The Fourier Transform properties

1. Duality of the Fourier Transform

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft}\, df \quad \& \quad X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt$$

Notice a certain symmetry in these two system transformations.

Say **y(t)** has a Fourier Transform **Y(f)**, then : $Y(f) = \int_{-\infty}^{\infty} y(\lambda)\, e^{-j2\pi\lambda f}\, d\lambda$

What is the transform of **Y(t)** ? Or, which signal on Inverse Fourier transformation would yield **Y(t)** ?

Recall the Inverse Transformation equation above, and put $\lambda = -\gamma$ in the equation for **Y**:

$$Y(t) = \int_{-\infty}^{\infty} y(-\gamma)\, e^{j2\pi\gamma t}\, d\gamma$$

Therefore, **y(-f)** is the Fourier transform of **Y(t)** (where Y(f) is the Fourier transform of y(t)) !

This remarkable relationship between a signal and its Fourier transform is called the Duality of the Fourier Transform.

i.e: $\boxed{x(.) \xrightarrow{FT} X(.) \;\Rightarrow\; X(.) \xrightarrow{FT} x(-.)}$

Duality implies a very remarkable relationship between the Fourier transform and its inverse. Notice the relationship between the Fourier Transform and the Fourier Inverse of X above:

$$x(.) \xleftarrow{FT^{-1}} X(.) \xrightarrow{FT} x(-.)$$

This gives us a very important insight into the nature of the Fourier transform. We will use it to prove many "dual" relationships: if some result holds for the Fourier Transform, a dual result will hold for the Inverse transform as well. We will encounter some examples soon.

2. Linearity

Both the Fourier transform and its inverse system are linear. Thus the Fourier transform of a linear combination of two signals is the same linear combination of their respective transforms. The same, of-course holds for the Inverse Fourier transform as well.

3.Memory

The independent variable for the input and output signals in these systems is not the same, so technically we can't talk of memory with respect to the Fourier transform and its inverse. But what we can ask is: if one changes a time signal locally, will only some corresponding local part of the transform change? Not quite.

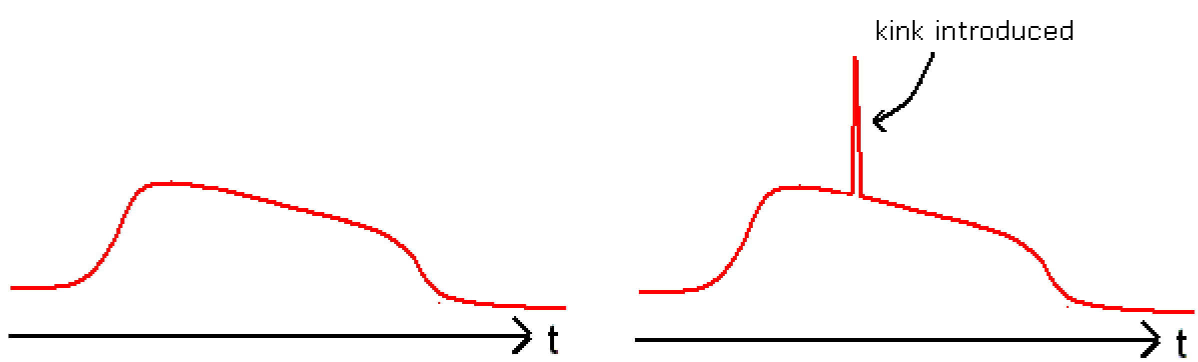

Figure 3.2

Introducing a local kink like in the above time-signal causes a large, spread-out distortion of the spectrum. In fact, the *more local the kink, the more spread-out the distortion*!

By duality,one can say the same about the inverse Fourier transform.

I.e: if **x(.)** has a Fourier transform **X(.)**, using Duality and the above discussion, we can say that introducing a local distortion in **X(.)** will cause a "wide-spread" distortion in **x(-.)**. But **x(.)** is also the inverse Fourier transform of this locally changed **X(.)**. Thus introducing a local kink in the spectrum of a signal changes it drastically.

4. Shift invariance

Again, we can't talk of shift variance/invariance with these systems as the independent variable for the input and output signals is not the same. But we can examine what happens to the spectrum of a signal on time-shifting it, and vice-versa.

$$\text{Say} \quad x(t) \xrightarrow{FT} X(f)$$

$$\therefore\ x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft}\, df$$

$$\therefore\ x(t-t_0) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f(t-t_0)}\, df = \int_{-\infty}^{\infty} \left(X(f)\, e^{-j2\pi ft_0}\right) e^{j2\pi ft}\, df$$

$$\Rightarrow \boxed{x(t-t_0) \xrightarrow{FT} X(f)\, e^{-j2\pi ft_0}}$$

Notice that nowhere has the magnitude of **X(f)** changed. **Only a phase (or argument) change** that is linear in frequency has taken place.

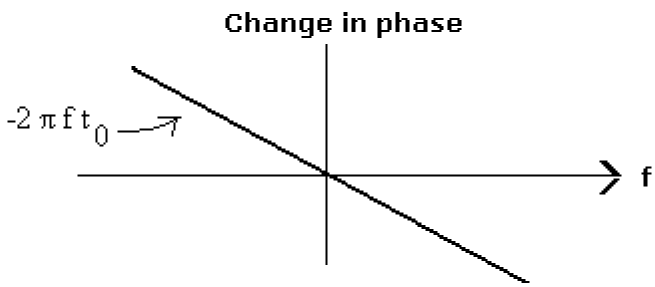

Figure 3.3

Let us, using Duality examine the effect of translating the spectrum on the time-signal.

$$\text{Assume } x(t) \xrightarrow{FT} X(f)$$
$$\text{then by duality, } X(t) \xrightarrow{FT} x(-f)$$
$$\therefore\ X(t - \alpha) \xrightarrow{FT} e^{-j2\pi\alpha f}\, x(-f)$$
$$\text{Applying Duality again,}$$
$$e^{-j2\pi\alpha t}\, x(-t) \xrightarrow{FT} X(-f - \alpha)$$
$$\Rightarrow e^{j2\pi\alpha t}\, x(t) \xrightarrow{FT} X(f - \alpha) \quad \text{or} \quad \boxed{X(f - f_0) \xrightarrow{FT^{-1}} e^{j2\pi f_0 t}\, x(t)}$$

5.Stability

Are our systems BIBO stable? i.e.: Will a bounded input necessarily give rise to a bounded output? No.

The integrals that describe the two systems need not converge for a bounded input signal. e.g.: they don't converge for a non-zero constant input signal.

Now that we have come to the issue of the Fourier transform and the Inverse Fourier transform not converging for a constant input signal, let us see what the Transform of the unit impulse is.

Note that the impulse, far from satisfying Dirichlet's conditions, is not even a function. It falls in the class of generalized functions. Thus what we are doing is extending our idea of the Fourier Transform. Why? Because we will find it useful.

$$\int_{-\infty}^{\infty} \delta(t)\, e^{-j2\pi f t}\, dt = 1 \text{ for all f}$$

That is, the Fourier transform of the unit impulse is the identity function. Thus, even though the inverse equation does not converge for the identity function, we say that that Fourier Transform of the unit impulse is the identity function.

$$\delta(t) \xrightarrow{FT} I(f) \quad \text{where } I(f) = 1\ \forall\ f$$

Why stop here? Consistent with duality, we say that the Fourier Transform of the identity function is the unit impulse:

$$1(t) \xrightarrow{FT} \delta(-f) = \delta(f)$$

We will even apply the time-shift and frequency-shift properties we have just proved to make further generalizations:

$$\delta(t - t_0) \xrightarrow{FT} e^{-j2\pi f t_0}$$

$$\&$$

$$e^{j2\pi f_0 t} \xrightarrow{FT} \delta(f - f_0)$$