EE634 HW2

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It is highly recommended that the notebook repository should be downloaded here and viewed as HTML.

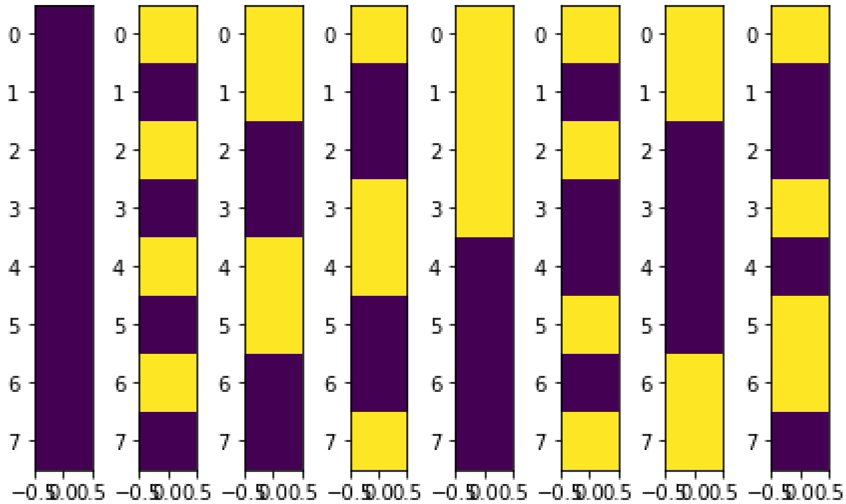
In [ ]:
```
import numpy as np
from numpy.linalg import eig
from matplotlib import pyplot as plt
from scipy.linalg import hadamard
from skimage.color import rgb2gray
from scipy.fft import fft, ifft, fft2, ifft2, dct, dctn
from mpl_toolkits.mplot3d import Axes3D
from matplotlib import cm
import itertools
%matplotlib inline
```

Q1

a

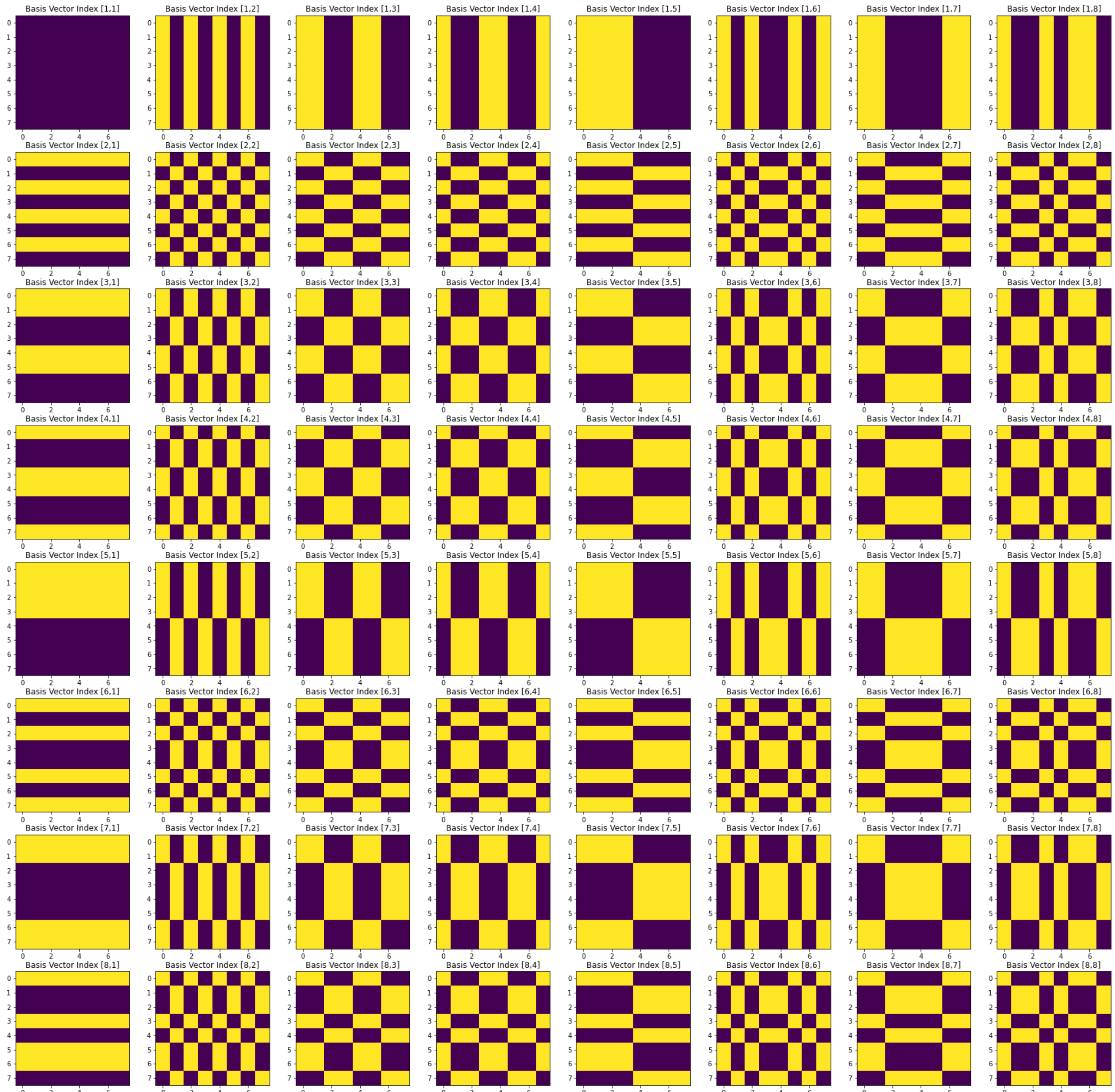
In [ ]:
```
def Hadamard_doubler(H):
    H_double_first_row = np.hstack((H,H))
    H_double_second_row = np.hstack((H,-1*H))
    return 1/np.sqrt(2) * np.vstack((H_double_first_row,H_double_second_row))

def HadamardMtx(N:int):
    H = 1/np.sqrt(2) * np.array([[1,1],[1,-1]])
    for _ in range(1,N):
        H = Hadamard_doubler(H)
    return H
```

In [ ]:
```
I = np.eye(8)
A = HadamardMtx(3)
B = np.empty(A.shape)
for i in range(8):
    basis_vector = I[:,i]
    transformed_basis_vec = np.expand_dims(A.dot(basis_vector),-1)

    plt.subplot(1,8,i+1)
    plt.imshow(transformed_basis_vec)
    B[:,i] = A.dot(basis_vector)

plt.tight_layout()
```



b

```
In [ ]: H = HadamardMtx(3)
        plt.figure(figsize=(30,30))
        for i in range(8):
            for j in range(8):
                basis_vec = np.zeros((8,8))
                basis_vec[i,j] = 1
                hadamard_basis_vec = H.dot(basis_vec).dot(H.T)#np.outer(B[:,i],B[:,j]) # 2D tr
                plt.subplot(8,8,8*(i)+(j+1))
                plt.imshow(hadamard_basis_vec)
                plt.title("Basis Vector Index ["+str(i+1)+","+str(j+1)+"]")
```



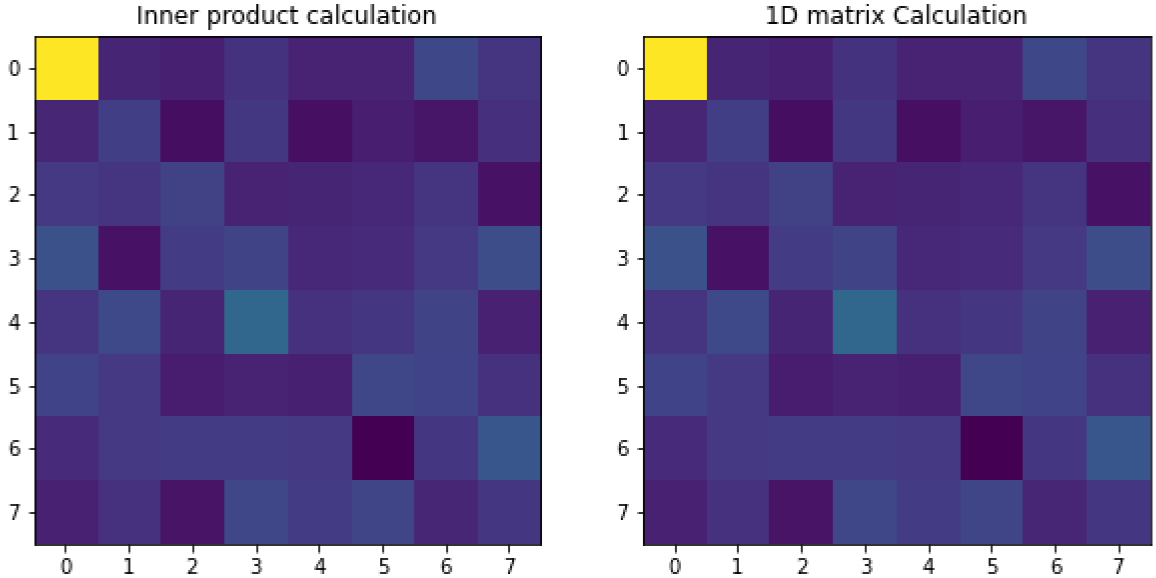
C

In [ ]:

```python
N = 3
for trial in range(50):
    H = HadamardMtx(N)
    T_inner_prod = np.zeros_like(H)
    I = np.random.randint(1,20,(2**N,2**N))
    for row in range(2**N):
        for col in range(2**N):
            basis_vec = np.zeros((2**N,2**N))
            basis_vec[row,col] = 1
            hadamard_basis_vec = H.dot(basis_vec).dot(H.T)
            T_inner_prod[row,col] = np.trace(hadamard_basis_vec.conj().T.dot(I))
    T_mtx = H.dot(I).dot(H)
    assert np.all(np.isclose(T_mtx,T_inner_prod))
print("All Transformed matrices are the same! The last trial: ")
plt.figure(figsize=(10,20))
plt.subplot(1,2,1)
plt.imshow(T_inner_prod)
```

```python
plt.title("Inner product calculation")
plt.subplot(1,2,2)
plt.imshow(T_mtx)
plt.title("1D matrix Calculation");
```

```
All Transformed matrices are the same! The last trial:
```



Q2

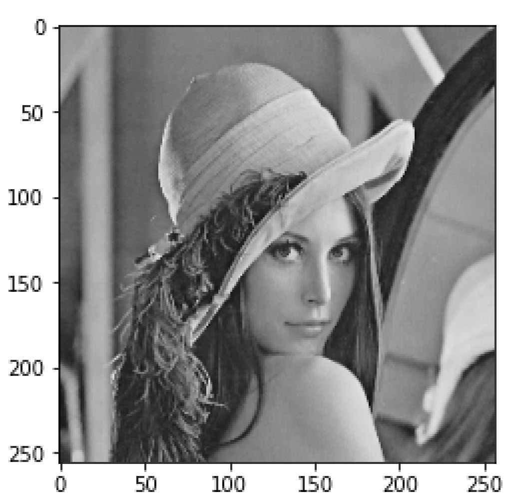
a

In [ ]:

```python
Lena = plt.imread("256by256grayscaleLena.png")
Lena = rgb2gray(Lena[...,:-1])
plt.imshow(Lena, cmap="gray")
```

Out[ ]:

```
<matplotlib.image.AxesImage at 0x21b2eec5b50>
```



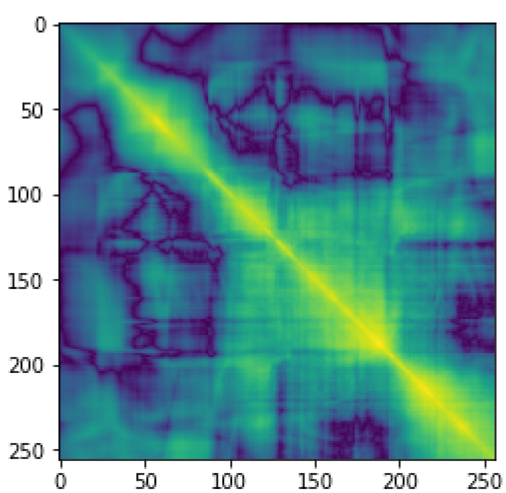
b

In [ ]:
```
def zero_mean_cols(img):
    return (img.T - img.mean(axis=1)).T

def cov(img):
    if len(img.shape) == 1:
        return np.outer(img,img.conj())
    img = zero_mean_cols(img)
    return img.dot(img.T.conj())

def KLT_mtx(img):
    C_img = cov(img)
    w_vl = eig(C_img)
    return w_vl[1].T

def diagonal_coef_ratio(cov_mtx):
    cov_mtx = np.abs(cov_mtx)
    print("\nDiagonal Coefficients Value Ratio:",(np.trace(cov_mtx)) / np.sum(cov_mtx)
```

In [ ]:
```
img = Lena
Cov_Lena = cov(img)
diagonal_coef_ratio((Cov_Lena))
plt.imshow(np.log(1+np.abs(Cov_Lena)))
```

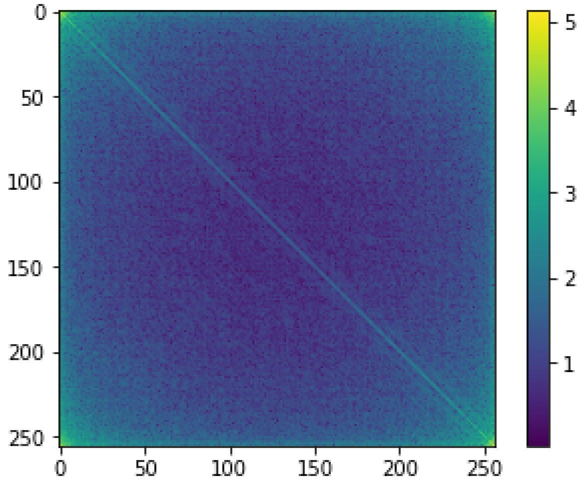
```
Diagonal Coefficients Value Ratio: 0.014061467
```

Out[ ]:
```
<matplotlib.image.AxesImage at 0x21b3012e940>
```



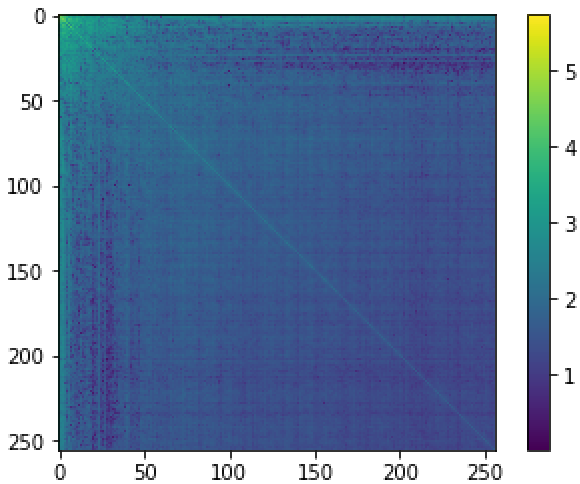
In [ ]:
```
img = fft(Lena,axis=0)
Cov_Lena = np.abs(cov((img)))
diagonal_coef_ratio((Cov_Lena))
plt.imshow(np.log10(1+np.abs((Cov_Lena))))
plt.colorbar();
```

```
Diagonal Coefficients Value Ratio: 0.1365569
```



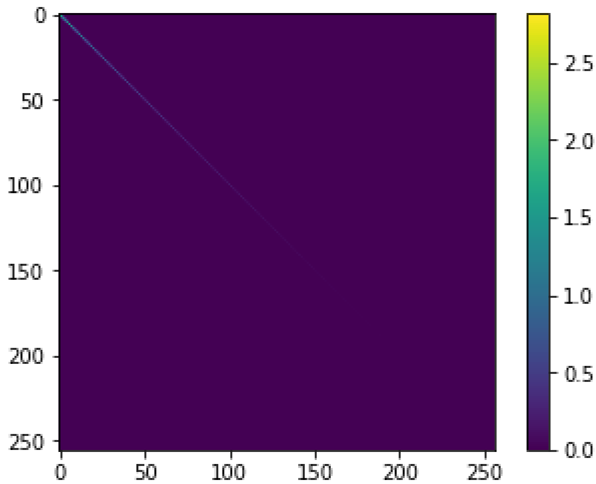
In [ ]:
```
img = np.abs(dct(Lena,axis=0))

Cov_Lena = cov(img)
diagonal_coef_ratio(Cov_Lena)
plt.imshow(np.log10(1+np.abs(Cov_Lena)))
plt.colorbar();
```

```
Diagonal Coefficients Value Ratio: 0.13431993
```



In [ ]:
```
A = KLT_mtx(Lena)
img = A.dot(Lena)
Cov_Lena = cov(img)
diagonal_coef_ratio(Cov_Lena)
plt.imshow(np.log10(1+np.abs(Cov_Lena)))
plt.colorbar();
```

```
Diagonal Coefficients Value Ratio: 0.99997926
```



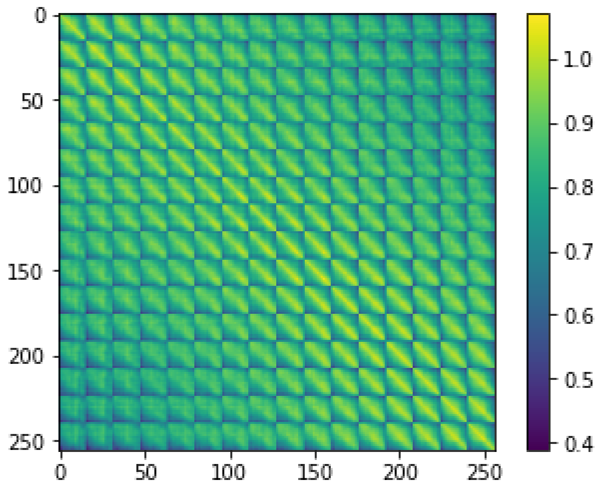
KLT concentrated the content of the covariance matrix in the diagonal of it, by finding the linear transform on image that diagonalizes its covariance matrix, *i.e.* decorrelating matrix.

C

In [ ]:

```
patch_size = 16
s1,_ = Lena.shape
num_horizontal = s1 // patch_size
num_vertical = num_horizontal
size = Lena.size
n_vectors = size // patch_size**2
step = s1 // patch_size
row_begin = 0
container = np.zeros((n_vectors,patch_size**2))
counter = 0
for row in range(patch_size):
    for col in range(patch_size):
        row_idx1 = row*patch_size
        row_idx2 = row*patch_size+step
        col_idx1 = col*patch_size
        col_idx2 = col*patch_size+step
        patch = Lena[row_idx1:row_idx2,col_idx1:col_idx2]
        container[:,counter] = patch.flatten()
        counter += 1
diagonal_coef_ratio(container)
plt.imshow(np.log10(1+np.abs(cov(container))))
plt.colorbar();
```

```
Diagonal Coefficients Value Ratio: 0.0039185826153480575
```

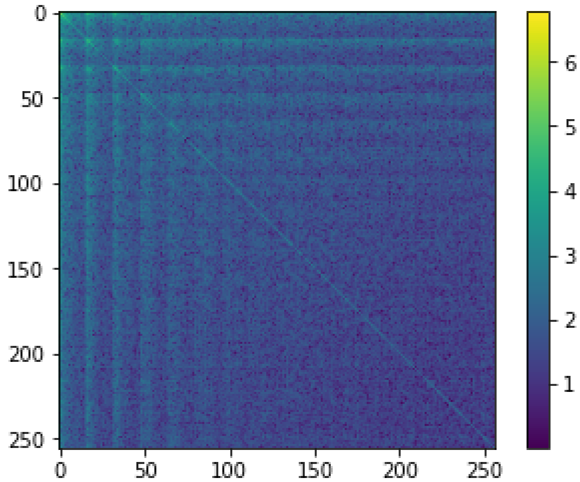


d

DCT

In [ ]:
```
patch_size = 16
s1,_ = Lena.shape
num_horizontal = s1 // patch_size
num_vertical = num_horizontal
size = Lena.size
n_vectors = size // patch_size**2
step = s1 // patch_size
row_begin = 0
container = np.zeros((n_vectors,patch_size**2))
counter = 0
new_image = np.zeros_like(Lena)
for row in range(patch_size):
    for col in range(patch_size):
        row_idx1 = row*patch_size
        row_idx2 = row*patch_size+step
        col_idx1 = col*patch_size
        col_idx2 = col*patch_size+step
        patch = Lena[row_idx1:row_idx2,col_idx1:col_idx2]
        patch = dctn(patch) # This transform is added
        new_image[row_idx1:row_idx2,col_idx1:col_idx2] = patch
        container[:,counter] = patch.flatten()
        counter += 1
diagonal_coef_ratio(cov(container))
plt.imshow(np.log10(1+np.abs(cov(container))))
plt.colorbar();
```

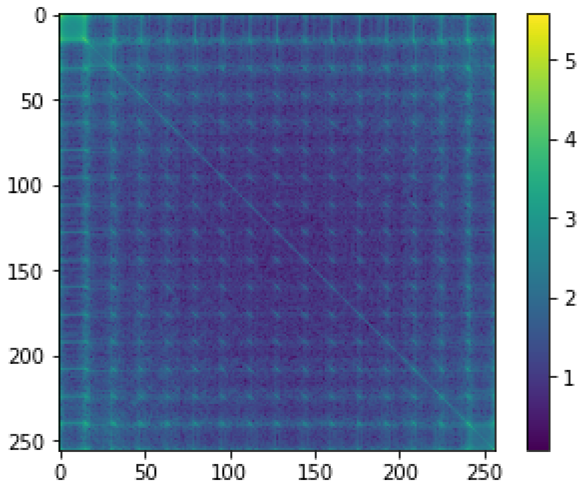
```
Diagonal Coefficients Value Ratio: 0.38328357729601065
```



DFT

In [ ]:
```
patch_size = 16
s1,_ = Lena.shape
num_horizontal = s1 // patch_size
num_vertical = num_horizontal
size = Lena.size
n_vectors = size // patch_size**2
step = s1 // patch_size
row_begin = 0
container = np.zeros((n_vectors,patch_size**2),dtype=complex)
counter = 0
for row in range(patch_size):
    for col in range(patch_size):
        row_idx1 = row*patch_size
        row_idx2 = row*patch_size+step
        col_idx1 = col*patch_size
        col_idx2 = col*patch_size+step
        patch = Lena[row_idx1:row_idx2,col_idx1:col_idx2]
        patch = fft2(patch) # This transform is added
        container[:,counter] = patch.flatten()
        counter += 1
diagonal_coef_ratio(cov(container))
plt.imshow(np.log10(1+np.abs(cov(container))))
plt.colorbar();
```

```
Diagonal Coefficients Value Ratio: 0.13403696211448685
```



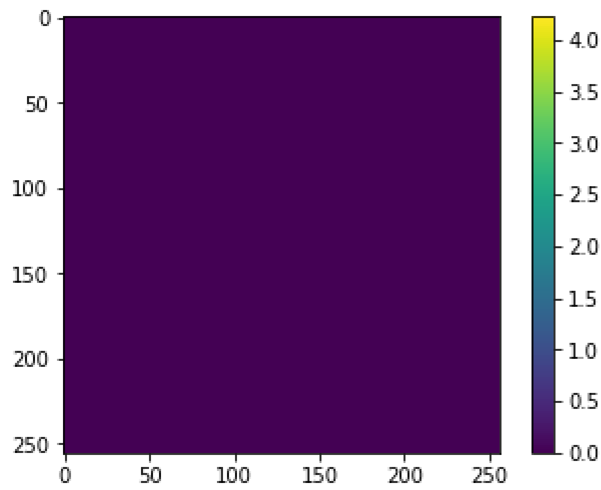
e

KLT

In [ ]:

```
patch_size = 16
s1,_ = Lena.shape
num_horizontal = s1 // patch_size
num_vertical = num_horizontal
size = Lena.size
n_vectors = size // patch_size**2
step = s1 // patch_size
row_begin = 0
container = np.zeros((n_vectors,patch_size**2),dtype=complex)
counter = 0
new_image = np.zeros_like(Lena)
for row in range(patch_size):
    for col in range(patch_size):
        row_idx1 = row*patch_size
        row_idx2 = row*patch_size+step
        col_idx1 = col*patch_size
        col_idx2 = col*patch_size+step
        patch = Lena[row_idx1:row_idx2,col_idx1:col_idx2]
        A = KLT_mtx(patch.flatten()) # This transform is added
        patch = A.dot(patch.flatten())
        container[:,counter] = patch.flatten()
        counter += 1
Cov_Lena = cov(container)
plt.imshow(np.log10(1+np.abs(Cov_Lena)))
diagonal_coef_ratio(Cov_Lena)
plt.colorbar();
```

```
Diagonal Coefficients Value Ratio: 0.9999998049779671
```



In [ ]:
```python
np.isclose(np.diag(Cov_Lena),0)
```

Out[ ]:
```
array([False,  True,  True,  True,  True,  True,  True,  True,  True,
        True,  True,  True,  True,  True,  True,  True,  True,  True,
        True,  True,  True,  True,  True,  True,  True,  True,  True,
        True,  True,  True,  True,  True,  True,  True,  True,  True,
        True,  True,  True,  True,  True,  True,  True,  True,  True,
        True,  True,  True,  True,  True,  True,  True,  True,  True,
        True,  True,  True,  True,  True,  True,  True,  True,  True,
        True,  True,  True,  True,  True,  True,  True,  True,  True,
        True,  True,  True,  True,  True,  True,  True,  True,  True,
        True,  True,  True,  True,  True,  True,  True,  True,  True,
        True,  True,  True,  True,  True,  True,  True,  True,  True,
        True,  True,  True,  True,  True,  True,  True,  True,  True,
        True,  True,  True,  True,  True,  True,  True,  True,  True,
        True,  True,  True,  True,  True,  True,  True,  True,  True,
        True,  True,  True,  True,  True,  True,  True,  True,  True,
        True,  True,  True,  True,  True,  True,  True,  True,  True,
        True,  True,  True,  True,  True,  True,  True,  True,  True,
        True,  True,  True,  True,  True,  True,  True,  True,  True,
        True,  True,  True,  True,  True,  True,  True,  True,  True,
        True,  True,  True,  True,  True,  True,  True,  True,  True,
        True,  True,  True,  True,  True,  True,  True,  True,  True,
        True,  True,  True,  True,  True,  True,  True,  True,  True,
        True,  True,  True,  True,  True,  True,  True,  True,  True,
        True,  True,  True,  True,  True,  True,  True,  True,  True,
        True,  True,  True,  True,  True,  True,  True,  True,  True,
        True,  True,  True,  True,  True,  True,  True,  True,  True,
        True,  True,  True,  True,  True,  True,  True,  True,  True,
        True,  True,  True,  True,  True,  True,  True,  True,  True,
        True,  True,  True,  True])
```

All of the transform performed well to decorrelate the image, they increased the fractional portion of diagonal coefficients in the covariance matrix. Although there is no remarkable difference 1D and 2D DFT, increasing the dimension in the DCT have significantly affected the energy compaction, from 0.13 to 0.38, which is around 200% increase.

1D KLT performed well in energy compaction, compacting 99% of the energy in the diagonal. In 2D KLT, it is observed that most of the energy is compacted in only one variable.

Q3

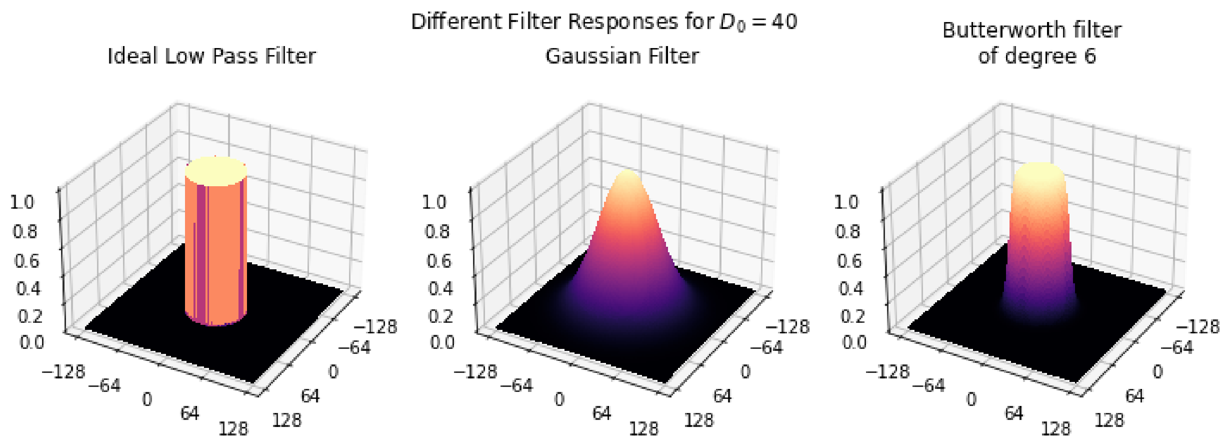
a

In [ ]:

```
maxsize = 128
minsize = -1*maxsize

X = np.arange(minsize, maxsize, 1)
Y = np.arange(minsize, maxsize, 1)
X, Y = np.meshgrid(X, Y)
D = np.sqrt(X**2+Y**2)
D0 = 40
N_butterworth = 6
sigma_gaussian = D0

H_ideal = (D<D0) * np.ones_like(D)
H_butterworth = 1 / (1 + (D/D0)**(2*N_butterworth))
H_gaussian = np.exp(-D*D/(2*sigma_gaussian**2))

# Plot the surface.
fig = plt.figure(figsize=plt.figaspect(.33))
ax = fig.add_subplot(1, 3, 1, projection='3d')
surf = ax.plot_surface(X, Y, H_ideal,rstride=1,
    cstride=1, cmap=cm.magma, linewidth=0, antialiased=False)
ax.set_zlim(0,1)
ax.view_init(30, 30)
ax.set_xticks(np.linspace(-128,128,5))
ax.set_yticks(np.linspace(-128,128,5))
plt.title("Ideal Low Pass Filter")

ax = fig.add_subplot(1, 3, 2, projection='3d')
surf = ax.plot_surface(X, Y, H_gaussian,rstride=1,
    cstride=1, cmap=cm.magma, linewidth=0, antialiased=False)
ax.set_zlim(0,1)
ax.view_init(30, 30)
ax.set_xticks(np.linspace(-128,128,5))
ax.set_yticks(np.linspace(-128,128,5))
plt.title("Gaussian Filter")

ax = fig.add_subplot(1, 3, 3, projection='3d')
surf = ax.plot_surface(X, Y, H_butterworth,rstride=1,
    cstride=1, cmap=cm.magma, linewidth=0, antialiased=False)
ax.set_zlim(0,1)
ax.view_init(30, 30)
ax.set_xticks(np.linspace(-128,128,5))
ax.set_yticks(np.linspace(-128,128,5))
plt.title("Butterworth filter \n of degree "+str(N_butterworth))

plt.suptitle(r"Different Filter Responses for $D_0 = 40$")
```

Out[ ]:

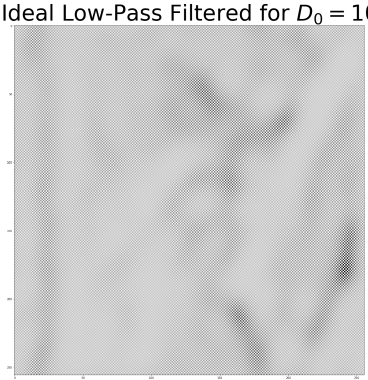
```
Text(0.5, 0.98, 'Different Filter Responses for $D_0 = 40$')
```



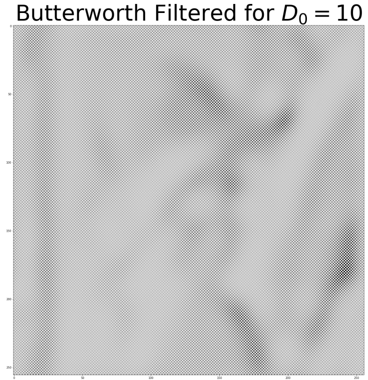
In [ ]:

```
fig = plt.figure(figsize=(80,80))
fontsize = 80
for i,D0 in enumerate([10,30,120]):
    N_butterworth = 6
    sigma_gaussian = D0
    H_ideal = (D<D0) * np.ones_like(D)
    H_butterworth = 1 / (1 + (D/D0)**(2*N_butterworth))
    H_gaussian = np.exp(-D*D/(2*sigma_gaussian**2))
    Lena_ideal = np.log10(1+np.real(ifft2(np.fft.fftshift(fft2(Lena))*H_ideal)))
    Lena_butterworth = np.log10(1+np.real(ifft2(np.fft.fftshift(fft2(Lena))*H_butterwo
    Lena_gaussian = np.log10(1+np.real(ifft2(np.fft.fftshift(fft2(Lena))*H_gaussian)))
    ax = fig.add_subplot(3, 3, (3*i+1))
    plt.imshow(Lena_ideal,cmap="gray")
    ax.set_title(r"Ideal Low-Pass Filtered for $D_0 = {}$".format(D0),fontsize=fontsiz

    ax = fig.add_subplot(3, 3, (3*i+2))
    plt.imshow(Lena_butterworth,cmap="gray")
    ax.set_title(r"Butterworth Filtered for $D_0 = {}$".format(D0),fontsize=fontsize)

    ax = fig.add_subplot(3, 3, (3*i+3))
    plt.imshow(Lena_gaussian,cmap="gray")
    ax.set_title(r"Gaussian Filtered for $D_0 = {}$".format(D0),fontsize=fontsize)
```

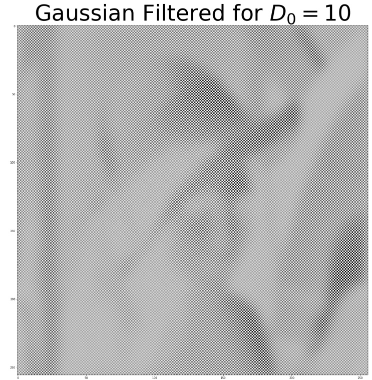
Ideal Low-Pass Filtered for $D_0 = 10$



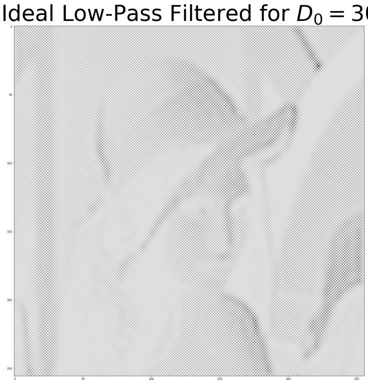
Butterworth Filtered for $D_0 = 10$



Gaussian Filtered for $D_0 = 10$



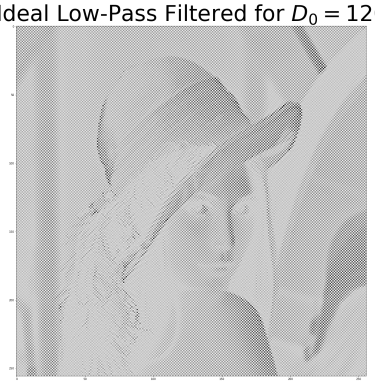
Ideal Low-Pass Filtered for $D_0 = 30$



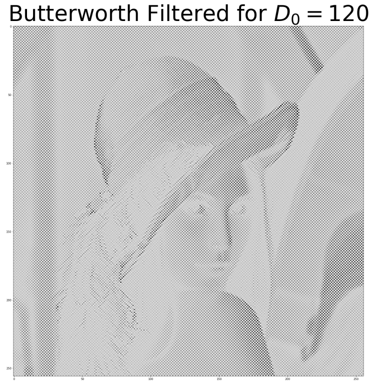
Butterworth Filtered for $D_0 = 30$

Gaussian Filtered for $D_0 = 30$

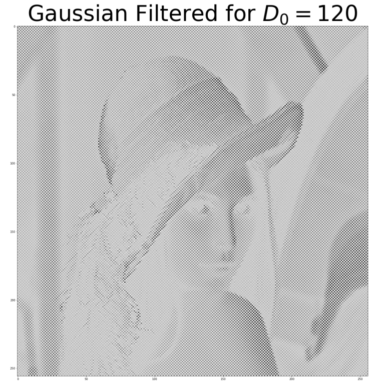
Ideal Low-Pass Filtered for $D_0 = 120$



Butterworth Filtered for $D_0 = 120$



Gaussian Filtered for $D_0 = 120$



Q4

a

In [ ]:
```
%pip install PyWavelets
import pywt
```

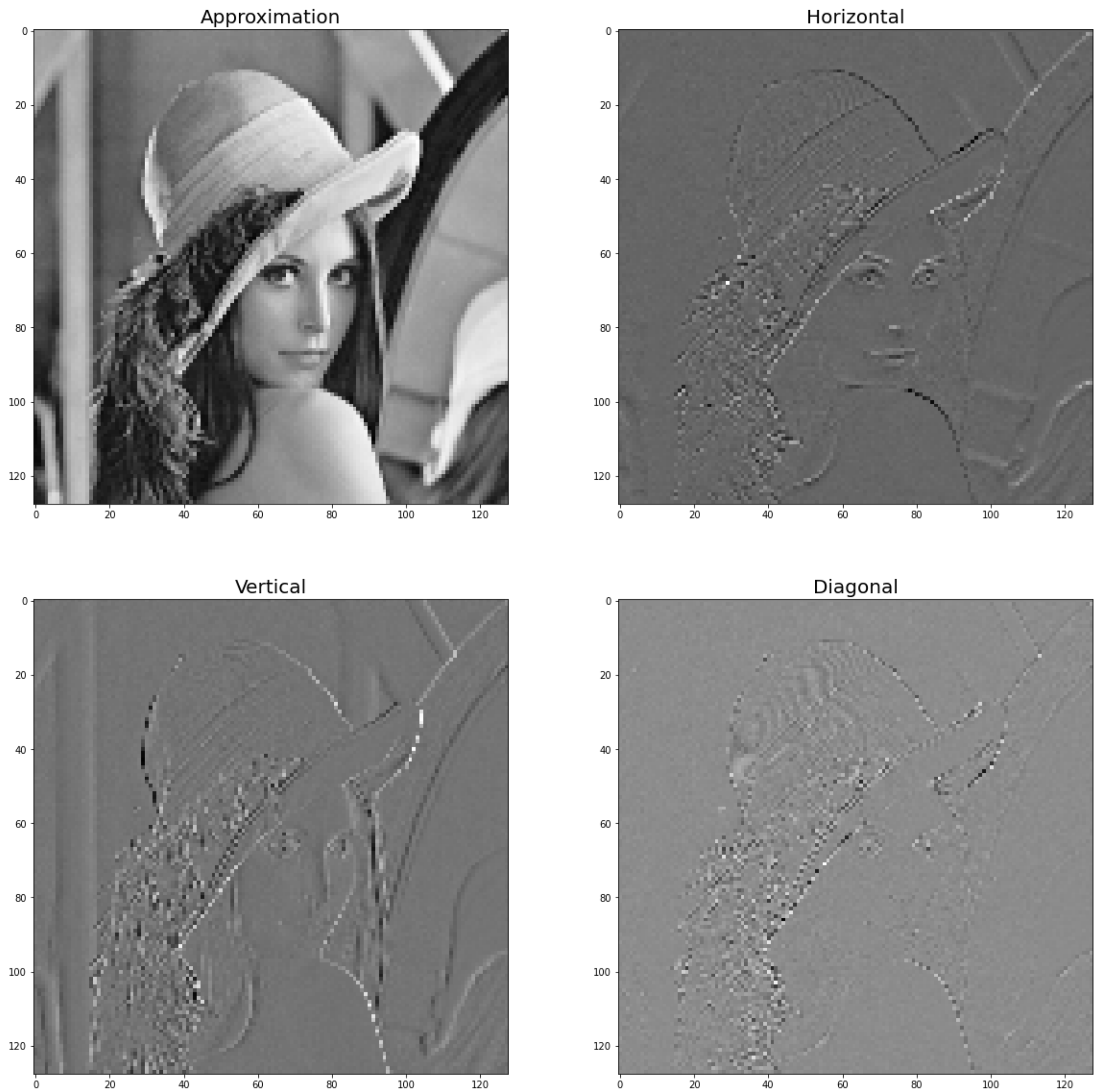
```
Requirement already satisfied: PyWavelets in c:\users\kutay\appdata\local\programs\py
thon\python39\lib\site-packages (1.3.0)
Requirement already satisfied: numpy>=1.17.3 in c:\users\kutay\appdata\local\programs
\python\python39\lib\site-packages (from PyWavelets) (1.19.5)
Note: you may need to restart the kernel to use updated packages.
WARNING: There was an error checking the latest version of pip.
```

In [ ]:
```
J1_coeffs = pywt.dwt2(Lena, "Haar", mode='symmetric', axes=(-2, -1))
cA, (cH, cV, cD) = J1_coeffs
plt.figure(figsize=(20,20))
```

```python
plt.subplot(2,2,1)
plt.imshow(cA,cmap="gray")
plt.title("Approximation",fontsize=20)
plt.subplot(2,2,2)
plt.imshow(cH,cmap="gray")
plt.title("Horizontal",fontsize=20)
plt.subplot(2,2,3)
plt.imshow(cV,cmap="gray")
plt.title("Vertical",fontsize=20)
plt.subplot(2,2,4)
plt.imshow(cD,cmap="gray")
plt.title("Diagonal",fontsize=20)
```

Out[ ]:

```
Text(0.5, 1.0, 'Diagonal')
```



In [ ]:

```python
J2_coeffs = pywt.wavedec2(Lena, "Haar", mode='symmetric', level=2, axes=(-2, -1))

(cA2, (cH2, cV2, cD2), (cH1, cV1, cD1)) = J2_coeffs
```

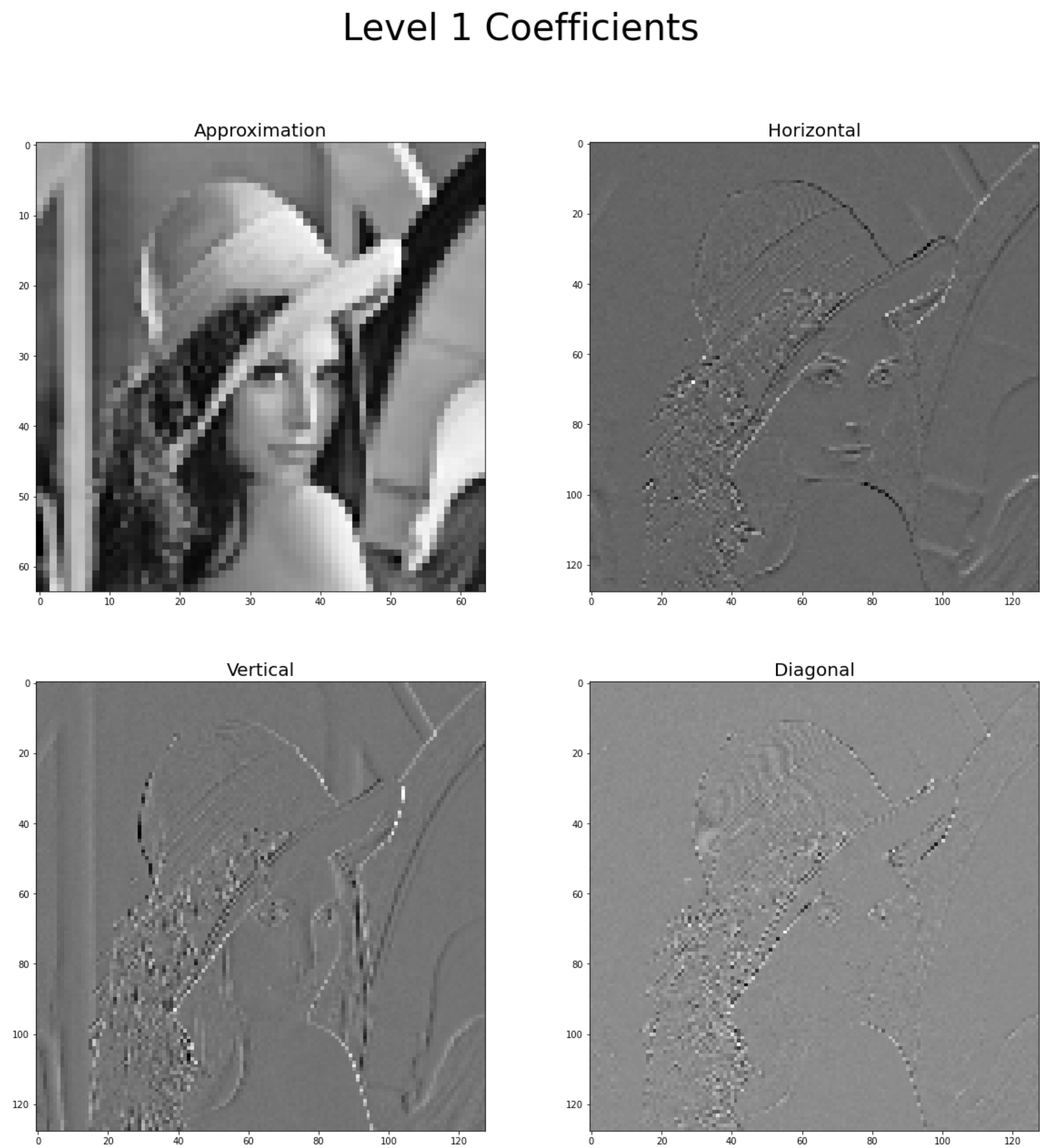
In [ ]:

```python
plt.figure(figsize=(20,20))
plt.subplot(2,2,1)
```

```
plt.imshow(cA2,cmap="gray")
plt.title("Approximation",fontsize=20)
plt.subplot(2,2,2)
plt.imshow(cH1,cmap="gray")
plt.title("Horizontal",fontsize=20)
plt.subplot(2,2,3)
plt.imshow(cV1,cmap="gray")
plt.title("Vertical",fontsize=20)
plt.subplot(2,2,4)
plt.imshow(cD1,cmap="gray")
plt.title("Diagonal",fontsize=20)
plt.suptitle("Level 1 Coefficients",fontsize=40)
```

Out[ ]:

```
Text(0.5, 0.98, 'Level 1 Coefficients')
```

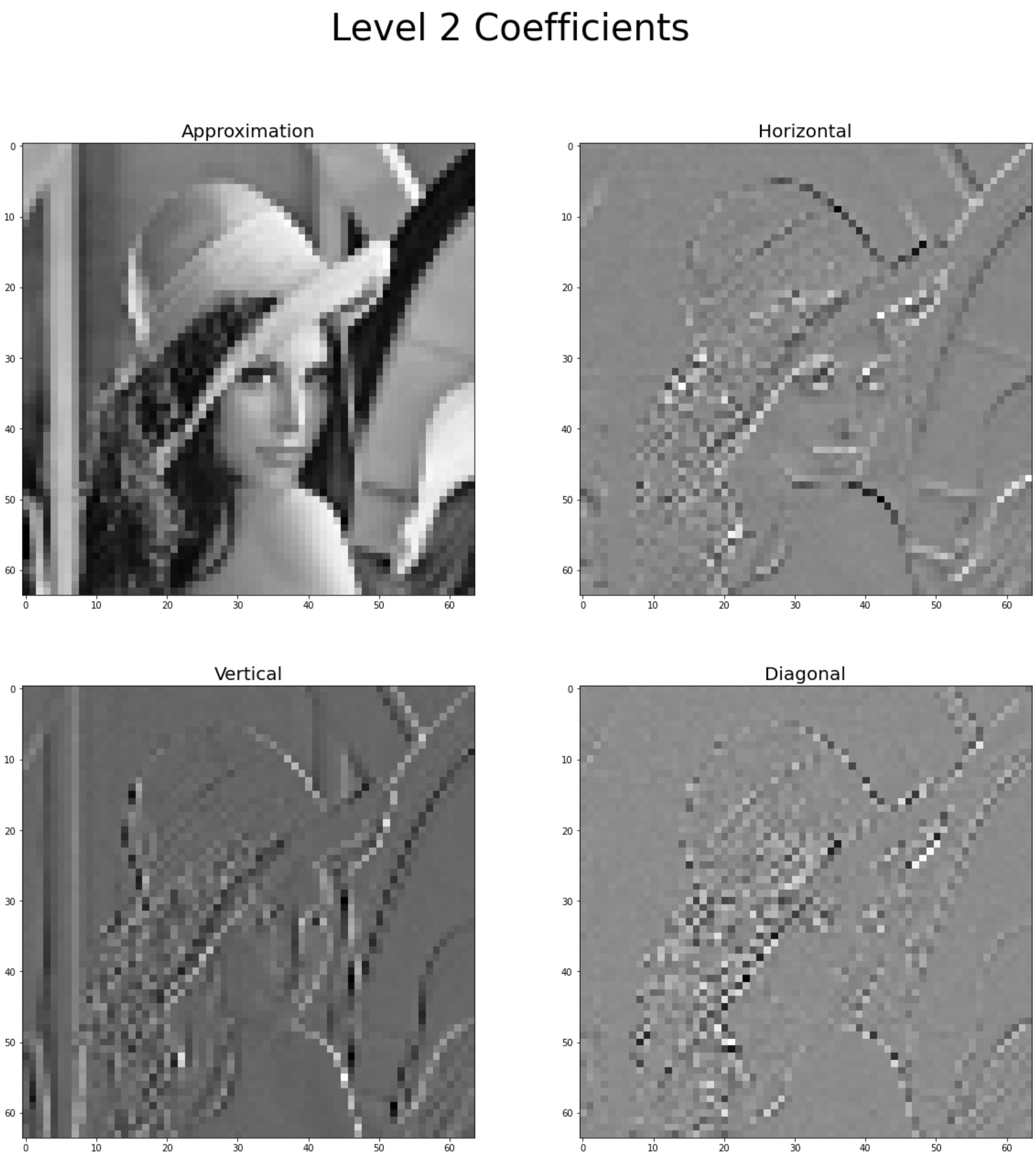


In [ ]:

```
plt.figure(figsize=(20,20))
plt.subplot(2,2,1)
```

```python
plt.imshow(cA2,cmap="gray")
plt.title("Approximation",fontsize=20)
plt.subplot(2,2,2)
plt.imshow(cH2,cmap="gray")
plt.title("Horizontal",fontsize=20)
plt.subplot(2,2,3)
plt.imshow(cV2,cmap="gray")
plt.title("Vertical",fontsize=20)
plt.subplot(2,2,4)
plt.imshow(cD2,cmap="gray")
plt.title("Diagonal",fontsize=20)
plt.suptitle("Level 2 Coefficients",fontsize=40)
```

Out[ ]:
```
Text(0.5, 0.98, 'Level 2 Coefficients')
```

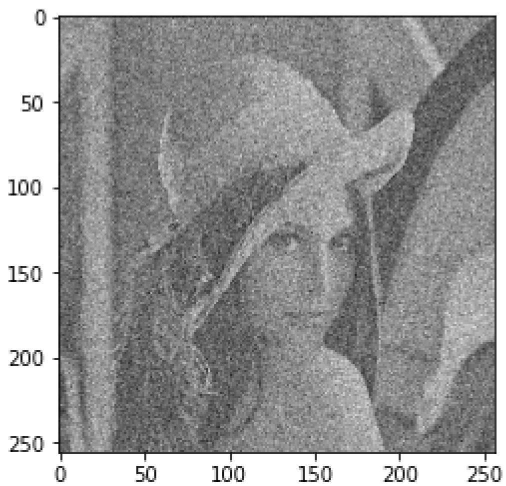


C

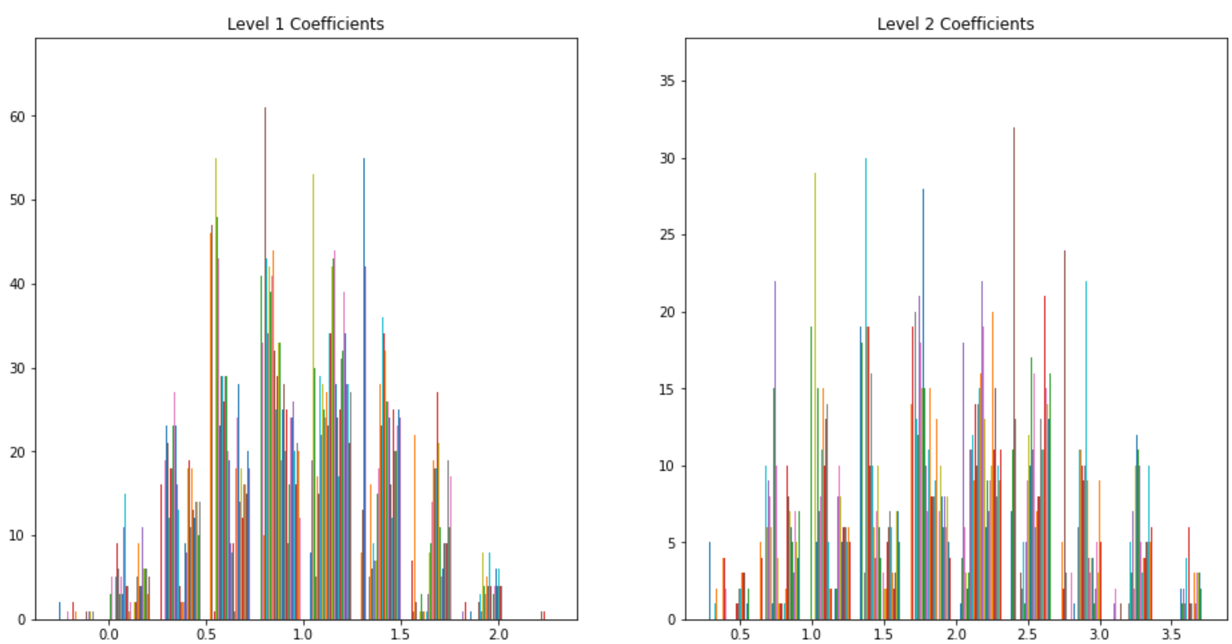
In [ ]:
```
s1,s2 = Lena.shape
Noisy_Lena = Lena + 0.2*np.random.randn(s1,s2)
```

In [ ]:
```
plt.imshow(Noisy_Lena,cmap="gray")
```

Out[ ]:
```
<matplotlib.image.AxesImage at 0x21b3ff26bb0>
```

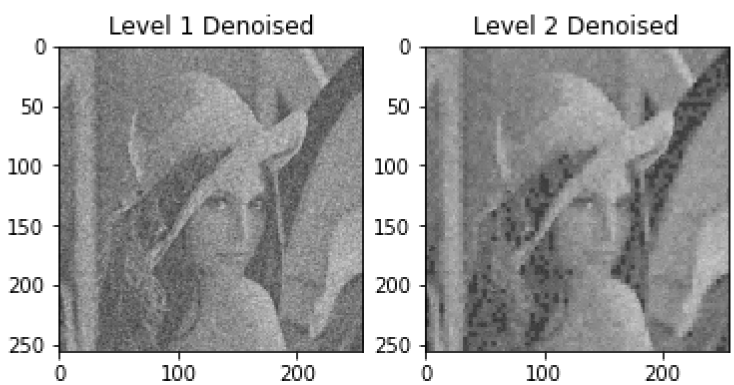


In [ ]:
```
plt.figure(figsize=(16,8))
J2_coeffs = pywt.wavedec2(Noisy_Lena, "Haar", mode='symmetric', level=2, axes=(-2, -1)
(cA2, (cH2, cV2, cD2), (cH1, cV1, cD1)) = J2_coeffs
J1_coeffs = pywt.dwt2(Noisy_Lena, "Haar", mode='symmetric', axes=(-2, -1))
cA, (cH, cV, cD) = J1_coeffs
plt.subplot(1,2,1)
plt.hist(cA);
# plt.xticks(np.arange(-1,3,0.25))
plt.title("Level 1 Coefficients")
plt.subplot(1,2,2)
plt.hist(cA2);
plt.title("Level 2 Coefficients");
```



```
In [ ]: threshold1 = 0.15; threshold2 = 0.8
cA_filtered = np.where(((cA>threshold1)),cA,np.zeros_like(cA))
cH_filtered = np.where(((cH>threshold1)),cH,np.zeros_like(cH))
cV_filtered = np.where(((cV>threshold1)),cV,np.zeros_like(cV))
cD_filtered = np.where(((cD>threshold1)),cD,np.zeros_like(cD))

cA2_filtered = np.where((cA2>threshold2) ,cA2,np.zeros_like(cA2))
cH2_filtered = np.where(((cH2>threshold2)),cH2,np.zeros_like(cH2))
cV2_filtered = np.where(((cV2>threshold2)),cV2,np.zeros_like(cV2))
cD2_filtered = np.where(((cD2>threshold2)),cD2,np.zeros_like(cD2))

J1_coeffs_filtered = (cA_filtered, (cH, cV, cD))
J2_coeffs_filtered = (cA2_filtered, (cH2_filtered, cV2_filtered, cD2_filtered),
                      (cH_filtered, cV_filtered, cD_filtered))
LenaBack_J1 = pywt.waverec2(J1_coeffs_filtered, "Haar", mode='symmetric', axes=(-2, -1
LenaBack_J2 = pywt.waverec2(J2_coeffs_filtered, "Haar", mode='symmetric', axes=(-2, -1
NNZ1 = np.sum([np.count_nonzero(item) for item in [cA_filtered,cV_filtered,cH_filtere
NNZ2 = np.sum([np.count_nonzero(item) for item in [cA_filtered,cV_filtered,cH_filtere
print("Number of nonzero elements:\n","NNZ1 = ",str(NNZ1),"| NNZ2 = ",str(NNZ2))

plt.subplot(1,2,1)
plt.imshow(LenaBack_J1,cmap="gray")
plt.title("Level 1 Denoised")
plt.subplot(1,2,2)
plt.imshow(LenaBack_J2,cmap="gray")
plt.title("Level 2 Denoised");
```

```
Number of nonzero elements:
 NNZ1 =  27440 | NNZ2 =  27462
```




```
In [ ]: from skimage.metrics import structural_similarity, peak_signal_noise_ratio
Lena_img = Lena.astype("float64")
ssim1 = structural_similarity(Lena_img,LenaBack_J1,);
psnr1 = peak_signal_noise_ratio(Lena_img,LenaBack_J1);
ssim2 = structural_similarity(Lena_img,LenaBack_J2);
psnr2 = peak_signal_noise_ratio(Lena_img,LenaBack_J2);
print("SSIM1 = ",str(ssim1),"| SSIM2 = ",str(ssim2))
print("PSNR1 = ",str(psnr1),"| PSNR2 = ",str(psnr2))
```

```
SSIM1 =  0.2523682469083722 | SSIM2 =  0.322883384073007O4
PSNR1 =  13.965766743870944 | PSNR2 =  16.560886472365937
```

With very close non-zero wavelet coefficients for 1 and 2 level DWTs, denoising via hard thresholding 2 level DWT transform performed better. This may be attributed to the fact that multilevel transform have better capability to decompose the low frequency content in finer scale. Hence, in both of the metrics(Peak Signal-to-Noise Ratio and Structural Similarity Index), multilevel transform exhibited more successful denoising.