Rudin-Keisler ordering of P-points

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Definition

For a non-empty set X we say that $\mathcal{U} \subseteq \mathcal{P}(X)$ is *an ultrafilter* on X if:

- $\emptyset \notin \mathcal{U}$;
- $a \cap b \in \mathcal{U}$ for every $a, b \in \mathcal{U}$;
- $b \in \mathcal{U}$ whenever there is an $a \in \mathcal{U}$ such that $a \subseteq b$;
- for each $a \subseteq X$, either $a \in \mathcal{U}$ or $X \setminus a \in \mathcal{U}$.

There is a natural topology on the space of all ultrafilters on $\mathbb{N}$, and with this topology it is usually denoted $\beta\mathbb{N}$.

As it turns out, $\beta\mathbb{N}$ coincides with the Cech-Stone compactification of a discrete countable space $\mathbb{N}$.

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$$a \in \mathcal{U} \Leftrightarrow n \in a.$$

The space of all non-principal ultrafilters on $\mathbb{N}$ is denoted $\mathbb{N}^*$. Hence

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One way to formalize this vague question is whether for any two non-principal ultrafilters on $\mathbb{N}$, say $\mathcal{U}$ and $\mathcal{V}$, there is a homeomorphism of the space $\mathbb{N}^*$ which maps $\mathcal{U}$ to $\mathcal{V}$?

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Definition (Gillman-Henriksen, 1954)

For a topological space X, we say that $x \in X$ is a *P-point* if the only prime ideal of $C(X, \mathbb{R})$ at x is the maximal ideal M_x.

Note that this is equivalent to the fact that intersection of any countably many neighborhoods of x is a neighborhood of x.

Again, in the space $\beta\mathbb{N}$, this is equivalent to saying that an ultrafilter $\mathcal{U}$ is a P-point iff for any collection $\{a_n : n \in \mathbb{N}\} \subseteq \mathcal{U}$ there is an $a \in \mathcal{U}$ such that $a \setminus a_n$ is finite for all $n \in \mathbb{N}$.

Typically, one would write $a \subseteq^* a_n$ instead of saying that $a \setminus a_n$ is finite.

Another equivalent condition for an ultrafilter $\mathcal{U}$ to be a P-point is that for every function $f : \mathbb{N} \to \mathbb{N}$ either f is finite-to-one on an element of $\mathcal{U}$ or f is constant on an element of $\mathcal{U}$.

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Theorem (W. Rudin, 1956)

Assume CH*. Then there is a P-point ultrafilter in* $\beta\mathbb{N}$*. Thus* $\mathbb{N}^*$ *is not a homogeneous space.*

Theorem (Shelah, 1970s)

There is a model of ZFC *with no P-point ultrafilters.*

$\mathrm{MA}(\sigma$-centered) ensures the existence of $2^{\mathfrak{c}}$ many P-points. Note that

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Let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters on $\mathbb{N}$. We say that $\mathcal{U}$ is *Rudin-Keisler reducible* to $\mathcal{V}$ if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for every $a \subseteq \mathbb{N}$:

$$a \in \mathcal{U} \Leftrightarrow f^{-1}[a] \in \mathcal{V}.$$

We also say that $\mathcal{U}$ is Rudin-Keisler below $\mathcal{V}$ and write $\mathcal{U} \leq_{RK} \mathcal{V}$.

Note that this is equivalent to the condition that $f[a] \in \mathcal{U}$ for each $a \in \mathcal{V}$. This motivates the notation $f(\mathcal{V}) = \mathcal{U}$ which is sometimes used.

Observation

For ultrafilters $\mathcal{U}$ and $\mathcal{V}$ there is a bijection $f : \mathbb{N} \to \mathbb{N}$ such that $a \in \mathcal{U} \Leftrightarrow f^{-1}[a] \in \mathcal{V}$ if and only if $\mathcal{U} \leq_{RK} \mathcal{V}$ and $\mathcal{V} \leq_{RK} \mathcal{U}$.

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The analysis of a variant of Rudin-Keisler ordering on ultrafilters led to the proof that (in ZFC) ω^* is not homogeneous.

Theorem (Kunen, 1970)

There are ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$ such that $\mathcal{U} \not\leq_{RK} \mathcal{V}$ and $\mathcal{V} \not\leq_{RK} \mathcal{U}$.

Note that if $\mathcal{V}$ is a P-point and $\mathcal{U} \leq_{RK} \mathcal{V}$, then $\mathcal{U}$ is also a P-point.

Note that no ultrafilter can have more that $\mathfrak{c}$ many *RK* predecessors (since there are only $\mathfrak{c}$ many functions from $\mathbb{N}$ to $\mathbb{N}$).

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An ultrafilter $\mathcal{U}$ is *selective* (Ramsey) iff for every function $f : \mathbb{N} \to \mathbb{N}$, either f is 1-1 on an element of $\mathcal{U}$ or f is constant on an element of $\mathcal{U}$.

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Under CH, there are $2^{\mathfrak{c}}$ pairwise RK incomparable selective ultrafilters.

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Conjecture

Assume $\mathrm{MA}(\sigma - \text{centered})$. Let $\mathbb{P}$ be a partial order of size at most $2^{\mathfrak{c}}$ where every element has at most $\mathfrak{c}$ many predecessors. Then $\mathbb{P}$ embeds into the set of P-points under the *RK* ordering (and under the Tukey ordering as well).

Theorem (Blass, 1973)

Both ω_1 and $(\mathbb{R}, <)$ embed into the set of P-points under the RK ordering.

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If a countable set $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of P-points has an upper bound which is a P-point, then there is an ultrafilter $\mathcal{U}$ such that $\mathcal{U} \leq_{RK} \mathcal{U}_n$ for each $n \in \mathbb{N}$.

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Theorem (Rosen, 1985)

Ordinal ω_1 embeds into the set of P-points under the RK ordering as an initial segment, i.e. there is a set of P-points $\{\mathcal{U}_\alpha : \alpha < \omega_1\}$ such that

- $\mathcal{U}_\alpha <_{RK} \mathcal{U}_\beta$ *for all* $\alpha < \beta < \omega_1$ *and*
- *for any ultrafilter* $\mathcal{U}$, *if there is* $\alpha < \omega_1$ *such that* $\mathcal{U} \leq_{RK} \mathcal{U}_\alpha$, *then there is some* $\gamma < \omega_1$ *such that* $\mathcal{U}_\gamma \equiv_{RK} \mathcal{U}$.

Theorem (Laflamme, 1989)

For each $1 \leq \alpha < \omega_1$, *there is an ultrafilter* $\mathcal{U}_\alpha$, *generic for a partial order* $\mathbb{P}_\alpha$ *with the following properties:*

- $\mathcal{U}_\alpha$ *is a rapid P-point ultrafilter.*
- *There is a sequence* $\langle \mathcal{V}_\gamma : \gamma < \alpha + 1 \rangle$ *of P-points such that* $\mathcal{V}_0 = \mathcal{U}_\alpha$, *that* $\mathcal{V}_\gamma <_{RK} \mathcal{V}_\beta$ *for all* $\beta < \gamma < \alpha + 1$, *and that for any* $\mathcal{U}$ *with* $\mathcal{U} \leq_{RK} \mathcal{U}_\alpha$ *there is* $\gamma < \alpha + 1$ *such that* $\mathcal{U} \equiv_{RK} \mathcal{V}_\gamma$.

Theorem (Raghavan-Shelah, 2014)

Assume $\mathrm{MA}(\sigma - \textit{centered})$*. Then* $(P(\mathbb{N})/\mathit{Fin}, \subseteq^*)$ *embeds into the set of P-points under the RK ordering (and under the Tukey ordering as well).*

In particular, this implies that every poset of size at most $\mathfrak{c}$ embeds.

Theorem (K-Raghavan, 2018)

Assume CH*. Then* $\mathfrak{c}^+$ *embeds into the set of P-points under the RK ordering (and under the Tukey ordering as well).*

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The language L consists of symbols for all relations and all functions on $\mathbb{N}$.

$\mathcal{N}$ is the standard model for this language.

$\mathcal{M}$ is an elementary extension of $\mathcal{N}$.

Note that if $a \in \mathcal{M}$, then the set $\left\{ {}^*f(a) : f \in \mathbb{N}^{\mathbb{N}} \right\}$ is the domain of an elementary submodel of $\mathcal{M}$. Submodels like this are called *principal*.

Then a principal submodel generated by a is isomorphic to the ultrapower of the standard model by the ultrafilter $\mathcal{U}_a = \{ b \subseteq \mathbb{N} : a \in {}^*b \}$, i.e.

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There is a reformulation of being a P-point in model theoretic terms.

Lemma

An ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is a P-point if and only if every nonstandard elementary submodel of $\mathcal{N}^{\mathbb{N}}/\mathcal{U}$ is cofinal with $\mathcal{N}^{\mathbb{N}}/\mathcal{U}$.

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Theorem (Blass, 1973)

If a countable set $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of P-points has an upper bound which is a P-point, then there is an ultrafilter $\mathcal{U}$ such that $\mathcal{U} \leq_{RK} \mathcal{U}_n$ for each $n \in \mathbb{N}$.

This theorem has two immediate consequences.

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Any RK-decreasing 'sequence' of P-points has an RK-lower bound.

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If two P-points have an upper bound, then they also have a lower bound.

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An ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is a $\mathrm{P}_{\mathfrak{c}}$-point if for any $\alpha < \mathfrak{c}$ and any collection $\{a_i : i < \alpha\} \subseteq \mathcal{U}$ there is some $a \in \mathcal{U}$ such that $a \subseteq^* a_i$ for each $i < \alpha$.

Lemma

Let $\alpha < \mathfrak{c}$ and $\{\mathcal{M}_i : i < \alpha\}$ be a collection of submodels of $\mathcal{M}$ such that:

- *each $\mathcal{M}_i$ is generated by a_i,*
- *$\mathcal{M}_j \subseteq \mathcal{M}_i$ whenever $i < j < \alpha$,*
- *each $\mathcal{M}_i$ is cofinal with $\mathcal{M}_0$.*

Suppose moreover that $\mathcal{U}_0 = \{b \subseteq \mathbb{N} : a_0 \in {}^*b\}$ is a $P_{\mathfrak{c}}$-point. Then there is a family $\{f_i : i < \alpha\} \subseteq \mathbb{N}^{\mathbb{N}}$ of finite-to-one maps such that:

- *${}^*f_i(a_0) = a_i$ for $i < \alpha$,*
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Definition

Let α be an ordinal, let $\mathcal{F} = \{f_i : i < \alpha\} \subseteq \mathbb{N}^{\mathbb{N}}$ be a family of functions, and let A be a subset of α. We say that a set $d \subseteq \mathbb{N}$ is $(A, \mathcal{F})$-*closed* if $f_i^{-1}[f_i[d]] \subseteq d$ for each $i \in A$.

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Lemma

Assume MA_α. *Let* $\mathcal{F} = \{f_i : i < \alpha\} \subseteq \mathbb{N}^\mathbb{N}$ *be a family of finite-to-one functions. Suppose that for each non-empty finite set* $A \subseteq \alpha$, *and each* $w \in \mathbb{N}$, *there is a finite* $(A, \mathcal{F})$*-closed set* $d \subseteq \mathbb{N}$ *containing* w *as an element. Then there is a finite-to-one function* $h \in \mathbb{N}^\mathbb{N}$, *and a collection* $\{e_i : i < \alpha\} \subseteq \mathbb{N}^\mathbb{N}$ *such that for each* $i < \alpha$, *there is* $k \in \mathbb{N}$ *such that* $h(n) = e_i(f_i(n))$ *whenever* $n \geq k$.

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Assume MA_α. *Let* $\mathcal{M}_i$ $(i < \alpha)$ *be a collection of pairwise cofinal submodels of* $\mathcal{M}$. *Suppose that* $\mathcal{M}_0$ *is principal, and that* $\mathcal{U}_0 = \{b \subseteq \mathbb{N} : a_0 \in {}^*b\}$ *is a* $P_\mathfrak{c}$*-point, where* a_0 *generates* $\mathcal{M}_0$. *Then there is an element* $c \in \bigcap_{i<\alpha} \mathcal{M}_i$ *which generates a principal model cofinal with all* $\mathcal{M}_i$.

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Assume MA. If a collection of fewer than $\mathfrak{c}$ many $P_\mathfrak{c}$-points has an upper bound which is a $P_\mathfrak{c}$-point, then it has a lower bound.

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