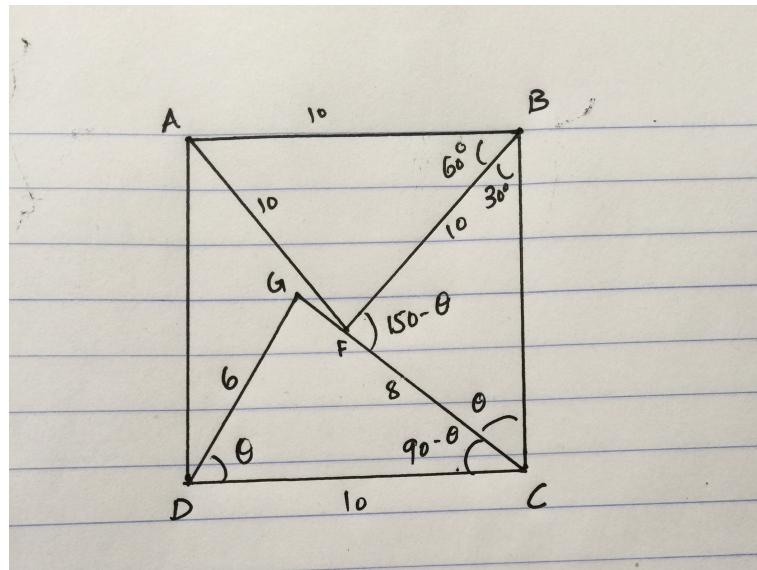


LSU Math Contest:Team 2019

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A. DeMorgan: "The moving power of mathematical invention is not reasoning but imagination "

1. Problem 1



Solution: Here:

$AB = DC = AF = FB = 10$, Property of rectangle and equilateral triangle.

$\angle DGF = 90^\circ$, let $\angle GDC = \theta$, then $\tan(\theta) = \frac{8}{6}$.

To find AD we find BC using sine law in $\triangle FBC$:

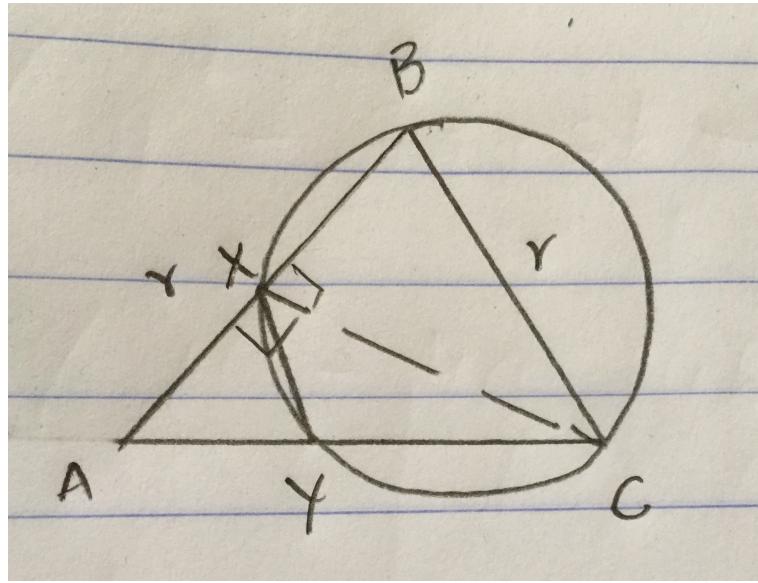
$$\frac{10}{\sin(\theta)} = \frac{BC}{\sin(150 - \theta)}$$

$$\text{So, } BC = \frac{10}{\sin(\theta)} \sin(150 - \theta) = 10 \left(\frac{\sin 150 \cos \theta - \cos 150 \sin \theta}{\sin \theta} \right) = 10 \left(\sin 30 \frac{\cos \theta}{\sin \theta} + \cos 30 \right)$$

$$BC = 10 \left(\frac{1}{2} \frac{6}{8} + \frac{\sqrt{3}}{2} \right) = \frac{15}{4} + 5\sqrt{3}$$

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2. Problem 2



Solution: There may be easier way to solve this problem using theorems in geometry. I will use brute force method here.

Construct a line joining XC so that $\angle BXC = 90$.

Using cosine law in $\triangle ABC$:

$$\cos A = \frac{r^2 + s^2 - r^2}{2rs} = \frac{s}{2r}$$

$$\cos 2A = 2\cos^2 A - 1 = 2\left(\frac{s}{2r}\right)^2 - 1 = \frac{s^2 - 2r^2}{2r^2}$$

Using cosine law in $\triangle AXC$:

$$[XY]^2 = [AX]^2 + [AY]^2 - 2[AX][AY] \cos A$$

$$\begin{aligned} [XY]^2 &= (r - BX)^2 + (s - CY)^2 - 2(r - BX)(s - CY) \cos A \\ &= \left(r + r \cos(\pi - 2A)\right)^2 + (s - r \cos A)^2 - 2\left(r - r \cos(\pi - 2A)\right)(s - r \cos A) \cos A \\ &= (r + r \cos 2A)^2 + (s - r \cos A)^2 - 2(r + r \cos 2A)(s - r \cos A) \cos A \\ &= \left(r + r\left(\frac{s^2 - 2r^2}{2r^2}\right)\right)^2 + (s - r\frac{s}{2r})^2 - 2\left(r + r\left(\frac{s^2 - 2r^2}{2r^2}\right)\right)(s - r\frac{s}{2r})\frac{s}{2r} \end{aligned}$$

$$= r^2\left(\frac{s}{2r}\right)^2 + \left(\frac{s}{2}\right)^2 - 2r\frac{s^2}{2r^2}\frac{s}{2}\frac{s}{2r}$$

$$= \frac{s^4}{4r^2} + \frac{s^2}{4} - \frac{s^4}{4r^2}$$

$$= \frac{s^2}{4}$$

$$\text{i.e. } [XY] = \frac{s}{2}$$

3. Problem 3

Solution: Note that $\frac{1}{7} = 0.142857$

We will use basic induction here. Given $a_k = \left\lfloor \frac{10^k}{7} \right\rfloor - 10 \left\lfloor \frac{10^{k-1}}{7} \right\rfloor$

$$a_1 = \left\lfloor \frac{10^1}{7} \right\rfloor - 10 \left\lfloor \frac{10^{1-1}}{7} \right\rfloor = 1 - 0 = 1$$

$$a_2 = \left\lfloor \frac{10^2}{7} \right\rfloor - 10 \left\lfloor \frac{10^{2-1}}{7} \right\rfloor = 14 - 10 = 4$$

$$a_3 = \left\lfloor \frac{10^3}{7} \right\rfloor - 10 \left\lfloor \frac{10^{3-1}}{7} \right\rfloor = 142 - 140 = 2$$

$$a_4 = \left\lfloor \frac{10^4}{7} \right\rfloor - 10 \left\lfloor \frac{10^{3-1}}{7} \right\rfloor = 1428 - 1420 = 8$$

$$a_5 = \left\lfloor \frac{10^5}{7} \right\rfloor - 10 \left\lfloor \frac{10^{4-1}}{7} \right\rfloor = 14285 - 14280 = 5$$

$$a_6 = \left\lfloor \frac{10^6}{7} \right\rfloor - 10 \left\lfloor \frac{10^{5-1}}{7} \right\rfloor = 142857 - 142850 = 7$$

$$\text{i. e } a_{2019} = a_{2019 \pmod 6} = a_3 = 2$$

4. Problem 4: How many numbers between 7,000,000 and 8,000,000 are there for which the millions digits equals the sum of the other six digits.

Solution: This is a standard counting (combinatorics) and number theory problem.

There are six positions and these all position has to add to 7. Consider the following equations:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 7; x_i \geq 0 \text{ for } i \in 1 \text{ to } 6$$

One solution of the equation is $x_1 = 3, x_2 = 3, x_3 = 1, x_4 = x_5 = x_6 = 0$. A possible interpretation for this solution is that we are distributing seven pennies (identical objects) among six children (distinct containers), and here we have given three pennies to the first children, and one to the third child and nothing to the last three children. Continuing with this interpretation, we see that each nonnegative integer solution of the equation corresponds to a selection with repetition of size 7 (identical pennies) from a collection of size 6 (the distinct children), so there are

$$C(6 + 7 - 1, 7) = C(12, 7) = C(12, 5) = 792$$

5. Problem 5: A certain positive integer is 12 times the sum of its digits (when written in base 10). What is the number?

Solution: This number can not be an one digit number. Assume it is a two digit number z with digits $z = ab; a, b \in 0 \text{ to } 9$, , then the following relation should hold:

$$12a + 12b = z = 10a + b$$

$$2a = 11b$$

This diophantine equation does not have solution in the given range of a and b .

Now Assume its a three digit number. The following relation should hold:

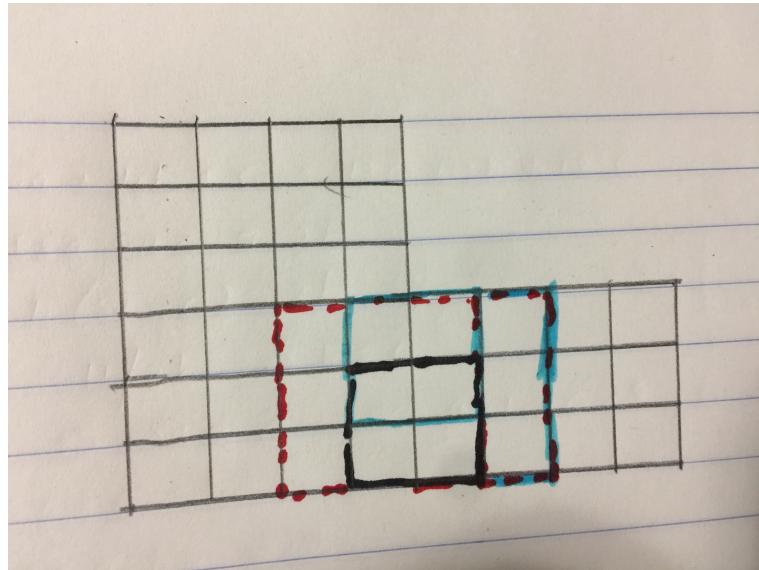
$$100a + 10b + c = 12a + 12b + 12c$$

$$88a - 11c = 2b$$

$$11(8a - c) = 2b$$

Since 11 and 2 are both prime , 11 divides b by divisibility rules. Now b = 0 because b is an integer from 0 to 9 only. This implies $8a - c = 0$. Plugging a = 1 and c = 8, we have the number. so the three digit number is 108.

6. Problem 6:



Solution:

Since this is a small square, we will use brute force method.

Number of 1-by-1 squares : $6 * 4 + 4 * 3 = 36$

Number of 2-by-2 squares : $(4 - 1)(6 - 1) + (4 - 1)(3 - 2) + 2 = 23$ The last two extra squares arises from sharing border.

Number of 3-by-3 squares : $(4 - 2)(6 - 2) + (4 - 2)(3 - 2) + 2 = 12$ The last two extra squares arises from sharing border.

Number of 4 by 4- squares : $(4 - 3)(6 - 3) = 3$. There is no 4-by-4 square on second half. Total squares = $36 + 23 + 12 + 3 = 74$ squares.

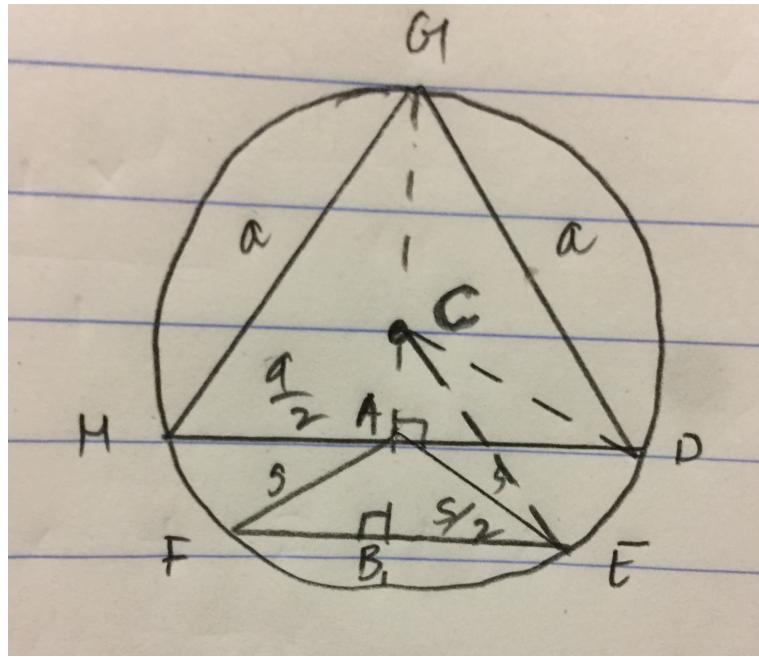
7. Problem 7:

Solution: Construct a straight line GAB passing through the center of a circle C . Construct radius $CD = CE = r$. Let this circle be unit circle so, $r = 1$. Let a be the side of equilateral triangle inscribed inside the circle and s be the side of another equilateral triangle.

Using trigonometry $r = a \frac{\sqrt{3}}{3}$ implies $a = \sqrt{3}$.

From the figure :

$HA = AD = \frac{a}{2}$, line passing through center of circle divides the chord into two halves.



$FB = BE = \frac{s}{2}$, same reason above.

Using trigonometry $AB = s\frac{\sqrt{3}}{2}$

Using figure again:

$$AB = CB - CA$$

$$s\frac{\sqrt{3}}{2} = \sqrt{1 - \frac{s^2}{4}} - \sqrt{1 - \frac{a^2}{4}}$$

$$s\frac{\sqrt{3}}{2} = \sqrt{1 - \frac{s^2}{4}} - \sqrt{1 - \frac{3}{4}}$$

$$s\frac{\sqrt{3}}{2} = \sqrt{1 - \frac{s^2}{4}} - \sqrt{12}$$

$$s\sqrt{3} + 1 = \sqrt{4 - s^2}$$

$$3s^2 + 2\sqrt{3}s + 1 = 4 - s^2$$

$$4s^2 + 2\sqrt{3}s - 3 = 0$$

Using quadratic formula and discarding negative value :

$$s = \sqrt{3} \left(\frac{\sqrt{5}-1}{4} \right)$$

Now the ratio of area of two equilateral triangle is given by :

$$\text{ratio} = \frac{HD \cdot GA}{FE \cdot AB}$$

$$= \frac{a \cdot a \frac{\sqrt{3}}{2}}{s \cdot s \frac{\sqrt{3}}{2}}$$

$$= \frac{a^2}{s^2}$$

$$= \frac{3}{3 \left(\frac{\sqrt{5}-1}{4} \right)^2}$$

$$= \frac{16}{6-2\sqrt{5}}$$

Rationalizing we get:

$$= 2(3 + \sqrt{5})$$

8. Problem 8: This is a number theory problem, an example of continued fractions.

$$? = \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{2 + \cfrac{1}{4 + \cdots}}}}}}}}$$

$$\text{Let } x = 2 + \cfrac{1}{4 + \cdots}$$

$$\text{Then you can write } x = 2 + \cfrac{1}{4 + \cfrac{1}{x}}.$$

Solution: $x = 2 + \cfrac{x}{4x+1}$

$$x = \cfrac{8x+2+x}{4x+1}$$

$$4x^2 + x = 8x + 2 + x$$

$$2x^2 - 4x - 1 = 0$$

Using the quadratic equation and discarding the negative answer, we get: $x = 1 + \sqrt{\frac{3}{2}}$

Now this continuous fraction is convergent to :

$$\cfrac{1}{1+\sqrt{\frac{3}{2}}}$$

rationalizing the denominator we get:

$$\sqrt{6} - 2$$

9. Problem 9: What does it mean to be a linear factor?

Solution:

Given $0 < a < b < c$ and let $p(x) = x(x - a)(x - b) - 2019$. $x - c$ is a linear factor of $p(x)$.

$$\text{i.e } p(c) = c(c - a)(c - b) - 2019 = 0$$

$$c - b < c - a < c \text{ because } 0 < a < b < c$$

$$c(c - a)(c - b) = 2019 = 673 * 3 * 1$$

$$\text{so } c = 673, c - b = 1 \text{ and } c - a = 3$$

$$\text{so } b = 672 \text{ and } a = 670$$

$$\text{Then } a + b + c = 673 + 672 + 670 = 2015.$$

10. A certain number M has a 3 as its unit digit when written in the decimal expansion. A new number N is formed from M by moving the unit digit to the left most position. For example, if $M = 213$ then $N = 321$. What is the smallest number M such that $N = 3M$?

Solution: This is an example of n - parasitic number. Let $M = a_n a_{n-1} \dots a_1 3$, a_i are digits of M . Then $N = 3a_n \dots a_1$. Since $N = 3M$

$$\begin{array}{r} a_n a_{n-1} \dots a_1 3 \\ \times 3 \\ \hline \end{array}$$

$$3 \times 3 = 9 = a_1$$

$$3 \times 9 = 27, \text{ so } a_2 = 7$$

$3 \times 7 = 21$, add $21 + 2 = 23$, this 2 is from the 27 above, so $a_3 = 3$

$$3 \times 3 = 9 \text{ add } 9 + 2 = 11, \text{ so } a_4 = 1$$

$$3 * 3 = 9$$

$$3 * 93 = 279$$

$$3 * 793 = 2379$$

$$3 * 3793 = 11379$$

$3 * 13793 = 41379$ We keep on doing this until it starts to repeat itself.