Sampling Theorem

What is Sampling?

Sampling is a methodology of representing a signal with less than the signal itself.

We can do better than just describing a signal by specifying the value of the dependent variable for each possible value of the independent variable. The concept is explained with the following examples where 'x(t)' is the dependent variable and 't' is the independent variable.

Let $$x(t) = A_o \sin(\omega t + \phi_o)$$

Here '**x(t)**' is defined by a sinusoidal relation with a phase constant , amplitude and angular frequency. Now the knowledge of these three parameters suffices to describe '**x(t)**' completely. Thus we are able to compute '**x(t)**' without depending on the independent variable '**t**'.

Consider another example given below:

$$x(t) = a_o + a_1 t + \ldots\ldots\ldots\ldots\ldots\ldots + a_N t^N$$

Here **x(t)** is a polynomial in 't' of degree 'N' and can be computed completely if we know the coefficients $a_0, a_1, a_2 \ldots a_N$.

Thus we observe that the **apriori** information we had that allowed us to represent these signals. In the first case we knew that '**x(t)**' is a pure sinusoid and in the second case we knew that it was a polynomial of degree '**N**'.

Thus, as a method of using Apriori information available to represent a signal economically is one way of defining sampling.

Sampling theorem:

If a signal has no frequency components than f_m, then it is completely described by instantaneous values $x(nT_s)$ uniformly spaced in time with period $T_s < 1/(2f_m)$.

A band-limited signal with band-width B may be reconstructed perfectly from its samples, if the signal is sampled uniformly at a rate greater than 2B.

A Band-limited signal is one whose Fourier Transform is non-zero on only a finite interval of the frequency axis.

Specifically, there exists a positive number **B** such that **X(f)** is non-zero only in $f \in [-B, B]$. **B** is also called the Bandwidth of the signal.

To start off, let us first make an observation about the class of Band-limited signals.

Lets consider a Band-limited signal **x(t)** having a Fourier Transform **X(f)**. Let the interval for which **X(f)** is non-zero be **-B** $\leq$ **f** $\leq$ **B**.

Then, $x(t) = \int_{-B}^{B} X(f)\, e^{j2\pi ft} df$ converges.

The RHS of the above equation is differentiable with respect to **t** any number of times as the integral is performed on a bounded domain and the integrand is differentiable with respect to t. Further, in evaluating the derivative of the RHS, we can take $\frac{d}{dt}$ inside the integral.

$$\frac{dx(t)}{dt} = \int_{-B}^{B} (j2\pi f)\, X(f)\, e^{j2\pi ft} df$$

In general,

$$\frac{d^n x(t)}{dt^n} = \int_{-B}^{B} (j2\pi f)^n X(f)\, e^{j2\pi ft} df$$

This implies that band limited signals are infinitely differentiable, therefore, very smooth .

We now move on to see how a Band-limited signal can be reconstructed from its samples.

Consider first a signal y(t) that is time-limited, i.e. it is non-zero only in [-T/2, T/2].

Its Fourier transform Y(f) is given by:

$$\mathbf{Y}(\mathbf{f}) = \int_{-T/2}^{+T/2} y(t) e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{+\infty} \tilde{y}(t) e^{-j2\pi f t}\, dt \qquad \rightarrow (1)$$

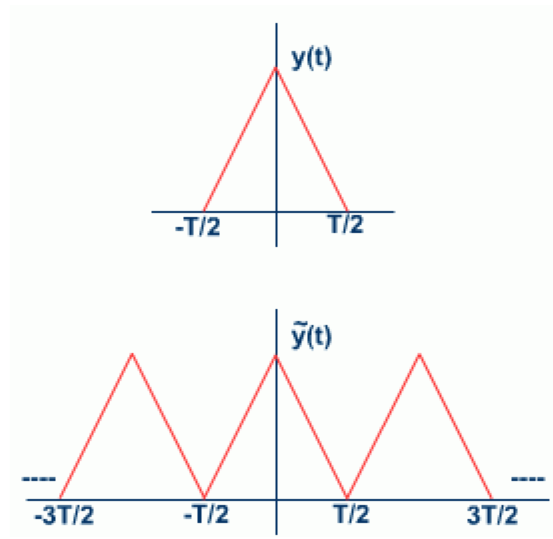

Figure 5.1

Where $\tilde{y}(t)$ is the periodic extension of y(t) as shown

Now, Recall that the coefficients of the Fourier series for a periodic signal x(t) are given by :

$$a_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, e^{-j\,2\pi n f_0 t}\, dt \qquad \text{where} \quad f_0 = \frac{1}{T} \qquad \text{——} \qquad (2)$$

Comparing (1) and (2), you will find

$$a_n = \frac{1}{\mathrm{T}} \mathrm{Y}\left(\frac{\mathrm{n}}{\mathrm{T}}\right)$$

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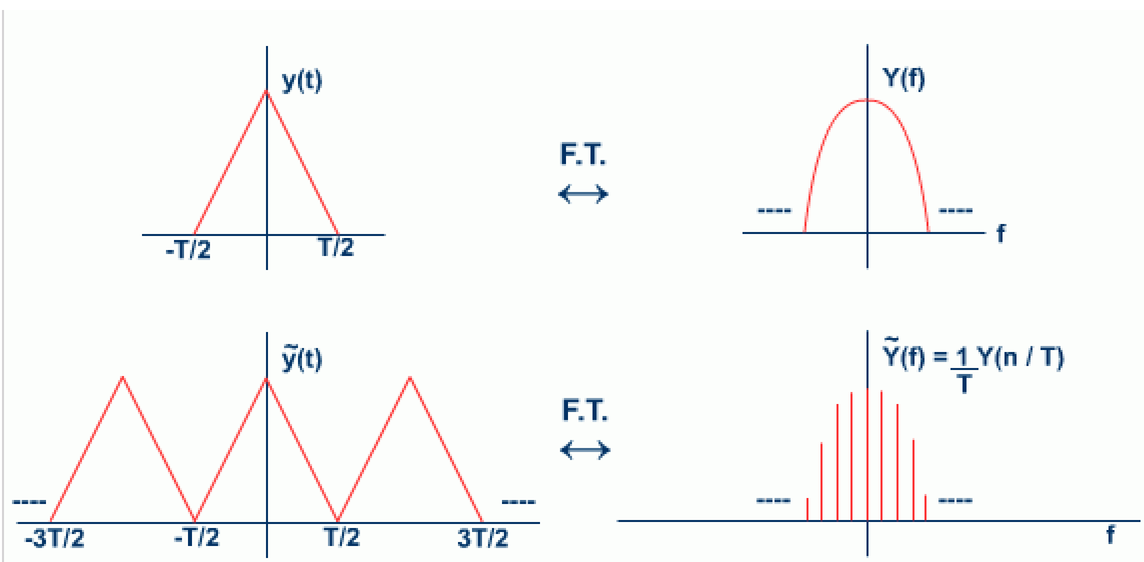

Figure 5.2

Therefore, given that; y(t) is time-limited in [-T/2, T/2] and periodic, the entire information about y(t) is contained in just equispaced samples of its Fourier transform! It is the dual of this result that is the basis of Sampling and Reconstruction of Band-limited signals .

Knowing the Fourier transform is limited to, say [-B, B], the entire information about the transform (and hence the signal) is contained in just uniform samples of the (time) signal !