Orthogonal Vector Space & Orhogonal signal space

Orthogonal Vector Space:

If unit vectors along these n mutually perpendicular coordinates are designated as $x_1, x_2, ...x_n$ and a general vector **A** in this n dimensional space has components $C_1, C_2, ...C_n$ respectively along these n coordinates, then

$$A = C_1 x_1 + C_2 x_2 + C_3 x_3 + \cdots + C_n x_n$$

All the vectors $x_1, x_2, ...x_n$ are mutually orthogonal, and the set must be complete in order for any general vector **A** to be represented by Eq.3.14. The condition of orthogonality implies that the dot product of any two vectors x_n and x_m must be zero, and the dot product of any vector with itself must be unity. This is the direct extension of Eq.3.13 and can be expressed as

$$x_m \cdot x_n = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

The constants $C_1, C_2, C_3 ....C_n$ in Eq.3.14 represent the magnitudes of the components of **A** along the vectors $x_1, x_2, x_3 ......x_n$ respectively.

It follows that

$$C_r = A \bullet x_r$$

Orhoganal signal space:

Approximation of a Function by a Set of Mutually Orthogonal Functions

$$f(t) \simeq C_1 g_1(t) + C_2 g_2(t) + \cdots + C_k g_k(t) + \cdots C_n g_n(t)$$

$$= \sum_{r=1}^{n} C_r g_r(t)$$

For the best approximation we must find the proper values of constants $C_1, C_2, C_3.......C_n$ such that ε, the mean square of $f_e(t)$, is minimized.

By definition,

$$f_e(t) = f(t) = \sum_{r=1}^{n} C_r g_r(t)$$

and

$$\varepsilon = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left[f(t) = \sum_{r=1}^{n} C_r g_r(t) \right]^2 dt$$

It is evident from Eq.3.26 that ε is a function of $C_1, C_2,.......C_n$ and to minimize ε, we must have

$$\frac{\partial \varepsilon}{\partial C_1} = \frac{\partial \varepsilon}{\partial C_2} = \cdots \frac{\partial \varepsilon}{\partial C_i} = \cdots \frac{\partial \varepsilon}{\partial C_{ni}} = 0$$

$$C_i = \frac{\int_{t_1}^{t_2} f(t) g_i(t) dt}{\int_{t_1}^{t_2} g_i^{\,2}(t) dt}$$

$$= \frac{1}{K_i} \int_{t_1}^{t_2} f(t) g_i(t) dt$$

We may summarize this result as follows. Given a set of n functions $g_1(t), g_2(t), \cdots g_n(t)$ mutually orthogonal over the interval (t_1, t_2), it is possible to approximate an arbitrary function $f(t)$ over this interval by a linear combination of these n functions.

$$f(t) \simeq C_1 g_1(t) + C_2 g_2(t) + \cdots + C_n g_n(t)$$

$$= \sum_{r=1}^{n} C_r g_r(t)$$

Evaluation of Mean Square Error

Let us now find the value of ε when optimum values of coefficients $C_1, C_2, ...C_n$ are chosen according to Eq.. By definition

$$\varepsilon = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} \left[f(t) = \sum_{r=1}^{n} C_r g_r(t) \right]^2 dt$$

$$= \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} \left[f^2(t)dt + \sum_{r=1}^{n} C_r{}^2 \int_{t_1}^{t_2} g_r{}^2(t)dt - 2\sum_{r=1}^{n} C_r \int_{t_1}^{t_2} f(t) g_r(t)dt \right]$$

But from the above equation $\int_{t_1}^{t_2} f(t) g_r(t)dt = C_r \int_{t_1}^{t_2} g_r{}^2(t)dt = C_r K_r$

Substituting we get

$$\varepsilon = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} \left[f^2(t)dt + \sum_{r=1}^{n} C_r{}^2 K_r - 2\sum_{r=1}^{n} C_r{}^2 K_r \right]$$

$$= \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} \left[f^2(t)dt - \sum_{r=1}^{n} C_r{}^2 K_r \right]$$

$$= \frac{1}{(t_2 - t_1)} \left[\int_{t_1}^{t_2} f^2(t)dt - (C_1{}^2 K_1 + C_2{}^2 K_2 + \cdots C_n{}^2 K_n) \right]$$