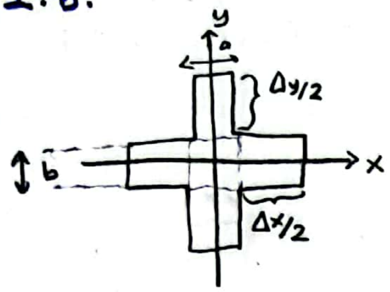
1.a.

$$f_1(x,y) = \text{rect}\left(\frac{x}{3},\frac{y}{3}\right) + \text{rect}\left(\frac{x}{2},\frac{y}{2}\right) - 2\,\text{rect}(x,y)$$

$$F_1(f_x,f_y) = 9\,\text{sinc}(3f_x,3f_y) + 4\,\text{sinc}(2f_x,2f_y) - 2\,\text{sinc}(f_x,f_y)$$

1.b.



$$g(x,y) = \text{rect}\left(\frac{x}{a+\Delta x},\frac{y}{b}\right) + \text{rect}\left(\frac{x}{a},\frac{y}{b+\Delta y}\right) - \text{rect}\left(\frac{x}{a},\frac{y}{b}\right)$$

$$G(f_x,f_y) = (a+\Delta x)b\,\text{sinc}\left((a+\Delta x)f_x, b f_y\right) + (b+\Delta y)a\,\text{sinc}\left(a f_x,(b+\Delta y)f_y\right) - ab\,\text{sinc}(a f_x, b f_y)$$

From the MATLAB Plots, it seems that suggestion is sound

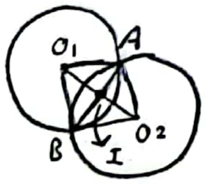
2.a

$\text{jinc}(\sqrt{x^2+y^2}) = \text{jinc}(r) = \text{jinc}(x,y)$ ⟶ since jinc is circularly symmetric

$\text{jinc}(x,y) ** \text{jinc}(x,y) \stackrel{?}{=} \text{jinc}(x,y) \longleftrightarrow \text{circ}(f_x,f_y)\cdot\text{circ}(f_x,f_y) \stackrel{?}{=} \text{circ}(f_x,f_y)$

$$\text{circ}(f_x,f_y) = \begin{cases} 1, & \sqrt{f_x^2+f_y^2} \le 1/4 \\ 0, & \text{o.w} \end{cases} \Rightarrow \text{circ}^2(f_x,f_y) = \text{circ}(f_x,f_y)\ \checkmark$$

2.b



$|O_1O_2| = d \Rightarrow |O_1I| = |IO_2| = d/2$

$\widehat{IO_1A} = \arccos(d) = \alpha$

$|AI| = \sqrt{1-d^2}/2$

$\sin\alpha = \sqrt{1-d^2}$

Area of the lens: $2\left[\underbrace{A(\overset{\frown}{AO_1B})}_{\substack{2\pi \to \pi r^2 = \frac{\pi}{4} \\ 2\alpha \to \frac{\alpha}{4}}} - \underbrace{A(\overset{\triangle}{AO_1B})}_{2\cdot\frac{1}{2}\cdot\frac{d}{2}\cdot\frac{1}{2}\sqrt{1-d^2}}\right] = 2\left(\frac{\alpha}{4} - \frac{d}{4}\sin\alpha\right) = \left(\frac{\alpha}{2} - \frac{d}{2}\sin\alpha\right)$

2.c

$$\mathrm{chat}(x,y) = \mathrm{chat}(\sqrt{x^2+y^2}) = \begin{cases} \frac{1}{2}\left[\left(\arccos(\sqrt{x^2+y^2}) - \sqrt{x^2+y^2}\sqrt{1-(x^2+y^2)}\right]\right., & \sqrt{x^2+y^2} \leq 1 \\ 0 & , \text{ o.w} \end{cases}$$

2.d

$$\mathrm{chat}(x,y) = \mathrm{circ}(x,y) ** \mathrm{circ}(x,y) \longleftrightarrow \mathrm{CHAT}(f_x,f_y) = \mathrm{jinc}(f_x,f_y)\,\mathrm{jinc}(f_x,f_y)$$

$$\mathcal{F}\{\mathrm{chat}(x,y)\} = \mathrm{jinc}^2(f_x\, f_y)$$

3)

$$S(f_x,f_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(x,y)\, e^{-j2\pi f_x x}\, e^{-j2\pi f_y y}\, dx\, dy$$

$$S(f_u,f_v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(\underbrace{a_1x+b_1y}_{u}, \underbrace{a_2x+b_2y}_{v})\, e^{-j2\pi f_x x}\, e^{-j2\pi f_y y}\, dx\, dy$$

Calculate jacobian for coordinate transformation $x,y \longrightarrow u,v$

$$J_{xy} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 \Rightarrow J_{uv} = \frac{1}{a_1b_2-a_2b_1}$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \xRightarrow{\text{given}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

$$S(f_u,f_v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(u,v)\, e^{-j2\pi f_x(A_1u+B_1v)}\, e^{-j2\pi f_y(A_2u+B_2v)}\, |J_{uv}|\, du\, dv$$

$$S(f_u,f_v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(u,v)\, e^{-j2\pi u(A_1f_x+A_2f_y)}\, e^{-j2\pi v(B_1f_x+B_2f_y)}\, |J_{uv}|\, du\, dv$$

$$= \frac{1}{|a_1b_2-a_2b_1|} S(A_1f_x+A_2f_y, B_1f_x+B_2f_y)$$

Special Case: $\begin{bmatrix} u \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}}_{A}\begin{bmatrix} x \\ y \end{bmatrix}$ $\quad A^{-1} = \frac{1}{\underbrace{\det(A)}_{1}}\begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$

$$s(a_1x+b_1y, a_2x+b_2y) \longleftrightarrow S(f_x\cos\phi - f_y\sin\phi, f_x\sin\phi + f_y\cos\phi)$$

4) Rayleigh Resolution is defined as the first zero crossing of impulse response

Input : $\delta(x,y) \longleftrightarrow 1(f_x,f_y)$

$H_1(f_x,f_y) = rect\left(\frac{f_x}{16},\frac{f_y}{4}\right) \longleftrightarrow 64\,sinc(16x,4y)$ $\qquad Res^x = \frac{1}{16}$, $Res^y = \frac{1}{4}$

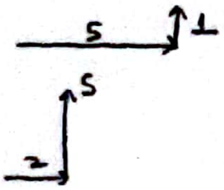
$H_2(f_x,f_y) = e^{-2\pi^2(f_x^2+f_y^2)} \longleftrightarrow h_2(x,y)$ is in the form of Gaussian $\Rightarrow$ $Res^x = \infty$, $Res^y = \infty$

$H_3(f_x,f_y) = jinc(f_x,f_y) \longleftrightarrow h_3(x,y) = circ(x,y) \Rightarrow Res^x = Res^y = 1/2$

$h_4(x,y) = \begin{cases} 1 & |x| \leq 1/2,\ |y| \leq 1/3 \\ 0 & else \end{cases}$ $\qquad Res^x = 1/2$, $Res^y = 1/3$

5) a) $f\left(\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}\right) = f\left(\begin{bmatrix} n_1 \\ n_2 \end{bmatrix} + \overbrace{\begin{bmatrix} \underline{v_1} & \underline{v_2} \end{bmatrix}}^{V} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}\right)$

$\underline{\underline{V_1}}$

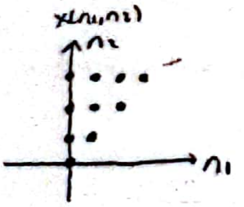


$\underline{\underline{\begin{bmatrix} 5 & 2 \\ 1 & 5 \end{bmatrix}}}$

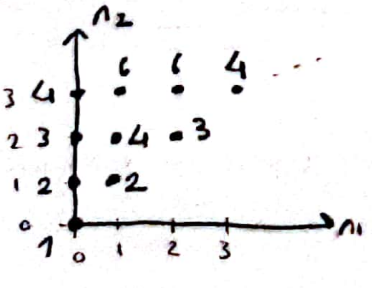
$\underline{\underline{V_2}}$

$\underline{\underline{\begin{bmatrix} -3 & 5 \\ 4 & 1 \end{bmatrix}}}$

b)



$y = x ** x$



$y(n_1,n_2) = \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} x(k_1,k_2)\,x(n_1-k_1,n_2-k_2)$ $\quad k_2 \geq k_1$, $k_1 \geq 0$

$= \sum_{k_1=0}^{\infty}\sum_{k_2=k_1}^{\infty} x(n_1-k_1,n_2-k_2)$, $\quad n_2-k_2 \geq n_1-k_1$, $k_1 \leq n_1$, $k_2 \leq n_2-n_1+k_1$

$= \sum_{k_1=0}^{n_1}\sum_{k_2=k_1}^{n_2-n_1+k_1} 1 = \sum_{k_1=0}^{n_1}(n_2-n_1+1) = \underline{\underline{(n_1+1)(n_2-n_1+1)}}$

$y(n_1,n_2) = \begin{cases} (n_1+1)(n_2-n_1+1), & n_1 \geq 0,\ n_2 \geq 1 \\ 0, & otherwise \end{cases}$

6) a) $y(n_1,n_2) = x(n_1,n_2)\,x(n_1-3,n_2)$

$\left.\begin{array}{l} y_1 = x_1(n_1,n_2)\,x_1(n_1-3,n_2) \\ y_2 = 2x_1(n_1,n_2)\,2x_1(n_1-3,n_2) = 4y_1 \end{array}\right\}$ Not linear

$\left.\begin{array}{l} y_3(n_1,n_2) = x(n_1-\Delta,n_2-\gamma)\,x(n_1-\Delta-3,n_2-\gamma) \\ = y_1(n_1-\Delta,n_2-\gamma) \end{array}\right\}$ S.I

b) $y(n_1,n_2) = \sum_{k_2} x(n_1,k_2)$

$\left.\begin{array}{l} y_1 = \sum_{k_2=-\infty}^{\infty} x_1(n_1,k_2) \\ ay_1 = \sum_{k_2} a x_1(n_1,k_2) = a\sum_{k_2} x_1(n_1,k_2) \\ \sum_{k_2}[x_1(n_1,k_2)+x_2(n_1,k_2)] = y_1+y_2 \end{array}\right\}$ Linear

$\left. y_3(n_1,n_2) = \sum_{k_2} x(n_1-l_1,k_2) = y_1(n_1-l_1,n_2) \right\}$ SI

c) $y(n_1,n_2) = \sum_{k_2=-1}^{1} x(n_1,k_2)$

$y(n_1,n_2) = x(n_1,-1) + x(n_1,0) + x(n_1,1)$

$$\left.\begin{aligned} \sum_{k_2=-1}^{1} a(x(n_1,k_2)) &= a\sum_{k_2=-1}^{1} x(n_1,k_2) = a\,y(n_1,n_2) \\ \sum_{k_2=-1}^{1} a(x_1+x_2) &= a\sum_{k_1=-1}^{1} x_1 + a\sum_{k_1=-1}^{1} x_2 = a(y_1+y_2) \end{aligned}\right\} \text{Linear}$$

$x[n_1,n_2-L] \rightarrow \boxed{H} \rightarrow y_1(n_1,n_2)$

$$\underbrace{y_1(n_1,n_2) = x(n_1,L-1) + x(n_1,L) + x(n_1,L+1)}_{\text{not S.I.}}$$

d) $y(n_1,n_2) = 255 - x(n_1,n_2)$

$$\underbrace{255 - 2x(n_1,n_2) \neq 2y(n_1,n_2)}_{\text{Not linear}}$$

$x[n_1-L_1, n_2-L_2] \rightarrow \boxed{H} \rightarrow y_1(n_1,n_2)$

$$y_1(n_1,n_2) = 255 - x(n_1-L_1, n_2-L_2)$$
$$\underbrace{= y(n_1-L_1, n_2-L_2)}_{\text{S.I}}$$

7) • $X(w_1,w_2) = \sum_{n_1}\sum_{n_2} x[n_1,n_2]\, e^{-jw_1n_1} e^{-jw_2n_2} \longleftrightarrow x[n_1,n_2]$

$$X^*(w_1,w_2) = \sum_{n_1}\sum_{n_2} \underbrace{x^*[n_1,n_2]}_{=x[n_1,n_2]} e^{jw_1n_1} e^{jw_2n_2} = X(-w_1,-w_2) \longleftrightarrow x[-n_1,-n_2]$$

↳ Real signals

If $x[n_1,n_2] = x[-n_1,-n_2] \Rightarrow X(w_1,w_2)$ is real

• $X(w_2,w_1) = \sum_{n_1}\sum_{n_2} x[n_1,n_2]\, e^{-jw_2n_1} e^{-jw_1n_2} = \sum_{u_1}\sum_{u_2} x[u_2,u_1]\, e^{-jw_1u_1} e^{-jw_2u_2}$

$n_1 = u_2$, $n_2 = u_1$

$$= \sum_{n_1}\sum_{n_2} x[n_1,n_2]\, e^{-jw_1n_1} e^{-jw_2n_2} \Longrightarrow x[n_1,n_2] = x[n_2,n_1]$$

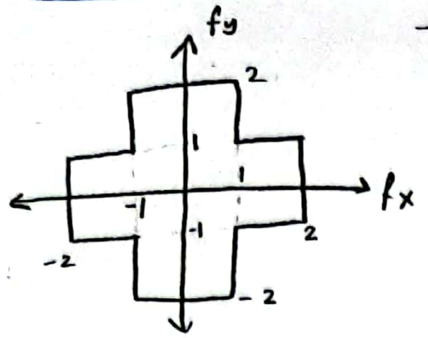
• $x[n_1,n_2] \longleftrightarrow X(w_1,w_2) = X(-w_1,w_2) \longleftrightarrow x[-n_1,n_2]$

$x[n_1,n_2] = x[-n_1,n_2]$

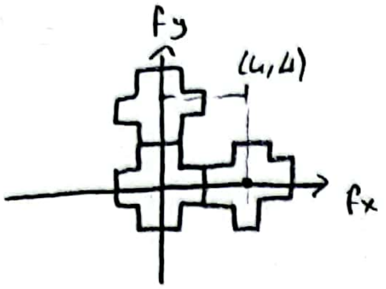
a) I, III   b) I, III   c) I, II

8

$\Omega = 2\pi f$

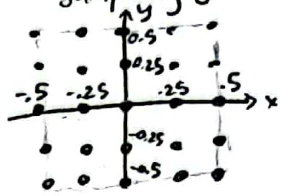


a)

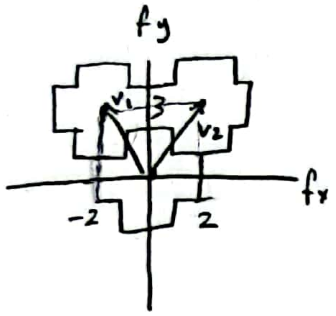


$\frac{1}{T_x} = 4, \ \frac{1}{T_y} = 4 \Rightarrow T_x = T_y = 1/4$

Sampling grid

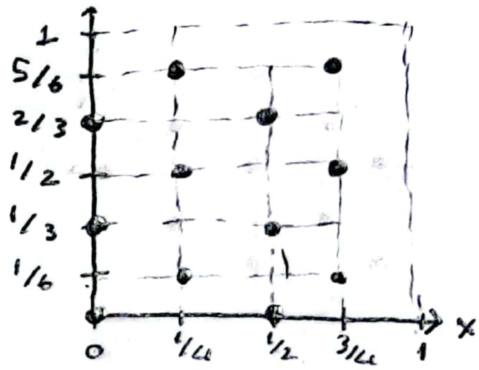


b)



$v_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \ v_2 \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \ V = [v_1, v_2]$

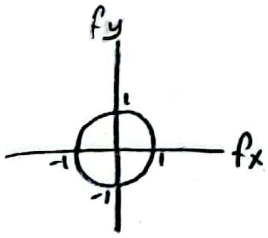
$U = V^{-T} = \begin{bmatrix} 1/4 & -1/4 \\ 1/6 & 1/6 \end{bmatrix}$



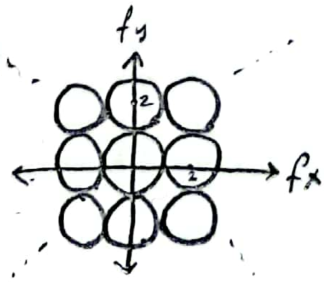
12 sampling points / m^2

c) Already chosen this strategy in b.

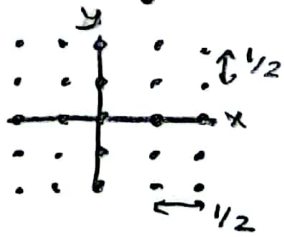
d)



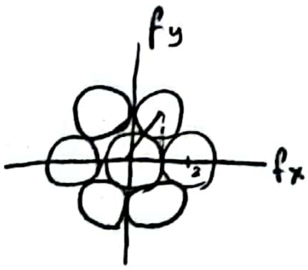
d.a)



$\frac{1}{T_x} = 2, \ \frac{1}{T_y} = 2 \Rightarrow T_x = T_y = 1/2$



d.b)



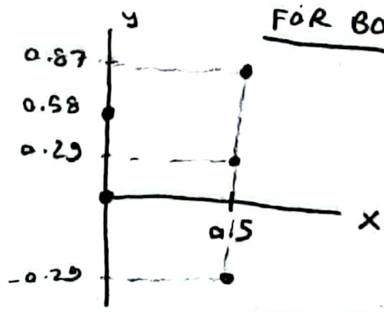
$v_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$

$U = V^{-T} = \begin{bmatrix} 0.5 & 0 \\ -\frac{\sqrt{3}}{6} & \frac{\sqrt{3}}{3} \end{bmatrix}$

$\approx \begin{bmatrix} 0.5 & 0 \\ -0.29 & 0.58 \end{bmatrix}$

Sampling Density: 4 samples / m^2

FOR BOTH CASES



CamScanner

$\det(V_a) = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4$ $\det(V_b) = \begin{vmatrix} 2 & 1 \\ 0 & \sqrt{3} \end{vmatrix} = 2\sqrt{3}$

Sampling density^1 = $4k$ Sampling density^2 = $2\sqrt{3}k$

$\frac{SD^2}{SD^1} \approx 0.87$, 13% more efficient