Math 1100 : College Algebra	**Spring 2021**

Lecture 6: Rational Functions and their Graphs

Notes: *Student* – – – – – – – – – – – – – –

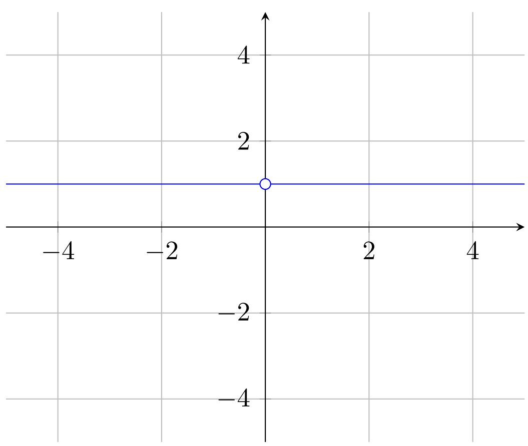
In this section, you will:
♣ Find the domain of Rational Functions.
♣ Identify Horizontal and Vertical Asymptotes.
♣ Graph rational functions
♣ Solved applied problems involving rational functions.

6.1 Introduction: What is Rational Function?

Similar to rational numbers, Rational function can be written as a fraction of polynomial over polynomial functions.
Consider a function

$$f(x) = \frac{x}{x}.$$

This function and the constant function $f(x) = 1$ are exactly same function except at one point $x = 0$.. The function $f(x) = \frac{x}{x}$ is not defined at $x = 0$, so it will have a hole at $x = 0$.



However, as x gets closer to 0, (from both sides: left side represented as 0^- and from right hand side represented as 0^+) y stays 1 everywhere. So we write :

$$As\ x \to 0^-, y \to 1$$

$$As\ x \to 0^+, y \to 1$$

One important thing to note is that Rational functions do not enjoy all real numbers as their domains. They have some restrictions so they are not continuous graphs. They may have holes and asymptotes.

Examples of Rational function:

1. $\frac{3(x-5)}{x-1}$
2. $\frac{1}{x}$
3. $\frac{2x^3}{1} = 2x^3$

The last example is both a polynomial and a rational function. In similar way, any polynomial is a rational function.

6.1.1 Implied Domain

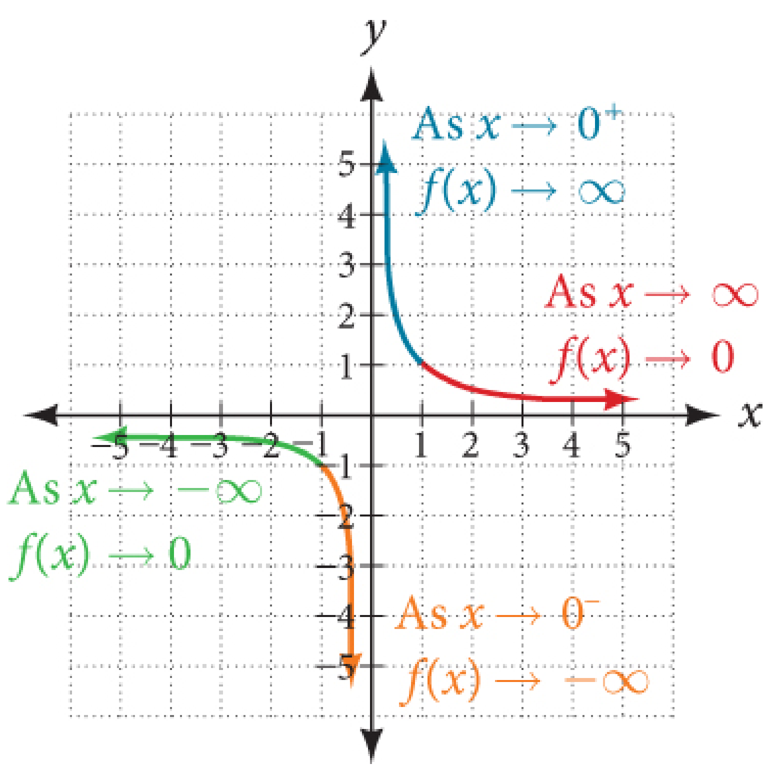
The implied domain of a rational function is the set of all real numbers except for the roots of the denominator. That's because it does not make sense to divide by 0.

Example: The implied domain of

$$\frac{-7(x-2)(x^2+1}{8(x-4)(x-6)}$$

is the set $D = \mathbb{R} - \{4,6\}$. That means $D = \{x : x \neq 4,6\}$. In interval notation $D = (-\infty,4) \cup (4,5) \cup (6,\infty)$.

6.1.2 Arrow Notation



6.2 Rational Functions

A rational function is a function of the form $R(x) = \frac{P(x)}{Q(x)}$ where P and Q are polynomials.The domain of a rational function consists of all real number x except those for which the denominator is zero.
The line $x = a$ is a **vertical asymptote** of the function f if

$$f(x) \to \pm\infty, \ as \ x \to a.$$

The line $y = b$ is a **horizontal asymptote** of the function f if

$$f(x) \to b \ as \ x \to \pm\infty.$$

To find vertical or horizontal asymptote, first you need to start by canceling the common factors of numerator and denominator if they have any.

Vertical Asymptotes: If the rational function $f(x) = \frac{p(x)}{q(x)}$ has been written in reduced form (so that p and q have no common factors), the vertical line $x = c$ is a vertical asymptote for f if and only if c is a real zero of polynomial q. In other words, f has vertical asymptotes at the x-intercepts of q.

Example: $G(x) = \frac{x^2+1}{x^2+2x-15}$
Here $q(x) = x^2 + 2x - 15 = (x + 5)(x - 3)$. Thus the equations of the two vertical asymptotes are $x = -5$ and $x = 3$.

Horizontal and Oblique Asymptotes Let $f(x) = \frac{p(x)}{q(x)}$ be a rational functions, where p is an nth degree polynomial with leading coefficient a_n and q is a mth degree polynomial with leading coefficient b_m, and $p(x)$ and $q(x)$ has no common factors other than constants. Then :

Case 1: If $n < m$:
The horizontal line $y = 0$ (the x-axis) is the horizontal asymptote for f.

For example:
$f(x) = \frac{6x+1}{5x^2-2x+3}$
The degree of denominator is larger than the degree of the numerator, and so we have any asymptote $y = 0$.

Case 2: If n=m:
The horizontal line $y = \frac{a_n}{b_n}$ is the horizontal asymptote for f.

For example:
$f(x) = \frac{6x^2-3x+2}{3x^2+5x-17}$.
In this case, the degrees of the numerator and denominator are the same , so the ratio of leading coefficient is the horizontal asymptote i.e $y = \frac{6}{3} = 2$.

Case 3: If $n = m + 1$:
The line $y = g(x)$ is an oblique asymptote for f, if g is the quotient polynomial obtained by dividing by p by q.

For Example:
$f(x) = \frac{7x^3+2x-1}{x^2+4x}$

The degree of the numerator is exactly 1 greater than the numerator, so we divide the numerator by denominator and the quotient is the oblique asymptote.

Case 4: If $n > m + 1$: There is no straight line horizontal or oblique asymptote for f.

For example:
$f(x) = \frac{3x^5-2x^3+7x^2-1}{4x^3+19x^2-3x+5}$
The degree of numerator is larger than the degree of denominator by more than one, so it has no horizontal asymptote.

TRY IT:

1. Let $f(x) = \frac{3x^2-13x+12}{3x^2+x-2}$. Find y intercept, x-intercept, vertical asymptote and horizontal asymptote if possible.

2. Let $f(x) = \frac{-3x-2x^3+3}{-5x^3-x^2+4}$. Find horizontal asymptote if possible.

3. Let $f(x) = \frac{3x^2-13x+12}{3x^2+20x+25}$. Find y intercept, x-intercept, vertical asymptote and horizontal asymptote if possible.

4. For each function determine the long run behavior:
a. $\frac{x^2+1}{x^3+2}$

b. $\frac{x^3+1}{x^2+2}$

c. $\frac{x^2+1}{x^2+2}$

6.3 How To graph a Rational Function

Given a rational function, sketch a graph.

1. Evaluate the function at 0 to find the y-intercept.
2. Factor the numerator and denominator.
3. For factors in the numerator not common to the denominator, determine where each factor of the numerator is zero to find the x-intercepts.
4. Find the multiplicities of the x-intercepts to determine the behavior of the graph at those points.
5. For factors in the denominator, note the multiplicities of the zeros to determine the local behavior. For those factors not common to the numerator, find the vertical asymptotes by setting those factors equal to zero and then solve.
6. For factors in the denominator common to factors in the numerator, find the removable discontinuities by setting those factors equal to 0 and then solve.

7. Compare the degrees of the numerator and the denominator to determine the horizontal or slant asymptotes.

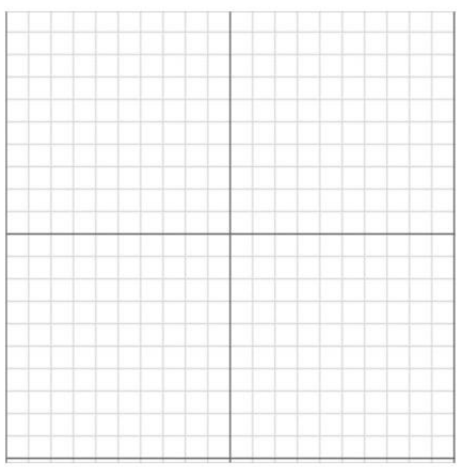
8. Sketch the graph

Example Graph $f(x) = \frac{(x+2)(x-3)}{(x+1)^2(x-2)}$

Find x-intercepts:
$x =$ ____________, ____________, ____________

To find y-intercept, plug $x =$ ______ into ______
$f($______$) = -$______

Leading term of Numerator= ______ , **Leading term of denominator=** ______



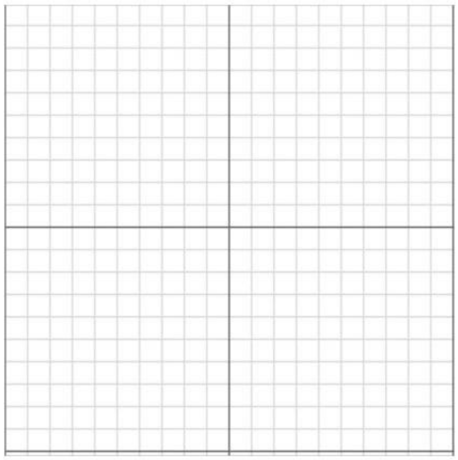
Key features
Vertical Asymptotes: ____________
Horizontal or slant Asymptotes ____________
Y-intercept ____________
X-intercepts ____________
Test points using x-intercept as a basis: ______,
______ , ______

Example Graph $f(x) = \frac{(x+2^2)(x-2)}{2(x-1)^2(x-3)}$

Find x-intercepts:
$x =$ ____________, ____________, ____________

To find y-intercept, plug $x =$ ______ into ______
$f($______$) = -$______

Leading term of Numerator= ______ , **Leading term of denominator=** ______



Key features

Vertical Asymptotes: ____________

Horizontal or slant Asymptotes ____________

Y-intercept ____________

X-intercepts ____________

Test points using x-intercept as a basis: ______,

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