LSU Math Contest:Team 2019

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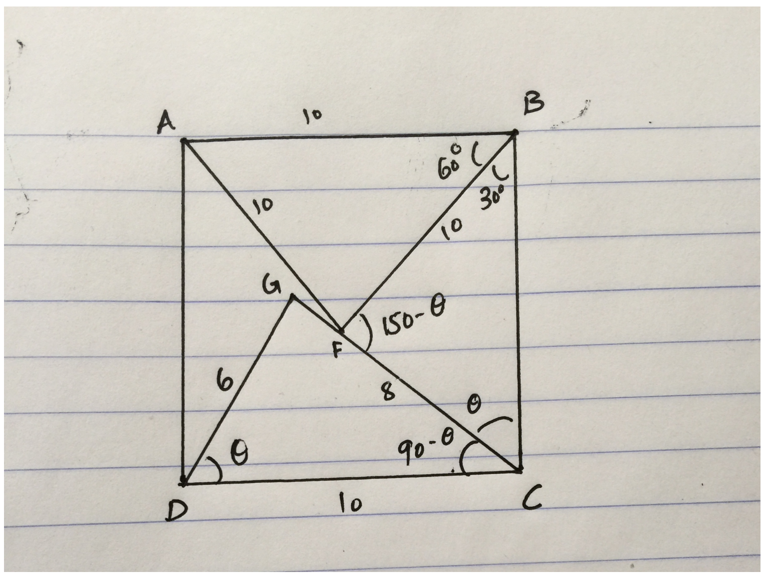
A. DeMorgan: "The moving power of mathematical invention is not reasoning but imagination "

1. Problem 1



Solution: Here:
$AB = DC = AF = FB = 10$, Property of rectangle and equilateral triangle.
$\angle DGF = 90^0$, let $\angle GDC = \theta$, then $\tan(\theta) = \frac{8}{6}$.
To find AD we find BC using sine law in $\triangle\ FBC$:
$\frac{10}{\sin(\theta)} = \frac{BC}{\sin(150-\theta)}$
So, $BC = \frac{10}{\sin(\theta)}\sin(150-\theta) = 10\left(\frac{\sin 150\cos\theta - \cos 150\sin\theta}{\sin\theta}\right) = 10\left(\sin 30\frac{\cos\theta}{\sin\theta} + \cos 30\right)$
$BC = 10\left(\frac{1}{2}\frac{6}{8} + \frac{\sqrt{3}}{2}\right) = \frac{15}{4} + 5\sqrt{3}$

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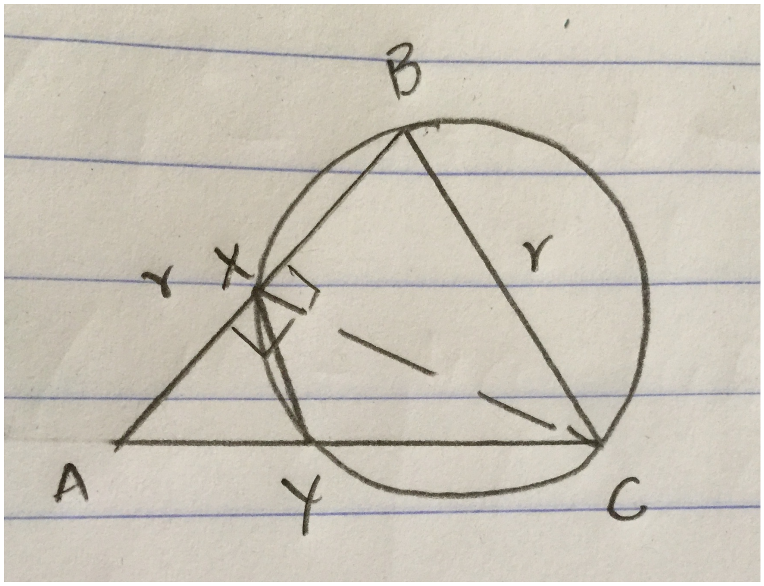
2. Problem 2



Solution: There may be easier way to solve this problem using theorems in geometry . I will use brute force method here.
Construct a line joining XC so that $\angle BXC = 90$.
Using cosine law in $\triangle\ ABC$:
$\cos A = \frac{r^2+s^2-r^2}{2rs} = \frac{s}{2r}$
$\cos 2A = 2\cos^2 A - 1 = 2(\frac{s}{2r})^2 - 1 = \frac{s^2-2r^2}{2r^2}$
Using cosine law in $\triangle\ AXC$:
$[XY]^2 = [AX]^2 + [AY]^2 - 2[AX][AY]\cos A$
$[XY]^2 = (r - BX)^2 + (s - CY)^2 - 2(r - BX)(s - CY)\cos A$
$= \left(r + r\cos(\pi - 2A)\right)^2 + (s - r\cos A)^2 - 2\left(r - r\cos(\pi - 2A)\right)(s - r\cos A)\cos A$
$= (r + r\cos 2A)^2 + (s - r\cos A)^2 - 2(r + r\cos 2A)(s - r\cos A)\cos A$
$= \left(r + r(\frac{s^2-2r^2}{2r^2})\right)^2 + (s - r\frac{s}{2r})^2 - 2\left(r + r(\frac{s^2-2r^2}{2r^2})\right)(s - r\frac{s}{2r})\frac{s}{2r}$
$= r^2(\frac{s}{2r})^2 + (\frac{s}{2})^2 - 2r\frac{s^2}{2r^2}\frac{s}{2}\frac{s}{2r}$
$= \frac{s^4}{4r^2} + \frac{s^2}{4} - \frac{s^4}{4r^2}$
$= \frac{s^2}{4}$
i.e $[XY] = \frac{s}{2}$

3. Problem 3

Solution: Note that $\frac{1}{7} = 0.142857$
We will use basic induction here. Given $a_k = \left\lfloor \frac{10^k}{7} \right\rfloor - 10 \left\lfloor \frac{10^{k-1}}{7} \right\rfloor$
$a_1 = \left\lfloor \frac{10^1}{7} \right\rfloor - 10 \left\lfloor \frac{10^{1-1}}{7} \right\rfloor = 1 - 0 = 1$
$a_2 = \left\lfloor \frac{10^2}{7} \right\rfloor - 10 \left\lfloor \frac{10^{2-1}}{7} \right\rfloor = 14 - 10 = 4$
$a_3 = \left\lfloor \frac{10^3}{7} \right\rfloor - 10 \left\lfloor \frac{10^{3-1}}{7} \right\rfloor = 142 - 140 = 2$
$a_4 = \left\lfloor \frac{10^4}{7} \right\rfloor - 10 \left\lfloor \frac{10^{3-1}}{7} \right\rfloor = 1428 - 1420 = 8$
$a_5 = \left\lfloor \frac{10^5}{7} \right\rfloor - 10 \left\lfloor \frac{10^{4-1}}{7} \right\rfloor = 14285 - 14280 = 5$
$a_6 = \left\lfloor \frac{10^6}{7} \right\rfloor - 10 \left\lfloor \frac{10^{5-1}}{7} \right\rfloor = 142857 - 142850 = 7$
i. e $a_{2019} = a_{2019(mod\ 6)} = a_3 = 2$

4. Problem 4: How many numbers between $7,000,000$ and $8,000,000$ are there for which the millions digits equals the sum of the other six digits.

Solution: This is a standard counting (combinatorics) and number theory problem.
There are six positions and these all position has to add to 7. Consider the following equations:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 7; x_i \geq 0\ for\ i \in 1\ to\ 6$$

One solution of the equation is $x_1 = 3, x_2 = 3, x_3 = 1, x_4 = x_5 = x_6 = 0$. A possible interpretation for this solution is that we are distributing seven pennies (identical objects) among six children (distinct containers) , and here we have given three pennies to the first children, and one to the third child and nothing to the last three children. Continuing with this interpretation , we see that each nonenegative integer solution of the equation corresponds to a selection with repetition of size 7 (identical pennies) from a collection of size 6 (the distinct children), so there are

$$C(6 + 7 - 1, 7) = C(12, 7) = C(12, 5) = 792$$

5. Problem 5: A certain positive integer is 12 times the sum of its digits (when written in base 10). What is the number?

Solution: This number can not be an one digit number. Assume it is a two digit number z with digits $z = ab; a, b \in 0\ to\ 9,$, then the following relation should hold:

$$12a + 12b = z = 10a + b$$

$$2a = 11b$$

This diophantine equation does not have solution in the given range of a and b.

Now Assume its a three digit number. The following relation should hold:

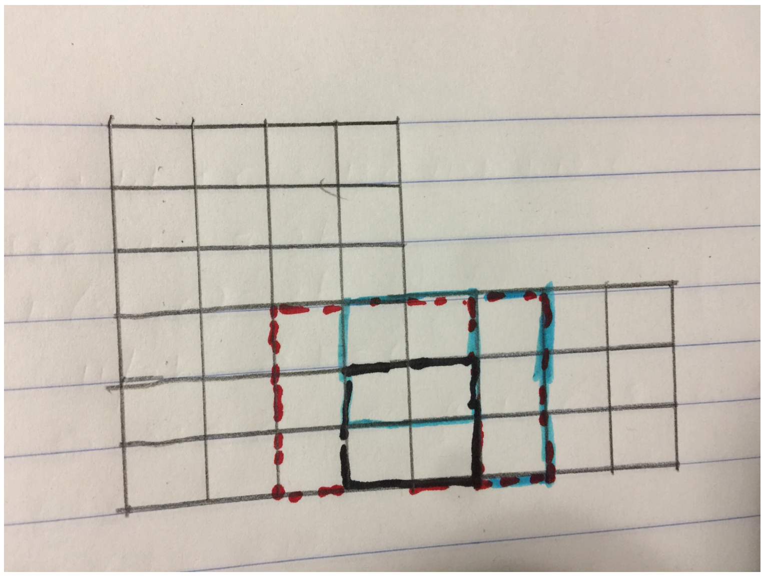
$$100a + 10b + c = 12a + 12b + 12c$$

$$88a - 11c = 2b$$

$$11(8a - c) = 2b$$

Since 11 and 2 are both prime , 11 divides b by divisibility rules. Now $b = 0$ because b in an integer from 0 to 9 only. This implies $8a - c = 0$. Plugging $a = 1$ and $c = 8$, we have the number. so the three digit number is 108.

6. Problem 6:



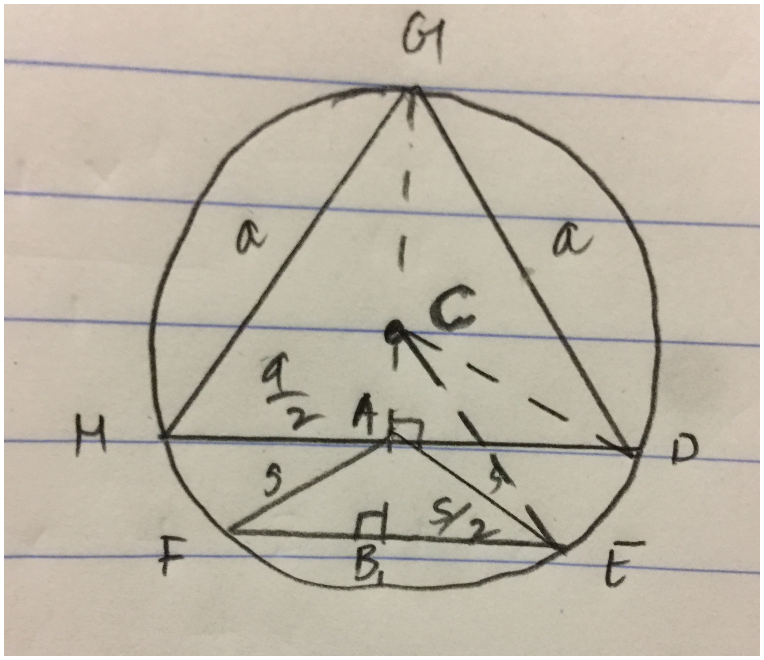
Solution:

Since this is a small square, we will use brutce force method.

Number of 1-by-1 squares : $6 * 4 + 4 * 3 = 36$
Number of 2-by-2 squares : $(4-1)(6-1)+(4-1)(3-2)+2 = 23$ The last two extra squares arises from sharing border.
Number of 3-by-3 squares : $(4-2)(6-2)+(4-2)(3-2)+2 = 12$ The last two extra squares arises from sharing border.
Number of 4 by 4- squares : $(4-3)(6-3) = 3$. There is no 4-by-4 square on second half. Total squares $= 36 + 23 + 12 + 3 = 74$ squares.

7. Problem 7:

Solution: Construct a straight line GAB passing through the center of a circle C. Construct radius $CD = CE = r$. Let this circle be unit circle so, $r = 1$. Let a be the side of equilateral triangle inscribed inside the circle and s be the side of another equilateral triangle.
Using trigonometry $r = a\frac{\sqrt{3}}{3}$ implies $a = \sqrt{3}$.
From the figure :
$HA = AD = \frac{a}{2}$, line passing through center of circle divides the chord into two halves.



$FB = BE = \frac{s}{2}$, same reason above.
Using trigonometry $AB = s\frac{\sqrt{3}}{2}$
Using figure again:
$AB = CB - CA$
$s\frac{\sqrt{3}}{2} = \sqrt{1 - \frac{s^2}{4}} - \sqrt{1 - \frac{a^2}{4}}$
$s\frac{\sqrt{3}}{2} = \sqrt{1 - \frac{s^2}{4}} - \sqrt{1 - \frac{3}{4}}$
$s\frac{\sqrt{3}}{2} = \sqrt{1 - \frac{s^2}{4}} - \sqrt{1}2$
$s\sqrt{3} + 1 = \sqrt{4 - s^2}$
$3s^2 + 2\sqrt{3}s + 1 = 4 - s^2$
$4s^2 + 2\sqrt{3}s - 3 = 0$
Using quadratic formula and discarding negative value :
$s = \sqrt{3}\left(\frac{\sqrt{5}-1}{4}\right)$

Now the ratio of area of two equilateral triangle is given by :

$ratio = \frac{HD*GA}{FE*AB}$
$= \frac{a*a\frac{\sqrt{3}}{2}}{s*s\frac{\sqrt{3}}{2}}$
$= \frac{a^2}{s^2}$
$= \frac{3}{3\left(\frac{\sqrt{5}-1}{4}\right)^2}$
$= \frac{16}{6-2\sqrt{5}}$
Rationalizing we get:
$= 2(3 + \sqrt{5})$

8. Problem 8: This is a number theory problem, an example of continued fractions.

$$? = \cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{2+\cfrac{1}{4+\cdots}}}}}}}}$$

Let $x = 2 + \cfrac{1}{4+\cdots}$

Then you can write $x = 2 + \cfrac{1}{4+\cfrac{1}{x}}$.

> **Solution:** $x = 2 + \frac{x}{4x+1}$
> $x = \frac{8x+2+x}{4x+1}$
> $4x^2 + x = 8x + 2 + x$
> $2x^2 - 4x - 1 = 0$
> Using the quadratic equation and discarding the negative answer, we get: $x = 1 + \sqrt{\frac{3}{2}}$
>
> Now this continuous fraction is convergent to :
>
> $\frac{1}{1+\sqrt{\frac{3}{2}}}$
> rationalizing the denominator we get:
> $\sqrt{6} - 2$

9. Problem 9: What does it mean to be a linear factor?

> **Solution:**
>
> Given $0 < a < b < c$ and let $p(x) = x(x-a)(x-b) - 2019$. $x - c$ is a linear factor of $p(x)$.
> i.e $p(c) = c(c-a)(c-b) - 2019 = 0$
> $c - b < c - a < c$ because $0 < a < b < c$
> $c(c-a)(c-b) = 2019 = 673 * 3 * 1$
>
> so $c = 673, c - b = 1$ and $c - a = 3$
> so $b = 672$ and $a = 670$
> Then $a + b + c = 673 + 672 + 670 = 2015$.

10. A certain number M has a 3 as its unit digit when written in the decimal expansion. A new number N is formed from M by moving the unit digit to the left most position. For example, if $M = 213$ then $N = 321$. What is the smallest number M such that $N = 3M$?

Solution: This is an example of $n-$ parasitic number. Let $M = a_n a_{n-1}......a_1 3$, a_i are digits of M Then $N = 3a_n...........a_1$ Since $N = 3M$
$a_n a_{n-1}......a_1 3$
$\times 3$

$3 \times 3 = 9 = a_1$
$3 \times 9 = 27$, so $a_2 = 7$
$3 \times 7 = 21$, add $21 + 2 = 23$, this 2 is from the 27 above, so $a_3 = 3$
$3 \times 3 = 9$ add $9 + 2 = 11$, so $a_4 = 1$

$3 * 3 = 9$
$3 * 93 = 279$
$3 * 793 = 2379$
$3 * 3793 = 11379$
$3 * 13793 = 41379$ We keep on doing this until it starts to repeat itself.