Parseval's theorem

We now prove another very important theorem using the Convolution Theorem. We first give its statement:

The Parseval's theorem states that the inner product between signals is preserved in going from time to the frequency domain.

i.e. $$\int_{-\infty}^{\infty} x(t)\overline{y(t)}\,dt = \int_{-\infty}^{\infty} X(f)\overline{Y(f)}\,df$$ *

where **X(f)**, **Y(f)**, are the Fourier Transforms of **x(t)**, **y(t)** respectively.

If we take **x(t) = y(t)**, $$\int_{-\infty}^{\infty} |x(t)|^2\,dt = \int_{-\infty}^{\infty} |X(f)|^2\,df$$

This is interpreted physically as *"**Energy calculated in the time domain is same as the energy calculated in the frequency domain**"*.

|X(.)|² *is called the "**Energy Spectral Density**".*

Proof:

From the convolution theorem, we have,

$$\int_{-\infty}^{+\infty} x(\lambda)h(t-\lambda)\,d\lambda = \int_{-\infty}^{+\infty} X(f)H(f)e^{j2\pi ft}\,df$$

Also,

$$y(-t) \xrightarrow{F.T.} Y(-f) \quad \text{and} \quad \overline{y(t)} \xrightarrow{F.T.} \overline{Y(-f)}$$

$$\therefore \overline{y(-t)} \xrightarrow{F.T.} \overline{Y(f)}$$

We perform the convolution of $x(t)$ and $\overline{y(-t)}$ and using convolution theorem,

$$\int_{-\infty}^{+\infty} x(\lambda)\overline{y(\lambda-t)}\,d\lambda = \int_{-\infty}^{+\infty} X(f)\overline{Y(f)}e^{j2\pi ft}\,df$$

Put $t = 0$ in the above equation, we get,

$$\int_{-\infty}^{+\infty} x(\lambda)\overline{y(\lambda)}\,d\lambda = \int_{-\infty}^{+\infty} X(f)\overline{Y(f)}\,df$$

$$\text{i.e. } \int_{-\infty}^{+\infty} x(t)\overline{y(t)}\,dt = \int_{-\infty}^{+\infty} X(f)\overline{Y(f)}\,df$$

Hence Proved.

Differentiation/Integration

$$X(f) = \int x(t)e^{-j2\pi ft}dt$$

$$\frac{dX(f)}{df} = \int x(t)\frac{d(e^{-j2\pi ft})}{df}dt$$

$$\frac{dX(f)}{df} = \int[-j2\pi t x(t)]e^{-j2\pi ft}dt$$

Hence if

$$x(t) \xrightarrow{FT} X(f)$$

then

$$-j2\pi t x(t) \xrightarrow{FT} X'(f)$$

Now,

$$x(t) = \int X(f)e^{j2\pi ft}df$$
$$\frac{dx(t)}{dt} = \int [j2\pi f X(f)]e^{j2\pi ft}df$$

Hence if,

$$x(t) \xrightarrow{FT} X(f)$$

then,

$$x'(t) \xrightarrow{FT} j2\pi f X(f)$$

The inverse operation of taking the derivative is running the integral : $\int_{-\infty}^{t} x(\lambda)d\lambda$

eg : $$\int_{-\infty}^{t} \delta(\lambda)d\lambda = u(t)$$

let

$$y(t) = \int_{-\infty}^{t} x(\lambda)d\lambda$$
$$\frac{dy(t)}{dt} = x(t)$$
$$x(t) \xrightarrow{FT} X(f)$$
$$\int_{-\infty}^{t} x(\lambda)d\lambda \xrightarrow{FT} \frac{X(f)}{j2\pi f}$$

This causes problem when

$$x(t) = \delta(t)$$
$$\delta(t) \xrightarrow{FT} 1$$

$u(t) \xrightarrow{FT} \frac{1}{j2\pi f} +$ impulse in frequency.

Example:

$$e^{-t}u(t) \xrightarrow{FT} \frac{1}{1+j2\pi f}$$

$$te^{-t}u(t) \xrightarrow{FT} \frac{j}{2\pi}\frac{d}{df}\left(\frac{1}{1+j2\pi f}\right)$$

$$te^{-t}u(t) \xrightarrow{FT} \frac{1}{(1+j2\pi f)^2}$$

Scaling of the independent variable by a real constant a

When a > 0 or a < 0

$$\int_{-\infty}^{\infty} x(at)\, e^{-j2\pi f t}\, dt$$

$$= \frac{1}{|a|}\int_{-\infty}^{\infty} x(\lambda)\, e^{-j2\pi f \frac{\lambda}{a}}\, d\lambda \qquad (\text{put } at = \lambda)$$

$$= \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

or

$$|a|^{1/2}\, x(at) \xrightarrow{FT} \frac{1}{|a|^{1/2}} X\left(\frac{f}{a}\right)$$

Hence the scaling of the independent variable is a self-dual operation.

Consider

$$\int_{-\infty}^{\infty} \left| |a|^{1/2}\, x(at) \right|^2 dt = |a| \times \int_{-\infty}^{\infty} \frac{|x(\lambda)|^2}{|a|}\, d\lambda$$

$$= \int_{-\infty}^{\infty} |x(\lambda)|^2\, d\lambda$$

Hence, x(t) and $|a|^{1/2}$ x(at) have the same energy. Therefore such scaling is called energy normalized scaling of the independent variable.