LSU Math Contest:Team 2018

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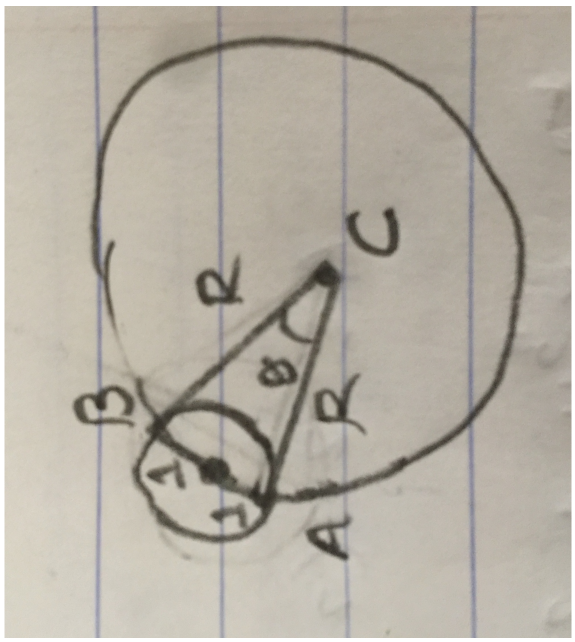
Felix Klein : " Plato Said, "God is a geometer." Jacobi changed this to, " God is an arithmetician." Then came Kronecker and faishioned the memorable expression, "God created the natural numbers, and all the rest is the work of man."

1. Problem 1



Solution:
Construct lines $AC = BC = R$ (radius of common circle). Line $AB = 2$ units, diameter of smaller circles.
$\theta = \frac{2\pi}{8} = 45^o$
Using cosine law in triangle ABC, we get
$2^2 = 2R^2 - 2R^2 \cos(45^o)$
$2 = R^2 - R^2 \frac{1}{\sqrt{2}} = \frac{R^2(\sqrt{2}-1)}{\sqrt{2}}$
$R^2 = \frac{2\sqrt{2}}{\sqrt{2}-1} = 2\sqrt{2}(\sqrt{2}+1) = 2(2+\sqrt{2})$

Now Area of a common circle $= \pi R^2 = 2\pi(2+\sqrt{2})$

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2. Problem 2 : Suppose x and y are positive integers satisfying

$$x^3y + 35y^4 = 2018$$

. Find xy

Solution:
$y(x^3 + 35y^3) = 2018$
the factors of 2018 are $1, 2, 1009, 2018$. Since x and y are $\mathbb{Z}$, $y = 2$. This will give $x = 9$. Hence $xy = 18$

3. Problem 3: What is the remainder if 10^{2018} is divided by 999?
This is a number theory congruence problem

Solution:
Observe $1000 = 1(mod\ 999)$
$10^3 = 1(mod\ 999)$
$(10^3)^{672} = 1^{672} = 1(mod\ 999)$
$10^{2016} = 1(mod\ 999)$
$10^2 * 10^{2016} = 10^2 * 1 = 100(mod\ 999)$

4. Problem 4: You roll a fair die five times and add the numbers that come up. What is the probability that the sum is 10?

Solution: Lets Partition *ten* using five numbers.
$P1 : 2, 2, 2, 2, 2,$
$P2 : 3, 2, 2, 2, 1$
$P3 : 3, 3, 2, 1, 1$
$P4 : 4, 2, 2, 1, 1$
$P5 : 4, 3, 1, 1, 1$
$P6 : 5, 2, 1, 1, 1$
$P7 : 6, 1, 1, 1, 1$

Permutations of Partitions:
$P1 = 1$
$P2 = \frac{5!}{3!} = 20$
$P3 = \frac{5!}{2!2!} = 30$
$P4 = \frac{5!}{2!2!} = 30$
$P5 = \frac{5!}{3!} = 20$
$P6 = \frac{5!}{3!} = 20$
$P7 = \frac{5!}{4!} = 5$

Total permutations of partitions is 126

Total sample space $= 6^5 = 7776$

Probablitility $= \frac{126}{7776} = \frac{7}{432}$

Another way to solve this is $\sum(x_i) = 10, 1 \leq x_i \leq 6, x_i \in \mathbb{Z}, i \in 1,2,3,4,5$

5. Problem 5: A certain positive integer is 12 times the sum of its digits (when written in base 10). What is the number?

Solution: This number can not be an one digit number. Assume it is a two digit number z with digits $z = ab; a, b \in 0 \ to \ 9,$, then the following relation should hold:

$$12a + 12b = z = 10a + b$$

$$2a = 11b$$

This diophantine equation does not have solution in the given range of a and b.

Now Assume its a three digit number. The following relation should hold:

$$100a + 10b + c = 12a + 12b + 12c$$

$$88a - 11c = 2b$$

$$11(8a - c) = 2b$$

Since 11 and 2 are both prime , 11 divides b by divisibility rules. Now $b = 0$ because b in an integer from 0 to 9 only. This implies $8a - c = 0$. Plugging $a = 1$ and $c = 8$, we have the number. so the three digit number is 108.

6. Problem 6: There are 46656 , 6-digit numbers that can be formed from the digits $1, 2, 3, 4, 5$, and 6, with repetition of digits allowed. If these numbers are listed in order what is the 2018th number in the list?

Solution: After first 6 changes, the 2nd last digit will change from 1 to 2. Similarly after $6*6 = 36$ changes the third last digit will change from 1 to 2. Similarly after $6*6*6*6 = 1296$, the 2nd digit from left changes to 1 to 2. so the first two digits should be 12.
Now $2018 - 1296 = 722$
$6*6*6 = 216$
$722 - 216 - 216 - 216 = 74$, 216 will cause change in third digit from left or 4th digit from right. Since 216 is subtracted three times the 4th digit from right will change from 1 to 4. So the first three digits will be 124.
$74 - 36 - 36 = 2$, 36 is subtracted twice so it will change the third last digit from 1 to 3. Since the remainder is only 2, it will cause change in last digit only. So, the final digit should be 124312.

7. Problem 7: . A box contains a collection of 9 cent stamps and 14 cent stamps. The total value of the 9 cent stamps is twice the total value of the 14 cent stamps and the total value of all the stamps is less than \$38.00. What is the maximum number of stamps in the box.

Solution: Let x be the number of 9 cents. Let y be the number of 14 cent stamps. Then the following equation holds:

$$9x = 2(14y) = 28y$$

$$9x + 14y < 3800$$

Using substitution we get

$$28y + 14y < 3800$$

$$y < \frac{3800}{42} \approx 90$$

This gives,

$$x = 280$$

$$x + y = 370$$

8. Problem 8: You are asked to go to the store and purchase apples, bananas, and oranges. Your mom says to buy a total of 20 pieces of fruit and your brother says that you must buy at least 1 of each fruit. How many ways can you make this purchase?

Solution:
Lets buy first 1 of each fruit, now we have to choose 17 total fruits from three fruits using repeatition. $x_1 + x_2 + x_3 = 17, x_i > 0, i \in 1, 2, 3$, so, the number of such solution are

$$C(17 + 3 - 1, 17) = C(19, 17) = 171$$

9. Problem 9: Express the number

$$\sqrt[3]{207 + 94\sqrt{5}}$$

in the form of $a + b\sqrt{5}$ with a and b as integers.

Solution:

$a + b\sqrt{5} = \sqrt[3]{207 + 94\sqrt{5}}$
cubing both sides, we get
$a^3 + 15ab^2 + \sqrt{5}(b^3 + 3a^2b) = 207 + 94\sqrt{5}$
so, we have two equations:
$a^3 + 15ab^2 = 207$ and $b^3 + 3a^2b = 94$
$a(a + 15b^2) = 3 * 3 * 23$ and $b(b^2 + 3a^2) = 2 * 47$
a is either 3 or 23 and b is either 2 or 47
by checking the combination $a = 3$ and $b = 2$.

10. Problem 10: Josephus Problem for $k = 3$
In Progress