River Parishes Community College

Math 1100: College Algebra

Equations and Inequalities

2.6 Other types of Equations

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Spring 2021

Department
Physical Science: Math

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Learning Objectives

In this section, you will learn:
♣ Solve equations involving rational exponents.
♣ Solve radical equations.
♣ Solve absolute value equations.
♣ Solve equations using factoring.
♣ Solve other types of equations.

1 Radical Exponents

Radical are often called roots. They are the tools to undo exponentiation. For example if a square of number is 4, then undoing this , i.e. taking the square root of 4, should give us our number 2 back. Another example:

$$4^3 = 64, \ \ so, \ \ \sqrt[3]{64} = 4$$

More Symbols:
forth root is written as $\sqrt[4]{\ }$
fifth root is written as $\sqrt[5]{\ }$
n-th root is written as $\sqrt[n]{\ }$

Properties :
For integer $n > 1$

$$(ab)^n = a^n b^n$$

$$\sqrt[n]{a^n} = a$$

$$\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$$

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

A rational exponent indicates a power in the numerator and a root in the denominator. There are multiple ways of writing an expression, a variable, or a number with a rational exponent:

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = \left(a^m\right)^{\frac{1}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

Example:

$$2^{\frac{3}{5}} = \left(2^{\frac{1}{5}}\right)^3 = \left(2^3\right)^{\frac{1}{5}} = \left(8\right)^{\frac{1}{5}} = \sqrt[5]{8} = (\sqrt[5]{2})^3$$

1.1 Solving equations involving rational exponents

The solutions of the radical equations

$$x^{\frac{m}{n}} = a$$

are as follows:
Case 1: **If m is an odd integer**:
There is only one solution

$$x = a^{\frac{n}{m}}$$

Case 2: **If m is an even integer**:
There are two solutions

$$x = \pm a^{\frac{n}{m}}$$

Examples:

Example 1 : Solve the following equation. If needed, Write your answer as a fraction reduced to lowest terms.

$$z^{3/2} - 27 = 0$$

Solution:
Isolate the expression with the radical exponent.
$z^{\frac{3}{2}} - 27 + 27 = 0 + 27$
$z^{\frac{3}{2}} = 27$
Raise the both sides of the equation to the $\frac{2}{3}$ power.
$z^{\frac{3}{2}\frac{2}{3}} = 3^{3\frac{2}{3}}$.
$z = 3^2 = 9$.

Example 2 : Solve the following equation. If needed, submit your answer as a fraction reduced to lowest terms.

$$z^{\frac{2}{3}} - \frac{49}{64} = 0$$

Solution:
Begin by isolating the radical terms, ie. add $\frac{49}{64}$ on both sides;
$z^{\frac{2}{3}} = \frac{49}{64}$
Raise both sides to power $\frac{3}{2}$ to get:
$z = \pm(\frac{49}{64})^{\frac{3}{2}}$.
$z = \pm(\frac{7^2}{8^2})^{\frac{3}{2}}$.
$z = \pm(\frac{7}{8})^3$

Example 3 : Solve the following equation:

$$\sqrt[3]{(x-2)^2} = 4.$$

Solution:

Rewrite the equation as : $(x-2)^{\frac{2}{3}} = 4$
$x - 2 = \pm 4^{\frac{3}{2}} = \pm \sqrt[2]{4^3} = \pm 8$
$x = 2 \pm 8$
$x = 10, -6$

Example 4 : Solve the equation $3x^{\frac{3}{4}} = x^{\frac{1}{2}}$
Solution:
Rewrite the equation as $3x^{\frac{3}{4}} - x^{\frac{1}{2}} = 0$
since $x^{\frac{1}{2}}$ is common in both of them, lets take it out and we get:
$x^{\frac{1}{2}}(3x^{\frac{3}{4}-\frac{1}{2}} - 1) = 0$
$x^{\frac{1}{2}}(3x^{\frac{3}{4}-\frac{2}{4}} - 1) = 0$
$x^{\frac{1}{2}}(3x^{\frac{3-2}{4}} - 1) = 0$
$x^{\frac{1}{2}}(3x^{\frac{1}{4}} - 1) = 0$
using the zero factor property,

$x^{\frac{1}{2}} = 0, (3x^{\frac{1}{4}} - 1) = 0$
$x^{\frac{1}{2}} = 0$ gives $x = 0$
Now we have $(3x^{\frac{1}{4}} - 1) = 0$
$3x^{\frac{1}{4}} = 1$
$x^{\frac{1}{4}} = \frac{1}{3}$
$x = \left(\frac{1}{3}\right)^4$
$x = \frac{1}{81}$
Thus we have $x = 0$ and $\frac{1}{81}$.

2 Solving Radical Equations

Extraneous Solution: While dealing with radical equations, some of the answers you get, may not satisfy the original equation and hence called extraneous solutions.

step 1 Consider the equation

$$\sqrt{x+4} = x - 2$$

First, let's solve it using usual algebra:

step 2 : Square both sides:
$\sqrt{x+4}^2 = (x-2)^2$
$x + 4 = x^2 - 4x + 4$

step 3 : Simplify :
$x^2 - 5x = 0$
$x(x-5) = 0$

step 4 : zero Property
$x = 0, 5$ are the two solutions.

Now let's plug each of them back into our original equation and see if they work:

For $x = 0$
$\sqrt{0+4} = 0 - 2$ will give $2 = -2$ which is not true. So, 0 can not be the solution

For $x = 5$
$\sqrt{5+4} = 5 - 2$ will give $3 = 3$ which is true. Hence 5 is the solution

General Solution Steps

Step 1. Isolate the Radical(s) and identify the index (n).
Step 2. Raise both sides of the equation to the "nth" power.
Step 3. Use algebraic techniques (i.e. factoring, combining like terms,...) to isolate the variable.
Repeat Steps 1 and 2 if necessary.
Step 4. Check answers. Eliminate any extraneous solutions from the final answer.

Example 5 : Solve the following radical equation. $\sqrt{2y+6}+4=y+3$
Solution:

Begin by isolating the radical expression on one side of the equation.
$\sqrt{2y+6}+4-4=y+3-4$
$\sqrt{2y+6}=y-1$
Raise both sides of the equation to squares to undo the isolated radical.
$2y+6=(y-1)^2$.
$2y+6=y^2-2y+1$; Foil the right side.
$2y-2y+6-6=y^2-2y-2y+1-6$.
$0=y^2-4y-5$.
$0=(y-5)(y+1)$.
Set the both factors to zero and solve for y.
$y=5,-1$.
Check your solutions in the original equation.
When $y=-1$, we get;
$\sqrt{2*(-1)+6}+4=-1+3$
$\sqrt{4}+4=2$.
$2+4=2$.
$6=2$.
Since $6\neq 2$, $y=-1$ is not the solution.

When $y=5$, we get;
$\sqrt{2*5+6}+4=5+3$
$\sqrt{16}+4=8$.
$4+4=8$.
$8=8$.
$y=5$. satisfies the equation.

Example 6 : Solve the following radical equation. If needed, write your answer as a fraction reduced to lowest terms. $\sqrt[3]{7y+16}-\sqrt[3]{19-5y}=0$
Solution:
Begin by isolating the radical
$\sqrt[3]{7y+16}=\sqrt[3]{19-5y}$
Raise both sides to third power.
$7y+16=19-5y$.
$7y+5y=19-16$
$12y=3$.
$y=\frac{3}{12}$.
$y=\frac{1}{4}$.

Example 7 : Solve the following radical equation. If needed, write your answer as a fraction reduced to lowest terms. $\sqrt[3]{6x-5}=5$.
Solution:
Since the radical term is already isolated, we begin by raising both sides to power third.
$6x-5=5^3$.
$6x-5=125$.
$6x=130$.
$x=\frac{130}{6}=\frac{65}{2}$.

Example 8 : Solve the following radical equation. $\sqrt[3]{5y^2+4y}=4$.

Solution:

Begin by raising both sides to power third;
$5y^2 + 4y = 4^3$.
$5y^2 + 4y = 64$.
$5y^2 + 4y - 64 = 0$.
No we use quadratic equation $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
$a = 5, b = 4, c = -64$.
$y = \frac{-4 \pm \sqrt{4^2 - 4*5*(-64)}}{2*5}$.
$y = \frac{-4 \pm \sqrt{16+1280}}{10}$.
$y = \frac{-4 \pm \sqrt{1296}}{10}$.
$y = \frac{-4 \pm 36}{10}$.
Either, $y = \frac{-4+36}{10} = \frac{32}{10} = \frac{16}{5}$.
or
$y = \frac{-4-36}{10} = -4$.

Check both $y = -4, \frac{16}{5}$, back into original equation, so both solution works this time.

Example 9 : Solve the following radical equation $\sqrt{x-6} - \sqrt{x+9} + 3 = 0$
Solution:
Begin by isolation on of the radical here:
$\sqrt{x-6} = \sqrt{x+9} - 3$
Square both sides
$x - 6 = x + 9 - 6\sqrt{x+9} + 9$
Use algebra to simplify and combine like terms:
$-24 = -6\sqrt{x+9}$
$\frac{-24}{-6} = \sqrt{x+9}$
$4 = \sqrt{x+9}$
Square both sides| $16 = x + 9$
$16 - 9 = x$
$x = 7$

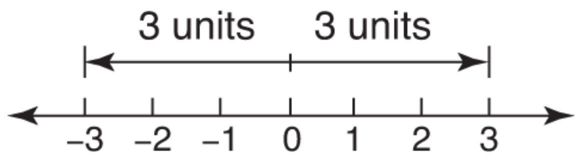
Example 10 : Solve the following equation : $\sqrt{z^2} = -1$
Solution:
There is no solution for this function because square root of any real number is always positive number.

3 Absolute value equations

Absolute value function is denoted by symbol $|x|$. It is a function which takes the entire real number line (both positive and negative) as a input and returns only positive number or zero as an output. Mathematically,

$$|x| = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$$

If $x < 0$, it is a negative number, so $-x$ will become a positive number. If $x > 0$, it is already a positive number.
Geometrically, $|x| = a$ represents a distance x is away from the origin. For example $|x| = 3$ represents a distance of 3 units away from the origin. If you go right hand side from origin, then $x = 3$ and if you go 3 distance left from origin, you will reach $x = -3$.



3.1 solving Absolute Value Equations

Example 11 Solve $|x| = 2$.
Solution:
$x = 2, -2$

Example 12 Solve $|x| = 2$
Solution:
No Solution because absolute value is a distance which is never negative

Example 13 Solve $|x - 2| = 3$.
Solution:
Note that this is asking you to find the points which are 3 units away from point 2 in the number line. If you go three units right of 2, you will get point 5 and if you go three units left of 2, you will get -1. Hence two solutions are: $x = -1, 5$

Example 14 $|3z - 2| = 23$

You begin by making it into two separate equations and then solving them separately.
$3z - 2 = 23$
$3z - 2 + 2 = 23 + 2$;add 2 on both sides.
$3z = 25$
$\frac{3z}{3} = \frac{25}{3}$; divide both sides by 3
$z = \frac{25}{3}$
or $3z - 2 = -23$
$3z - 2 + 2 = -23 + 2$;add 2 on both sides.
$3z = -21$
$\frac{3z}{3} = \frac{-21}{3}$; divide both sides by 3
$z = -7$

Example 15 $|3y + 3| - 11 = 0$

Before we solve two equations for absolute valve; it is easier if we separate $|3y + 3|$ from 11 first.
$|3y + 3| - 11 + 11 = 0 + 11$; add 11 on both sides.
$|3y + 3| = 11$

You begin by making it into two separate equations and then solving them separately.

$3y + 3 = 11$
$3y + 3 - 3 = 11 - 3$; subtract 3 on both sides.
$3y = 8$
$\frac{3x}{3} = \frac{8}{3}$; divide both sides by 3
$x = \frac{8}{3}$
or

$3y + 3 = -11$
$3y + 3 - 3 = -11 - 3$; subtract 3 on both sides.

$3y = -14$
$\frac{3x}{3} = \frac{-14}{3}$; divide both sides by 3
$x = \frac{-14}{3}$

Example 16 $|3x - 9| = 0$

You begin by making it into two separate equations and then solving them separately.
$3x - 9 = 0$
$3x - 9 + 9 = 0 + 9$;add 9 on both sides.
$3x = 9$
$\frac{3x}{3} = \frac{9}{3}$; divide both sides by 3
$x = 3$

or
$3x - 9 = -0$
But $-0 = 0$, so $x = 3$ in this case as well.

Example 17 $|15y + 9| + 9 = 7$

Before we solve two equations for absolute valve; it is easier if we separate $|15y + 9|$ from 9 first.
$|15y + 9| + 9 - 9 = 7 - 9$; sbutract 9 on both sides.
$|15y + 9| = -2$

Since the output of absolute function is always positive, We have NO SOLUTIONS.

Example 18 $|7y - 4| = |6y + 3|$

You begin by making it into two separate equations and then solving them separately.
$7y - 4 = 6y + 3$
$7y - 6y - 4 = 6y - 6y + 3$; Subtract $6y$ on both sides
$y - 4 = 3$
$y - 4 + 4 = 3 + 4$; add 4 on both sides
$y = 7$

or

$7y - 4 = -(6y + 3)$
$7y - 4 = -6y - 3$
$7y + 6y - 4 = -6y + 6y - 3$; add $6y$ on both sides
$13y - 4 = -3$
$13y - 4 + 4 = -3 + 4$; add 4 on both sides
$13y = 1$
$\frac{13y}{13} = \frac{1}{13}$; divide both sides by 13;
$y = \frac{1}{13}$

4 Solving Equations by Factoring

Example 19 : Solve the following polynomial equation : $7x^3 + 12x^2 = 4x$
Solution:
$7x^3 + 12x^2 - 4x = 4x - 4x$; subtract $4x$ on both sides
$7x^3 + 12x^2 - 4x = 0$
$x(7x^2 + 12x - 4) = 0$
You can use quadratic formula or you can first multiply the 7 with -4 to get -28. Now you can find two numbers such that you get -28 when you multiply and 12 when you add them.
14 and -2 are two numbers.

$x(7x + 14)(7x - 2) = 0$
Either $x = 0$ or $7x + 14 = 0$ or $7x - 2 = 0$.
If $7x + 14 = 0$, then $7x = -14$ so $x = \frac{-14}{7} = -2$.
If $7x - 2 = 0$, then $7x = 2$, so $x = \frac{2}{7}$.
So $x = 0, 2$ and $\frac{2}{7}$.

Example 20 : $x^3 + 4x^2 - 16xy - 64 = 0$
Solution:
x^2 is common is first two terms and -16 is common in last two terms.
$x^2(x + 4) - 16(x + 4) = 0$
Now $(x + 4)$ is common in both terms.
$(x + 4)(x^2 - 16) = 0$.
$(x + 4)(x^2 - 16) = 0$.
Now use the identity $a^2 - b^2 = (a - b)(a + b)$.
$(x + 4)(x + 4)(x - 4) = 0$.
$y = 4, -4$.

Example 21 Solve by factoring $12x^4 = 3x^2$
Solution:
Bring $3x^2$ on left hand side:
$12x^4 - 3x^2 = 0$
Since $3x^2$ is a common factor, lets take it out:
$3x^2(4x^2 - 1) = 0$
Use identity $a^2 - b^2 = 0$ to get $4x^2 - 1 = (2x)^2 - 1 = (2x + 1)(2x - 1) = 0$
$3x^2(2x + 1)(2x - 1) = 0$
Set each factor to zero i.e. $3x^2 = 0, 2x + 1 = 0, 2x - 1 = 0$ to get $x = 0, -\frac{1}{2}$ and $\frac{1}{2}$

5 Quadratic Like Equations

Example 22 : Solve the following quadratic equation like $(y - 7)^2 - 11(y - 7) + 30 = 0$.
Solution:
$(y - 7)^2 - 11(y - 4) + 30 = 0$.
Think of this as $A^2 - 11A + 30 = 0$.
Find two numbers such that you get 30 when you multiply and you get -11 when you add them.
$(A - 5)(A - 6) = 0$.
$(y - 7 - 5)(y - 7 - 6) = 0$.
$(y - 12)(y - 13) = 0$.
Either $y - 12 = 0$ or $y - 13 = 0$.
$y = 12$. or $y = 13$.

Example 23 : Solve the following quadratic equation like $(x - 5)^2 - 5(x - 5) - 14 = 0$.
Solution:
$(x - 5)^2 - 5(x - 5) - 14 = 0$.
Think of this as $A^2 - 5A - 14 = 0$.
Find two numbers such that you get -14 when you multiply and you get -5 when you add them.
$(A - 7)(A + 2) = 0$.
$(x - 5 - 7)(x - 5 + 2) = 0$.
$(x - 12)(x - 3) = 0$.
Either $x - 12 = 0$ or $x - 3 = 0$.
$x = 12$. or $x = 3$.

Example 24 : Solve $x^4 = 9$.
Solution:

Subtract 9 on both sides to get;
$x^4 - 9 = 0$.
$(x^2)^2 - 3^2 = 0$.
Now use the identity $a^2 - b^2 = (a-b)(a+b)$.
$(x^2 - 3)(x^2 + 3) = 0$.
Either $x^2 + 3 = 0$ or $x^2 - 3 = 0$.
$x^2 = -3$. or $x^2 = 3$.
Taking square roots on both sides, we get;
$x = \pm\sqrt{-3}$ or $x = \pm\sqrt{3}$
$x = \pm i\sqrt{3}$ or $x = \pm\sqrt{3}$.

Example 25 : Solve the the following polynomial equation. $z^4 + 3z^2 - 4 = 0$.
Solution:
Let $A = z^2$, then $A^2 = z^4$. We can rewrite the polynomial as
$A^2 + 3A - 4 = 0$
Find two numbers such that you get -4 when you multiply, and 3 when you add. The two numbers are -1 and 4.
Then $(A+4)(A-1) = 0$. Then $(z^2+4)(z^2-1) = 0$
Either $z^2 + 4 = 0$ or $z^2 - 1 = 0$
That means $z^2 = -4$ or $z^2 = 1$ Taking square root on both sides, we get $z = \pm\sqrt{-4}$ or $z = \pm\sqrt{1}$
$z = \pm i\sqrt{4} = \pm i2$ or $z = \pm 1$.

6 Solving Rational Equations

Example 26 : Solve the following rational equation and simplify your answer

$$\frac{6}{x-7} - \frac{7}{x+6} = 1$$

Solution:
$\frac{(x+6)}{(x+6)}\frac{6}{x-7} - \frac{(x-7)}{(x-7)}\frac{7}{x+6} = \frac{(x+6)(x-7)}{(x+6)(x-7)}$

Since we have the same denominator, we will get rid of it:
$(x+6)6 - (x-7)7 = (x+6)(x-7)$
$6x + 36 - 7x + 49 = x^2 + 6x - 7x - 42$
If you have same terms on both sides, you can cancel them;
$36 + 49 = x^2 - 42$
$85 = x^2 - 42$
$85 + 42 = x^2$
$127 = x^2$
$\pm\sqrt{127} = x$.

Example 27 : Solve the following rational equation and simplify your answer

$$\frac{x}{x+2} - \frac{1}{x-4} = 1$$

Solution:
$\frac{(x-4)}{(x-4)}\frac{x}{x+2} - \frac{(x+2)}{(x+2)}\frac{1}{x-4} = \frac{(x+2)(x-4)}{(x+2)(x-4)}$

Since we have the same denominator, we will get rid of it:
$(x-4)x - (x+2)1 = (x+2)(x-4)$
$x^2 - 4x - x - 2 = x^2 + 2x - 4x - 8$

If you have same terms on both sides, you can cancel them;
$-x - 2 = 2x - 8$
$-x - 2x = -8 + 2$
$-3x = -6$
$x = 2$

Example 28 : Solve the following rational equation and simplify your answer

$$\frac{y}{y-1} - \frac{9}{y+3} = \frac{y^2}{y^2+2y-3}$$

Solution:
First we factor $y^2 + 2y - 3$ into $(y+3)(y-1)$.
we now rewrite the expression as:
$\frac{y}{y-1} - \frac{9}{y+3} = \frac{y^2}{(y+3)(y-1)}$
$\frac{(y+3)}{(y+3)}\frac{y}{y-1} - \frac{(y-1)}{(y-1)}\frac{9}{y+3} = \frac{y^2}{(y+3)(y-1)}$
Since we have the same denominator, we will get rid of it:
$(y+3)y - (y-1)9 = y^2$
$y^2 + 3y - 9y + 9 = y^2$
If you have same terms on both sides, you can cancel them;
$3y - 9y + 9 = 0$
$-6y = -9$
$y = \frac{9}{6} = \frac{3}{2}$

Example 29 : Solve the following rational equation and simplify your answer

$$-\frac{4}{x-4} = 1 - \frac{3}{x+10}$$

Solution:
$-\frac{(x+10)}{(x+10)}\frac{4}{x-4} = \frac{(x-4)(x+10)}{(x-4)(x+10)} - \frac{(x-4)}{(x-4)}\frac{3}{x+10}$
Since we have the same denominator, we will get rid of it:
$-(x+10)4 = (x-4)(x+10) - (x-4)3$
$4x - 40 = x^2 - -4x + 10x - 40 - 3x + 12$
If you have same terms on both sides, you can cancel them;
$0 = x^2 + 10x - 3x + 12$
$0 = x^2 + 7x + 12$
$0 = (x+3)(x+4)$
$x = -3, -4$