s) \cdot d\overrightarrow{W}(s)$$

Then the equation

$$\Delta L(t,x) = \left[\frac{\partial}{\partial x} L(t,x) + L(t,x)\vec{\gamma}(t,x) \cdot \vec{\sigma}(t,x) + \frac{\delta L^2(t,x)}{1+\delta L^2(t,x)} |\vec{\gamma}(t,x)|^2\right] \Delta t + L(t,x)\vec{\gamma}(t,x) \cdot \Delta\overrightarrow{W}(t)$$

admits a unique non-negative solution $L(t,x)$ for any

$$t \geq 0$$

and any non-negative initial condition

$$L(0, x) = L_0$$

If

$$L_0 > 0$$

then

$$L(t, x) > 0 \; \forall \; t > 0$$

If

$$L_0 \in \mathbb{C}^k(\mathbb{R}_+) \; \forall \; t \geq 0$$

$$\vec{\gamma}(t, x) \in \mathbb{C}^k(\mathbb{R}_+)$$

$$M(t, x) \in \mathbb{C}^k(\mathbb{R}_+)$$

$$\left. \frac{\partial^j}{\partial x^j} \vec{\gamma}(t, x) \right|_{x=0} = 0 \; \forall \; j = 0, 1, \dots, k$$

then

$$L(t, x) \in \mathbb{C}^k(\mathbb{R}_+) \; \forall \; t \geq 0$$

4. <u>Proof</u>:
    a. Mild Solution to $\Delta L(t, x)$ => By the solution to $\Delta L(t, x)$ we mean the so-called mild solution (Da Prato and Zabczyk (1992)), i.e., $L(t, x)$ is a solution if

$$L(t,x) = L(0,x+t) + \int_0^t L(s,x+t-s)\vec{\gamma}(s,x+t-s)\cdot\vec{\sigma}(s,x+t-s)ds$$

$$+\int_0^t \frac{\delta L^2(s,x+t-s)}{1+\delta L(s,x+t-s)}|\vec{\gamma}(s,x+t-s)|^2 ds$$

$$+\int_0^t L(s,x+t-s)\vec{\gamma}(s,x+t-s)\cdot d\vec{W}(s)\ \forall\, x,t \geq 0$$

b. Validity of the Solution to $0 \leq t \leq x$ => The above integral form holds true for

$$0 \leq x < \delta$$

because the process

$$L(t,x-t), 0 \leq t \leq x, x > 0$$

is a solution to

$$\Delta y(t,x) = y(t,x)\vec{a}(t,x)\cdot\left\{\left[\frac{\delta y(t,x)}{1+\delta y(t,x)}\vec{a}(t,x)+\vec{\xi}(t,x)\right]\Delta t + \Delta\vec{W}(t)\right\}$$

$$y(0,x) > 0$$

with

$$\vec{a}(t,x) = \vec{\gamma}(t,(x-t)\vee 0)$$

and

$$\vec{\xi}(t,x) = 0$$

c. Validity for $\delta \leq x \leq 2\delta$ => For

$$\delta \leq x \leq 2\delta$$

the process

$$L(t, x - t), 0 \leq t \leq x$$

satisfies the solution to $\Delta y(t, x)$ with

$$\vec{a}(t, x) = \vec{\gamma}(t, (x - t) \vee 0)$$

and

$$\vec{\xi}(t, x) = \frac{\delta L(t, (x - \delta - t) \vee 0)}{1 + \delta L(t, (x - \delta - t) \vee 0)}$$

Thus by induction, we prove that $\Delta y(t, x)$ admits a unique solution for any

$$x > 0$$

and

$$0 \leq t \leq x$$

d. Using Induction to Complete the Smoothness Proof => Further by induction, from the recursion relation for $\vec{\sigma}(t, x)$, we deduce that the corresponding $\vec{a}(t, x)$ and $\vec{\xi}(t, x)$ satisfy the assumptions of regularity in the statement for uniqueness and existence, and hence $L(t, x)$ is smooth as well.

5. Application of the above Result to $r(t, x)$: If for some

$$k \in \mathbb{N}$$

and all

$$t \geq 0$$

$$\vec{\gamma}(t, x) \in \mathbb{C}^{k}(\mathbb{R}_{+})$$

and

$$\frac{\partial^{j}}{\partial x^{j}} \vec{\gamma}(t, x)\bigg|_{x=0} = 0 \;\; j = 0, 1, \dots, k$$

then

$$\Delta r(t, x) = \frac{\partial}{\partial x}\left[\left\{r(t, x) + \frac{1}{2}|\vec{\sigma}(t, x)|^{2}\right\}\Delta t + \vec{\sigma}(t, x) \cdot \Delta\overrightarrow{W}(t)\right]$$

has a unique solution

$$r(t, x) \in \mathbb{C}^{k-1}(\mathbb{R}_{+})$$

for any positive initial condition

$$r(0, x) \in \mathbb{C}^{k-1}(\mathbb{R}_{+})$$

6. <u>Proof of Smooth, Unique Solution to $r(t, x)$</u>: Consider the $\Delta r(t, x)$ evolution equation above with the fixed volatility process $\vec{\sigma}(t, x)$ given by

$$\vec{\sigma}(t, x) = \sum_{k=1}^{\left\lfloor\frac{x}{\delta}\right\rfloor} \frac{\delta L(t, x - k\delta)}{1 + \delta L(t, x - k\delta)} \vec{\gamma}(t, x - k\delta)$$

$$\Delta L(t,x) = \left[\frac{\partial}{\partial x}L(t,x) + L(t,x)\vec{\gamma}(t,x)\cdot\vec{\sigma}(t,x) + \frac{\delta L^2(t,x)}{1+\delta L^2(t,x)}|\vec{\gamma}(t,x)|^2\right]\Delta t$$
$$+ L(t,x)\vec{\gamma}(t,x)\cdot\Delta\vec{W}(t)$$

Applying the above result, the proof follows.

7. Non-smooth $\vec{\gamma}(t,x)$: The volatility $\vec{\sigma}(t,x)$ given by the recurrence relation may not be differentiable with respect to x for some functions $\vec{\gamma}$ (for example, piece-wise constant with respect to x). In such cases, the term structure dynamics cannot be analyzed in the HJM framework of

$$\Delta r(t,x) = \frac{\partial}{\partial x}\left[\left\{r(t,x) + \frac{1}{2}|\vec{\sigma}(t,x)|^2\right\}\Delta t + \vec{\sigma}(t,x)\cdot\Delta\vec{W}(t)\right]$$

8. Validity of the Savings Account Numeraire No-Arbitrage: The difficulty presented above, however, is just technical, since the relation

$$\frac{P(t,T)}{\beta(t)} = P(0,T)e^{-\int_0^t \vec{\sigma}(s,T-s)\cdot d\vec{W}(s) - \frac{1}{2}\int_0^t |\vec{\sigma}(s,T-s)|^2 ds}$$

is still sufficient to eliminate arbitrage. By setting

$$T = t$$

we may also define the savings account numeraire purely in terms of the price volatility $\vec{\sigma}(t,x)$.

9. Recurrence for the Spot Account Numeraire: It is also easy to see that

$$P(t,t+\delta) = \beta(t)P(0,t+\delta)e^{-\int_0^t \vec{\sigma}(s,t+\delta-s)\cdot d\vec{W}(s) - \frac{1}{2}\int_0^t |\vec{\sigma}(s,t+\delta-s)|^2 ds}$$
$$= \beta(t)P(0,t+\delta)e^{-\int_0^{t+\delta} \vec{\sigma}(s,t+\delta-s)\cdot d\vec{W}(s) - \frac{1}{2}\int_0^{t+\delta} |\vec{\sigma}(s,t+\delta-s)|^2 ds} = \frac{\beta(t)}{\beta(t+\delta)}$$

since

$$\vec{\sigma}(t, x) = 0$$

for

$$0 \leq x < \delta$$

10. <u>Establishment of the No-Arbitrage Criterion</u>: Solving the recurrence relationship

$$\frac{\beta(t)}{P(t, t + \delta)} = \beta(t + \delta)$$

we get

$$\beta(t) = \prod_{k=0}^{\left\lfloor\frac{t}{\delta}\right\rfloor} \frac{1}{P(\{t - [k + 1]\delta\}^{+}, t - k\delta)}$$

Thus, the zero coupon prices

$$\{P(t, T); 0 \leq t \leq T\}$$

discounted by

$$\{\beta(t); t \geq 0\}$$

satisfy the evolution dynamics for $\Delta r(t, x)$, and hence there is no arbitrage.

Upper/Lower Bounds for the LIBOR Rate

1. Non-Markovian Nature of all Rates: The regularity of $\vec{\gamma}$ has an important influence on the short rate $r(t,0)$ dynamics. If the process

$$\{r(t,0); t \geq 0\}$$

is a semi-martingale, then it satisfies

$$\Delta r(t,0) = \frac{\partial}{\partial x} r(t,x)\Big|_{x=0} \Delta t$$

Thus, the short rate is a process of finite variation, and therefore it cannot be strong Markov, except in the deterministic case (Cinlar and Jacod (1981)). The LIBOR process

$$\{L(t,0); t \geq 0\}$$

satisfies an equivalent $\Delta L(t,0)$ relation as well.

2. Bounding $|\vec{\gamma}(s, x+t-s) \cdot \vec{\sigma}(s, x+t-s)|$: It follows from the existence/uniqueness criterion and the relation

$$\Delta y(t,x) = y(t,x)\vec{a}(t,x) \cdot \left\{\left[\frac{\delta y(t,x)}{1+\delta y(t,x)}\vec{a}(t,x) + \vec{\xi}(t,x)\right]\Delta t + \Delta\overrightarrow{W}(t)\right\}; \quad y(0,x) > 0; \quad \delta > 0$$

that the process

$$\{L(t,x); t,x \geq 0\}$$

satisfies

$L(t, x)$

$$= L(0, x+t)e^{\int_0^t \vec{\gamma}(s,x+t-s)\cdot d\vec{W}(s)+\int_0^t \vec{\gamma}(s,x+t-s)\cdot\left[\frac{\delta L(s,x+t-s)}{1+\delta L(s,x+t-s)}\vec{\gamma}(s,x+t-s)+\vec{\sigma}(s,x+t-s)-\frac{1}{2}\vec{\gamma}(s,x+t-s)\right]ds}$$

and

$$|\vec{\gamma}(s, x+t-s)\cdot\vec{\sigma}(s, x+t-s)| \le \sum_{k=1}^{\left\lfloor\frac{x+t-s}{\delta}\right\rfloor} |\vec{\gamma}(s, x+t-s)||\vec{\gamma}(s, x+t-s-k\delta)|$$

3. <u>Explicit Lower and Upper Bounds for $L(t, x)$</u>: Therefore

$$L_1(t, x) \le L(t, x) \le L_2(t, x)$$

where

$$L_1(t, x) = L(0, x+t)e^{\int_0^t \vec{\gamma}(s,x+t-s)\cdot d\vec{W}(s)-\int_0^t\left[\alpha(s,x+t-s)+\frac{1}{2}|\vec{\gamma}(s,x+t-s)|^2\right]ds}$$

and

$$L_2(t, x) = L(0, x+t)e^{\int_0^t \vec{\gamma}(s,x+t-s)\cdot d\vec{W}(s)+\int_0^t\left[\alpha(s,x+t-s)+\frac{1}{2}|\vec{\gamma}(s,x+t-s)|^2\right]ds}$$

where

$$\alpha(s, x+t-s) = \sum_{k=1}^{\left\lfloor\frac{x+t-s}{\delta}\right\rfloor} |\vec{\gamma}(s, x+t-s)||\vec{\gamma}(s, x+t-s-k\delta)|$$

4. <u>Application of the Bound above</u>: Consequently, the LIBOR rate is bounded from below and above by log-normal processes. The estimate from the above can be used to show that the Euro-dollar futures price is well-defined. The most common Euro-dollar contract relates to

the LIBOR rate. The futures payoff at time T is equal to $\delta L(T, 0)$, and hence the Euro-dollar futures price at time

$$t \le T$$

is

$$\mathbb{E}[\delta L(T, 0)|\mathcal{F}_t]$$

Because

$$L(T, 0) \le L_2(T, 0)$$

and

$$\mathbb{E}[\delta L_2(T, 0)] = L(T, 0)e^{\int_0^T [\alpha(s, T-s) + |\vec{\gamma}(s, T-s)|^2] ds} < \infty$$

we conclude that the expectation is finite.

5. Stochastic System of Equations for Rates: For

$$n = 1, 2, \cdots$$

and

$$t \ge 0$$

define

$$y_n(t) = L(t, [n\delta - t \ \vee 0]), \vec{\gamma}_n(t) = \vec{\gamma}(t, [n\delta = t \ \vee 0])$$

and assume that

$$\vec{\gamma}(t, 0) = 0$$

It easily follows that the processes satisfy the following closed system of stochastic equations

$$\Delta y_n(t) = y_n(t)\vec{\gamma}_n(t) \cdot \left[\sum_{j=\left\lfloor \frac{t}{\delta} \right\rfloor + 1}^{n} \frac{\delta y_j(t)}{1 + \delta y_j(t)} \vec{\gamma}_j(t)\Delta t + \Delta\overrightarrow{W}(t) \right]$$

6. Mean Reversion of Rates: We now examine whether the model for $y_n(t)$ above implies mean-reverting behavior. For this, we assume the following;

$$|\vec{\gamma}(t, x)| \leq \beta(x)$$

where

$$\underset{0 \leq x \leq \delta}{sup} \sum_{k=0}^{\infty} \beta(x + k\delta) < \infty$$

wnd

$$\int_{0}^{\infty} (x + 1)\beta^2(x) dx < \infty$$

7. Bounding the p^{th} Raw Moment of LIBOR: Under the $|\vec{\gamma}(t, x)|$ bounding conditions above, for any

$$p \geq 1$$

and any deterministic initial condition

$$L(0,x) \in \mathbb{C}_b(\mathbb{R}_+)$$

we have

$$\sup_{t \geq 0} \sup_{x \geq 0} \mathbb{E}[L^p(t,x)] < \infty$$

a. Proof => We use α and L_2 as defined above. By the $|\vec{\gamma}(t,x)|$ bounding conditions,

$$\sup_{t \geq 0} \sup_{x \geq 0} \int_0^t \left[\alpha(s, x+t-s) + \frac{1}{2}|\vec{\gamma}(s, x+t-s)|^2\right] ds < \infty$$

and

$$\mathbb{E}\left[\left\{\int_0^t \vec{\gamma}(s, x+t-s) \cdot d\vec{W}(s)\right\}^2\right] \leq \int_0^\infty \beta^2(x+s)dx \leq \int_0^\infty \beta^2(x)dx < \infty$$

Since $\log L_2$ is Gaussian,

$$\sup_{t \geq 0} \sup_{x \geq 0} \mathbb{E}[{L_2}^p(t,x)] < \infty$$

for any

$$p \geq 1$$

Since

$$L \leq L_2$$

$$\sup_{t \geq 0} \sup_{x \geq 0} \mathbb{E}[L^p(t,x)] < \infty$$

Invariant Measure for the LIBOR Rate

1. Additional Bounding Assumptions: Additionally, we assume

$$\vec{\gamma}(t,x) = \vec{\gamma}(x)$$

$$\int_0^\infty x \left|\frac{\partial \vec{\gamma}(x)}{\partial x}\right|^2 dx < \infty$$

$$\sup_{0 \leq x \leq \delta} \sum_{k=0}^{\infty} \left|\frac{\partial \vec{\gamma}(x + k\delta)}{\partial x}\right| < \infty$$

$$\int_0^\infty |\vec{\gamma}(x)| dx = C < \frac{1}{K}$$

2. Existence of an Invariant Measure - Approach: The assumption

$$\vec{\gamma}(t,x) = \vec{\gamma}(x)$$

   implies that L is a time-homogenous Markov process. Hence we can examine the notion of invariant measures. The proof of the existence of an invariant measure will follow the standard Krylov-Bogoliubov scheme – the Feller property and the tightness of the family of distributions $\mathcal{L}[L(t)]_{t \geq 0}$ implies the existence of an invariant measure (Da Prato and Zabczyk (1992)).
3. Setup and Definitions: Let

$$C_0(\mathbb{R}) = \{u \in C(\mathbb{R}) : u(x) \to 0 \; as \; x \to \infty\}$$

and let

$$C^\alpha(\mathbb{R}) = \{u \in C(\mathbb{R}) : |u(x) - u(z)| \leq C|x - z|^\alpha\}$$

for any

$$0 < \alpha \leq 1$$

We represent the Holder norm in $C^\alpha(\mathbb{R})$ by $\|\cdot\|_\alpha$.

4. Relative Compactness Criteria: A family of functions

$$\Gamma \subset C_0(\mathbb{R}_+)$$

is relatively compact in $C_0(\mathbb{R}_+)$ if and only if the following conditions are satisfied:

   a. The family Γ is equi-continuous on any bounded set;
   b. There exists a function

$$R : \mathbb{R}_+ \to \mathbb{R}_+$$

such that

$$R(u) \to 0$$

as

$$u \to \infty$$

and

$$|f(u)| \leq R(u)$$

for any

$$f \in \Gamma$$

and

$$u \geq 0$$

5. <u>Existence of a Concentrated/Tight Invariant Measure - Statement</u>: We use the additional bounds above, and let $L(t, x)$ be the solution to

$$\Delta L(t,x) = \left[\frac{\partial}{\partial x}L(t,x) + L(t,x)\vec{\gamma}(t,x)\cdot\vec{\sigma}(t,x) + \frac{\delta L^2(t,x)}{1+\delta L^2(t,x)}|\vec{\gamma}(t,x)|^2\right]\Delta t + L(t,x)\vec{\gamma}(t,x)\cdot\Delta\overrightarrow{W}(t)$$

and

$$\sup_{0 \leq x < \infty} |\log L(0,x)| < \infty$$

Then

$$\sup_{t \geq 0} \mathbb{E}[\|\log L(t)\|] < \infty$$

If, moreover,

$$\int_0^\infty |\vec{\gamma}(x)|dx = C < \frac{1}{K}$$

is satisfied, then there exists an invariant measure for the process $L(t,x)$ concentrated on the closed set

$$U = \{u \in C(\mathbb{R}) : u > 0, and\ u(x) \to 0\ as\ x \to \infty\}$$

6. <u>Proof</u>:
   a. Step #1 - Log Representation of $L(t,x)$ => Consider the process

$$l(t,x) = \log L(t,x)$$

which can be represented as

$$l(t,x) = l_0(x+t) + \int_0^t \vec{F}\big(l(t-s)\big)(x+t-s)\cdot\vec{\gamma}(x+t-s)ds$$
$$-\frac{1}{2}\int_0^t |\vec{\gamma}(x+t-s)|^2 ds + M(t,x)$$

for any

$$t \geq 0$$

where M is defined by

$$M(t,x) = \int_0^t \vec{\gamma}(x+t-s)\cdot d\overrightarrow{W}(s)$$

and

$$\vec{F}(l)(x) = \sum_{k=0}^{\left\lfloor \frac{x}{\delta} \right\rfloor} \frac{\delta e^{l(x-k\delta)}}{1 + \delta e^{l(x-k\delta)}} \vec{\gamma}(x - k\delta)$$

for any

$$l \in C(\mathbb{R})$$

b. Step #2 - l as a Feller Process => Using

$$\underset{0 \leq x \leq \delta}{sup} \sum_{k=0}^{\infty} \beta(x + k\delta) < \infty$$

we can see that

$$\vec{\gamma} \cdot \vec{F} : \ C_0(\mathbb{R}_+) \to C_0(\mathbb{R}_+)$$

is a Lipschitz-continuous function. By the standard fixed-point method, $l(t)$ depends continuously on the initial condition in the space $C_0(\mathbb{R}_+)$. Therefore the process l is a Feller process.

c. Step #3 - Bounding $\{M(t,x)\}^2$ and $\left[\frac{\partial M(t,x)}{\partial x}\right]^2$ => Notice that

$$\frac{\partial M(t,x)}{\partial x} = \int_0^t \frac{\partial \vec{\gamma}(x + t - s)}{\partial x} \cdot d\vec{W}(s)$$

By the Ito formula, we have

$$\mathbb{E}\left[\int_0^{\infty} \{M(t,x)\}^2 dx\right] \leq \int_0^{\infty} \int_0^{\infty} |\vec{\gamma}(x+s)|^2 dx\, ds = \sqrt{2} \int_0^{\infty} x|\vec{\gamma}(x)|^2 dx$$

and

$$\mathbb{E}\left[\int_0^\infty \left\{\frac{\partial M(t,x)}{\partial x}\right\}^2 dx\right] \le \int_0^\infty \int_0^\infty \left|\frac{\partial \vec{\gamma}(x+s)}{\partial x}\right|^2 dx\, ds = \sqrt{2}\int_0^\infty x\left|\frac{\partial \vec{\gamma}(x)}{\partial x}\right|^2 dx$$

d. Step #4 - Bounding $\underset{x \ge u}{sup}\{M(t,x)\}^2$ => Using the expressions for $\mathbb{E}\left[\int_0^\infty \{M(t,x)\}^2 dx\right]$ and $\mathbb{E}\left[\int_0^\infty \left\{\frac{\partial M(t,x)}{\partial x}\right\}^2 dx\right]$, using Sobolev embedding along with the additional bounds above, we get

$$\mathbb{E}\left[\underset{x \ge u}{sup}\{M(t,x)\}^2\right] \le C_1\left\{\int_0^\infty \int_0^\infty |\vec{\gamma}(x+s)|^2 dx\, ds + \int_0^\infty \int_0^\infty \left|\frac{\partial \vec{\gamma}(x+s)}{\partial x}\right|^2 dx\, ds\right\}$$
$$\le R(u) < \infty$$

and for

$$\alpha < \frac{1}{2}$$

$$\mathbb{E}\left[\|M(t,x)\|_\alpha{}^2\right] \le C_1\left\{\int_0^\infty \int_0^\infty |\vec{\gamma}(x+s)|^2 dx\, ds + \int_0^\infty \int_0^\infty \left|\frac{\partial \vec{\gamma}(x+s)}{\partial x}\right|^2 dx\, ds\right\} \le R(u)$$
$$< \infty$$

where R and $R(u)$ are independent of t, and

$$R(u) \to 0$$

as

$$u \to \infty$$

e. Step #5 - Bounding $\mathbb{E}[\|l(t)\|]$ => From

$$l(t,x) = l_0(x+t) + \int_0^t \vec{F}\big(l(t-s)\big)(x+t-s)\cdot\vec{\gamma}(x+t-s)ds$$
$$-\frac{1}{2}\int_0^t |\vec{\gamma}(x+t-s)|^2 ds + M(t,x)$$

for any

$$t \geq 0$$

we get

$$\mathbb{E}[\|l(t)\|] \leq \|l(0)\| + K\int_0^\infty |\vec{\gamma}(x)|dx + \frac{1}{2}\int_0^\infty |\vec{\gamma}(x)|^2 dx + \mathbb{E}\left[\sup_{x \geq 0} |M(t,x)|\right]$$

Using the bounds for

$$\sup_{t \geq 0} \mathbb{E}\left[\sup_{x \geq 0} |M(t,x)|\right]$$

above, we see that

$$\sup_{t \geq 0} \mathbb{E}[\|l(t)\|] < \infty$$

f. Step #6 - Approach for Proving Tightness of $\mathbb{E}[l(t)]_{t\geq 0}$ => From

$$\int_0^\infty |\vec{\gamma}(x)|dx = C < \frac{1}{K}$$

and assuming that

$$l_0 = 0$$

in order to prove the existence of an invariant measure for the process l, we will need to prove that the family of laws $\mathbb{E}[l(t)]_{t\geq 0}$ is tight.

g. Step #7 - Bounding $|l(t,x)|_{l_1}$ Unconditionally => Again from the expression for $l(t,x)$, we get

$$\begin{matrix} sup \\ t \geq 0 \end{matrix} \mathbb{E}\left[\begin{matrix} sup \\ x \geq u \end{matrix} |l(t,x)|\right] \leq K \int_u^\infty |\vec{\gamma}(x)| dx + \frac{1}{2} \int_u^\infty |\vec{\gamma}(x)|^2 dx + \mathbb{E}\left[\begin{matrix} sup \\ x \geq u \end{matrix} |M(t,x)|\right] \rightarrow 0$$

as

$$u \rightarrow \infty$$

h. Step #8 - Lipschitz Criterion on $\vec{F}(\psi)$ => From the Lipschitz criterion listed earlier, we can see that for any

$$\psi \in C(\mathbb{R})$$

$$\frac{\left|\vec{F}(\psi)(x) - \vec{F}(\psi)(u)\right|}{|x-u|^\alpha} \leq C_1 + CK \begin{matrix} sup \\ x, u \geq 0 \end{matrix} \frac{|\psi(x) - \psi(u)|}{|x-u|^\alpha}$$

for a certain constant C_1.

i. Step #9 - Explicit Bounds for $\mathbb{E}[\|l(t)\|_\alpha]$ => From

$$\int_0^\infty (x+1)\beta^2(x)dx < \infty$$

$$\int_0^\infty x\left|\frac{\partial\vec{\gamma}(x)}{\partial x}\right|^2 dx < \infty$$

$$\int_0^\infty |\vec{\gamma}(x)|dx = C < \frac{1}{K}$$

we get

$$\sup_{t \geq 0} \mathbb{E}[\|l(t)\|_\alpha] \leq C_1 + CK \sup_{t \geq 0} \mathbb{E}[\|l(t)\|_\alpha]$$

Since

$$CK < 1$$

we see that

$$\sup_{t \geq 0} \mathbb{E}[\|l(t)\|_\alpha] \leq \frac{C_2}{1 - CK}$$

j. Step #10 - Proof that the Family is Tight => Using

$$\sup_{t \geq 0} \mathbb{E}\left[\sup_{x \geq u} |l(t,x)|\right] \leq K\int_u^\infty |\vec{\gamma}(x)|dx + \frac{1}{2}\int_u^\infty |\vec{\gamma}(x)|^2 dx + \mathbb{E}\left[\sup_{x \geq u} |M(t,x)|\right]$$
$$\to 0$$

as

$$u \to \infty$$

and

$$\sup_{t \geq 0} \mathbb{E}[\|l(t)\|_\alpha] \leq \frac{C_2}{1 - CK}$$

since

$$CK < 1$$

as well as the relative compactness criterion, we can see that the family $\mathcal{L}[L(t)]_{t \geq 0}$ is tight on $C_0(\mathbb{R})$.

k. Step #11 - Existence of Invariant Measures on $L(t, x)$ => Since $l(t, x)$ is a Feller process, by the standard Krylov-Bogoliubov technique there exists an invariant measure for the process l, concentrated on $C_0(\mathbb{R})$. Existence of invariant measures for l on $C_0(\mathbb{R})$ is equivalent to existence of invariant measures for L on U.

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Application of BGM to Derivatives Pricing

Cap/Floor Pricing

1. Payer Forward Swap Fixing: Consider a payer forward swap on principal \$1 settled quarterly in arrears at times

$$T_j = T_0 + j\delta \; \forall \; j = 1, \cdots, n$$

The LIBOR rate received at T_j is set at T_{j-1} at the level

$$L(T_{j-1}, 0) = \frac{1}{\delta}\left[\frac{1}{P(T_{j-1} - T_j)} - 1\right]$$

2. Payer Forward Swap Cash-flows and Pricing: The swap cash flows at $T_j, j = 1, \cdots, n$ are $\delta L(T_{j-1}, 0)$ and $-\kappa\delta$ and hence the time $t(t \leq T_0)$ value of the swap is (Brace and Musiela (1994b))

$$\mathbb{E}\left[\sum_{j=1}^{n} \frac{\beta(t)}{\beta(T_j)}\{L(T_{j-1}, 0) - \kappa\}\delta|\mathcal{F}_t\right] = P(t - T_0) - \sum_{j=1}^{n} C_j P(t - T_j)$$

where

$$C_j = \kappa\delta$$

for $j = 1, \cdots, n - 1$ and

$$C_n = 1 + \kappa\delta$$

3. The Par Forward Swap Rate: The forward swap rate $\omega_{T_0}(t, n)$ at time t for the forward/futures maturity/expiry T_0 is that value of the fixed rate κ which makes the value of the forward swap zero, i.e.,

$$\omega_{T_0}(t, n) = \frac{P(t, T_0) - P(t, T_n)}{\delta \sum_{j=1}^{n} P(t, T_j)}$$

4. Cap/Floor Pricing Formulation: In a forward cap (resp. floor) on principal \$1 settled in arrears at times T_j, $j = 1, \cdots, n$, the cash flows at times T_j are $\delta\{L(T_{j-1}, 0) - \kappa\}^+$ (resp. $\delta\{\kappa - L(T_{j-1}, 0)\}^+$). The cap price at time $t \le T_0$ is

$$Cap(t) = \mathbb{E}\left[\sum_{j=1}^{n} \frac{\beta(t)}{\beta(T_j)}\{L(T_{j-1}, 0) - \kappa\}^+ \delta | \mathcal{F}_t\right] = \sum_{j=1}^{n} P(t, T_j)\, \mathbb{E}_{T_j}\left[\{L(T_{j-1}, 0) - \kappa\}^+ \delta | \mathcal{F}_t\right]$$

where $\mathbb{E}_T$ stands for the expectation under the forward measure $\mathbb{P}_T$ defined by (Musiela (1995))

$$\mathbb{P}_T = \mathbb{P}_0 e^{-\int_0^t \vec{\sigma}(s, T-s) \cdot d\overrightarrow{W}(s) - \frac{1}{2}\int_0^t |\vec{\sigma}(s, T-s)|^2 ds} = \frac{\mathbb{P}_0}{P(0, T)\beta(t)}$$

5. The LIBOR Rate Process Equation: The LIBOR rate process equation

$$K(t, T) = L(t, T - t), 0 \le t \le T$$

satisfies

$$\Delta K(t,T) = K(t,T)\vec{\gamma}(t,T-t)\cdot\left\{\left[\vec{\sigma}(t,T-t)+\frac{\delta K(t,T)}{1+\delta K(t,T)}\vec{\gamma}(t,T-t)\right]\Delta t+\Delta\vec{W}(t)\right\}$$
$$= K(t,T)\vec{\gamma}(t,T-t)\cdot\left[\vec{\sigma}(t,T+\delta-t)\Delta t+\Delta\vec{W}(t)\right]$$

6. <u>$K(t,T)$ Under the Forward Measure</u>: The process

$$\overrightarrow{W_T}(t) = \vec{W}(t)+\int_0^t \vec{\sigma}(s,T-s)\cdot d\vec{W}(s)$$

is a Brownian motion under $\mathbb{P}_T$. Consequently

$$\Delta K(t,T) = K(t,T)\vec{\gamma}(t,T-t)\cdot\overrightarrow{W_{T+\delta}}(t)$$

and hence $K(t,T)$ is log-normally distributed under $\mathbb{P}_{T+\delta}$.

7. <u>The Caplet Pricing Relation</u>: From the above, it follows that

$$\mathbb{E}_{T+\delta}[\{L(T,0)-K\}^+\delta|\mathcal{F}_t] = \mathbb{E}_{T+\delta}[\{K(T,T)-K\}^+\delta|\mathcal{F}_t]$$
$$= K(t,T)N[h(t,T)]-KN[h(t,T)-\varsigma(t,T)]$$

where

$$h(t,T) = \frac{\log\frac{K(t,T)}{K}+\frac{1}{2}\varsigma^2(t,T)}{\varsigma(t,T)}$$

and

$$\varsigma^2(t,T) = \int_t^T |\vec{\gamma}(s,T-s)|^2 ds$$

8. <u>The Cap Pricing Relation</u>: The cap price at time $t \le T_0$ is

$$Cap(t) = \sum_{j=1}^{n} \delta P(t, T_j)\{K(t, T_{j-1})N[h(t, T_{j-1})] - KN[h(t, T_{j-1}) - \varsigma(t, T_{j-1})]\}$$

9. Comparison of the Caplet Price with Black's Formula: The preceding $Cap(t)$ formula corresponds to the market Black futures formula with discount from settlement date. It was originally derived using a different approach and model setup by Miltersen, Sandmann, and Sondermann (1994).

Payer Swap Option Pricing

1. Payer Swap Option Details Recap: The payer swap option at strike κ maturing at T_0 gives the right to receive at T_0 the cash flows corresponding to the payer swap settled in arrears, or alternately, discounted from the settlement dates

$$T_j = T_0 + j\delta \; \forall \; j = 1, \cdots, n$$

to T_0 the value of the cash flows defined by $\{\omega_{T_0}(t, n) - K\}^+$, where

$$\omega_{T_0}(T_0, n) = \frac{P(t, T_0) - P(t, T_n)}{\delta \sum_{j=1}^{n} P(t, T_j)}$$

2. Payer Swap Option Pricing: Hence at time $t \leq T_0$ the price of the option is

$$\mathbb{E}\left[\frac{\beta(t)}{\beta(T_0)}\mathbb{E}\left[\sum_{j=1}^{n}\frac{\beta(T_0)}{\beta(T_j)}\{\omega_{T_0}(T_0,n)-\kappa\}^{+}\delta|\mathcal{F}_{T_0}\right]|\mathcal{F}_t\right]$$

$$=\mathbb{E}\left[\frac{\beta(t)}{\beta(T_0)}\left\{1-\sum_{j=1}^{n}C_jP\left(T_0,T_j\right)\right\}^{+}|\mathcal{F}_t\right]$$

$$=\mathbb{E}\left[\frac{\beta(t)}{\beta(T_0)}\mathbb{E}\left[\sum_{j=1}^{n}\frac{\beta(T_0)}{\beta(T_j)}\left\{L\left(T_{j-1},0\right)-\kappa\right\}^{+}\delta|\mathcal{F}_{T_0}\right]|\mathcal{F}_t\right]$$

where

$$C_j=\kappa\delta$$

for $j=1,\cdots,n-1$ and

$$C_n=1+\kappa\delta$$

(Brace and Musiela (1994b)).

3. Spot Measure Option in-the-money Probability: Let

$$A=\left\{\omega_{T_0}(T_0,n)\geq\kappa\right\}$$

be the event that the swaption ends in the money. Then the payer swap option price is

$$P_{Swaption}(t) = \mathbb{E}\left[\frac{\beta(t)}{\beta(T_0)}\left\{1 - \sum_{j=1}^{n} C_j P\left(T_0, T_j\right)\right\}^{+} |\mathcal{F}_t\right]$$

$$= \mathbb{E}\left[\frac{\beta(t)}{\beta(T_0)}\left\{1 - \sum_{j=1}^{n} C_j P\left(T_0, T_j\right)\right\}\mathcal{I}_A|\mathcal{F}_t\right]$$

$$= \mathbb{E}\left[\frac{\beta(t)}{\beta(T_0)}\mathcal{I}_A|\mathcal{F}_t\right] - \mathbb{E}\left[\frac{\beta(t)}{\beta(T_0)}\left\{\sum_{j=1}^{n} C_j P\left(T_0, T_j\right)\right\}\mathcal{I}_A|\mathcal{F}_t\right]$$

$$= P(t, T_0)\mathbb{P}_{T_0}(A|\mathcal{F}_t) - \sum_{j=1}^{n} C_j \mathbb{E}\left[\frac{\beta(t)}{\beta(T_0)}\mathbb{E}\left[\frac{\beta(T_0)}{\beta(T_j)}|\mathcal{F}_{T_0}\right]\mathcal{I}_A|\mathcal{F}_t\right]$$

$$= P(t, T_0)\mathbb{P}_{T_0}(A|\mathcal{F}_t) - \sum_{j=1}^{n} C_j P\left(t, T_j\right)\mathbb{P}_{T_j}(A|\mathcal{F}_t)$$

4. Forward Measure Option in-the-money Probability: Also for all $j = 1, \cdots, n$,

$$P\left(t, T_{j-1}\right)\mathbb{P}_{T_{j-1}}(A|\mathcal{F}_t) = \mathbb{E}\left[\frac{\beta(T_0)}{\beta\left(T_{j-1}\right)}\mathcal{I}_A|\mathcal{F}_t\right] = \mathbb{E}\left[\frac{\beta(T_0)}{\beta\left(T_j\right)}\frac{1}{P\left(T_{j-1}, T_j\right)}\mathcal{I}_A|\mathcal{F}_t\right]$$

$$= \mathbb{E}\left[\frac{\beta(T_0)}{\beta\left(T_j\right)}\{1 + \delta K\left(T_{j-1}, T_{j-1}\right)\}\mathcal{I}_A|\mathcal{F}_t\right]$$

$$= P\left(t, T_j\right)\mathbb{P}_{T_j}(A|\mathcal{F}_t) + \delta P\left(t, T_j\right)\mathbb{E}\left[K\left(T_{j-1}, T_{j-1}\right)\mathcal{I}_A|\mathcal{F}_t\right]$$

$$= P\left(t, T_j\right)\mathbb{P}_{T_j}(A|\mathcal{F}_t) + \delta P\left(t, T_j\right)\mathbb{E}\left[K\left(T_0, T_{j-1}\right)\mathcal{I}_A|\mathcal{F}_t\right]$$

where the last equality holds because the process

$$\left\{K\left(t, T_{j-1}\right); 0 \le t \le T_{j-1}\right\}$$

is a martingale under the $\mathbb{P}_{T_j}$ measure, and the event A is $\mathcal{F}_{T_0}$-measurable.

5. Payer Swap Option Pricing Formula: The payer swap option price at time $t \le T_0$ is

$$P_{Swaption}(t) = \delta \sum_{j=1}^{n} P(t, T_j) \mathbb{E}_{T_j}\left[\{K(T_0, T_{j-1}) - \kappa\} \mathcal{I}_A | \mathcal{F}_t\right]$$

Payer Swap Option Pricing Simplification

1. <u>The Approach</u>: To simplify the payer swap option pricing formula, we need to first analyze the relationships between the forward measures $\mathbb{P}_{T_j}$ given from

$$\mathbb{P}_{T_j} = \mathbb{P}_0 e^{-\int_0^t \vec{\sigma}(s, T_j - s) \cdot d\vec{W}(s) - \frac{1}{2}\int_0^t \left|\vec{\sigma}(s, T_j - s)\right|^2 ds} = \frac{\mathbb{P}_0}{P(0, T_j)\beta(t)}$$

as well as the corresponding forward Brownian motions $\overrightarrow{W_T}(t)$ given by

$$\overrightarrow{W_T}(t) = \vec{W}(t) + \int_0^t \vec{\sigma}(s, T - s) \cdot d\vec{W}(s)$$

for $j = 1, 2, \cdots$.

2. <u>Consequent Period Measure Change</u>: We have

$$\Delta\overrightarrow{W_{T_j}}(t) = \Delta\vec{W}(t) + \vec{\sigma}(s, T_j - s)\Delta t = \Delta\overrightarrow{W_{T_{j-1}}}(t) + \left[\vec{\sigma}(t, T_j - t) - \vec{\sigma}(t, T_{j-1} - t)\right]\Delta t$$

$$= \Delta\overrightarrow{W_{T_{j-1}}}(t) + \frac{\delta K(t, T_{j-1})}{1 + \delta K(t, T_{j-1})} \vec{\gamma}(t, T_{j-1} - t)\Delta t$$

3. <u>Change in the LIBOR PV $\Delta\left[\frac{\delta K(t, T_{j-1})}{1 + \delta K(t, T_{j-1})}\right]$</u>: Also because the process

$$\{K(t, T_{j-1}); 0 \le t \le T_{j-1}\}$$

satisfies

$$\Delta K(t, T_{j-1}) = K(t, T_{j-1}) \vec{\gamma}(t, T_{j-1} - t) \cdot \Delta \overrightarrow{W_{T_j}}(t)$$

we have

$$\begin{aligned}
\Delta \left[\frac{\delta K(t, T_{j-1})}{1 + \delta K(t, T_{j-1})} \right] \\
&= \frac{\delta K(t, T_{j-1})}{\left[1 + \delta K(t, T_{j-1})\right]^2} \vec{\gamma}(t, T_{j-1} - t) \cdot \Delta \overrightarrow{W_{T_j}}(t) \\
&- \frac{\delta^2 K^2(t, T_{j-1})}{\left[1 + \delta K(t, T_{j-1})\right]^3} \left| \vec{\gamma}(t, T_{j-1} - t) \right|^2 \Delta t \\
&= \frac{\delta K(t, T_{j-1})}{\left[1 + \delta K(t, T_{j-1})\right]^2} \vec{\gamma}(t, T_{j-1} - t) \cdot \Delta \overrightarrow{W_{T_{j-1}}}(t)
\end{aligned}$$

and hence

$$\left\{ \frac{\delta K(t, T_{j-1})}{1 + \delta K(t, T_{j-1})} ; 0 \le t \le T_{j-1} \right\}$$

is a super-martingale under the measure $\mathbb{P}_{T_j}$ and a martingale under the measure $\mathbb{P}_{T_{j-1}}$.

4. Forward Bond Pricing: Let for $t \le T_0$

$$F_{T_0}(t, T_k) = \frac{P(t, T_k)}{P(t, T_0)}$$

denote the forward price at time t for settlement at time T_0 on a T_k maturity zero coupon bond.

5. Option in-the-money Probability in Terms of LIBOR: Because we have

$$F_{T_0}(t,T_k) = \prod_{i=1}^{k} F_{T_{i-1}}(t,T_i) = \frac{1}{\prod_{i=1}^{k}[1+\delta K(t,T_{i-1})]}$$

$$A = \left\{\sum_{k=1}^{n} C_k P(T_0,T_k) \leq 1\right\}$$

becomes

$$A = \left\{\sum_{k=1}^{n} \frac{C_k}{\prod_{i=1}^{k}[1+\delta K(t,T_{i-1})]} \leq 1\right\}$$
$$= \left\{\sum_{k=1}^{n} \frac{C_k}{\prod_{i=1}^{k}[1+\delta K(t,T_{i-1})]e^{\int_t^T \vec{\sigma}(s,T_{i-1}-s)\cdot d\overrightarrow{W_{T_i}}(s)+\frac{1}{2}\int_0^t |\vec{\sigma}(s,T_{i-1}-s)|^2 ds}} \leq 1\right\}$$

6. <u>Coupon Period Measure Change</u>: Using the period T_0 left pivot for both $i, j = 1, \cdots, n$, for $t \leq T_0$, we get

$$\Delta\overrightarrow{W_{T_i}}(t) = \Delta\overrightarrow{W_{T_j}}(t) + \sum_{l=0}^{i-1} \frac{\delta K(t,T_l)}{1+\delta K(t,T_l)}\vec{\gamma}(t,T_l-t)\,\Delta t - \sum_{l=0}^{j-1} \frac{\delta K(t,T_l)}{1+\delta K(t,T_l)}\vec{\gamma}(t,T_l-t)\,\Delta t$$

7. <u>Simplification of the $\Delta\overrightarrow{W_{T_i}}(t)$ Component of A under $\Delta\overrightarrow{W_{T_j}}(t)$</u>: Using the above expression for $\Delta\overrightarrow{W_{T_i}}(t)$, we can write

$$X_i = \int_t^T \vec{\gamma}(s, T_{i-1} - s) \cdot d\overrightarrow{W}_{T_i}(s)$$

$$= \int_t^T \vec{\gamma}(s, T_{i-1} - s) \cdot d\overrightarrow{W}_{T_j}(s)$$

$$+ \sum_{l=1}^{i} \int_t^T \frac{\delta K(t, T_{l-1})}{1 + \delta K(t, T_{l-1})} \vec{\gamma}(s, T_{i-1} - s) \cdot \vec{\gamma}(t, T_{l-1} - t) ds$$

$$- \sum_{l=1}^{j} \int_t^T \frac{\delta K(t, T_{l-1})}{1 + \delta K(t, T_{l-1})} \vec{\gamma}(s, T_{i-1} - s) \cdot \vec{\gamma}(t, T_{l-1} - t) ds$$

8. Approximation of X_i by ${X_i}^j$: We will approximate the conditional on $\mathcal{F}_t$ distribution of $X_1, \cdots, X_n$ under the measure $\mathbb{P}_{T_j}$ (for each $j = 1, \cdots, n$) by the distribution of the random vector ${X_1}^j, \cdots, {X_n}^j$ where

$$ {X_i}^j = \int_t^T \vec{\gamma}(s, T_{i-1} - s) \cdot d\overrightarrow{W}_{T_j}(s) + \sum_{l=1}^{i} \frac{\delta K(t, T_{l-1})}{1 + \delta K(t, T_{l-1})} \Delta_{li} - \sum_{l=1}^{j} \frac{\delta K(t, T_{l-1})}{1 + \delta K(t, T_{l-1})} \Delta_{li} $$

and

$$\Delta_{li} = \int_t^T \vec{\gamma}(s, T_{i-1} - s) \cdot \vec{\gamma}(t, T_{l-1} - t) ds$$

9. Approximation of the Change in LIBOR PV: In view of the expression for the change in LIBOR PV, i.e.,

$$\Delta\left[\frac{\delta K(s, T_l)}{1 + \delta K(s, T_l)}\right] = \frac{\delta K(s, T_l)}{[1 + \delta K(s, T_l)]^2} \vec{\gamma}(s, T_l - s) \cdot \Delta\overrightarrow{W}_{T_l}(s)$$

the approximations above correspond to Wiener chaos 0 approximation of the process

$$\frac{\delta K(s,T_l)}{1+\delta K(s,T_l)}, s \leq T_l$$

under the measure $\mathbb{P}_{T_l}$.

10. Higher Order Approximation of X_i: A more accurate approximation involving Wiener chaos of order 0 and 1 may be used as well. However, since the order 1 Wiener chaos contribution is not significant, so $K(s,T_l)$ can be replaced by its t value $K(t,T_l)$, or because of

$$\Delta\left[\frac{\delta K(t,T_l)}{1+\delta K(t,T_l)}\right] = \frac{\delta K(t,T_l)}{[1+\delta K(t,T_l)]^2}\vec{\gamma}(t,T_l - t)\cdot\Delta\overrightarrow{W}_{T_l}(t)$$

by the conditional expectation under $\mathbb{P}_{T_l}$ given $\mathcal{F}_t$.

11. Distribution of ${X_1}^j,\cdots,{X_n}^j$ under $\mathbb{P}_{T_t}$: The conditional-on-$\mathcal{F}_t$ distribution of ${X_1}^j,\cdots,{X_n}^j$ under the measure $\mathbb{P}_{T_j}$ is $N\left(\mu^j,\Delta\right)$, where

$${\mu_i}^j = \sum_{l=1}^{i}\frac{\delta K(t,T_{l-1})}{1+\delta K(t,T_{l-1})}\Delta_{li} - \sum_{l=1}^{j}\frac{\delta K(t,T_{l-1})}{1+\delta K(t,T_{l-1})}\Delta_{li}$$

12. Estimation of μ and Δ for the Distribution: In practice, the first eigenvalue for the matrix Δ is approximately 50 times larger than the second, and therefore we can assume that Δ is of rank 1, or equivalently

$$\Delta_{li} = \Gamma_l\Gamma_i$$

for some positive constants $\Gamma_1,\cdots,\Gamma_n$. Setting

$$\mathrm{d}_0 = 0$$

and for $i \geq 1$

$$\mathrm{d}_i = \sum_{l=1}^{i} \frac{\delta K(t, T_{l-1})}{1 + \delta K(t, T_{l-1})} \Gamma_l$$

it follows that

$$\mu_i{}^j = \Gamma_i(\mathrm{d}_i - \mathrm{d}_j)$$

13. <u>Solution to the Lower Cutoff for A</u>: For all $j = 1, \cdots, n$, the function

$$\mathrm{f}_j(x) = 1 - \sum_{k=1}^{n} \frac{C_k}{\prod_{i=1}^{k} [1 + \delta K(t, T_{i-1})] e^{\Gamma_i(x + \mathrm{d}_i - \mathrm{d}_j) - \frac{1}{2}\Gamma_i{}^2}}$$

satisfies

$$\mathrm{f}_j{}'(x) > 0$$

$$\mathrm{f}_j(-\infty) = -n\delta\kappa$$

and

$$\mathrm{f}_j(\infty) = 1$$

Hence there is a unique point s_j such that

$$\mathrm{f}_j(\mathrm{s}_j) = 0$$

Moreover, if s_0 is the solution with $j = 0$, clearly

$$\mathrm{s}_j = \mathrm{s}_0 + \mathrm{d}_j$$

14. <u>In-the-money Probability in the T_j-Measure</u>: Also

$$f_j(x) \geq 0$$

for

$$x \geq s_j$$

and therefore, using

$$A = \left\{\sum_{k=1}^{n} \frac{C_k}{\prod_{i=1}^{k}[1 + \delta K(t, T_{i-1})] e^{\int_t^{T_0} \vec{\gamma}(s, T_{i-1} - s) \cdot d\overrightarrow{W}_{T_i}(s) - \frac{1}{2}\int_t^{T_0} |\vec{\gamma}(s, T_{i-1} - s)|^2 ds}} \leq 1\right\}$$

$$X_i{}^j = \int_t^T \vec{\gamma}(s, T_{i-1} - s) \cdot d\overrightarrow{W}_{T_j}(s) + \sum_{l=1}^{i} \frac{\delta K(t, T_{l-1})}{1 + \delta K(t, T_{l-1})} \Delta_{li} - \sum_{l=1}^{j} \frac{\delta K(t, T_{l-1})}{1 + \delta K(t, T_{l-1})} \Delta_{li}$$

and

$$\mu_i{}^j = \Gamma_i(d_i - d_j)$$

we deduce that

$$\mathbb{P}_{T_j}(A|\mathcal{F}_t) = \mathbb{P}_{T_j}\left(X_j{}^j \geq \Gamma_j s_j\right) = N\left(-s_0 - d_j\right)$$

15. <u>T_j-Measure in-the-money LIBOR Expectation</u>: From standard arguments

$$\mathbb{E}_{T_j}\left[K\left(T_0, T_{j-1}\right)\mathcal{I}_A|\mathcal{F}_t\right] = \mathbb{E}_{T_j}\left[K\left(t, T_{j-1}\right) e^{\int_t^{T_0} \vec{\gamma}(s, T_{j-1} - s) \cdot d\overrightarrow{W}_{T_i}(s) - \frac{1}{2}\int_t^{T_0} |\vec{\gamma}(s, T_{j-1} - s)|^2 ds} \mathcal{I}_A|\mathcal{F}_t\right]$$
$$= K\left(t, T_{j-1}\right) N\left(-s_0 - d_j + \Gamma_j\right)$$

which leads to the approximated formula for the payer swaption.

16. Payer Swap Option Formula Approximation: The price at time $t \geq T_0$ of the payer swaption can be approximated by

$$P_{Swaption,Approximation}(t) = \delta \sum_{j=1}^{n} P(t, T_j)\left[K(t, T_{j-1})N(-\mathrm{s}_0 - \mathrm{d}_j + \Gamma_j) - \kappa N(-\mathrm{s}_0 - \mathrm{d}_j)\right]$$

where s_0 is given by

$$\sum_{k=1}^{n} \frac{C_k}{\prod_{i=1}^{k}[1 + \delta K(t, T_{i-1})]e^{\Gamma_i(x + \mathrm{d}_i - \mathrm{d}_j) - \frac{1}{2}\Gamma_i{}^2}} = 1$$

$$C_k = \kappa\delta \; \forall \; k = 1, \cdots, n-1$$

$$C_n = 1 + \kappa\delta$$

while Γ_i is given from

$$\int_t^T \vec{\gamma}(s, T_{i-1} - s) \cdot \vec{\gamma}(t, T_{l-1} - t)ds = \Gamma_i \Gamma_l$$

and

$$\mathrm{d}_i = \sum_{l=1}^{i} \frac{\delta K(t, T_{l-1})}{1 + \delta K(t, T_{l-1})} \Gamma_l$$

Mismatched Periods Cap/Swaption Pricing

1. Introduction: In the US, the UK, and the Japanese markets, caps correspond to rates compounded quarterly, while swaptions are semi-annual. In the German market caps are quarterly and swaptions annual. We deal with these mismatched periods by assuming lognormal volatility structure on the quarterly rates.
2. Lognormal Volatility Swaption Formulation: The forward swap rate at time $t \leq T_0$ is

$$
{\omega_{T_0}}^{(k)}(t, n) = \frac{P(t, T_0) - P(t, T_{kn})}{k\delta \sum_{j=1}^{n} P\left(t, T_{kj}\right)}
$$

and hence the time $t \leq T_0$ price of a payer swaption at strike κ maturing at T_0 is

$$
\begin{aligned}
P_{Swaption,k}(t) &= \mathbb{E}\left[\sum_{j=1}^{n} \frac{\beta(t)}{\beta\left(T_{jk}\right)} \left\{{\omega_{T_0}}^{(k)}(T_0, n) - \kappa\right\}^{+} k\delta | \mathcal{F}_t\right] \\
&= \mathbb{E}\left[\frac{\beta(t)}{\beta(T_0)} \left\{1 - \sum_{j=1}^{n} {C_j}^{(k)} P\left(T_0, T_{jk}\right)\right\}^{+} | \mathcal{F}_t\right] \\
&= P(t, T_0) \mathbb{P}_{T_0}(A | \mathcal{F}_t) - \sum_{j=1}^{n} {C_j}^{(k)} P\left(T_0, T_{jk}\right) \mathbb{P}_{T_{jk}}(A | \mathcal{F}_t)
\end{aligned}
$$

where

$$
{C_j}^{(k)} = \kappa k \delta \; \forall \; j = 1, \cdots, n - 1
$$

and

$$
{C_n}^{(k)} = 1 + \kappa k \delta
$$

while

$$A = \{{\omega_{T_0}}^{(k)}(T_0, n) \geq \kappa\} = \left\{\sum_{j=1}^{n} {C_j}^{(k)} P(T_0, T_{jk}) \leq 1\right\}$$

3. Mismatched Periods in-the-money Probability in the Forward Measure: Extending the formulation

$$P(t, T_{j-1}) \mathbb{P}_{T_{j-1}}(A|\mathcal{F}_t) = P(t, T_j) \mathbb{P}_{T_j}(A|\mathcal{F}_t) + \delta P(t, T_j) \mathbb{E}_{T_j}[K(T_0, T_{j-1}) \mathcal{J}_A | \mathcal{F}_t]$$

it follows that for all j

$$P(t, T_{k(j-1)}) \mathbb{P}_{T_{k(j-1)}}(A|\mathcal{F}_t)$$
$$= P(t, T_{kj}) \mathbb{P}_{T_{kj}}(A|\mathcal{F}_t)$$
$$+ \delta \sum_{i=1}^{k} P(t, T_{k(j-1)+i}) \mathbb{E}_{T_{k(j-1)+i}}[K(T_0, T_{k(j-1)+i-1}) \mathcal{J}_A | \mathcal{F}_t]$$

and so

$$P_{Swaption,k}(t) = \delta \sum_{j=1}^{n} \left\{ \sum_{i=k(j-1)+1}^{kj} P(t, T_i) \mathbb{E}_{T_i}[K(T_0, T_{i-1}) \mathcal{J}_A | \mathcal{F}_t] - k\kappa P(t, T_{jk}) \mathbb{P}_{T_{jk}}(A|\mathcal{F}_t) \right\}$$

4. Mismatched Payer Swap Option Approximation: The full set of arguments used for the matched period option pricing above may be employed here.
5. Mismatched Period Swaption Approximation Formula: Let k and δ be such that $\frac{1}{k\delta}$ is the compounding frequency per year of the swap rate ${\omega_{T_0}}^{(k)}(t, n)$ given by

$${\omega_{T_0}}^{(k)}(t, n) = \frac{P(t, T_0) - P(t, T_{kn})}{k\delta \sum_{j=1}^{n} P(t, T_{kj})}$$

The time $t \leq T_0$ price of a payer swaption can be approximated by the formula

$$P_{Swaption\ Approximation,k}(t)$$
$$= \delta \sum_{j=1}^{n} \left\{ \sum_{i=k(j-1)+1}^{kj} P(t,T_i)K(t,T_{i-1})N\left(-{s_0}^{(k)} - d_i - \Gamma_i\right) \right.$$
$$\left. - k\kappa P\left(t,T_{jk}\right)N\left(-{s_0}^{(k)} - d_{kj}\right) \right\}$$

where ${s_0}^{(k)}$ is given by

$$\sum_{j=1}^{n} \frac{{C_j}^{(k)}}{\prod_{i=1}^{kj}[1+\delta K(t,T_{i-1})]e^{\Gamma_i\left({s_0}^{(k)}+d_i\right)-\frac{1}{2}{\Gamma_i}^2}} = 1$$

where

$${C_j}^{(k)} = \kappa k\delta\ \forall\ j = 1,\cdots,n-1$$

and

$${C_n}^{(k)} = 1 + \kappa k\delta$$

and Γ_i is defined from

$$\int_t^T \vec{\gamma}(s,T_{i-1}-s)\cdot\vec{\gamma}(t,T_{l-1}-t)ds = \Gamma_i\Gamma_l$$

and

$$\mathrm{d}_i = \sum_{l=1}^{i} \frac{\delta K(t,T_{l-1})}{1+\delta K(t,T_{l-1})}\Gamma_l$$

6. Mismatched Period Joint Volatility Calibration: If one chooses $\delta = 0.25$, for example, in a market with quarterly and semi-annual caps and swaptions, then the mismatched period payer swaption formula above can be used to price the semi-annual caps and the swaptions, and hence it can also be used to jointly calibrate both quarterly and semi-annual volatility inputs.

Approximate vs. Full Simulation Comparisons

1. Cross-Verification of Simulation against Approximation: To analyze the differences between the exact swaption value computed by simulation, and the approximation for the mismatched payer swaption formula with $k = 1$ and $t = 0$, Brace, Gatarek, and Musiela (1997) fit a one-factor model to the US cap and swaption date for 12 July 1994, generating a typical volatility structure.
2. Generation of the Simulation Prices: Simulation prices were generated under the $\mathbb{P}_{T_n}$ measure using the exact formula

$$P(0, T_n)\mathbb{E}_{T_n}\left[\left\{\sum_{j=0}^{n-1} C_j \prod_{i=j+1}^{n} \{1 + \delta K(T_0, T_{i-1})\} + C_n\right\}^+\right]$$

with

$$C_0 = 1$$

$$C_j = -\kappa\delta \; \forall \; j = 1, \cdots, n-1$$

$$C_n = -(1 + \kappa\delta)$$

and

$$K(t, T_{i-1}) = K(0, T_{i-1}) e^{\int_0^t \vec{\gamma}(s, T_{i-1}-s) \cdot d\overrightarrow{W}_{T_i}(s) - \frac{1}{2}\int_0^t |\vec{\gamma}(s, T_{i-1}-s)|^2 ds}$$

$$\overrightarrow{W_{T_{i-1}}}(t) = \overrightarrow{W_{T_i}}(t) - \int_0^t \frac{\delta K(s, T_{i-1})}{1 + \delta K(s, T_{i-1})} \vec{\gamma}(s, T_{i-1} - s) ds$$

3. <u>Simulation of the T_n-Brownians</u>: The simulation equations above permit the recursive calculation of the Brownian motions

$$\overrightarrow{W_{T_0}}(t), \cdots, \overrightarrow{W_{T_{n-1}}}(t)$$

for

$$0 \leq t \leq T_0$$

For each simulation of $\overrightarrow{W_{T_i}}(t)$ on $[0, T_0]$ that gives values of

$$K(T_0, T_{i-1}) \; \forall \; i = 1, \cdots, n$$

substitution into the exact formula above provides the value of the swaption.

4. <u>Validation of the Simulation Sequence</u>: The simulation procedure, which involves Reimann and stochastic integration steps, was checked by back-calculating the cap prices used in the parametrization. The simulation prices coincided with the closed form prices calculated using

$$Cap(0) = \sum_{j=1}^{n} \delta P(0, T_j) \{ K(t, T_{j-1}) N[h(t, T_{j-1})] - \kappa N[h(t, T_{j-1}) - \varsigma(t, T_{j-1})] \}$$

5. <u>Comparison with Lognormal Black Closed Form</u>: Comparison can also be done with the approximate/simulated swaption formula, along with the market formula based on assuming that the underlying swap-rate is lognormal, and given as

$$P_{Swaption,Market}(t) = \delta \sum_{j=1}^{n} P(t, T_j)\{\omega_{T_0}(t, n)N(h) - \kappa N(h - \gamma T_0)\}$$

where

$$h = \frac{\log \frac{\omega_{T_0}(t, n)}{\kappa} + \frac{1}{2}\gamma^2 T_0}{\gamma\sqrt{T_0}}$$

6. Reduction to the Black Lognormal Closed Form: Note that because

$$\mathbb{E}\left[\sum_{j=1}^{n} \frac{1}{\beta(T_j)} \delta\{\omega_{T_0}(T_0, n) - \kappa\}^+\right] = \delta \sum_{j=1}^{n} P(0, T_j)\mathbb{E}_{T_j}\left[\{\omega_{T_0}(T_0, n) - \kappa\}^+\right]$$

the market seems to identify the forward measures

$$\mathbb{P}_{T_j} \ \forall\, j = 1, \cdots, n$$

with the forward measure $\mathbb{P}_{T_0}$ and assumes log-normality of the swap rate processes

$$\omega_{T_0}(t, n), 0 \leq t \leq T_0$$

under the measure $\mathbb{P}_{T_0}$. In fact

$$P_{Swaption\ Approximation,k}(0)$$
$$= \delta \sum_{j=1}^{n} \left\{ \sum_{i=k(j-1)+1}^{kj} P(0,T_i)K(0,T_{i-1})N\left(-s_0{}^{(k)} - d_i - \Gamma_i\right) - k\kappa P\left(0,T_{jk}\right)N\left(-s_0{}^{(k)} - d_{kj}\right) \right\}$$

where $s_0{}^{(k)}$ is given by

$$\sum_{j=1}^{n} \frac{C_j{}^{(k)}}{\prod_{i=1}^{kj}[1+\delta K(0,T_{i-1})]e^{\Gamma_i\left(s_0{}^{(k)}+d_i\right)-\frac{1}{2}\Gamma_i{}^2}} = 1$$

reduces to

$$P_{Swaption,Market}(0) = \delta \sum_{j=1}^{n} P\left(0,T_j\right)\left\{\omega_{T_0}(0,n)N(h) - \kappa N(h-\gamma T_0)\right\}$$

where

$$h = \frac{\log \frac{\omega_{T_0}(0,n)}{\kappa} + \frac{1}{2}\gamma^2 T_0}{\gamma\sqrt{T_0}}$$

if

$$d_i = 0$$

$$\Gamma_i = \sqrt{\Delta_{ii}} = \gamma\sqrt{T_0}$$

and

$$K(0, T_i) = \kappa$$

Typical Model Calibration Results

1. Two-Factor Calibration: Brace, Gatarek, and Musiela (1997) calibrate the model above to the data from the UK market using a two-factor model with piecewise constant volatility structure

$$\gamma(t, x) = f(t)\gamma(x)$$

where

$$\gamma(x) = [\gamma_1(x), \gamma_2(x)]$$

and

$$f: \mathbb{R}_+ \to \mathbb{R}$$

Thus if

$$f \equiv 1$$

the volatility is time-homogenous, so f represents the term structure of volatility (Brace and Musiela (1994a)).

2. Normal HJM Fit: The normal HJM model can almost always be fit to the UK and the US caps and the swaptions data with a one-factor homogenous volatility; fitting the correlation with the second factor improves the overall fit.

3. Log-normal HJM Fit: The log-normal HJM model frequently cannot fit a term structure of volatility in the log-normal case, and may this indicate that the price volatility of the normal HJM is more stable than the yield volatility of the log-normal HJM.
4. Comparison of the Implied Black Volatilities: The implied Black volatilities of the caps and the swaptions for both models are quiet similar, with log-normal volatilities being 1.0% to 1.5% greater at longer swaption maturities – possibly reflecting the different impact of correlation on the two models.

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The SABR Model

Introduction

1. Definition: SABR model, or “Stochastic Alpha, Beta, Rho” model, is a stochastic volatility model for forward LIBOR rates. Consider the forward rate

$$f_t = F(t; S, T)$$

Under the T-forward measure Q^T with numeraire $P(t,T)$ this forward rate is a martingale. In addition, we also assume its volatility is also a martingale under Q^T.

2. The SDE: The SDE is specified as (Wang (2010)):

$$\Delta f_t = \sigma_t (f_t)^{\beta} \Delta z_t$$

$$\Delta \sigma_t = \nu \sigma_t \Delta w_t$$

$$\mathbb{E}^{\mathbb{Q}^T}[\Delta z_t \Delta w_t] = \rho \Delta t$$

$$f_{t=0} = f_0$$

and

$$\sigma_{t=0} = \sigma_0$$

where the current forward price $f_{t=0} = f_0$ is observed in the market.

3. Model Parameters: The model has four parameters;

$$\sigma_0 > 0$$

$$0 \leq \beta \leq 1$$

$$-1 < \rho < 1$$

$$v \geq 0$$

In terms of the model name, stochastic alpha stands for σ_t, beta and rho for their respective parameters. If

$$\beta = 0$$

the forward rate is normal; if

$$\beta = 1$$

the forward rate is log-normal. If the *Volatility of Volatility Parameter*

$$v = 0$$

the model is reduced to the constant elasticity of variance (CEV) model.

4. SABR Model Closed Form: SABR models the implied volatility curve directly, which is then used to obtain the European option prices using the Black-76 model. The Black implied volatility is modeled as

$$\sigma_{Model}(K, f_0) = A.\left(\frac{z}{\chi(z)}\right).B$$

where

$$A = \frac{\sigma_0}{(f_0 K)^{\frac{1-\beta}{2}} \left[1 + \left\{\frac{(1-\beta)^2}{24} log^2 \frac{f_0}{K} + \frac{(1-\beta)^4}{1920} log^4 \frac{f_0}{K} + \cdots\right\}\right]}$$

$$B = \left[1 + \left\{\frac{(1-\beta)^2}{24} - \frac{{\sigma_0}^2}{(f_0 K)^{1-\beta}} + \frac{\rho\beta v \sigma_0}{4(f_0 K)^{\frac{1-\beta}{2}}} + \frac{2-3\rho^2}{24} v^2\right\} + \cdots\right]$$

$$z = \frac{v}{\sigma_0} (f_0 K)^{\frac{1-\beta}{2}} \log \frac{f_0}{K}$$

$$\chi(z) = \log \frac{\sqrt{z^2 - 2\rho z + 1} + z - \rho}{1 - \rho}$$

5. <u>SABR Approximation</u>: Except for the special cases of $\beta = 0$ and $\beta = 1$, the no closed form approximation is known. But the approximation is very accurate as long as the option is not too out-of-the-money, or T is not too large.

Parameter Estimation

1. <u>Implied Volatility Curve Shape</u>: The four parameters influence the shape of the implied volatility curve differently.

Parameter	Curve Property	Definition
σ_0	Level	The curve shifts upwards as it increases
β	Slope	The curve steepens as it decreases
ρ	Slope	The curve steepens as it decreases
v	Curvature	The curvature increases as it increases

2. <u>Calibration Steps</u>: To estimate the parameters, or in other words, to calibrate the model, it usually takes 3 steps; a) Estimate β; b) Imply σ_0 from ρ and v; c) Calibrate ρ and v.

According to the parameter table, β and ρ both control the shape of the volatility curve. Thus to some degree, the model is over-determined. A common industry practice is to skip step 1 by choosing directly

$$\beta = 0.5$$

Another way of calibration combines steps 2 and 3 together, and calibrates these 3 parameters directly.

3. Step 1 - Estimate β: For the ATM options, the expression for $\sigma_{ATM,Market}$ can be re-written as

$$\sigma_{ATM,Market} \approx \sigma_{Model}(f_0, f_0) = \frac{\sigma_0}{f_0^{\,1-\beta}}\left\{1 + \left[\frac{(1-\beta)^2}{24}\frac{\sigma_0^{\,2}}{f_0^{\,2-2\beta}} + \frac{\rho\beta v\sigma_0}{4f_0^{\,1-\beta}} + \frac{2-3\rho^2}{24}v^2\right]T\right\}$$

Taking logs (Hagan, Kumar, and Lesniewski (2002)), we get

$$\log \sigma_{ATM,Market} = \log \sigma_0 - (1-\beta)\log f_0$$

Therefore β can be estimated from a linear regression between the log ATM volatilities and the log forward rate time series.

4. Step 2 - Imply σ_0 from ρ and v: Given the current ATM market volatility $\sigma_{ATM,Market}$, we can invert the above equation to obtain the following cubic equation in σ_0:

$$\left[\frac{(1-\beta)^2 T}{24 f_0^{\,2-2\beta}}\right]\sigma_0^{\,3} + \left[\frac{\rho\beta v T}{4 f_0^{\,1-\beta}}\right]\sigma_0^{\,2} + \left[1 + \frac{2-3\rho^2}{24}v^2 T\right]\sigma_0 - \sigma_{ATM,Market} f_0^{\,1-\beta} = 0$$

5. Smallest Positive Solution σ_0: So σ_0, as the smallest positive real root to this equation, is then explicitly calibrated to the ATM volatility $\sigma_{ATM,Market}$. It is expressed as a function of the parameters ρ and v, which will be calibrated in the next step. The Tartaglia approach to the cubic equation solution can be found in Flannery, Press, Teukolsky, and Vetterling (1992).

6. Step 3 - Calibrate ρ and v: After step 2, there remain only 2 parameters to be calibrated - ρ and v. The calibration process is a fairly standard one. We choose the parameters that bring the model volatilities down to the market quote implied volatilities. That is

$$(\rho, v) = \underset{\rho, v}{\arg min} \sum_i \left[\hat{\sigma}_i(\rho, v) - \sigma_{i,Market}\right]^2$$

7. Step 3 Full Surface Calibration: Following steps 1, 2, and 3, SABR model in its primitive form can be relatively straightforward to calibrate. In general, if one tries to calibrate a model to a volatility surface $\sigma(K, T)$ (or a volatility cube $\sigma(K, T_\alpha, T_\beta)$ in the case of swap options), the process is usually more complicated. So SABR fixes the forward rate (and time T) and calibrates itself to a smile/skew curve with respect to the strike K.
8. Forward Rate Dynamics under the T-Forward Measure: The forward rate

$$f_t = F(t; S, T)$$

is treated in its own K-forward measure and does not interact with other forward rates. Compare this with the cap volatility calibration where the ATM volatility curve with respect to time T is considered. In both cases, the curve is one dimensional. In sum, as long as we don't consider the forward rate and its volatility dynamics under other T-forward measures, the calibration process demands much less effort.

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Section XII: LMM Extensions, Calibration, and Greeks

LMM Calibration and Greeks Overview

Motivation for Robust LMM Calibration

1. Impact on Pricing and Greeks: The LIBOR market model has established itself as the benchmark model for interest rate derivatives. If the observed correlation and the volatility surfaces cannot be reproduced by a model, it is hopeless to be able to get meaningful prices. The crucial task, therefore, before it comes to pricing, is to calibrate the model to the given market data.
2. Drawback of the Basic LMM: Schatz (2011) provides an overview of the LIBOR market mode and shows how to obtain a robust calibration. The big drawback of the model is that it cannot reproduce the typically observed implied volatility smile.

Robust LMM Calibration Approaches Overview

1. Stochastic Volatility Smile/Skew Extensions: LMM can be extended to model market smile by making use of stochastic volatility. It has been observed that a time homogenous parametrization of these stochastic volatility models has the capability of fitting the market very well.
2. Approximate Terminal Correlation Estimation Formula: Schatz (2011) develops a new approximate terminal correlation estimation formula based on the same parameter averaging techniques used for pricing in a stochastic volatility LMM.
3. Robust Local/Global Optimization Procedures: Schatz (2011) also looks at a very robust calibration procedure and shows how to use it if there are no starting values available, and introduce some new approximations that make it possible to use global optimizers.
4. Optimal Local/Global Optimizer Choice: The optimal choice if the local and global optimizers for each step of the calibration must be dealt with, especially in the context of the

algorithms that have attracted interest in the financial community in the recent times (e.g., differential evolution algorithms).

5. Stability of the Calibrated Parameters: Furthermore, an analysis of the robustness of the calibrated parameters over the specified period of interest must also be performed.

Cross Currency LIBOR Market Model

1. Cross Currency Extensions to LMM: The extensions to the cross currency LIBOR market model introduced by Schlogl (2002) must also be incorporated.
2. Displacement/Stochastic Volatility Fits: These extensions make use of the displacement and stochastic volatility to fit the observed smile and skew respectively.

LMM Based Greeks Calculation Approaches

1. Monte Carlo Based LMM Greeks: The calculation of greeks for exotic interest rate derivatives is crucial for hedging and therefore for trading these products. For a LIBOR market model this task can only be fulfilled by means of non-trivial Monte Carlo based methods.
2. Transition Densities Using Proxy Simulation: One possibility is to use the proxy simulation scheme method, which makes use of the transition densities of the discretized processes.
3. Fourier Techniques for Transition Densities: When it comes to the calculation of these transition densities in the stochastic volatility models, they can only be done by means of Fourier inversion.
4. Conditional Independence of Stochastic Volatility: Techniques are available for the calculation of the weights by making use of the conditional independence of the underlying from the stochastic volatility process, which can be applied for any process for which this conditional independence holds. This is also the case for the extension to the stochastic volatility cross currency model.

5. <u>Simplicity of Transition Density Estimator</u>: The developed estimators are very similar to the standard estimators for an LMM, and are fast enough to be employable in everyday practice.
6. <u>Simulation Schemes/Random Sequence Impact</u>: Schatz (2011) shows how these greeks calculated by these proxy simulation methods perform compared to the finite differences approximation and shows how the use of Sobol sequences influences the results.

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LMM Extensions Overview and Literature

LMM Approach Advantages and Drawbacks

1. LMM as a Market Standard: The LIBOR market model, which was introduced by Brace, Gatarek, and Musiela (1997), Miltersen, Sandmann, and Sondermann, and Jamshidian (1997) has not only gained acceptance in the academia and industry, but it has become a market standard and a benchmark model today.
2. Comparison between LMM and HJM: Basically it is a discretized version of the famous Heath, Jarrow, and Merton (HJM) model. Brigo and Mercurio (2006) contain the details.
3. Main Advantages of the LMM: The main advantage of the LIBOR market model (LMM) in comparison to the HJM is that it models the market observable forward LIBOR rates instead of the mathematically abstract instantaneous forward rates and derivatives pricing is much easier in the framework. In particular caplets and swaptions can be priced consistently with the long used formulas of Black, which are still a market standard. The market quotes caps and swaptions in Black volatilities.

Major Extensions to the LMM

1. Drawback of the LMM: A major drawback of the LMM is that it cannot capture the market smile.
2. Impact of LMM Short-comings: Not incorporating skew or smile may result in severe losses when using such a model to price exotic products. Therefore many extensions have been proposed to incorporate the skew or smile in the existing literature.
3. Overview of the LMM Extensions: Displacements have been introduced, CEV, Levy, and stochastic volatility models have been proposed to deal with the problem. The main

approaches in the smile and the skew modeling for the LIBOR market model are the following.

4. Local Volatility LIBOR Market Model Extensions: The first steps towards incorporating the skew features in a LMM were made by Andersen and Andreasen (2000) by using the local volatility processes for the LIBOR rates.
5. Discontinuous Jump Diffusion Levy Processes: Attempts to deal with the smile problem by making use of the jump diffusion processes were undertaken, for example, by Glasserman and Kou (2003) and Belomestny and Schoenmakers (2006).
6. Challenges with the Levy Approaches: General Levy LIBOR models were studied by Eberlein and Ozkan (2005), but the calibration, especially a robust calibration, of these models is a tough task.
7. Market Smiles Using Stochastic Volatility: Models making use of stochastic volatility to make use of market smiles were introduced among others by Joshi and Rebonato (2003), Piterbarg (2003), Andersen and Brotherton-Ratcliffe (2005), and Wu and Zhang (2006).
8. Forward LIBOR – Stochastic Volatility Correlation: A major problem with many stochastic volatility models is that, in order to introduce a skew, they correlate the SDE's underlying the forward LIBOR rates and the stochastic volatility.
9. Differential Skew vs. Smile Approaches: Piterbarg (2003) and Andersen and Brotherton-Ratcliffe (2005) mix the approaches of displaced diffusion for the skew and stochastic volatility for the smile. Furthermore, many researchers feel that skews and smiles are caused different market phenomena, and should therefore be modelled separately.
10. Simplicity of the Displaced Diffusion Setting: The displaced diffusion setting results in more simplified formulation than the CEV setting, and works perfectly in combination with stochastic volatility.
11. Forward LIBOR Stochastic Volatility Independence: From a calibration point-of-view, the methods of Andersen and Brotherton-Ratcliffe and Piterbarg are preferred, as in these cases the stochastic volatility processes are independent of the forward LIBOR rates.

Derivative Fair Value Pricing Challenge

1. Inefficiency in First Generation Models: The first step we have to take in order to address the derivative pricing/hedging question is to choose a model. Many exotic products, like products with callable features, are difficult to handle, and therefore the simple first-generation interest-rate models like the Hull-White or the Black-Kaminsky models should not be used. The problem is that they cannot be calibrated to a rich enough set of market instruments.
2. Second Generation LMM Type Models: One has to use the second generation models like the LIBOR market models which are capable of calibrating a whole volatility surface.
3. Addressing the Derivatives Pricing Challenge: A well-calibrated model that is capable of capturing the market features with a Monte-Carlo pricing engine (if analytical pricing approaches cannot be used) takes care of the derivatives pricing question.

Hedging the Derivatives Cash Flow

1. Hedging with Monte-Carlo Greeks: In order to hedge the derivative cash flows, one faces the problem that for many of the cash flows there are no natural hedges available, and therefore hedging usually comes down to calculating the Greeks and trying to neutralize the Greeks of the portfolio.
2. Greeks Smoothness - Models and Methodology: The calculation of the Greeks by means of Monte-Carlo methods is treated in detail in Schatz (2011). Therefore a property a good model should fulfill is the possibility of smooth Greek calculation.
3. Stochastic Volatility Monte Carlo Greeks: While there is a broad literature on how to calculate Greeks in a LIBOR market model (LMM) and the displaced diffusion LIBOR market model (DDLMM) setups, there is still a lack of these methods for LMM's with smile features.
4. Forward/Stochastic Volatility Correlation Complications: The correlation between the LIBOR rates and the stochastic volatility processes in the stochastic volatility LIBOR market models (SVLMM's) poses severe problems when we actually want to calculate the Greeks, and therefore one of the uncorrelated models, e.g., Piterbarg (2003) or Andersen and Brotherton-Ratcliffe (2005) are preferred.

5. Advantage of the Piterbarg (2003) Approach: From a calibration view-point the Piterbarg approach is preferable, since the calibration can be split up into a 2-step process. Therefore considerable focus is paid to this approach in the smile section.
6. Stability of the Calibrated Parameters: If the calibrated parameters vary too much on a daily basis, the hedging portfolio needs to be adjusted very often, therefore we should make sure that the calibration options robust parameters.
7. Approaches Behind Cross Currency LMM: Another topic that needs to be considered is the Cross Currency Model. The model introduced by Schlogl (2002) needs to be reviewed n detail, and some generic extensions to arrive at a joint model where the individual markets are in the spirit of Piterbarg model needs to be developed. The ability to calculate Greeks should also be maintained.

Basic LMM and its Calibration

1. Parametrization of Volatility and Correlation: First, the interest rate market is introduced before a closer examination of the LIBOR market model is carried out. The correlations and the volatility parameterizations are examined, and these are used for the displaced diffusion as well as the stochastic volatility LIBOR market models.
2. Terminal Forward Rate Correlation Estimation: The concepts of instantaneous and terminal correlation are introduced, and an approximation for the terminal correlation is given.
3. Analytical Pricing for LMM Calibration: Furthermore, the analytical pricing for the products that are used in the LMM calibration is demonstrated, and used to get a robust calibration. These treatments serve as a basis for the extended models later on, and can be viewed as a collection of the main literature available to date.

LMM Skew and its Calibration

1. Market Skew Incorporation Using Displacements: The incorporation of the skew then needs to be considered, where the LIBOR market model is extended using 2 different versions of displacements, and the calculation of the prices for the basic products is shown again.
2. Piterbarg and Andersen and Brotherton-Ratcliffe Approaches: The equivalence between the Piterbarg and the Andersen and Brotherton-Ratcliffe approaches is demonstrated, since in a sense both the models introduce the same dynamics. Pricing of the basic products is discussed, and hints are provided on how to calibrate the displaced model to the market data.

LMM Smile and its Calibration

1. Smile Incorporation through Stochastic Volatility: The extension of the LIBOR market model to incorporate smile features is then considered. A new time homogenous skew parametrization is introduced, as are the parameter averaging techniques introduced to calculate the effective skew and the effective volatility.
2. Challenges with Stochastic Volatility Pricing: After that some common problems that occur when one uses stochastic volatility models are looked into, and with all that in hand the pricing of the basic products is dealt with.
3. Stochastic Volatility Terminal Correlation Pricing: A new approximation formula for the stochastic volatility models is introduced, and used in the calibration. In the final part, a method to approximate the effective volatility to speed up the global or the initial calibration is shown.
4. Global Stochastic Volatility Parameters Calibration: While Piterbarg (2013) presents a calibration routine that works perfectly well in case we have good starting parameters, the routine is too slow for global calibration.
5. Swaption Calibration Using Global Optimizer: The approximation presented above enables the use of global optimizers to find optimal parameters. A detailed description of the calibration to swaptions is covered at the end.

Cross Currency Extensions to LMM

1. CCLMM and Displaced Diffusion CCLMM: The cross currency LIBOR market model (CCLMM) introduced by Schlogl (2002) is presented before being extended to displaced diffusion cross-currency LIBOR market model (DDCCLMM).
2. SVCCLMM Stochastic Volatility CCLMM Extensions: Finally the CCLMM is extended to a new stochastic volatility setting in the spirit of the Piterbarg model seen earlier, but the FX following a similar process as the forward LIBOR rates.
3. CCLMM Extensions Under Terminal Measure: All the necessary dynamics are derived under the terminal measure to simulate the underlyings later on and to calculate the sensitivities.

LMM Monte Carlo Methods and Greeks

1. Monte Carlo Methods for LMM: Monte Carlo techniques for LIBOR market models and its extensions are then considered. Transition densities are calculated for the models, and are used to calculate the Monte Carlo Greeks later.
2. Monte Carlo Greeks for LMM: An overview of the Monte Carlo methods for Greek estimation and their application to LIBOR market model is given next. After that, the application of the proxy Greek method to all of the introduced models is shown.
3. Independent Stochastic Volatility Process Case: In particular, application of the above techniques to the stochastic volatility LMM in case of an independent stochastic volatility process is shown.

Numerical Methods for LMM Calibration

1. Local and Global Optimizer Comparison: Numerical optimizers for the calibration of the models will then be introduced. Different local and global optimizers will be looked at and their features compared.

2. Differential Evolution Algorithm SV Calibration: The differential evolution algorithm is presented in more detail as it has been introduced to solve financial problems only recently, and the application to the calibration of a stochastic volatility model is demonstrated.

LMM Calibration and Greeks – Results

1. Numerical Results – Forward Rate Curve: First, the initial forward rate curve needs to be obtained before the initial/global calibration of the stochastic volatility LIBOR market model is dealt with. The actual fitting qualities of the model will be analyzed and the robustness of the obtained parameters examined.
2. Finite Differences vs. Proxy Greeks: To effectively calculate the Greeks, the finite differences method needs to be compared against the proxy Greeks method. The effect of using Sobol sequences will be looked at closely, and the new stochastic volatility estimator analyzed.
3. Displacement/Stochastic Volatility Calibration Quality: Finally the impact of the displacement and the stochastic volatility on the calibration quality will be looked at.

First Generation LMM Treatment Literature

1. Stochastic Calculus and Financial Mathematics: Excellent introductions to the basic concepts of stochastic calculus and financial mathematics can be found in Oksendahl (2003), Bingham and Kiesel (2004), and Shreve (2004).
2. Treatment of the Basic LMM: Quiet a few articles and books have dealt with the LIBOR market model, e.g., Rebonato (2002, 2004), Schoenmakers (2005), Brigo and Mercurio (2006), Brace (2007), Fries (2007), and Gatarek, Bachert, and Maksymiuk (2007).
3. LMM Analysis Explored in Detail: The books of Rebonato, Fries, and especially Brigo and Mercurio are very good starting points. Introductory treatment here mainly follows Rebonato (2002), Schoenmakers (2005), and Brigo and Mercurio (2006).

Smile Extensions to the LMM

1. Overview of the LMM Extensions: A good overview of this topic can be found in Meister (2004), Brigo and Mercurio (2006), and Svoboda-Greenwood (2007).
2. LMM Details and Further Extensions: Some specific and detailed promising approaches are given in Joshi and Rebonato (2003), Piterbarg (2003), Andersen and Brotherton-Ratcliffe (2005), and Wu and Zhang (2006).

Numerical Methods in Calibration/Greeks

1. Comprehensive Numerical Algorithms Suite Guide: Both Jackel (2002) and Press (2007) contain very good overview (and implementation) on many topics.
2. Monte Carlo and Optimization Algorithms: Both Nocedal (2000) and Press (2007) contain an introduction to the optimization algorithms. Glasserman (2004), Kloeden, Schurz, and Platen (2007) as well as Duffy and Kienitz (2009) contain details on Monte Carlo simulation methods.
3. Theoretical Background and Greeks Estimation: Details on the calculation of Greeks is contained in the works and the papers of Glasserman and Zhao (1999), Fries (2005), Fries and Kampen (2006), Fries (2007), Fries and Joshi (2008), and the papers of Joshi, especially Denson and Joshi (2009). Kloeden and Platen (1999) and Protter (2005) contain a theoretical background on these topics.

Object Based Financial Valuation Models

1. Quantlib Design and Architecture: For the implementation of these models, Schatz (2011) used the open source library Quantlib (http://quantlib.org) as a basis. As there is virtually no documentation available for Quantlib, the reader should get familiar with the advanced object

concepts first. Some recommended books are Duffy (2004, 2006), Joshi (2008), and Duffy and Kienitz (2009).

2. DRIP Analytics Design and Architecture: Another very commonly used fixed income library for valuation, risk, and scenario analysis is DRIP (http://creditanalytics.codeplex.com). DRIP is built in Java, has no external dependencies, and is therefore easy to install/work with. It contains a very elaborate and customizable curve construction suite (Forward, Funding, Overnight, Treasury, FX, Credit, etc.). From a valuation point of view, it covers a wide suite of interest rate, credit, and FX products. Finally, it has an elaborate mechanism for risk calculation and historical horizon analysis.

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Section XIII: Algorithmic Differentiation

Algorithmic Differentiation

Glossary

4. Wengert List: List of all the non over-writable program variables (Wengert (1964)) – can also be seen as a linearization of the computational graph. By construction, it is an intermediate variable.
3. Intermediate Wengert Canonical Variable: These are intermediate financial variables those are fixed from the point-of-view of the output Jacobians and the input parameters that serve as computation graph parsimonious optimizers (Figures 8 and 9).
4. Wengert fan-in and fan-out: Reduction of a set of initial/intermediate Wengert variates onto the subsequent set is called fan-in; the opposite is fan-out.
5. Wengert funneling: Same as Wengert fan-in.
5. Micro-Jacobian: Change in the calibrated instrument measure coefficients to unit change in the quoted instrument measures.
6. Self-Jacobian: Self-Jacobian refers to the Jacobian of the Objective Function at any point in the variate to the Objective Function at the segment nodes, i.e., $\frac{\partial Y(t)}{\partial Y(t_K)}$. Self-Jacobian is a type of micro-Jacobian.
7. Derivative Entity: The entity whose dynamics are determined by the evolution of a stochastic variate, and whose specific facets/measures are observable.
8. Path-wise Derivative Estimator: $\frac{\partial V}{\partial X_i(0)}$, where V is the value of the derivative, and $X_i(0)$ is the starting value for a specific stochastic variate.
9. Non-Parsimonized Parameters: Parameters that map one-to-one with the input instrument set, e.g., typical curve bootstrapping.
10. Parsimonization: Reduction of the parameter space from the input measure space.

Overview

1. AD History: Iri (1991)
2. Mathematical Foundations: Griewank (2000)
3. Survey: Berz (1996)
4. Implementation Tools, Methodologies, Processes, and Techniques (Bischof, Hovland, and Norris (2005))
5. AD Resource: *http://www.autodiff.org/*

Algorithmic Differentiation in Finance

1. Focus has been primarily on Monte-Carlo methodologies.
2. Although path-wise optimized sensitivity generation had been employed earlier (Glasserman (2004)), Giles and Glasserman (2006) first discussed adjoint methods in path-wise sensitivity generation.
3. Full extension to LMM based stochastic variate evolution and a corresponding exotic (in this case Bermudan) swap option evaluation (Leclerc, Liang, and Schneider (2009)), as well as to correlated defaults and their sensitivities (Capriotti and Giles (2011)).
4. Capriotti (2011) covers automated Greek generation, but with a focus on automatic differentiation, and in the context of Monte-Carlo methods.
5. Finally, algorithmic differentiation has also been applied to addressing the issue of calibration along with sensitivity generation (Schlenkirch (2011)).

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Algorithmic Differentiation - Basics

Motivation and Advantages

1. Definition: Automatic differentiation is a set of techniques for transforming a program that calculates the numerical values of a function into a program that calculates numerical values for derivatives of that function with about the same accuracy and efficiency as the function values themselves (Bartholomew-Biggs, Brown, Christianson, and Dixon (2000)).
2. Symbolic Derivatives: Calculate the local symbolic derivatives rather than the a) divided differences, or b) numerical differentials (*Automatic Differentiation - Wikipedia Entry*).
3. Calculation Speed: Same number of Objective Function Calculation as the original; however, potential "chain rule" multiplication factor effects.
4. Accuracy vs. Performance: Due to the usage of symbolics, accuracy of Automatic Differentiation always better than numerical differentials; however, due to the chain-rule issue, may not be always faster.
5. Scalability at infinitesimal variates: Since Automatic Differentiation is always symbolic and therefore infinitesimal, it will automatically scale to arbitrarily small variate infinitesimals – reduced errors due to bit cancellation etc.
6. Higher-order derivatives: Automatic Differentiation does not need additional objective function evaluations for higher order derivative calculations (beyond the chain-rule issues); therefore, those are infinitesimally correct too.

Program Sequence Construction Modes

1. Forward Automatic Differentiation: Express the final and the intermediate variables as a consequence of a computed forward graph, and derive the symbolic forward derivative graph.

- Effectively computes the gradient of the intermediate variables to the variates or the "independent variables" and transmits them up the graph.

2. Reverse Automatic Differentiation: Express the intermediate variables and the input variates as nodes in the computed reverse graph, and derive the symbolic reverse derivative graph.
    - Often may still need the forward path to store the calculated intermediates needed on the way back.
    - Effectively computes the gradient of the intermediate variables to the "dependent variables" and transmits them down the graph.
3. Speed:
    - Forward Mode => Speed proportional to n, the number of "independent" variables
    - Reverse Mode => Speed proportional to m, the number of "dependent" variables
4. Memory Usage (Ghaffari, Li, Li, and Nie (2007)):
    - Forward Mode => a) Each Wengert variable, b) Forward Jacobian for each Wengert, c) Forward Dependency Graph
    - Reverse Mode => a) Each Wengert Adjoint, b) Reverse Jacobian for each Wengert, c) Forward/Backward Dependency graph
5. When the difference is minimal: When the dependence of the final Jacobian sensitivity step is the dominating factor, and the adjointing step is not the rate-determining part, then the performance will always be $\Theta(n)$, where n is the number of sensitivities – for e.g., if

$$\mathrm{y} = \sum_{i=1}^{n} x_i$$

given that $\frac{\partial y}{\partial x_i}$ is trivial to calculate, the performance will always be $\Theta(n)$.

    - For instance, given a univariate objective function (as in constrained/unconstrained optimization (e.g., maximization/minimization) problems), either forward or reverse Automatic Differentiation is an equally good choice for sensitivity generation, owing to its performance.

Canonicalization - Program Statements Simplification by Decomposition

1. Program Line-level decomposition: Canonicalization decomposes the program/statement units into specific analysis bits.
    - Canonicalization is commonly used in many areas of computer science, e.g., in compiler design/code generation, SKU formulation/synthesis/customization etc.
2. Canonicalization Implementation: In general, canonicalization and other related Automatic Differentiation Source Code generation/transformation techniques should go hand in hand with optimizing compiled code emission techniques, program active variable activity analysis.
    - Canonicalization sequence should include steps (Bischof, Hovland, and Norris (2005)) where you would be able to mark the mathematical "Automatically Differentiable" code segments to separate from the others during, for instance, pre-processing etc.
    - For true program transformation effectiveness, Hot-Spot type dynamic run-time analysis is needed in addition to static compile time data flow analysis etc.
    - In VM oriented languages like Java, the run-time GC already works, so would it might make a candidate for embedding AD execution/selective sensitivity generation in.
3. Equivalence with Wengert Structuring: Given that canonicalization consists of hoisting all the l-value updates separately without side effects, it is effectively the same as Wengert un-rolling and DAG linearization.
4. Limitations with the implementation: For many of the reasons above, automated implementations of canonicalization (like other automated code generation/re-structuring) might result in "invisible inefficiencies", and the had-drafted techniques those are based upon essentially the same principles may be more optimal.
5. Post canonicalized Enhancement Cost: Given that the worst case operation is division, going from

$$c = \frac{a}{b}$$

to

$$dc = \frac{da}{b} - \frac{a}{b^2} db$$

results in going from 1 function unit execution cost to 4 automatic differentiation execution unit costs. Typically due to “weird” functions, the worst-case addition to a single post-canonicalized statement is a factor of 5, not 4.

6. Divided Differences based Differentiation Fall back:

$$\frac{\partial^n y}{\partial x^n} \cong \sum_{i=1}^{n} \frac{(-1)^i C_i{}^n y\big(x + \delta(n - 2i)\big)}{(2\delta)^n}$$

Challenges of Automating the Differentiation

1. Deep-dig perspective: Re-purposed Automatic Differentiation perspective forces the visualization of the computation at the granularity of the symbolic functional forms of the objective function.
    a. Objective Function evaluator over-loading => This requires propagation of the inner most symbolic graph nodes through the graph chain => causes additional cognitive SKU export!!
    b. Objective Function Neighborhood Behavior => With every Wengert variable, calculation of the set of forward sensitivities and the reverse Jacobians builds a local picture of the Objective Function without having to evaluate it.
2. Block-level View Fixation: Source code transformation techniques are very invasive, and require highly locally frozen view fixation, and are therefore less cognitive. Operator overloading techniques enable retention of the domain focus, and are therefore more cognitive.
    a. Naïve operator overloading would simply generate a block-level (or function call level) adjoint. This can explode the required storage, in addition to generating sub-optimal reverse-mode code. Needless to mention, source code transformation

techniques can be built to overcome this – in practice, however, many may not quite do it.

3. Complied language Automatic Differentiation implementation: Without the usage of obfuscating "versatile" templates, auto-generation of very generic forward/reverse accumulation code is impossible. Therefore source level function overloading and automated program instrumentation techniques are very hard.
    a. Further, compiled language source code transformation appears to be a vestige of "smart compiler" efforts of the '90s – classic instance of where a simple idea is "intellectually transmitted" than "built out-of-the-box".
4. Symbolic Differentiation Challenges with certain Unit Functional Forms: When you consider functions such as

$$y = \frac{1}{f(x)}$$

and you seek $\frac{dy}{dx}$ symbolically, the higher order symbolic differentiations become much more challenging:

$$\frac{dy}{dx} = -\frac{1}{f^2(x)}\frac{df(x)}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{2}{f^3(x)}\frac{df(x)}{dx} - \frac{1}{f^2(x)}\frac{d^2f(x)}{dx^2}$$

and so on for higher orders. Thus symbolically handling these series this way gets out of control fast!

Wengert Representation and Optimal Program Structure Synthesis

1. Combination of Forward/Reverse Modes: Forward (n inputs) and reverse (m outputs) mode represent just two possible (extreme) ways of recursing through the chain rule. For

$$n > 1$$

and

$$m > 1$$

there is a golden mean, but finding the optimal way is probably an NP-hard problem (Berland (2006)) – optimal Jacobian accumulation is NP-complete (Naumann (2008)).

2. Wengert Intermediate Fan-in and possibly fan-out: See Figures 8 to 10 for illustrate this.
    - Wengert Intermediate Performance Enhancement => If there exists an intermediate quantity that is fixed from the point-of-view of the output Jacobians and the input parameters, the performance may be improved (see Figure 1).
    - Reusable Intermediate Performance Improvement => If the input/output computation leads to sufficient commonality among the Wengert intermediate calculation, that may also reduce computation by promoting reuse, thereby improving efficiency.
    - Wengert Funneling Criterion => For non-optimizing, non-parsimonized Wengert funnels,

$$\frac{\partial P_i}{\partial MI_j} \to \delta_{ij}$$

    for the Wengert fan to be a funneling fan-in – otherwise rippling out causes huge non-diagonal state evolution matrices. This is true for

$$I \to P$$

$$P \to W$$

and

$$W \to 0$$

3. Standardized Computational Finance Structures: In computational finance (esp. computational fixed income finance), the payout/product/pricer object serves the function of the intermediate Wengert variate indicated above. From below this variate you have the inputs/parameters rippling up, and from above you have the Jacobians/output measure adjoints feeding down (Figure 9).
   - Reactive Tree Upticks => Every intermediate element in Figure 9 is a reactive tree dependent node from the entity below, so forwarding/adjointing should happen with every real-time uptick.
   - Automatic Differentiation for the Wengert Canonicals => This involves the following:
     a. Identifying the abstractable financial canonical/reusable common object structures (market parameters, product parameters, pricer parameters, etc.)
     b. Working out their forward differentials and the reverse adjoints.
   - One Financial Automatic Differentiation view => The Intermediate Wengert Canonical View is the conceptual parsimonisation of the variate parameters space and the Jacobian measure space.

Optimization using Pre-accumulation and Check Pointing

1. Pre-accumulation: Aggregation (and possibly caching) of the sensitivity Jacobian over all the intermediate Wengert's inside a routine/block/module – thereby only exposing $\frac{\partial Output_i}{\partial Input_j}$ for the group unit (not each Wengert inside).
   a. Pre-accumulation also provides a suitable boundary for parallelization.
   b. It may also be looked at as the appropriate edge at which the source code transformation technique and operator overloading technique may "merge".

2. Cross-country Accumulation: Same as pre-accumulation, but pre-accumulation occurs in a specified (forward/reverse), Cross-country accumulation need not – in fact it may be guided by program analysis using Optimal Wengert intermediate composition techniques.
    a. This is also referred to as check pointing.
    b. This typically also requires snapshotting the program global and other execution context parameters at the checkpoint boundaries.
    c. Works best when the program state is easily and minimally savable, and quickly recoverable.
    d. Will also work well in conjunction with traditional kernel level check pointing schemes for fail-over etc.

Algorithmic Differentiation Financial Application Space Customization

1. Math Modules:
    - Forward differentials and auto-adjointing of math modules => May be needed for most of them.
    - Every block, compute the base "value", forward differential, and reverse adjoint.
    - In fact, for every active double-precision variable v, source code transformation automatic differentiation techniques recursively automatically generate the doublet $(v, \dot{v})$. Further, this calculation may also be parallelized.
        - This particular calculation may also be propagated at the function call level, so that the assignment outputs are automatically generated for the doublet/multiple.
        - Computational structures => Design restrictions may also be imposed by the computability of the AD of a math module, i.e., would the financial **MarketParamsContainer** be broken down into further parameter units?
2. Stochastic Variate Automatic Differentiation: Evolution of stochastic variates and their derivative entities may be further optimized by exploiting sparse-ness of the multi-factor co-variance matrix, thereby evolving the variate/derivative matrix that is sparse optimally (as opposed to blind delta bumps that may happen when computing differentials).

- Variance Reduction along the forward path => If a specific forward path a) does not need to be traveled, or b) certain forward Wengert intermediates automatically compute to zero, then these produce zero path derivatives. Further, external pre-computations can be done during the adjoint generation.
- Delta effects on the Optimal Exercise Dates => This imposes restrictions on how the path derivatives maybe computed using automatic differentiation. This may also be used in conjunction with regression analysis for estimating optimal exercise times. That certainly enables adjoint automatic differentiation techniques to be used.
- Tangent multi-mode arc derivatives =>
    a. Identifying the circumstances under which they are re-usable
    b. Arc derivatives extraction intermediates may also be re-used
    c. Depends (as always) on the speed up and memory used.

3. Quasi-analytic Computation Models: No Monte-Carlo evolution needed at all, but still Wengert intermediate level reformulation necessary to enhance the quasi-analytics analysis (e.g., Copula methods).
    - Adjoint-Natural Formulation Mode => Typical formulation works out the Wengerts backwards from the final measure (e.g., say from PV), so they are automatically amenable to the adjoint mode of automatic differentiation.
4. Latent State Calibration from Observed Manifest Measures:
    - Formulation of the de-convolution the latent state from the observed manifest measure is necessary for the extraction of the latent state parameter set (this is accomplished by the calibration process).
    - Of course, latent state calibration occurs among the elastic and the inelastic dimensions, and the inelastics are parameter set!
    - Latent state calibration/parameterization etc. inherently involve parsimonization – this is where the models come in.

Reference

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Sensitivity Generation During Curve Construction

Introduction

1. Advantages: In addition to the usual advantage that Automatic Differentiation provides on doing accurate Greeks on the same run as pricing, there is no need for multiple bumped curves anymore – but the proper Jacobians need to be calculated.
   - Further speed up => The segment micro-Jacobian needs to be pre-calculated right during the calibration - we need to calculate the Jacobian $\frac{\partial C_i}{\partial q_j}$ where C_i is the ith coefficient, and q_j is the jth input.
2. Curve Calibration Deltas: Typical deltas are with respect to the
   - dynamical latent state stochastic variates (e.g., the forward rates)
   - calibrated parameters (e.g., the segment spline coefficients)
   - unit change in the quoted instrument measures (e.g., 1 bp change) - here the Jacobians need to ripple upwards from the quoted instrument manifest measures.
3. Span/Segment Elastic Variates: Consider the situation where the latent state in itself (not its transformation) is explicitly measured. There are 5 different kinds of latent state proxies to consider:
   - Φ => Span stochastic latent state evolution variate.
   - Φ_k => Stochastic latent state evolution variate for segment k.
   - ϕ => Implied Span Quoted Instrument Manifest Measure.
   - ϕ_k => Implied Quoted Instrument Manifest Measure for Segment k.
   - φ_k => Observed Quoted Instrument Manifest Measure for Segment k at precisely a single variate point – typically, the observations are done at the anterior/posterior terminal ends of the segment.
4. Span/Segment variate relations: For a given calculated/formulated output manifest measure Ξ, the following are true by definition:

$$\Phi_k(t = t_k) = \Phi(t = t_k)$$

implies that

$$\left|\frac{\partial \Xi}{\partial \Phi}\right|_{t=t_k} = \left|\frac{\partial \Xi}{\partial \Phi_k}\right|_{t=t_k}$$

$$\varphi_k = \phi_k(t = t_k) = \phi(t = t_k)$$

implies that

$$\frac{\partial \Xi}{\partial \varphi_k} = \left|\frac{\partial \Xi}{\partial \phi}\right|_{t=t_k} = \left|\frac{\partial \Xi}{\partial \phi_k}\right|_{t=t_k}$$

5. Sensitivities to the elastic variates:
   - Sensitivity to Stochastic Evolution Variate => $\frac{\partial \Xi}{\partial \Phi}$
   - Sensitivity to Implied Span Quoted Instrument Measure => $\frac{\partial \Xi}{\partial \phi}$
   - Sensitivity to Observed Span Quoted Instrument Measure => $\frac{\partial \Xi}{\partial \varphi_k}$
   - $\frac{\partial \Xi}{\partial \varphi_k}$ (Case c) above) is what you need to calculate the hedge ratio
6. Piece-wise constant segment variate: In this case,

$$\frac{\partial \Xi}{\partial \Phi_k} = \frac{\partial \Xi}{\partial \phi_k} = \frac{\partial \Xi}{\partial \varphi_k}$$

7. Splined segment variate: Recall that segment spline coefficient calibration is simply a problem of matching to a terminal node (which is the quoted instrument measure at the terminal node). Thus, for a formulated output Ξ, at node k, it is obvious that

$$\frac{\partial \Xi}{\partial \Phi_k} \neq \frac{\partial \Xi}{\partial \phi_k}$$

- Stochastic Evolution Variate Derivative => For the case where Ξ refers to the discount factor, it can be shown that

$$D_F(t) = e^{-\int \Phi(t)dt} = e^{-\sum_{i=0}^{j} \int_{t_i}^{t_{i+1}} \Phi_i(t)dt - \int_{t_j}^{t} \Phi_j(t)dt}$$

where

$$t_j < t < t_{j+1}$$

Thus

$$\frac{\partial D_F(t)}{\partial \Phi_k} = -D_F(t) \times \begin{cases} t_{k+1} - t_k & k < j \\ t - t_k & k = j \\ 0 & k > j \end{cases}$$

- Quoted Instrument Manifest Measure Derivative => This depends on the actual details of the quadrature. Thus

$$\frac{\partial D_F(t)}{\partial \phi_k} = -D_F(t) \times \begin{cases} \displaystyle\int_{t_k}^{t_{k+1}} \frac{\partial \Phi}{\partial \varphi_k} dt & k < j \\ \displaystyle\int_{t_k}^{t} \frac{\partial \Phi}{\partial \varphi_k} dt & k = j \\ 0 & k > j \end{cases}$$

8. <u>Linear Dependence of Integrand Quadrature</u>: For many functional formulations in finance, the calculated product measure (Ξ) has a linear dependence on the stochastic evolution variate, i.e.,

$$\Xi \Longrightarrow \Psi\left(\int_{t_a}^{t_b} \Phi(t)\, dt\right)$$

This implies that

$$\frac{\partial \Xi}{\partial \Phi_k} = \delta_{ik} \frac{\partial \Xi}{\partial \Psi_i}(t_{i+1} - t_i)$$

i.e.,

$$\frac{\partial \Xi}{\partial \Phi_k} = \alpha \delta_{ik}$$

only, and not on the quadrature details.

Curve Jacobian

1. Representation Jacobian: Every Curve implementation needs to generate the Jacobian of the following latent state metric using its corresponding latent state quantification metric:
   - Forward Rate Jacobian to Quote Manifest Measure
   - Discount Factor Jacobian to Quote Manifest Measure
   - Zero Rate Jacobian to Quote Manifest Measure
2. Calibration Jacobian vs. Monte-Carlo Automatic Differentiation Delta: Both of these are actually path-wise, the difference being that:
   - Jacobian generated during calibration is part of inference, therefore iterative.
   - Jacobian of Monte-Carlo Automatic Differentiation is typically path-wise and non-iterative, therefore it is technically part of prediction.

3. Importance of the representation Self-Jacobian: Representation Self-Jacobian computation efficiency is critical, since Jacobian of any function $F(Y)$ is going to be dependent on the self-Jacobian $\frac{\partial Y(t)}{\partial Y(t_k)}$ because of the chain rule.
4. Forward Rate->DF Jacobian: Using $D_f(t_k)$ to represent the discount factor at t_k and to represent the forward rate $F(t_A, t_B)$ between times t_A and t_B, we get

$$F(t_A, t_B) = \frac{1}{t_B - t_A} \log \frac{\partial D_f(t_A)}{\partial D_f(t_B)}$$

$$\frac{\partial F(t_A, t_B)}{\partial D_f(t_k)} = \frac{1}{t_B - t_A} \left\{ \frac{1}{D_f(t_A)} \frac{\partial D_f(t_A)}{\partial D_f(t_k)} - \frac{1}{D_f(t_B)} \frac{\partial D_f(t_B)}{\partial D_f(t_k)} \right\}$$

5. Zero Rate to Forward Rate Equivalence: This equivalence may be used to construct the Zero Rate Jacobian From the Forward Rate Jacobian. Thus the above equation may be used to extract the Zero Rate micro-Jacobian.
6. Zero Rate->DF Jacobian: Using $Z(t)$ to represent the discount factor at t, we get

$$\frac{\partial Z(t)}{\partial D_f(t_k)} = \frac{1}{t - t_0} \left\{ \frac{1}{D_f(t)} \frac{\partial D_f(t)}{\partial D_f(t_k)} \right\}$$

7. Quote->Zero Rate Jacobian:

$$\frac{\partial Q_j(t)}{\partial D_f(t_k)} = (t_k - t_0) \left\{ D_f(t_k) \frac{\partial Q_j(t)}{\partial D_f(t_k)} \right\}$$

8. PV->Quote Jacobian:

$$\frac{\partial PV_j(t)}{\partial Q_k} = \sum_{i=1}^{n} \left\{ \frac{\partial PV_j(t)}{\partial D_f(t_i)} \div \frac{\partial Q_j(t)}{\partial D_f(t_i)} \right\}$$

9. Cash Rate DF micro-Jacobian: Using r_j to represent the Cash Rate Quote for the j[th] Cash instrument, we get

$$\frac{\partial r_j}{\partial D_f(t_k)} = \frac{1}{t_j - t_{START}} \left\{ \frac{1}{D_f(t_j)} \frac{\partial D_f(t_j)}{\partial D_f(t_k)} \right\}$$

10. Cash Instrument PV-DF micro-Jacobian:

$$\frac{\partial PV_{CASH,j}}{\partial D_f(t_k)} = \frac{1}{D_f(t_{j,SETTLE})} \frac{\partial D_f(t_j)}{\partial D_f(t_k)}$$

    There is practically no performance impact on construction of the PV-DF micro-Jacobian in then adjoint mode as opposed for forward mode, due to the triviality of the adjoint.

11. Euro-dollar Future DF micro-Jacobian: Setting Q_j to be the quote for the j[th] EDF with start date of $t_{j,START}$ and maturity of t_j, we get

$$\frac{\partial Q_j}{\partial D_f(t_k)} = \frac{\partial D_f(t_j)}{\partial D_f(t_k)} \frac{1}{D_f(t_{j,START})} - \frac{D_f(t_j)}{{D_f}^2(t_{j,START})} \frac{\partial D_f(t_{j,START})}{\partial D_f(t_k)}$$

12. Euro-dollar Future PV-DF micro-Jacobian:

$$\frac{\partial PV_{EDF,j}}{\partial D_f(t_k)} = \frac{\partial D_f(t_j)}{\partial D_f(t_k)} \frac{1}{D_f(t_{j,START})} - \frac{D_f(t_j)}{{D_f}^2(t_{j,START})} \frac{\partial D_f(t_{j,START})}{\partial D_f(t_k)}$$

    There is practically no performance impact on construction of the PV-DF micro-Jacobian in then adjoint mode as opposed for forward mode, due to the triviality of the adjoint.

13. Interest Rate Swap DF micro-Jacobian: Setting Q_j to be the quote for the j[th] IRS maturing at t_j, $DV01_j$ to be the $DV01$ of the swap, and $PV_{Floating,j}$ as the floating PV of the swap, we get

$$Q_j \cdot DV01_j = PV_{Floating,j}$$

$$\frac{\partial\left(Q_j \cdot DV01_j\right)}{\partial D_f(t_k)} = \frac{\partial\left(PV_{Floating,j}\right)}{\partial D_f(t_k)}$$

$$\frac{\partial\left(Q_j \cdot DV01_j\right)}{\partial D_f(t_k)} = \frac{\partial\left(Q_j\right)}{\partial D_f(t_k)} \cdot DV01_j + \frac{\partial\left(DV01_j\right)}{\partial D_f(t_k)} \cdot Q_j$$

$$\frac{\partial\left(DV01_j\right)}{\partial D_f(t_k)} = \sum_{i=1}^{j} N(t_i)\Delta_i \frac{\partial D_f(t_i)}{\partial D_f(t_k)}$$

$$PV_{Floating,j} = \sum_{i=1}^{j} \mathcal{L}_i N(t_i)\Delta_i D_f(t_i)$$

$$\frac{\partial PV_{Floating,j}}{\partial D_f(t_k)} = \sum_{i=1}^{j} \frac{\partial \mathcal{L}_i}{\partial D_f(t_k)} N(t_i)\Delta_i D_f(t_i) + \sum_{i=1}^{j} \mathcal{L}_i N(t_i)\Delta_i \frac{\partial D_f(t_i)}{\partial D_f(t_k)}$$

14. Interest Rate Swap PV-DF micro-Jacobian:

$$\frac{\partial PV_{IRS,j}}{\partial D_f(t_k)} = -\sum_{i=1}^{j} \frac{\partial \mathcal{L}_i}{\partial D_f(t_k)} N(t_i)\Delta_i D_f(t_i) + \sum_{i=1}^{j} \left[c_j - \mathcal{L}_i\right] N(t_i)\Delta_i \frac{\partial D_f(t_i)}{\partial D_f(t_k)}$$

There is no performance impact on construction of the PV-DF micro-Jacobian in then adjoint mode as opposed for forward mode, due to the triviality of the adjoint. Either way the performance is $\Theta(n \times k)$, where n is the number of cash flows, and k is the number of curve factors.

15. Credit Default Swap DF micro-Jacobian: Setting c_j to be the coupon for the j[th] CDS maturing at t_j, $PV_{CDS,j}$ to be the PV of the CDS contract, $PV_{Coupon,j}$ as the Coupon Leg of the CDS,

$PV_{LOSS,j}$ as the PV of the Loss Leg of the CDS, and $PV_{ACCRUED,j}$ as the PV of the Accrual Paid on Default, we have

$$PV_{CDS,j} = PV_{Coupon,j} - PV_{LOSS,j} + PV_{ACCRUED,j}$$

$$PV_{Coupon,j} = c_j \sum_{i=1}^{j} N(t_i)\Delta_i S_P(t_i) D_f(t_i)$$

$$\frac{\partial PV_{Coupon,j}}{\partial D_f(t_k)} = c_j \sum_{i=1}^{j} N(t_i)\Delta_i S_P(t_i) \frac{\partial D_f(t_i)}{\partial D_f(t_k)} + \frac{\partial c_j}{\partial D_f(t_k)} \sum_{i=1}^{j} N(t_i)\Delta_i S_P(t_i) D_f(t_i)$$

$$PV_{LOSS,j} = \int_0^{t_j} N(t)[1 - R(t)] D_F(t) dS_P(t)$$

$$\frac{\partial PV_{LOSS,j}}{\partial D_f(t_k)} = \int_0^{t_j} N(t)[1 - R(t)] \frac{\partial D_F(t)}{\partial D_f(t_k)} dS_P(t)$$

$$PV_{ACCRUED,j} = c_j \sum_{i=1}^{j} \int_{t_{i-1}}^{t_i} N(t)\Delta(t, t_{i-1}) D_F(t) dS_P(t)$$

$$\frac{\partial PV_{ACCRUED,j}}{\partial D_f(t_k)} = \frac{\partial c_j}{\partial D_f(t_k)} \sum_{i=1}^{j} \int_{t_{i-1}}^{t_i} N(t)\Delta(t, t_{i-1}) D_F(t) dS_P(t) + c_j \sum_{i=1}^{j} \int_{t_{i-1}}^{t_i} N(t)\Delta(t, t_{i-1}) \frac{\partial D_F(t)}{\partial D_f(t_k)} dS_P(t)$$

16. <u>Credit Default Swap DF micro-Jacobian</u>:

$$\frac{\partial PV_{CDS,j}}{\partial D_f(t_k)} = \frac{\partial c_j}{\partial D_f(t_k)} \sum_{i=1}^{j} \left\{ N(t_i)\Delta_i S_P(t_i) D_f(t_i) + \int_{t_{i-1}}^{t_i} N(t)\Delta(t, t_{i-1}) D_F(t) dS_P(t) \right\}$$

$$+ c_j \sum_{i=1}^{j} \left[N(t_i)\Delta_i S_P(t_i) \frac{\partial D_f(t_i)}{\partial D_f(t_k)} + \int_{t_{i-1}}^{t_i} N(t)\Delta(t, t_{i-1}) \frac{\partial D_F(t)}{\partial D_f(t_k)} dS_P(t) \right]$$

$$- \int_{0}^{t_j} N(t)[1 - R(t)] \frac{\partial D_F(t)}{\partial D_f(t_k)} dS_P(t)$$

There is no performance impact on construction of the PV-DF micro-Jacobian in then adjoint mode as opposed for forward mode, due to the triviality of the adjoint. Either way the performance is $\Theta(n \times k)$, where n is the number of cash flows, and k is the number of curve factors.

Stochastic Entity Evolution

Stochastic Entity Evolution – Sensitivity Formulation

1. Evolution Dynamics: Simplest evolution of stochastic variables $\mathcal{L}_i(t)$ will be ones with constant forward volatilities. Once the dynamics is formulated according to

$$\Delta\mathcal{L}_i(t) = \mu_i(\mathcal{L}_i, t)\Delta t + \sum_j \sigma_{ij}(\mathcal{L}_i, t)\Delta W_j$$

   where $\mu_i(\mathcal{L}_i, t)$ is the component drift, and σ_{ij} is the component co-variance to the factor W_j, subsequent evolution can be determined.

   - The Eulerized version of the above is

$$\Delta x_j(t) = h\mu_j(\vec{x}, t)\Delta t + \sqrt{h}\sum_l \sigma_{jl}(\vec{x}, t)\Delta Z_l$$

     where h is the time-step, and $\vec{Z}$ is the Weiner random variable.
   - In the case of forward rates, e.g., the drifts can be established by a no-arbitrage condition binding the forward rate drifts to their variances.
2. Evolution of the derivative entity: Once the stochastic variate dynamics is established, the dynamics of the observed derivative entity can be progressively determined.
3. Derivative Entity Measure path-wise evolution: Evolution sequence can be determined for the individual pay-off measures as well. These measures may further be dependent on the differentials of the derivative entity, so those may also need to be evolved using automatic differentiation.
4. Derivative Entity Computational efficiency enhancement:
   - Using the adjoint automatic differentiation methods
   - Using optimal combination of forward and adjoint automatic differentiation methods

- Further optimizations using sparse-ness of the multi-factor co-variance matrix, thereby evolving the variate/derivative matrix that is sparse optimally (as opposed to blind delta bumps that may happen when computing differentials).
- Quasi-analytic computation models and automatic differentiation techniques => No Monte-Carlo evolution needed at all, but still Wengert intermediate level reformulation necessary to enhance the quasi-analytics analysis (e.g., Copula methods).

5. Derivative Entity Measure Calculation: $[Instrument, Manifest\ Measure]$ input to $[Instrument, Manifest\ Measure]$ output is equivalently maintained in the Jacobian. Alternately, the computation may also hold $[Latent\ State\ Calibrated\ Parameter]$ to $[Instrument, Manifest\ Measure]$ Output map.

Sensitivities to Stochastic State Variates and Dynamical Parameters

1. State Variates: These are base stochastic entities that characterize the actual system statics/dynamics.
    - Sensitivities to the state variates are typically sensitivities to the "current" (or starting) realization of these variates – e.g., delta, gamma.
2. Dynamic Parameters**:** Model parameters that govern the evolution/equilibrium behavior of the state variates, and thereby the system dynamics.
    - Examples would be sensitivities to volatility, correlation, etc.
3. Segment/Span Coefficients: These are the additional coefficients serve act as the interpolated "PROXY" for the segment latent state at the unobserved points in the segment.
    - Sensitivities may also be sought to the coefficients.

Stochastic Variate Evolution Constrained by Splines

1. The forward rates (or indeed any term instrument measures) need to evolve such that
    - They are continuous at the boundaries
    - The first (and possibly the second) derivatives are continuous at the boundaries

- The boundary conditions (either financial or tensional) are retained intact

2. For e.g., the evolution dynamics of the forward rates (or indeed any term instrument measures) can still be via LMM, but splines may still be applicable to the intermediate nodes, as the segment spline coefficients adjust to the forward rate nodes.
3. Splines may also be used for any term instrument measure determinant (e.g., the volatility surface maybe also be interpolatively constructed using splines), so as to preserve the continuity/smoothness, as opposed to piece-wise discreteness.

Formulation of the Evolution of Stochastic Variate Self-Jacobian

1. Evolution Formulation:

$$\Delta x_j(t) = \mu_j(x_1, \cdots, x_n, t)\Delta t + \sum_{l=1}^{m} \sigma_{jl}(x_1, \cdots, x_n, t)\Delta W_l(t)$$

2. Definition of Self-Jacobian Delta:

$$J_{ij} = \frac{\partial x_i(t)}{\partial x_j(0)}$$

3. Evolution Sensitivity Formulation:
   a. i => Index over the number of underliers $(1, \cdots, n)$
   b. l => Index over the number of independent stochastic factors $(1, \cdots, m)$
   c. Then

$$\frac{\partial \Delta x_j(t)}{\partial x_k(0)} = \sum_{i=1}^{n} \left[\frac{\partial \mu_j(x_1, \cdots, x_n, t)}{\partial x_i(t)} \frac{\partial x_i(t)}{\partial x_k(0)}\right] \Delta t + \sum_{l=1}^{m} \left[\frac{\partial \sigma_{jl}(x_1, \cdots, x_n, t)}{\partial x_i(t)} \frac{\partial x_i(t)}{\partial x_k(0)}\right] \Delta W_l(t)$$

   d. Eulerized version of the above is:

$$\frac{\partial \Delta x_j(t)}{\partial x_k(0)} = \sum_{i=1}^{n}\left[\frac{\partial \mu_j(x_1,\cdots,x_n,t)}{\partial x_i(t)}\frac{\partial x_i(t)}{\partial x_k(0)}\right]h + \sqrt{h}\sum_{l=1}^{m}\left[\frac{\partial \sigma_{jl}(x_1,\cdots,x_n,t)}{\partial x_i(t)}\frac{\partial x_i(t)}{\partial x_k(0)}\right]Z_l(t)$$

e. First re-jig:

$$\frac{\partial x_j(t+h)}{\partial x_k(0)} = \sum_{i=1}^{n}\left[\delta_{ij} + h\frac{\partial \mu_j(t)}{\partial x_i(t)} + \sqrt{h}\sum_{l=1}^{n}\left\{Z_l(t+h)\frac{\partial \sigma_{jl}(t)}{\partial x_i(t)}\right\}\right]\frac{\partial x_i(t)}{\partial x_k(0)}$$
$$= \sum_{i=1}^{n} D_{ij}(k,t)\frac{\partial x_i(t)}{\partial x_k(0)}$$

where

$$D_{ij}(k,t) = \delta_{ij} + h\frac{\partial \mu_j(t)}{\partial x_i(t)} + \sqrt{h}\sum_{l=1}^{n}\left\{Z_l(t+h)\frac{\partial \sigma_{jl}(t)}{\partial x_i(t)}\right\}$$

f. Second re-jig:

$$\left[\frac{\partial x(t+h)}{\partial x_k(0)}\right] = [D(k,t)]\left[\frac{\partial x(t)}{\partial x_k(0)}\right]$$

where $\left[\frac{\partial x(t+h)}{\partial x_k(0)}\right]$ and $\left[\frac{\partial x(t)}{\partial x_k(0)}\right]$ are column matrices, and $[D(k,t)]$ is an $n \times n$ square matrix.

g. Third re-jig:

$$\left[\frac{\partial x(t+h)}{\partial x_k(0)}\right] = [D(k,t)][D(k,t-h)]\cdots[D(k,0)]\left[\frac{\partial x(0)}{\partial x_k(0)}\right]$$

- This is still forward automatic differentiation mode and is $\Theta(n)$, but you can optimize this using specific techniques shown in Glasserman and Zhao (1999).
- Another significant optimization can be achieved by adjointing techniques [Griewank (2000), Giles and Pierce (2000)].

- To achieve further significant optimization, transpose this, to get the following adjoint form:

$$\left[\frac{\partial x(t+h)}{\partial x_k(0)}\right]^{\mathrm{T}} = \left[\frac{\partial x(0)}{\partial x_k(0)}\right]^{\mathrm{T}} [D(k,0)]^{\mathrm{T}}[D(k,h)]^{\mathrm{T}} \cdots [D(k,t-h)]^{\mathrm{T}}[D(k,t)]^{\mathrm{T}}$$

which actually reduces to vector/matrix as opposed to matrix/matrix in the non-transposed version – this would be $\Theta(n^2)$, as opposed to $\Theta(n^3)$.

h. The matrix nature of $[D(k,t)]$ simply arises from the chain rule summation over i. Similar chain rules may be set for the different cash flow Jacobians, etc.

i. Re-casting $D_{ij}(k,t)$ from above as

$$D_{ij}(k,t) = D_{ij,PRIOR}(k,t) + D_{ij,DRIFT}(k,t) + D_{ij,VOLATILITY}(k,t)$$

we can separate out the different contributions to $D_{ij}(k,t)$. a) The term

$$D_{ij,PRIOR}(k,t) = \delta_{ij}(k,t)$$

is the contribution due to the previous D, i.e., $D_{ij}(k,t-h)$. b) The term

$$D_{ij,DRIFT}(k,t) = h\frac{\partial \mu_j(t)}{\partial x_i(t)}$$

is the contribution from the derivative of the drift term. c) The term

$$D_{ij,VOLATILITY}(k,t) = \sqrt{h}\sum_{l=1}^{n}\left\{Z_l(t+h)\frac{\partial \sigma_{jl}(t)}{\partial x_i(t)}\right\}$$

is the contribution from the volatility derivative.

4. <u>Definition of Self-Jacobian Gamma</u>:

$$\Gamma_{ij} = \frac{\partial^2 x_C(t)}{\partial x_A(0)\partial x_B(0)}$$

$$\frac{\partial^2 x_C(t+h)}{\partial x_A(0)\partial x_B(0)} = \sum_{i=1}^{n}\sum_{j=1}^{n}\left[S_{ij}(C,\vec{x}(t),t)\frac{\partial x_i(t)}{\partial x_A(0)}\frac{\partial x_j(t)}{\partial x_B(0)}\right] + \sum_{i=1}^{n}\left[M_i(C,\vec{x}(t),t)\frac{\partial^2 x_i(t)}{\partial x_A(0)\partial x_B(0)}\right]$$

$$S_{ij}(C,\vec{x}(t),t) = h\frac{\partial^2 \mu_C(\vec{x}(t),t)}{\partial x_i(t)\partial x_j(t)} + \sqrt{h}Z_C(t+h)\frac{\partial^2 \sigma_C(\vec{x}(t),t)}{\partial x_i(t)\partial x_j(t)}$$

$$M_i(C,\vec{x}(t),t) = h\frac{\partial \mu_C(\vec{x}(t),t)}{\partial x_i(t)} + \sqrt{h}Z_C(t+h)\frac{\partial \sigma_C(\vec{x}(t),t)}{\partial x_i(t)}$$

Correlated Stochastic Variables Evolution

1. Continuous Evolution of LMM-type Quantities: Let $\vec{X}$ be the vector of financial variables that need to be mapped to the corresponding Weiner variates $\vec{Z}$. In LMM, for e.g., start with

$$\vec{X}(0) = \{X_1(0), X_2(0), \cdots, X_n(0)\}$$

then the LMM evolutionary techniques generate $\vec{Z}$ and update $\vec{X}(t)$.

- Any continuous entity can be chosen to model correlations, not just LMM-type asset movements. For instance, if a default process can be correspondingly transformed to an asset indicator variable, that may be correlated with the other asset variables too.
- For a set of correlated variates, the stochastic evolution equation is

$$X_j(t+h) = X_j(t) + h\mu_j(\vec{X},t) + \sqrt{h}\sigma_j(\vec{X},t)\sum_{l=1}^{m}\rho_{jl}(\vec{X},t)Z_l(t+h)$$

- Here $\sigma_j(\vec{X}, t)$ is the variance, and $\rho_{jl}(\vec{X}, t)$ is the correlation matrix – the variance is factored out of the covariance matrix to produce the correlation grid. $Z_l(t+h)$ is in the usual i.i.d. $\mathcal{N}(0, 1)$.
- The corresponding delta is

$$\left[\frac{\partial \vec{X}(t+h)}{\partial X_k(0)}\right] = [D]\left[\frac{\partial \vec{X}(t)}{\partial X_k(0)}\right]$$

- The entry in matrix D is given as

$$D_{ij} = \delta_{ij} + h\frac{\partial \mu_j(\vec{X}, t)}{\partial X_i(t)} + \sqrt{h}\sum_{l=1}^{m} Z_l(t+h)\left\{\rho_{jl}(\vec{X}, t)\frac{\partial \sigma_j(\vec{X}, t)}{\partial X_i(t)} + \sigma_j(\vec{X}, t)\frac{\partial \rho_{jl}(\vec{X}, t)}{\partial X_i(t)}\right\}$$

- The corresponding parameter sensitivity is:

$$\left[\frac{\partial \vec{X}(t+h)}{\partial \alpha}\right] = [D]\left[\frac{\partial \vec{X}(t)}{\partial \alpha}\right]$$

  This may be simplified in cases where α is an explicit function ONLY of the state evolution variables as

$$\frac{\partial X_j(t+h)}{\partial \alpha} = \frac{\partial X_j(t)}{\partial \alpha} + h\frac{\partial \mu_j(\vec{X}, t)}{\partial \alpha} + \sqrt{h}\sum_{l=1}^{m} Z_l(t+h)\left\{\rho_{jl}(\vec{X}, t)\frac{\partial \sigma_j(\vec{X}, t)}{\partial \alpha} + \sigma_j(\vec{X}, t)\frac{\partial \rho_{jl}(\vec{X}, t)}{\partial \alpha}\right\}$$

2. Correlated Default Times: Unlike the continuous variables above, if we are to consider the correlations between default times ONLY, it is much more efficient to draw correlated

default times – again this correlation is different from that of continuous asset value times that results in default.

3. Generation of Correlated Default Times:
   - Generate the vector $\vec{Z}_{INDEPENDENT}$.
   - Factorize the correlation matrix ρ_{jk} to create the Cholesky diagonal matrices C and C^{T}.
   - Use the Cholesky transformation to create $\vec{Z}_{CORRELATED}$ from $\vec{Z}_{INDEPENDENT}$ using

$$\vec{Z}_{CORRELATED} = C\vec{Z}_{INDEPENDENT}$$

   - For each entity $\tilde{z}_i$ in $\vec{Z}_{CORRELATED}$:
     i. Evaluate the cumulative normal

$$y_i = \int_{x=-\infty}^{\tilde{z}_i} \mathcal{N}(0,1)dx$$

     where $\mathcal{N}(0,1)$ is a Normal distribution with unit mean and zero variance.

     ii.

$$\tau_{i,DEFAULT} = S_i^{-1}(y_i)$$

     where S_i is the survival probability for the entity i.

     iii. More generally remember that

$$X_i = M_i^{-1}(y_i)$$

LMM Forward Rate Evolution

1. Importance of the LMM Formulation: 2 reasons why it is important:

- LMM is one of the most popularly used formulation, and it is essential to evaluate the impact the no-arbitrage constrained drift has on the evolution and the impact on the greeks.
- The lognormal nature of the forward rate $\vec{L}(t)$ is important in its own right.

2. No-arbitrage constraint specification:

$$\mathcal{L}_j(t+h) = \mathcal{L}_j(t) + h\mu_j\left(\vec{\mathcal{L}}(t), t\right) + \sqrt{h}Z_l(t+h)\sigma_j\left(\vec{\mathcal{L}}(t), t\right)$$

where

$$\mu_j\left(\vec{\mathcal{L}}(t), t\right) = b_j\mathcal{L}_j(t) \sum_{p=\eta(t)}^{j} \frac{b_p\Delta\left(t_{p-1}, t_p\right)\mathcal{L}_p(t)}{1 + \Delta\left(t_{p-1}, t_p\right)\mathcal{L}_p(t)}$$

$$\sigma_j\left(\vec{\mathcal{L}}(t), t\right) = b_j\mathcal{L}_j(t)$$

and $\eta(t)$ is the maturity of the first instrument that matures after t [Brace, Gatarek, and Musiela (1997), Jamshidian (1997)].

3. Forward Rate Volatility vs. At-the-Money Swap Option Volatility: LMM uses forward rate volatilities, so there needs to be a conversion step that involves converting the market observed at-the-money swap option volatility onto LMM forward rate volatility [Brigo and Mercurio (2001)].

- Self-Jacobian of the extended LMM Formulation: As shown in Denson and Joshi (2009a) and Denson and Joshi (2009b)

$$\frac{\partial\mathcal{L}_j(t+h)}{\partial\mathcal{L}_k(0)} = \frac{\partial\mathcal{L}_j(t)}{\partial\mathcal{L}_k(0)} + \sum_{i=1}^{n}\left\{h\frac{\partial\mu_j\left(\vec{\mathcal{L}}(t), t\right)}{\partial\mathcal{L}_i(t)} + \sqrt{h}Z_l(t+h)\frac{\partial\sigma_j\left(\vec{\mathcal{L}}(t), t\right)}{\partial\mathcal{L}_i(t)}\right\}\frac{\partial\mathcal{L}_i(t)}{\partial\mathcal{L}_k(0)}$$

where

$$\frac{\partial \sigma_j\left(\vec{\mathcal{L}}(t), t\right)}{\partial \mathcal{L}_i(t)} = \delta_{ij} b_i$$

$$\frac{\partial \mu_j\left(\vec{\mathcal{L}}(t), t\right)}{\partial \mathcal{L}_i(t)}(\eta(t) \le i) = \delta_{ij} b_j \sum_{p=\eta(t)}^{j} \frac{b_p \Delta\left(t_{p-1}, t_p\right) \mathcal{L}_p(t)}{1 + \Delta\left(t_{p-1}, t_p\right) \mathcal{L}_p(t)} + \frac{b_j \mathcal{L}_j(t)}{[1 + \Delta(t_{i-1}, t_i) \mathcal{L}_i(t)]^2}$$

$$\frac{\partial \mu_j\left(\vec{\mathcal{L}}(t), t\right)}{\partial \mathcal{L}_i(t)}(\eta(t) > i) = \delta_{ij} b_j \sum_{p=\eta(t)}^{j} \frac{b_p \Delta\left(t_{p-1}, t_p\right) \mathcal{L}_p(t)}{1 + \Delta\left(t_{p-1}, t_p\right) \mathcal{L}_p(t)}$$

4. Forward-Rate Evolution Matrix: As expected

$$\left[\frac{\partial x(t+h)}{\partial x_k(0)}\right]^{\mathrm{T}} = \left[\frac{\partial x(0)}{\partial x_k(0)}\right]^{\mathrm{T}} [D(k, 0)]^{\mathrm{T}} [D(k, h)]^{\mathrm{T}} \cdots [D(k, t-h)]^{\mathrm{T}} [D(k, t)]^{\mathrm{T}}$$

where

$$D_{ij}\left(\vec{\mathcal{L}}(t), t\right)(\eta(t) \le i) = \delta_{ij} \left\{1 + h b_j \sum_{p=\eta(t)}^{j} \frac{b_p \Delta\left(t_{p-1}, t_p\right) \mathcal{L}_p(t)}{1 + \Delta\left(t_{p-1}, t_p\right) \mathcal{L}_p(t)} + \sqrt{h} b_j Z_j(t+h)\right\} + \frac{h b_j \mathcal{L}_j(t)}{[1 + \Delta(t_{i-1}, t_i) \mathcal{L}_i(t)]^2}$$

and

$$D_{ij}\left(\vec{\mathcal{L}}(t), t\right)(\eta(t) > i) = \delta_{ij} \left\{1 + h b_j \sum_{p=\eta(t)}^{j} \frac{b_p \Delta\left(t_{p-1}, t_p\right) \mathcal{L}_p(t)}{1 + \Delta\left(t_{p-1}, t_p\right) \mathcal{L}_p(t)} + \sqrt{h} b_j Z_j(t+h)\right\}$$

5. Variate Jacobian Parameter Sensitivity:

$$\frac{\partial \mathcal{L}_j(t+h)}{\partial \alpha} = \frac{\partial \mathcal{L}_j(t)}{\partial \alpha} + h\frac{\partial \mu_j\left(\vec{\mathcal{L}}(t),t\right)}{\partial \alpha} + \sqrt{h}Z_j(t+h)\frac{\partial \sigma_j\left(\vec{\mathcal{L}}(t),t\right)}{\partial \alpha} + \sum_{i=1}^{n} D_{ij}\left(\vec{\mathcal{L}}(t),t\right)\frac{\partial \mathcal{L}_i(t)}{\partial \alpha}$$

where $D_{ij}\left(\vec{\mathcal{L}}(t),t\right)$ is available from above for the two scenarios. Re-casting the above, we get

$$\frac{\partial \mathcal{L}_j(t+h)}{\partial \alpha} = B_j\left(\vec{\mathcal{L}}(t),t\right) + \sum_{i=1}^{n} D_{ij}\left(\vec{\mathcal{L}}(t),t\right)\frac{\partial \mathcal{L}_i(t)}{\partial \alpha}$$

where

$$B_j\left(\vec{\mathcal{L}}(t),t\right) = \mathcal{L}_j(t)\left\{\sqrt{h}Z_j(t+h)\frac{\partial b_j}{\partial \alpha} + h\sum_{p=\eta(t)}^{j}\frac{b_p\Delta\left(t_{p-1},t_p\right)\mathcal{L}_p(t)}{1+\Delta\left(t_{p-1},t_p\right)\mathcal{L}_p(t)}\left[b_p\frac{\partial b_j}{\partial \alpha} + b_j\frac{\partial b_p}{\partial \alpha}\right]\right\}$$

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Formulation of Sensitivities for Pay-off Functions

Formulation of Pay-off Function Stochastic Evolution

1. Monte-Carlo Path-wise Derivatives: Path-wise derivatives are typically forward derivatives, not adjoint [Giles and Glasserman (2006)]. Therefore computation time is proportional to the number of inputs. Further, not easy to accommodate these in complex payouts [Capriotti (2011)].
2. Payoff Expectation Formulation:

$$V = \mathbb{E}^{Q}\left[P\left(\vec{X}\right)\right]$$

   [Harrison and Kreps (1979)], where $\vec{X}$ is the vector of financial variables.

   - Path Payoff Expectation [Kallenberg (1997)] =>

$$V = \frac{1}{N_{MC}} \sum_{i=1}^{N_{MC}} P\left(\vec{X}[i]\right)$$

     and

$$Variance = \frac{{N_{MC}}^{2} \sum_{i=1}^{N_{MC}} \left\{P\left(\vec{X}[i]\right)\right\}^{2} - \left\{\sum_{i=1}^{N_{MC}} P\left(\vec{X}[i]\right)\right\}^{2}}{{N_{MC}}^{2}}$$

Path Greek

1. Unbiased Estimate of Path Sensitivity: Estimate is unbiased [Kunita (1990), Broadie and Glasserman (1996), Glasserman (2004)] if

$$\langle\frac{\partial Y(x)}{\partial x(0)}\rangle = \frac{\partial}{\partial x(0)}\langle Y(x)\rangle$$

where $x(0)$ is the starting point for the variate.

2. Monte-Carlo Greek Definition: Greek is defined at the change in Y with respect to the starting value of x, i.e., $x(0)$.

$$\frac{\partial Y\big(x(t)\big)}{\partial x(0)} = \frac{\partial Y\big(x(t)\big)}{\partial x(0)}\frac{\partial x(t)}{\partial x(0)}$$

If x is a multi-component vector $\vec{X}$, then

$$\frac{\partial Y\left(\vec{X}(t)\right)}{\partial x_j(0)} = \sum_{j=1}^{n}\frac{\partial Y\left(\vec{X}(t)\right)}{\partial x_j(0)}\frac{\partial x_j(t)}{\partial x_j(0)}$$

3. Pay-off Function Delta:

$$\frac{\partial V(t)}{\partial x_k(0)} = \sum_{j=1}^{n}\frac{\partial V(t)}{\partial x_j(0)}\frac{\partial x_j(t)}{\partial x_k(0)}$$

Now use the earlier formulation for $\left[\frac{\partial x(t)}{\partial x_k(0)}\right]$ to establish the path delta. In particular, using above,

$$\left[\frac{\partial V(t)}{\partial x_k(0)}\right]^{\mathrm{T}} = \left[\frac{\partial V(0)}{\partial x_k(0)}\right]^{\mathrm{T}}[D(k,0)]^{\mathrm{T}}[D(k,h)]^{\mathrm{T}}\cdots[D(k,t-h)]^{\mathrm{T}}[D(k,t)]^{\mathrm{T}}$$

so all the speed up advantages associated with the adjoint formulation above follows.

4. Variance in the Greeks in addition to the base Greeks:

- Cluster all the Path-wise Greeks calculated for a given input (either $x_k(0)$ or a parameter α).
- Within that cluster estimate the corresponding Greek.
- Usual population sampling variance techniques applied to compute the variance in the Greek.

5. Path Parameter α Sensitivity:

$$\frac{dV(t)}{d\alpha} = \frac{\partial V(t)}{\partial \alpha} + \sum_{j=1}^{n} \frac{\partial V(t)}{\partial x_j(0)} \frac{\partial x_j(t)}{\partial \alpha}$$

Now use the earlier formulation for $\left[\frac{\partial x(t)}{\partial \alpha}\right]$ to establish the path parameter sensitivity.

6. Explicit Pay-off Greek Formulation:

$$\frac{\partial x_j(t+h)}{\partial \alpha} = h \frac{\partial \mu_j(t)}{\partial \alpha} + \sqrt{h} \sum_{l=1}^{m} Z_l(t+h) \frac{\partial \sigma_{jl}(t)}{\partial \alpha} + \sum_{i=1}^{n} \left\{ \delta_{ij} + h \frac{\partial \mu_j(t)}{\partial x_i(t)} + \sqrt{h} \sum_{l=1}^{m} Z_l(t+h) \frac{\partial \sigma_{jl}(t)}{\partial x_i(t)} \right\} \frac{\partial x_i(t)}{\partial \alpha}$$

- Notice that it has additional terms since the explicit dependence of μ and σ on α is, in general, non-zero: otherwise

$$B_j(t, \alpha) = 0$$

and the pay-off parameter sensitivity formulation proceeds precisely along the same lines as delta formulation.

- First re-jig:

$$\frac{\partial x_j(t+h)}{\partial \alpha} = B_j(t, \alpha) + \sum_{i=1}^{n} D_{ij}(t, \alpha) \frac{\partial x_i(t)}{\partial \alpha}$$

where $D_{ij}(t, \alpha)$ is exactly the same as earlier, and

$$B_j(t, \alpha) = h\frac{\partial \mu_j(t)}{\partial \alpha} + \sqrt{h}\sum_{l=1}^{m} Z_l(t+h)\frac{\partial \sigma_{jl}(t)}{\partial \alpha}$$

- Second Re-jig:

$$\left[\frac{\partial \vec{x}(t+h)}{\partial \alpha}\right] = [B(t, \alpha)] + [D(t, \alpha)]\left[\frac{\partial \vec{x}(t)}{\partial \alpha}\right]$$

where $\left[\frac{\partial \vec{x}(t+h)}{\partial \alpha}\right]$, $\left[\frac{\partial \vec{x}(t)}{\partial \alpha}\right]$, and $[B(t, \alpha)]$ are $n \times 1$ column matrices, and $[D(t, \alpha)]$ is an $n \times n$ square matrix.

- Third Re-jig: Generalizing over all the j's, we get

$$\left[\frac{\partial \vec{x}(t+h)}{\partial \alpha}\right] = \sum_{e=0}^{s}\left(\left\{\prod_{f=1}^{e}[D(t-fh)]\right\}[B(t-eh, \alpha)]\right) + [D(t, \alpha)]\left[\frac{\partial \vec{x}(t)}{\partial \alpha}\right] + \left\{\prod_{e=1}^{s}[D(t-eh)]\right\}\left[\frac{\partial \vec{x}(t)}{\partial \alpha}\right]$$

- Fourth re-jig: Transposing the above we get

$$\left[\frac{\partial \vec{x}(t+h)}{\partial \alpha}\right]^{\mathrm{T}} = \sum_{e=s}^{1}\left([B(t-eh, \alpha)]^{\mathrm{T}}\left\{\prod_{f=e}^{1}[D(t-fh)]^{\mathrm{T}}\right\}\right) + \left[\frac{\partial \vec{x}(t)}{\partial \alpha}\right]^{\mathrm{T}}[D(t, \alpha)]^{\mathrm{T}} + \left\{\prod_{e=s}^{1}\left[\frac{\partial \vec{x}(t)}{\partial \alpha}\right]^{\mathrm{T}}[D(t-eh)]^{\mathrm{T}}\right\}$$

- Implications of the re-jig: Given that $[B(t,\alpha)]^{\mathrm{T}}$ and $\left[\frac{\partial\vec{x}(t)}{\partial\alpha}\right]^{\mathrm{T}}$ are now row matrices, and that they are the preceding terms in the series, all the adjoint advantages indicated earlier continue to be valid. Further the previous formulations for $[D(t,\alpha)]$ can be re-used at the same Eulerian time step.
- Adjoint Storage Demands: Remember that $[B(t,\alpha)]$ and $[D(t,\alpha)]$ still need to be retained in memory during the forward evolutionary sweep for $\left[\frac{\partial\vec{x}(t)}{\partial\alpha}\right]$, so this represents a corresponding increase on the storage requirements.

Payoff Sensitivity to the Correlation Matrix

1. Payoff Sensitivity Formulation: Irrespective of where the stochastic process is diffusive or not,

$$\frac{\partial V}{\partial\rho_{jk}} = \sum_{i=1}^{n}\frac{\partial V}{\partial\tilde{z}_j}\frac{\partial\tilde{z}_i}{\partial\rho_{jk}}$$

   where ρ_{jk} is the correlation matrix.
2. Financial Variable to Correlated Random Partial:
   - Recall that if

$$y(z) = \int_{x=-\infty}^{x=z}\Phi(x)dx$$

   then

$$\frac{\partial y}{\partial z} = \Phi(z)$$

- From this, and using

$$X_i = M_i^{-1}(y_i)$$

you can derive

$$\frac{\partial X_i}{\partial \tilde{z}_i} = \Phi(\tilde{z}_i)\frac{\partial X_i}{\partial \Phi(X_i)}$$

3. Differential of the Cholesky Factorization Matrix:

$$\frac{\partial \tilde{z}_i}{\partial \rho_{jk}} = \sum_{l=1}^{n}\sum_{m=1}^{n}\frac{\partial \tilde{z}_i}{\partial C_{lm}}\frac{\partial C_{lm}}{\partial \rho_{jk}}$$

where $\frac{\partial C_{lm}}{\partial \rho_{jk}}$ is readily computed. Therefore

$$\frac{\partial V}{\partial \rho_{jk}} = \sum_{i=1}^{n}\frac{\partial V}{\partial \tilde{z}_i}\frac{\partial \tilde{z}_i}{\partial \rho_{jk}} = \sum_{i=1}^{n}\frac{\partial V}{\partial X_i}\frac{\partial X_i}{\partial \tilde{z}_i}\frac{\partial \tilde{z}_i}{\partial \rho_{jk}} = \sum_{i=1}^{n}\frac{\partial V}{\partial X_i}\Phi(\tilde{z}_i)\frac{\partial X_i}{\partial \Phi(X_i)}\frac{\partial \tilde{z}_i}{\partial \rho_{jk}}$$

where $\frac{\partial \tilde{z}_i}{\partial \rho_{jk}}$ is given from above.

Algorithmic Differentiation in Payoff Sensitivities Calculation

1. Monte-Carlo Path-wise Derivatives: Path-wise derivatives are typically forward derivatives, not adjoint (Giles and Glasserman (2006)). Therefore computation time is proportional to the number of inputs.
2. Forward Monte-Carlo evolution variates: The full set forward evolution variates is still needed for extracting the fields/parameters required for the delta estimation of the adjoint path.

3. Corresponding storage requirements: All the variates set (the transition matrices etc.) still need to be maintained, so this represents an increase in the storage needed.
4. Adjointing vs. Reverse Mode: Typically adjoint refers **ONLY** to the intermediate/dynamical matrices [Giles (2007), Giles (2009)], whereas **REVERSE** refers to calculation of only the relevant outputs and their sensitivities [Griewank (2000)].
    - Adjointing deals with the evolved latent state space parameters left to right, therefore technically it is still forward in the time sense – and achieves optimization by minimizing the matrix<->matrix computations.
    - In the non-matrix sense (as in adjoint automatic differentiation), the term reverse and adjoint are analogous, i.e., adjoint/reverse refer to a scan backwards from right to left inside the SAME step, for e.g., a time step.
    - Finally, formalized pure "forward" and pure "reverse" is often theoretical constructs. Just like hand-rolled code can beat generic optimizers, hand-rolled algorithmic differentiation code will be better – even for Monte-Carlo sensitivity runs. However, development productivity gains to be attained by using automated AD tools are well documented.
5. Systematic Design Paradigm for using Automatic Differentiation for Path-wise Monte-Carlo Derivatives: Capriotti and Giles (2011) detail several techniques for this.
6. Cost:
    - B => Base; F => Forward; R => Reverse.
    - Forward Automatic Differentiation Cost =>

$$\frac{Cost[B+F]}{Cost[B]} = [2, 2.5]$$

    - Reverse Automatic Differentiation Cost =>

$$\frac{Cost[B+F+R]}{Cost[B]} = [4, 5]$$

7. Calibration along with Automatic Sensitivities Generation: Automatic Differentiation is natural performance fit in these situations (Kaebe, Maruhn, and Sachs (2009), Schlenkirch

(2011)). Many approaches in this regard end up utilizing intermediate value theorem to facilitate the formulation (Christianson (1998), Giles and Pierce (2000)).

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Bermudan Swap Option Sensitivities

Base Formulation

1. Option Valuation under Monte-Carlo: Unlike typical closed forms (such Black-Scholes, Black etc.), volatility does not explicitly show up in the PV generation part for options. Instead, it features intrinsically, through the evolution dynamics, and from the valuation of the underlying that needs to be valued under a specific exercise scenario.
2. $H \times M$ Bermudan Swap Option Details:
   - Define the M swap exercise/pay date tenor grids

$$T_0 < T_1 < \cdots < T_M$$

   - Option exercise dates T_r start from date T_H onwards, i.e.

$$T_r \in \{T_H, T_{H+1}, \cdots, T_{M-1}\}$$

   - The cash flow stream after the exercise is the payment stream

$$\vec{X} \in \{X_r, X_{r+1}, \cdots, X_M\}$$

3. $H \times M$ Exercised Bermudan Swap Valuation:

$$X_i = N(T_i)\Delta(t_{i-1}, t_i)[\mathcal{L}_i - \mathcal{R}]$$

   where $\mathcal{R}$ is the fixed rate. The Bermudan Swap PV is

$$PV_{Berm}(T_r) = \mathbb{E}\left[\sum_{i=r}^{M} D_f(t_i)X_i\right]$$

where $\mathbb{E}[\cdots]$ is the expectation operator.

4. $H \times M$ Bermudan Swap Valuation SKU:
   - Simulate a single path sequence of $\vec{\mathcal{L}}$.
   - For this path, evaluate $PV(X_i)$ for each $\vec{X} \in \{X_r, X_{r+1}, \cdots, X_M\}$.
   - For this path, generate a vector of $PV_{Berm}(T_p)$ corresponding to each possible exercise date $T_p \in \{T_H, T_{H+1}, \cdots, T_{M-1}\}$.
   - Find T_r that maximizes $PV_{Berm}(T_p)$.
   - Record $\{T_r, PV_{Berm}(T_p)\}$.

Greek Estimation

1. $H \times M$ Exercised Bermudan Swap Option Delta/Parameter Sensitivity [Piterbarg (2004), Capriotti and Giles (2011)]:

$$\frac{\partial PV_{Berm}(T_p)}{\partial \mathcal{L}_k(0)} = \frac{\partial\{\mathbb{E}[\sum_{i=r}^{M} D_f(t_i)X_i]\}}{\partial \mathcal{L}_k(0)} = \mathbb{E}\left[\sum_{i=r}^{M} \frac{\partial\{D_f(t_i)X_i\}}{\partial \mathcal{L}_k(0)}\right]$$

$$\frac{\partial PV_{Berm}(T_p)}{\partial \alpha} = \frac{\partial\{\mathbb{E}[\sum_{i=r}^{M} D_f(t_i)X_i]\}}{\partial \alpha} = \mathbb{E}\left[\sum_{i=r}^{M} \frac{\partial\{D_f(t_i)X_i\}}{\partial \alpha}\right]$$

2. Individual Cash-flow PV and Greeks [Leclerc, Liang, and Schneider (2009)]:

$$PV_j = D_f(t_j)\Delta(t_{j-1}, t_j)[\mathcal{L}_j - \mathcal{R}]$$

$$D_f(t_j) = \prod_{p=1}^{j} \frac{1}{1 + \Delta(t_{p-1}, t_p)\mathcal{L}_p}$$

implies that

$$PV_j = \left\{\prod_{p=1}^{j} \frac{1}{1 + \Delta(t_{p-1}, t_p)\mathcal{L}_p}\right\} \Delta(t_{j-1}, t_j)[\mathcal{L}_j - \mathcal{R}]$$

Remember that

$$\frac{\partial PV_j(t)}{\partial \mathcal{L}_k(0)} = \sum_{i=1}^{n} \frac{\partial PV_j(t)}{\partial \mathcal{L}_i(0)} \frac{\partial \mathcal{L}_i(t)}{\partial \mathcal{L}_k(0)}$$

where $\frac{\partial \mathcal{L}_i(t)}{\partial \mathcal{L}_k(0)}$ is given by the LMM formulation presented earlier.

3. Cash-flow PV Delta:

$$\frac{\partial PV_j(t)}{\partial \mathcal{L}_i(t)} = \frac{\partial}{\partial \mathcal{L}_i(t)} \left\{ \left\{\prod_{p=1}^{j} \frac{1}{1 + \Delta(t_{p-1}, t_p)\mathcal{L}_p}\right\} \Delta(t_{j-1}, t_j)[\mathcal{L}_j - \mathcal{R}] \right\}$$

$$\frac{\partial PV_j(t)}{\partial \mathcal{L}_i(t)}[j \geq i] = \left[\delta_{ij} - \frac{\Delta(t_{i-1}, t_i)[\mathcal{L}_j - \mathcal{R}]}{1 + \Delta(t_{i-1}, t_i)\mathcal{L}_i(t)}\right] \Delta(t_{j-1}, t_j) D_f(t_j)$$

$$\frac{\partial PV_j(t)}{\partial \mathcal{L}_i(t)}[j < i] = 0$$

LSM Methodology

1. Curve-Fitting to Extract Optimal Exercise: Since the simple model of maximizing $PV_{Berm}(T_r)$ across T_r gets too cumbersome if the exercise dates are numerous – LSM based optimal exercise determination laid out in [Longstaff and Schwartz (2001)] can be used – regress T_r against $PV_{Berm}(T_r)$.
2. Continuous or Fine-grained Call Schedules: LSM is highly effective in these situations. Sampling is reduced to a few evenly spaced-out grid points – such that the full sample scoping is eliminated.
3. Interpolation between Sampled Nodes: Any appropriate inter-nodal interpolating/splining technique to determine $PV_{Berm}(T_r)$ as a function of T_r is valid – e.g., constant $PV_{Berm}(T_r)$ over T_r, linear/quadratic/polynomial $PV_{Berm}(T_r)$ over T_r, or even exponential/hyperbolic tension spline-based $PV_{Berm}(T_r)$ over T_r.

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Basket Sensitivities

NTD Product Formulation

1. Running Index Details: Let

$$p = 1, \cdots, n$$

refer to the number of components in the basket,

$$j, k = 1, \cdots, n$$

identify the row/column index of the correlation matrix for each of the n components,

$$l, m = 1, \cdots, n$$

represent the factorized Cholesky diagonal matrix for the n components, r correspond to the r^{th} component in the current draw of ordered default times; it corresponds to the current n^{th}-to-default, and N point to the "N" in NTD, i.e.,

$$\tau_N \equiv \tau_r$$

2. Base NTD Pricing:

$$V_{NTD} = V_{LOSS} + V_{PREMIUM} + V_{ACCRUED}$$

$$V_{LOSS} = [1 - R_r(\tau_r)] D_f(\tau_r) N(\tau_r)$$

$$V_{PREMIUM} = c \sum_{i=1}^{n} N(t_i) D_f(t_i) \Delta(t_{i-1}, t_i) \mathbb{I}_{t_i \leq \tau_r}$$

$$V_{ACCRUAL} = c \sum_{i=1}^{n} N(\tau_r) D_f(\tau_r) \Delta(t_{i-1}, \tau_r) \mathbb{I}_{t_i \leq \tau_r} \mathbb{I}_{t_i \geq \geq \tau_r}$$

$\mathbb{I}_{t \leq \tau_r}$ represents the default indicator that is 1 if

$$t \leq \tau_r$$

and 0 otherwise.

- To make the computation convenient [Capriotti and Giles (2010), Capriotti and Giles (2011), Giles (2009), Chen and Glasserman (2008)] $\mathbb{I}_{t \leq \tau_r}$ is regularized and smeared out using an appropriate proxy, i.e.,

$$\mathbb{I}_{t \leq \tau_r} \cong \mathcal{H}(t \leq \tau_r)$$

- $\mathcal{H}(t \leq \tau_r)$ can be the Heaviside function.
- The proxy v has a bias, but it can be designed to be much tighter than the Monte-Carlo accuracy.

3. <u>NTD Sensitivity</u>:

$$\frac{\partial V_{NTD}}{\partial \rho_{jk}} = \sum_{p=1}^{n} \frac{\partial V_{NTD}}{\partial \tau_p} \frac{\partial \tau_p}{\partial \rho_{jk}}$$

$$\frac{\partial V_{NTD}}{\partial \tau_p} = \frac{\partial V_{LOSS}}{\partial \tau_p} + \frac{\partial V_{PREMIUM}}{\partial \tau_p} + \frac{\partial V_{ACCRUED}}{\partial \tau_p}$$

$$\frac{\partial V_{LOSS}}{\partial \tau_p} = \delta_{rp} \frac{\partial \left\{ [1 - R_r(\tau_r)] D_f(\tau_r) N(\tau_r) \right\}}{\partial \tau_p}$$

$$\frac{\partial V_{PREMIUM}}{\partial \tau_p} = c\delta_{rp} \sum_{i=1}^{n} N(t_i) D_f(t_i) \Delta(t_{i-1}, t_i) \frac{\partial \mathcal{H}(t_i \leq \tau_r)}{\partial \tau_p}$$

$$\frac{\partial V_{ACCRUAL}}{\partial \tau_p} = c\delta_{rp} \sum_{i=1}^{n} \frac{\partial \left\{ N(\tau_r) D_f(\tau_r) \Delta(t_{i-1}, \tau_r) \mathcal{H}(t_i \leq \tau_r) \mathcal{H}(t_i \geq \tau_r) \right\}}{\partial \tau_p}$$

Basket Options

1. Base Pricing Formulation:

$$V = D_f(T) \sum_{i=1}^{n} [W_i X_i(T) - S]^+$$

2. Basket Options Delta:

- Remember from earlier that

$$\frac{\partial V(t)}{\partial x_k(0)} = \sum_{j=1}^{n} \frac{\partial V(t)}{\partial x_j(t)} \frac{\partial x_j(t)}{\partial x_k(0)}$$

Here

$$t = T$$

$$V(t) = V_{BO}(t)$$

$$\frac{\partial V_{BO}\left(\vec{X}(T),T\right)}{\partial X_i(T)}$$

$$= \frac{\partial D_f(T)}{\partial X_i(T)}\sum_{p=1}^{n}\left[W_p X_p(T) - S\right]^+ + W_i D_f(T)$$

$$\times\ Black\_Scholes\_Delta\left(Strike_p, T\right)$$

where

$$Strike_p = \frac{S - \sum_{\substack{p \neq i, \\ p=1}}^{n} W_p X_p(T)}{W_i}$$

Reference

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Section XIV: Asset Backed Models and Analytics

PeerIQ Analytics Credit Model Methodology

Overview of Credit Model Methodology

1. Risk Estimation in MPL Platforms: PeerIQ (2015) has introduced version 2.0 of the PeerIQ Credit Model for estimating prepayment and default risk on a pool of US unsecured consumer loans issued by a select set of marketplace lending ("MPL") platforms.
2. Valuation, Prepay, and Default Drivers: This extends the version 1.0 PeerIQ Credit Model published previously in August 2015. In both cases the goal is to enable the market participants understand the drivers behind valuation, prepayment, and default risk across their holdings in a transparent and robust manner.
3. Usage of Markov Logit Models: A main objective is to apply the rigor and forward looking tools of mortgage analytics, namely the Multinomial Logit and the Markov Chain approach to modeling the MPL collateral.
4. Simulation of Prepay and Default: The second objective is to have the ability apply simulations to prepayment and default rates, and ultimately the cash flows.
5. Usage of Statistical Learning Techniques: The final objective is to use machine learning techniques such as regularization and cross-validation to improve the robustness of the modeling.

Credit Methodology – Purpose and Introduction

1. MPL Growth/Institutional Investors Participation: The dramatic growth in Marketplace Lending (MPL) has been paired with the transition of the consumer and the SME credit to the

capital markets. Institutional investors are increasingly funding credit risk in whole loan, structured credit, or warehouse formats.

2. MPL Risk and Analytics Tools: A new set of analytics and risk pricing tools are necessary to enable institutional capital efficiently access, price, and exchange risk. A core component of risk pricing are independent 3rd party credit models that forecast cash flows on pools of loans.
3. Improvement in Liquidity and Transparency: By releasing a new credit model, PeerIQ aims to improve transparency and standards in MPL by providing risk management tools promoting independent pricing and whole loan and ABS market liquidity.
4. Valuation and Risk MPL Common Language: Independent credit models increase activity by allowing market participants to speak a common language in reference to valuation and risk.
5. Independent 3rd Party Credit Models: Several sources have described on various occasions the robust scaling and growth in the US and global marketplace lending sector. Large institutional investors require a 3rd party credit model to understand their credit risk exposures, and to participate in size.
6. Short Duration, High Yield Risk: Further, credit spreads have generally tightened since the 2008 credit crisis in a backdrop of quantitative easing, healthy global economy, and stringent regulation. As a result, investors see MPL offering an attractive short-duration, high-yield credit risk compared to the alternatives.
7. Engagement in the MPL Space: Investors have been engaged in the market through a variety of activities including, but not limited to:
    a. Investing in marketplace lending platform equity
    b. Facilitating funding for the platforms via a provision of credit facilities
    c. Investing in MPL securitizations
    d. Directly lending to borrowers.
8. Proliferation Among MPL Asset Classes: In addition, there has been a proliferation of marketplace lenders across a multitude of asset classes including consumer, purchase finance, education finance, real estate, merchant cash advance, and small businesses.
9. Asset Pricing and Regulation Obligation: As a consequence, institutional investors, diversified financial services firms, and funding providers require an independent 3rd party to help them price their holdings or satisfy other regulatory obligations.

10. MPL Valuation Standards Methodology Enhancement: Finally it is imperative that 3rd parties commit to improving and enhancing their standards and methodologies to serve what is a rapidly growing and evolving market. The PeerIQ 2.0 Credit Model is an attempt at enhancements over several key areas foe projections of cash flows on historical data above 1.0.
11. Additional Loan Specific Risk Factors: 2.0 addresses the need to incorporate additional loan specific measures of risk. Version 1.0 segments the loan by 6 factors:
    a. The originator
    b. Loan credit quality grade as provided by the originator
    c. Origination vintage
    d. Loan term
    e. Loan status
    f. Loan age
12. Incorporating Macro-economic Driving Factors: 2.0 provides a structure for incorporating macro-economic factors that drive the estimates of default and prepayment.
13. Reducing Dependence on Recent Issuance: Reducing the reliance on the most recently issued set of loans to drive expectation of prepay and default is another objective (1.0 applied the default and prepay experience of the most recently issued cohort corresponding to the risk factors above).
14. Usage of Statistical Learning Techniques: Version 2.0 applies advanced statistical and machine learning techniques to develop a predictive model for prepay and default. Thus PeerIQ Model Version 2.0 aims to address and overcome the shortcomings from the Model 1.0 above, and bring rigorous techniques to an expanding asset class with a growing investor base.

Scope of PeerIQ Model 2.0

1. Loans Originated by Lending Club: For the purposes of demonstration, PeerIQ (2015) illustrate the construction and performance of Model Version 2.0 on public data from loans originated by Lending Club ("LC") with reporting months from 1 January 2010 to 1 July 2015.
2. The PeerIQ Data Model: PeerIQ's data model is proprietary and unified in its methodology for cleaning, enriching, and housing data across all MPL originators, and gets expanded as additional asset classes and originators are on-boarded.
3. Similarity with Lending Club Model: Although PeerIQ does tailor the Model to specific data classes and originators, the methodology and the model structure is substantially similar to the Lending Club model.

PeerIQ Data Model Construction Rules

1. Amendments to Originator Generated Payments: PeerIQ has made specific amendments (or transformations) on the raw originator-generated payments and balances for loans.
2. Consistency and Accuracy Across Cohorts: These changes have been made in a rule-based fashion based on conversations with marketplace lenders to ensure the calculation of the cohort payments and balances in a consistent and accurate manner.
3. Reconciliation Between Borrowers and Originators: Many of the above rules help reconcile between a borrower snapshot file (the borrower file) and a cumulative payments file (the payments file) published by the originators.

PeerIQ Loan Data Quality Rules

1. Inaccurate Originator Loan Level Record: Some originators publish inaccurate records for loans that have previously charged or paid off. These records are excluded from the cleaned data set and calculations.
2. Identification and Removal of Duplicates: If the loan has more than one record with the same originator loan ID, loan month, month on book, and outstanding principal BOP balance, it is assumed that the subsequent records are duplicates and that they must be removed.
3. Combine Payments for a Given Month: If the loan as multiple payments for a given month, then these must be combined to form a single payment. The formula applied is: combine all rows where count of loan month, originator loan ID > 1.
4. Entry for Maximum Loan Month: Each loan that has a non-zero EOP balance for a month prior to the maximum loan month should have a record for the maximum loan month. For example, if the maximum loan month on file is February 2015, and the loan is current in January 2015 but does not have a February 2015 record, a record will need to be created.
5. Loan Month Issue Date Consistency: Each loan ID should have a record where the loan month equals the issue date. Further, a record must exist for each loan between the issue date and the current file date, charge off date, or fully paid date, whichever is earlier.
6. Loan Age/Days Past Due: Days Past Due value should never be negative. The expected loan age is calculated as the loan month minus the issue date.
7. Charge Off/Fully Paid Fields:
    a. All instances where the

    $$CO\ Amount\ != 0$$

    should have the charge off flag set to 1.
    b. All loans that have a status of fully paid should have an end-of-period balance > 0.
    c. BOP principal minus principal received and charge-off amount should equal to 0.
8. Principal, Interest, and Fee Payments:
    a. Interest paid for the given loan on a given month should equal between the borrower and the originator
    b. Amount paid should equal the sum of principal, interest, and fees paid
    c. All principal payments, interest payments, and fee payments should be positive.

9. Field Unchanged Through the Loan Life: The following fields should remain unchanged and populated through the loan life: loan purpose, loan interest rate, loan grade, loan term, loan state, original principal, issuance date.
10. Consistency of the Recovery Fields: All recoveries should be positive, and should be recorded at the month the loan charges off.

Lending Club Loan Level Data

1. Lending Club Loan Types Considered: As a starting point for the demonstration of the modeling approach, PeerIQ uses the loan level public data from Lending Club. As such, the loan products considered are fixed rate, fixed term, fully amortizing 36 month and 60 month loans issued by Lending Club.
2. Number and Size of Loans: In all, PeerIQ (2015) uses over 9 million loan months of Lending Club data in constructing the model. The table below contains high level descriptive statistics for select items from the dataset.
3. Descriptive Statistics for LC Data: Source: PeerIQ Research

Field	Mean	Standard Deviation	Minimum	Maximum	First Quartile	Median	Third Quartile
Age (Months on Balance)	10.1	8.9	0.0	60.0	3.0	8.0	15.0
Vintage	February 2013	NA	February 2007	February 2015	April 2012	March 2013	February 2014
Original Principal	$14,254	$8,254	$500	$35,000	$8,000	$12,000	$20,000

Monthly Gross Income	$6,066	$4,599	$250	$725,549	$3,750	$5,167	$7,333
Term (Months)	42.7	10.8	36.0	60.0	36.0	36.0	60.0
Coupon	13.7%	4.3%	5.3%	29.0%	10.6%	13.5%	16.3%
FICO Origination	699	31	612	847	677	692	717
DTI (ex-mortgage)	16.6%	7.8%	0.0%	39.0%	11.0%	16.0%	22.0%
Total Borrower Accounts	25	11	1	162	16	23	31
Revolving Utilization Rate	57%	24%	0%	892%	40%	58%	75%
Inquiries in Last 6 Months	0.9	1.2	0.0	33.0	0.0	0.0	1.0
DQ Accounts in Last 2 Years	0.3	0.8	0.0	39.0	0.0	0.0	0.0
Months since Last DQ	34	22	0	188	16	31	50
Months since Last	76	29	0	129	55	79	102

Public Record							
Total Open Credit Lines	11	5	0	90	8	10	14

4. Period of LC Loan Origination: Overall, the sample used contains 245,243 distinct loans originated between February 2007 and February 2015. Average loan size is a little over $14,000, varying between $500 and $35,000 for Lending Club.
5. Variation Among the Underwriting Parameters: There is also considerable variation among other under-writing information, including DTI and revolving debt utilization rates, for example.
6. Defaults/Prepayments by Origination Year: Ultimately the goal is derive insight into termination events (defaults and prepays) from loan pools, and the table below lists some simple summaries of defaults and prepayments by origination year.
7. LC Prepay and Default Exits: Source: PeerIQ Research.

Origination Year	**Origination Volume ($mm)**	**Prepays ($mm)**	**Cumulative Defaults ($mm)**	**Cumulative Defaults to Date (%)**
2010	132	26	12	8.82%
2011	262	62	27	10.49%
2012	718	164	75	10.39%
2013	1,982	405	135	6.83%
2014	3,504	387	88	2.51%

8. Peaking of Defaults by Vintage: The table above shows that the defaults in the current cycle have peaked for the 2011 and the 2012 vintages.

PeerIQ Loan Credit Model Implementation

1. Discrete Loan Level Status Codes: The approach to modeling starts with the observation that at a given point in time, a loan has several status codes, or 'state's: current, delinquent, fully prepaid, or charged off.
2. Explicit Transition Between Loan States: The key conceptual idea is to model transitions between the various states explicitly. For example, a loan may move from a state of 'current' to '30-day delinquent', or move further down the delinquency queue (e.g., move from 30 day to 60 day delinquent). Alternatively, a loan may 'cure' and move from a status of current to delinquent.
3. Exit From the Transition Graph: The only exits for a loan from this network are a transition to prepay or default, which are of course the end statuses we are most interested in. The graph table below illustrates the transition network.
4. Directed State Transition Network Graph: Source: PeerIQ Research.

State	C	P	D3	D6	D6+	D
C	Y	Y	Y	N	N	Y
P	N	Y	N	N	N	N
D3	Y	Y	Y	Y	N	N
D6	N	N	Y	Y	Y	Y
D6+	N	N	N	Y	Y	Y
D	N	N	N	N	N	Y

5. State Transition Network Graph Annotation: Y's indicate directionality of the possible transitions. Note that prepay and default here are "absorbing" states, i.e., states from which exits are not possible. For modeling purposes, and due to data considerations, the 90-180 day delinquent states have been combined into the D6+ category.
    a. C => Current
    b. P => Prepay
    c. D3 => 30 day Delinquent
    d. D6 => 60 day Delinquent
    e. D6+ => 90 and more days Delinquent
    f. D => Default
6. Cross-State Transition Probabilities: Mathematically the probabilities of these moves can be represented in a transition matrix thereby describing the propensity of the borrower to move from one state to another.
7. State -> State Transition Setup:

$$P_{i,j}(t) = \begin{bmatrix} P_{C,C}(t) & \cdots & P_{C,D}(t) \\ \vdots & \ddots & \vdots \\ P_{6+,C}(t) & \cdots & P_{6+,D}(t) \end{bmatrix}$$

   Each entry in the matrix represents the probability of the borrower moving from the row state to the column state in a particular month. For example, $P_{C,C}(t)$ represents the probability of the borrower moving from current to current (that is, staying current) over month t, and so on.
8. Transition Probabilities From MNL Frameworks: The next step is to decide on the parameterization of the entries in the transition matrix, i.e., how does one model the transition probabilities $P_{i,j}(t)$ (commonly referred to as the 'roll rate's?).
9. Transition Matrix in Markov Process: The MNL framework has features prominently in the mortgage modeling literature in modeling prepay and default. Such a modeling setup is known as the *Markov* process, which is practice means that each monthly observation of a loan's transitions is independent of any prior observations.

10. MNL Formulation in Logit Framework: Under the logit assumption inherent in MNL, the transition probabilities assume the following form:

$$P_{i,j}(t) = \frac{e^{\overrightarrow{\beta_0}+\overrightarrow{\beta_1}\cdot\overrightarrow{X_{\iota,J}}(t)}}{1 + e^{\overrightarrow{\beta_0}+\overrightarrow{\beta_1}\cdot\overrightarrow{X_{\iota,J}}(t)}}$$

11. Loan Variables as Regressor Factors: Here $\overrightarrow{X_{\iota,J}}$ represents the matrix of regressors (or predictors) observed at a particular time (such as consumer credit variables, loan age, cohort, for example, and macro-economic variables such as inherent rates). Note that we are not restricted to using the same set of predictors for every transition.
12. Normalization Under the MNL Framework: The parametrization above implies that the probabilities $P_{i,j}(t)$ will lie between zero and one, as they should. Further, since the probabilities across a given row should sum up to one, there will always be a status s for which the probabilities are determined as

$$P_{i,s}(t) = 1 - \sum_{j} P_{i,j}(t) = \frac{1}{1 + \sum_k e^{\overrightarrow{\beta_0}+\overrightarrow{\beta_1}\cdot\overrightarrow{X_{\iota,k}}(t)}}$$

13. Reduction of the Transition Probabilities: In estimating the model, one does not generally estimate all the probabilities from one state to another, as this would make the models unnecessarily complex, especially if the transitions are rare.
14. Current to Charge-Off Transition: While there are various reasons a loan can theoretically transition from current to charge-off (skipping intermediate statuses such as delinquency), such as due to the death of the borrower, these tend to be rare empirically.
15. Current-to-Default Transition Likelihood: This, it is quite intuitive that the probability of going from a state of current to a state of charge-off in a month should be quiet low. As shown in PeerIQ (2015) this monthly transition rate can be seen to be at most 0.03% for 60 month loans. Therefore this probability is not estimated.
16. Estimation of the Sparse Matrix: Thus, a sparse matrix, which is a subset of all the possible transitions – is estimated. The grid below gives the transitions that are estimated in the model, and those that are not.

17. Sparse Transition Matrix and Determinants: Source: PeerIQ Research.

States	C	D3	D6	D6+	D	P
C	Y	Y	N	N	N	Y
D3	Y	Y	Y	N	Y	Y
D6	N	Y	Y	Y	N	N
D6+	N	N	Y	Y	Y	N

Clearly the number of transitions to be estimated will vary depending on the originator and the observed frequency.

18. Transition Probabilities Using Separate Models: Since there is no restriction to using the same set of regressors for each probability, it is instructive to think of each of the entries estimated in the sparse matrix as separate models.

PeerIQ Credit Model Selection Methodology

1. Cross Validation Based Model Selection: PeerIQ (2015) applies a rigorous out-of-sample based testing procedure to estimate each of the models discussed above. The procedure for the model selection is described as follows.
2. Candidate Variables for Model Selection: PeerIQ (2015) has identified a set of candidate variables for selection. Those variables and their data types are listed in the table below.
3. Candidate Variables and their Types: Source: PeerIQ Research.

Variable	Implementation Type

Age (Months Remaining)	Evenly Spaced Linear Splines
Term (36 or 60 months)	Categorical Variable
Interaction Variable (Age Spline * Term)	Linear Spline * Categorical Variable
DTI	Continuous Variable
FICO at Origination	Categorical Variable
Vintage (Origination Year)	Categorical Variable
Seasonality (Month of Year)	Categorical Variable
Original Loan Size Bucket	Categorical Variable
Coupon Stack Bucket	Categorical Variable
Loan Purpose	Categorical Variable
Employment Length	Categorical Variable
Inquiries in the Past 6 Months	Untransformed
Monthly Gross Income	Untransformed
Total Outstanding Accounts	Untransformed
Revolving Credit Utilization	Untransformed
Delinquent Accounts in last 2 Years	Untransformed
Total Open Credit Lines	Untransformed

4. Model Variables from Candidate Regressors: Altogether the combinations of the categorical and the numerical variables generate 86 candidate regressors from which the best choice for each transition probability is optimized. The details of the procedure for model variable selection follows.
5. Likelihood Estimation for Traditional MNL: Variable selection was achieved via a regularized logistic regression procedure. In traditional logistic regression one typically

optimizes for the values of $\overrightarrow{\beta_0}$ and $\overrightarrow{\beta_1}$ in the equation above. Applying maximum likelihood across borrowers, the formulation for the transition probability between states (for e.g., borrowers moving from current to prepay) follows, as shown below.

6. Example Current -> Prepay MLE Setup:

$$Max\left[L\left(\overrightarrow{\beta_0},\overrightarrow{\beta_1}\right)\right] = \sum_{m=1}^{N} \log P_{c,p,m}$$

where $P_{c,p,m}$ represents the probability of the m^{th} borrower moving from current to prepay (dropping time subscripts for convenience) with the probabilities given from

$$P_{i,j}(t) = \frac{e^{\overrightarrow{\beta_0}+\overrightarrow{\beta_1}\cdot\overrightarrow{X_{\iota,J}}(t)}}{1 + e^{\overrightarrow{\beta_0}+\overrightarrow{\beta_1}\cdot\overrightarrow{X_{\iota,J}}(t)}}$$

7. Penalty Based Regularized Logistic Regression: Regularized logistic regression is similar to the equation above, but it imposes a penalty on $\overrightarrow{\beta_0}$ and $\overrightarrow{\beta_1}$ to ensure that the coefficients that are not predictive of the transition probability are not unnecessarily added (and thus eliminated from the regression).
8. Current Prepay Penalized MLE Setup:

$$Max\left[L\left(\overrightarrow{\beta_0},\overrightarrow{\beta_1}\right)\right] = \sum_{m=1}^{N} \log P_{c,p,m} + \alpha \sum_{j=1}^{p} \left|\beta_j\right|$$

where β_j represents the set of predictors in $\overrightarrow{\beta_1}$, α represents the parameter that determines the strength of the penalty function.

9. Optimal Penalty Loading Selection Algorithm: The penalty loading parameter α is selected using the following step:
    a. Select a possible range of values for α

b. For each value of α in the range above, train the logistic regression model on a training dataset, and test the predictive power of the model on a separate test dataset
c. The test data is defined by sampling every third observation to minimize the risk of over-fitting the data
d. Choose the value of α which results in the 'best' performance on the testing dataset

10. Classifier Performance Using Sample AUC: The AUC is a well-known measure of predicting accuracy of a classification technique (Receiver Operating Characteristic (Wiki)). Thus, the 'best' in the algorithm above is decided by looking at that value of α that gives the best out-of-sample AUC, e.g., on classifying borrowers who move from current to prepay in a given month.

Empirical Results – Regressor Contribution Weights

1. C -> P Predictor Loading Strength Order:

Strong Positive	**Strong Inverse**	**Weak/Neutral**
a. Age_k0.0 b. Total_accounts c. Inq_6m	a. Revolving_utilization b. Total_open_credit_lines c. Age_k0.0_term_60 d. Age_k18.0 e. Dti_ex_mortgage	a. Coupon_20+ b. Cohort_2014 c. Cohort_2013 d. Fico_750_800 e. Lp_debt_consolidation f. Coupon_15-20 g. Cohort_2015 h. Mo_3 i. Mo_5 j. Mo_2 k. Mo_4 l. Size_15_20

		m. Mo_10 n. El_10+years o. El_<1year p. Mo_8 q. Mo_9 r. Mo_11 s. Size_gte20 t. Fico_650_700 u. Lp_small_business v. Coupon_<10 w. Cohort_2011 x. Cohort_2009 y. Cohort_2008 z. Cohort_2010 aa. El_n/a bb. Mo_12 cc. Term_60 dd. Age_k12.0

2. C -> D3 Predictor Loading Strength Order:

Strong Positive	**Strong Inverse**	**Weak/Neutral**
a. Age_k0.0 b. Inq_6m	a. Age_k6.0 b. Monthly_gross_income c. Age_k12.0	a. Coupon_20+ b. Coupon_15-20 c. Dti_ex_mortgage d. Dq_accounts_past_2_years e. Lp_small_business f. Lp_educational g. Cohort_2008 h. Lp_moving

		i. El_n/a j. Mo_10 k. Lp_other l. Fico_650_700 m. Mo_7 n. Lp_medical o. El_<1year p. Mo_6 q. Mo_11 r. Size_5_10 s. Mo_9 t. El_10+years u. Lp_major_purchase v. Lp_debt_consolidation w. Mo_3 x. Size_lte5 y. Cohort_2015 z. Mo_2 aa. Mo_5 bb. Fico_750_800 cc. Lp_wedding dd. Cohort_2014 ee. Total_accounts ff. Fico_800_850 gg. Term_60 hh. Lp_home_improvement ii. Age_k24.0 jj. Cohort_2012 kk. Cohort_2013 ll. Age_k24.0term60 mm. Coupon_<10

		nn. Total_open_credit_lines oo. Revolving_utilization

3. D3 -> D3 Predictor Loading Strength Order:

Strong Positive	Strong Inverse	Weak/Neutral
a. Age_k0.0	a. Cohort_2015	a. Coupon_15-20
b. Dq_accounts_past_2_years	b. Cohort_2014	b. Size_15_20
c. Cohort_2009	c. Age_k24.0term_60	c. Fico_650_700
d. Cohort_2008	d. Cohort_2013	d. El_4years
e. Cohort_2010	e. Size_lte5	e. Mo_7
f. Revolving_utilization	f. Total_open_credit_lines	f. Mo_10
g. Size_gte20	g. Age_k18.0term_60	g. Mo_8
h. Coupon_20+	h. Cohort_2012	h. Fico_750_800
	i. Term_60	i. Mo_12
	j. Size_5_10	j. El_3years
	k. El_7years	k. Mo_9
		l. Mo_5
		m. Coupon_<10
		n. Mo_11
		o. Lp_credit_card
		p. El_5years
		q. El_6years
		r. El_8years

4. D3 -> C Predictor Loading Strength Order:

Strong Positive	Strong Inverse	Weak/Neutral
a. Monthly_gross_income	a. Total_open_credit_lines	a. Revolving_utilization
b. Dq_accounts_past_2_years	b. Cohort_2014	b. Age_k12.0term_60

	c. Cohort_2015 d. Dti_ex_mortgage e. Cohort_2013	c. Mo_3 d. Mo_4 e. Lp_moving f. Fice_650_700 g. Size_lte5 h. Lp_car i. Cohort_2008 j. Lp_medical k. Cohort_2010 l. Size_5_10 m. Coupon_<10 n. Size_gte20 o. Cohort_2009 p. Lp_other q. Age_k12.0 r. Lp_debt_consolidation s. El_7years t. Mo_5 u. El_5years v. El_8years w. Size_15_20 x. El_9years y. Mo_6 z. Inq_6m aa. El_10+years bb. Mo_12 cc. Mo_7 dd. Term_60 ee. Cohort_2012 ff. Age_k30.0 gg. Mo_11

		hh. El_n/a

5. D3 -> P Predictor Loading Strength Order:

Strong Positive	Strong Inverse	Weak/Neutral
a. Total_accounts	a. Age_k30.0	a. Size_gte20
b. Age_k6.0	b. Age_k0.0term_60	b. Mo_10
c. Monthly_gross_income	c. Dti_ex_mortgage	c. Lp_credit_card
d. Fico_800_850	d. Lp_car	d. Mo_6
e. Cohort_2009	e. Mo_8	e. Mo_7
f. El_10+years	f. Lp_small_business	f. Cohort_2014
g. El_8years	g. Fico_700_750	g. Cohort_2013
h. El_5years	h. Size_lte5	h. Size_15_20
i. Cohort_2015		i. Cohort_2010
j. El_6years		j. Term_60

6. D3 -> D Predictor Loading Strength Order:

Strong Positive	Strong Inverse	Weak/Neutral
a. Age_k0.0	a. Coupon_15_20	a. Size_5_10
b. Total_accounts	b. El_3years	b. Cohort_2012
c. Age_k6.0	c. Coupon_20+	c. Mo_5
d. Dti_ex_mortgage		d. Mo_6
e. Size_lte5		e. El_7years
f. Cohort_2013		f. Fico_700_750
g. Cohort_2014		g. Term_60
h. El_10+years		h. El_<1year
		i. Mo_12
		j. Mo_2

		k. Size_15_20 l. Size_gte20 m. El_4years

7. D6 -> D6 Predictor Loading Strength Order:

Strong Positive	Strong Inverse	Weak/Neutral
a. Age_k0.0 b. Age_k6.0 c. Revolving_utilization d. Coupon_20+ e. Total_accounts f. El_4years	a. Cohort_2014 b. Cohort_2015 c. Cohort_2013 d. Cohort_2012 e. Cohort_2011 f. Age_k24.0term_60 g. Age_k30.0term_60 h. Size_lte5 i. Lp_major_purchase j. Term_60	a. Cohort_2009 b. Coupon_15-20 c. El_<1year d. Fico_750_800 e. Mo_5 f. Size_gte20 g. Lp_home_improvement h. Mo_2 i. Lp_other j. Lp_debt_consolidation k. Mo_3 l. Mo_4 m. El_10+years n. Age_k18.0term_60 o. Fico_700_750 p. Lp_credit_card q. Mo_10 r. Mo_12 s. Mo_11 t. Size_15_20 u. El_7years v. El_9years w. Mo_7

		x. El_6years y. Size_5_10 z. El_8years aa. Dti_ex_mortgage

8. D6 -> D6+ Predictor Loading Strength Order:

Strong Positive	**Strong Inverse**	**Weak/Neutral**
a. Age_k30.0term_60 b. Age_k24.0term_60 c. Age_k18.0term_60 d. Total_open_credit_lines	a. Age_k30.0 b. Total_accounts c. Age_k0.0 d. Cohort_2008 e. Age_k6.0	a. Cohort_2012 b. Coupon_20+ c. Mo_11 d. El_9years e. Cohort_2015 f. Size_gte20 g. Mo_7 h. Fico_650_700 i. El_4years j. Cohort_2013 k. Mo_8 l. Lp_credit_card m. Age_k36.0term_60 n. Mo_12 o. Mo_2 p. Coupon_<10 q. Monthly_gross_income r. El_<1year s. Mo_6 t. Fico_750_800 u. Mo_9 v. Term_60

		w. El_10+years x. Cohort_2009 y. El_n/a z. Cohort_2010 aa. Mo_4 bb. Age_k24.0 cc. Size_5_10 dd. Revolving_utilization ee. Size_lte5 ff. Mo_3

9. D6+ -> D6+ Predictor Loading Strength Order:

Strong Positive	**Strong Inverse**	**Weak/Neutral**
a. Age_k12.0 b. Cohort_2008 c. Cohort_2009	a. Age_k0.0 b. Cohort_2013 c. Cohort_2014 d. Cohort_2012 e. Cohort_2011	a. Mo_11 b. Mo_9 c. Revolving_utilization d. Coupon_20+ e. Fico_650_700 f. Size_gte20 g. Mo_7 h. Coupon_15-20 i. Age_k18.0 j. Mo_12 k. Lp_debt_consolidation l. El_3years m. El_6years n. Mo_6 o. Size_5_10 p. Lp_other

		q. Mo_4 r. Mo_10 s. Mo_3 t. Size_lte5 u. Age_k0.0term_60

10. D6+ -> D Predictor Loading Strength Order:

Strong Positive	Strong Inverse	Weak/Neutral
a. Cohort_2012 b. Cohort_2013 c. Cohort_2014 d. Mo_7 e. Term_60 f. Age_k24.0term_60 g. Mo_12 h. Mo_6 i. Cohort_2011	a. Age_k6.0 b. Age_k12.0 c. Dq_accounts_past_2_years d. Cohort_2008 e. Cohort_2009 f. Total_accounts g. Mo_2	a. Lp_car b. Size_15_20 c. Dti_ex_mortgage d. Mo_10 e. El_6years f. El_9years g. El_n/a h. Mo_5 i. El_5years j. El_7years k. El_3years l. Coupon_15_20 m. Lp_debt_consolidation n. Mo_9 o. El_10+years p. Mo_8 q. Mo_3 r. Size_lte5 s. Coupon_<10 t. Cohort_2010

Empirical Analysis of Seasoning Effects

1. Default and Prepay over Time: Perhaps one of the most important predictors of a loan's prepayment and default rate is just the passage of time. The empirical exhibits of the conditional prepayment C-P probabilities over age (months on balance) by PeerIQ (2015) illustrate that monthly prepayment probabilities generally increase with loan seasoning.
2. Time Impact on the Delinquency: Conversely, monthly rolls to delinquency (and ultimately to default) generally fall as a loan seasons. Conditional on the fact that a loan has not defaulted to a certain point in time, it is more generally likely to prepay and not default in the future.
3. Actual vs. Fitted C-P Probabilities: PeerIQ (2015) find that the fitted C-P probabilities correspond quiet well to the actual C-P probabilities indicating that the age splines are quiet explanatory (however less so at later ages where there is some noise due to the thinning of the sample size). Nonetheless the general uptrend of prepayment by loan age is evident in both the actual and the fitted C-P probabilities.
4. Hump in the Delinquency Curve: Conversely, rolls into delinquency peak early in the life of a loan (within the first 15 months of age), followed by a gradual decline through the life of the loan. The result is particularly true for the 60 month loans which (due to their longer term) season meaningfully towards the back end.

Analysis of the Vintage/Cohort Effects

1. Vintage/Cohort Impact on Loan: Vintage and cohort effects are designed to capture the difference in prepayment and default experience of loans identical in all respects except for the fact that they were originated at different times.
2. Impact on Prepay/Delinquency Rolls: While the monotonicity of the prepayment rates are likely due to the seasoning effects described above (for e.g., the relative lower prepayment probability of the younger 2015 vintage loans), the peak in fitted delinquency rolls around

the 2012 vintage for the 60 month loans and the2013 vintage for the loan 36 month loans is both interesting and consistent with the prepay/default exist in LC data seen before.

Analysis of Empirical Seasonality Effects

1. Usage of Origination DTI Metric: Once again the DTI metrics used in the model are at origination. Again, the results are quite intuitive and fit the data well.
2. DTI Impact on Prepay/Delinquency: Borrowers with higher DTI tend to prepay slower and tend to roll to delinquency at a higher rate, albeit not with the same effect as that observed on prepayment.

CPR And CDR Curve Estimation

1. Definition of CPR and CDR: The fit probabilities (and therefore the sparse transition matrices) obtained from the above are used to derive CPR and CDR curves. CPR and CDR curves are defined as

$$CDR(t) = 1 - \left[1 - \frac{Default(t)}{BeginBalance(t)}\right]^{12}$$

and

$$CPR(t) = 1 - \left[1 - \frac{Prepay(t)}{BeginBalance(t)}\right]^{12}$$

2. CPR/CDR for Loan Pools: CPR and CDR therefore represent an annualized measure of prepay and default for every dollar of principal outstanding at the start of a period. Clearly, in order to estimate the CPR and CDR, we need to project the cash flows for a given pool of loans.

3. Projecting Each Loan in Pool: A first step to projecting the cash flows is to project the status of each loan in the pool at all future points in time. This is achieved by using the transition matrix

$$P_{i,j}(t) = \begin{bmatrix} P_{c,c}(t) & \cdots & P_{c,d}(t) \\ \vdots & \ddots & \vdots \\ P_{6+,c}(t) & \cdots & P_{6+,d}(t) \end{bmatrix}$$

4. Using MNL Loan Transition Probabilities: Probabilities are estimated at all points in time, as per the earlier section. Suppose, for example, that at a given time t the loan status is current. Aw: The multinomial distribution for where the loan can transition to at time $t+1$ is used, and it is simply by the first row of the transition matrix.
5. Loan Target Status Random Draw: Therefore a random draw is made from this distribution to project the status of the loan at time $t+1$. If, in that draw, the loan ended up in a status of 30-day delinquent, a simulation is done using the second row of $P_{i,j}(t+1)$ and so on, until the earliest of one of maturity, default, or prepay.
6. Projection of Loan Cash Flows: After having obtained the future status of all the loans in the pool, we can compute the cash flows appropriate to the product under consideration (in this case the 36 month vs. the 60 month fixed rate loans).
7. Loan State Dependent Cash Flow: For example, in the case of Lending Club, one can continue to apply the monthly fixed payment on the loan if the loan is in current status, and accrue interest if the loan becomes delinquent, or discontinue further payments altogether if the loan voluntarily prepays or defaults.
8. CPR/CDR from Cash Flows: Such logic allows the computation of the loan balances from period to period, and ultimately the CPR and the CDR above.
9. PeerIQ CPR and CDR Simulation: As a final example, PeerIQ (2015) project loan status, derive cash flows, and compute CPR and CDR for the entire population of outstanding Lending Club loans – outstanding as of January 2013 – and examine the results. PeerIQ generates the results of one simulation on the portfolio containing 14,000 Lending Club loans as at 1 January 2013.

10. Actual vs. Simulated CPR/CDR: As evidenced from the above, the projection for the CPR and the CDR agrees with the realized values for much of the projection period. For CDR, there is some volatility towards the latter end of the projection period where the sample thins out, and where the number of defaults (as a rare transition) can have a meaningful impact.
11. Multi-Path CPR/CDR Simulation: In addition to such 'single path' projections of CPR and CDR on a portfolio, because the calibration produces a *distribution* of transitions, one can generate multiple paths for the CPR and CDR on a loan portfolio. Using the model PeerIQ simulated 10 paths for the LC portfolio resulting in a distribution of CPR and CDR for the portfolio.

PeerIQ Credit Model Future Enhancements

1. Inclusion of Macro-economic Regressors: For the purposes of keeping the analysis simple, PeerIQ's model excludes macro-economic regressors from the analysis. However economic variables can be integrated quiet easily into the structure of the model.
2. Stochastic Simulation of Market Variables: In future versions we plan to add in carefully selected market variables such as interest rates or unemployment, which can be used as scenario variables, or stochastically simulated via parameters calibrated from the prices of traded instruments (e.g., interest rate options).

References

- PeerIQ (2015): PeerIQ Analytics Credit Model Methodology, Version 2.0.
- Receiver Operating Characteristic (Wiki).

Section XV: Asset Allocation and Portfolio Construction

Modern Portfolio Theory

Introduction, Background, Focus, and Motivation

1. Scope of Modern Portfolio Theory: Modern Portfolio (MPT), or Mean Variance Analysis, is a mathematical framework for assembling assets such that the expected return is maximized for a given level of risk, defined as variance (Modern Portfolio Theory (Wiki)).
2. Asset Risk and Return Assessment: MPT's key insight is that an asset's risk and return should not be assessed by itself, but by how it contributes to the portfolio's overall risk and return (Markowitz (1952)).

Mathematical Model

1. Investor Risk and Return Preferences: MPT assumes that investors are risk averse, meaning that, given two portfolios that offer the same expected returns, investors will prefer the less risky one. Thus an investor will take on increased risks only if compensated with higher expected returns. Conversely, an investor who wants higher expected returns must accept more risk.
2. Investor Risky Portfolio Choice Assessment: The exact trade-off curve will be the same for all the investors, but different investors will evaluate the trade-off differently based on the individual risk aversion characteristics. The implication is that a rational investor will not invest in a portfolio if a second portfolio exists with a more favorable risk-to-expected return profile – i.e., if for that level of risk an alternate portfolio exists that has better expected returns.
3. Portfolio Properties Used by MPT: Under the MPT, the portfolio return is the proportion-weighted combination of the constituent assets' returns. Portfolio volatility is a function of the correlations ρ_{ij} of the component assets for all asset pairs (i, j).

4. Multi Asset Portfolio Expected Return: The expected portfolio return is

$$\mathbb{E}[R_p] = \sum_i w_i \mathbb{E}[R_i]$$

where R_p is the return on the portfolio, R_i is the return on the asset i, and w_i is the weight of the asset i. The portfolio variance is

$${\sigma_p}^2 = \sum_i {w_i}^2 {\sigma_i}^2 + \sum_i \sum_{j \neq i} w_i w_j \sigma_i \sigma_j \rho_{ij}$$

where ρ_{ij} is the correlation coefficient between the returns on assets i and j. Alternatively the above expression can be re-cast as

$${\sigma_p}^2 = \sum_i \sum_j w_i w_j \sigma_i \sigma_j \rho_{ij}$$

where

$$\rho_{ij} = 1$$

for

$$i = j$$

The corresponding portfolio return volatility (standard deviation) is

$$\sigma_p = \sqrt{{\sigma_p}^2}$$

5. Two Asset Portfolio Returns/Variance: Portfolio Returns is

$$\mathbb{E}[R_p] = w_A\mathbb{E}[R_A] + w_B\mathbb{E}[R_B] = w_A\mathbb{E}[R_A] + (1 - w_A)\mathbb{E}[R_B]$$

The Portfolio Variance is

$$\sigma_p{}^2 = w_A{}^2\sigma_A{}^2 + w_B{}^2\sigma_B{}^2 + 2w_Aw_B\rho_{AB}\sigma_A\sigma_B$$

6. Three Asset Portfolio Returns/Variance: Portfolio Returns is

$$\mathbb{E}[R_p] = w_A\mathbb{E}[R_A] + w_B\mathbb{E}[R_B] + w_C\mathbb{E}[R_C]$$

The Portfolio Variance is

$$\sigma_p{}^2 = w_A{}^2\sigma_A{}^2 + w_B{}^2\sigma_B{}^2 + w_C{}^2\sigma_C{}^2 + 2w_Aw_B\rho_{AB}\sigma_A\sigma_B + 2w_Bw_C\rho_{BC}\sigma_B\sigma_C + 2w_Cw_A\rho_{CA}\sigma_C\sigma_A$$

7. Portfolio Diversification - Inversely Correlated Assets: An investor can reduce the portfolio risk simply by holding combinations of instruments that are not perfectly positively correlated, i.e., correlation coefficient

$$-1 \le \rho_{ij} < 1$$

In other words, investors can reduce their exposure to individual asset risk by holding a diversified portfolio of assets. Diversification may allow for the same portfolio expected return with reduced risk.

8. Special Case - Completely Uncorrelated Assets: If all asset pairs have correlations of 0, i.e., they are perfectly uncorrelated, the portfolio's return variance is the sum over all assets of the square of the fraction held in the asset times the asset's return variance.
9. Frontier Without Risk-Free Asset: Every possible combination of the risk assets, without including any holdings of the risk-free assets, can be plotted in the risk-expected returns space, and a collection of all such possible portfolios defines a region in this space. The left boundary of this region is a hyperbola (Merton (1972)), and the upper edge of this region is

the efficient frontier in the absence of a risk-free asset (sometimes called that Markowitz bullet).

10. Portfolio at the Efficient Frontier: Combinations along this upper edge represent portfolios (including no holdings of the risk-free asset) for which there is lowest risk for a given level of expected return. Equivalently, a portfolio lying on the efficient frontier represents the combination offering the best possible expected return for a given risk level. The tangent to the hyperbola at the tangency point indicates the best possible Capital Allocation Line (CAL).
11. Determination of the Efficient Frontier: For a given risk tolerance

$$q \in [0, \infty)$$

the efficient frontier is found by minimizing $w^T \Xi w - q R^T w$ where w is the vector of portfolio weights, and

$$\sum_i w_i = 1$$

(the weights can be negative, which means that the investors can short that security), Ξ is the covariance matrix for the returns on the assets in the portfolio, R is the vector of expected returns, $w^T \Xi w$ is the variance of the portfolio returns, and $R^T w$ is the expected return on the portfolio.
12. The Risk Tolerance Parameter Interpretation:

$$q \geq 0$$

is a "risk tolerance" factor, where 0 results in a portfolio with minimum risk, and ∞ results in the portfolio being infinitely far out in the frontier with both the expected return and the risk unbounded. The above optimization finds the point on the frontier at which the inverse of the slope of the frontier would be q if the portfolio return variance instead of the standard deviation were plotted horizontally. The frontier in its entirety is parametric on q. Many software packages, including MATLAB, Microsoft Excel, Mathematica, and R provide optimization routines suitable for the above problem.

13. Alternate Efficient Frontier Approach Specification: An alternate approach to specifying the efficient frontier is to do so parametrically on the expected portfolio return $R^T w$. This version of the problem requires that we minimize $w^T \Xi w$ subject to

$$R^T w = \mu$$

for a parameter μ. This problem is easily solved using a Lagrange multiplier.

14. Two Mutual Fund Theorem - Concept: One key result of the above analysis is the two mutual fund theorem (Merton (1972)). The theorem states that any portfolio on the efficient frontier can be generated by any two give portfolios on the frontier; the latter two portfolios are the mutual funds in the theorem's name. So in the absence of a risk-free asset, the investor can achieve any desired efficient portfolio even if all that is accessible is a pair of efficient mutual funds.
15. Two Mutual Fund Theorem - Allocation: If the location of the desired portfolio on the frontier is between the location of the two mutual funds, both mutual funds will be held in positive quantities. If the desired portfolio is outside the range spanned by the two mutual funds, then one of the mutual funds must be sold short (held in negative quantity) while the size of the investment in the other mutual fund must be greater than the amount available for investment – the excess being funded by borrowing from the other mutual fund.
16. Characteristics of the Risk-Free Asset: The risk free asset is a hypothetical asset that pays a risk free rate. In practice, short term government securities such as UST bills are used as risk-free asset because they pay a fixed coupon and have an exceptionally low default risk. The risk-free rate has zero variance in its returns (hence risk-free); it is also uncorrelated with any other asset (by definition, since its variance is zero).As a result, when combined with any other asset or portfolio of assets, the change in return is linearly related to the change in risk as the proportions in the combination vary.
17. Risk Free Asset Efficient Frontier: When a risk-free asset is introduced the corresponding half-line becomes the new efficient frontier – this line is the tangent to the hyperbola at the pure risky portfolio with the highest Sharpe ratio.

18. Corresponding Efficient Frontier Portfolio Weights: As demonstrated in Modern Portfolio Theory (Wiki), the vertical intercept represents a portfolio with 100% of holdings in the risk-free asset; the tangency with the hyperbola represents a portfolio with no risk-free holdings and 100% of the assets held in the portfolio occurring at the tangency point; and points on the half-line beyond the tangency point are leveraged portfolios involving negative holdings of the risk free asset (the latter has been sold short – in other words the investor has borrowed at the risk-free rate) and an amount invested in the tangency portfolio equal to more than 100% of the investor's initial capital.
19. The Capital Allocation Line - Definition: The efficient half-line is called the Capital Allocation Line (CAL) and its formula can be shown to be

$$\mathbb{E}[R_C] = R_F + \frac{\sigma_C}{\sigma_P}(\mathbb{E}[R_P] - R_F)$$

    Here P is the sub-portfolio of the risky assets at the tangency point with the Markowitz bullet, F is the risk-free asset, and C is a combination of portfolios P and F.
20. Improved Risk Adjusted Returns Profile: Thus the introduction of the risk-free asset as a possible component of the portfolio has improved the range of the risk-expected returns available, because everywhere except at the tangency portfolio the half-line gives a higher expected returns than the hyperbola does at every possible risk level.
21. The One Mutual Fund Theorem: The fact that all points on the linear efficient locus can be achieved by a combination of holdings of the risk-free asset and the tangency portfolio is known as the One Mutual Fund Theorem (Merton (1972)), where the mutual fund referred to is the tangency portfolio.

Asset Pricing

1. MPT Basis For Asset Pricing: The analysis above describes the optimal behavior of an individual investor. Asset pricing theory builds on this analysis as follows. Since everyone holds risky assets in identical proportions to each other – namely in proportions given by the

tangency portfolio – in market equilibrium, the risky assets' prices, and therefore their expected returns, will adjust so that the ratios in the tangency portfolio are the same as in the ratios in which the risky assets are supplied to the market.

2. Asset Supply/Demand Re-adjustment: Thus the relative supplies will equal relative demands. MPT derives the required expected return for a correctly priced asset in this context.
3. The Diversifiable Idiosyncratic/Specific Risk: Specific risk is the risk associated with the individual assets – within a portfolio these risks can be reduced through diversification (i.e., specific risks cancel out). Thus specific risk is also called diversifiable, unique, unsystematic, or idiosyncratic risk.
4. The Non-Diversifiable Systematic Risk: Systematic risk (a. k. a portfolio risk or market risk) refers to the risk common to all securities – except for selling short, systematic risk cannot be diversified away in one market. Within the market portfolio, asset specific risk will be diversifiable away to the extent possible. Systematic risk is therefore equated with the risk (standard deviation) of the market portfolio.
5. Rational Approach to Security Purchase: Since a security will be purchased only if it improves the risk-expected return characteristics of the market portfolio, the relevant measure of the risk of a security is the risk it adds to the market portfolio, and not its risk in isolation.
6. Conditional Asset Pricing Model Approaches: In this context, the volatility of the asset, and its correlation with the market portfolio, are historically observed, and are therefore given. There are several approaches to asset pricing that attempt to price assets by modeling the stochastic properties of the moments of the assets' returns – these are broadly referred to as conditional asset pricing models.
7. Long/Short Market Model Portfolios: Systematic risks within one market can be managed through a strategy of using both long and short portfolios within one portfolio, creating a market neutral portfolio. Market neutral portfolios will therefore have correlations of zero.
8. Capital Asset Pricing Model - Definition: The asset returns depend on the amount paid for the asset today. The price paid must ensure that the market portfolio's risk/return characteristics improve when the asset is added to it. The CAPM is a model that derives the theoretical required expected return (i.e., discount rate) for an asset in the market, given the risk-free rate available to investors and the risk of the market as a whole.

9. CAPM - Mathematical Framework Specification: The CAPM is usually expressed as

$$\mathbb{E}[R_i] = R_i + \beta_i\left(\mathbb{E}[R_m] - R_f\right)$$

Here β_i is the measure of the asset sensitivity to a movement in the overall market; β_i is usually estimated through regression on historical data. β_i exceeding 1 signifies more than average riskiness in the sense of the asset's contribution to the overall portfolio risk; β_i below one indicate a lower than average risk combination. $\mathbb{E}[R_m] - R_f$ is the market premium, the expected excess return of the market portfolio's expected return over the risk-free rate.

10. CAPM Derivation Market Portfolio Risk: The incremental impact on risk and expected return when an additional risky asset a is added to the market portfolio m follows the formula for a two-asset portfolio. The results are used to derive the asset appropriate discount rate. Market portfolio's risk is ${w_m}^2{\sigma_m}^2 + ({w_a}^2{\sigma_a}^2 + 2w_a w_m \rho_{am}\sigma_a\sigma_m)$ Hence the risk added to the portfolio is ${w_a}^2{\sigma_a}^2 + 2w_a w_m \rho_{am}\sigma_a\sigma_m$, but since the weight of the asset is relatively low

$${w_a}^2 \approx 0$$

i.e., the additional risk is $2w_a w_m \rho_{am}\sigma_a\sigma_m$.

11. CAPM Derivation Market Portfolio Return: The market portfolio's expected return is $w_m\mathbb{E}[R_m] + (w_a\mathbb{E}[R_a])$ Hence the additional expected return is $w_a\mathbb{E}[R_a]$.

12. CAPM Derivation - Risk/Return Portfolio: If an asset a is correctly priced, the improvement in the risk-to-expected return ratio achieved by adding it to a market portfolio m will at least match the gains of spending that amount on an increased stake in the market portfolio. The assumption is that the investor will purchase the asset with funds borrowed at the risk free rate R_f; this is rational if

$$\mathbb{E}[R_a] > R_f$$

13. CAPM Derivation Excess Returns/Risk: Thus

$$\frac{w_a\left(\mathbb{E}[R_a] - R_f\right)}{2w_a w_m \rho_{am}\sigma_a\sigma_m} = \frac{w_m\left(\mathbb{E}[R_m] - R_f\right)}{2w_a w_m \sigma_m\sigma_m}$$

i.e.

$$\mathbb{E}[R_a] = R_f + \frac{\rho_{am}\sigma_a\sigma_m}{\sigma_m\sigma_m}\left(\mathbb{E}[R_m] - R_f\right)$$

i.e.

$$\mathbb{E}[R_a] = R_f + \frac{\sigma_{am}}{\sigma_{mm}}\left(\mathbb{E}[R_m] - R_f\right)$$

14. <u>CAPM Derivation Asset-to-Market</u>: $\frac{\sigma_{am}}{\sigma_{mm}}$ is the beta, - the covariance between the asset return and the market return divided by the variance of the market return – i.e., the sensitivity of the asset price to movement in the market's portfolio value.
15. <u>The Security Characteristic Line (SCL)</u>: The excess asset returns can be estimated statistically using the following regression equation:

$$R_i(t) - R_f = \alpha_i + \beta_i\left(R_M(t) - R_f\right) + \epsilon_i(t)$$

where α_i is called the asset's alpha, and β_i is the asset's beta coefficient.
16. <u>Expected Returns as Asset Numeraire</u>: Once an asset's expected return $\mathbb{E}[R_a]$ is calculated using CAPM the future cash flows of the asset can be discounted to their present values using this rate to establish the correct price for the asset. A riskier stock will have a higher beta and be discounted at a higher rate, and a less sensitive stock will have a lower beta and be discounted at a lower rate. In theory an asset is correctly priced when its observed price is the same as its value calculated using the CAPM derived discount rate. If the observed price is higher than the valuation, the asset is over-valued; it is under-valued for too low a price.

Criticisms

1. Historical vs Future Statistical Inputs: The risk, the return, and the correlation measures used by MPT are based on expected, i.e., they are mathematical values about the future. In practice, investors must substitute predictions based on historical measurements of asset returns and volatility for these values. Very often such expected values fail to take into account new circumstances that did not exist when the historical data were first generated (Damghani (2013)).
2. Structural Approach to Risk Management: More fundamentally investors are stuck with estimating key parameters from past market data because MPT attempts to model risk in terms of likelihood of losses, but says nothing about why those losses might occur. The risk measurements used are probabilistic in nature, not structural. This is a major difference compared to many engineering approaches to risk management (Hubbard (2007)).
3. Better Measures for Investors' Preferences: MPT uses the mathematical concept of variance to quantify risk, and this may be justified under elliptically distributed returns such as normal returns, but for general return distributions, other risk measures (e.g. coherent risk measures) might reflect investors' true preferences.
4. Fundamentally Asymmetric Nature of Risk: In particular, variance is a symmetric measure that counts abnormally high returns as just as risky as abnormally low returns. In reality, many investors are only concerned about losses, and do not care about the dispersion or tightness of above-average returns. According to this view, the intuitive concept of risk is fundamentally asymmetric in nature.
5. Gaussian Nature of Returns Distribution: MPT has also been criticized because it assumes that returns follow a Gaussian distribution (Taleb (2007)). Researchers have built on previous work that proposed using stable distributions instead, and have presented strategies for deriving optimal portfolios in such settings (Rachev and Mittnik (2000), Risk Manager Journal (2006), Doganoglu, Hartz, and Mittnik (2007)).
6. Extensions by Black and Litterman: The Black-Litterman model is an extension of the unconstrained Markowitz optimization that incorporates relative and absolute "views" on inputs of risk and returns.

Other Applications

1. MPT Applied in Regional Science: In a series of seminal works, Michael Conroy modeled the labor force in economy using portfolio-theoretic methods to examine growth and variability in the labor force. This was followed by a long literature on the relationship between economic growth and volatility (Chandra (2003)).
2. Modeling of the Concept of "Self": MPT has been used to model the concept of "self" in social psychology. When the self attributes comprising the self-concept constitute a well-diversified portfolio, the psychological outcomes at the level of an individual such as mood and self-esteem should be more stable than when the self-concept is undiversified (Chandra and Shadel (2007)). This prediction has been confirmed in studies involving human subjects.
3. Correlation/Uncertainty in Information Retrieval: Recently MPT has been applied to modeling the uncertainty and correlation between documents in information retrieval. Given a query, the aim is to maximize the overall relevance of a ranked list of documents and at the same time minimize the overall uncertainty of the ranked list (Wang (2009)).
4. Project Portfolios/Non-Financial Assets: When MPT is applied outside of traditional financial portfolios, some differences may need to be considered, e.g., the discrete nature of projects (i.e., projects may be all/nothing, have inseparable logical units, etc.), launch/closure limitations (e.g., no recovery/salvage of a half complete project) (Hubbard (2007), Sabbadini (2010)).
5. Re-defining the Investment Boundaries: Some of the simplest elements of MPT are applicable to virtually any kind of portfolio. As an example, for major projects the MPT investment boundary can be defined in more general terms like "choice of an ROI less than the cost of capital" or "chance of losing more than half of the investment" as opposed to the well-defined historical variance used in MPT. When risk is specified in terms of the uncertainty of forecasts and possible losses, the concept is then transferrable to various types of investments (Hubbard (2007)).

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The Black-Litterman Model

Overview

1. Black Litterman Chronology and Taxonomy: This section surveys the literature on the Black-Litterman model. The survey, drawn heavily from Walters (2014), is provided as both a chronology and as a taxonomy, as there are many claims on the model in the literature.
2. The Canonical Model and Derivations: A complete description of the canonical model including full derivations from the underlying principles using both Theil's Mixed Estimation Model and the Bayes' Theory is provided.
3. Model Parameters Calibration and Computation: The various parameters of the model are considered, along with the information on the calibration and the computation. Further treatment is given to several of the key papers, including worked samples illustrating the concept.

Introduction

1. Black Litterman Model and Evolution: The Black-Litterman model was first published by Fischer Black and Robert Litterman in 1990. During the 20+ years since the original paper, many authors have published research referring to their model as Black-Litterman. This has led to a variety of models being labeled as Back-Litterman even though they may be very different from the original model.
2. Variants in Literature and Taxonomy: In the chronological survey of the literature, Walters (2014) introduces several papers that make significant contributions to the literature on the Black-Litterman model. He also documents the taxonomy of models that have been described as Black-Litterman, and further refers to the Black-Litterman Model as described in the original papers as the Canonical Black-Litterman Model.

3. Equilibrium Market Portfolio as Prior: The Canonical Model of Black-Litterman makes two significant contributions to the problem of asset allocation. First it provides an intuitive prior – the equilibrium market portfolio – as a starting point for the estimation of the asset returns.
4. Alternative Priors in Earlier Work: Previous similar work started with the uninformative uniform prior distribution or with the global minimum variance portfolio. The latter method, studied by Frost and Savarino (1986) and Jorion (1986), took a shrinkage approach to describe the final asset allocation. Neither of these methods have an intuitive connection back to the market.
5. Shrinkage Based on "Reverse Optimization": The idea one could use "reverse optimization" to generate a stable distribution of returns from the equilibrium market portfolio as a starting point for shrinkage is a significant improvement to the process of returns estimation.
6. Incorporating Investors' Views on Returns: Second, the Black-Litterman model provides a clear way for the investors to specify views on returns and to blend those views with prior information. The investors' views' are allowed to be partial or complete, and the views can span arbitrary and overlapping sets of assets. The model outputs estimates of excess returns and the corresponding precision.
7. Blending Prior with the Investors' Views: Prior to Black and Litterman (1991a) nothing similar had been proposed for the portfolio selection problem. Theil's mixing model had been developed, but nobody had applied it to the problem of estimating asset returns. No research linked the process of specifying the views to the blending of the prior and the investors' views. The Black-Litterman model provides a quantitative framework for specifying the investors' views, and a clear way to combine those investors' views with an intuitive prior to arrive at a new combined distribution.
8. Enhancements to the Original Model: The state of the art in the portfolio selection problem has significantly changed during the time since the publication of the original Black-Litterman model. Because of its rich theoretical basis it can be applied alongside several modern portfolio selection approaches as can be seen in the literature.

Historical Taxonomy and Literature Survey

1. Black-Litterman Model Classification Traits: The primary dimensions used by Walters (2014) to classify the models will be: does it specify the parameters as distributions or as point estimates, and does it include the parameter τ. In order for an author's model to be considered as the Canonical Black-Litterman it would need to match both these criteria.
2. Canonical Black-Litterman Reference Model: Walters (2014) collects the efforts into three different Reference Models based on the dimensions provided above. The situation where both of the above conditions are met is called the Canonical Reference Model, i.e., the model described by the original authors.
3. Alternative Black-Litterman Reference Model: The Alternate Reference Models describe models that use point estimates, but for some reason include τ, which now becomes a scaling factor with no quantitative basis. These authors typically treat the model as just a shrinkage/mixing estimator, thereby losing the explicit connection to Bayesian statistics. As Walters (2014) shows, using the Bayes' law formula – but substituting in different variables – is not theoretically tenable.
4. "Beyond Black Litterman" Reference Model: Finally the "Beyond Black Litterman" Reference Model uses point estimates and no longer includes τ at all. This makes it a pure shrinkage/mixing model.
5. Black-Litterman Model First Version: The Black-Litterman Model was first published by Fischer Black and Robert Litterman of Goldman Sachs in an internal Goldman Sachs Fixed Income Research Note (Black and Litterman (1990)). This research note was extended into a paper (Black and Litterman (1991a)). The paper provides a good overview of the features of then model, but does not provide all of the formulas used in the model.
6. Black-Litterman Model Second Version: A second internal Goldman Sachs Fixed Income research Note was published in the same year (Black and Litterman (1991b)). This paper was later extended and published in the Financial Analysis Journal (Black and Litterman (1992)). It provides the rationale for the methodology, and some steps involved in the derivation, but does not show all the formulas or a full derivation. It also includes a rather complex worked example based on global equilibrium – see Litterman (2003) on the details of the methods required to address this problem.
7. Black- Litterman Model Third Version: Unfortunately, because of the merging of the two problems, the second version of the Black-Litterman Model produces results that are difficult

to reproduce. He and Litterman (1999) provide more details on the workings of the model, but not a complete set of formulas. They do provide, however, a much simpler to reproduce working model.

8. Goldman Sachs Asset Allocation Process: Bevan and Winkelmann provide details on how they use the Black-Litterman Model as part of their broader asset allocation process at Goldman Sachs, including some calibrations of the model which they perform. This is useful to anyone planning on building a Black-Litterman approach onto an ongoing asset allocation process.
9. The Satchell and Scowcroft Model: Satchell and Scowcroft (2000) attempt to de-mystify the Black-Litterman Model, but instead introduce a new non-Bayesian expression of the model. Their model uses point estimates for the prior and the views, and use τ and Ω only to control the shrinkage of the views on the prior. Because they use point estimates instead of distributions, their model does not include any information on the precision of the estimate. This allows them to recommend setting

$$\tau = 1$$

10. Use of a Stochastic τ: Satchell and Scowcroft (2000) also introduce a stochastic τ, but because they use point estimates, this really becomes a model with stochastic covariance of returns. The model was occasionally used in the literature after this point, but was largely replaced by Meucci's models in the mid-2000's.
11. Black Litterman Model Confidence Estimation: Drobetz (2001) provides a further description of the Black-Litterman Model including a good description of how to interpret the confidence in the estimates including a diagram – he also works out an example.
12. The Fusai and Meucci Model: Fusai and Meucci (2003) introduced yet another non-Bayesian variant of the model that removed the parameter τ altogether. Meucci (2005) followed up on this paper and coined the phrase "Beyond Black Litterman". Once the model is viewed as only a shrinkage model, Ω provides enough degrees of freedom to the shrinkage and the parameter τ becomes superfluous and confusing. Since the mid-2000's, there has been a mixture of the "Canonical" and the "Beyond Black Litterman" Models used in the literature

(a more comprehensive survey can be found at http://www.blacklitterman.org/methods.html). However these hybrid models were later debunked by Michaud et al (2013).

13. The Firoozye and Blamont Model: Firoozye and Blamont (2003) provide a good overview of the Canonical Reference Model and present an intuition into the parameter τ. They also illustrate the reduction in the variance of the posterior estimate as a result of the mixing.
14. The Herold Model - Active Portfolios: Herold (2003) provides an alternate view of the problem where he examines optimizing alpha generation for active portfolio management, essentially specifying that the sample distribution has zero mean. He uses the Alternate Reference Model with point estimates and tracking error to determine how much shrinkage to allow. The two significant contributions by his paper are: a) application of the model to active portfolio management, and b) designing some additional measures that can be used to validate that the views are reasonable.
15. The Koch Model - 100% Certainty: Koch (2008) includes derivations of the "master formula" and the alternative form under 100% certainty. He does not mention posterior variance, or show the alternate form of master formula under uncertainty (the general case). He does include a slide on the sensitivity of the posterior estimate of τ, but he uses the Alternate Reference Model, so this treatment is not valid for the Canonical Model.
16. The Idzorek Model - Shrinkage Impact: Idzorek (2005) introduced a technique for specifying Ω such that the impact of the shrinkage was specified in terms of percentage of change in the weights between 0 and the change caused by 100% confidence. His paper uses the Alternative Reference Model but his techniques can be applied to the Canonical Black-Litterman Model because it is sensitive to the value of τ specified by the investor.
17. The Krishnan and Mains Model: Krishnan and Mains (2005) provide an extension to the Black-Litterman model to account for an additional factor that is uncorrelated with the market. They call this the two-factor Black-Litterman Model and they demonstrate an example of extending the Black-Litterman model with a recession factor. They show how it intuitively impacts the expected returns computed from the model.
18. The Mankert Model - Sampling Theory: Mankert (2006) provides a detailed walk-through of the model and works the elaborate transformation between the two specifications of the Black-Litterman "master formula" for the estimated asset returns. She approaches the

problem from the point of view of Sampling Theory and provides additional intuition on the value of τ in the context of the Alternate Reference Model.

19. The Meucci Model - Non-Normality: Meucci (2006) provides a method for using non-normal views in the Black-Litterman model. Meucci (2008) extends this model to a wider range of parameters, and allows for the usage of the full distribution as well as for scenario analysis.
20. The Beach and Orlov Model: Beach and Orlov (2007) illustrate using GARCH models to generate the views. They use a model for international equities across 20 countries. They show how the results change as the investor uses different values for τ. They do not provide exact details for the uncertainty of the views and appear to be using an Alternative Reference Model even though their techniques can be applied to the Canonical Reference Model.
21. The Braga and Natale Model: Braga and Natale (2007) describe a model for calibrating the uncertainty in the views using the Tracking Error Volatility (TEV). They provide sensitivities for the posterior estimates to the various views. The TEV metric is well-known for its use in active portfolio management. They use the Alternative Reference Model, but their work could be used for the Canonical Reference Model as well.
22. The Martellini and Ziemann Model: Martellini and Ziemann (2007) describe an approach to the active management of a fund of hedge funds. They use VaR as their objective function for the reverse optimization, and they use a variant of the CAPM model extended to include skewness and kurtosis in determining the market portfolio. They use a factor model to generate the rankings, and then convert the rankings into their confidence in the views. They do not use Bayesian features of the model, but rather use point estimates, and thus do not have information on the precision of their estimates.
23. Stable Paretian Distributions for Measures: Giacometti, Bertocci, Rachev, and Fabozzi (2007) provide an approach to computing the neutral portfolio using stable Paretian distributions rather than the normal distribution described in the original Black and Litterman model. They also use multiple different measures of risk for their portfolio selection model – variance VaR and CVaR.
24. The Cheung Model: Cheung (2009) introduces the concept of an augmented model. This version of the model integrates a factor model and does a joint estimation of the factor returns. Cheung uses an Alternate Reference Model, his variant of the model can easily be used with the Canonical Reference Model.

25. Bertsimas, Gupta, and Paschalidis Model: Bertsimas, Gupta, and Paschalidis (2012) introduce a way to measure the alignment of views with prior estimate by comparing the view portfolio weights with the eigenvalues of the prior covariance matrix.
26. Michaud, Esch, and Michaud Model: Michaud, Esch, and Michaud (2012) provide a blistering critique of the Alternative Reference Model. Owing to their focus on the Alternative Reference Model, a significant part is devoted to problems with point estimates, and much of it is not relevant to the Canonical Reference Model. Further, the basic arguments are on the elementary statistical properties of time series, essentially ignoring non-stationary and auto-regressive models, as well as the richer, state-of-the-art econometric models.
27. Model Interoperability: Much of the work presented with any of the reference models can be used with the Canonical Reference Model, even if it has been initially formulated for the Alternative Reference Model (for additional notes, see Christodoulakis (2002)).

Canonical Black-Litterman Reference Model

1. The Black-Litterman Reference Model: The reference model for the returns is the base upon which the rest of the Black-Litterman Model is built. It includes assumptions on which variables are random, and which ones are not. It also defines which parameters are modeled, and which ones are not modeled. Most importantly, some authors of papers on the Black-Litterman model use reference models, e.g., Alternate Reference Models, or Beyond Black-Litterman Model, and these do not have the same theoretical basis as the Canonical one which was initially specified in Black and Litterman (1992).
2. Normal Distribution for Expected Returns: The normally distributed expected returns can be expressed as

$$r \sim \mathcal{N}(\mu, \Psi)$$

The fundamental goal of the Black-Litterman model is to model the expected returns, which are assumed to be normally distributed with a mean μ and variance Ψ. Note that we will need at least these values – the expected returns and the covariance matrix later as inputs into a portfolio selection model.

3. <u>The Random Unknown Mean Return</u>: We define μ the unknown men return, as a random variable itself distributed as

$$\mu \sim \mathcal{N}(\varkappa, \Psi_{\varkappa})$$

$\varkappa$ is our estimate of the mean and $\Psi_{\varkappa}$ is the variance of the unknown mean μ about the estimate. Another way to view this simple relationship is shown in the formula below:

$$\mu = \varkappa + \epsilon_{\varkappa}$$

i.e., the prior returns are normally distributed around $\varkappa$ with a disturbance value $\epsilon_{\varkappa}$.

4. <u>Relation Between $\epsilon_{\varkappa}$ and ϵ</u>: $\epsilon_{\varkappa}$ is normally distributed with zero mean and $\Psi_{\varkappa}$ variance, and is assumed to be uncorrelated with ϵ, where ϵ is defined as

$$r = \mu + \epsilon$$

5. <u>Relationship between all the Variances</u>: The Reference Model can be completed by defining Ψ_r as the variance of the returns about our estimate $\varkappa$. From

$$\mu = \varkappa + \epsilon_{\varkappa}$$

and the assumption above that $\varkappa$ and μ are not correlated, the formula to compute Ψ_r is

$$\Psi_r = \Psi + \Psi_{\varkappa}$$

The formula indicates that the proper relationship between the variances is

$$\Psi_r \geq \max(\Psi, \Psi_{\varkappa})$$

6. Boundary Conditions Check for Ψ_r: A check at the boundaries ensures that the Reference Model is correct. In the absence of estimation error, e.g.

$$\epsilon = 0$$

then

$$\Psi_r = \Psi$$

As the estimate gets worse, i.e., as $\Psi_\varkappa$ increases, Ψ_r increases as well.

7. Alternate vs. Canonical μ Estimate: The Canonical Reference Model for the Black-Litterman Model expected return is

$$r \sim \mathcal{N}(\varkappa, \Psi_r)$$

A common misconception about the Canonical Black-Litterman Reference Model is that

$$r \sim \mathcal{N}(\mu, \Psi)$$

is a Reference Model, and that μ is a point estimate. This approach is what is labeled the Alternate Reference Model.

8. Impact of the Reference Model: Several authors approach the model from this point of view so it cannot be neglected, but it is a fundamentally different model. When considering the results from the various Black-Litterman implementations it is important to understand how the various parameters will impact the results.

Computing the Equilibrium Returns

1. Using the General Equilibrium Theory: The Black-Litterman Model starts with a market neutral equilibrium portfolio for the prior estimate of returns. The model relies on the General Equilibrium Theory to state of the aggregate portfolio is at equilibrium, each of the sub-portfolio must be at equilibrium.
2. Quadratic Utility for CAPM Portfolio: The Black-Litterman Model can be used with any utility function, which makes it very flexible. In practice most practitioners use the Quadratic Utility Function and assume a risk-free asset, and thus the equilibrium model reduces to the Capital Asset Pricing Model (CAPM). The neutral portfolio in this situation is the CAPM Market Portfolio.
3. Consistency across the Utility Functions: Some authors have used other utility functions – most notably Bertsimas, Gupta, and Paschalidis (2012) – but others have used CVaR and the other measures of portfolio risk without applying the same theoretical basis. In order to preserve the symmetry of the model the practitioners should use the same utility function to both identify the neutral portfolio as well as in the portfolio selection area.
4. The Standard Formulation Setting Choices: In his paper Walters (2014) uses the Quadratic Utility Function, the CAPM, and the Unconstrained Mean-Variance Optimizer because it is a well-understood model.
5. CAPM Market Excess Returns Prior: Given the previous assumption, the prior distribution for the Black-Litterman portfolio is the estimated mean excess return from the CAPM market portfolio. The process of computing the CAPM equilibrium excess returns is straightforward.
6. The Basic CAPM Market Model: CAPM is based on the concept that there is a linear relationship between risk (as measured by the standard deviation of returns) and expected returns. Further it requires returns to be normally distributed. The model is of the form

$$\mathbb{E}[r] = r_f + \beta r_m + \alpha$$

where r_f is the risk free rate, r_m is the excess returns of the market portfolio, β is a regression coefficient computed as

$$\beta = \rho \frac{\sigma_p}{\sigma_m}$$

and α is the residual – or asset specific (idiosyncratic) – excess return.

7. Idiosyncratic Risk Reduction through Diversification: Under CAPM the idiosyncratic risk associated with an asset is uncorrelated with that from the other assets, and this risk can be reduced through diversification. Thus the investor is rewarded for taking systematic risk measured by β but is not rewarded for the idiosyncratic risk associated with α.
8. CAPM Market Portfolio Special Features: In the CAPM world all investors should hold the same risky portfolio – the CAPM market portfolio. Because all investors hold risky assets only in their market portfolio, at equilibrium the market capitalization of the various assets will determine their weights in the market portfolio. Defining the Sharpe ratio as the excess returns divided by the excess risk, i.e. $\frac{r-r_f}{\sigma}$, it can be seen that the CAPM market portfolio has the maximum Sharpe Ration of any portfolio on the efficient frontier.
9. Challenges Computing the Optimized Market Portfolio: Note that the CAPM Market Portfolio contains all investable assets, which makes it very hard to actually specify. Because we are in equilibrium all sub markets should also be in equilibrium so that any sub-market we choose is part of the global equilibrium. While this allows us to reverse optimize the excess returns from the market capitalization and the covariance matrix, forward optimization from this point to identify the investors' optimal portfolio within CAPM is problematic as we do not have information for the entire market portfolio.
10. Optimal Allocation among the Assets: In general, however, the investor usually selects an investible universe and the desired optimal asset allocation within that universe, so the theoretical problem with the market portfolio can be ignored.
11. The Capital Market Line Construction: The Capital Market Line is the line through the risk-free rate and the CAPM market portfolio. The Two-Fund Separation Theorem, closely related to the CAPM, states that all investors should hold portfolios on the Capital Market Line.
12. Significance of the Capital Market Line: Any portfolio on the Capital Market Line dominates all the portfolios on the Efficient Frontier, the CAPM being the only point on both the Efficient Frontier and the Capital Market Line. Depending on their risk aversion the investor will hold arbitrary fractions of wealth in the risk-free asset and/or the CAPM market portfolio.

13. Risk-Free Asset in the Portfolio: Starting with the market portfolio, the starting point will be a set of weights which are all greater than zero and naturally sum to one. The market portfolio only includes risk assets, because by definition, the investor is rewarded only for taking systematic risk. Thus, in CAPM, the risk-free asset with

$$\beta = 0$$

will not be included in the market portfolio. Walters (2014) examines the case where the Bayesian investor may invest in the risk-free asset based on their confidence on their returns estimates.
14. Historical Returns Covariance Matrix Estimation: One assumption here is that the covariance matrix of the returns Ψ is known. In practice, this covariance matrix is estimated from historical returns data. It is often computed from higher frequency data and scaled up to the time frame required for the asset allocation problem. By computing it from the actual historical data, we can ensure that the covariance matrix is positive definite.
15. Other Returns Covariance Estimation Techniques: Without basing the estimation process on actual data, there are significant issues involved in ensuring that the covariance matrix is positive definite. One could, however, apply shrinkage or modern random matrix theory filters to make it more robust.
16. Note on the Notation Used: This section follows Walters (2014) where the notation used is similar to that used in He and Litterman (1999) for all the terms in the formulas. The notation is a little different from (and conflicts with) the notation used later in the section on Bayesian Theory.
17. Equation for the "Reverse Optimization": One can derive the equation for reverse optimization starting from the quadratic utility function

$$U = w^T \Pi - \frac{\delta}{2} w^T \Psi w$$

where represents the investors' utility – this is the objective function used for the Mean-Variance Optimization, w is the vector of weights invested in each asset, Π is the vector or equilibrium excess returns for each asset, δ is the risk aversion parameters, and Ψ is the covariance matrix of the excess returns of the assets.

18. Maxima for the Utility Function: U is a convex function, so it will have a single maxima. If we maximize the utility with no constraints, there is a closed form solution. The exact solution is found by taking the first derivative with respect to the weights w and setting it to zero.

$$\frac{\partial U}{\partial w} = \Pi - \delta \Psi w = 0$$

19. Estimation of the Excess Returns Vector: Solving this for Π – the vector of excess returns – yields

$$\Pi = \delta \Psi w$$

20. Estimating the Risk Aversion Factor: In order to use the above formula to solve for the CAPM Market Portfolio, there needs to be a value for δ - the risk aversion coefficient of the market. One way of finding δ is by multiplying both sides by w^T and replacing the vector terms with scalar terms

$$r - r_f = \delta \sigma^2$$

21. δ Estimate for Equilibrium Portfolio: The expression at equilibrium above states that the excess returns on the portfolio is equal to the risk aversion parameter scaled by the variance of the portfolio. Conversely

$$\delta = \frac{r - r_f}{\sigma^2}$$

where r is the total return on the Market Portfolio, i.e.

$$r = w^T \Pi + r_f$$

r_f is the risk-free rate, and σ^2 is the variance of the Market Portfolio, i.e.

$$\sigma^2 = w^T \Psi w$$

22. <u>δ Values used in Literature</u>: Most authors specify that value of δ that they used. Bevan and Winkelmann (1998) describe their process of calibrating their returns to an average Sharpe Ratio based on their experience. For global fixed income – their area of expertise – they use a Sharpe Ratio of 1.0. Black and Litterman (1992) use a Sharpe Ratio closer to 0.5 in the example in their paper.
23. <u>Risk Aversion from Sharpe Ratio</u>: Given the Sharpe Ratio one can write the formula for δ in terms of Sharpe Ratio as

$$\delta = \frac{SharpeRatio}{\sigma_m}$$

Thus one can calibrate the returns in terms of

$$\delta = \frac{r - r_f}{{\sigma_m}^2}$$

or

$$\delta = \frac{SharpeRatio}{\sigma_m}$$

24. <u>Sharpe Ratio of Market Portfolio</u>: In order to use

$$\delta = \frac{r - r_f}{\sigma^2}$$

one needs to have an implied return for the market portfolio which may be harder to estimate than the Sharpe Ratio of the market portfolio.

25. No-Constraint Reverse Portfolio Optimization: Once there is a value for

$$\Pi = \delta\Psi w$$

plugging in w, δ, and Ψ into

$$\Pi = \delta\Psi w$$

generates the set of equilibrium returns.

$$\Pi = \delta\Psi w$$

is the closed form solution to the reverse optimization problem for computing asset returns given an mean-variance portfolio in the absence of constraints. Re-arranging

$$\Pi = \delta\Psi w$$

produce a formula for the closed form calculation of the optimal portfolio weights in the absence of constraints:

$$w = (\delta\Psi)^{-1}\Pi$$

26. Reverse Optimization under Simple Constraints: Herold (2005) provides insights into how implied returns can be computed in the presence of simple equality constraints such as the budget constraint of the full investment

$$\sum w = 1$$

constraint. He illustrates how errors can be introduced during the reverse optimization if the constraints are assumed to be non-binding when they are actually binding for a given portfolio. Note that since one is dealing with the market portfolio which has only positive weights that sum to 1, one can safely assume that there are no binding constraints on the reverse optimization.

27. Incorporating the Prior Distribution Variance: The only piece missing is the variance of our estimate of the mean. Looking back at the Reference Model what is needed is $\Psi_{\varkappa}$. Black and Litterman made the simplifying assumption that the structure of the covariance matrix of the estimate is proportional to the covariance of the returns Ψ. They created a parameter τ as the constant of proportionality. Given the assumption

$$\Psi_{\varkappa} = \tau\Psi$$

the prior distribution is

$$Prior(A) \sim \mathcal{N}(\varkappa, \tau\Psi)$$

or

$$r_A \sim \mathcal{N}(Prior(A), \Psi)$$

This is the prior distribution of the Black Litterman Model. It represents the estimate of the mean, which is expressed as a distribution of the actual unknown mean about the estimate.

28. Incorporating Prior and Conditional Variance: Using

$$r \sim \mathcal{N}(\varkappa, \Psi_r)$$

one can re-write the above formula in terms of $\varkappa$ as

$$r_A \sim \mathcal{N}(\varkappa, [1 + \tau]\Psi)$$

29. No Views Unconstrained Portfolio Weights: It is often written that an investor with no views and using an unconstrained mean-variance portfolio selection model will invest 100% in the neutral portfolio, but this is only true if they apply a budget constraint. Because of the uncertainty in the estimates they will invest $\frac{1}{1+\tau}$ in the neutral portfolio an $\frac{\tau}{1+\tau}$ in the risk-free asset. This can be seen from the following:

$$\Pi = \delta\Psi w$$

$$w = (\delta\Psi)^{-1}\Pi$$

$$\hat{w} = ([1+\tau]\delta\Psi)^{-1}\Pi$$

$$\hat{w} = \frac{1}{1+\tau}(\delta\Psi)^{-1}\Pi$$

$$\hat{w} = \frac{1}{1+\tau}w$$

Walters (2014) illustrates the concept graphically.

30. Confidence Shift on the Efficient Frontier: One can alternatively view the Bayesian Efficient Frontier as a shift to the right if they plot the generated Efficient Frontier with an increased covariance matrix and a budget constraint. In this case the uncertainty just pushes each point further to the right in the risk/return space.

Specifying the Views

1. Investors Views as Conditional Distribution: This section describes the process of specifying the investors' views on the estimated mean excess returns. The combination of the investors' views is defined as the conditional distribution.

2. <u>Zero Correlation among the Views</u>: First, by construction, each view will be unique and uncorrelated with the other views. This gives the conditional distribution the property that the covariance matrix will be diagonal, with off-diagonal entries equaling zero. This constraint on the problem simplifies it and improves the stability of the result. Estimating co-variances between views is more error prone and complicated than estimating the view variances.
3. <u>Fully Invested Absolute/Relative Views</u>: The views are also further constrained to be fully invested, i.e., the sum of the weights in the view is either zero (a relative view) or one (an absolute view). Views are not required on any or all assets. In addition, the views can conflict – the mixing process will merge the views based on the confidence on them, and confidence on the prior.
4. <u>View Representation Matrix - Asset Weights</u>: The investors' k views on n assets is represented as follows. First, $\mathbb{P}$ refers to the $k \times n$ matrix of the asset weights within each view. For a relative view the sum of the weights will be zero, and for an absolute view the sum will be 1. Different authors compute the weights within the views differently – He and Litterman (1999) and Idzorek (2005) use a market capitalization weighted scheme, whereas Satchell and Scowcroft (2000) use an equi-weighted scheme in their examples. In practice, the weights will be a mixture depending upon the process used to estimate the view returns.
5. <u>View Representation Matrix - Excess Returns</u>: Second, $\mathbb{Q}$ is a $k \times 1$ vector of weights for each view.
6. <u>View Representation Matrix - Returns Variance</u>: Finally Ω is a $k \times k$ matrix of the covariance of the views. Ω is diagonal, as the views are required to be independent and uncorrelated. Ω^{-1} is known as the confidence in the investors' views. The i^{th} diagonal element of Ω is represented w_i.
7. <u>Invertibility of the $\mathbb{P}$ Matrix</u>: $\mathbb{P}$ is not required to be invertible. Meucci (2006) describes a method of augmenting the matrices to make the $\mathbb{P}$ matrix invertible, while not changing the net results.
8. <u>Invertibility of the Ω Matrix</u>: Ω is symmetric and zero on all non-diagonal elements, but may also be zero on a diagonal if the investor is certain of the given view. This means that Ω may or may not be invertible. At a practical level one may require

$$w_i > 0$$

so that Ω is made invertible, but from a formulation point of view Ω is not required to be invertible.

9. Relative and Absolute Views Specification: As an example of how these matrices would be populated, Walters (2014) examines some investors' views, using four assets and two views. The first is a relative view in which the investor believes that asset #1 will outperform asset #2 by 2% with a confidence of w_1. The second is an absolute view in which the investor believes that asset #2 will return 3% with a confidence w_2.
10. Sample Absolute/Relative View Matrices: Note that the sample above does not express a view on asset #4, and thus its returns should not be directly adjusted. The views are specified as follows:

$$\mathbb{P} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\mathbb{Q} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\Omega = \begin{bmatrix} w_1 & 0 \\ 0 & w_2 \end{bmatrix}$$

11. Conditional Distribution in the View Space: Given the above specification of the views, the conditional distribution mean and variance can be formulated as

$$\mathcal{P}(\mathbb{B}|\mathbb{A}) \sim \mathcal{N}(\mathbb{Q}, \Omega)$$

In general this cannot be converted into a useful expression in the asset space because of the mixture of relative and absolute views, and because the $\mathbb{P}$ matrix is not required to be of full rank.

12. View Representation in Asset Space: In the asset space the conditional distribution can be represented as

$$\mathcal{P}(\mathbb{B}|\mathbb{A}) \sim \mathcal{N}(\mathbb{P}^{-1}\mathbb{Q}, [\mathbb{P}^T\Omega^{-1}\mathbb{P}])$$

This representation, however, is not of much practical use. Incomplete views and relative views make the variance non-invertible, and relative views also impact the mean term. Also $\mathbb{P}$ may not be invertible, and even if it is $[\mathbb{P}^T\Omega^{-1}\mathbb{P}]$ may not be, making the above expression impossible to evaluate in practice. Luckily to work with the Black-Litterman Model, one does not need to explicitly evaluate $\mathcal{P}(\mathbb{B}|\mathbb{A})$ above. It is, however, of interest to see how the views are projected onto the asset space.

View Distribution in the Asset Space – Derivation

1. Expected View Return Conditional Distribution: The starting point is the definition of the views

$$\hat{Q} = Q + \epsilon$$

where $\hat{Q}$ is the $k \times 1$ vector of the unknown mean returns to the views, Q is the $k \times 1$ vector of the estimated mean returns to the views, and ϵ is a $n \times 1$ matrix of the residuals from the regression where

$$\mathbb{E}[\epsilon] = 0$$

$$\mathbb{V}[\epsilon] = \mathbb{E}[\epsilon\epsilon^T] = \Omega$$

and Ω is non-singular. The expression for $\hat{Q}$ above can be re-written as an explicit distribution of $\hat{Q}$ as

$$\hat{Q} \sim \mathcal{N}(Q, \Omega)$$

2. Projections onto the Asset Space: Using the definition above the unknown mean returns of the views based on the unknown mean returns of the assets and the portfolio pick matrix P may be re-written as

$$P\hat{\Pi} = \hat{Q}$$

where P is the $k \times n$ matrix of weights for the view portfolios and $\hat{\Pi}$ is the $n \times 1$ vector of the unknown returns of the assets. Substitution of

$$\hat{Q} = Q + \epsilon$$

onto $\hat{Q}$ above yields

$$P\hat{\Pi} = Q + \epsilon$$

3. View Estimated Mean in Asset Space: Assuming P is invertible, which requires it to be of full rank, both sides above may be multiplied by P^{-1}. This is the projection of the view-estimated means onto the asset space representing the Black-Litterman conditional distribution. If P is not invertible a slightly different formulation is needed, adding another term on the right hand side

$$\hat{\Pi} = P^{-1}Q + P^{-1}\epsilon$$

4. View Covariance in Asset Space: The next task is to represent the above formula as a distribution. For this purpose the covariance of the random term needs to be computed. The variance of the unknown asset means about the estimated view means projected onto the asset space is calculated as follows:

$$Variance = \mathbb{E}[P^{-1}\epsilon\epsilon^T(P^{-1})^T] = P^{-1}\mathbb{E}[\epsilon\epsilon^T](P^{-1})^T = P^{-1}\Omega(P^{-1})^T$$

Using

$$(P^T)^{-1} = (P^{-1})^T$$

and

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$Variance = P^{-1}\Omega(P^{-1})^T = P^{-1}\Omega(P^T)^{-1} = (P^T\Omega^{-1}P)^{-1}$$

5. Consolidated Asset Space View Projection: Thus one arrives at the projection of the views onto the asset space as

$$\widehat{\Pi} \sim \mathcal{N}(\mathbb{P}^{-1}\mathbb{Q}, [\mathbb{P}^T\Omega^{-1}\mathbb{P}])$$

The covariance term here is the covariance of the unknown mean returns about the estimated returns from the views, not the covariance of the expected returns.

Specifying Ω

1. Benchmark of Investor Confidence: Ω, the variance of the views, is inversely related to the investor confidence in the view. However, the basic Black-Litterman Model does not provide an intuitive way of quantifying this relationship. It is up to the investor to compute this variance of the views.
2. Typical Methods to calculate Ω: The common ways to calculate Ω are:
   a. Choose an Ω proportional to the variance of the prior
   b. Specify Ω from a confidence interval
   c. Use the variance of residuals in a factor model
   d. Use Idzorek's method to specify confidence along the weight dimension

Ω Proportional to the Variance of the Prior

1. <u>He and Litterman (1999) Ω Specification</u>: Here the assumption is that the variance of the views is just proportional to the variance of the asset returns, just as the variance of the prior distribution is. Both He and Litterman (1999) and Meucci (2006) use this method, though they use it differently. He and Litterman (1999) set the variance of the views as

$$w_{ij} = P(\tau\Psi)P^T \ \forall \ i = j$$

$$w_{ij} = 0 \ \forall \ i \neq j$$

   or

$$\Omega = Diagonal[P(\tau\Psi)P^T]$$

2. <u>Implication of the Above Specification</u>: The above specification of the variance or the uncertainty of the views essentially equally weights investors' views and the market equilibrium weights. By adding τ in the expression the posterior estimate of the returns becomes independent of τ as well. This seems to be the most common method used in the literature.
3. <u>Meucci's Specification – Non-diagonal Ω</u>: Meucci (2006) doesn't bother with diagonalization at all – he just sets

$$\Omega = \frac{1}{c} P\Psi P^T$$

   He further sets

$$c > 1$$

and one obvious choice for c is τ^{-1}. As will be seen later this form of the variance of views lends itself to some simplifications of the Black-Litterman formulas.

Using Confidence Interval for Ω

1. Ω as One Standard Deviation: The investor can specify the variance using a confidence interval around an estimated mean return, e.g., an asset that has an estimated mean return of 3% with an expectation that it is 68% likely to be within the interval $(2\%, 4\%)$. Knowing that 68% of the outcomes falls within one standard deviation of the mean allows one to translate this into a variance for the view as $(1\%)^2$
2. Ω as a Confidence Metric: Note that Ω is the estimate of the uncertainty around the mean, and not the variance of the returns around the mean. This formulation of the variance of the view is consistent with the Canonical Reference Model.

Ω as the Variance of Residuals from a Factor Model

1. Generalized Factor Model for Returns: If the investor is using a factor model to compute the views they can use the variance of the residuals from the model to drive the variance of the return estimates. The general expression for the factor model of returns is

$$r = \sum_{i=1}^{n} \beta_i f_i + \epsilon$$

   where r is the return on the asset, β_i is the factor loading for the factor i, f_i is the return due to factor i, and ϵ is an independent normally distributed residual.
2. Variance of Factor Model Returns: The general expression for the variance of the returns from a factor model is

$$\mathbb{V}[r] = B\mathbb{V}[f]B^T + \mathbb{V}[\epsilon]$$

where B is the factor loading matrix, and F is the vector of returns due to the various factors. Given

$$r = \sum_{i=1}^{n} \beta_i f_i + \epsilon$$

and the assumption that ϵ is independent and normally distributed, one can compute the variance of ϵ directly as part of the regression. While the regression may yield a full covariance matrix, the mixing model will be more robust if only the diagonal elements are used.

3. GARCH Style View Factor Models: Beach and Orlov (2006) describe their work using GARCH style factor models to generate their views for the Black-Litterman model. They generate their precision of views using GARCH models.

Using Idzorek's Method for Ω

1. Asset Weights as Confidence Metric: Idzorek (2005) describes a method for specifying the confidence in the views in terms of a percentage move of the weights on the interval from 0% confidence to 100% confidence. This algorithm is examined in the section on extensions.

The Estimation Model

1. Theil and Bayesian Estimation Approaches: The original Black-Litterman Model reference Theil's Mixed Estimation Model rather than a Bayesian Estimation Model, though similar results are obtained from both methodologies. Theil's method serves as a starting point

because it is simpler and cleaner. The Bayesian method and its derivation are also worked out for completeness.

2. Mean and Estimated Returns Distribution: With either approach the Canonical Black Litterman Reference Model will be used. In this reference model the estimation model is used to compute the distribution of returns about the mean return. This distinction is important in understanding the values used for τ and Ω, and for computing the variance of the prior and the posterior distributions of returns.
3. Posterior Estimate of the Means: The posterior estimate of the mean generated by the estimation model is more precise than either the prior estimate or the investors' views. Note that one would not expected changes in the variance of the distribution of returns because the estimate of the mean is more precise.
4. Posterior Means Estimate - Sample Illustration: A prototypical illustration of the above would be to blend the distributions

$$\mathbb{P}(A) \sim \mathcal{N}(10\%, 20\%)$$

and

$$\mathbb{P}(B|A) \sim \mathcal{N}(12\%, 20\%)$$

If one applies either estimation model in a straightforward fashion

$$\mathbb{P}(A|B) \sim \mathcal{N}(11\%, 10\%)$$

Clearly with the data the variance of the returns distribution did not get cut in half just because one has a slightly better estimate of the mean. In this case the mean is a random variable, and the variance of the posterior corresponds to the variance of the estimated mean around the mean return, not the variance of the distribution of returns around the mean return. In this case the posterior

$$\mathbb{P}(A) \sim \mathcal{N}(11\%, 10\%)$$

makes sense.

5. Shrinkage from Blending of Distributions: By blending these two estimates of the mean, one has an estimate of the mean with much less uncertainty (less variance) than either of the estimates, even though there is no improvement in the estimate of the actual distribution of returns around the mean.

Theil's Mixed Estimation Model

1. Theil's Estimation Model – Setup: Theil's mixed estimation model was created for the purpose of estimating parameters from a mixture of complete prior data and partial conditional data. This is a good problem fit in the current case as it allows one to express views on all of them. The views can be expressed on a single asset, or on a combination of assets. The views do not even to be consistent, the estimation model will take each into account based on the investors' confidence.
2. Linear Models for Parameters' Estimation: Theil's Mixed Estimation Model starts from a linear model for the parameters to be estimated. One can use

$$\mu = \varkappa + \epsilon_{\varkappa}$$

   from the reference model as a starting point.
3. Linear Models for Prior Returns: A simple linear factor model can be expressed as

$$x\beta = \varkappa + u$$

   where $\varkappa$ is the $n \times 1$ vector of equilibrium returns for the assets, x is the $n \times n$ matrix of the factor loadings I_n for the model, β is the $n \times 1$ vector of unknown means for the asset return process, and u is an $n \times 1$ matrix of the residuals from the regression where

$$\mathbb{E}[u] = 0$$

$$\mathbb{V}[u] = \mathbb{E}[u^T u] = \Phi$$

and Φ is non-singular.

4. Linear Model Returns Variance Estimator: The Black-Litterman Model uses a very simple linear model where the expected return for each asset is modeled by a single factor which has a coefficient of 1. Thus x becomes the identity matrix. Given that β and u are independent, and that x is constant, one can model the variance of $\varkappa$ as

$$\mathbb{V}[\varkappa] = x^T \mathbb{V}[\beta] x + \mathbb{V}[u]$$

which can be simplified to

$$\mathbb{V}[\varkappa] = \Psi + \Phi$$

where Ψ is the historical covariance matrix of the returns as used earlier, and Φ is the covariance of the residuals or of the estimate about the actual mean.

5. Total Variance of Estimated Return: This ties back to the formula

$$\Psi_r = \Psi + \Psi_\varkappa$$

used in the reference model. The total variance of the estimated return is the form of the variance of the actual return process plus the variance of the estimate of the mean. This relation will be re-visited later.

6. Conditional Linear Models for Returns: The next consideration is the additional information that needs to be combined with the prior. This information can be subjective views, or it can be derived from statistical data. This is also allowed to be incomplete, meaning that there may not be an estimate for each asset. As before, a simple linear factor model can be expressed as

$$p\beta = q + v$$

where q is the $k \times 1$ vector of equilibrium returns for the assets, p is the $k \times n$ matrix of the factor loadings I_n for the model, β is the $n \times 1$ vector of unknown means for the asset return process, and v is an $k \times 1$ matrix of the residuals from the regression where

$$\mathbb{E}[v] = 0$$

$$\mathbb{V}[v] = \mathbb{E}[v^T v] = \Omega$$

and Ω is non-singular.

7. Combining Prior and Conditional Models: One can combine prior and conditional models as

$$\begin{bmatrix} x \\ p \end{bmatrix} \hat{\beta} = \begin{bmatrix} \varkappa \\ q \end{bmatrix} + \begin{bmatrix} u \\ v \end{bmatrix}$$

where the expected value of the residual is 0, and the expected value of the variance of the residual is

$$\mathbb{V}\begin{bmatrix} v \\ u \end{bmatrix} = \mathbb{E}\left[\begin{bmatrix} v \\ u \end{bmatrix}\begin{bmatrix} u^T & v^T \end{bmatrix}\right] = \begin{bmatrix} \Phi & 0 \\ 0 & \Omega \end{bmatrix}$$

8. Estimation Using Generalized Lease Squares: One can then apply the generalized least squares procedure, which leads to estimating $\hat{\beta}$ as

$$\hat{\beta} = \left[\begin{bmatrix} x^T & p^T \end{bmatrix}\begin{bmatrix} \Phi & 0 \\ 0 & \Omega \end{bmatrix}^{-1}\begin{bmatrix} x \\ p \end{bmatrix}\right]^{-1}\left[\begin{bmatrix} x^T & p^T \end{bmatrix}\begin{bmatrix} \Phi & 0 \\ 0 & \Omega \end{bmatrix}^{-1}\begin{bmatrix} \varkappa \\ q \end{bmatrix}\right]$$

This can be re-written without the matrix notation as

$$\hat{\beta} = [x^T\Phi^{-1}x + p^T\Omega^{-1}p]^{-1}[x^T\Phi^{-1}\varkappa + p^T\Omega^{-1}q]$$

9. Simplification to the Black-Litterman Case: For the Black-Litterman Model – which is a single factor per asset - x can be dropped as it is an identity matrix. If one wanted to use a multi-factor model for the equilibrium then x would be the equilibrium factor loading matrix

$$\hat{\beta}_{BL} = [\Phi^{-1} + p^T\Omega^{-1}p]^{-1}[\Phi^{-1}\varkappa + p^T\Omega^{-1}q]$$

10. Interpretation of the $\hat{\beta}$ Estimate: This new $\hat{\beta}$ is the weighted average of the estimates. The precision is inverse of the variance. The posterior estimate $\hat{\beta}$ is also the best linear unbiased estimate given the data, and has the property that it minimizes the variance of the residual. Note that given a new $\hat{\beta}$ an updated expectation for the variance of the residual should also be available.
11. Multi-Factor Prior Returns Model: If one were using a factor model for the prior, one would retain x – the factor weightings – in the formulas. This results in a multi-factor model where all the factors will be priced into the equilibrium.
12. Variation of the Estimation Residual: One can reformulate the combined relationship in terms of the estimate of $\hat{\beta}$ and a new residual $\tilde{u}$ as

$$\begin{bmatrix} x \\ p \end{bmatrix}\hat{\beta} = \begin{bmatrix} \varkappa \\ q \end{bmatrix} + \tilde{u}$$

Once again

$$\mathbb{E}[\tilde{u}] = 0$$

leading to the expression for the variance of the new residual as

$$\mathbb{V}[\tilde{u}] = \mathbb{E}[\tilde{u}^T\tilde{u}] = [\Phi^{-1} + p^T\Omega^{-1}p]^{-1}$$

and the total variance is

$$\mathbb{V}[y \quad \varkappa] = \mathbb{V}\left[\hat{\beta}\right] + \mathbb{V}[\tilde{u}]$$

13. Simplification of the Residual Variance: This section began with the assertion that the variance of the returns process is a known quantity. Improved estimation of the quantity $\hat{\beta}$ does not alter the estimate of the variance of the returns distribution Ψ. Because of the improved estimate one expects that the variance of the residual estimate has decreased, thereby the total variance has changed. This simplifies the variance formula

$$\mathbb{V}[\varkappa] = \Psi + \Phi$$

to

$$\mathbb{V}[y \quad \varkappa] = \Psi + \mathbb{V}[\tilde{u}]$$

14. Interpretation of the Posterior Variance: This is a clearly intuitive result consistent with the realities of financial time series. The two estimates of the mean have been combined to arrive at a better estimate of the mean. The variance of this estimate has been reduced, but the actual variance of the underlying process has not changed. Given the uncertain estimate of the process, the total variance of the estimated process has also improved incrementally, but it has the asymptotic limit that it cannot be less than the variance of the underlying process.
15. Consistency with Canonical Reference Model: The above is the convention for computing the covariance of the posterior distribution of the Canonical reference Model as shown in He and Litterman (1999).
16. Simplification under Absence of Views: In the absence of views the above formula simplifies to

$$\mathbb{V}[y \quad \varkappa] = \Psi + \Phi$$

which is the variance of the prior distribution of returns.

A Quick Introduction to Bayes' Theorem

1. Application of Bayes' Theorem/Nomenclature: This section provides a quick overview of the relevant portions of the Bayes' Theory in order to create a common vocabulary that can be used in analyzing the Black-Litterman model from a Bayesian point of view.
2. Bayes' Theory – Basic Statement: Bayes' theory states

$$P[A|B] = \frac{P[B|A]P[A]}{P[B]}$$

3. LHS - The Conditional Posterior Distribution: $P[A|B]$ is the conditional probability of A given B. This is also referred to as the posterior distribution.
4. RHS - The Conditional Sampling Distribution: $P[B|A]$ is the conditional probability of B given A. This is also known as the sampling distribution, thus will be referred to as the conditional distribution from here on.
5. RHS - The Unconditional Prior Distribution: $P[A]$ is the unconditional probability of A – also known as the prior distribution. This will be referred to as the prior distribution from here on.
6. RHS - The Unconditional Normalizing Distribution: $P[B]$ is the unconditional probability of B, and is known as the normalizing constant. When applying the Bayes' formula and solving for the posterior distribution the normalizing constant typically disappears inside the constants of the integration.
7. Use of Normal Distributions throughout: A general problem in applying Bayes' Theory is the identification of an intuitive and tractable prior distribution. As seen earlier one of the core assumptions of the Black-Litterman model and the Mean Variance Optimization is that the asset returns are normally distributed. For that reason the treatment here is confined to the case of normally distributed prior and conditional distributions. Give that these inputs are normal distributions it then follows that the posterior will also be normally distributed.
8. Using non-Normal Probability Distributions: When the prior and the posterior distributions have the same functional form the prior distribution is referred to as the conjugate prior. Given interest there is nothing that prevents the construction of the Black-Litterman model

using different distributions, however the normal distribution is generally the most straightforward.

9. Unknown Mean and Known Variance: Another core assumption of the Black-Litterman model is that the variances of the prior and the conditional distributions about the actual mean are known, but the actual mean itself is unknown. This scenario corresponds to the *Unknown Mean and Known Variance* case and is well-documented in the Bayesian literature. This matches the model used by Theil, where the estimate of the mean is uncertain, but the variance is known.
10. Specification of the Prior Distribution: The prior distribution is given as

$$P[A] \sim \mathcal{N}\left(x, \frac{S}{n}\right)$$

where S is the sample variance about the mean of the distribution with n samples and $\frac{S}{n}$ is the variance of x about the mean.

11. Specification of the Conditional Distribution: The conditional distribution is specified as

$$P[B|A] \sim \mathcal{N}(\mu, \Omega)$$

Here is Ω is the uncertainty of the estimate of μ in the mean, and is *not* the variance of the full distribution about its mean.

12. Specification of the Posterior Distribution: From the above the posterior can be specified as

$$P[A|B] \sim \mathcal{N}([\Omega^{-1} + nS^{-1}]^{-1}[\Omega^{-1}\mu + nS^{-1}x]^{T}, [\Omega^{-1} + nS^{-1}]^{-1})$$

The variance term above is the variance of the estimated mean about the actual mean.

13. Precision in Bayesian Distributions: In Bayesian statistics the inverse of the variance is known as the precision. The posterior mean can be described as the weighted mean of the prior and conditional means, where the weighting factor is the corresponding precision. The expression for the posterior distribution $P[A|B]$ above requires that the precisions of both the prior and the conditional be non-infinite and that the sum be non-zero. Infinite precision

corresponds to a variance of 0 or absolute confidence. Zero precision corresponds to infinite variance or complete uncertainty.

The PDF Based Approach

1. PDF Based Black-Litterman Derivation: This section contains a derivation of the Black-Litterman master formula using the standard Bayesian approach for modeling the posterior of two normal distributions. An alternate derivation is presented in Mankert (2006) where the author derives the Black-Litterman *Master Formula* from Sampling Theory, and also shows the detailed transformation between the two forms of this formula.
2. PDF Based Posterior Bayesian Estimation*:* This approach follows a Bayesian approach to computing the PDF of the posterior distribution when the prior and the conditional distributions are both normal distributions. The derivation here is based on the proof shown in De Groot (1970). A similar approach is taken by Scowcroft and Satchell (2000).
3. PDF Based Bayesian Approach Outline: The method of this proof is to examine all the terms in the PDF of each distribution that depend on $\mathbb{E}[r]$ neglecting the other terms as they have no dependence on $\mathbb{E}[r]$ and are thus constant with respect to $\mathbb{E}[r]$.
4. PDF for the Prior Distribution*:* Starting from the prior distribution one can derive an expression proportional to the value of the PDF

$$P[A] \sim \mathcal{N}\left(x, \frac{S}{n}\right)$$

with n samples from the population. Thus the PDF $\xi(x)$ of $P[A]$ satisfies

$$\xi(x) \propto e^{-\left[\frac{S}{n}\right]^{-1}(\mathbb{E}[r]-x)^2}$$

5. PDF for the Conditional Distribution: One next considers the PDF of the conditional distribution

$$P[B|A] \sim \mathcal{N}(\mu, \Omega)$$

The PDF $\xi(\mu|x)$ of $P[B|A]$ satisfies

$$\xi(\mu|x) \propto e^{-\Omega^{-1}(\mathbb{E}[r]-\mu)^2}$$

6. Combining Prior and Conditional Distributions: Combining the distributions $\xi(x)$ and $\xi(\mu|x)$ from above the expression for the PDF of the posterior distribution satisfies

$$\xi(x|\mu) \propto e^{-\Omega^{-1}(\mathbb{E}[r]-\mu)^2 - \left[\frac{S}{n}\right]^{-1}(\mathbb{E}[r]-x)^2}$$

or

$$\xi(x|\mu) \propto e^{-\Phi}$$

7. Simplification of the Combined Exponent: Looking at the quantity in the exponent Φ and simplifying it yields

$$\Phi = \Omega^{-1}(\mathbb{E}[r] - \mu)^2 + \left[\frac{S}{n}\right]^{-1} (\mathbb{E}[r] - x)^2$$

$$\Phi = \Omega^{-1}\{(\mathbb{E}[r])^2 - 2\mu\mathbb{E}[r] + \mu^2\} + \left[\frac{S}{n}\right]^{-1} \{(\mathbb{E}[r])^2 - 2x\mathbb{E}[r] + x^2\}$$

$$\Phi = (\mathbb{E}[r])^2 \left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\} - 2\mathbb{E}[r]\left\{\mu\Omega^{-1} + x\left[\frac{S}{n}\right]^{-1}\right\} + \mu^2\Omega^{-1} + x^2\left[\frac{S}{n}\right]^{-1}$$

8. Introducing the Precision Weighted Mean: On introducing

$$y = \frac{\mu\Omega^{-1} + x\left[\frac{S}{n}\right]^{-1}}{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}}$$

and substituting it in the second term

$$\Phi = (\mathbb{E}[r])^2\left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\} - 2y\mathbb{E}[r]\left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\} + \mu^2\Omega^{-1} + x^2\left[\frac{S}{n}\right]^{-1}$$

9. Simplification Using the Posterior Mean: On adding

$$0 = y^2\left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\} - \left\{\mu\Omega^{-1} + x\left[\frac{S}{n}\right]^{-1}\right\}^2\left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\}^{-1}$$

$$\Phi = (\mathbb{E}[r])^2\left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\} - 2y\mathbb{E}[r]\left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\} + \mu^2\Omega^{-1} + x^2\left[\frac{S}{n}\right]^{-1}$$
$$+ y^2\left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\} + \left\{\mu\Omega^{-1} + x\left[\frac{S}{n}\right]^{-1}\right\}^2\left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\}^{-1}$$

$$\Phi = (\mathbb{E}[r])^2\left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\} - 2y\mathbb{E}[r]\left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\} + y^2\left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\} + \mu^2\Omega^{-1}$$
$$+ x^2\left[\frac{S}{n}\right]^{-1} - \left\{\mu\Omega^{-1} + x\left[\frac{S}{n}\right]^{-1}\right\}^2\left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\}$$

$$\Phi = \left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\}\{(\mathbb{E}[r])^2 - 2y\mathbb{E}[r] + y^2\} + \mu^2\Omega^{-1} + x^2\left[\frac{S}{n}\right]^{-1}$$
$$- \left\{\mu\Omega^{-1} + x\left[\frac{S}{n}\right]^{-1}\right\}^2\left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\}^{-1}$$

$$\Phi = \left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\}\{(\mathbb{E}[r])^2 - 2y\mathbb{E}[r] + y^2\} - \left\{\mu\Omega^{-1} + x\left[\frac{S}{n}\right]^{-1}\right\}^2 \left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\}^{-1} + \left\{\mu^2\Omega^{-1} + x^2\left[\frac{S}{n}\right]^{-1}\right\}\left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\}\left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\}^{-1}$$

$$\Phi = \left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\}\{(\mathbb{E}[r])^2 - 2y\mathbb{E}[r] + y^2\} - \left\{\mu^2\Omega^{-2} + 2\mu x\Omega^{-1}\left[\frac{S}{n}\right]^{-1} + x^2\left[\frac{S}{n}\right]^{-2}\right\}\left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\}^{-1} + \left\{\Omega^{-2}\mu^2 + x^2\left[\frac{S}{n}\right]^{-1}\Omega^{-1} + \mu^2\Omega^{-1}\left[\frac{S}{n}\right]^{-1} + x^2\left[\frac{S}{n}\right]^{-2}\right\}\left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\}^{-1}$$

$$\Phi = \left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\}\{(\mathbb{E}[r])^2 - 2y\mathbb{E}[r] + y^2\} + \left\{\left[\frac{S}{n}\right]^{-1}\Omega^{-1}x^2 + \mu^2\Omega^{-1}\left[\frac{S}{n}\right]^{-1} - 2\mu x\Omega^{-1}\left[\frac{S}{n}\right]^{-1}\right\}\left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\}^{-1}$$

$$\Phi = \left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\}\{(\mathbb{E}[r])^2 - 2y\mathbb{E}[r] + y^2\} + \left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\}^{-1}(x - \mu)\left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\}$$

10. <u>The Posterior Mean and Variance</u>: The second term has no dependency on $\mathbb{E}[r]$ and thus can be included in the proportionality factor. One is left with

$$\xi(x|\mu) \propto e^{-\left\{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}\right\}^{-1}(\mathbb{E}[r] - y)^2}$$

Thus the posterior mean is y as defined in

$$y = \frac{\mu\Omega^{-1} + x\left[\frac{S}{n}\right]^{-1}}{\Omega^{-1} + \left[\frac{S}{n}\right]^{-1}}$$

and the corresponding variance is $\left\{\Omega^{-1}+\left[\frac{S}{n}\right]^{-1}\right\}^{-1}$

Using Bayes' Theorem for the Estimation Model

1. Specification of the Prior Distribution: In the Black-Litterman model the prior distribution is based on the equilibrium implied excess returns. One of the major assumption made by the Black-Litterman model is that the covariance of the prior estimate is proportional to the covariance of the actual returns, but the two quantities are independent. The parameter τ will serve as the constant of proportionality. The prior distribution for the Black-Litterman model was specified using

$$P[A] \sim \mathcal{N}(\Pi, \tau\mathcal{H})$$

and

$$r_A \sim \mathcal{N}(P[A], \mathcal{H})$$

2. Specifying Views Conditional on Prior: The conditional distribution is based on the investors' views. The investors' views are specified as returns to portfolio of assets, and each view has an uncertainty that will impact the overall mixing process. The conditional distribution from the investors' views was specified in

$$P[B|A] \sim \mathcal{N}(PQ, [P^T\Omega^{-1}P]^{-1})$$

3. Blending Conditional and Prior Views: The posterior distribution from the Bayes' Theorem is the precision weighted mean of the prior estimate and the conditional estimate. Therefore one can readily apply Bayes' Theory to the problem of blending the prior and the conditional distributions to create new posterior distribution of asset returns.

4. <u>Posterior Distribution of Asset Returns</u>: Given

$$P[A] \sim \mathcal{N}(\Pi, \tau\mathcal{H})$$

$$r_A \sim \mathcal{N}(P[A], \mathcal{H})$$

and

$$P[B|A] \sim \mathcal{N}(P^{-1}Q, [P^T\Omega^{-1}P]^{-1})$$

for the prior and the conditional distributions respectively, one can apply Bayes' Theorem and derive the following expression for the posterior distributions of the asset returns.

$$P[B|A] \sim \mathcal{N}(\{[\tau\mathcal{H}]^{-1}\Pi + P^T\Omega^{-1}Q\}\{[\tau\mathcal{H}]^{-1} + P^T\Omega^{-1}P\}^{-1}, \{[\tau\mathcal{H}]^{-1} + P^T\Omega^{-1}P\}^{-1})$$

This is sometimes referred to as the Black-Litterman master formula.

5. <u>Alternate Representation of the Posterior Returns</u>: An alternate representation of the same expression for the mean returns $\widehat{\Pi}$ and covariance M is

$$\widehat{\Pi} = \Pi + \tau\Psi P^T[P\tau\Psi P^T + \Omega]^{-1}[Q - P\Pi]$$

and

$$M = [(\tau\Psi)^{-1} + P^T\Omega^{-1}P]^{-1}$$

To re-iterate the posterior M is the variance of the posterior mean about the actual mean. It is the uncertainty in the posterior mean estimate, and not the gross variance of the returns.

6. <u>Derivation of the Alternate Representation</u>: This part contains the derivation of the alternate representation of the Black-Litterman master formula for the posterior expected returns. Starting from

$$P[A|B]\sim\mathcal{N}([(\tau\mathit{\Psi})^{-1}\mathit{\Pi}+P^T\Omega^{-1}Q][(\tau\mathit{\Psi})^{-1}+P^T\Omega^{-1}P]^{-1},[(\tau\mathit{\Psi})^{-1}+P^T\Omega^{-1}P]^{-1})$$

one derives

$$\mathbb{E}[r]=[(\tau\mathit{\Psi})^{-1}+P^T\Omega^{-1}P]^{-1}[(\tau\mathit{\Psi})^{-1}\mathit{\Pi}+P^T\Omega^{-1}Q]$$

7. Separating the Covariance Term: Separating parts of the second term

$$\mathbb{E}[r]=[(\tau\mathit{\Psi})^{-1}+P^T\Omega^{-1}P]^{-1}(\tau\mathit{\Psi})^{-1}\mathit{\Pi}+[(\tau\mathit{\Psi})^{-1}+P^T\Omega^{-1}P]^{-1}P^T\Omega^{-1}Q$$

8. Replacing Precision in the First Term: Replacing the precision term in the first term with the alternate form one gets

$$\mathbb{E}[r]=\{(\tau\Psi-\tau\Psi P^T[P\tau P^T+\Omega]^{-1}P\tau\Psi)(\tau\mathit{\Psi})^{-1}\mathit{\Pi}\}+\{[(\tau\mathit{\Psi})^{-1}+P^T\Omega^{-1}P]^{-1}P^T\Omega^{-1}Q\}$$

$$\mathbb{E}[r]=\{\mathit{\Pi}-(\tau\Psi P^T[P\tau\Psi P^T+\Omega]^{-1}P\mathit{\Pi})\}+\{[(\tau\mathit{\Psi})^{-1}+P^T\Omega^{-1}P]^{-1}P^T\Omega^{-1}Q\}$$

$$\mathbb{E}[r]=\{\mathit{\Pi}-(\tau\Psi P^T[P\tau\Psi P^T+\Omega]^{-1}P\mathit{\Pi})\}+\{(\tau\mathit{\Psi})(\tau\mathit{\Psi})^{-1}[(\tau\mathit{\Psi})^{-1}+P^T\Omega^{-1}P]^{-1}P^T\Omega^{-1}Q\}$$

$$\mathbb{E}[r]=\{\mathit{\Pi}-(\tau\Psi P^T[P\tau\Psi P^T+\Omega]^{-1}P\mathit{\Pi})\}+\{(\tau\mathit{\Psi})[I_n+P^T\Omega^{-1}P\tau\mathit{\Psi}]^{-1}P^T\Omega^{-1}Q\}$$

$$\mathbb{E}[r]=\{\mathit{\Pi}-(\tau\Psi P^T[P\tau\Psi P^T+\Omega]^{-1}P\mathit{\Pi})\}+\{\tau\mathit{\Psi}[I_n+P^T\Omega^{-1}P\tau\mathit{\Psi}]^{-1}[\Omega(P^T)^{-1}]^{-1}Q\}$$

$$\mathbb{E}[r]=\{\mathit{\Pi}-(\tau\Psi P^T[P\tau\Psi P^T+\Omega]^{-1}P\mathit{\Pi})\}+\{\tau\mathit{\Psi}[\Omega(P^T)^{-1}+P\tau\mathit{\Psi}]^{-1}Q\}$$

$$\mathbb{E}[r]=\{\mathit{\Pi}-(\tau\Psi P^T[P\tau\Psi P^T+\Omega]^{-1}P\mathit{\Pi})\}+\{\tau\mathit{\Psi}P^T(P^T)^{-1}[\Omega(P^T)^{-1}+P\tau\mathit{\Psi}]^{-1}Q\}$$

$$\mathbb{E}[r]=\{\mathit{\Pi}-(\tau\Psi P^T[P\tau\Psi P^T+\Omega]^{-1}P\mathit{\Pi})\}+\{\tau\mathit{\Psi}P^T[\Omega+P\tau\mathit{\Psi}P^T]^{-1}Q\}$$

9. Alternate Black Litterman Formula Form: Thus the alternate form of the Black-Litterman expression expected returns is

$$\mathbb{E}[r] = [(\tau\Psi)^{-1} + P^T\Omega^{-1}P]^{-1}[(\tau\Psi)^{-1}\Pi + P^T\Omega^{-1}Q]$$

10. Posterior Covariance of Returns: Computing the posterior covariance of the returns requires adding the variance of the estimate about the mean to the variance of the distribution about the estimate as seen in

$$V([y \quad \Pi]) = \Psi + V[\tilde{u}]$$

This is mentioned in He and Litterman (1999) but not in the other papers.

$$\Psi_P = \Psi + M$$

Substituting the posterior variance from

$$M = [(\tau\Psi)^{-1} + P^T\Omega^{-1}P]^{-1}$$

results in

$$\Psi_P = \Psi + [(\tau\Omega)^{-1} + P^T\Omega^{-1}P]^{-1}$$

11. Posterior Covariance without Views: In the absence of views this reduces to

$$\Psi_P = \Psi + \tau\Psi$$

or

$$\Psi_P = (1 + \tau)\Psi$$

Thus when applying the Black-Litterman model in the absence of views the variance of the estimated returns will be greater than the prior distribution variance. The impact of this formula can be seen in the results shown in He and Litterman (1999) where the investors' weights sum to less than 1 if they have no views. Idzorek (2005) and most other authors do not compute a new posterior variance but instead use a known input variance of the returns about the mean.

12. Views on Asset Subset: In the event that the investor only has partial views – that is views on a subset of assets – usage of the posterior estimate of the variance tilts the posterior weights towards assets with lower variance and higher precision of the estimated mean, and away from higher variance – lower precision of the estimated mean. Thus the existence of views and of the updated covariance will tilt the optimizer towards using or not using those assets. Thus tilt may not be very large if τ is small, but will still be measurable.
13. Estimation of the Prior Covariance: Since one often builds the unknown covariance matrix of returns Ψ from historical data, basic methods of statistics may be used to compute τ. τ may also be estimated from the confidence of the prior distribution. Both of these provide an intuition for selecting a value of τ that is closer to 0 than to 1.
14. Choice of τ used in the Literature: Black and Litterman (1992), He and Litterman (1999), and Idzorek (2005) all indicate that they used small values of τ in their calculations, on the order of $0.025 - 0.050$ Satchell and Scowcroft (2000) state that many investors use a τ around 1, which has no intuitive connection to data, and in fact demonstrates that their paper uses the Alternate Reference Model.
15. Boundary Conditions 100% View Uncertainty: The results can be checked to see if they correspond to the intuition around the boundary conditions. Given

$$\hat{\Pi} = \Pi + \tau\Psi P^T[P\tau\Psi P^T + \Omega]^{-1}[Q - P\Pi]$$

it is easy to see that letting

$$\Omega \to 0$$

corresponding to 100% certainty of views results in

$$\hat{\Pi} = \Pi + \Psi P^T[P\Psi P^T]^{-1}[Q - P\Pi]$$

Thus under 100% certainty of views the estimated return is insensitive to the value of τ used. Further if P is invertible – which means that a view has been offered on every asset – then

$$\hat{\Pi} = P^{-1}Q$$

16. <u>Boundary Conditions 100% View Uncertainty</u>: If the investor is not sure about the views than

$$\Omega \to \infty$$

and

$$\hat{\Pi} = \Pi + \tau\Psi P^T[P\tau\Psi P^T + \Omega]^{-1}[Q - P\Pi]$$

reduces to

$$\hat{\Pi} = \Pi$$

17. <u>Boundary Conditions - Posterior Covariance</u>: An alternate form of the posterior covariance

$$M = [(\tau\Psi)^{-1} + P^T\Omega^{-1}P]^{-1}$$

derived using the Woodbury Identity Matrix is

$$M = \tau\Psi - \tau\Psi P^T[\Omega + P\tau\Psi P^T]^{-1}P\tau\Psi$$

If

$$\Omega \to 0$$

– total confidence in views, plus every asset is in at least one view, the above reduces to

$$M = 0$$

If on the other hand the investor is not confident on the views

$$\Omega \to \infty$$

and the above reduces to

$$M = \tau\Psi$$

The Alternate Reference Model

1. The Satchell and Scowcroft Model: The most common Alternate Reference Model is the one used in Satchell and Scowcroft (2000) and in the work of Meucci prior to his introduction of *Beyond Black Litterman*.

$$\mathbb{E}[r] = \sim\mathcal{N}(\mu, \Psi)$$

In this model $\mathbb{E}[r]$ is normally distributed with a variance Ψ While μ is estimated, it is not considered a random variable. This is commonly referred to as having

$$\tau = 1$$

but the estimate is a point estimate, thus the parameter τ gets eliminated.

2. Interpretation of the Model Ω: In this model Ω becomes the covariance of returns to the views around the unknown mean return, just as Ψ is the covariance of the prior around its mean. Given that point estimates are now being used, the posterior now is a point estimate,

and the posterior covariance is no longer being estimated. Corresponding there is no posterior precision to use downstream in the portfolio selection model.

3. Scaling the Covariance by τ: On re-writing

$$M = [(\tau\Psi)^{-1} + P^T\Omega^{-1}P]^{-1}$$

and moving τ around one gets

$$\hat{\Pi} = \Pi + \Psi P^T \left[P\tau\Psi P^T + \frac{\Omega}{\tau}\right]^{-1} [Q - P\Pi]$$

Here τ appears on only term in the formula. Because the Alternate Reference Model does not include updating the covariance of the estimates this is the only formula.

4. Recast of the Posterior Mean: Given that the investor selects both Ω and τ to control the blending of the prior and the views, one of these terms can be eliminated. Since τ is a single term for all the views and Ω has a separate term for each view Ω will be retained. The posterior estimate of the mean can be re-written as

$$\hat{\Pi} = \Pi + \tau\Psi P^T[P\tau\Psi P^T + \Omega]^{-1}[Q - P\Pi]$$

5. Elimination of the Posterior Variance: As a note none of the authors prior to Meucci (2008) except for Black and Litterman (1992) and He and Litterman (1999) make any mention of the details of the Canonical Reference Model or of the fact that different authors actually use quiet different reference models.

6. Updated Posterior Covariance Estimate: In the Canonical Reference Model the updated posterior covariance of the unknown mean about the estimate will be smaller than the covariance of either the prior or the conditional estimates indicating that the addition of more estimates will reduce the uncertainty of the model. The posterior variance of the returns

$$\Psi_p = \Psi + M$$

will never be lesser than the prior variance of the returns.

7. Improved Estimate of the Returns Variance: This matches intuition as adding more information reduces the uncertainty of the estimates. Given that there is some uncertainty in the value M

$$\Psi_p = \Psi + M$$

provides a better estimate of the variance of returns than the prior variance of the returns.

The Impact of τ

1. Canonical/Alternate Reference Models Usage: The meaning and impact of the parameter τ causes a great deal of confusion for many users of the Black-Litterman Model. It is obvious that investors using the Canonical Reference Model use τ and that it has a very precise meaning in that model. An author that selects essentially a random value for τ is not using the Canonical Reference Model, but is instead using the Alternate Reference Model.
2. τ in the Canonical Reference Model: Given the Canonical Reference Model one can still perform an exercise to understand the impact of τ on the results. Starting with an expression for Ω similar to the one used by He and Litterman (1999), rather than using only the diagonal, the entire structure of the covariance matrix is retained.
3. He and Litterman Posterior Returns: Substituting

$$\Omega = P\tau\Psi P^T$$

into

$$\hat{\Pi} = \Pi + \tau\Psi P^T[P\tau\Psi P^T + \Omega]^{-1}[Q - P\Pi]$$

one gets

$$\widehat{\Pi} = \Pi + \tau\Psi P^T[P\tau\Psi P^T + \Omega]^{-1}[Q - P\Pi] = \Pi + \tau\Psi P^T[P\tau\Psi P^T + P\tau\Psi P^T]^{-1}[Q - P\Pi]$$

$$= \Pi + \tau\Psi P^T[2P\tau\Psi P^T]^{-1}[Q - P\Pi] = \Pi + \frac{1}{2}\tau\Psi P^T(P^T)^{-1}[P\tau\Psi]^{-1}[Q - P\Pi]$$

$$= \Pi + \frac{1}{2}\tau\Psi[\tau\Psi]^{-1}P^{-1}[Q - P\Pi] = \Pi + \frac{1}{2}P^{-1}[Q - P\Pi]$$

i.e.

$$\widehat{\Pi} = \Pi + \frac{1}{2}[P^{-1}Q - \Pi]$$

4. He and Litterman Proportional τ: Clearly using

$$\Omega = P[\tau\Psi]P^T$$

is a simplification, but shows that setting Ω proportional to τ eliminates it from the final formula for $\widehat{\Pi}$. However in the Canonical Reference Model it does not eliminate τ from the posterior covariance expression

$$\Psi_p = \Psi + M$$

5. Proportional τ-Generated Posterior Returns: In general if Ω is formulated as

$$\Omega = P[\alpha\tau\Psi]P^T$$

then

$$\widehat{\Pi} = \Pi + \frac{1}{2}[P^{-1}Q - \Pi]$$

can be written as

$$\hat{\Pi} = \Pi + \frac{1}{1+\alpha}[P^{-1}Q - \Pi]$$

6. <u>Proportional τ Posterior Covariance</u>: Similarly substituting

$$\Omega = P[\tau\Psi]P^T$$

into

$$M = \tau\Psi - \tau\Psi P^T[\Omega + P\tau\Psi P^T]^{-1}P\tau\Psi$$

produces a similar result

$$M = \tau\Psi - \tau\Psi P^T[P\tau\Psi P^T + P\tau\Psi P^T]^{-1}P\tau\Psi = \tau\Psi - \tau\Psi P^T[2P\tau\Psi P^T]^{-1}P\tau\Psi$$
$$= \tau\Psi - \frac{1}{2}\tau\Psi P^T[P^T]^{-1}[\tau\Psi]^{-1}P^{-1}P\tau\Psi = \tau\Psi - \frac{1}{2}\tau\Psi = \frac{1}{2}\tau\Psi$$

Thus

$$M = \frac{1}{2}\tau\Psi$$

in this case.

7. <u>τ Impact on the Posterior Covariance</u>: Note that τ is not eliminated from M. It may also be observed that if τ is of the order of 1 and one were to use

$$\Psi_p = \Psi + M$$

the uncertainty on the estimate of means would be a significant part of the variance of returns. With Alternate Reference Model no posterior variance calculations are performed and the mixing is weighted by the variance of the returns.

8. He and Litterman Symmetric τ: In both cases the choice for τ has evenly weighted prior and conditional distributions in the estimate of the posterior distribution. This matches the intuition considering that two impacts have been blended and both have the same level of uncertainty. The posterior will be an average of the two distributions.
9. Meuccis's Scaling of Prior τ: If one solves for the more general case of

$$\Omega = P[\alpha\tau\Psi]P^T$$

where

$$\alpha \geq 0$$

substitution into

$$\hat{\Pi} = \Pi + \tau\Psi P^T[P\tau\Psi P^T + \Omega]^{-1}[Q - P\Pi]$$

and following the same logic used to derive

$$M = \frac{1}{2}\tau\Psi$$

results in

$$\hat{\Pi} = \Pi + \frac{1}{1+\alpha}[P^{-1}Q - \Pi]$$

10. Impact on Stability of Results: This parametrization of the uncertainty is specified in Meucci (2005) and offers an option between using the same uncertainty for the prior and the views versus having to specify a separate and unique uncertainty for each view. Given that the prior covariance matrix is essentially being multiplied by a constant, this parametrization of the uncertainty of views does not have a negative impact on the stability of the results.

11. Impact on the View Covariance: Note that this specification of the uncertainty in the views changes the assumption from the views being uncorrelated to one where the views have the same correlations as the prior returns.
12. Cross Model τ Usage – Summary: In summary if the investor uses the Alternate Reference Model and makes Ω proportional to Ψ then they only need to calibrate the constant of proportionality α that indicates their relative confidence in their views versus the equilibrium. If they use the Canonical Reference Model and set Ω proportional to $\tau\Psi$ then the returns estimate does not depend on τ but the posterior covariance does.

Calibration of τ

1. τ from Daily Returns Series: The first method for calibrating τ relies on falling back to basic statistics. When estimating the mean of a distribution the uncertainty of the mean estimate will be proportional to the number of samples. Given that the covariance matrix is estimated from historical data

$$\tau = \frac{1}{T}$$

results from the maximum likelihood estimator;

$$\tau = \frac{1}{T - k}$$

results from the best quadratic unbiased estimator. Here T is the number of samples and k is the number of assets.
2. Typical Estimates for τ Values: There are a number of other estimators, but usually the maximum likelihood estimator above is the one used. Given that one usually aims for a sample number of around 60 - 5 years of monthly samples - τ is on the order of 0.02. This is

consistent with several papers that indicate they used τ values in the range of $0.025 - 0.050$.

3. Using Intuition in τ Calibration: The easiest and the most intuitive way to calibrate τ is as part of a confidence interval for the prior mean estimates. This concept can be illustrated using a simple example. The scenario used is one where

$$\tau = 0.05$$

with a single asset having a prior estimate of 7.0% for the excess returns and 15.0% is the known standard deviation of the returns around the mean. The confidence interval used is $(1.0\%, 5.0\%)$ with 68.0% confidence.

4. Prior Distribution Impact - First Scenario: Two scenarios respectively with

$$\tau = 0.05$$

and

$$\tau = 1.00$$

are considered, with the ratio of View Precision to Prior Precision held constant across them. Even though the first scenario uses a seemingly small

$$\tau = 0.05$$

the prior estimate has a relatively low precision based on the width of the confidence interval, and the posterior estimate will be heavily weighted towards the view.

5. Prior Distribution Impact - Second Scenario: In the second scenario

$$\tau = 1$$

the prior confidence interval is so worthless as to make the prior estimate worthless. In order to keep the posterior estimate the same across the scenarios the view estimate also has a wide confidence interval indicating that the investor is not really confident in any of their estimates.

6. Table Depicting Prior/View Confidence:

τ	**Prior at 68% Confidence**	**Prior Precision**	**View σ**	**View Precision**	**View at 68% Confidence**	**View/Prior Precision**
0.05	(4.60%, 9.40%)	888	2.00%	2500	(1.00%, 5.00%)	2.81
1.00	(−8.00%, 22,00%)	44.40	8.90%	125	(−5.90%, 11.90%)	2.81

7. View/τ Variance Interplay: Given such wide intervals for the

$$\tau = 1$$

scenario - 16% chance that the asset has mean returns less than −8.00% - it is hard to imagine having much convection in using the final asset allocation. Thus the interplay between the selection of τ and the specification of the variance of the views is critical.

8. Model Specific τ Usage Impact: Further this example illustrates the difference between using the Alternate Reference Model and the Canonical Reference Model. Specifying

$$\tau = 1$$

is the Alternate Reference Model, but results in very uncertain outputs in the Canonical Reference Model.

9. Calibration from the CAPM Market Weights: Finally τ could be calibrated from the amount invested in the risk free asset given the prior distribution. It can be seen here that he portfolio invested in risky assets given the prior weights will be

$$w = \Pi[\delta(1+\tau)\Psi]^{-1}$$

Thus the weights allocated to the assets are smaller by $\frac{1}{1+\tau}$ than the CAPM market weights. This is because the Bayesian investor is uncertain in the estimate of the prior and does not want to be 100% invested in the risky assets.

Black Litterman Model Implementation Steps

1. CAPM Asset Class Equilibrium Weights: w_{eq} is the Equilibrium Weight for each asset class. It is derived from the capitalization weighted CAPM market portfolio.
2. Historical Asset Returns Covariance: Ψ is the matrix of covariance between the asset classes, and is computed from the historical data.
3. Risk Free Asset Class Yield: r_f is the risk-free rate for the base currency.
4. Market Portfolio Risk Aversion Coefficient: δ is the risk aversion coefficient of the market portfolio. This can be computed using the returns and the standard deviation of the market portfolio, or can be extraneously supplied.
5. Prior Returns Distribution Covariance: τ represents the measure of uncertainty of the equilibrium variance. This is usually set to a small number of the order of $0.025 - 0.050$
6. Setting P, Ω, and Q: First the vector of equilibrium returns Π is computed using reverse optimization from

$$\Pi = \delta\Psi w$$

Then the investors views are formulated, and P, Ω, and Q are specified. Given k views and n assets, P is a $k \times n$ matrix where each row sums to 0 for a relative view and to 1 for an absolute view. Q is a $k \times 1$ vector of excess returns for each view. Ω is a diagonal $k \times k$ matrix of the variance of the views, i.e., the confidence in the views. As a starting point most authors call for the values of w_i to be set equal to $p^T \tau \Psi_i p$ where p is the row from P for the specific view.

7. Applying the Black Litterman Master Formula: Assuming uncertainty in all the views the Black Litterman master formula may be applied to compute the joint returns:

$$\widehat{\Pi} = \Pi + \tau\Psi P^T[P\tau\Psi P^T + \Omega]^{-1}[Q - P\Pi]$$

8. Estimation of the Joint Returns Covariance: The joint covariance is computed from

$$M = \tau\Psi - \tau\Psi P^T\left[\frac{\Omega}{\tau} + P\tau\Psi P^T\right]^{-1} P\Psi$$

9. Computation of the Posterior Covariance: After the above step the sample posterior covariance is computed from

$$\Psi_p = \Psi + M$$

10. Computing the Optimal Portfolio Weights: Finally the weights are computed for the optimal portfolio on the unconstrained (or with constraints, as the case may be) efficient frontier using

$$\widehat{w} = \left[\delta\Psi_p\right]^{-1}\widehat{\Pi}$$

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Incorporating User-Specified Confidence Levels

Overview

1. The Black-Litterman Model Motivation: As seen before the Black Litterman model enables users to combine their unique views regarding the performance of the various assets with the market equilibrium in a manner the results in intuitive diversified portfolios.
2. Tilt Based User Confidence Specification: Idzorek (2005) introduces a new method for controlling the tilts and the final portfolio weights caused by views. The method asserts that the magnitude of the tilt should be controlled by the user-specified confidence level based on an intuitive 0% to 100% confidence level. This is an intuitive method for specifying one of the most abstract mathematical parameters of the Black Litterman model.

Introduction

1. Problem with the Markowitz Paradigm: The Black Litterman asset allocation model is a highly sophisticated portfolio construction method that overcomes the problems of unintuitive highly concentrated portfolios, high input sensitivity, and high estimation error. These three related and well-documented problems with the mean-variance optimization are the most likely reasons that practitioners do not use the Markowitz paradigm, in which the returns are maximized for a given level of risk.
2. Basis behind the Black Litterman Model: The Black Litterman model uses a Bayesian approach to combine the subjective views of an investor with respect to one or more assets with the market equilibrium vector of expected returns – the prior distribution – to form a new, mixed estimate of the expected returns.
3. Basic Black Litterman Model Literature: The model was introduced by Black and Litterman (1990), expanded in Black and Litterman (1991, 1992), and treated in greater detail by Bevan

and Winkelmann (1998), He and Litterman (1999), and Litterman (2003). Other key works on the model were done by Lee (2000), Satchell and Scowcroft (2000), and for the mathematically more inclined, by Christodoulakis (2002).

4. Conceptual Components behind the Model: The Black Litterman Model combines CAPM (Sharpe (1964)), reverse optimization (Sharpe (1974)), mixed estimation (Theil (1971, 1978)), universal hedge ratio/Black's global CAPM (Black (1989a, 1989b), Litterman (2003)), and mean-variance optimization (Markowitz (1952)).

Estimating the Excess Returns Distribution

1. Advantages of the Black Litterman Model: The Black Litterman Model creates stable, mean-variance efficient portfolios based on the investors' unique insights, which overcome the problem of input sensitivity. According to Lee (2000) the Black Litterman model also largely mitigates the problem of estimation error-maximization (Michaud (1989)) by spreading the errors throughout the vector of expected returns.
2. Small Changes in Expected Returns: The most important input in mean-variance optimization is the vector of expected returns. However best and Grauer (1991) show that a small change in the expected returns of one of the portfolio's assets can force half of the assets from the portfolio.
3. Starting Point for Expected Returns: In a search for a reasonable starting point for expected returns Black and Litterman (1992), He and Litterman (1999), and Litterman (2003) all explore several alternative forecasts, historical returns, equal "mean" returns for all assets, and risk-adjusted equal mean returns.
4. Persistence of the Extreme Portfolios: They demonstrate that these alternative forecasts lead to extreme portfolios – when unconstrained portfolios with large long and short positions, and when subject a long-only constraint, portfolios that are concentrated in only a relatively small number of assets.

Reverse Optimization of Expected Returns

1. Basic Idea behind Reverse Optimization: The Black Litterman Model uses the market equilibrium returns as a neutral starting point. Equilibrium returns are the set of returns that clear the market. The equilibrium returns are derived using a reverse optimization method in which the vector of implied excess equilibrium returns is extracted from known information using reverse optimization.
2. Reverse Optimization Equilibrium Returns Expression:

$$\mathit{\Pi} = \lambda \Psi w_{MKT}$$

   where $\mathit{\Pi}$ is the Implied Equilibrium Return Vector - $N \times 1$ column vector, λ is the risk-aversion coefficient; Ψ is the covariance matrix of excess returns - $N \times N$ matrix, and w_{MKT} is the market capitalization weights of the assets - $N \times 1$ column vector. Possible alternatives to the market capitalization weights include a presumed benchmark and a set of float adjusted capitalization weights.
3. The Risk-Return Tradeoff: The risk aversion coefficient λ characterizes the expected risk-return tradeoff. It is the rate at which an investor will forego expected excess return for less variance.
4. Excess Return Reverse Optimization Estimate: In the reverse optimization process the risk aversion coefficient acts as a scaling factor for the reverse optimization estimate of the excess returns; the weighted reverse optimization excess returns equals the specified market risk premium.
5. The Implied Risk Aversion Coefficient: More excess returns per unit risk – larger λ – increases the estimated excess returns. The *implied* risk aversion coefficient λ for a portfolio can be estimated by dividing the expected excess returns by the variance of the portfolio (Grinold and Kahn (1999))

$$\lambda = \frac{\mathbb{E}[R] - r_f}{\sigma^2}$$

where $\mathbb{E}[R]$ is the expected market or the benchmark total returns, r_f is the risk-free rate, and

$$\sigma^2 = w_{MKT}{}^T \Psi w_{MKT}$$

is the total variance of the market – or benchmark – excess returns.

6. Currency Returns Incorporation in the Model: To illustrate the model and keep the scope manageable Idzorek (2005) presents an eight asset example in addition to the general model, and avoids discussing currencies. Currency returns in the model are treated in detail by Black (1989a, 1989b), Black and Litterman (1991, 1992), Grinold (1996), Meese and Crownover (1999), Grinold and Meese (2000), and Litterman (2003).
7. Expected Excess Returns Vector Table

Asset Class	**Historical** $\boldsymbol{\mu_{HIST}}$	**CAPM GSMI** $\boldsymbol{\mu_{GSMI}}$	**CAPM Portfolio** $\boldsymbol{\mu_P}$	**Implied Equilibrium Returns Vector** $\boldsymbol{\Pi}$
US Bonds	3.15%	0.02%	0.08%	0.08%
International Bonds	1.75%	0.18%	0.67%	0.67%
US Large Growth	-6.39%	5.57%	6.41%	6.41%
US Large Value	-2.86%	3.39%	4.08%	4.08%
US Small Growth	-6.75%	6.59%	7.43%	7.43%
US Small Value	-0.54%	3.16%	3.70%	3.70%
International Developed Equity	-6.75%	3.92%	4.80%	4.80%
International Emerging Equity	-5.26%	5.60%	6.60%	6.60%

All four estimates are based on 60 months of excess returns over the risk-free rate. The two CAPM estimates are based on a risk premium of 3. Dividing the risk premium by the variance of the market – or benchmark – excess returns σ^2 results in a risk aversion coefficient λ of approximately 3.07.

8. Asset Set Expected Excess Returns: The table presents four estimates of expected excess returns for the eight assets – US Bonds, International Bonds, US Large Growth, US Large

Value, US Small Growth, US Small Value, International Developed Equity, and International Emerging Equity.

9. UBS Global Securities Market Index: The GSMI CAPM excess returns vector is calculated relative to the UBS Global Securities Market Index GSMI – a global index and a good proxy for the world market portfolio.
10. CAPM Implied Equilibrium Vector: The second CAPM excess-returns vector is calculated relative to the market capitalization weighted portfolio using *implied bets* and is identical to the Implied Equilibrium Returns Vector Π.
11. CAPM Returns Based Implied Betas: Literature on the Black Litterman Model often refers to the reverse optimized Implied Returns Equilibrium Vector Π as the CAPM returns, which can be confusing. CAPM returns based on regression betas can be significantly different from the CAPM based on *implied betas*. Idzorek (2005) uses the procedure in Grinold and Kahn (1999) to calculate the implied betas.
12. Market Capitalization Weighted Portfolio Beta: Just as one is able to use the market capitalization weights and the covariance matrix to infer the Implied Equilibrium Returns Vector, one can extract a vector of implied betas. These implied betas are the betas of the N assets relative to the market capitalization weighted portfolios. As one would expect the market capitalization weighted beta of the market portfolio is 1.
13. Implied Beta Formulation and Components:

$$\beta = \frac{\Psi w_{MKT}}{{w_{MKT}}^T \Psi w_{MKT}} = \frac{\Psi w_{MKT}}{\sigma^2}$$

where β is the vector of implied betas, Ψ is the covariance matrix of excess returns, w_{MKT} is the vector of market capitalization weights, and

$$\sigma^2 = {w_{MKT}}^T \Psi w_{MKT} = \frac{1}{\beta^T \Psi^{-1} \beta}$$

is the variance of the market – or the benchmark – excess returns. The vector of the CAPM returns is the same as the vector of the reverse optimized returns when the CAPM returns are based on implied betas relative to the market capitalization weighted portfolio.

14. Historical vs. GSMI Capital Returns: The Historical Returns Vector has a larger standard deviation and range compared to the other vectors. The GSMI CAPM Returns Vector is quiet similar to the Implied Equilibrium Returns vector Π - the correlation is 98%.
15. Solution to the Unconstrained Variance Minimization: Rearranging

$$\Pi = \lambda \Psi w_{MKT}$$

and substituting μ – representing any vector of excess returns – for Π – the vector of Implied Equilibrium Excess Returns – leads to the solution to the unconstrained minimization problem

$$\max_{w} w^T \mu - \frac{\lambda w^T w}{2}$$

results in

$$w = [\lambda \Psi]^{-1} \mu$$

If μ does not equal Π w will not equal w_{MKT}.
16. The Corresponding Optimal Portfolio Weights: The Table on Recommended Portfolio Weights uses

$$w = [\lambda \Psi]^{-1} \mu$$

to find the optimum weights for these portfolios based on the returns vectors from the first Table. The market capitalization weights are presented in the last column.
17. Table of Reverse Optimization Weights:

Asset Class	Weights Based on	Weights Based on	Weights Based on	Weights Based on Implied

	Historical μ_{HIST}	**CAPM GSMI** μ_{GSMI}	**CAPM Portfolio** μ_P	**Equilibrium Returns Vector** Π
US Bonds	1144.32%	21.33%	19.34%	19.34%
International Bonds	-104.59%	5.19%	26.13%	26.13%
US Large Growth	54.99%	10.80%	12.09%	12.09%
US Large Value	-5.29%	10.82%	12.09%	12.09%
US Small Growth	-60.52%	3.73%	1.34%	1.34%
US Small Value	81.47%	-0.49%	1.34%	1.34%
International Developed Equity	-104.36%	17.10%	24.18%	24.18%
International Emerging Equity	14.59%	14.59%	3.49%	3.49%
High	1144.32%	21.33%	26.13%	26.13%
Low	-104.59%	-0.49%	1.34%	1.34%

18. Historical Return Vector Optimal Portfolio: Not surprisingly the Historical Return Vector produces an extreme portfolio.
19. GSMI vs. Equilibrium Returns Comparison: Those not familiar with mean-variance optimization may expected two highly correlated Returns Vectors to lead to similarly correlated vectors of portfolio holdings. Nevertheless despite the similarity between the CAPM GSMI Returns vector and the Implied Equilibrium Returns Vector Π the returns vectors produce two rather distinct weight vectors – the correlation coefficient is 66%.
20. GSMI vs. Equilibrium Weights Comparison: Most of the weights of the CAPM GSMI based portfolios are different significantly than the benchmark market capitalization weighted portfolio, especially the allocation to International Bonds. As one would expect, since the process of extracting Implied Equilibrium Returns from using the Market Capitalization is reversed, the Implied Equilibrium Returns Vector Π leads back to the Market Capitalization weighted portfolio.
21. Black-Litterman Model Starting Point: In the absence of views that differ from the Implied Equilibrium Returns, investors should hold the market portfolio. The Implied Equilibrium Vector Π is the market-neutral starting point for the Black Litterman Model.

The Black Litterman Model

1. The Fundamental Black-Litterman Expression: K is used to represent the number of views, and N is used to express the number of assets in the formula. The expression for the New Combined Returns Vector $\mathbb{E}[R]$ is

$$\mathbb{E}[R] = [(\tau\Psi)^{-1} + P^T\Omega^{-1}P]^{-1}[(\tau\Psi)^{-1}\Pi + P^T\Omega^{-1}Q]$$

2. The Black Litterman Expression Parameters: Here $\mathbb{E}[R]$ is the Posterior Combined Return Vector - $N \times 1$ column vector; τ is a scalar; Ψ is the covariance matrix of excess returns - $N \times N$ matrix; P is a matrix that identifies the assets involved in the views - $K \times N$ matrix, or $1 \times N$ in the special case of one view; Ω is a diagonal covariance matrix of error terms from the expressed view representing the uncertainty in each view - $K \times K$ matrix; Π is the Implied Equilibrium Returns Vector - $N \times 1$ column vector; and Q is the View Returns Vector - $K \times 1$ column vector.
3. First Idzorek Projection - Absolute View: International Developed Equity will have an absolute return of 5.25% - Confidence in this view is 5.25%. This is an example of the absolute view. From the final column of the First Table the Implied Equilibrium Returns for International Developed Equity is 4.80%, 45 bp lower than this view of 5.25%.
4. Second Idzorek Projection - Relative View: International Bonds will outperform US Bonds by 25 bp – Confidence of the View is 50%.
5. Assessing the Second View Impact: The second view says that the return of International Bonds will be 0.25% greater than that of the US Bonds. In order to gauge whether the second view will have a positive or a negative effect on the International Bonds relative to the US Bonds it is necessary to evaluate the respective Implied Equilibrium Returns of the two assets in the View.
6. View Based Cross Asset Tilt: The first Table indicates that the Implied Equilibrium Returns for the International and the US Bonds is 0.67% and 0.08% respectively, a difference of 0.59%. The second view of 0.25% is less than the 0.59% by which the returns of the

International Bonds exceeds that of the US Bonds; thus one would expect the model to tilt the portfolio away from the International Bonds in favor of the US Bonds.

7. View Returns Lesser than Implied: In general – and in the absence of constraints and additional views – if the view is lesser than the difference of the two Implied Equilibrium returns, the model tilts the portfolio towards the underperforming asset.
8. View Returns Greater than Implied: Likewise if the view is greater than the two implied equilibrium returns, the model tilts the portfolio toward the underperforming assets.
9. Relative Under/Over Performing Assets: The third view demonstrates a scenario involving multiple assets and that the terms *outperforming* and *underperforming* are relative. The number of outperforming assets need not match the number of assets underperforming.
10. Long/Short Sub-portfolio Components: The results from views that involve multiple assets with a range of different Implied Equilibrium Returns can be less intuitive. The assets in a relative view form two separate mini sub-portfolios – a long portfolio and a short portfolio.
11. Relative Weighting of Each Asset: The relative weighting of each nominally outperforming asset is proportional to the asset's market capitalization divided by the sum of the market capitalizations of the other nominally outperforming assets of that view. Likewise the relative weighting of each nominally underperforming asset is proportional to that asset's market capitalization divided by the market capitalizations of the other underperforming assets. The net long position less the net short positions equal 0.
12. Equilibrium View Returns Differential Impact: The mini-portfolio that actually receives the positive view may not be the nominally outperforming assets from the expressed view. In general if the view return is greater than the weighted average Implied Equilibrium differential return the model will tend to overweight the "outperforming" assets.
13. View 3 Table of Nominally Outperforming Assets:

Asset Class	**Market Capitalization (Billions)**	**Relative Weight**	**Implied Equilibrium Returns Vector Π**	**Weighted Excess Return**
US Large Growth	$5174	90.00%	6.41%	5.77%
US Small Growth	$575	10.00%	7.43%	0.74%

TOTAL	$5749	100.00%	-	6.52%

14. US Large and Small Growth: Idzorek's View 3 shows that the nominally outperforming assets are the US Large Growth and the US Small Growth and the nominally underperforming assets are the US Large Value and the US Small Value. The Table above demonstrates that the weighted average Implied Equilibrium Returns of the mini-portfolio formed from US Large Growth and the US Small Growth is 6.52%.
15. View 3 Table of Nominally Underperforming Assets:

Asset Class	**Market Capitalization (Billions)**	**Relative Weight**	**Implied Equilibrium Returns Vector *Π***	**Weighted Excess Return**
US Large Value	$5174	90.00%	4.08%	3.67%
US Small Value	$575	10.00%	3.70%	0.37%
TOTAL	$5749	100.00%	-	4.04%

16. US Large and Small Value: Using the Table above the weighted average Implied Equilibrium Return of the mini-portfolio formed from the US Large Value and the US Small Value is 4.04%. The weighted average Implied Equilibrium Return differential is 2.47%.
17. Impact of the Return Differentials: Because View 3% states that the US Large Growth and the US Small Growth will outperform the US Large Value and the US Small Value by only 2% - a reduction from the current weighted average Implied Equilibrium Return differential of 2.47% - the view appears to actually represent a reduction in the performance of US Large Growth and US Small Growth relative to US Large Value and US Small Value.
18. Corresponding Increments in the Allocation: The effect is numerically demonstrated later, where the nominally outperforming assets of the third View – US Large Growth and US Small Growth – receive reductions in their allocations, and the nominally underperforming assets – US Large Value and US Small Value – receive increases in their allocations.

Building the Inputs

1. <u>P Matrix Specification Litterman's Approach</u>: Methods for specifying the P matrix vary. Litterman (2003) assigns a percentage value to the assets in question.
2. <u>Satchell and Scowcroft P Matrix</u>: Satchell and Scowcroft (2000) use an equal weighting scheme. Under this system the weightings are proportional to 1 divided by the number of respective assets underperforming or outperforming. View 3 has 2 nominally underperforming assets, each of which receives a -0.5 weighting. View 3 also contains 2 nominally outperforming assets, each of which receives a +0.5 weighting. This weighting scheme ignores the market capitalization of the assets involved in the view.
3. <u>Satchell and Scowcroft Approach Impact</u>: The market capitalizations of the US Large Growth and the US Large Value asset classes are 9 times the market capitalizations of the US Small Growth and the US Small Value asset classes. Yet Satchell and Scowcroft affects the relative weights equally causing large changes in the two smaller asset classes.
4. <u>P Matrix Specification Idzorek Approach</u>: In contrast to Satchell and Scowcroft (2000) equal weighting scheme Idzorek prefers to use a market capitalization weighting scheme. More specifically the relative weighting of each asset is proportional to the asset's market capitalization divided by the total capitalization of either the outperforming or the underperforming assets of that particular view.
5. <u>Idzorek's P Matrix Scoping Loadings</u>: As shown in the previous tables the relative market capitalization weights of the nominally outperforming assets are 0.9 for US Large Growth and 0.1 for US Small Growth while the relative market capitalization weights for the nominally underperforming assets are -0.9 for US Large Value and -0.1 for US Small Value. These figures are used to create a new P matrix that is used for all of the subsequent calculations.
6. <u>Market Capitalization Method P Matrix</u>:

$$P = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ -1.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.9 & -0.9 & 0.1 & -0.1 & 0.0 & 0.0 \end{bmatrix}$$

7. Computation of the View Variances: Once the matrix P is defined one can calculate the variance of each individual view portfolio. The variance of an individual view portfolio is ${p_k}^T \Psi p_k$ where p_k is a single $1 \times N$ row vector from matrix P that corresponds to the k^{th} view and Ψ is the covariance of matrix of excess returns.
8. Covariance of the Projection Uncertainty: The respective variance of each individual view portfolio is an important source of information regarding the certainty – or lack thereof – of the level of confidence that should be placed on a view. This information is shortly used to re-visit the error variances ω that form the diagonal elements of Ω.
9. Variance of the View Portfolios:

View	**Expression**	**Value**
1	${p_1}^T \Psi p_1$	2.836%
2	${p_2}^T \Psi p_2$	0.563%
3	${p_3}^T \Psi p_3$	3.462%

10. Specifying Scoping and Projection Uncertainties: Conceptually the Black-Litterman Model is a complex weighted average of the Implied Equilibrium Returns Vector Π and the View Returns Vector Q in which the relative weightings are a function of the scalar τ and the uncertainty Ω. Unfortunately the scalar and the uncertainty in the views are the most abstract and difficult to specify parameters of the model.
11. Enhanced Scoping/Projection Uncertainty Impact: The greater the level of confidence – certainty – in the expressed views, the closer the new return vector will be to the views. If the investor is less confident on the expressed views, the new returns vector should be closer to the Implied Equilibrium Returns Vector Π.
12. Estimating the Scoping Uncertainty Parameter: The scalar τ is more or less inversely proportional to the relative weight given to the Implied Equilibrium Returns Vector Π. Unfortunately guidance in the literature for setting the scalar's value is scarce.
13. Estimate of Means vs. Variance: Both Black and Litterman (1992) and Lee (2000) address this issue; since the uncertainty in the mean is less than the uncertainty in the returns the scalar τ is close to zero. One would expect Equilibrium Returns to less volatile than Historical Returns.

14. Lee (2000) Estimation of the Scalar τ: Lee, who has considerable experience working with a variant of the Black Litterman model typically sets the value of the scalar τ between 0.01 and 0.05, and then calibrates the model based upon a target level of the tracking error.
15. Satchell and Scowcroft τ Estimate: Conversely Satchell and Scowcroft (2000) indicate that the value of τ is often set to 1. They also include an advanced mathematical discussion of one method for establishing a conditional value for τ.
16. Blamont and Firoozye τ Estimate: Finally Blamont and Firoozye (2003) interpret $\tau\Psi$ as the standard error of the estimate of the Implied Equilibrium Vector Π; thus τ is approximately 1 divided by the number of observations.
17. Departure from the Asset's Prior Weight: In the absence of constraints the Black Litterman model only recommends departure from the market capitalization weight only if the asset is a subject of a view. For assets that are the subject of a view, the magnitude of their departure from the market capitalization weight is controlled by the ratio of τ to the variance of the error term ω of the view in question.
18. Impact of the Additional Views: The magnitude of the departure from the market capitalization weights is also affected by the other views. Additional views lead to a different Combined Returns Vector $\mathbb{E}[R]$, which in turn leads to a new vector of recommended weights.
19. He and Litterman τ Estimate: The easiest way to calibrate the Black Litterman Model is to make an assumption on the value of τ. He and Litterman (1999) calibrate the confidence of a view so that the ratio $\frac{\omega}{\tau}$ is equal to the variance of the view portfolio ${p_k}^T\Psi p_k$.
20. Explicit Calculation of the Ω Elements: Assuming

$$\tau = 0.025$$

and using the individual variances of the view portfolios ${p_k}^T\Psi p_k$ from the previous Table the covariance matrix of the error term has the following form. The general case is

$$\Omega = \begin{pmatrix} ({p_1}^T\Psi p_1)\tau & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & ({p_k}^T\Psi p_k)\tau \end{pmatrix}$$

and in the case of Idzorek's example

$$\Omega = \begin{bmatrix} 0.000709 & 0.000000 & 0.000000 \\ 0.000000 & 0.000141 & 0.000000 \\ 0.000000 & 0.000000 & 0.000866 \end{bmatrix}$$

21. Impact of $\frac{\omega}{\tau}$ on Returns: When the covariance matrix of the error term is calculated using this method the actual value of the scalar τ becomes irrelevant, as only the ratio $\frac{\omega}{\tau}$ enters the model. For example changing the assumed value of the scalar τ from 0.025 to 15 dramatically changes the value of the diagonal elements Ω but the new Combined Returns Vector $\mathbb{E}[R]$ is unaffected.

22. Returns Vectors and Portfolio Weights:

Asset Class	**New Combined Returns Vector $\mathbb{E}[\boldsymbol{R}]$**	**Implied Equilibrium Returns Vector $\boldsymbol{\Pi}$**	**Difference $\mathbb{E}[\boldsymbol{R}] - \boldsymbol{\Pi}$**	**New Weight $\hat{\mathbf{w}}$**	**Market Capitalization Weight $\mathbf{w}_{MKT}$**	**Difference $\hat{\mathbf{w}} - \mathbf{w}_{MKT}$**
US Bonds	0.07%	0.08%	-0.02%	29.88%	19.34%	10.54%
International Bonds	0.50%	0.67%	-0.17%	15.59%	26.13%	-10.54%
US Large Growth	6.50%	6.41%	0.08%	9.35%	12.09%	-2.73%
US Large Value	4.32%	4.08%	0.24%	14.82%	12.09%	2.73%
US Small Growth	7.59%	7.43%	0.16%	1.04%	1.34%	-0.30%
US Small Value	3.94%	3.70%	0.23%	1.65%	1.34%	0.30%
International Developed Equity	4.93%	4.80%	0.13%	27.81%	24.81%	3.63%
International Emerging Equity	6.84%	6.60%	0.24%	3.49%	3.49%	0.00%
TOTAL	-	-	-	103.63%	100.00%	3.63%

23. Deviation from the Asset's Equilibrium Weight: Even though the expressed views only involve 7 out of the 8 asset classes, the table above shows that the individual returns of all assets changed from their respective Implied Equilibrium Returns. A single view causes the return of every asset in the portfolio to change from its Implied Equilibrium Return since each individual return is linked to the other returns via the covariance matrix of excess returns Ψ.
24. Strength of Black Litterman Model: The new weight vector $\hat{\omega}$ in the Table is based upon the Combined Excess Return Vector $\mathbb{E}[R]$. The Table also illustrates one of the strongest features of the Black Litterman model. Only the weights for the 7 assets for which the views were expressed changed from their original market capitalization weights and the directions of their changes are intuitive. No views were expressed on International Emerging Equity and its weights are unchanged.
25. Original Portfolio Plus Long/Short: From a macro perspective the new portfolio can be viewed as a sum of two portfolios where Portfolio 1 is the original market capitalization weighted portfolio and Portfolio 2 is a series of long and short positions based on the views.
26. Long/Short Estimations from Projections: Portfolio 2 can be sub-divided into mini-portfolios with offsetting long and short positions that sum to 0. View 1, the absolute view, increases the weight of the International Developed Equity without an offsetting position, resulting in portfolio positions that no longer sum to 1.
27. Black Litterman Model with Constraints: The intuitiveness of the Black Litterman model is less apparent when used investment constraints such as constraints on unity, risk, beta, and short selling. He and Litterman (1999) and Litterman (2003) suggest that, in the presence of constraints, the investor input the new Combined Returns Vector $\mathbb{E}[R]$ into a mean-variance optimizer.

Fine Tuning the Model

1. Principle behind Model Fine Tuning: One can fine tune the Black Litterman Model by studying the New Combined Return Vector $\mathbb{E}[R]$, calculating the anticipated risk-return

characteristics of the new portfolio, and then adjusting the scalar τ and the individual variances of the error term ω that form the diagonal elements of the covariance matrix of the error term Ω.

2. Bevan and Winkelmann Fine Tuning Approach: Bevan and Winkelmann (1998) offer guidance in setting the weight given to the View Returns Vector Q. After deriving an initial Combined Return Vector $\mathbb{E}[R]$ and the subsequent portfolio weights, they calculate the anticipated Information Ratio of the new portfolio.
3. Maximum Anticipated Information Ratio Metric: They recommend a Maximum Anticipated Information Ratio of 2.0. If the Information Ratio is greater than 2.0 they recommend decreasing the weight given to the views, i.e., decreasing the value of τ and leaving the diagonal elements of Ω unchanged.
4. Table on Posterior Portfolio Statistics:

Measure	**Market Capitalization Weighted Portfolio $\mathbf{w}_{MKT}$**	**Black Litterman Portfolio $\hat{\mathbf{w}}$**
Excess Return	3.000%	3.101%
Variance	0.00979	0.01012
Standard Deviation	9.893%	10.058%
Beta	1	1.01256
Residual Return	-	0.063%
Residual Risk	-	0.904%
Active Return	-	0.101%
Active Risk	-	0.913%
Sharpe Ratio	0.3033	0.3083
Information Ratio	-	0.0699

5. Anticipated Risk-Return Characteristics Comparison: Idzorek (2005) compares the anticipated risk-return characteristics of the market capitalization weighted portfolio with the Black Litterman portfolio using the new weights produced by the New Combined Vector.

6. Idzorek's Calculations of Portfolio Statistics: The computations by Idzorek are based on the implied betas derived from the covariance matrix of historical excess returns and the mean-variance data of the market capitalization weighted benchmark portfolio.
7. Grinold and Kahn Measure Expressions: From Grinold and Kahn (1999); The residual return is

$$\theta_P = \mathbb{E}[R_P] - \beta_P \mathbb{E}[R_B]$$

The Residual Risk is

$$\omega_P = \sqrt{{\sigma_P}^2 - {\beta_P}^2 {\sigma_B}^2}$$

The Active Return is

$$\mathbb{E}[R_{PA}] = \mathbb{E}[R_P] - \mathbb{E}[R_B]$$

The Active Risk is

$$\psi_P = \sqrt{{\omega_P}^2 + {\beta_{PA}}^2 {\sigma_B}^2}$$

The Active Portfolio Beta is

$$\beta_{PA} = \beta_P - 1$$

8. Grinold and Kahn Measure Dictionary: $\mathbb{E}[R_P]$ is the expected returns of the posterior portfolio, $\mathbb{E}[R_B]$ is the expected returns of the benchmark market capitalization weighted portfolio based on the New Combined Expected Returns Vector $\mathbb{E}[R]$, ${\sigma_B}^2$ is the variance of the benchmark portfolio, and ${\sigma_P}^2$ is the variance of the posterior portfolio.

9. Posterior Sharpe/Information Ratio Estimates: Overall the views have very little effect on the expected risk-return characteristics of the new portfolio. However both the Sharpe Ratio and the Information Ratio have increased slightly. The ex-ante information ratio is well below the recommended maximum of 2.0.
10. Verifying Absence of Unintended Results: Next the outputs from the views should be evaluated to confirm that there are no unintended results. For instance investors confined to unity may want to remove absolute views, such as View 1.
11. Sources for Asset Projection Views: Investors should evaluate their ex-post Information Ratios for guidance when setting their confidence on various views. An investment manager could the set the confidence on a particular view based in part on the corresponding analyst's Information Coefficient.
12. The Investment Manager's Information Coefficient: According to Grinold and Kahn (1999) a manager's Information Coefficient is the correlation of his forecasts with the actual results. This gives greater relative importance to the more skillful analysts.
13. Accurate Estimation of the Asset Variance: Most of the examples in the literature, including the eight asset sample presented by Idzorek (2005), use a simple covariance matrix of historical returns. However investors should use the best possible estimate of the covariance matrix of the excess returns. Litterman and Winkelmann (1998) and Litterman (2003) outline the methods they prefer for estimating the covariance matrix of returns, as well as several alternate methods of estimation.
14. The Qian and Gorman Extension: Qian and Gorman (2001) extend the Black-Litterman Model, enabling investors to express views on volatilities and correlations in order to derive a conditional estimate of the covariance matrix of returns. They assert that the conditional covariance matrix stabilizes the results of mean-variance optimization.

Method for Incorporating User-Specified Confidence Levels

1. Difficulty in the Specification of Ω: As the discussion above illustrates Ω is one of the most abstract mathematical parameters of the Black-Litterman Model. Unfortunately, according to

Litterman (2003), how to specify the diagonal elements of Ω, representing the uncertainty of the views, is a common question without an universal answer.

2. View Specific Probability Density Function: Regarding Ω Herold (2003) indicates that the major difficulty of the Black-Litterman model is that it forces the user to specify a probability density function for each view, which makes Black-Litterman model suitable only for quantitative managers.
3. Idzorek's Implied Confidence Estimation Method: Idzorek presents a new method for determining the implied confidence levels in the views and how an implied confidence level framework can be coupled with an intuitive 0% to 100% user specified confidence level in each view to determine the values of Ω which simultaneously removes the need for specifying a value for τ.

Implied Confidence Levels

1. Specifying an Intuitive Confidence Level: It was seen earlier that the individual variances of the error terms ω that form the diagonal elements of the covariance matrix Ω of the error term ω were based on the view portfolios' variances ${p_k}^T \Psi p_k$ multiplied by the scalar τ. However Idzorek (2005) contends that there are other sources of variances in addition ${p_k}^T \Psi p_k$ that affect the investor's confidence in a view. When each view was stated an intuitive level of confidence from 0% to 100% was assigned to each view.
2. Factors affecting Confidence in the View: Presumably additional factors can affect an investor's confidence in a view, such as historical accuracy or *score* of the model, the screen, or the analyst the produced the view, as well as the difference between the view and the implied market equilibrium.
3. Accommodation of the Multiple Factors: These factors, and possibly others, should then be combined with the variance of the view portfolio ${p_k}^T \Psi p_k$ to produce the best possible estimate of the confidence levels on the views. Doing so would enable the Black-Litterman model maximize across much of the investors' information.

4. <u>100% Confidence in a View</u>: Setting all of the diagonal elements of Ω equal to zero is equivalent to specifying 100% confidence in each of the K views. *Ceteris paribus* doing so would produce the greatest departure from the benchmark capitalization weights for the assets named in the views.
5. <u>Expected Returns Under 100% Confidence</u>: When 100% confidence is specified for all of the views the Black Litterman expression for the New Combined Return Vector under 100% certainty $\mathbb{E}[R_{100\%}]$ is

$$\mathbb{E}[R_{100\%}] = \Pi + \tau\Psi P^T[P^T\Psi P]^{-1}[Q - P\Pi]$$

To distinguish the above expression from the Black Litterman expression

$$\mathbb{E}[R] = [(\tau\Psi)^{-1} + P^T\Omega^{-1}P]^{-1}[(\tau\Psi)^{-1}\mathit{\Pi} + P^T\Omega^{-1}Q]$$

the subscript 100% is added.
6. <u>Allocation under Different Confidence Levels</u>: Substituting $\mathbb{E}[R_{100\%}]$ for μ in

$$w = [\lambda\Psi]^{-1}\mu$$

leads to $w_{100\%}$ the weight vector based on 100% confidence on the views. Idzorek (2005) illustrates the relative weights w_{MKT}, $\widehat{w}$, and $w_{100\%}$ for each asset.
7. <u>100% Confidence for Single Asset</u>: When an asset is named in only one view the vector of portfolio weights with 100% confidence $w_{100\%}$ enables one to calculate an intuitive 0% to 100% confidence for each view. In order to do so one must solve the unconstrained maximization problem twice – once using $\mathbb{E}[R]$ and once using $\mathbb{E}[R_{100\%}]$.
8. <u>Weights under Different Confidence Levels</u>: The New Combined Returns Vector $\mathbb{E}[R]$ based on the covariance matrix of the error term Ω leads to vector $\widehat{w}$ while the New Combined Return Vector $\mathbb{E}[R_{100\%}]$ based on 100% confidence levels leads to the vector $w_{100\%}$.
9. <u>Implied Confidence Using Allocation Deviations</u>: The departure of these new weight vectors from the vector of market capitalization weights w_{MKT} are $\widehat{w} - w_{MKT}$ and $w_{100\%} - w_{MKT}$ respectively. It is then possible to determine the *implied* level of confidence in these views by

dividing each weight difference $\hat{w} - w_{MKT}$ by the corresponding maximum weight difference $w_{100\%} - w_{MKT}$.

10. Implied Confidence Level of Views:

Asset Class	Market Capitalization Weight $\mathbf{w}_{MKT}$	New Weight $\hat{\mathbf{w}}$	Difference $\hat{\mathbf{w}} - \mathbf{w}_{MKT}$	New Weights Based on 100% Confidence $\mathbf{w}_{100\%}$	Difference $\mathbf{w}_{100\%} - \mathbf{w}_{MKT}$	Implied Confidence Level $\frac{\hat{\mathbf{w}} - \mathbf{w}_{MKT}}{\mathbf{w}_{100\%} - \mathbf{w}_{MKT}}$
US Bonds	19.34%	29.88%	10.54%	43.82%	24.48%	43.06%
International Bonds	26.13%	15.59%	-10.54%	1.65%	-24.48%	43.06%
US Large Growth	12.09%	9.35%	-2.73%	3.81%	-8.28%	33.02%
US Large Value	12.09%	14.82%	2.73%	20.37%	8.28%	33.02%
US Small Growth	1.34%	1.04%	-0.30%	0.42%	-0.92%	33.02%
US Small Value	1.34%	1.65%	0.30%	2.26%	0.92%	33.02%
International Developed Equity	24.81%	27.81%	3.63%	35.21%	11.03%	32.94%
International Emerging Equity	3.49%	3.49%	-	3.49%		

11. Idzorek's Empirical Implied Confidence Estimate: The implied level of confidence in a view based on the scaled variance of the individual view portfolios is shown in the Table above. The implied confidence levels in View 1, View 2, and View 3 in the instance above are 32.94%, 43.06%, and 33.02% respectively.
12. Principle Driver behind Idzorek's Approach: Given the discrepancy between the stated confidence levels and the implied confidence levels one could experiment with different ω's and recalculate the New Combined Return Vector $\mathbb{E}[R]$ and the new set of recommended portfolio weights. Idzorek's approach formalizes this algorithmically.

The Tilt Based Intuitive Approach

1. Tilt Induced by Specified Confidence: Idzorek (2005) proposes that the diagonal elements of be derived in a manner that is based on the user-specified confidence levels and that results in portfolio tilts, which approximate $w_{100\%} - w_{MKT}$ multiplied by the user-specified confidence level C.

$$Tilt_k \approx (w_{100\%} - w_{MKT}) \times C_k$$

   where $Tilt_k$ is the approximate tilt caused by the k^{th} view - $N \times 1$ column vector, and C_k is the confidence in the k^{th} view.
2. Incorporation of the Estimated Tilt: Furthermore in the absence of the other views the approximate recommended weight vector resulting from the view is

$$w_{k\%} \approx w_{MKT} + Tilt_k$$

   where $w_{k\%}$ is the target weight vector based on the tilt caused by the k^{th} view - $N \times 1$ column vector.
3. Step #1 - 100% Confidence Returns: For each view k calculate the New Combined Return Vector $\mathbb{E}[R_{k,100\%}]$ using the Black Litterman formula under 100% certainty treating each view as if it were the only one.

$$\mathbb{E}[R_{k,100\%}] = \Pi + \tau\Psi {p_k}^T [{p_k}^T \Psi p_k]^{-1} [Q_k - p_k\Pi]$$

   where $\mathbb{E}[R_{k,100\%}]$ is the Expected Return Vector based on 100% confidence on the k^{th} view - $N \times 1$ column vector, p_k identifies the assets involved in the k^{th} view - $1 \times N$ row vector, and Q_k is the k^{th} view returns.

4. Adjustment Applied to Absolute Views: If the view in question is an absolute view and the view is specified as a total return rather than as an excess return, subtract the risk-free rate from Q_k.
5. Step #2 - 100% Confidence Weights: Calculate $w_{k,100\%}$ the weight vector based on 100% confidence in the k^{th} view, using the unconstrained maximization formula

$$w_{k,100\%} = [\lambda\Psi]^{-1}\mathbb{E}[R_{k,100\%}]$$

6. Step #3 - Departure from Market: Calculate using pair-wise subtraction the maximum departures from the market capitalization weights caused by 100% confidence on the k^{th} view.

$$D_{k,100\%} = w_{k,100\%} - w_{MKT}$$

   where $D_{k,100\%}$ is the departure from the market capitalization weight based on 1005 confidence in k^{th} view - $N \times 1$ column vector.
7. Assets not in the View: The asset classes of $w_{k,100\%}$ that are not part of the k^{th} view retain their original weight leading to a value of 0 for the elements of $D_{k,100\%}$ that are not part of the k^{th} view.
8. Step #4 - 100% Confidence Tilt: Compute using pair-wise multiplication the N elements of $D_{k,100\%}$ by the unspecified confidence C_k in the k^{th} view to estimate the desired tilt caused by the k^{th} view.

$$Tilt_k = D_{k,100\%} \times C_k$$

   where $Tilt_k$ is the desired active weights caused by the k^{th} view - $N \times 1$ column vector – and C_k is an $N \times 1$ column vector where the assets that are part of the view receive the user-specified confidence level of the k^{th} view and the assets that are not part of the view are set to 0.

9. Step #5 - Tilt Based Weight: Estimate using pair-wise addition the target weight vector $w_{k,\%}$ based on the tilt:

$$w_{k,\%} = w_{MKT} + Tilt_k$$

10. Step #6 - Variance Error Minimization: Find the value of ω_k - the k^{th} diagonal element of Ω representing the uncertainty in the k^{th} view that minimizes the sum of the squared differences between $w_{k,\%}$ and w_k.

$$\min \sum \left(w_{k,\%} - w_k\right)^2$$

subject to

$$\omega_k > 0$$

where

$$w_k = [\lambda\Psi]^{-1}[(\tau\Psi)^{-1} + {p_k}^T {\omega_k}^{-1} p_k]^{-1}[(\tau\Psi)^{-1}\Pi + {p_k}^T {\omega_k}^{-1} Q_k]$$

11. Active Risk of the k^{th} View: Having just determined the weight vector associated with a specific view w_k it may be useful to calculate the active risk associated with the specific view in isolation as $\sqrt{{w_A}^T \Psi w_A}$
12. Active Risk Estimation Measure Dictionary: Here

$$w_A = w_k - w_{MKT}$$

is the active portfolio weights;

$$w_k = [\lambda\Psi]^{-1}[(\tau\Psi)^{-1} + {p_k}^T {\omega_k}^{-1} p_k]^{-1}[(\tau\Psi)^{-1}\Pi + {p_k}^T {\omega_k}^{-1} Q_k]$$

is the weight vector of the portfolio based on the k^{th} view and the user-specified confidence level, and Ψ is the covariance matrix of excess returns.

13. Constructing the Complete Ω Matrix: Repeat the above six steps for the k views, build a diagonal $K \times K$ matrix in which the diagonal elements of Ω are the ω_k values calculated in step #6, and solve for the New Combined Return Vector $\mathbb{E}[R]$ using

$$\mathbb{E}[R] = [(\tau\Psi)^{-1} + P^T\Omega^{-1}P]^{-1}[(\tau\Psi)^{-1}\Pi + P^T\Omega^{-1}Q]$$

14. Independence of the Prior Confidence Parameter: Throughout this process the value of the scalar τ is held constant and does not explicitly affect the New Combined Return Vector $\mathbb{E}[R]$ which eliminates the difficulties associated with specifying it.
15. Advantages of the Idzorek Approach: Despite the relative complexities of the steps for specifying the diagonal elements of Ω the key advantage of this new method is that it enables the user to determine the values of Ω based on an intuitive 0% to 100% confidence scale.
16. Alternate Methods for specifying Ω: Alternative methods for specifying the diagonal elements of Ω require one to specify these abstract values directly. Some of these approaches are explored in Zimmermann, Drobetz, and Oertmann (2002), Fusai and Meucci (2003), and Litterman (2003).

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Leibnitz Integral Rule

1. <u>Differentiation of the Limits of Integrals</u>:

$$\frac{\partial}{\partial y}\int_{a(y)}^{b(y)} f(x,y)dx = \frac{\partial b(y)}{\partial y}f(x,y) - \frac{\partial a(y)}{\partial y}f(x,y) + \int_{a(y)}^{b(y)} \frac{\partial f(x,y)}{\partial y}dx$$

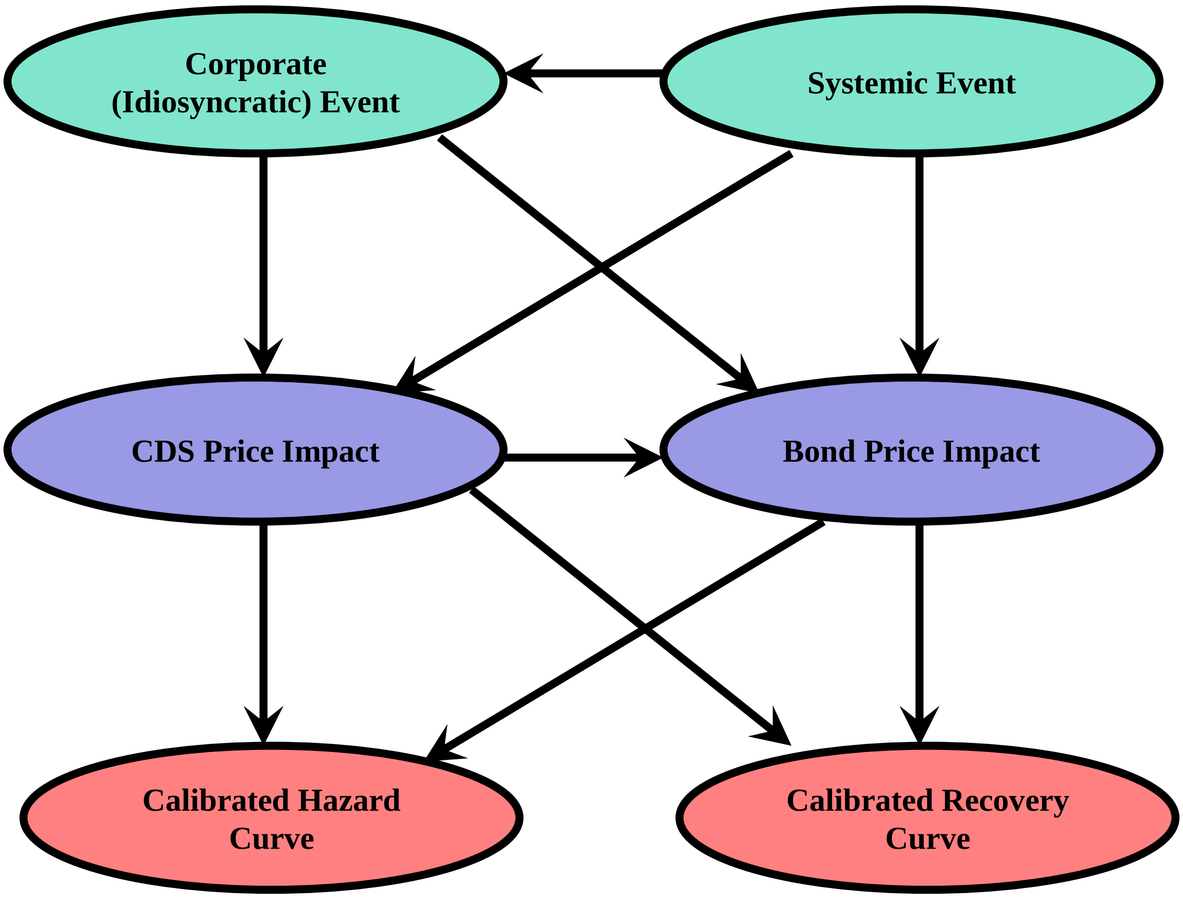

Figure 1: Causality Bayesian Network DAG For Credit Curve Building

Figure 2: Transition Splines – Low Width Transition Stretch

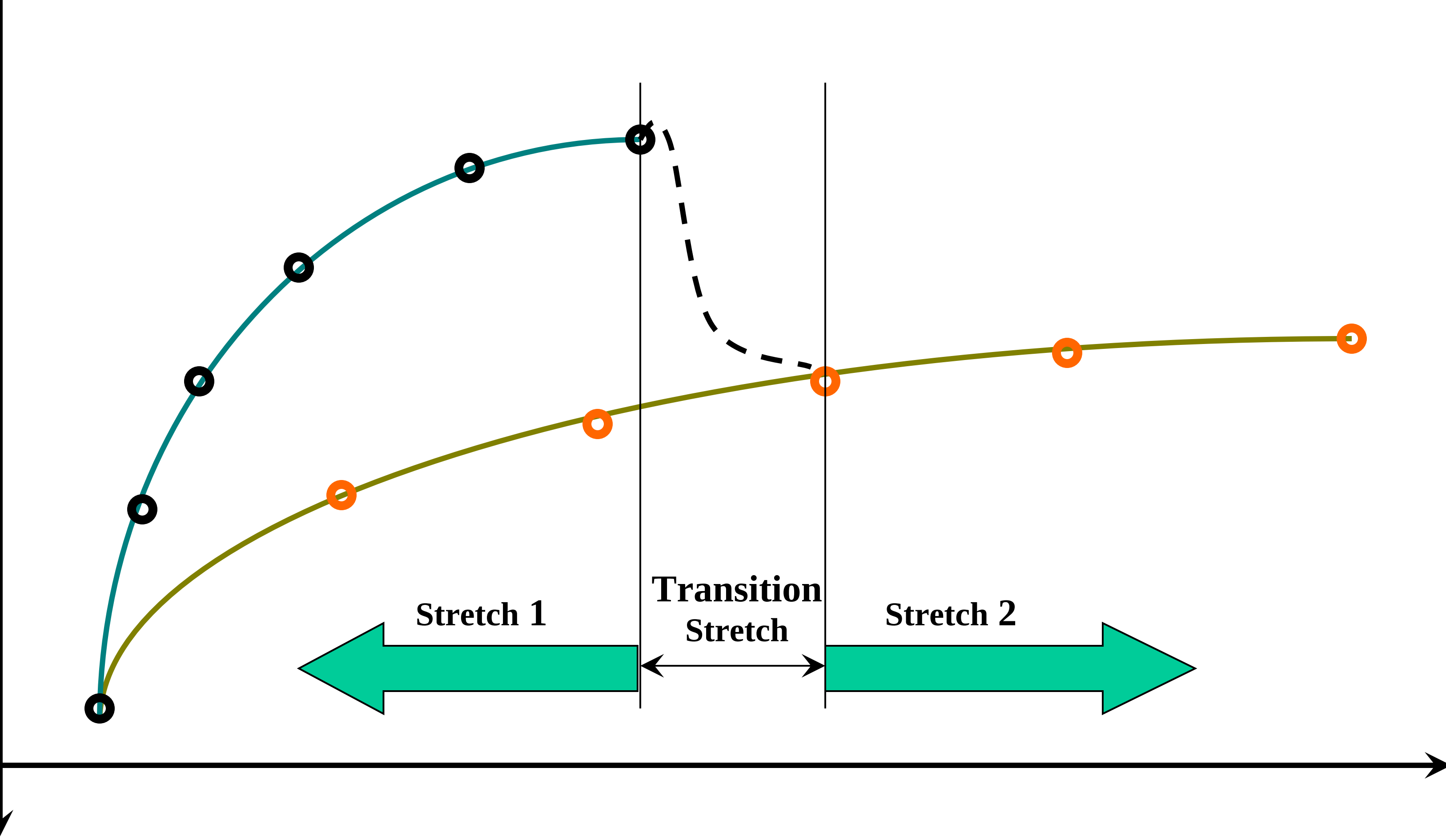

Figure 3: Transition Splines – High Width Transition Stretch

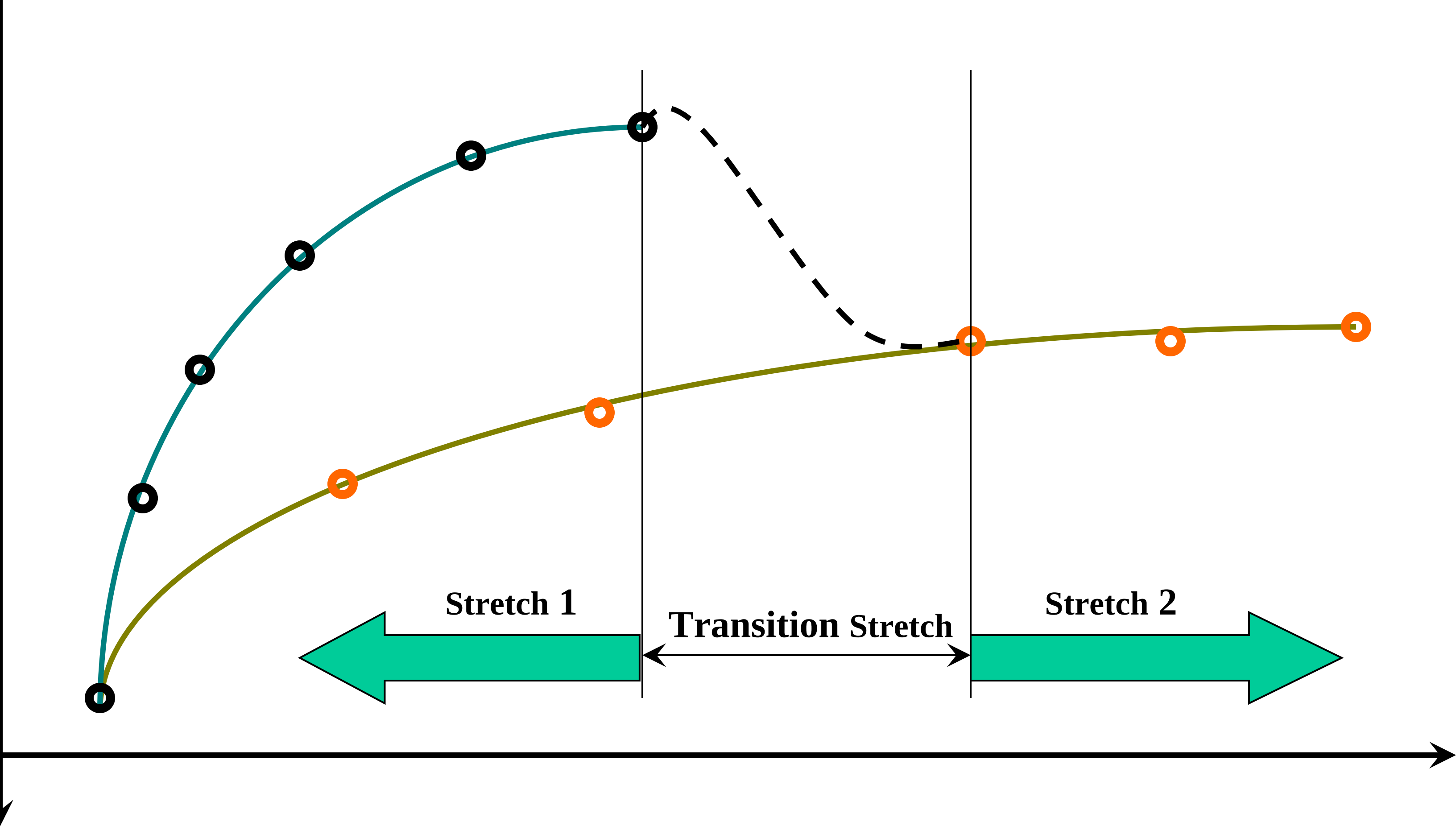

Figure 4: Transition Splines – Segment <-> Stretch Layout

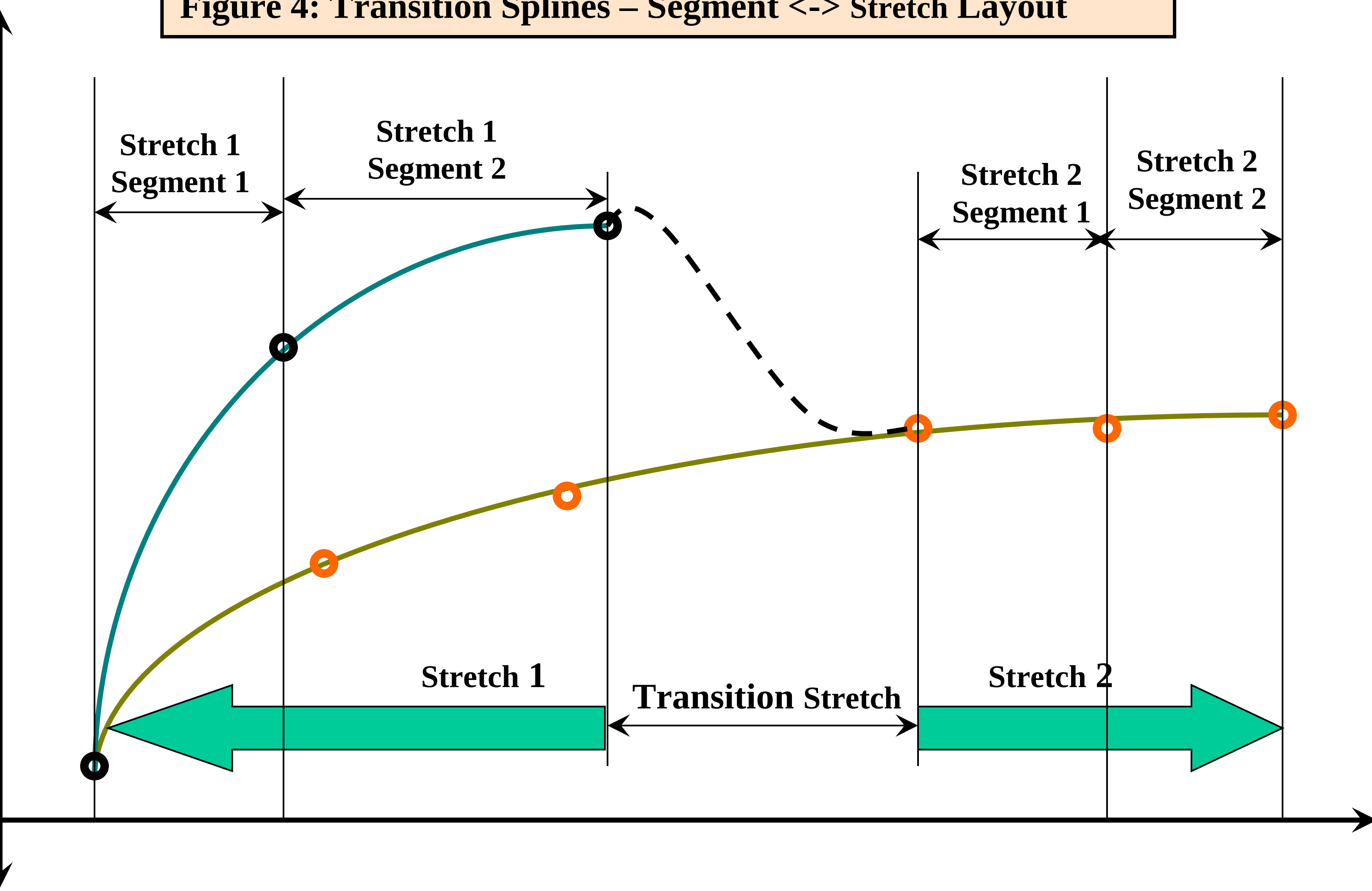

Figure 5: Float-Float Swap Set-up

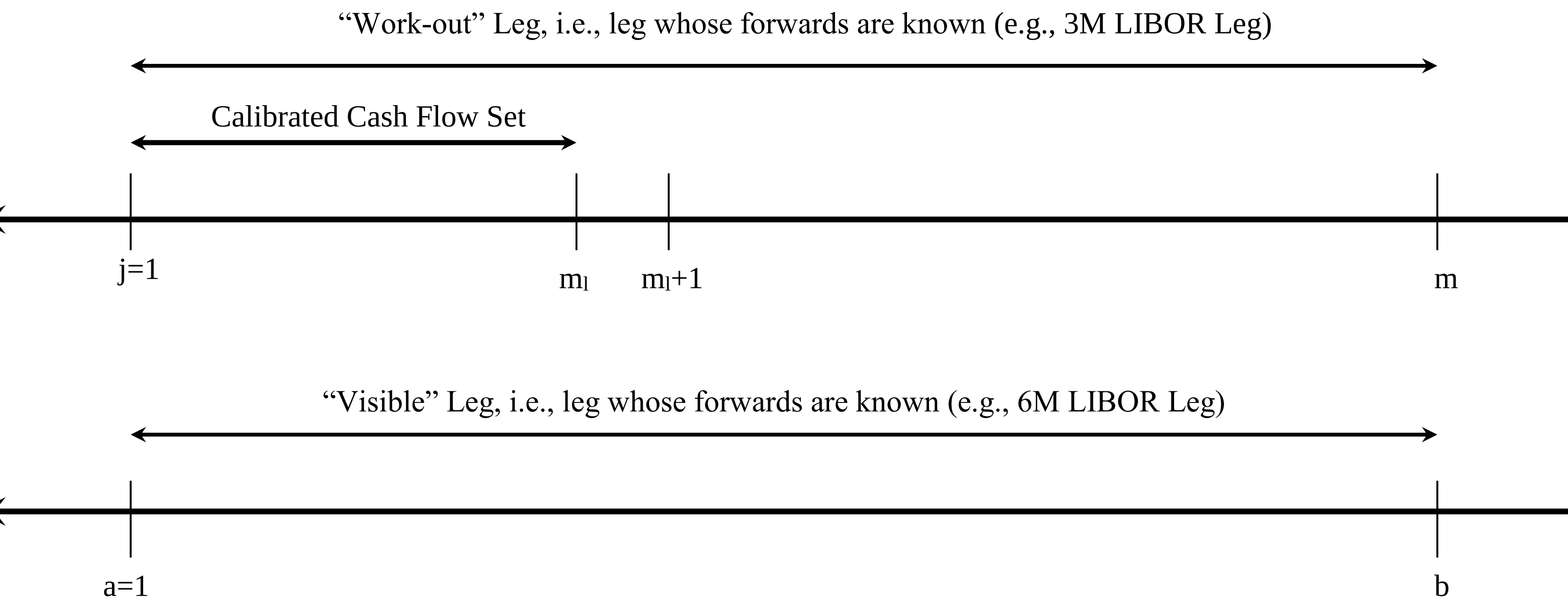

Figure 6: Latent State Quote Sensitivity

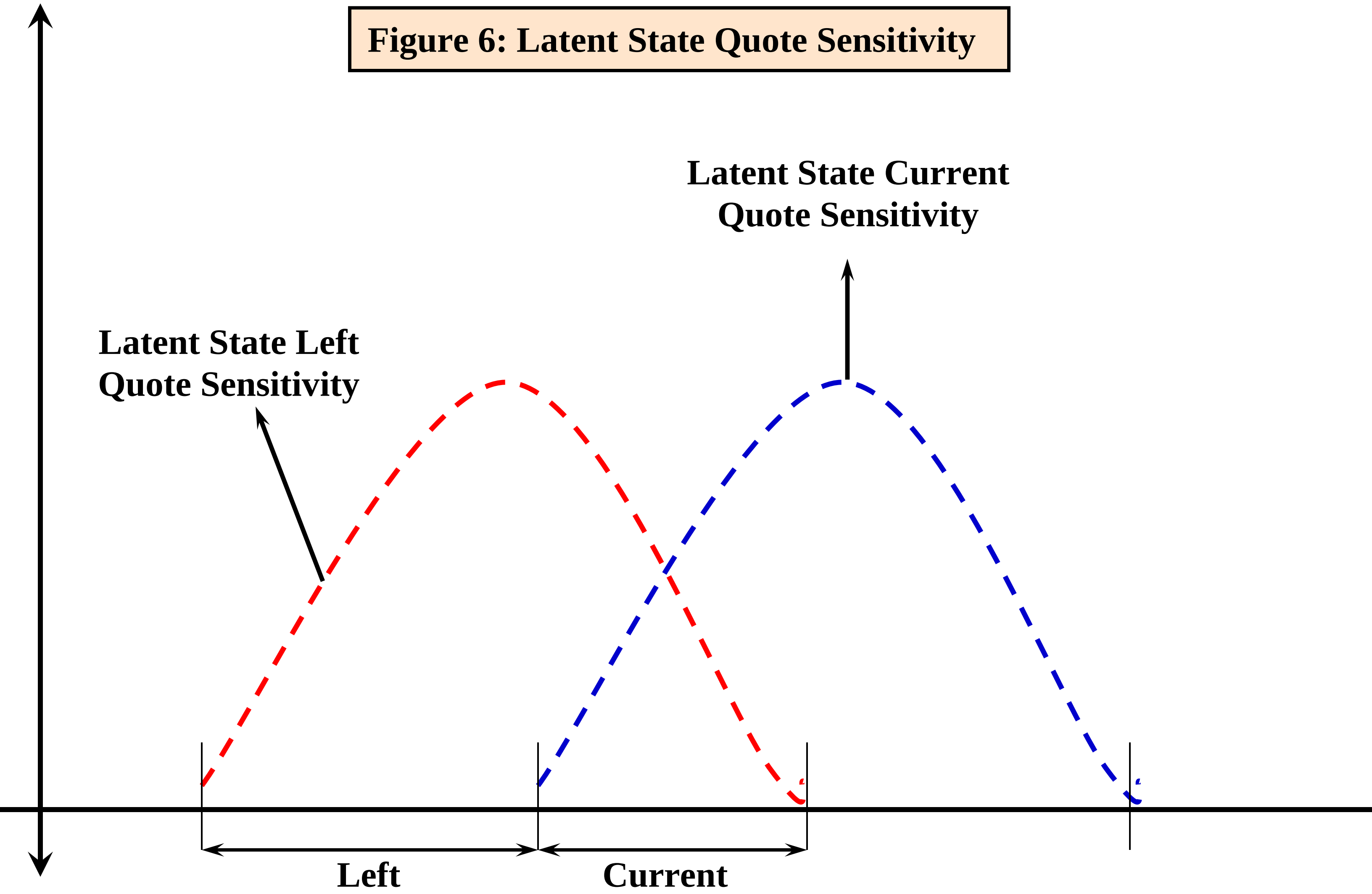

Figure 7: Measure Change Setup

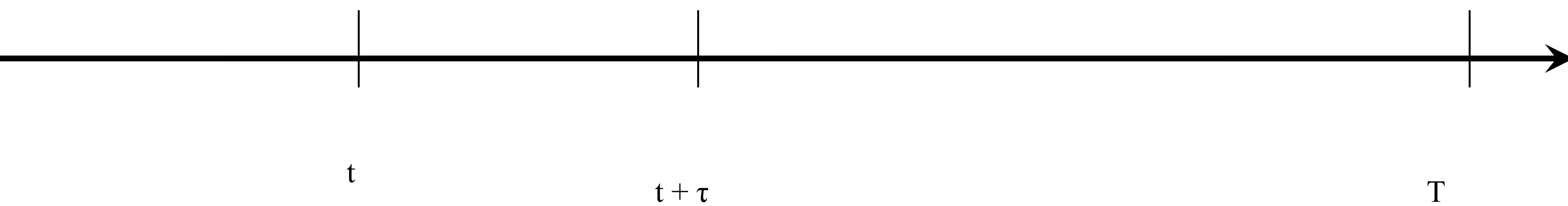

Figure 8: Optimal Intermediate Wengert Variable

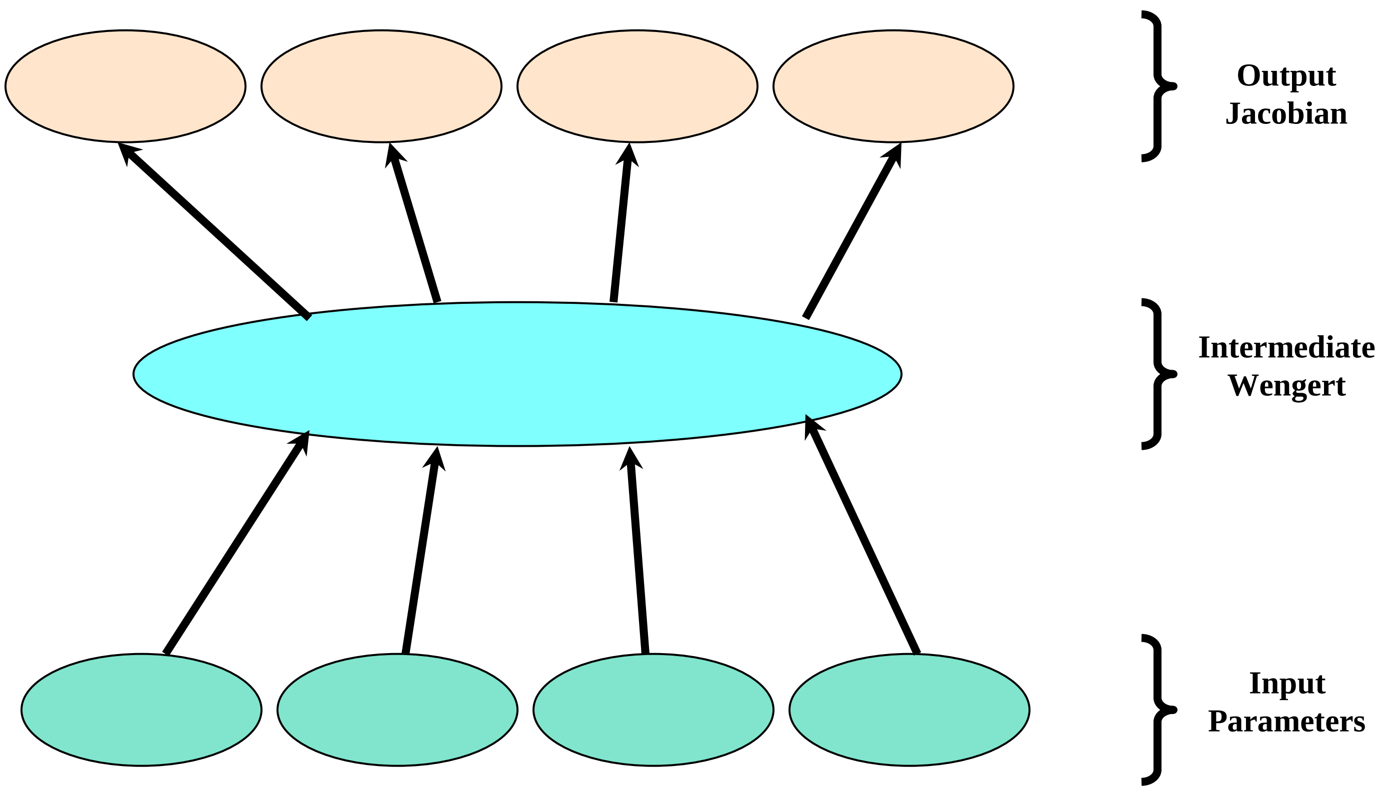

Figure 9: Computation Financial Object Scheme

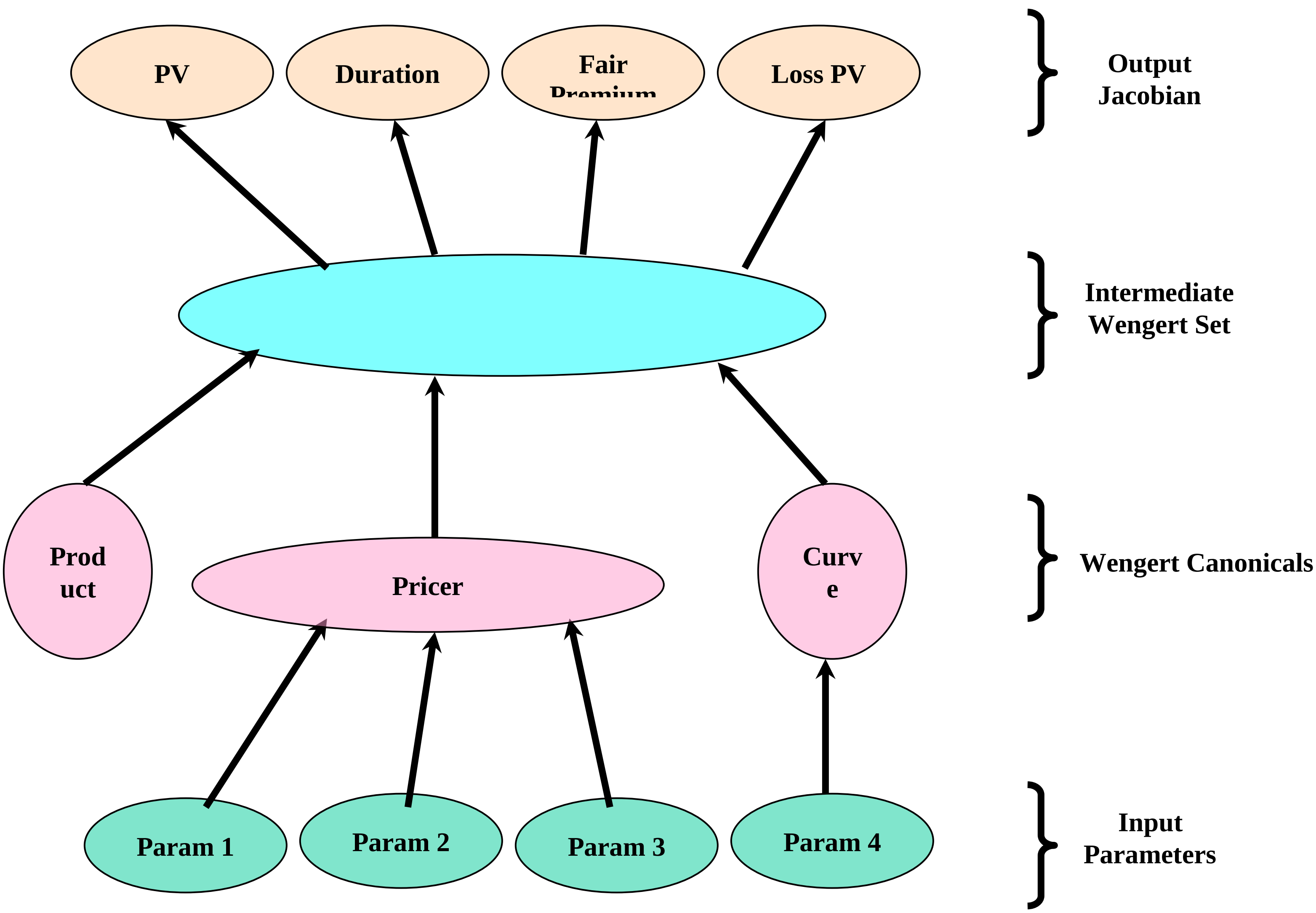

Figure 10: Wengert Fan-in and fan-out