EE634 HW1

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The notebook can be accessed here. It is highly recommended to download the repo files as a zip and view the files in HTML format.

In [ ]:
```
import numpy as np
from scipy.fft import fft, ifft, fft2, ifft2, fftshift
from scipy.signal import convolve2d
from scipy.linalg import toeplitz
from matplotlib import pyplot as plt
from matplotlib import cm
from numpy import pi as pi
from mpl_toolkits.mplot3d import Axes3D
%matplotlib inline
```

Q1a

$$x(n_1, n_2) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$x * h(n_1, n_2) = \begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & 1 & 1 & \frac{1}{4} \\ \frac{1}{4} & 1 & 1 & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 \end{bmatrix}$$

Using the linear convolutions dimension expression one can conclude that the filter is 3×3. So let

$$h(-n_1, -n_2) = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Using the corner elements, one can deduce that the corner elements of the filter is 0. With this configuration:

$$
\begin{aligned}
i &= 0 \\
h + i &= \frac{1}{4} \implies h = \frac{1}{4} \\
g + h &= \frac{1}{4} \implies g = 0 \\
f + i &= \frac{1}{4} \implies f = \frac{1}{4} \\
e + h + f + i &= 1 \implies e = \frac{1}{2} \\
d + g + e + h &= 1 \implies d = \frac{1}{4} \\
&\vdots
\end{aligned}
$$

By using the symmetry in input and output, one can also conclude that:

$$
\begin{aligned}
a = c &= 0 \\
b &= \frac{1}{4}
\end{aligned}
$$

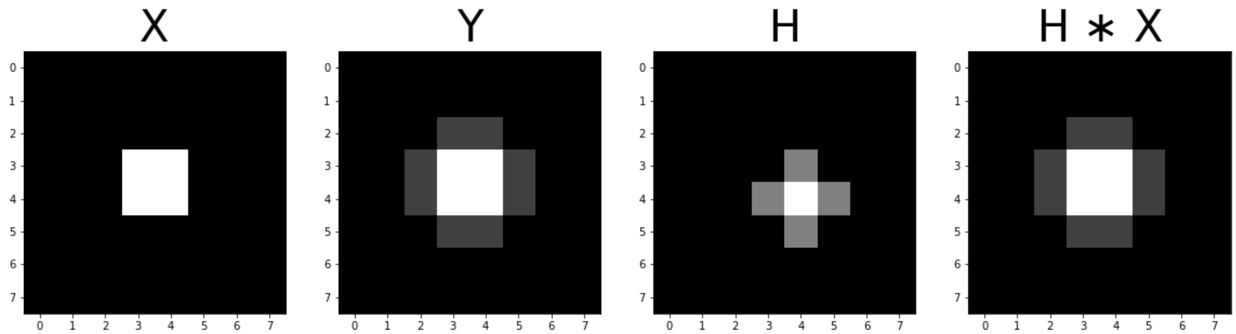
Hence

$$
h(n_1, n_2) = \begin{bmatrix} 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 \end{bmatrix}
$$

In [ ]:
```
x = np.zeros((8,8))
x[3:5,3:5] = 1
y = np.zeros_like(x)
y[2:6,2:6] = np.array([[0,.25,.25,0],[.25,1,1,.25],[.25,1,1,.25],[0,.25,.25,0]])
h = np.zeros_like(x)
h[3:6,3:6] = np.array([[0,.25,0],[.25,.5,.25],[0,.25,0]])
y_prime = np.abs(fftshift(ifft2(fft2(x)*fft2(h))))
fsize = 40
plt.figure(figsize=(20,80))
plt.subplot(1,4,1)
plt.imshow(np.abs(x))
plt.title('X',fontsize=fsize)
plt.set_cmap(cmap="gray")
plt.subplot(1,4,2)
plt.imshow(np.abs(y))
plt.title('Y',fontsize=fsize)
plt.set_cmap(cmap="gray")
plt.subplot(1,4,3)
plt.imshow(np.abs(h))
plt.title('H',fontsize=fsize)
plt.set_cmap(cmap="gray")
plt.subplot(1,4,4)
plt.imshow(y_prime)
plt.title(r'H $\ast$ X',fontsize=fsize)
plt.set_cmap(cmap="gray")
```

```python
assert np.isclose(np.sum(y_prime-y),0)
print("Resultant convolution matches the given.")
```

```
Resultant convolution matches the given.
```



Q1b

In [ ]:
```python
H = h[3:6,3:6]
H
```

Out[ ]:
```
array([[0.  , 0.25, 0.  ],
       [0.25, 0.5 , 0.25],
       [0.  , 0.25, 0.  ]])
```

In [ ]:
```python
u = np.linalg.svd(H)
singular_values = u[1]
singular_values
```

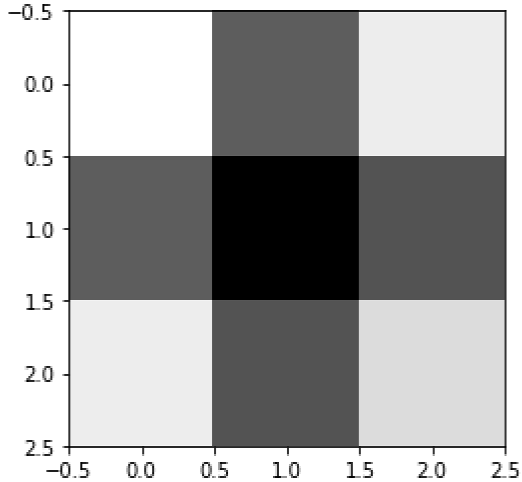
Out[ ]:
```
array([6.83012702e-01, 1.83012702e-01, 1.27795495e-17])
```

As can be seen above, we have 3 distinct singular values for the convolution kernel. For a kernel to be separable, it should be expressed as one outer product. However, when we use the SVD to decompose the matrix into outer products, we see that it has two nonzero singular values. One separable filter example can be seen below. The kernel is defined as an outer product and it has only one nonzero singular value.

In [ ]:
```python
a = np.random.randint(0,100,(3,1))
b = np.outer(a,a)
u = np.linalg.svd(b)
singular_values = u[1]
print(singular_values)
plt.imshow(b)
```

```
[1.61870000e+04 9.22945517e-13 2.22565261e-14]
```

Out[ ]:
```
<matplotlib.image.AxesImage at 0x1643af82430>
```



Q1d

$$
\begin{aligned}
H(w_1, w_2) &= \sum_{n_1=0}^{2}\sum_{n_2=0}^{2} h(n_1, n_2) e^{-jw_1 n_1} e^{-jw_2 n_2} \qquad (1)\\
&= \frac{1}{4}(e^{-jw_1} + e^{-jw_2} + e^{-jw_1}e^{-j2w_2} + e^{-j2w_1}e^{-jw_2}) + \frac{1}{4}(e^{-jw_1}e^{-jw_2})\\
&= \frac{1}{4}e^{-jw_1}(1 + e^{-j2w_2}) + \frac{1}{4}e^{-jw_2}(1 + e^{-j2w_1}) + \frac{1}{2}(e^{-jw_1}e^{-jw_2})\\
&= \frac{1}{4}e^{-jw_1}e^{-jw_2}(cos(\frac{w_1}{2}) + cos(\frac{w_2}{2})) + \frac{1}{2}e^{-jw_1}e^{-jw_2}
\end{aligned}
$$

$$
\implies |H(w1, w2)| = \frac{1}{2} + cos(\frac{w_1}{2}) + cos(\frac{w_2}{2})
$$

This filter acts as low pass filter, since its magnitude have higher values around the origin.

Q1c

$$
H(k_1, k_2) = \sum_{n_1=0}^{2}\sum_{n_2=0}^{2} h(n_1, n_2) e^{-j\frac{2\pi}{N_1}k_1 n_1} e^{-j\frac{2\pi}{N_2}k_2 n_2} \qquad (2)
$$

Since image has n1-n2 symmetry, its DFT has k1-k2 symmetry in the frequency domain, *i.e.* $H(k_1, k_2) = H(k_2, k_1)$

$$
\begin{aligned}
H(0,0) &= \sum_{n_1=0}^{2}\sum_{n_2=0}^{2} h(n_1,n_2) = 1.5 \\
H(1,0) &= \sum_{n_1=0}^{2}\sum_{n_2=0}^{2} h(n_1,n_2)e^{-j\frac{2\pi n_1}{3}} \\
&= (h(0,1)+h(0,2)+h(0,3)) + (h(1,0)+h(1,1)+h(1,2))e^{-j\frac{2\pi}{3}} + (h(2,1)+ \\
&= \frac{1}{4} + e^{-j\frac{2\pi}{3}} + \frac{1}{4}e^{-j\frac{4\pi}{3}} \\
H(0,1) &= H(1,0) \\
H(2,0) &= \sum_{n_1=0}^{2}\sum_{n_2=0}^{2} h(n_1,n_2)e^{-j\frac{4\pi n_1}{3}} \\
&= (h(0,1)+h(0,2)+h(0,3)) + (h(1,0)+h(1,1)+h(1,2))e^{-j\frac{4\pi}{3}} + (h(2,1)+ \\
&= \frac{1}{4} + e^{-j\frac{4\pi}{3}} + \frac{1}{4}e^{-j\frac{2\pi}{3}} \\
H(0,2) &= H(2,0) \\
H(1,1) &= \sum_{n_1=0}^{2}\sum_{n_2=0}^{2} h(n_1,n_2)e^{-j\frac{2\pi(n_1+n_2)}{3}} \\
&= \sum_{n_2=0}^{2} e^{-j\frac{2\pi n_2}{3}}\left(h(0,n_2) + h(1,n_2)e^{-j\frac{2\pi}{3}} + h(2,n_2)e^{-j\frac{4\pi}{3}}\right) \\
&= 0 \;\; (\textit{See calculation below}) \\
H(2,2) &= \sum_{n_1=0}^{2}\sum_{n_2=0}^{2} h(n_1,n_2)e^{-j\frac{2\pi(2n_1+2n_2)}{3}} \\
&= \sum_{n_2=0}^{2} e^{-j\frac{4\pi n_2}{3}}\left(h(0,n_2) + h(1,n_2)e^{-j\frac{2\pi}{3}} + h(2,n_2)e^{-j\frac{4\pi}{3}}\right) \\
&= 0 \;\; (\textit{See calculation below})
\end{aligned}
$$

In [ ]:

```
sum = 0
for n2 in range(3):
    sum += np.exp(-1j*2*pi*n2/3) * (h[0,n2] + h[1,n2] * np.exp(-1j*2*pi/3) + h[2,n2] *
print("H(1,1) =",sum)
sum = 0
for n2 in range(3):
    sum += np.exp(-1j*4*pi*n2/3) * (h[0,n2] + h[1,n2] * np.exp(-1j*2*pi/3) + h[2,n2] *
print("H(2,2) =",sum)
```

```
H(1,1) = 0j
H(2,2) = 0j
```

Cross Term calculation example:

In [ ]:

```
(0.25 + 1*np.exp(-1j*2*pi/3) + 0.25*np.exp(-1j*4*pi/3))
```

Out[ ]:

```
(-0.37499999999999999-0.6495190528383291j)
```

In [ ]:
```
fft2(H)
```

Out[ ]:
```
array([[ 1.5  -0.j        , -0.375-0.64951905j, -0.375+0.64951905j],
       [-0.375-0.64951905j,  0.   +0.j        ,  0.   -0.j        ],
       [-0.375+0.64951905j,  0.   +0.j        ,  0.   -0.j        ]])
```

As expected, we again obtained an low pass convolution filter. Higher frequency terms "at the edges" of the filter are zero, whereas center terms have higher magnitude. This is totally expected, since DFT is the sampled version of DTFT where $w = \frac{2\pi k}{N}$.

Q1 e

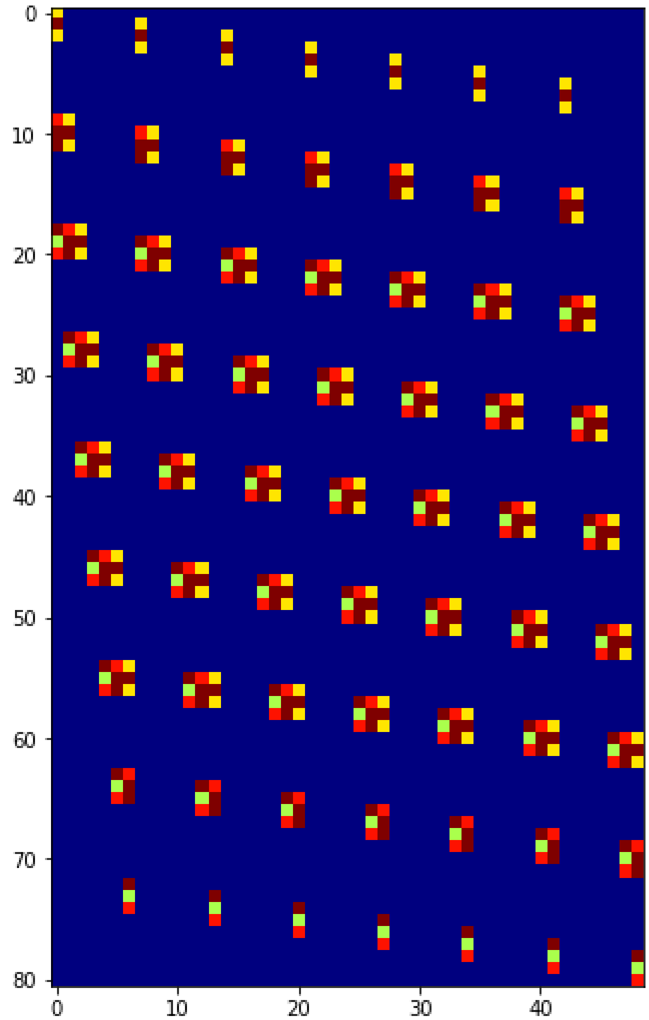
In [ ]:
```
def linear_conv_mat(h:np.array,output_size):
    L = h.size
    h_ex = np.zeros(output_size)
    h_ex[0:L] = h
    first_row = np.roll(np.flip(h_ex),1)
    return toeplitz(h_ex.T,first_row[0:(output_size-L+1)]) # H + X - 1 = output_size

def linear_conv2_mat(h:np.ndarray, image_shape):
    L1,L2 = image_shape
    N1,N2 = h.shape
    H = np.empty(((N1+L1-1)*(N2+L2-1),0))
    image_size = L1*L2
    for i in range(image_size):
        row = np.mod(i,L1)
        col = i//L1
        basis_vec = np.zeros((L1,L2))
        basis_vec[row,col] = 1
        basis_vec_output = convolve2d(h,basis_vec)
        H = np.column_stack((H, basis_vec_output.flatten()))
    return H
```

In [ ]:
```
for _ in range(50):
    N1 = np.random.randint(0,20)
    N2 = np.random.randint(0,20)
    L1 = np.random.randint(0,20)
    L2 = np.random.randint(0,20)
    H = np.random.randint(0,100,(N1,N2))
    x = np.random.randint(0,100,(L1,L2))
    H_mat = linear_conv2_mat(h=H,image_shape=x.shape)
    y_prime = convolve2d(H,x)
    y_prime_vec = H_mat.dot(x.flatten())
    y_prime_back = np.reshape(y_prime_vec,y_prime.shape,order="F")

    assert np.isclose(np.sum(y_prime-y_prime_back),0) # Check if they are the same
print("linear_conv2_mat works")
```

```
linear_conv2_mat works
```

In [ ]:
```
H = np.random.randint(5,10,(3,3))
x = np.random.randint(5,10,(7,7))
H_mat = linear_conv2_mat(h=H,image_shape=x.shape)
plt.figure(figsize=(18,9))
plt.imshow(H_mat,cmap="jet")
```

Out[ ]: <matplotlib.image.AxesImage at 0x16434f61a30>



Q1f

Let the i^{th} column of the image to convolved is called x_i. Then, the matrix product Cx can be considered as "convolution with matrices s_i and the vectors x_i". Let the output image be y and the i^{th} column of it is called y_i. Then:

$$y_i = \sum_{k=0}^{L_2} s_{[i-k]} x_k$$

where sk corresponds to the matrix-vector product with indices higher than P_2 and negative indices of s correspond to zero multiplication. Since s matrix is a Toeplitz matrix, this product is equal to a convolution with a vector which circulates in the columns of this Toeplitz matrix, *i.e.* the columns of the filter kernel.

In [ ]:

```
def conv2_by_fft(x:np.ndarray, h:np.ndarray):
    if h.shape[0] > x.shape[0] and h.shape[1] > x.shape[1]:
        x,h = h,x
    L1,L2 = x.shape
    P1,P2 = h.shape
```

```
    rows = L1+P1-1
    cols = L2+P2-1
    Y = np.empty((rows,cols))
    for i in range(cols):
        output_col = np.zeros((rows,))
        for p in range(L2): # travel through image columns
            if i-p >= 0 and i-p < P2:
                product = np.multiply(fft(x[:,p],rows),fft(h[:,i-p],rows)) # splitted
                output_col += np.real(ifft(product))
                Y[:,i] = output_col
    return Y
```

Test the function

In [ ]:
```
for _ in range(1500):
    L1 = np.random.randint(3,15)
    L2 = np.random.randint(3,15)
    N1 = np.random.randint(1,L1-1)
    N2 = np.random.randint(1,L2-1)
    H = np.random.randint(0,100,(N1,N2))
    x = np.random.randint(0,100,(L1,L2))
    y = conv2_by_fft(x,H)
    y_prime = convolve2d(x,H)
    assert np.isclose(np.sum(y_prime-y),0) # Check if they are the same
print("conv2_by_fft works")
```

```
conv2_by_fft works
```

Q1g

The energy of the signal is defined as

$$\sum_{n=0}^{\infty} x^2[n]$$

Using Parsevals relation, one can write

$$\sum_{n=0}^{\infty} x^2[n] = \sum_{k=0}^{\infty} X^2[k]$$

where $X(k)$ is the DFT of $x[n]$. The same relation is also valid for 2D signals.

$$\begin{align}
y[m,n] &= h[m,n] ** x[m,n] \tag{1}\\
Y[k,l] &= H[k,l]\ X[k,l] \tag{2}\\
E_x &= \sum_k \sum_l X^2[k,l] \tag{3}\\
E_y &= \sum_k \sum_l Y^2[k,l] = \sum_k \sum_l H^2[k,l] X^2[k,l] \tag{4}
\end{align}$$

To have $E_x = E_y$, right hand side of Eqn. 4 should be normalized with $\frac{E_x}{E_y}$, which corresponds to replacing filter $H[k,l]$ with $\sqrt{\frac{E_x}{E_y}}H[k,l]$.

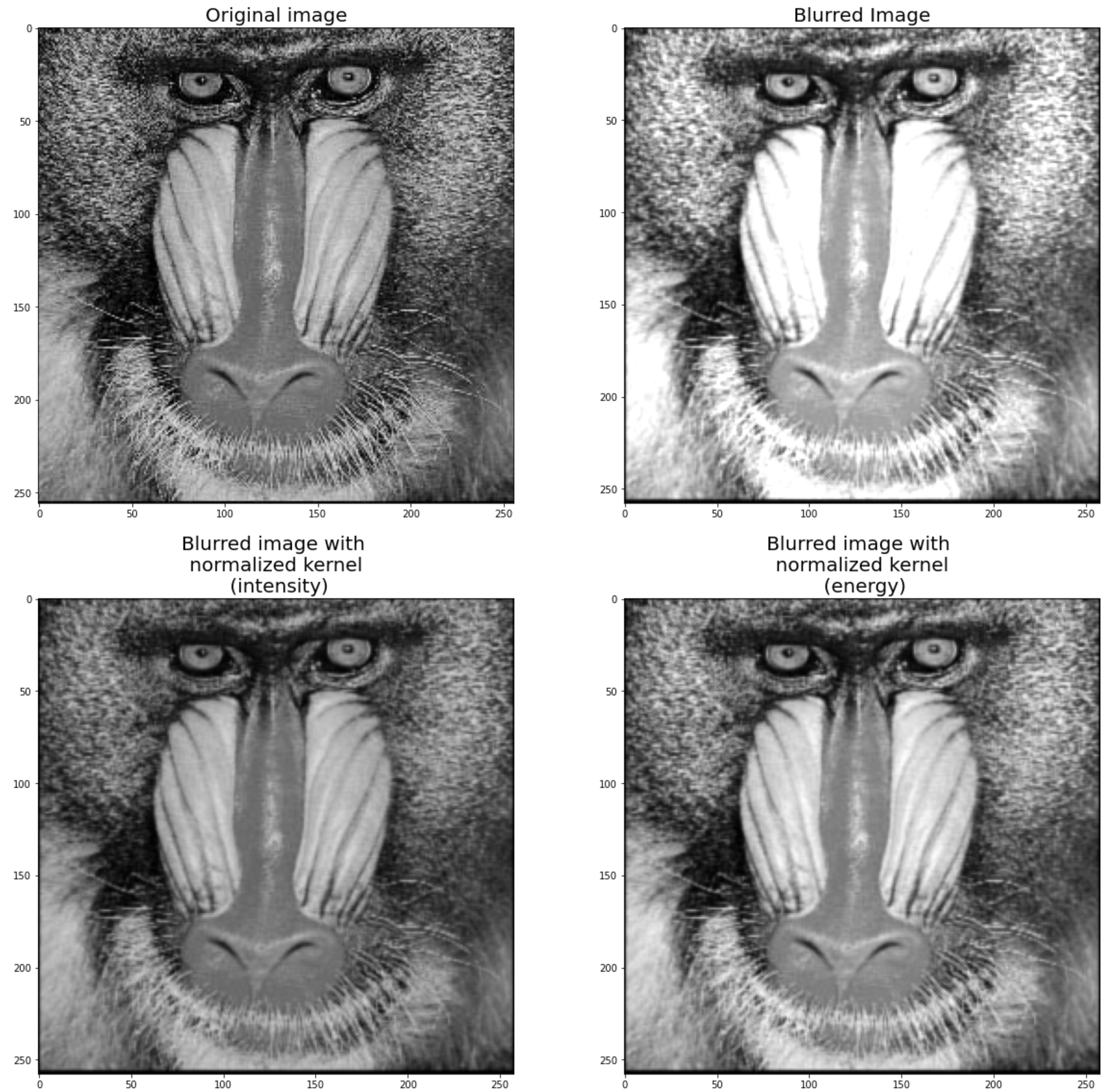
In [ ]:

```
image = plt.imread("mandrill.bmp")
h = np.array([[0,.25,0],[.25,.5,.25],[0,.25,0]])
blurred_image = conv2_by_fft(h,image)
E_x = np.trace(image.conj().T.dot(image))
E_y = np.trace(blurred_image.conj().T.dot(blurred_image))
print("Energy ratio:",np.sqrt(E_x/E_y))
modified_h = np.sqrt(E_x/E_y) * h
plt.figure(figsize=(20,20))
plt.subplot(2,2,1)
plt.imshow(image)
plt.title("Original image",fontsize=20)
plt.subplot(2,2,2)
plt.imshow(blurred_image,vmin=np.min(image), vmax=np.max(image))
blurred_image_normalized_intesity = conv2_by_fft(h/np.sum(h),image) # Done normalizati
blurred_image_normalized_energy = conv2_by_fft(modified_h,image)
plt.title("Blurred Image",fontsize=20)
plt.subplot(2,2,3)
plt.imshow(blurred_image_normalized_intesity,vmin=np.min(image), vmax=np.max(image))
plt.title("Blurred image with \n normalized kernel \n (intensity)",fontsize=20)
plt.subplot(2,2,4)
plt.imshow(blurred_image_normalized_energy,vmin=np.min(blurred_image_normalized_energy
plt.title("Blurred image with \n normalized kernel \n (energy)",fontsize=20)
```

```
Energy ratio: 0.004278659326852085
```

Out[ ]:

```
Text(0.5, 1.0, 'Blurred image with \n normalized kernel \n (energy)')
```



In the last image, we observe that the hairy part of the cheeks of the mandrill got blurred and is not distinctive as it is in the original image anymore.

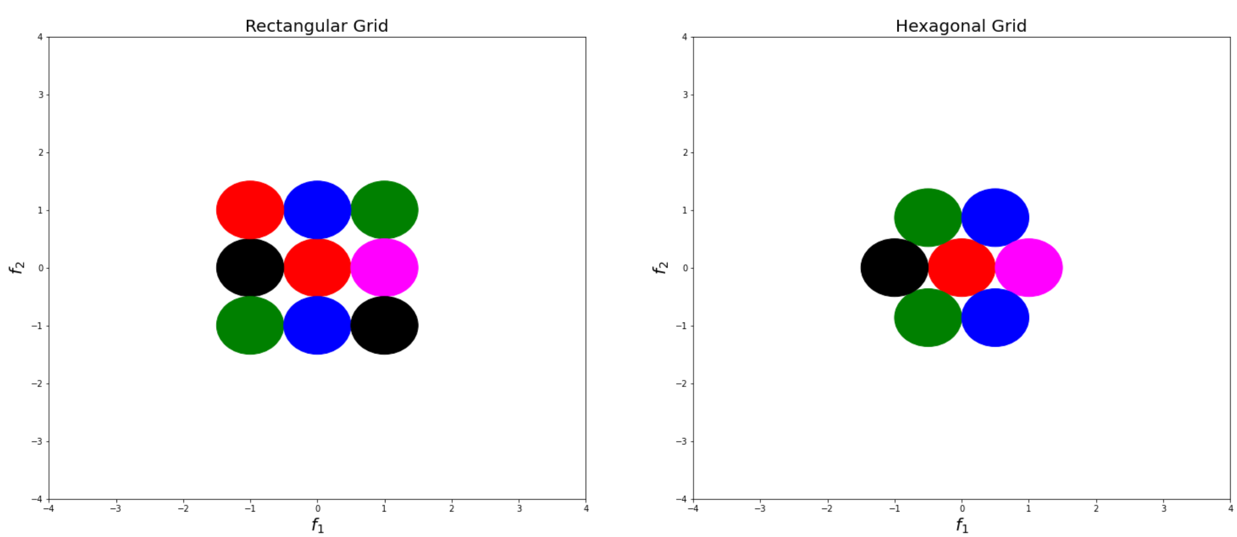
Q2

In [ ]:
```
plt.figure(figsize=(25,10))

plt.subplot(1,2,1)
ax = plt.gca()
ax.cla() # clear things for fresh plot
circle1 = plt.Circle((0, 0), .5, color='r')
circle2 = plt.Circle((1, 1), .5, color='g')
circle3 = plt.Circle((0, 1), .5, color='b')
circle4 = plt.Circle((1, 0), .5, color='magenta')
circle5 = plt.Circle((-1, 0), .5, color='k')
circle6 = plt.Circle((-1, -1), .5, color='g')
circle7 = plt.Circle((0, -1), .5, color='b')
circle8 = plt.Circle((1, -1), .5, color='k')
```

```
circle9 = plt.Circle((-1, 1), .5, color='r')
# change default range so that new circles will work
ax.set_xlim((-4, 4))
ax.set_ylim((-4, 4))
ax.add_patch(circle1)
ax.add_patch(circle2)
ax.add_patch(circle3)
ax.add_patch(circle4)
ax.add_patch(circle5)
ax.add_patch(circle6)
ax.add_patch(circle7)
ax.add_patch(circle8)
ax.add_patch(circle9)
plt.title("Rectangular Grid",fontsize=20)
plt.xlabel(r'$f_1$',fontsize=20)
plt.ylabel(r'$f_2$',fontsize=20)


plt.subplot(1,2,2)
ax = plt.gca()
ax.cla() # clear things for fresh plot
circle1 = plt.Circle((0, 0), .5, color='r')
circle2 = plt.Circle((-0.5, 0.5*np.sqrt(3)), .5, color='g')
circle3 = plt.Circle((0.5, 0.5*np.sqrt(3)), .5, color='b')
circle4 = plt.Circle((1, 0), .5, color='magenta')
circle5 = plt.Circle((-0.5, -0.5*np.sqrt(3)), .5, color='g')
circle6 = plt.Circle((0.5, -0.5*np.sqrt(3)), .5, color='b')
circle7 = plt.Circle((-1, 0), .5, color='k')
# change default range so that new circles will work
ax.set_xlim((-4, 4))
ax.set_ylim((-4, 4))
ax.add_patch(circle1)
ax.add_patch(circle2)
ax.add_patch(circle3)
ax.add_patch(circle4)
ax.add_patch(circle5)
ax.add_patch(circle6)
ax.add_patch(circle7)
plt.title("Hexagonal Grid",fontsize=20)
plt.xlabel(r'$f_1$',fontsize=20)
plt.ylabel(r'$f_2$',fontsize=20)
```

Out[ ]:

```
Text(0, 0.5, '$f_2$')
```



To recover the signal exactly from the frequency spectrum, we should conduct sampling avoiding aliasing. The minimum sampling frequency for this in regular grid turned out to be 1 $cycles/meter$, that is maximum 1 meter period. The most efficient sampling grid that we can utilize for circular support functions is the hexagonal grid. It corresponds to the sampling matrix V in the frequency domain where

$$V = \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

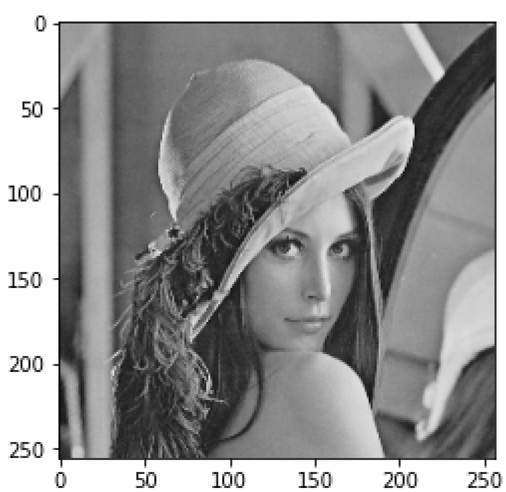
The sampling grid in space is given by

$$V^{-T} = \begin{bmatrix} 1 & 1 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

Q3a

In [ ]:
```python
from skimage.color import rgb2gray
image = plt.imread("256by256grayscaleLena.png")
image = rgb2gray(image[...,1:4])
minsize = 0
maxsize = image.shape[1]
X = np.arange(minsize, maxsize, 1)
Y = np.arange(minsize, maxsize, 1)
X, Y = np.meshgrid(X, Y)
plt.imshow(image,cmap="gray")
```

Out[ ]:
```
<matplotlib.image.AxesImage at 0x16436e21bb0>
```

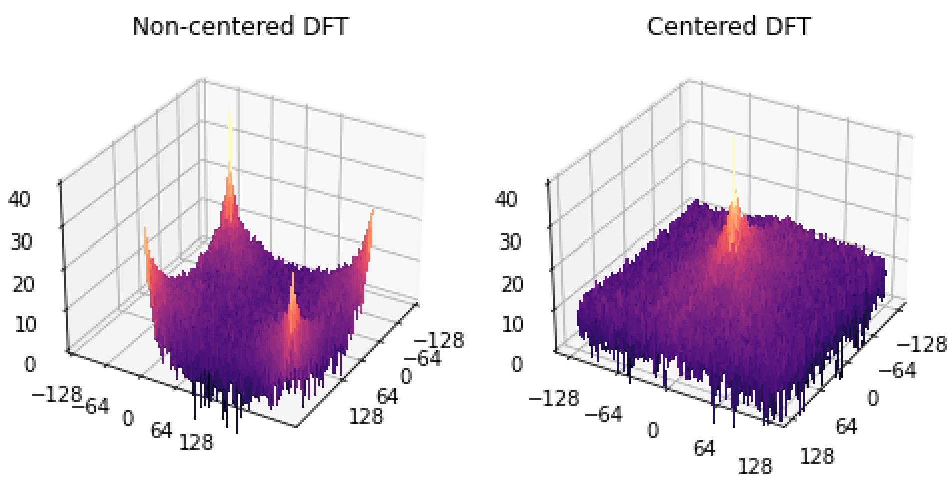


In [ ]:
```python
# Plot the surface.
fig = plt.figure(figsize=plt.figaspect(.5))
ax = fig.add_subplot(1, 2, 1, projection='3d')
surf = ax.plot_surface(X, Y, 10*np.log10(np.abs(fft2(image))),rstride=1,
    cstride=1, cmap=cm.magma, linewidth=0, antialiased=False)
ax.set_zlim(0,40)
ax.view_init(30, 30)
```

```
ax.set_xticks(np.linspace(-128,128,5))
ax.set_yticks(np.linspace(-128,128,5))
plt.title("Non-centered DFT")

ax = fig.add_subplot(1, 2, 2, projection='3d')
surf = ax.plot_surface(X-128, Y-128, 10*np.log10(np.abs(fftshift(fft2(image)))),rstric
    cstride=1, cmap=cm.magma, linewidth=0, antialiased=False)
ax.set_zlim(0,40)
ax.view_init(30, 30)
ax.set_xticks(np.linspace(-128,128,5))
ax.set_yticks(np.linspace(-128,128,5))
plt.title("Centered DFT")
```

Out[ ]:

```
Text(0.5, 0.92, 'Centered DFT')
```



Q3b

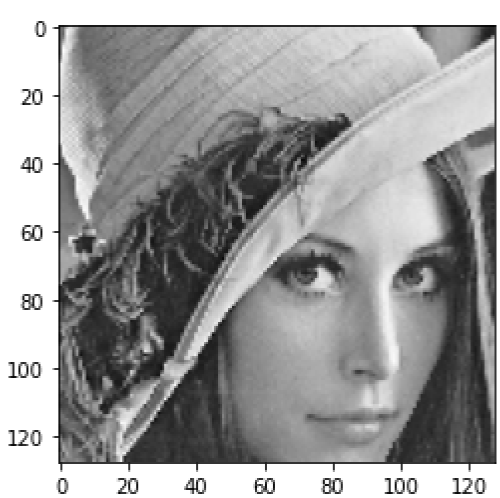
Take the image and upsample with zeroes in every 2 elements.

In [ ]:

```
cropped_image = image[60:188,60:188]
plt.imshow(cropped_image)
```

Out[ ]:

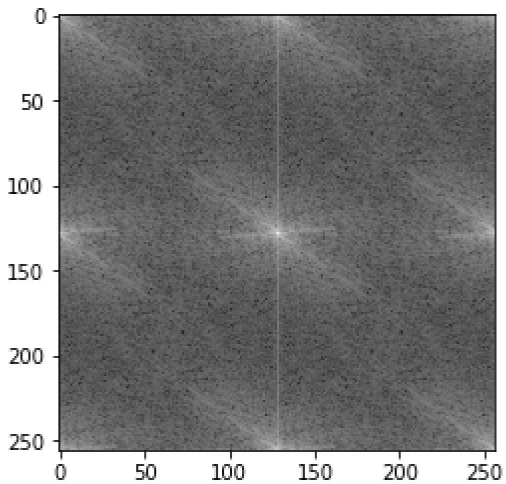
```
<matplotlib.image.AxesImage at 0x1644651e190>
```



Calculate its 2D Fourier transform.

In [ ]:
```python
s1,s2 = cropped_image.shape
upsampled_image = np.zeros((2*s1,2*s2))
upsampled_image[::2,::2] = cropped_image
FFT_upsampled = fft2(upsampled_image)
plt.imshow(np.log10(np.abs(FFT_upsampled)))
```

Out[ ]:
```
<matplotlib.image.AxesImage at 0x1644a3b4160>
```

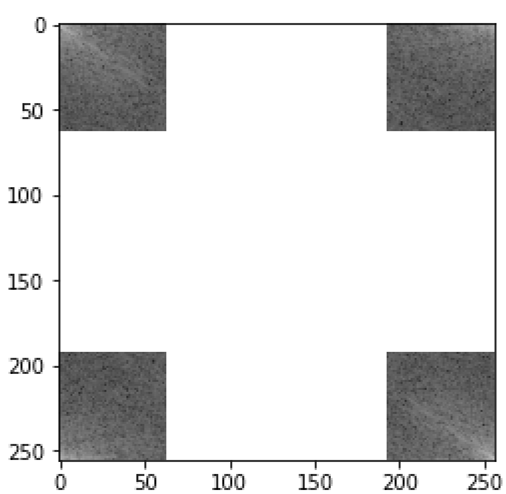


Filter the "central" portion of the frequency spectrum out with an ideal low pass filter.

In [ ]:
```python
## Take the center portion, equate remaining to 0
shifted = fftshift(FFT_upsampled)
Filtered_FFT = np.zeros_like(FFT_upsampled)
Filtered_FFT[64:192,64:192] = FFT_upsampled[64:192,64:192]
Filtered_FFT = fftshift(Filtered_FFT)
plt.imshow(np.log10(np.abs(Filtered_FFT)))
```

```
C:\Users\Kutay\AppData\Local\Temp/ipykernel_15656/1597208387.py:6: RuntimeWarning: di
vide by zero encountered in log10
  plt.imshow(np.log10(np.abs(Filtered_FFT)))
```

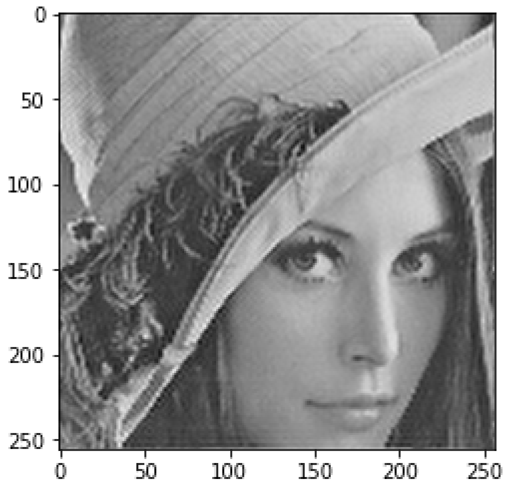
Out[ ]:
```
<matplotlib.image.AxesImage at 0x1644a9031f0>
```



Take the inverse 2D FFT to obtain the interpolated image.

In [ ]:
```python
interpolated_image = np.real(ifft2(Filtered_FFT))
plt.imshow(interpolated_image)
```

Out[ ]: <matplotlib.image.AxesImage at 0x16446d31a60>



In [ ]: