A dichotomy for transitive lists

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Joint work with Roy Shalev and Stevo Todorčević

Trees

Let κ be a regular uncountable cardinal, and $(T, <_T)$ a κ-tree, i.e. a partially ordered set such that:

- for every $t \in T$, the set $t_\downarrow = \{x \in T : x <_T t\}$ is well-ordered;
- $|\{t \in T : \mathrm{otp}(t_\downarrow, <_T) = \alpha\}| < \kappa$ for each ordinal α;
- the height of T is κ, i.e. $\kappa = \min\{\alpha : (\forall t \in T)\ \mathrm{otp}(t_\downarrow, <_T) < \alpha\}$.

Denote $T_\alpha = \{t \in T : \mathrm{otp}(t_\downarrow, <_T) = \alpha\}$.

Let $(\kappa, <_T)$ be a κ-tree so that $(\forall \alpha, \beta < \kappa)\, \alpha <_T \beta \Rightarrow \alpha < \beta$. Then:

- $(\forall \delta < \kappa)\, \delta_\downarrow \subseteq \delta$;
- $(\forall \alpha < \kappa)\ |\{\delta < \kappa : \mathrm{otp}(\delta_\downarrow, <_T) = \alpha\}| < \kappa$;
- $(\forall \gamma, \delta < \kappa)\, (\gamma \in \delta_\downarrow \Rightarrow \gamma_\downarrow \subseteq \delta_\downarrow)$.

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For the remainder of the talk $\lambda < \kappa$ are regular infinite cardinals.

Transitive lists

Definition

A sequence $\vec{L} := \langle L_\delta \mid \delta \in \Delta \rangle$ is a κ-*list* with support Δ if

- $\Delta \in [\kappa]^\kappa$;
- $L_\delta \subseteq \delta$ for every $\delta \in \Delta$;
- $|\{\delta \in \Delta \mid \mathrm{otp}(L_\delta) = \alpha\}| < \kappa$ for every $\alpha < \kappa$.

If a κ-list $\vec{L}$ additionally satisfies

- for every $\gamma < \delta$ in Δ, if $\gamma \in L_\delta$, then $L_\gamma \subseteq L_\delta$,

then we say that $\vec{L}$ is a *transitive* κ*-list*.

For $\alpha < \kappa$ denote $\mathrm{Lev}_\alpha(\vec{L}) = \{\delta \in \Delta \mid \mathrm{otp}(L_\delta) = \alpha\}$.

We denote by $\mathsf{TL}(\kappa)$ the class of all transitive κ-lists.

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Trees and transitive lists

For a transitive κ-list $\vec{L} = \langle L_\delta : \delta < \kappa \rangle$ define the relation $\prec_{\vec{L}}$ on κ by

$$\gamma \prec_{\vec{L}} \delta \text{ iff } \gamma \in L_\delta.$$

Lemma

If $\vec{L}$ is a transitive κ-list, then $(\kappa, \prec_{\vec{L}})$ is a tree if and only if $\langle L_\gamma : \gamma \in L_\delta \rangle$ is injective for each $\delta < \kappa$.

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Transitive lists, cont'd

Given $\vec{L} := \langle L_\delta \mid \delta \in \Delta\rangle$ and $A \subseteq \Delta$, we define $\vec{L} \upharpoonright A := \langle L_\delta \mid \delta \in \Delta \cap A\rangle$.

For $X \subseteq \kappa$, we measure the minimal size of a family needed to cover X:

$$\rho_{\vec{L}}(X) := \min\{\theta \leq \kappa \mid \kappa = \theta \text{ or } \left(\exists F \in [\Delta]^\theta\right)\ X \subseteq \bigcup_{\beta \in F} L_\beta\}.$$

Lemma

$\vec{L} = \langle L_\delta : \delta \in \Delta\rangle$ belongs to $\mathsf{TL}(\kappa)$ iff it satisfies the following conditions:

(1) $\Delta \in [\kappa]^\kappa$;

(2) $L_\delta \subseteq \delta$ *for every* $\delta \in \Delta$;

(3') $C(\vec{L}) := \{\gamma \in \mathrm{acc}(\kappa) \mid (\forall \delta \in \Delta \setminus \gamma)\ \sup(L_\delta \cap \gamma) = \gamma\} \in [\kappa]^\kappa$;

(4) *for every* $\gamma < \delta$ *in* Δ, *if* $\gamma \in L_\delta$, *then* $L_\gamma \subseteq L_\delta$,

Transitive lists, cont'd

Given $\vec{L} := \langle L_\delta \mid \delta \in \Delta\rangle$ and $A \subseteq \Delta$, we define $\vec{L} \restriction A := \langle L_\delta \mid \delta \in \Delta \cap A\rangle$.

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The list dichotomy

Definition

For cardinals θ_0, θ_1, denote by

$$\mathsf{TL}\left(\lambda^+\right) \to \begin{pmatrix} \lambda^+ & ; & \lambda\text{-partition} \\ \theta_0 & & \theta_1 \end{pmatrix}$$

the assertion that for every transitive λ^+-list $\vec{L} := \langle L_\delta \mid \delta \in \Delta \rangle$ either:

- there is $A \in [\lambda^+]^{\lambda^+}$ such that $\rho_{\vec{L}}(A \cap \delta) < \theta_0$ for every $\delta < \lambda^+$, or
- a partition $\bigcup_{i<\lambda} A_i = \Delta$ so that for $i < \lambda$ and $X \in [A_i]^{\theta_1}$: $\rho_{\vec{L}}(X) \geq \theta_1$.

Let $\mathsf{LD}\left(\lambda^+\right)$ denote $\mathsf{TL}\left(\lambda^+\right) \to \begin{pmatrix} \lambda^+ & ; & \lambda\text{-partition} \\ 2 & & \lambda \end{pmatrix}$.

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A slightly reformulated dichotomy:

$\mathsf{LD}(\lambda^+)$: For every $\vec{L} \in \mathsf{TL}(\lambda^+)$ either

- $(\exists A \in [\lambda^+]^{\lambda^+})(\forall \delta < \lambda^+)(\exists \gamma \in \Delta)\, A \cap \delta \subseteq L_\gamma$ or
- we can partition $\Delta = \bigcup_{i<\lambda} A_i$ so that

$$(\forall i < \lambda)(\forall X \in [A_i]^{\lambda})(\forall \Sigma \in [\Delta]^{<\lambda})\, X \not\subseteq \bigcup_{\delta \in \Sigma} L_\delta.$$

The local list dichotomy

Definition

For cardinals θ_0, θ_1, denote by

$$\mathsf{TL}\left(\lambda^+\right) \to \begin{pmatrix} \lambda^+ & ; & \lambda\text{-partition, local} \\ \theta_0 & & \theta_1 \end{pmatrix}$$

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- a partition $\bigcup_{i<\lambda} A_i = \Delta$ so that $\rho_{\vec{L} \restriction A_i}(X) \geq \theta_1$ for $i < \lambda$, $X \in [A_i]^{\theta_1}$.

And let $\mathsf{LD}_{part,loc}\left(\lambda^+\right)$ denote $\mathsf{TL}\left(\lambda^+\right) \to \begin{pmatrix} \lambda^+ & ; & \lambda\text{-partition, local} \\ 2 & & \lambda \end{pmatrix}$.

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For cardinals θ_0, θ_1, denote by

$$\mathsf{TL}\left(\lambda^+\right) \to \left(\begin{array}{c} \lambda^+ \\ \theta_0 \end{array} ; \begin{array}{c} \lambda\text{-partition, local} \\ \theta_1 \end{array}\right)$$

the assertion that for every transitive λ^+-list $\vec{L} := \langle L_\delta \mid \delta \in \Delta \rangle$ either:

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Dichotomy for many lists

Definition

For λ an infinite regular cardinal, let $\mathsf{LD}(\lambda^+, \lambda)$ be the assertion that for all λ-many lists $\langle \vec{L}_j := \langle L_{\delta,j} \mid \delta \in \Delta_j \rangle \mid j < \lambda \rangle$ in $\mathsf{TL}(\lambda^+)$ either:

- there exists $A \in [\lambda^+]^{\lambda^+}$ such that for every $\delta < \lambda^+$ there exists $j < \lambda$, $\rho_{\vec{L}_j}(A \cap \delta) = 1$, or
- there exists a sequence $\langle A_i \mid i < \lambda \rangle$ such that $\bigcup_{i<\lambda} A_i = \bigcup_{j<\lambda} \Delta_j$ and, for all $i, j < \lambda$ and $X \in [A_i \cap \Delta_j]^\lambda$ we have $\rho_{\vec{L}_j}(X) \geq \lambda$.

Then $\mathsf{LD}(\lambda^+, \lambda) \Rightarrow \mathsf{LD}(\lambda^+) \Rightarrow \mathsf{LD}_{part,loc}(\lambda^+)$.

Application to trees

Lemma

Let $\vec{L} = \langle L_\delta \mid \delta \in \Delta \rangle$ be a transitive λ^+-list satisfying the second alternative of $\mathsf{LD}_{part,loc}(\lambda^+)$. Then, the list is special, i.e. there exists a map $c : \Delta \to \lambda$ such that $\gamma \notin L_\delta$ for all $\gamma, \delta \in \Delta$ satisfying $c(\gamma) = c(\delta)$.

A κ-tree T is Aronszajn if it has no branch, i.e. there is no linearly ordered subset of $(T, <_T)$ intersecting all nonempty levels of T. Note that such a tree never satisfies the first alternative of our dichotomies.

Corollary

Assume $\mathsf{LD}(\lambda^+)$. Then every λ^+-Aronszajn tree is special.
In particular λ^+-Souslin's hypothesis holds.

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Assume $\mathsf{LD}(\lambda^+)$*. Then every* λ^+*-Aronszajn tree is special.*
In particular λ^+*-Souslin's hypothesis holds.*

Application to lattices

Definition

A lower semi-lattice P is called λ^+-*Aronszajn* if it is well-founded and

- P is of size λ^+;
- For all $\alpha < \lambda^+$, $1 \leq |P_\alpha| \leq \lambda$;
- P does not contain any chain of size λ^+.

A lower semi-lattice is called λ^+-*Souslin* if it is λ^+-Aronszajn and every set of pairwise incomparable elements is of size $< \lambda^+$.

Theorem (Raghavan – Yorioka, 2014)

The following hold:

- $\mathrm{MA}_{\aleph_1}$ *implies there are no* $\aleph_1$*-Souslin lower semi-lattices.*
- $\mathrm{PID} + \mathfrak{p} > \omega_1$ *imply there are no* $\aleph_1$*-Souslin lower semi-lattices.*
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Application to lattices, cont'd

A λ^+-Aronszajn lower semi-lattice $(P, <)$ is called *special* if there exists a map $c : P \to \lambda$ such that for all $i < \lambda$, the set $c^{-1}(\{i\})$ is a set of p.i.e.

Theorem

Suppose $\mathrm{LD}(\lambda^+)$ *holds. Then there is no* λ^+*-Souslin lower semi-lattice. Moreover, every* λ^+*-Aronszajn lower semi-lattice is special.*

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Application to strongly unbounded colorings

Definition (Rinot – Shalev – Todorčević, 2024)

Suppose that $c : \lambda \times \lambda^+ \to \lambda$ is a coloring (let $c_\beta(\eta) := c(\eta, \beta)$).

- c is *unbounded* if for every cofinal $B \subseteq \lambda^+$, there is an $\eta < \lambda$ such that $\sup\{c_\beta(\eta) \mid \beta \in B\} = \lambda$;
- c is *strongly unbounded* if for every cofinal $B \subseteq \lambda^+$, there are $\eta < \lambda$ and $t : \eta \to \lambda$ so that $\sup\{c_\beta(\eta) \mid \beta \in B \ \& \ t \subseteq c_\beta\} = \lambda$.

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Corollary

Assuming $\mathsf{LD}(\omega_1)$, *then* $\mathfrak{b} > \aleph_1$.

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Application to towers

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A sequence $\mathcal{T} := \langle T_\alpha \mid \alpha < \lambda^+ \rangle$ of unbounded subsets of λ is a λ^+-*tower* in $(\mathcal{P}(\lambda), \subseteq^*)$ if $T_\alpha \subseteq^* T_\beta$ iff $\alpha < \beta$.

- It is *Souslin* if it contains no $\subseteq$-antichains of cardinality λ^+,
- It is *Hausdorff* if there is a map $c : \lambda^+ \to \lambda$ such that for $i, j < \lambda$ and $\alpha \in c^{-1}[\{i\}]$, the set $\{\xi \in c^{-1}[\{i\}] \cap \alpha \mid T_\xi \setminus T_\alpha \subseteq j\}$ is of size $< \lambda$.

Theorem (Borodulin-Nadzieja – Chodounský, 2015)

Assuming $\mathsf{MA}_{\aleph_1}$*, every* ω_1*-tower is Hausdorff, in particular non-Souslin.*

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Let λ *be a regular infinite cardinal and assume* $\mathsf{LD}(\lambda^+, \lambda)$*. Then every* λ^+*-tower in* $(\mathcal{P}(\lambda), \subseteq^*)$ *is Hausdorff.*

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A sequence $\langle (a_\alpha, b_\alpha) \mid \alpha < \lambda^+ \rangle$ in $[\lambda]^\lambda$ is a (λ^+, λ^+)-*pregap* in $\mathcal{P}(\lambda)$ if:

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A pregap $\mathcal{G} = \langle (a_\alpha, b_\alpha) \mid \alpha < \lambda^+ \rangle$ is a *gap* in $\mathcal{P}(\lambda)$ if there is no $c \in [\lambda]^\lambda$ such that $a_\alpha \subseteq^* c$ and $|b_\alpha \cap c| < \lambda$ for all $\alpha < \lambda^+$.

Recall that a pregap $\mathcal{G}$ is a gap iff for every $I \in [\lambda^+]^{\lambda^+}$, there are α and β in I so that $(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) \neq \emptyset$.

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Application to gaps, cont'd

Let $\mathcal{G} = \langle (a_\alpha, b_\alpha) \mid \alpha < \lambda^+ \rangle$ be a (λ^+, λ^+)-gap in $\mathcal{P}(\lambda)$.

- It is an S-gap iff for every $I \in [\lambda^+]^{\lambda^+}$ there are two distinct ordinals α and β in I such that $(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha) = \emptyset$;
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Clearly, any T-gap is an S-gap.

Theorem (Lopez – Todorčević, 2017)

It is consistent that there is an S-gap and that there are not T-gaps.

Theorem (Kunen)

If $\mathsf{MA}_{\aleph_1}$ holds, then there are no S-gaps.

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Assuming $\mathsf{LD}(\lambda^+)$ *there are no* (λ^+, λ^+)*-T-gaps in* $\mathcal{P}(\lambda)$*.*

Theorem (Shalev–Todorčević)

Suppose that κ *is a weakly compact cardinal and* $2^\kappa = \kappa^+$*. Then there is a partial order* $\mathbb{P}$ *such that in the forcing extension by* $\mathbb{P}$ *we have that* CH *holds and every* (ω_2, ω_2)*-gap in* $(\mathcal{P}(\omega_1), \subseteq^*)$ *is Hausdorff.*

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Application to Tukey order of directed sets

We say that a directed set D is λ^+-*nice* if $\mathsf{D} := (\lambda^+, \leq_{\mathsf{D}})$ is such that $\mathsf{D} \leq_T [\lambda^+]^{<\lambda}$, for each $\alpha <_{\mathsf{D}} \beta$ in λ^+ we have $\alpha < \beta$, and for every subset $X \in [\lambda^+]^{\lambda^+}$ there exists an unbounded subset $Y \subseteq X$ of size λ.

Let $\mathsf{D} = (\lambda^+, \leq_{\mathsf{D}})$ be a λ^+-nice directed set. Define a forcing notion $\mathbb{A}_{\mathsf{D}}$, where a condition q in $\mathbb{A}_{\mathsf{D}}$ is of the form $q = (A_q, B_q)$ for $A_q, B_q \in [\lambda^+]^{<\lambda}$. The order $\leq_{\mathbb{A}_{\mathsf{D}}}$ is defined by $q_1 \leq_{\mathbb{A}_{\mathsf{D}}} q_0$ iff:

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Lemma

Let $\mathsf{D} = (\lambda^+, \leq_{\mathsf{D}})$ be a λ^+-nice directed set, then $\mathbb{A}_{\mathsf{D}}$ is a $< \lambda$-closed partial order which adds $Y \subseteq \lambda^+$ of size $(\lambda^+)^V$ so that every λ-sized subset of Y is $\leq_{\mathsf{D}}$-unbounded. In particular, it is forced that $\mathsf{D} \equiv_T [(\lambda^+)^V]^{<\lambda}$.

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Application to Tukey theory of directed sets, cont'd

Theorem

Let $\lambda^{<\lambda} = \lambda$ *and* $\mathsf{LD}(\lambda^+)$. *Suppose* $\mathsf{D} = (D, <_D)$ *is a directed set so that* $\mathsf{D} <_T [\lambda^+]^{<\lambda}$ *and every* $X \in [D]^{\lambda^+}$ *contains an unbounded subset of size* λ. *Then there is a* λ^+*-c.c. forcing so that in the extension* $\mathsf{D} \equiv_T [\lambda^+]^{<\lambda}$.

Corollary (Todorčević, 1985)

Assuming $\mathsf{MA}_{\aleph_1}$, *every directed set* D *of size* $\aleph_1$ *in which every uncountable set contains a countable subset unbounded in* D *is Tukey equivalent to* $[\omega_1]^{<\omega}$.

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Consistency at ω_1

Definition (Chodounský – Zapletal, 2015)

A poset P satisfies Y-c.c. if for every countable elementary submodel $M \prec H_\theta$ containing P end every condition $q \in P$ there is a filter $F \in M$ on the completion $\mathrm{ro}(P)$ such that $\{p \in \mathrm{ro}(P) \cap M : p \geq q\} \subseteq P$.

This notion is preserved under the finite support iteration.

Theorem (Chodounský – Zapletal, 2015)

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Consistency at ω_1, cont'd

Definition

P is *strong* Y-c.c. if there exists a function w defined on P such that

- for every $p \in P$, $w(p)$ is a finite set;
- if $p, q \in P$ are compatible, then they have a lower bound $r \leq p, q$ such that $w(r) = w(p) \cup w(q)$;
- whenever $\langle p_\alpha \mid \alpha < \omega_1 \rangle$ and $\langle q_\alpha \mid \alpha < \omega_1 \rangle$ are subsets of P such that $\langle w(p_\alpha) \mid \alpha < \omega_1 \rangle$ and $\langle w(q_\alpha) \mid \alpha < \omega_1 \rangle$ are Δ-systems with the same root, there are ordinals $\alpha, \beta < \omega_1$ so that p_α and q_β are compatible.

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Theorem

$\mathsf{MA}_{\aleph_1}(Y - \textit{c.c.})$ *implies* $\mathsf{LD}(\omega_1, \omega)$

$\mathsf{MA}_{\aleph_1}(Y-\text{c.c.})$ implies $\mathsf{LD}(\omega_1,\omega)$

Assume that $\vec{L}_j := \langle L_{\delta,j} \mid \delta \in \Delta_j \rangle$ is in $\mathsf{TL}(\omega_1)$ for every $j < \omega$.

Let condition (1) of Definition of $\mathsf{LD}(\omega_1,\omega)$ fail: for every $A \in [\omega_1]^{\omega_1}$, there is $\delta < \omega_1$ such that $\rho_{\vec{L}_j}(A \cap \delta) > 1$ for every $j < \omega$.

We define a forcing notion $\mathbb{A}_{\vec{L}}$ where a condition p is as follows:

- p is a finite function with domain a subset of ω;
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the order defined by $q \leq p$ iff:

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$\mathbb{A}_{\vec{L}}$ *is strong Y-c.c.*

Central claim

For $j < \omega$, uncountable subsets $R, T \subseteq \omega_1$, and two sequences of distinct elements $\langle a_\alpha \mid \alpha \in R\rangle \subseteq \Delta_j$ and $\langle b_\beta \mid \beta \in T\rangle \subseteq \Delta_j$, there are $R' \in [R]^{\omega_1}$ and $T' \in [T]^{\omega_1}$ such that $a_\alpha \notin L_{b_\beta,j}$ for every $\alpha \in R'$ and $\beta \in T'$.

Proof: Consider the set $Z(R) := \{\beta < \omega_1 \mid |\{\alpha \in R \mid \beta \in L_{a_\alpha,j}\}| = \aleph_1\}$. To see it is uncountable, fix $\beta < \omega_1$. As $C(\vec{L}_j) = \{\gamma \in \mathrm{acc}(\kappa) \mid \forall \delta \in \Delta \setminus \gamma \left(\sup(L_{\delta,j} \cap \gamma) = \gamma\right)\}$, let $\gamma := \min(C(\vec{L}_j) \setminus (\beta + 1))$ and $\eta := \sup\{\alpha \in R \mid a_\alpha \leq \gamma\}$. As $\gamma \in C(\vec{L}_j)$, for $\alpha \in R \setminus (\eta + 1)$ we have $\sup(L_{a_\alpha,j} \cap \gamma) = \gamma$. Hence there is $c_\alpha \in (\beta, \gamma)$ such that $c_\alpha \in L_{a_\alpha,j}$. So there is $c \in (\beta, \gamma)$ so that $\{\alpha \in R \mid c = c_\alpha \in L_{a_\alpha,j}\}$ is uncountable.

As $Z(R)$ is uncountable and (1) fails, there is $\delta < \omega_1$ so that $\rho_{\vec{L}_j}(Z(R) \cap \delta) > 1$. We claim that for $d \in Z(R) \cap \delta$, the set $\{\beta \in T \mid d \notin L_{b_\beta,j}\}$ is uncountable. Otherwise for all $d \in Z(R) \cap \delta$ the set $\{\beta \in T \mid d \notin L_{b_\beta,j}\}$ is countable. As $Z(R) \cap \delta$ is countable and T is uncountable, for some $\beta \in T$, for all $d \in Z(R) \cap \delta$ we have $d \in L_{b_\beta,j}$ which contradicts the assumption that $\rho_{\vec{L}_j}(Z(R) \cap \delta) > 1$.

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For $j < \omega$, uncountable subsets $R, T \subseteq \omega_1$, and two sequences of distinct elements $\langle a_\alpha \mid \alpha \in R \rangle \subseteq \Delta_j$ and $\langle b_\beta \mid \beta \in T \rangle \subseteq \Delta_j$, there are $R' \in [R]^{\omega_1}$ and $T' \in [T]^{\omega_1}$ such that $a_\alpha \notin L_{b_\beta, j}$ for every $\alpha \in R'$ and $\beta \in T'$.

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Consistency at ω_1, cont'd

A *c.c.c. partition* $[X]^{<\omega} = K_0 \cup K_1$ is defined as follows: for all sequences $\langle a_\xi \mid \xi < \omega_1 \rangle$ of finite subsets of X, either $[a_\xi]^{<\omega} \not\subseteq K_0$ for some $\xi < \omega_1$, or there are $\xi \neq \eta$ so that $[a_\xi \cup a_\eta]^{<\omega} \subseteq K_0$. Similarly for $[X]^n = K_0 \cup K_1$.

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Theorem (Larson-Todorčević, 2002)

Both $\mathsf{MA}_{\aleph_1}(S)[S]$ *and* $\mathscr{K}_2'$ *imply that* $\mathscr{K}_2'(rec)$ *holds.*

Theorem

$\mathscr{K}_2'(rec)$ *implies* $\mathsf{LD}_{\omega_1,loc}(\omega_1)$*, while* $\mathscr{K}_2'$ *implies* $\mathsf{LD}_{part,loc}(\omega_1)$*.*

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Definition

Let $\kappa < \omega$, $t : k \to \{<, >\}$, and $a, b \in [\mathbb{R}]^k$. We say that (a, b) realizes t if $a(i)t(i)b(i)$ for $i < k$. By $T(a, b)$ we denote the unique t realized by (a, b).

Definition (Abraham – Shelah, 1981)

Let $E \in [\mathbb{R}]^{\omega_1}$ and $k < \omega$. E is k*-entangled* if for each uncountable family $\mathcal{A} \subseteq [E]^k$ of pairwise disjoint sets and $t : k \to \{<, >\}$ there are $a, b \in \mathcal{A}$ so that $T(a, b) = t$. E is *entangled* if it is k-entangled for each $k < \omega$.

Theorem (Todorčević, 1989)

If $\theta = \theta^{<\theta} > \aleph_0$, then there is a finite support c.c.c. iteration $\mathbb{P}_\theta$ such that $V^{\mathbb{P}_\theta}$ satisfies $2^{\aleph_0} = \theta$ + MA(productive c.c.c.) + "every Aronszajn tree is special" + "there exists an entangled set of reals".

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Consistency at ω_2

For an ordinal α, V_α is the set all well-founded sets of rank smaller than α.

For a set M, a formula φ is Π^1_1 over M if it is of the form $(\forall X)\psi$ where X is a second order variable (interpreted as a subset of M) and ψ may have subsets and elements of M as parameters.

For a regular uncountable cardinal κ, a filter $\mathcal{F}$ over κ is *normal* if it does not contain any bounded subset of κ and for every regressive function $f : S \to \kappa$ for $S \in \mathcal{F}$, there is $\alpha < \kappa$ such that $f^{-1}[\{\alpha\}] \in \mathcal{F}$.

κ is *weakly compact* if for every Π^1_1 formula φ over V_κ and every $R \subseteq V_\kappa$, if $(V_\kappa, \in, R) \models \varphi$, then there is an $\alpha < \kappa$ such that $(V_\alpha, \in, R \cap V_\alpha) \models \varphi$.

Then the family $\{U_{R,\varphi} \mid R \subseteq V_\kappa$ and φ is a Π^1_1-formula over $V_\kappa\}$, where $U_{R,\varphi} := \{\alpha < \kappa \mid$ if $(V_\kappa, \in, R) \models \varphi$, then $(V_\alpha, \in, R \cap V_\alpha) \models \varphi\}$, generates a normal and $< \kappa$-complete filter on κ (denoted $\mathcal{U}$). Then

- The set of all strongly inaccessible cardinals below κ is in $\mathcal{U}$;
- Every club in κ is an element of $\mathcal{U}$.

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κ is *weakly compact* if for every Π^1_1 formula φ over V_κ and every $R \subseteq V_\kappa$, if $(V_\kappa, \in, R) \models \varphi$, then there is an $\alpha < \kappa$ such that $(V_\alpha, \in, R \cap V_\alpha) \models \varphi$.

Then the family $\{U_{R,\varphi} \mid R \subseteq V_\kappa$ and φ is a Π^1_1-formula over $V_\kappa\}$, where $U_{R,\varphi} := \{\alpha < \kappa \mid$ if $(V_\kappa, \in, R) \models \varphi$, then $(V_\alpha, \in, R \cap V_\alpha) \models \varphi\}$, generates a normal and $< \kappa$-complete filter on κ (denoted $\mathcal{U}$). Then

- The set of all strongly inaccessible cardinals below κ is in $\mathcal{U}$;
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Consistency at ω_2

For an ordinal α, V_α is the set all well-founded sets of rank smaller than α.

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Consistency at ω_2, cont'd

Theorem

Suppose that κ is a weakly compact cardinal and $2^\kappa = \kappa^+$. Then there is a partial order $\mathbb{P}$ such that both CH *and* $\mathsf{LD}(\omega_2, \omega_1)$ *hold in the forcing extension by $\mathbb{P}$.*

The proof uses a modification of the method developed by Laver and Shelah for a proof of the consistency of SH_{ω_2} and CH.

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The poset

Let $\vec{L} := \langle \vec{L}_j := \langle L_{\delta,j} \mid \delta \in \Delta_j \rangle \mid j < \omega_1 \rangle$ be a sequence of ω_1-many transitive ω_2-lists. We define forcing $\mathbb{A}_{\vec{L}}$ where a condition p is as follows:

- p is a partial map with countable domain which is a subset of ω_1;
- for $i \in \mathrm{dom}(p)$, $p(i) = (A_i^p, B_i^p)$ for $A_i^p \in [\bigcup_{j \leq i} \Delta_j]^{\leq \omega}$, $B_i^p \in [\omega_2]^{\leq \omega}$.

The order is defined by letting $q \leq p$ iff:

- $\mathrm{dom}(q) \supseteq \mathrm{dom}(p)$ and for every $i \in \mathrm{dom}(p)$ and $j < \mathrm{ssup}(\mathrm{dom}(p))$:
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Then $\mathbb{A}_{\vec{L}}$ is a σ-closed partial order such that in the forcing extension there exists a partition $\langle A_i \mid i < \omega_1 \rangle$ of $\bigcup_{j<\omega_1} \Delta_j$ such that for all $i, j < \omega_1$ and $X \in [A_i \cap \Delta_j]^{\omega_1}$ we have $\rho_{\vec{L}_j}(X) \geq \omega_1$.

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Iteration

Assume GCH, that κ is weakly compact, and $\mathcal{U}$ a normal, $<\kappa$-closed filter on κ. Let $\mathcal{P}$ be the set of pairs $(\mathbb{R}, \tau)$ so that $\mathbb{R}$ is a forcing in H_{κ^+}, and τ an $\mathbb{R}$-very nice name for an element of $\mathsf{TL}(\kappa, \omega_1)$. Fix an enumeration $\langle(\mathbb{R}_\xi, \tau_\xi) \mid \xi < \kappa^+\rangle$ of $\mathcal{P}$ so that each pair is listed cofinally often.

We force with $\mathbb{P}_{\kappa^+}$, where $(\langle\mathbb{P}_\xi \mid \xi \leq \kappa^+\rangle, \langle\dot{\mathbb{Q}}_\xi \mid \xi < \kappa^+\rangle)$ is the countable support iteration satisfying that for every $\xi < \kappa^+$:

- $\mathbb{Q}_0 := \mathrm{Col}(\omega_1, \kappa)$.
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If $\mathbb{P}_{\kappa^+}$ is κ-c.c., then $\mathbb{1}_{\mathbb{P}_{\kappa^+}}$ forces $2^{\aleph_0} = \aleph_1$ and $\mathsf{LD}(\omega_2, \omega_1)$.

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We assume that for $\xi \leq \kappa^+$ the forcing $\mathbb{P}_\xi$ consists of ξ-sequences whose each element is either a map with domain subset of ω_1 and the value at each point is a pair of countable subsets of κ or the empty set (except the first element which is an element of $\mathrm{Col}(\omega_1, \kappa)$).

For $\xi < \kappa^+$ enumerate it as $\vec{\xi} = \{\xi_\gamma \mid \gamma \in |\xi|\}$. Now fix $\xi < \kappa^+$.

Given a condition $p \in \mathbb{A}_{\vec{L}}$ and α an ordinal, we define $p|_\alpha$ to be the map with domain $\mathrm{dom}(p) \cap \alpha$ such that for every $i \in \mathrm{dom}(p) \cap \alpha$ we have $p|_\alpha(i) := (A_i^p \cap \alpha, B_i^p \cap \alpha)$.

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Technical setup for the inductive hypothesis, cont'd

Let $\alpha < \kappa$, $\eta \leq \xi$, and $f \in \mathbb{P}_\eta$. Let $f|_\alpha^\xi$ be the function h with domain η such that

- If $\nu \in \{\xi_\gamma \mid \gamma < \alpha\} \cap \eta$, then
  - $\nu = 0 \implies h(\nu) = f(0) \restriction (\alpha \times \omega_1)$;
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- otherwise $h(\nu) = \emptyset$.

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For simplification, we write $\#_\alpha^{\xi,\eta}(f, h)$ to denote $\#_\alpha^{\xi,\eta}(f, f, h)$ and $\#_\alpha^\xi(f, g, h)$ to denote $\#_\alpha^{\xi,\xi}(f, g, h)$.

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Inductive hypothesis

In order to prove that $\mathbb{P}_\xi$ satisfies the κ-c.c. for every $\xi \leq \kappa^+$, we will need the following inductive hypothesis.

Inductive hypothesis: For all $\eta < \xi$, the poset $\mathbb{P}_\eta$ satisfies the following property, called the $\mathcal{U}^+$-*cross condition*: For all $W \in \mathcal{U}^+$ and all sequences $\langle \langle f_\alpha, g_\alpha, h_\alpha \rangle \mid \alpha \in W \rangle$ such that $\#^\eta_\alpha(f_\alpha, g_\alpha, h_\alpha)$ for every $\alpha \in W$, there exists a $U \in \mathcal{U}^+$ subset of W such that for every $\alpha < \beta$ in U conditions f_α and g_β are $\mathbb{P}_\eta$-compatible.

Lemma

Suppose $\mathbb{P}_\eta$ satisfies the $\mathcal{U}^+$-cross condition. Then $\mathbb{P}_\eta$ satisfies the $\mathcal{U}^+$-Knaster condition and in particular the κ-chain condition.

Proof: We fix a sequence of conditions $\langle p_\alpha \mid \alpha < \kappa \rangle$ in $\mathbb{P}_\eta$. Since $\kappa \in \mathcal{U}^+$ and $\#^\eta_\alpha(p_\alpha, p_\alpha, p_\alpha|^\eta_\alpha)$ trivially holds for all $\alpha < \kappa$. By the assumption there exists $U \in \mathcal{U}^+$ such that for every $\alpha < \beta$ in U the conditions p_α and p_β are $\mathbb{P}_\eta$-compatible which is as sought.

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