Memory and linear systems

We shall look at different system properties like

- **memory**
- **linearity**
- **shift-invariance**
- **stability**
- **causality**

Memory:

Memory is a property relevant only to systems whose input and output signals have the *same independent variable*. A system is said to be memoryless if its output for each value of the independent variable is dependent only on the input signal at that value of independent variable.

For example the system with description : y(t) = 5x(t) (y(t) is the output signal corresponding to input signal x(t)) is memoryless. In the physical world a resistor can be considered to be a memoryless system (with voltage considered to be the input signal, current the output signal).

By definition, a system that does not have this property is said to have memory.

How can we identify if a system has memory?

For a memoryless system, changing the input at any instant can change the output *only at that instant*. If, in some case, a change in input signal at some instant changes the output at some other instant, we can be sure that the system has memory.

Note: Consider a system whose output Y(t) depends on input X(t) as: Y(t) = X(t-5) + { X(t) - X(t-5) }

While at first glance, the system might appear to have memory, it does not. This brings us to the idea that given the description of a system, it need not at all be the most economical one. The same system may have more than one description.

Examples:

Assume **y[n]** and **y(t)** are respectively outputs corresponding to input signals **x[n]** and **x(t)**

1. The identity system **y(t) = x(t)** is of-course **Memoryless**
2. System with description **y[n] = x[n-5] has memory**. The input at any "instant" depends on the input 5 "instants" earlier.
3. System with description

$$y(t) = \int_{-\infty}^{t} x(\lambda)\, d\lambda$$

also **has memory**. The output at any instant depends on all past and present inputs.

Linearity:

Now we come to one of the most important and revealing properties systems may have - Linearity. Basically, the principle of linearity is equivalent to the principle of *superposition*, i.e. a system can be said to be linear if, for any two input signals, their linear combination yields as output the same linear combination of the corresponding output signals.

Definition:

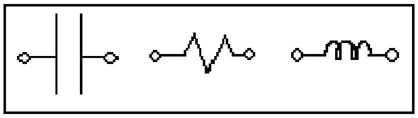
Given any two inputs , acted on by the system as follows $X_1(t) -> S -> Y_1(s)$; $X_2(t) -> S -> Y_2(s)$ The system is said to be linear if for any two constants and the signal $X(t) = \alpha X_1(t) + \beta X_2(t)$ The system acts on it follows

$$X(t) -> S -> Y(s) \text{ , where } Y(s) = \alpha Y_1(s) + \beta Y_2(s)$$

(It is not necessary for the input and output signals to have the same independent variable for linearity to make sense. The definition for systems with input and/or output signal being discrete-time is similar.)

Example of linearity

A capacitor, an inductor, a resistor or any combination of these are all linear systems, if we consider the voltage applied across them as an input signal, and the current through them as an output signal. This is because these simple passive circuit components follow the principle of superposition within their ranges of operation.



Additivity and Homogeneity:

Linearity can be thought of as consisting of two properties:

- **Additivity**

  system is said to be additive if for any two input signals $\mathbf{x_1(t)}$ and $\mathbf{x_2(t)}$,
  $X_1(t) \rightarrow S \rightarrow Y_1(s)\ ;\ X_2(t) \rightarrow S \rightarrow Y_2(s)$
  $\rightarrow [X_1(t) + X_2(t)] \rightarrow S \rightarrow [Y_1(s) + Y_2(s)]$
  i.e. the output corresponding to the sum of any two inputs is the sum of the two outputs.

- **Homogeneity (Scaling)**

A system is said to be homogenous if, for any input signal **X(t)**,

$$X(t) \rightarrow S \rightarrow Y(s) \Rightarrow \alpha X(t) \rightarrow S \rightarrow \alpha Y(s) \quad \forall \alpha$$

i.e. scaling any input signal scales the output signal by the same factor.

A system is linear is equivalent to saying the system obeys both additivity and homogeneity.

Additivity and homogeneity are independent properties**.**

We can prove this by finding examples of systems which are additive but not homogeneous, and vice versa.

Again, **y(t)** is the response of the system to the input **x(t)**.

Example of a system which is additive but not homogeneous:

$$Y(t) = \overline{X(t)}$$

[It is homogeneous for real constants but not complex ones - consider $\alpha = i = \sqrt{-1}$

nal and the output may be the same signal played backwards. This is clearly not causal

Example of a system which is homogeneous but not additive: $Y(t) = \dfrac{\{X(t)\}^2}{X(t-1)}$

Examples of Linearity: Assume **y[n]** and **y(t)** are respectively outputs corresponding to input signals **x[n]** and **x(t)**

1) System with description $\mathbf{y(t) = t \cdot x(t)}$ is linear.

Consider any two input signals, $\mathbf{x_1(t)}$ and $\mathbf{x_2(t)}$, with corresponding outputs $\mathbf{y_1(t)}$ and $\mathbf{y_2(t)}$.

a and **b** are arbitrary constants. The output corresponding to $\mathbf{a.x_1(t) + b.x_2(t)}$ is

$\mathbf{= t\,(a.x_1(t) + b.x_2(t))}$

$\mathbf{= t.a.x_1(t) + t.b.x_2(t)}$, which is the same linear combination of $\mathbf{y_1(t)}$ and $\mathbf{y_2(t)}$.

Hence proved.

2) The system with description $y(t) = \{x(t)\}^2$ is not linear.