River Parishes Community College

Math 1100: College Algebra

Equations and Inequalities

2.5 Solving Quadratic Equations

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Department
Physical Science: Math

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Learning Objectives

In this section, you will learn:
♣ Solve Quadratic Equation by Factor Method
♣ Solve Quadratic Equation by Square root method.
♣ Learn to use Quadratic Formula.

This is a long topic. The only way to master this topic is to actually do lots of example. For our class, we will limit ourselves to three ways of solving quadratic: Factor Method, Square root method and Quadratic Formula. Know that Factor Method and Square root may be lot more faster than using quadratic formula in a particular example, but their scope is limited. On the other hand, quadratic formula may be time consuming, but it works for all quadratic equations. For example:
$x^2 - 7x - 18 = 0$ This problem can be solved using the quadratic formula, but it would be faster to solve the problem by factoring. Generally, factoring should be considered if the leading coefficient is a 1, meaning the problem is in the form $x^2 + bx + c = 0$.

$2(x-5)^2 - 7 = 0$ This problem could be solved using the quadratic formula, but we would need to simplify the problem first in order to get the problem in the correct form. A better option would be to use the square root principle.

1 Complex number i

With only real numbers in hand, we can not solve a quadratic equation like $x^2 + 1 = 0$. So we need to go beyond the realm of real numbers and introduce another domain of numbers called complex numbers.
Think of real numbers as points in an infinite straight line where as complex numbers as points in plane (sheet of graph paper). Only one number is enough to represent a point in a line whereas you need two numbers (order pair (x, y)) in a graph paper (plane). This way you can think complex number as an extension or big brother of a number line. One benefit of complex number is now we can write solutions for this quadratic equations as well:

$$x^2 = -1$$

The solution of this special quadratic is a a special complex number and we denote it with i and we define :

$$i = \sqrt{-1} \text{ and } i^2 = -1$$

Understand that there is no real number that when you square it will give -1, and this complex number i is just a symbol to denote the solution of $x^2 + 1 = 0$. Complex numbers are usually written as (x, y) as an order pair or $x + iy$ as a complex form. The beauty of using this symbol is that we can tell that there will always be exactly two solutions for quadratic equation because of degree 2.

1.1 Multiplying By the Conjugate

The complex solution to a quadratic equation with negative imaginary part is just as good a solution as the one with positive imaginary part, thus these two solutions are called conjugates.

Suppose a point (x, y) is given in a plane and we wish to find the distance (magnitude)of this point from origin. One way to find its magnitude is to use pythagorean theorem by constructing a right-angled triangle. Another method is to actually define a complex conjugate.
Instead of $z = a + bi$, think about a number $z^* = a - bi$, called the "complex conjugate". It has the same real part, but is the "mirror image" in i direction. The conjugate or "i-axis reflection" has the same magnitude, but the opposite angle!

What happens if you multiply by the conjugate? What is z times z^*?
We got a real number, like we expected! The math fans can try the algebra also:

$$(a + bi)(a - bi) = a^2 + abi - abi - b^2 i^2 = a^2 + b^2$$

Tada! The result has no imaginary parts, and is the magnitude squared. This says that there is the function that gives magnitude of this complex number just by multiplication with another complex number. The further study of complex number can be very interesting and insightful.

2 Factor Method

Zero Factor Property:

If $ab = 0$ then either $a = 0$ or $b = 0$ or both $a = 0$ and $b = 0$.

Note that this property says if a product of two terms is zero then at least one of the terms has to be zero. To solve a quadratic equation by factoring, we first must move all the terms over to one side of the equation and keep zero on the other side. Then we use zero factor principle and set each factors to zero and solve for the variable.

Example 1 : $(2x + 3)(x + 1) = 0$
Solution:

Either
$2x + 3 = 0$
$2x = -3$
$x = -\frac{3}{2}$

or,
$x + 1 = 0$
$x = -1$

Example 2 : $(a + 1)(a + 5) = 0$
Solution:
Either
$a + 1 = 0$
$a = -1$

or
$a + 5 = 0.$
$a = -5.$

Example 3 $(2m + 3)(4m + 3) = 0$
Solution:

$2m + 3 = 0.$
$2m = -3$
$m = -\frac{3}{2}.$

or
$4m + 3 = 0.$
$4m = -3.$
$m = -\frac{3}{4}.$

Example 4 $x^2 - 11x + 24 = 0$
Solution:
Find two numbers such that sum is -11 and product is 24.
-8 and -3 works.
$(x-8)(x-3) = 0.$

Either
$x - 8 = 0$
$x = 8.$

or
$x - 3 = 0.$
$x = 3.$

Example 5 $n^2 + 7n + 10 = 0$
Solution:
Find two numbers such that sum is 7 and the product is 10.
5 and 2 works.
Either
$n + 2 = 0.$
$n = -2.$

or
$n + 5 = 0.$
$n = -5.$

Example 6 : $n^2 + 8n + 15 = 0$
Solution:
Similarly as above, write
$(n+5)(n+3) = 0.$
$n = -5, -3.$

Example 7 : $n^2 + 3n - 18 = 0.$
Solution:
$(n+6)(n-3) = 0.$
$n = -6, 3.$

Example 8 : $3r^2 - 16r - 12 = 0$
Solution:
Multiply first $-12 * 3 = -36$ then you have two numbers which would multiply to -36 and add to -16.
-18 and 2 are the two numbers. This is a different than the previous numbers.Factor as:

(3r-18)(3r+2)=0
Either
$3r - 18 = 0.$
$r = 6.$

or
$3r + 2 = 0.$
$3r = -2.$
$r = -\frac{2}{3}.$

Example 9 : $7x^2 + 2x = 0$.
Solution:
x is common to both $7x^2$ and $2x$
$x(7x + 2) = 0$.

Either, $x = 0$.

or
$7x + 2 = 0$.
$7x = -2$.
$x = -\frac{2}{7}$

3 Solve by taking square roots.

The second method of solving quadratics we'll be looking at uses the **square root property**:

$$\text{If } x^2 = a, \text{ then } x = \pm\sqrt{a}$$

The symbol $\pm$ is read as plus or minus and it means both $x = \sqrt{a}$ and $x = -\sqrt{a}$ are the solutions. It is a short hand notation to write both solutions at once.

Example 10 : $r^2 = 7$.
Solution:
$r = \pm\sqrt{7}$

Example 11 : $x^2 - 1 = 80$
Solution:
$x^2 = 80 + 1$.
$x^2 = 81$.
$x = \pm\sqrt{81}$.
$x = \pm 9$.

Example 12 : $4x^2 - 6 = 7$
Solution:
$4x^2 = 7 + 6$
$4x^2 = 13$.
$x^2 = \frac{13}{4}$
$x = \pm\sqrt{\frac{13}{4}}$
$x = \pm\frac{\sqrt{13}}{2}$.

Example 13 : $(p - 4)^2 = 16$.
Solution:
$p - 4 = \pm\sqrt{16}$.
$p - 4 = \pm 4$.
We have,

$p - 4 = 4$
$p = 8$.

and
$p - 4 = -4$.
$p = 0$.

Example 14 : $(6x+2)^2+4=28$.
Solution:
$(6x+2)^2 = 28-4 = 24$.
$6x+2 = \pm\sqrt{24}$.
$6x+2 = \pm\sqrt{4*6}$.
$6x+2 = \pm 2\sqrt{6}$.
$6x = \pm 2\sqrt{6} - 2$.

$x = \pm\frac{2\sqrt{6}}{6} - \frac{2}{6}$.
$x = \pm\frac{\sqrt{6}}{3} - \frac{1}{3}$.

Example 15 : $(7x+5)^2 = 10$.
Solution:
$7x+5 = \pm\sqrt{10}$.
$7x = \pm\sqrt{10} - 5$.
$x = \pm\frac{\sqrt{10}}{7} - \frac{5}{7}$.

Example 16 : $2(y+8)^2 = -18$.
Solution:
$(y+8)^2 = -9$.
$y+8 = \pm\sqrt{-9}$.
$y+8 = \pm i\sqrt{9}$.
$y+8 = \pm i3$.
$y = -8 \pm i3$.

4 Using Quadratic Formulas

The quadratic formula is a technique that can be used to solve quadratics, but in order to solve a quadratic using the quadratic formula the problem must be in the correct form. To solve a quadratic using the quadratic formula the quadratic must be in the form $ax^2+bx+c=0$. In other words, the quadratic must be in descending order (highest power to lowest power) and equal to zero making it easy to identify the values of a, b, and c to plug into the quadratic formula. Here is the quadratic formula :

If $ax^2+bx+c=0$, then $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Three cases:
1. If $b^2-4ac>0$, both roots will be real roots . for example : $x^2-5x-36=0$
2. If $b^2-4ac=0$, there is only one root (repeated twice) for example $x^2-4x+4=0$.
3. If $b^2-4ac=0$, the roots are complex numbers and they are complex conjugates.

Example 17 $y^2+8y+24=0$
Solution:

$a=1, b=8, c=24$
$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x = \frac{-8\pm\sqrt{8^2-4*1*24}}{2}$

$x = \frac{-8\pm\sqrt{64-96}}{2}$

$x = \frac{-8 \pm \sqrt{-32}}{2}$

$x = \frac{-8 \pm 4i\sqrt{2}}{2}$

$x = -4 \pm i2\sqrt{2}$

In this example the radical disappeared and the final answers were simple fractions which means that this problem could have been solved by factoring, but since we already have the answer at this point it makes little difference.

Example 18 $12x^2 + 7x = 12$

Solution:

Rewrite the equation by subtracting 12 on both sides into $12x^2 + 7x - 12 = 0$

$a = 12, b = 7, c = -12$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-7 \pm \sqrt{7^2 - 4*12*(-12)}}{2(12)}$

$x = \frac{-7 \pm \sqrt{625}}{24}$

$x = \frac{-7 \pm 25}{24}$

$x = \frac{3}{4}, -\frac{4}{3}$

Example 19 $-4x^2 - 9x = -12$

Solution:

Rewrite the equation by adding 12 in both sides

$-4x^2 - 9x + 12 = 0$

$a = -4, b = -9, c = 12$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-(-9) \pm \sqrt{(-9)^2 - 4*(-4)*(12}}{2(-4)}$

$x = \frac{9 \pm \sqrt{273}}{-8}$

This problem cannot be simplified.

Example 20 : Solve $x^2 + 25 = 8x$

Solution:

Rewrite the equation by subtracting $8x$ in both sides

$x^2 - 8x + 25 = 0$

$a = 1, b = -8, c = 25$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4*1*25}}{2(1)}$

$x = \frac{8 \pm \sqrt{-36}}{2}$

$x = \frac{8 \pm 6i}{2}$

$x = 4 \pm 3i$

Example 21 : Solve $-3x^2 + 8x = 9$
Solution:
Rewrite the equation as:
$-3x^2 + 8x - 9 = 0$
$a = -3, b = 8, c = -9$
$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$
$x = \frac{-8 \pm \sqrt{(8)^2-4(-3)(-9)}}{2(-3)}$
$x = \frac{-8 \pm \sqrt{-44}}{-6}$
$x = \frac{-8 \pm \sqrt{44}i}{-6}$
Simplify the radical and reduce/simplify to get the final answer.
$x = \frac{-8 \pm 2\sqrt{11}i}{-6}$
$x = \frac{-4 \pm \sqrt{11}i}{-3}$
$x = \frac{4 \pm \sqrt{11}i}{3}$

Example 22 : Solve $(4x - 3)^2 = 5x$
Solution:
We need to foil first and bring $5x$ on the left side
$16x^2 - 24x + 9 = 5x$
$16x^2 - 29x + 9 = 0$
$a = 16, b = -29, c = 9$
$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$
$x = \frac{-(-29) \pm \sqrt{(-29)^2-4(16)(9)}}{2(16)}$
$x = \frac{-29 \pm \sqrt{265}}{32}$

This problem cannot be simplified.