MIDDLE EAST TECHNICAL UNIVERSITY (METU)
Department of Electrical and Electronics Engineering

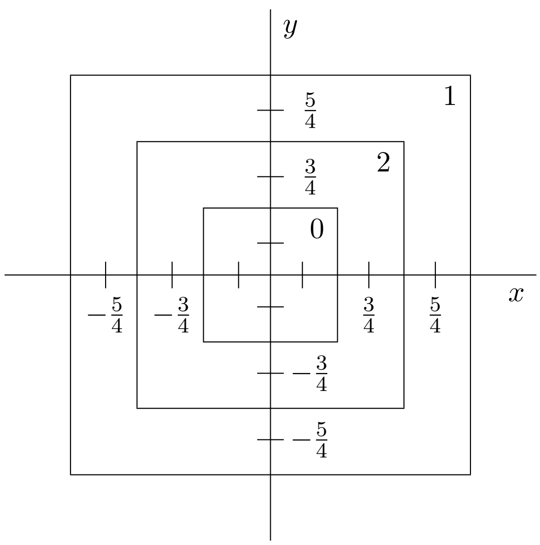
EE 798 Theory of Remote Image Formation
2021-2022 Spring Semester

Problem Set 1

Issued: Monday, April 4, 2022 **Due:** 23:59, Thursday, April 14, 2022

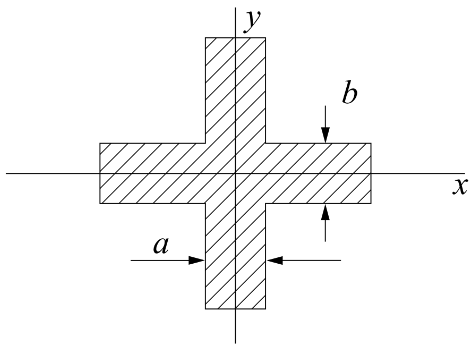
Problem 1.1 (FT of basic 2D continuous signals)
a. Compute the Fourier Transform of the following 2-D signal:

$f_1(x, y)$



Note that inner “box” has amplitude 0, middle “box” has amplitude 2 and outer most (largest) ”box” has amplitude 1.

b. Suppose that $g(x, y)$ is a two-valued function, assuming only the values zero and one. The region of the x, y plane where $g(x, y)$ equals one is called the *aperture* (see the figure below). One wishes to design an aperture whose total area is small but whose Fourier transform has a narrow (two-dimensional) main lobe. A cross configuration has been suggested with the argument that the wide width in x should give a narrow transform in the f_x direction, and the wide width in y should give a narrow transform in the f_y direction. Calculate the Fourier transform and decide whether the suggestion is sound. Also plot the resulting Fourier transform in Matlab.



Problem 1.2 (FT of common 2D continuous signals)
a. Verify that

$$\mathrm{jinc}\left(\sqrt{x^2+y^2}\right) ** \mathrm{jinc}\left(\sqrt{x^2+y^2}\right) = \mathrm{jinc}\left(\sqrt{x^2+y^2}\right)$$

where the jinc function is defined as in class.

b. Given two overlapping circles of unit radius with centers at distance d, find the area of the lens-shaped intersection. (Hint: Use simple geometric arguments).

c. The two-dimensional convolution of two circle functions is called the two-dimensional *hat function* (or the *Chinese hat function*) and is denoted $\mathrm{chat}(x, y)$,

$$\mathrm{chat}(x, y) = \mathrm{circ}(x, y) ** \mathrm{circ}(x, y)$$

Find an expression for $\mathrm{chat}(x, y)$. Also plot in Matlab and verify that it looks like Chinese hat. (Hint: Use your answer to the previous part.)

d. What is the Fourier transform of $\mathrm{chat}(x, y)$?

Problem 1.3 (Property of 2D FT)
Prove that if

$$s(x, y) \Leftrightarrow S(f_x, f_y)$$

then

$$s(x\cos\psi - y\sin\psi, x\sin\psi + y\cos\psi) \Leftrightarrow S(f_x\cos\psi - f_y\sin\psi, f_x\sin\psi + f_y\cos\psi)$$

and more generally,

$$s(a_1x + b_1y, a_2x + b_2y) \Leftrightarrow \frac{1}{|a_1b_2 - a_2b_1|} S(A_1f_x + A_2f_y, B_1f_x + B_2f_y)$$

where

$$\begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}^{-1}$$

Problem 1.4 (Resolution)

Consider the general case of imaging a scene $s(x,y)$ which has the Fourier Transform $S(f_x, f_y)$ using the point-spread function $h(x,y)$ with associated Fourier Transform $H(f_x, f_y)$. This results in

$$G(f_x, f_y) = H(f_x, f_y)\, S(f_x, f_y)$$

where $G(f_x, f_y)$ is Fourier Transform of the imaged scene. Determine the Rayleigh resolution of the system when using the following point-spread functions.

$$
\begin{aligned}
H_1(f_x, f_y) &= \begin{cases} 1 & |f_x| \leq 8, |f_y| \leq 2 \\ 0 & else \end{cases} \\
H_2(f_x, f_y) &= e^{-2\pi^2\left(f_x^2 + f_y^2\right)} \\
H_3(f_x, f_y) &= \mathrm{jinc}(f_x, f_y) \\
h_4(x, y) &= \begin{cases} 1 & |x| \leq \frac{1}{2}, |y| \leq \frac{1}{3} \\ 0 & else \end{cases}
\end{aligned}
$$

Problem 1.5 (2D discrete-space signals and convolution)

(a) Determine a periodicity matrix that describes the periodicity of the sequence shown above. Also determine a second periodicity matrix for this sequence.

(b) Consider the sequence x defined by

$$x(n_1, n_2) = \begin{cases} 1, & n_1 \geq 0, n_2 \geq n_1 \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

Determine the convolution of x with itself.

Problem 1.6 (System properties)

Determine whether each of the systems below is linear and shift-invariant:

(a) $y(n_1, n_2) = x(n_1, n_2)x(n_1 - 3, n_2)$
(b) $y(n_1, n_2) = \sum_{k_2=-\infty}^{\infty} x(n_1, k_2)$
(c) $y(n_1, n_2) = \sum_{k_2=-1}^{1} x(n_1, k_2)$
(d) $y(n_1, n_2) = 255 - x(n_1, n_2)$ which represents the point-wise intensity transformation for obtaining the negative of an image.

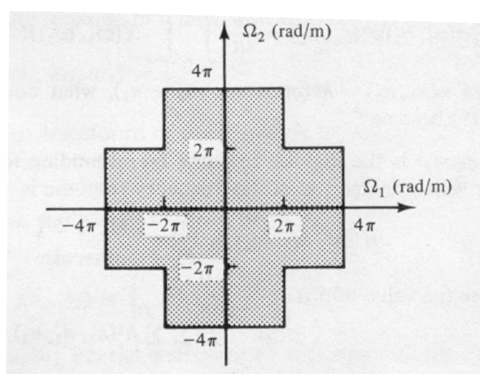
Problem 1.7 (Discrete-Space FT)

Consider 2-D signals with regions of support as shown in the figure below. If each sequence is 1 on the samples marked with heavy dots and zero on the samples with the light dots, for which sequences is

(a) $X(\omega_1, \omega_2)$ real?
(b) $X(\omega_1, \omega_2) = X(\omega_2, \omega_1)$?
(c) $X(\omega_1, \omega_2) = X(-\omega_1, \omega_2)$?

Justify your answers.

Problem 1.8 (Sampling)
Consider an analog signal that is bandlimited. Its Fourier transform is nonzero over the shaded region in the following figure. Determine and **sketch** the sampling grid that will yield the minimum sampling density (in samples per square meter) and will permit an exact reconstruction of the analog signal



(a) if the signal is sampled on a rectangular grid.
(b) if the signal is sampled with any sampling strategy.
(c) The minimum sampling density possible with any sampling strategy is 12 samples per square meter. Sketch the sampling raster that corresponds to this optimal case.
(d) Repeat parts (a)-(b) for a signal with circular support in the frequency domain and verify that hexagonal sampling is optimal as stated in the class. Find also the minimum sampling density in each case and compare.