<u>Show:</u> $\sum_x |f(x)|^2 = \sum_w |F(x)|^2$

<u>Proof:</u> $\sum_x |f(x)|^2$

$= \sum_x f(x) \cdot \overline{f(x)}$

complex conjugated: $|x|^2 = x \cdot \overline{x}$ with $x \in \mathbb{C}$

$= \sum_x \left(\frac{1}{\sqrt{N}} \sum_w F(w) e^{iwx}\right) \overline{\left(\frac{1}{\sqrt{N}} \sum_{w'} F(w') e^{iw'x}\right)}$

<u>Definition:</u> $f(x) = \frac{1}{\sqrt{N}} \sum_w F(w) e^{iwx}$

$= \sum_x \frac{1}{N} \sum_w F(w) e^{iwx} \sum_{w'} \overline{F(w')} e^{-iw'x}$

$= \frac{1}{N} \sum_w \sum_{w'} F(w) \overline{F(w')} \sum_x e^{i(w-w')x}$

$= \frac{1}{N} \sum_h \sum_{h'} F\left(\frac{2\pi h}{N}\right) \overline{F\left(\frac{2\pi h'}{N}\right)} \sum_x e^{i\left(\frac{2\pi h}{N} - \frac{2\pi h'}{N}\right)x}$

<u>Definition:</u> discrete frequencies $w = \frac{2\pi h}{N}$, $k = 0, \dots, N-1$ and $x = 0, \dots, N-1$

$= \frac{1}{N} \sum_{h=0}^{N-1} \sum_{h'=0}^{N-1} F\left(\frac{2\pi h}{N}\right) \overline{F\left(\frac{2\pi h'}{N}\right)} \underbrace{\sum_{x=0}^{N-1} e^{i2\pi(h-h')x/N}}$

<u>case 1:</u> $h = h'$

$\sum_{x=0}^{N-1} e^{i2\pi(h-h)x/N}$

$= \sum_{x=0}^{N-1} e^0$

$= \sum_{x=0}^{N-1} 1$

$= N$

<u>case 2:</u> $h \neq h'$

$\sum_{x=0}^{N-1} e^{i2\pi(h-h')x/N}$

$= \frac{1-\left(e^{i2\pi(h-h')/N}\right)^N}{1-e^{i2\pi(h-h')}}$

<u>geometric series sum:</u>

$r = e^{i2\pi(h-h')/N}$ $\quad \sum_{x=0}^{N-1} r^x = \frac{1-r^N}{1-r}$

$$= \frac{1 - e^{i2\pi(h-h')}}{1 - e^{i2\pi(h-h')/N}}$$

$$\underset{\uparrow}{=} \frac{1-1}{1 - e^{i2\pi(h-h')/N}} = 0$$

Euler Identity:

$$e^{i\theta} = \cos(\theta) + i \cdot \sin(\theta)$$

$$\theta = 2\pi(h-h') \quad \text{with} \quad n = (h-h') \in \mathbb{N}/\{0\}$$

$$\Rightarrow e^{in \cdot 2\pi} = \underbrace{\cos(n2\pi)}_{1} + \underbrace{i \cdot \sin(n2\pi)}_{=0} = 1$$

$\Rightarrow$ Combining both cases:

$$\sum_{x=0}^{N-1} e^{i(w_h - w_{h'})x}$$

$$= \begin{cases} N & \Rightarrow h = h' \\ 0 & \Rightarrow h \neq h' \end{cases}$$

$$= N \delta_{hh'}$$

Kronecker Delta

$$\delta_{h,h'} = \begin{cases} 1 & \Rightarrow h = h' \\ 0 & \Rightarrow h \neq h' \end{cases}$$

$$= \frac{1}{N} \sum_{h=0}^{N-1} \sum_{h'=0}^{N-1} F\left(\frac{2\pi h}{N}\right) \overline{F\left(\frac{2\pi h'}{N}\right)} N \cdot \delta_{h,h'}$$

$$= \underbrace{\frac{N}{N}}_{=1} \underbrace{\sum_{h=0}^{N-1} \sum_{h'=0}^{N-1} F_h \overline{F_{h'}} \delta_{h,h'}}$$

$$= \begin{cases} F_h F_{h'} & \Rightarrow h = h' \\ 0 & \Rightarrow h \neq h' \end{cases}$$

$$= \sum_{h=0}^{N-1} F_h \overline{F_h}$$

$$= \sum_w F_w \overline{F_w}$$

$$= \sum_w |F(w)|^2 \quad \square$$