

RIVER PARISHES COMMUNITY COLLEGE

MATH 1100: COLLEGE ALGEBRA

EQUATIONS AND INEQUALITIES

2.6 Other types of Equations

Semester
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Department
Physical Science: MATH

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Learning Objectives

In this section, you will learn:

- ♣ Solve equations involving rational exponents.
- ♣ Solve radical equations.
- ♣ Solve absolute value equations.
- ♣ Solve equations using factoring.
- ♣ Solve other types of equations.

1 Radical Exponents

Radical are often called roots. They are the tools to undo exponentiation. For example if a square of number is 4, then undoing this , i.e. taking the square root of 4, should give us our number 2 back. Another example:

$$4^3 = 64, \text{ so, } \sqrt[3]{64} = 4$$

More Symbols:

forth root is written as $\sqrt[4]{}$

fifth root is written as $\sqrt[5]{}$

n-th root is written as $\sqrt[n]{}$

Properties :

For integer $n > 1$

$$(ab)^n = a^n b^n$$

$$\sqrt[n]{a^n} = a$$

$$\sqrt[n]{ab} = \sqrt[n]{a} \sqrt[n]{b}$$

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

A rational exponent indicates a power in the numerator and a root in the denominator. There are multiple ways of writing an expression, a variable, or a number with a rational exponent:

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = \left(a^m\right)^{\frac{1}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

Example:

$$2^{\frac{3}{5}} = \left(2^{\frac{1}{5}}\right)^3 = \left(2^3\right)^{\frac{1}{5}} = \left(8\right)^{\frac{1}{5}} = \sqrt[5]{8} = (\sqrt[5]{2})^3$$

1.1 Solving equations involving rational exponents

The solutions of the radical equations

$$x^{\frac{m}{n}} = a$$

are as follows:

Case 1: **If m is an odd integer:**

There is only one solution

$$x = a^{\frac{n}{m}}$$

Case 2: **If m is an even integer:**

There are two solutions

$$x = \pm a^{\frac{n}{m}}$$

Examples:

Example 1 : Solve the following equation. If needed, Write your answer as a fraction reduced to lowest terms.

$$z^{3/2} - 27 = 0$$

Solution:

Isolate the expression with the radical exponent.

$$z^{\frac{3}{2}} - 27 + 27 = 0 + 27$$

$$z^{\frac{3}{2}} = 27$$

Raise the both sides of the equation to the $\frac{2}{3}$ power.

$$z^{\frac{3}{2} \cdot \frac{2}{3}} = 3^{3 \cdot \frac{2}{3}}$$

$$z = 3^2 = 9.$$

Example 2 : Solve the following equation. If needed, submit your answer as a fraction reduced to lowest terms.

$$z^{\frac{2}{3}} - \frac{49}{64} = 0$$

Solution:

Begin by isolating the radical terms, ie. add $\frac{49}{64}$ on both sides;

$$z^{\frac{2}{3}} = \frac{49}{64}$$

Raise both sides to power $\frac{3}{2}$ to get:

$$z = \pm \left(\frac{49}{64}\right)^{\frac{3}{2}}$$

$$z = \pm \left(\frac{7^2}{8^2}\right)^{\frac{3}{2}}$$

$$z = \pm \left(\frac{7}{8}\right)^3$$

Example 3 : Solve the following equation:

$$\sqrt[3]{(x-2)^2} = 4.$$

Solution:

Rewrite the equation as : $(x-2)^{\frac{2}{3}} = 4$

$$x-2 = \pm 4^{\frac{3}{2}} = \pm \sqrt[2]{4^3} = \pm 8$$

$$x = 2 \pm 8$$

$$x = 10, -6$$

Example 4 : Solve the equation $3x^{\frac{3}{4}} = x^{\frac{1}{2}}$

Solution:

Rewrite the equation as $3x^{\frac{3}{4}} - x^{\frac{1}{2}} = 0$

since $x^{\frac{1}{2}}$ is common in both of them, lets take it out and we get:

$$x^{\frac{1}{2}}(3x^{\frac{3}{4} - \frac{1}{2}} - 1) = 0$$

$$x^{\frac{1}{2}}(3x^{\frac{3}{4} - \frac{2}{4}} - 1) = 0$$

$$x^{\frac{1}{2}}(3x^{\frac{3-2}{4}} - 1) = 0$$

$$x^{\frac{1}{2}}(3x^{\frac{1}{4}} - 1) = 0$$

using the zero factor property,

$$x^{\frac{1}{2}} = 0, (3x^{\frac{1}{4}} - 1) = 0$$

$$x^{\frac{1}{2}} = 0 \text{ gives } x = 0$$

Now we have $(3x^{\frac{1}{4}} - 1) = 0$

$$3x^{\frac{1}{4}} = 1$$

$$x^{\frac{1}{4}} = \frac{1}{3}$$

$$x = \left(\frac{1}{3}\right)^4$$

$$x = \frac{1}{81}$$

Thus we have $x = 0$ and $\frac{1}{81}$.

2 Solving Radical Equations

Extraneous Solution: While dealing with radical equations, some of the answers you get, may not satisfy the original equation and hence called extraneous solutions.

step 1 Consider the equation

$$\sqrt{x+4} = x-2$$

First, let's solve it using usual algebra:

step 2 : Square both sides:

$$\sqrt{x+4}^2 = (x-2)^2$$

$$x+4 = x^2 - 4x + 4$$

step 3 : Simplify :

$$x^2 - 5x = 0$$

$$x(x-5) = 0$$

step 4 : zero Property

$$x = 0, 5 \text{ are the two solutions.}$$

Now let's plug each of them back into our original equation and see if they work:

For $x = 0$

$$\sqrt{0+4} = 0-2 \text{ will give } 2 = -2 \text{ which is not true. So, } 0 \text{ can not be the solution}$$

For $x = 5$

$$\sqrt{5+4} = 5-2 \text{ will give } 3 = 3 \text{ which is true. Hence } 5 \text{ is the solution}$$

General Solution Steps

- Step 1. Isolate the Radical(s) and identify the index (n).
- Step 2. Raise both sides of the equation to the "nth" power.
- Step 3. Use algebraic techniques (i.e. factoring, combining like terms,...) to isolate the variable.
- Repeat Steps 1 and 2 if necessary.
- Step 4. Check answers. Eliminate any extraneous solutions from the final answer.

Example 5 : Solve the following radical equation. $\sqrt{2y+6} + 4 = y + 3$

Solution:

Begin by isolating the radical expression on one side of the equation.

$$\sqrt{2y+6} + 4 - 4 = y + 3 - 4$$

$$\sqrt{2y+6} = y - 1$$

Raise both sides of the equation to squares to undo the isolated radical.

$$2y + 6 = (y - 1)^2$$

$$2y + 6 = y^2 - 2y + 1; \text{ Foil the right side.}$$

$$2y - 2y + 6 - 6 = y^2 - 2y - 2y + 1 - 6.$$

$$0 = y^2 - 4y - 5.$$

$$0 = (y - 5)(y + 1).$$

Set the both factors to zero and solve for y .

$$y = 5, -1.$$

Check your solutions in the original equation.

When $y = -1$, we get;

$$\sqrt{2 * (-1) + 6} + 4 = -1 + 3$$

$$\sqrt{4} + 4 = 2.$$

$$2 + 4 = 2.$$

$$6 = 2.$$

Since $6 \neq 2$, $y = -1$ is not the solution.

When $y = 5$, we get;

$$\sqrt{2 * 5 + 6} + 4 = 5 + 3$$

$$\sqrt{16} + 4 = 8.$$

$$4 + 4 = 8.$$

$$8 = 8.$$

$y = 5$. satisfies the equation.

Example 6 : Solve the following radical equation. If needed, write your answer as a fraction reduced to lowest terms. $\sqrt[3]{7y+16} - \sqrt[3]{19-5y} = 0$

Solution:

Begin by isolating the radical

$$\sqrt[3]{7y+16} = \sqrt[3]{19-5y}$$

Raise both sides to third power.

$$7y + 16 = 19 - 5y.$$

$$7y + 5y = 19 - 16$$

$$12y = 3.$$

$$y = \frac{3}{12}.$$

$$y = \frac{1}{4}.$$

Example 7 : Solve the following radical equation. If needed, write your answer as a fraction reduced to lowest terms. $\sqrt[3]{6x-5} = 5$.

Solution:

Since the radical term is already isolated, we begin by raising both sides to power third.

$$6x - 5 = 5^3.$$

$$6x - 5 = 125.$$

$$6x = 130.$$

$$x = \frac{130}{6} = \frac{65}{2}.$$

Example 8 : Solve the following radical equation. $\sqrt[3]{5y^2+4y} = 4$.

Solution:

Begin by raising both sides to power third;

$$5y^2 + 4y = 4^3.$$

$$5y^2 + 4y = 64.$$

$$5y^2 + 4y - 64 = 0.$$

No we use quadratic equation $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$$a = 5, b = 4, c = -64.$$

$$y = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 5 \cdot (-64)}}{2 \cdot 5}.$$

$$y = \frac{-4 \pm \sqrt{16 + 1280}}{10}.$$

$$y = \frac{-4 \pm \sqrt{1296}}{10}.$$

$$y = \frac{-4 \pm 36}{10}.$$

$$\text{Either, } y = \frac{-4 + 36}{10} = \frac{32}{10} = \frac{16}{5}.$$

or

$$y = \frac{-4 - 36}{10} = -4.$$

Check both $y = -4, \frac{16}{5}$, back into original equation, so both solution works this time.

Example 9 : Solve the following radical equation $\sqrt{x - 6} - \sqrt{x + 9} + 3 = 0$

Solution:

Begin by isolation on of the radical here:

$$\sqrt{x - 6} = \sqrt{x + 9} - 3$$

Square both sides

$$x - 6 = x + 9 - 6\sqrt{x + 9} + 9$$

Use algebra to simplify and combine like terms:

$$-24 = -6\sqrt{x + 9}$$

$$\frac{-24}{-6} = \sqrt{x + 9}$$

$$4 = \sqrt{x + 9}$$

$$\text{Square both sides} \mid 16 = x + 9$$

$$16 - 9 = x$$

$$x = 7$$

Example 10 : Solve the following equation : $\sqrt{z^2} = -1$

Solution:

There is no solution for this function because square root of any real number is always positive number.

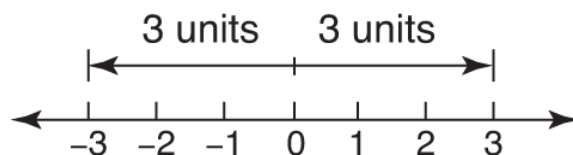
3 Absolute value equations

Absolute value function is denoted by symbol $|x|$. It is a function which takes the entire real number line (both positive and negative) as a input and returns only positive number or zero as an output. Mathematically,

$$|x| = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$$

If $x < 0$, it is a negative number, so $-x$ will become a positive number. If $x > 0$, it is already a positive number.

Geometrically, $|x| = a$ represents a distance x is away from the origin. For example $|x| = 3$ represents a distance of 3 units away from the origin. If you go right hand side from origin, then $x = 3$ and if you go 3 distance left from origin, you will reach $x = -3$.



3.1 solving Absolute Value Equations

Example 11 Solve $|x| = 2$.

Solution:

$$x = 2, -2$$

Example 12 Solve $|x| = -2$

Solution:

No Solution because absolute value is a distance which is never negative

Example 13 Solve $|x - 2| = 3$.

Solution:

Note that this is asking you to find the points which are 3 units away from point 2 in the number line.

If you go three units right of 2, you will get point 5 and if you go three units left of 2, you will get -1 .

Hence two solutions are: $x = -1, 5$

Example 14 $|3z - 2| = 23$

You begin by making it into two separate equations and then solving them separately.

$$3z - 2 = 23$$

$$3z - 2 + 2 = 23 + 2 \text{ ;add 2 on both sides.}$$

$$3z = 25$$

$$\frac{3z}{3} = \frac{25}{3} \text{ ; divide both sides by 3}$$

$$z = \frac{25}{3}$$

$$\text{or } 3z - 2 = -23$$

$$3z - 2 + 2 = -23 + 2 \text{ ;add 2 on both sides.}$$

$$3z = -21$$

$$\frac{3z}{3} = \frac{-21}{3} \text{ ; divide both sides by 3}$$

$$z = -7$$

Example 15 $|3y + 3| - 11 = 0$

Before we solve two equations for absolute value; it is easier if we separate $|3y + 3|$ from 11 first.

$$|3y + 3| - 11 + 11 = 0 + 11 \text{ ; add 11 on both sides.}$$

$$|3y + 3| = 11$$

You begin by making it into two separate equations and then solving them separately.

$$3y + 3 = 11$$

$$3y + 3 - 3 = 11 - 3 \text{ ; subtract 3 on both sides.}$$

$$3y = 8$$

$$\frac{3y}{3} = \frac{8}{3} \text{ ; divide both sides by 3}$$

$$y = \frac{8}{3}$$

or

$$3y + 3 = -11$$

$$3y + 3 - 3 = -11 - 3 \text{ ; subtract 3 on both sides.}$$

$$\begin{aligned}
 3y &= -14 \\
 \frac{3x}{3} &= \frac{-14}{3}; \text{ divide both sides by 3} \\
 x &= \frac{-14}{3}
 \end{aligned}$$

Example 16 $|3x - 9| = 0$

You begin by making it into two separate equations and then solving them separately.

$$3x - 9 = 0$$

$$3x - 9 + 9 = 0 + 9; \text{ add 9 on both sides.}$$

$$3x = 9$$

$$\frac{3x}{3} = \frac{9}{3}; \text{ divide both sides by 3}$$

$$x = 3$$

or

$$3x - 9 = -0$$

But $-0 = 0$, so $x = 3$ in this case as well.

Example 17 $|15y + 9| + 9 = 7$

Before we solve two equations for absolute value; it is easier if we separate $|15y + 9|$ from 9 first.

$$|15y + 9| + 9 - 9 = 7 - 9; \text{ subtract 9 on both sides.}$$

$$|15y + 9| = -2$$

Since the output of absolute function is always positive, We have NO SOLUTIONS.

Example 18 $|7y - 4| = |6y + 3|$

You begin by making it into two separate equations and then solving them separately.

$$7y - 4 = 6y + 3$$

$$7y - 6y - 4 = 6y - 6y + 3; \text{ Subtract } 6y \text{ on both sides}$$

$$y - 4 = 3$$

$$y - 4 + 4 = 3 + 4; \text{ add 4 on both sides}$$

$$y = 7$$

or

$$7y - 4 = -(6y + 3)$$

$$7y - 4 = -6y - 3$$

$$7y + 6y - 4 = -6y + 6y - 3; \text{ add } 6y \text{ on both sides}$$

$$13y - 4 = -3$$

$$13y - 4 + 4 = -3 + 4; \text{ add 4 on both sides}$$

$$13y = 1$$

$$\frac{13y}{13} = \frac{1}{13}; \text{ divide both sides by 13;}$$

$$y = \frac{1}{13}$$

4 Solving Equations by Factoring

Example 19 : Solve the following polynomial equation : $7x^3 + 12x^2 = 4x$

Solution:

$$7x^3 + 12x^2 - 4x = 4x - 4x; \text{ subtract } 4x \text{ on both sides}$$

$$7x^3 + 12x^2 - 4x = 0$$

$$x(7x^2 + 12x - 4) = 0$$

You can use quadratic formula or you can first multiply the 7 with -4 to get -28 . Now you can find two numbers such that you get -28 when you multiply and 12 when you add them.

14 and -2 are two numbers.

$$x(7x + 14)(7x - 2) = 0$$

Either $x = 0$ or $7x + 14 = 0$ or $7x - 2 = 0$.

If $7x + 14 = 0$, then $7x = -14$ so $x = \frac{-14}{7} = -2$.

If $7x - 2 = 0$, then $7x = 2$, so $x = \frac{2}{7}$.

So $x = 0, 2$ and $\frac{2}{7}$.

Example 20 : $x^3 + 4x^2 - 16xy - 64 = 0$

Solution:

x^2 is common in first two terms and -16 is common in last two terms.

$$x^2(x + 4) - 16(x + 4) = 0$$

Now $(x + 4)$ is common in both terms.

$$(x + 4)(x^2 - 16) = 0.$$

$$(x + 4)(x^2 - 16) = 0.$$

Now use the identity $a^2 - b^2 = (a - b)(a + b)$.

$$(x + 4)(x + 4)(x - 4) = 0.$$

$$y = 4, -4.$$

Example 21 Solve by factoring $12x^4 = 3x^2$

Solution:

Bring $3x^2$ on left hand side:

$$12x^4 - 3x^2 = 0$$

Since $3x^2$ is a common factor, let's take it out:

$$3x^2(4x^2 - 1) = 0$$

Use identity $a^2 - b^2 = 0$ to get $4x^2 - 1 = (2x)^2 - 1 = (2x + 1)(2x - 1) = 0$

$$3x^2(2x + 1)(2x - 1) = 0$$

Set each factor to zero i.e. $3x^2 = 0, 2x + 1 = 0, 2x - 1 = 0$ to get $x = 0, -\frac{1}{2}$ and $\frac{1}{2}$

5 Quadratic Like Equations

Example 22 : Solve the following quadratic equation like $(y - 7)^2 - 11(y - 7) + 30 = 0$.

Solution:

$$(y - 7)^2 - 11(y - 7) + 30 = 0.$$

Think of this as $A^2 - 11A + 30 = 0$.

Find two numbers such that you get 30 when you multiply and you get -11 when you add them.

$$(A - 5)(A - 6) = 0.$$

$$(y - 7 - 5)(y - 7 - 6) = 0.$$

$$(y - 12)(y - 13) = 0.$$

Either $y - 12 = 0$ or $y - 13 = 0$.

$$y = 12. \text{ or } y = 13.$$

Example 23 : Solve the following quadratic equation like $(x - 5)^2 - 5(x - 5) - 14 = 0$.

Solution:

$$(x - 5)^2 - 5(x - 5) - 14 = 0.$$

Think of this as $A^2 - 5A - 14 = 0$.

Find two numbers such that you get -14 when you multiply and you get -5 when you add them.

$$(A - 7)(A + 2) = 0.$$

$$(x - 5 - 7)(x - 5 + 2) = 0.$$

$$(x - 12)(x - 3) = 0.$$

Either $x - 12 = 0$ or $x - 3 = 0$.

$$x = 12. \text{ or } x = 3.$$

Example 24 : Solve $x^4 = 9$.

Solution:

Subtract 9 on both sides to get;

$$x^4 - 9 = 0.$$

$$(x^2)^2 - 3^2 = 0.$$

Now use the identity $a^2 - b^2 = (a - b)(a + b)$.

$$(x^2 - 3)(x^2 + 3) = 0.$$

Either $x^2 + 3 = 0$ or $x^2 - 3 = 0$.

$$x^2 = -3. \text{ or } x^2 = 3.$$

Taking square roots on both sides, we get;

$$x = \pm\sqrt{-3} \text{ or } x = \pm\sqrt{3}$$

$$x = \pm i\sqrt{3} \text{ or } x = \pm\sqrt{3}.$$

Example 25 : Solve the the following polynomial equation. $z^4 + 3z^2 - 4 = 0$.

Solution:

Let $A = z^2$, then $A^2 = z^4$. We can rewrite the polynomial as

$$A^2 + 3A - 4 = 0$$

Find two numbers such that you get -4 when you multiply, and 3 when you add. The two numbers are -1 and 4 .

$$\text{Then } (A + 4)(A - 1) = 0. \text{ Then } (z^2 + 4)(z^2 - 1) = 0$$

$$\text{Either } z^2 + 4 = 0 \text{ or } z^2 - 1 = 0$$

That means $z^2 = -4$ or $z^2 = 1$ Taking square root on both sides, we get $z = \pm\sqrt{-4}$ or $z = \pm\sqrt{1}$

$$z = \pm i\sqrt{4} = \pm i2 \text{ or } z = \pm 1.$$

6 Solving Rational Equations

Example 26 : Solve the following rational equation and simplify your answer

$$\frac{6}{x-7} - \frac{7}{x+6} = 1$$

Solution:

$$\frac{(x+6)}{(x+6)} \frac{6}{x-7} - \frac{(x-7)}{(x-7)} \frac{7}{x+6} = \frac{(x+6)(x-7)}{(x+6)(x-7)}$$

Since we have the same denominator, we will get rid of it:

$$(x+6)6 - (x-7)7 = (x+6)(x-7)$$

$$6x + 36 - 7x + 49 = x^2 + 6x - 7x - 42$$

If you have same terms on both sides, you can cancel them;

$$36 + 49 = x^2 - 42$$

$$85 = x^2 - 42$$

$$85 + 42 = x^2$$

$$127 = x^2$$

$$\pm\sqrt{127} = x.$$

Example 27 : Solve the following rational equation and simplify your answer

$$\frac{x}{x+2} - \frac{1}{x-4} = 1$$

Solution:

$$\frac{(x-4)}{(x-4)} \frac{x}{x+2} - \frac{(x+2)}{(x+2)} \frac{1}{x-4} = \frac{(x+2)(x-4)}{(x+2)(x-4)}$$

Since we have the same denominator, we will get rid of it:

$$(x-4)x - (x+2)1 = (x+2)(x-4)$$

$$x^2 - 4x - x - 2 = x^2 + 2x - 4x - 8$$

If you have same terms on both sides, you can cancel them;

$$-x - 2 = 2x - 8$$

$$-x - 2x = -8 + 2$$

$$-3x = -6$$

$$x = 2$$

Example 28 : Solve the following rational equation and simplify your answer

$$\frac{y}{y-1} - \frac{9}{y+3} = \frac{y^2}{y^2 + 2y - 3}$$

Solution:

First we factor $y^2 + 2y - 3$ into $(y+3)(y-1)$.

we now rewrite the expression as:

$$\frac{y}{y-1} - \frac{9}{y+3} = \frac{y^2}{(y+3)(y-1)}$$
$$\frac{(y+3)}{(y+3)} \frac{y}{y-1} - \frac{(y-1)}{(y-1)} \frac{9}{y+3} = \frac{y^2}{(y+3)(y-1)}$$

Since we have the same denominator, we will get rid of it:

$$(y+3)y - (y-1)9 = y^2$$

$$y^2 + 3y - 9y + 9 = y^2$$

If you have same terms on both sides, you can cancel them;

$$3y - 9y + 9 = 0$$

$$-6y = -9$$

$$y = \frac{9}{6} = \frac{3}{2}$$

Example 29 : Solve the following rational equation and simplify your answer

$$-\frac{4}{x-4} = 1 - \frac{3}{x+10}$$

Solution:

$$-\frac{(x+10)}{(x+10)} \frac{4}{x-4} = \frac{(x-4)(x+10)}{(x-4)(x+10)} - \frac{(x-4)}{(x-4)} \frac{3}{x+10}$$

Since we have the same denominator, we will get rid of it:

$$-(x+10)4 = (x-4)(x+10) - (x-4)3$$

$$4x - 40 = x^2 - 4x + 10x - 40 - 3x + 12$$

If you have same terms on both sides, you can cancel them;

$$0 = x^2 + 10x - 3x + 12$$

$$0 = x^2 + 7x + 12$$

$$0 = (x+3)(x+4)$$

$$x = -3, -4$$