Shift variant and causal systems

Shift Invariance

This is another important property applicable to systems with the same independent variable for the input and output signal. We shall first define the property for continuous time systems and the definition for discrete time systems will follow naturally.

Definition:

Say, for a system, the input signal **x(t)** gives rise to an output signal **y(t)**. If the input signal **$x(t - t_0)$** gives rise to output **$y(t - t_0)$**, for every **t_0**, and every possible input signal, we say the system is shift invariant.

i.e. for every permissible **x(t)** and every **t_0** $x(t) \to y(t) \Rightarrow x(t - t_0) \to y(t - t_0)$

In other words, for a shift invariant system, shifting the input signal shifts the output signal by the same offset.

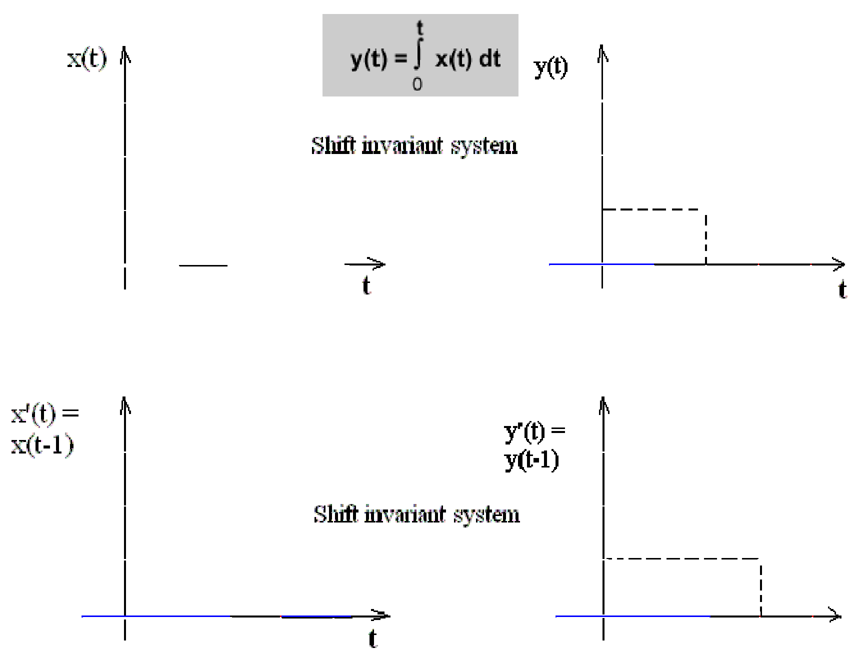


Figure 4.2

Note this is not to be expected from every system. **x(t)** and **x(t - t_0)** are different (related by a shift, but different) input signals and a system, which simply maps one set of signals to another, need not at all map **x(t)** and **x(t - t_0)** to output signal also shift by **t_0**.

A system that does not satisfy this property is said to be **shift variant.**

amples of Shift Invariance:

sume **y[n]** and **y(t)** are respectively outputs corresponding to input signals **x[n]** and **x(t)**

..The system with description

$\!\!\!\,$[n] = x[n] + x[n-5]

s shift invariant as input x[n-n_0] gives rise to output x[n-n_0] + x[n-5- n_0] which equals y[n-n_0].

:.The system with description

(t) = x(2t)

s shift variant.

The output corresponding to input signal x(t-t_0) is x(2t-t_0).

f this is confusing, let us say the input signal is g(t) = x(t-t_0). The output signal is clearly

;(2t), which is the same as x(2t-t_0).

f the system were shift-invariant, the output "should" have been x(2(t-t_0)) !

Similarly, you can verify that systems with descriptions y(t) = x(t^2), or y[n] = n.x[n]

re also examples of shift variant systems.

Stability

Let us learn about one more important system property known as stability. Most of us are familiar with the word stability, which intuitively means resistance to change or displacement. Broadly speaking a stable system is a one in which small inputs lead to predictable responses that do not diverge, i.e. are bounded. To get the qualitative idea let us consider the following physical example.

Example

Consider an ideal mechanical spring (elongation proportional to tension). If we consider tension in the spring as a function of time as the input signal and elongation as a function of time to be the output signal, it would appear intuitively that the system is stable. A small tension leads only to a finite elongation.

There are various ideas/notions about stability not all of which are equivalent. We shall now introduce the notion of **BIBO** Stability, i.e. **BOUNDED INPUT-BOUNDED OUTPUT STABILITY.**

Statement:

A system is said to be BIBO stable if for any bounded input signal, the output signal is bounded.

Given a bounded input, i.e. $\exists\, M_x \geq 0 \ni |x(t)| \leq M_x \ \forall t,$

then $\exists\, M_y \geq 0 \ni |y(t)| \leq M_y \ \forall t.$

Note: This should be true for all bound inputs **x(t)**

It is not necessary for the input and output signal to have the same independent variable for this property to make sense. It is valid for continuous time, discrete time and hybrid systems

Examples

,for t > 0.

as x-->

Consider systems with the following descriptions. y(t) is the output signal corresponding to the input signal x(t).

BIBO Stable system : In a BIBO stable system, every bounded input is assured to give a bounded output. An unbounded input can give us either a bounded or an unbounded output, i.e. nothing can be said for sure.

BIBO Unstable system: In a BIBO unstable system, there exists at least one bounded input for which output is unbounded. Again, nothing can be said about the system's response to an unbounded input.

Causality

Causality refers to cause and effect relationship (the effect follows the cause). In a causal system, the value of the output signal at any instant depends only on "past" and "present" values of the input signal (i.e. only on values of the input signal at "instants" less than or equal to that "instant"). Such a system is often referred to as being **non-anticipative**, as the system output does not anticipate future values of the input (remember again the reference to time is merely symbolic). As you might have realized, causality as a property is relevant only for systems whose input and output signals have the **same independent**

variable. Further, this independent variable must be **ordered** (it makes no sense to talk of "past" and "future" when the independent variable is not ordered).

What this means mathematically is that If two inputs to a causal (continuous-time) system are identical up to some time to, the corresponding outputs must also be equal up to this same time (we'll define the property for continuous-time systems; the definition for discrete-time systems will then be obvious).

Definition

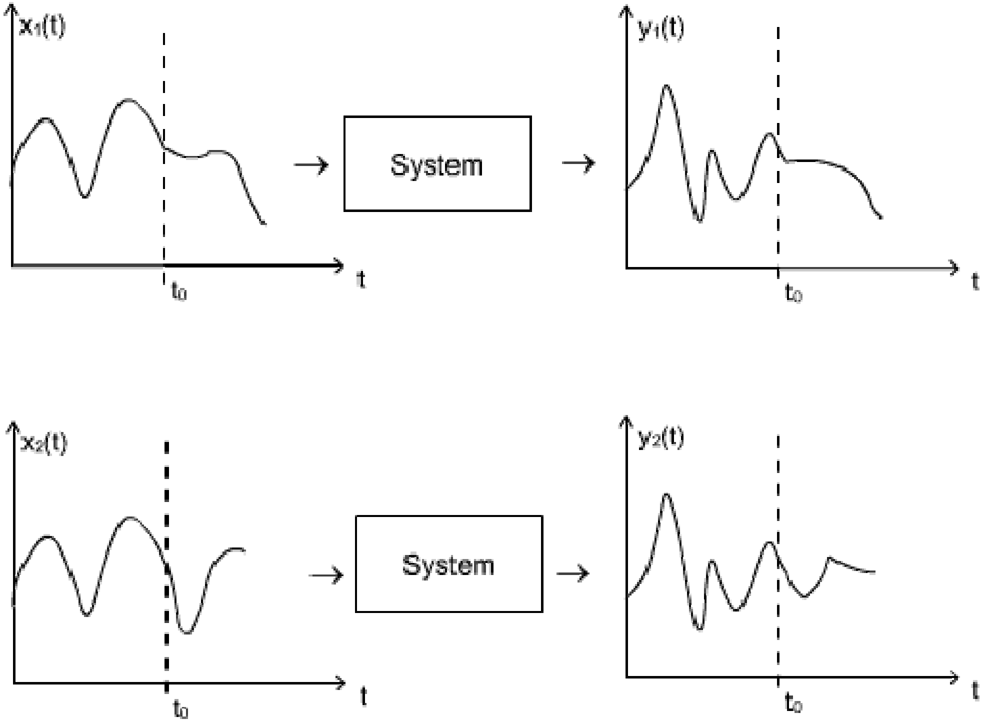
Let $\mathbf{x_1(t)}$ and $\mathbf{x_2(t)}$ be two input signals to a system and $\mathbf{y_1(t)}$ and $\mathbf{y_2(t)}$ be their respective outputs.

The system is said to be causal if and only if:

$$\begin{array}{llll} & x_1(t) = x_2(t) & \forall & t \leq t_0 \\ \Rightarrow & y_1(t) = y_2(t) & \forall & t \leq t_0 \quad \text{for every } t_0 \text{ and every } x_1(t) \text{ and } x_2(t) \end{array}$$

This of course is only another way of stating what we said before: for **any** $\mathbf{t_0}$ **:** $\mathbf{y(t_0)}$ depends only on values of $\mathbf{x(t)}$ for $\mathbf{t <= t_0}$

As an example of the behavior of causal systems, consider the figure below:



CAUSAL SYSTEM

Figure 4.3

The two input signals in the figure above are identical to the point $\mathbf{t = t_0}$, and the system being a causal system, their corresponding outputs are also identical till the point $\mathbf{t = t_0}$.

Examples of Causal systems

Assume y[n] and y(t) are respectively the outputs corresponding to input signals x[n] and x(t)

1. System with description **y[n] = x[n-1] + x[n]** is clearly causal, as output "at" n depends on only values of the input "at instants" less than or equal to **n** (in this case n and n-1).

2. Similarly, the continuous-time system with description $y(t) = \int_{-\infty}^{t} x(\lambda)\, d\lambda$ is causal, as value of output at any time $\mathbf{t_0}$ depends on only value of the input at $\mathbf{t_0}$and before.

3. But system with description **y[n] = x[n+1]** is not causal as output at **n** depends on input one instant later.

Note:

If you think the idea of non-causal systems is counter intuitive, i.e: if you think no system can "anticipate the future", remember the independent variable need not be time. Visualizing non-causal systems with , say one-dimensional space as the independent variable is not difficult at all ! Even if the independent variable is time, we need not always be dealing with real-time, i.e. with the time axes of the input and output signals synchronized. The input signal may be a recorded audio.