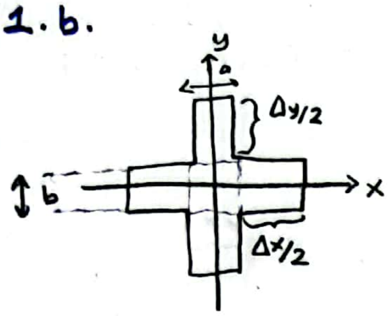
1.a.

$$f_1(x,y) = \mathrm{rect}\left(\frac{x}{3},\frac{y}{3}\right) + \mathrm{rect}\left(\frac{x}{2},\frac{y}{2}\right) - 2\,\mathrm{rect}(x,y)$$

$$F_1(f_x,f_y) = 9\,\mathrm{sinc}(3f_x,3f_y) + 4\,\mathrm{sinc}(2f_x,2f_y) - 2\,\mathrm{sinc}(f_x,f_y)$$

1.b.



$$g(x,y) = \mathrm{rect}\left(\frac{x}{a+\Delta x},\frac{y}{b}\right) + \mathrm{rect}\left(\frac{x}{a},\frac{y}{b+\Delta y}\right) - \mathrm{rect}\left(\frac{x}{a},\frac{y}{b}\right)$$

$$G(f_x,f_y) = (a+\Delta x)b\,\mathrm{sinc}\left((a+\Delta x)f_x, b f_y\right) + (b+\Delta y)a\,\mathrm{sinc}(a f_x,(b+\Delta y)f_y) - ab\,\mathrm{sinc}(a f_x, b f_y)$$

From the MATLAB Plots, it seems that suggestion is sound

2.a

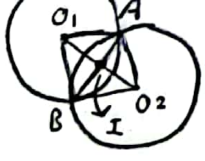
$\mathrm{jinc}(\sqrt{x^2+y^2}) = \mathrm{jinc}(r) = \mathrm{jinc}(x,y)$

$\hookrightarrow$ since jinc is circularly symmetric

$\mathrm{jinc}(x,y) ** \mathrm{jinc}(x,y) \stackrel{?}{=} \mathrm{jinc}(x,y) \longleftrightarrow \mathrm{circ}(f_x,f_y).\mathrm{circ}(f_x,f_y) \stackrel{?}{=} \mathrm{circ}(f_x,f_y)$

$$\mathrm{circ}(f_x,f_y) = \begin{cases} 1, & \sqrt{f_x^2+f_y^2} \le 1/4 \\ 0, & \text{o.w} \end{cases} \Rightarrow \mathrm{circ}^2(f_x,f_y) = \mathrm{circ}(f_x,f_y) \checkmark$$

2.b



$|O_1O_2| = d \Rightarrow |O_1I| = |IO_2| = d/2$

$\widehat{IO_1A} = \arccos(d) = \alpha$

$|AI| = \sqrt{1-d^2}/2$

$\sin\alpha = \sqrt{1-d^2}$

Area of the lens: $2\left[\underbrace{A(\overset{\frown}{AO_1B})}_{\substack{2\pi \to \pi r^2 = \frac{\pi}{4} \\ 2\alpha \to \frac{\alpha}{4}}} - \underbrace{A(\overset{\triangle}{AO_1B})}_{2.\frac{1}{2}.\frac{d}{2}.\frac{1}{2}\sqrt{1-d^2}}\right]$

$= 2\left(\frac{\alpha}{4} - \frac{d}{4}\sin\alpha\right)$

$= \left(\frac{\alpha}{2} - \frac{d}{2}\sin\alpha\right)$

2.c

$$\text{chat}(x,y) = \text{chat}(\sqrt{x^2+y^2}) = \begin{cases} \frac{1}{2}\left[\left(\arccos(\sqrt{x^2+y^2}) - \sqrt{x^2+y^2}\sqrt{1-(x^2+y^2)}\right)\right], & \sqrt{x^2+y^2} \leq 1 \\ 0, & \text{o.w} \end{cases}$$

2.d

$$\text{chat}(x,y) = \text{circ}(x,y) ** \text{circ}(x,y) \longleftrightarrow \text{CHAT}(f_x,f_y) = \text{jinc}(f_x,f_y)\,\text{jinc}(f_x,f_y)$$

$$\mathcal{F}\{\text{chat}(x,y)\} = \text{jinc}^2(f_x\, f_y)$$

3)

$$S(f_x,f_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(x,y)\, e^{-j2\pi f_x x}\, e^{-j2\pi f_y y}\, dx\,dy$$

$$S(f_u,f_v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(\underbrace{a_1x+b_1y}_{u}, \underbrace{a_2x+b_2y}_{v})\, e^{-j2\pi f_x x}\, e^{-j2\pi f_y y}\, dx\,dy$$

Calculate jacobian for coordinate transformation $x,y \longrightarrow u,v$

$$J_{xy} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 \Rightarrow J_{uv} = \frac{1}{a_1b_2-a_2b_1}$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \overset{\text{given}}{\Longrightarrow} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

$$S(f_u,f_v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(u,v)\, e^{-j2\pi f_x(A_1u+B_1v)}\, e^{-j2\pi f_y(A_2u+B_2v)}\, |J_{uv}|\, du\,dv$$

$$S(f_u,f_v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(u,v)\, e^{-j2\pi u(A_1f_x+A_2f_y)}\, e^{-j2\pi v(B_1f_x+B_2f_y)}\, |J_{uv}|\, du\,dv$$

$$= \frac{1}{|a_1b_2-a_2b_1|} S(A_1f_x+A_2f_y, B_1f_x+B_2f_y)$$

Special Case: $\begin{bmatrix} u \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}}_{A}\begin{bmatrix} x \\ y \end{bmatrix}$ $\quad A^{-1} = \frac{1}{\underbrace{\det(A)}_{1}}\begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$

$$s(a_1x+b_1y, a_2x+b_2y) \longleftrightarrow S(f_x\cos\phi - f_y\sin\phi, f_x\sin\phi + f_y\cos\phi)$$

CamScanner

4)

Rayleigh Resolution is defined as the first zero crossing of impulse response

Input : $\delta(x,y) \longleftrightarrow 1(f_x, f_y)$

$H_1(f_x, f_y) = rect\left(\frac{f_x}{16}, \frac{f_y}{4}\right) \longleftrightarrow 64\ sinc(16x, 4y)$ $\qquad Res^x = \frac{1}{16},\ Res^y = \frac{1}{4}$

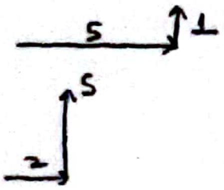
$H_2(f_x, f_y) = e^{-2\pi^2(f_x^2 + f_y^2)} \longleftrightarrow h_2(x,y)$ is in the form of Gaussian $\Rightarrow$ $Res^x = \infty$, $Res^y = \infty$

$H_3(f_x, f_y) = jinc(f_x, f_y) \longleftrightarrow h_3(x,y) = circ(x,y) \Rightarrow Res^x = Res^y = 1/2$

$$h_4(x,y) = \begin{cases} 1 & |x| \le 1/2,\ |y| \le 1/3 \\ 0 & else \end{cases} \qquad Res^x = 1/2,\ Res^y = 1/3$$

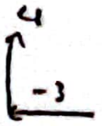
5) a) $f\left(\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}\right) = f\left(\begin{bmatrix} n_1 \\ n_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \underline{v_1} & \underline{v_2} \end{bmatrix}}_{V} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}\right)$

$\underline{V_1}$



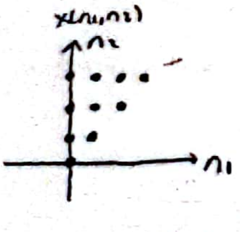
$\begin{bmatrix} 5 & 2 \\ 1 & 5 \end{bmatrix}$

$\underline{V_2}$

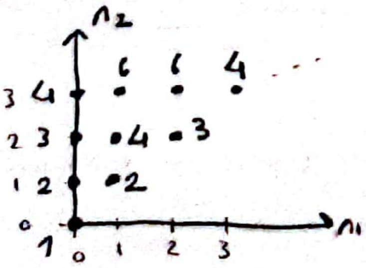


$\begin{bmatrix} -3 & 5 \\ 4 & 1 \end{bmatrix}$

b)



$y = x ** x$



$$y(n_1,n_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x(k_1,k_2)\, x(n_1-k_1, n_2-k_2) \qquad k_2 \ge k_1,\ k_1 \ge 0$$

$$= \sum_{k_1=0}^{\infty} \sum_{k_2=k_1}^{\infty} x(n_1-k_1, n_2-k_2), \qquad n_2-k_2 \ge n_1-k_1,\ k_1 \le n_1,\ k_2 \le n_2-n_1+k_1$$

$$= \sum_{k_1=0}^{n_1} \sum_{k_2=k_1}^{n_2-n_1+k_1} 1 = \sum_{k_1=0}^{n_1} (n_2-n_1+1) = \underline{(n_1+1)(n_2-n_1+1)}$$

$$y(n_1,n_2) = \begin{cases} (n_1+1)(n_2-n_1+1), & n_1 \ge 0,\ n_2 \ge 1 \\ 0, & otherwise \end{cases}$$

6) a) $y(n_1,n_2) = x(n_1,n_2)\, x(n_1-3, n_2)$

$y_1 = x_1(n_1,n_2)\, x_1(n_1-3, n_2)$

$y_2 = 2x_1(n_1,n_2)\, 2x_1(n_1-3,n_2) = 4y_1$ $\}$ Not linear

$y_3(n_1,n_2) = x(n_1-\Delta, n_2-\gamma)\, x(n_1-\Delta-3, n_2-\gamma)$

$= y_1(n_1-\Delta, n_2-\gamma)$ $\}$ S.I

b) $y(n_1,n_2) = \sum_{k_2} x(n_1, k_2)$

$y_1 = \sum_{k_2=-\infty}^{\infty} x_1(n_1, k_2)$

$ay_1 = \sum_{k_2} a x_1(n_1,k_2) = a \sum_{k_2} x_1(n_1,k_2)$ $\}$ Linear

$\sum_{k_2} [x_1(n_1,k_2) + x_2(n_1,k_2)] = y_1 + y_2$

$y_3(n_1,n_2) = \sum_{k_2} x(n_1-l_1, k_2) = y_1(n_1-l_1, n_2)$ $\}$ SI

c) $y(n_1,n_2) = \sum_{k_2=-1}^{1} x(n_1,k_2)$

$y(n_1,n_2) = x(n_1,-1) + x(n_1,0) + x(n_1,1)$

$\left.\begin{aligned} \sum_{k_2=-1}^{1} a(x(n_1,k_2)) &= a\sum_{k_2=-1}^{1} x(n_1,k_2) = a\,y(n_1,n_2) \\ \sum_{k_2=-1}^{1} a(x_1+x_2) &= a\sum_{k_1=-1}^{1} x_1 + a\sum_{k_1=-1}^{1} x_2 = a(y_1+y_2) \end{aligned}\right\}$ Linear

$x[n_1,n_2-L] \rightarrow \boxed{H} \rightarrow y_1(n_1,n_2)$

$\underbrace{y_1(n_1,n_2) = x(n_1,L-1) + x(n_1,L) + x(n_1,L+1)}_{\text{not S.I.}}$

d) $y(n_1,n_2) = 255 - x(n_1,n_2)$

$\underbrace{255 - 2x(n_1,n_2) \neq 2y(n_1,n_2)}_{\text{Not linear}}$

$x[n_1-L_1, n_2-L_2] \rightarrow \boxed{H} \rightarrow y_1(n_1,n_2)$

$y_1(n_1,n_2) = 255 - x(n_1-L_1, n_2-L_2)$

$\underbrace{= y(n_1-L_1, n_2-L_2)}_{\text{S.I}}$

7)

- $X(w_1,w_2) = \sum_{n_1}\sum_{n_2} x[n_1,n_2]\, e^{-jw_1n_1} e^{-jw_2n_2} \longleftrightarrow x[n_1,n_2]$

$X^*(w_1,w_2) = \sum_{n_1}\sum_{n_2} \underbrace{x^*[n_1,n_2]}_{=x[n_1,n_2]}\, e^{jw_1n_1} e^{jw_2n_2} = X(-w_1,-w_2) \longleftrightarrow x[-n_1,-n_2]$

$\hookrightarrow$ Real signals

If $x[n_1,n_2] = x[-n_1,-n_2] \Rightarrow X(w_1,w_2)$ is real

- $X(w_2,w_1) = \sum_{n_1}\sum_{n_2} x[n_1,n_2]\, e^{-jw_2n_1} e^{-jw_1n_2} = \sum_{u_1}\sum_{u_2} x[u_2,u_1]\, e^{-jw_1u_1} e^{-jw_2u_2}$

$n_1 = u_2$, $n_2 = u_1$

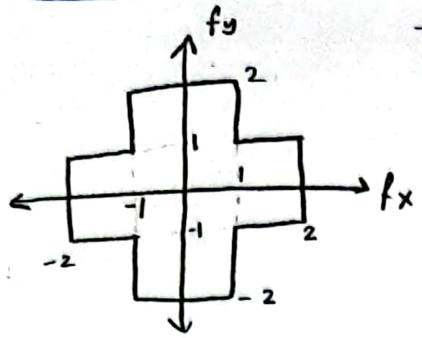
$= \sum_{n_1}\sum_{n_2} x[n_1,n_2]\, e^{-jw_1n_1} e^{-jw_2n_2} \Longrightarrow x[n_1,n_2] = x[n_2,n_1]$

- $x[n_1,n_2] \longleftrightarrow X(w_1,w_2) = X(-w_1,w_2) \longleftrightarrow x[-n_1,n_2]$

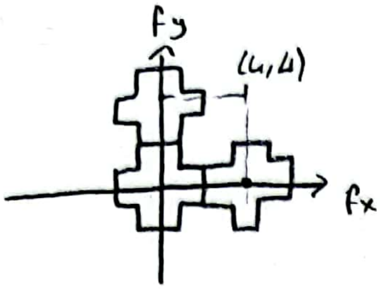
$x[n_1,n_2] = x[-n_1,n_2]$

a) I, III b) I, III c) I, II

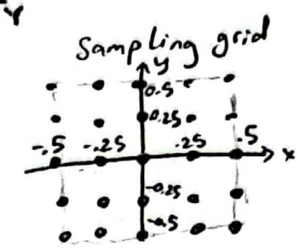
8 $\Omega = 2\pi f$



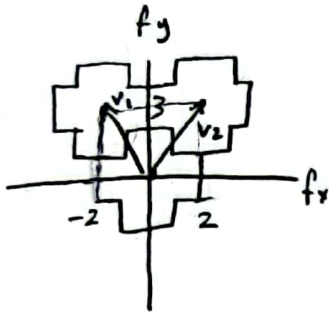
a)



$\frac{1}{T_x} = 4, \ \frac{1}{T_y} = 4 \Rightarrow T_x = T_y = 1/4$

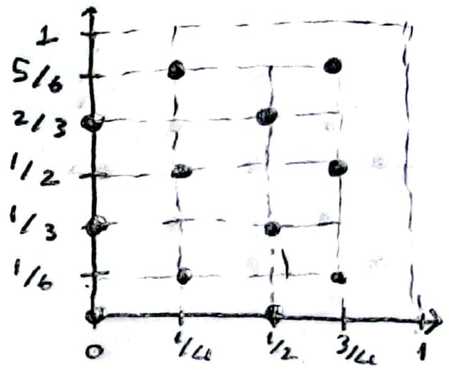


b)



$v_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \ v_2 \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \ V = [v_1, v_2]$

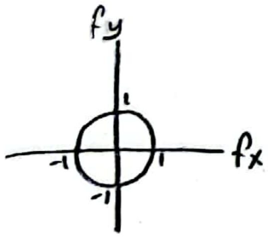
$U = V^{-T} = \begin{bmatrix} 1/4 & -1/4 \\ 1/6 & 1/6 \end{bmatrix}$



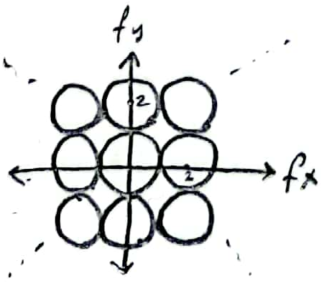
12 sampling points/m^2

c) Already chosen this strategy in b.

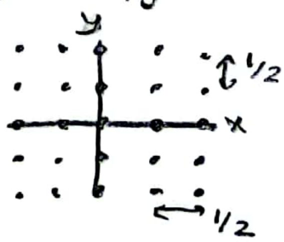
d)



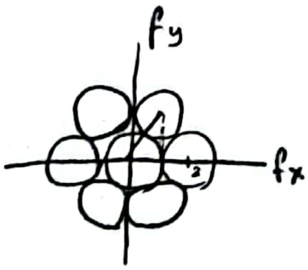
d.a)



$\frac{1}{T_x} = 2, \ \frac{1}{T_y} = 2 \Rightarrow T_x = T_y = 1/2$



d.b)



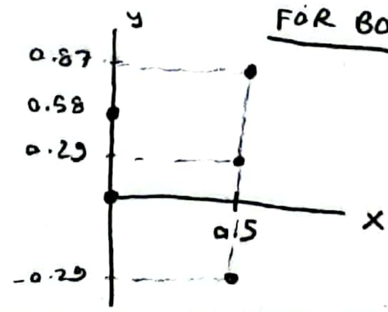
$v_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$

$U = V^{-T} = \begin{bmatrix} 0.5 & 0 \\ -\frac{\sqrt{3}}{6} & \frac{\sqrt{3}}{3} \end{bmatrix}$

$\approx \begin{bmatrix} 0.5 & 0 \\ -0.29 & 0.58 \end{bmatrix}$

Sampling Density: 4 samples/m^2

FOR BOTH CASES



CamScanner

$\det(V_a) = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4$ $\det(V_b) = \begin{vmatrix} 2 & 1 \\ 0 & \sqrt{3} \end{vmatrix} = 2\sqrt{3}$

Sampling density^1 = $4k$ Sampling density^2 = $2\sqrt{3}k$

$\frac{SD^2}{SD^1} \approx 0.87$, 13% more efficient

CamScanner

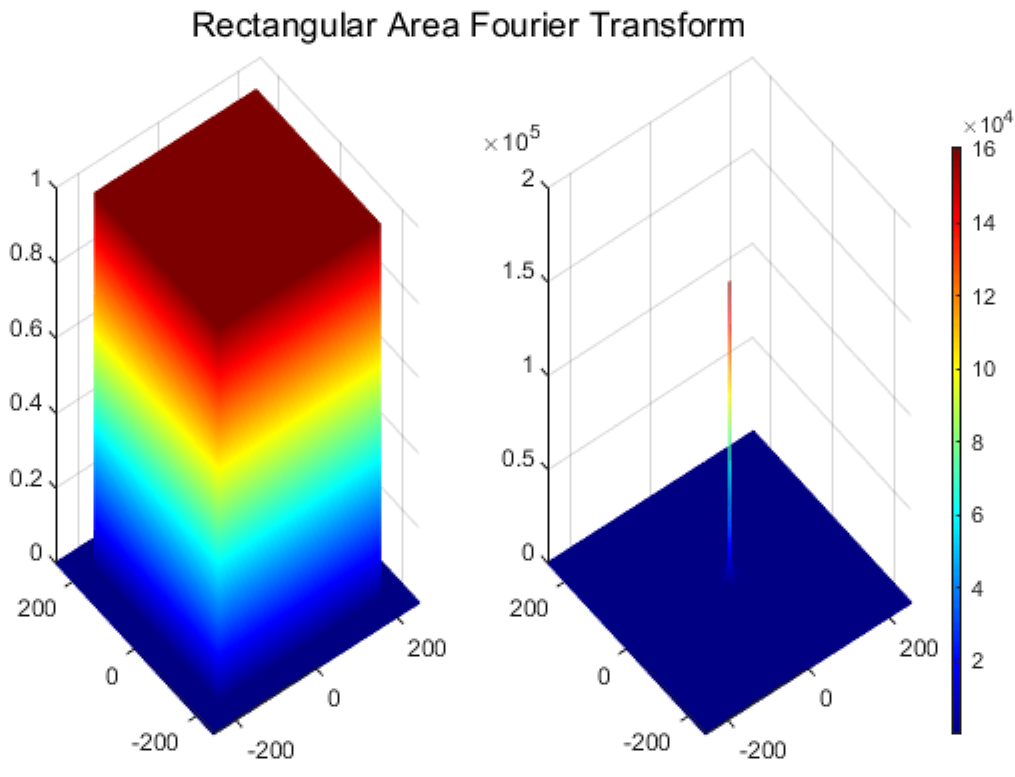
Q1

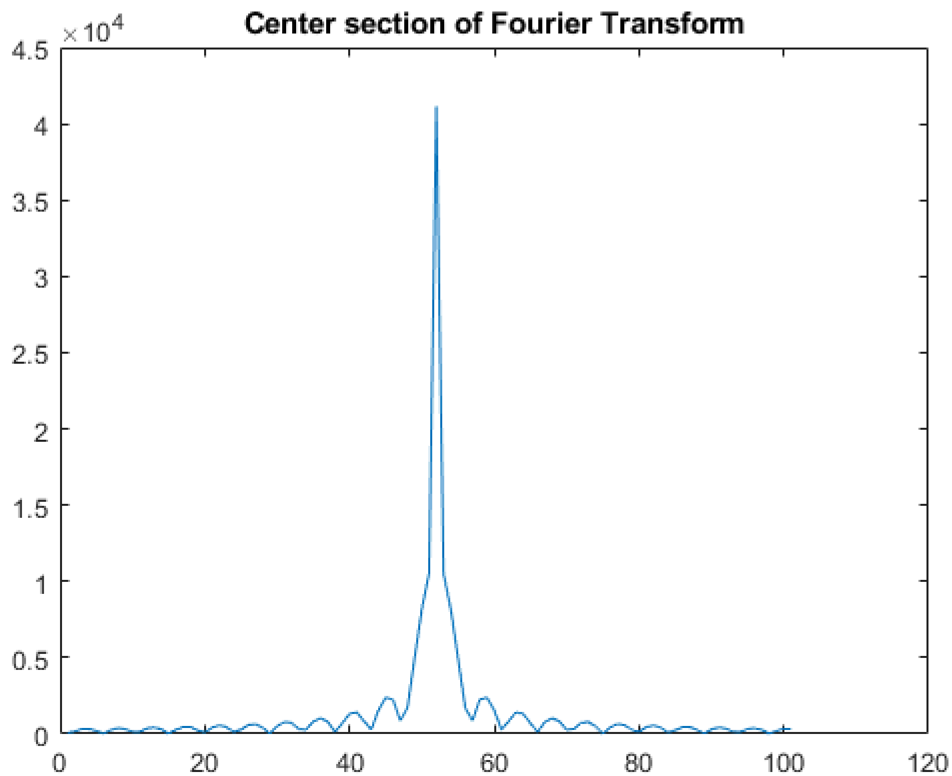
The proposed method created a narrower main lobe in the frequency domain. Hence the suggestion is sound.

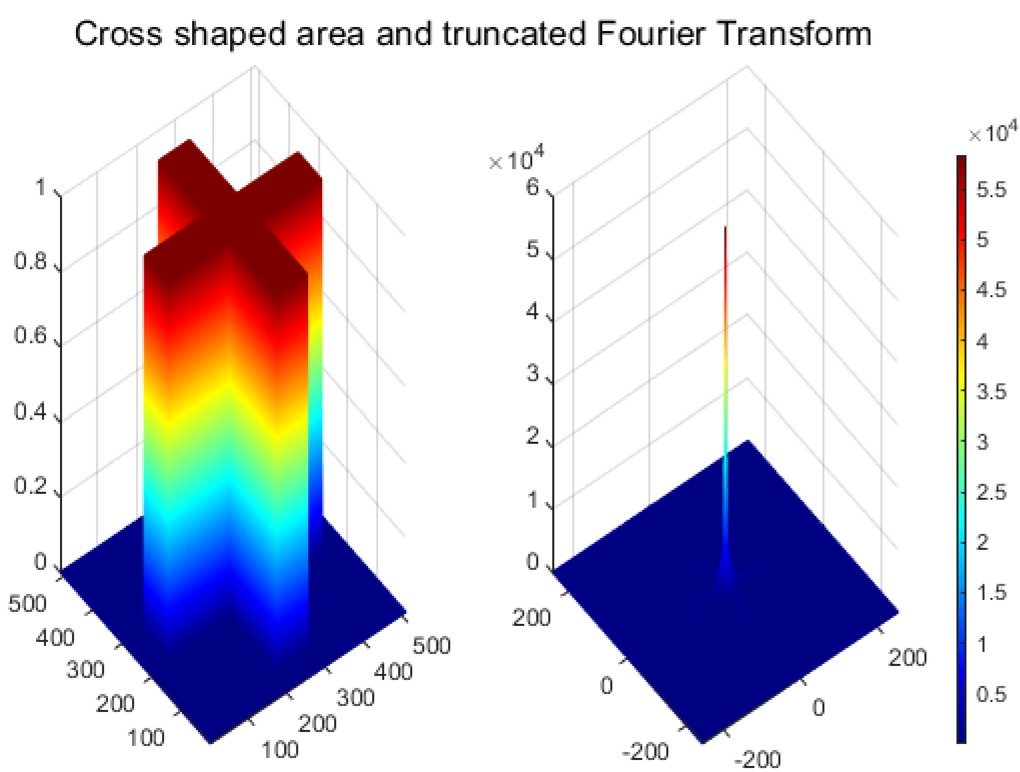
```
close all; clear; clc;
s = 512; ks = 40;
A = zeros(s,s);
A(0.5*s-5*ks:0.5*s+5*ks,0.5*s-5*ks:0.5*s+5*ks) = 1;
[X,Y] = meshgrid(1:s,1:s);
[fft_X,fft_Y] = meshgrid(-s/2+1:s/2,-s/2+1:s/2);
f = figure
subplot(1,2,1)
surf(fft_X,fft_Y,A,'EdgeColor','interp'), colormap jet
% xlim([-s/2+1,s/2])
% ylim([-s/2+1,s/2])
h = subplot(1,2,2);
fft_mag = abs(fftshift(fft2(A)));
surf(fft_X,fft_Y,fft_mag,fft_mag,'EdgeColor','interp'), colormap jet
% xlim([1,s])
% ylim([1,s])
c = colorbar(h,'Position',[0.93 0.11 0.01 0.7]);
sgtitle('Rectangular Area Fourier Transform')
figure
plot(fft_mag(s/2,(s/2-50):(s/2+50)))
title('Center section of Fourier Transform')
A = zeros(s,s);
A(0.5*s-5*ks:0.5*s+5*ks,0.5*s-ks:0.5*s+ks) = 1;
A(0.5*s-ks:0.5*s+ks,0.5*s-5*ks:0.5*s+5*ks) = 1;
[X,Y] = meshgrid(1:s,1:s);
f = figure
subplot(1,2,1)
surf(X,Y,A,'EdgeColor','interp'), colormap jet
xlim([1,s])
ylim([1,s])
h = subplot(1,2,2);
fft_mag = abs(fftshift(fft2(A)));
targetSize = [256 256];
surf(fft_X,fft_Y,fft_mag,fft_mag,'EdgeColor','interp'), colormap jet
% xlim([1,s])
% ylim([1,s])
c = colorbar(h,'Position',[0.93 0.11 0.01 0.7]);
sgtitle('Cross shaped area and truncated Fourier Transform')
figure
plot(fft_mag(s/2,(s/2-50):(s/2+50)))
title('Center section of Fourier Transform')
```

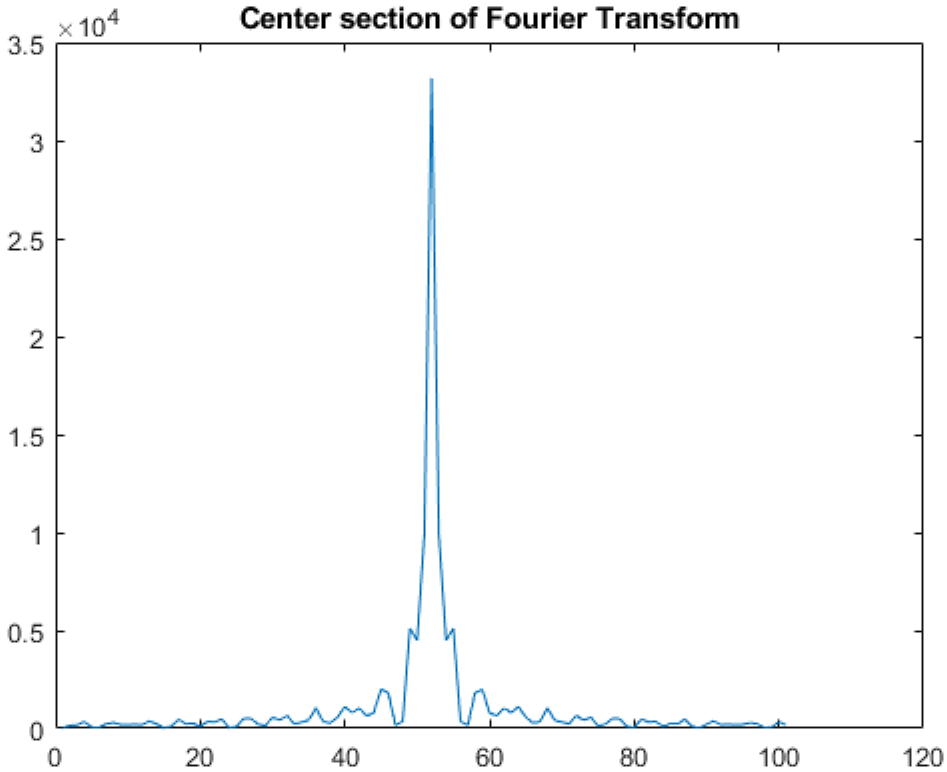
```
f = 

  Figure (1) with properties:

      Number: 1
```

```
        Name: ''
       Color: [0.9400 0.9400 0.9400]
    Position: [680 558 560 420]
       Units: 'pixels'

  Use GET to show all properties


f =

  Figure (3) with properties:

      Number: 3
        Name: ''
       Color: [0.9400 0.9400 0.9400]
    Position: [680 558 560 420]
       Units: 'pixels'

  Use GET to show all properties
```




Center section of Fourier Transform
×10^4
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
0
20
40
60
80
100
120

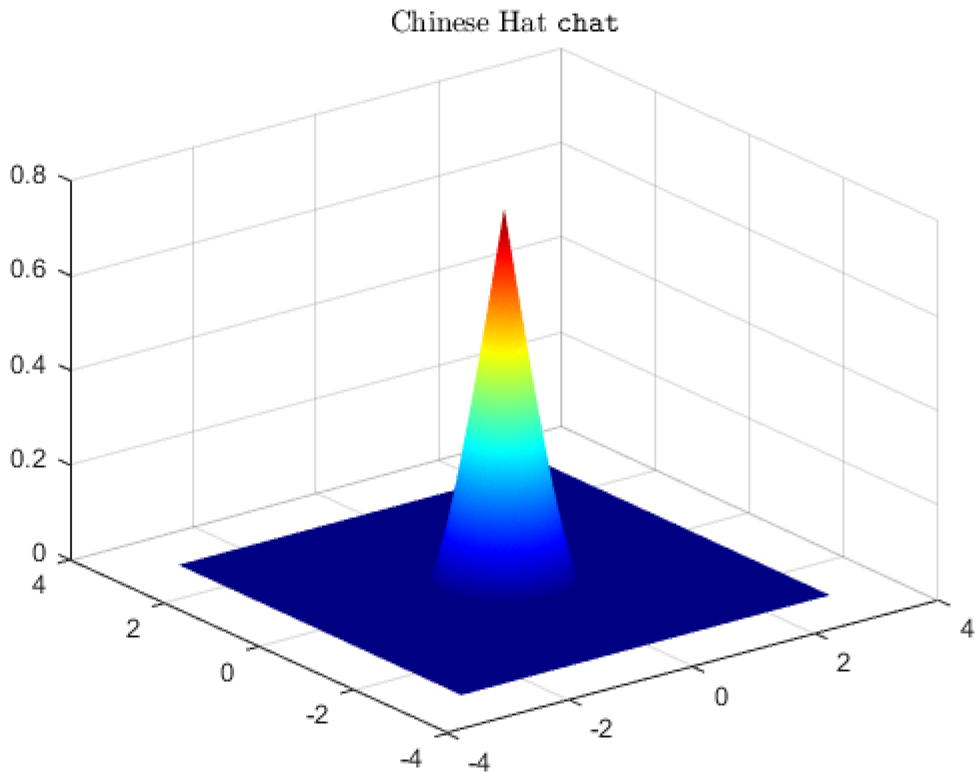

Cross shaped area and truncated Fourier Transform
1
0.8
0.6
0.4
0.2
0
500
400
300
200
100
500
400
300
200
100
×10^4
6
5
4
3
2
1
0
200
0
-200
200
0
-200
×10^4
5.5
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5



Chinese Hat

```
[X,Y] = meshgrid(-3:0.01:3,-3:0.01:3);
f = figure
D = (X.^2+Y.^2);
A = zeros(size(D));
A(D<=1) = 0.5*(acos(sqrt(D(D<=1))) - sqrt(D(D<=1)).*sqrt(1-D(D<=1)));
surf(X,Y,A,'EdgeColor','none'), colormap jet
title('Chinese Hat \texttt{chat}','Interpreter','latex')
```

```
f = 

  Figure (5) with properties:

      Number: 5
        Name: ''
       Color: [0.9400 0.9400 0.9400]
    Position: [680 558 560 420]
       Units: 'pixels'

  Use GET to show all properties
```



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