# Dirichlet Conditions for Fourier Series

Let $$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi f_0 kt}$$

be the Fourier series expansion corresponding to the periodic signal **x(t)** (i.e: the $c_k$'s are as calculated by the formula in the previous lecture). Then the above summation may or may not converge to the actual signal **x(t)**.

We shall discuss the convergence of the Fourier series representation of a periodic signal in two contexts, namely **Pointwise convergence** and **Convergence in squared norm**. We will first see what each of these terms means and then discuss the conditions under which each kind of convergence takes place.

For the subsequent discussion let,

$$\tilde{x}_N(t) = \sum_{k=-N}^{N} c_k e^{j2\pi f_0 kt} \qquad \Rightarrow \qquad \tilde{x}(t) = \lim_{N\to\infty} \tilde{x}_N(t)$$

**Pointwise Convergence**

Pointwise convergence implies the series converges to the original function at any point, i.e: the Fourier Series representation of a signal x(t) is said to converge pointwise to the signal x(t) if:

$$\lim_{N\to\infty} \tilde{x}_N(t) = x(t) \qquad \forall t$$

i.e to say $$\tilde{x}(t) \equiv x(t).$$

**Convergence in squared norm**

The Fourier Series representation is said to converge in the sense of squared norm to the signal x(t) if

$$\lim_{N\to\infty}\int_0^T \left|x(t) - x_N(t)\right|^2 dt = 0$$

Pointwise convergence implies convergence in squared norm. As convergence in squared norm is a more relaxed condition than pointwise convergence, convergence in the squared norm sense covers a much larger domain of signals than pointwise convergence.

Finally, we now move on to the conditions for these forms of convergence

**Dirichlet Conditions For Pointwise Convergence**

Consider the following 3 conditions that may be imposed on a periodic signal **x(t)** :

**1) x(t)** should be **absolutely integrable** over a period.

A signal that does not satisfy this condition is **x(t) = tan(t)** as:- $\int_{-\pi/2}^{\pi/2} |tan(x)|\,dx$ does not exist

**2) x(t)** should have only a **finite number of discontinuities** over one period. Furthermore, each of these discontinuities must be finite. An example of a function which has infinite number of discontinuities is illustrated below. The function is shown over one of the periods.

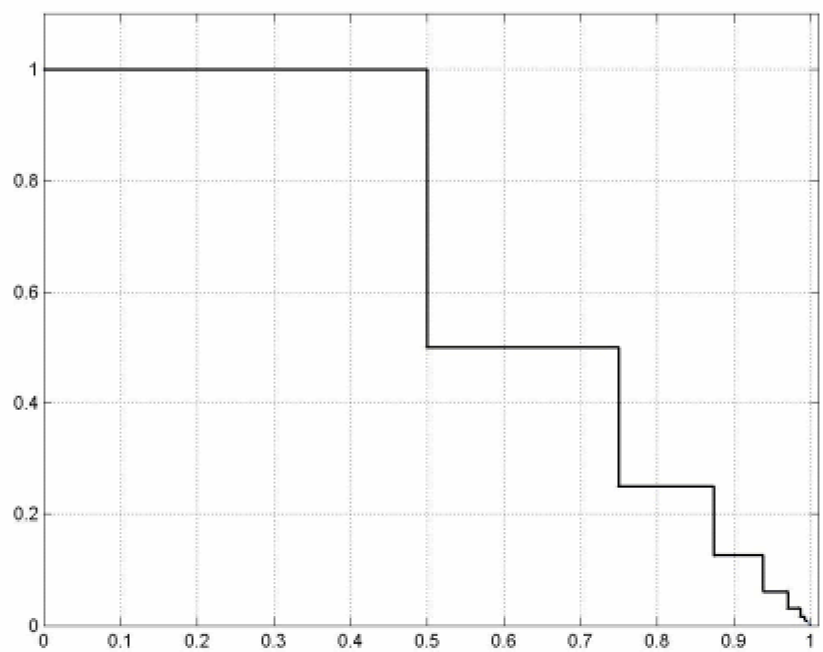


**Figure 2.1**

**3)** The signal **x(t)** should have only a **finite number of maxima and minima** in one period. An example of a function which has infinite number of maxima and minima is: a periodic signal with period 1, defined on (0,1] as: $x(t) = \sin(\frac{1}{t})$ .

If the signal satisfies the above conditions, then at all points where the signal is continuous, the Fourier Series converges to the signal. However, at points where the signal is discontinuous (Dirichlet conditions allow finite number of discontinuities in a period), the Fourier Series converges to the average of the left and the right hand limits of the signal. Mathematically, at a point of discontinuity $t_o$,

$$\tilde{x}(t_o) = \frac{\lim_{x \to t_o^+} x(t) + \lim_{x \to t_o^-} x(t)}{2}$$

ss

In practice, the restrictions imposed on signals by the Dirichlet conditions are not very severe, and most of the signals we will deal with satisfy these conditions.

**Condition for convergence in squared norm sense**

If, for a periodic signal **x(t)** with period **T**, $\int_0^T |x(t)|^2 \, dt$ converges, then its Fourier Series converges to it in the **squared norm** sense. As is expected, this is a far more relaxed constraint than the Dirichlet conditions.

At this point let us define some terms which will be of use to us later in the course:

is called the **instantaneous power** or **energy density** of the signal **x(t)**.

If **x(t)** is **periodic** with period **T**, and $\frac{1}{T}\int_0^T |x(t)|^2 dt$ converges & is finite, **x(t)** is called a **finite power signal**, and the value of the integral is called the **power** of the signal.

(Thus we can say, if a periodic signal has finite power, we are guaranteed of convergence in squared norm of its Fourier Series)

If **x(t)** is **non-periodic**, and $\int_{-\infty}^{\infty} |x(t)|^2 dt$ converges, **x(t)** is said to be a **finite energy signal**, and the value of the integral is called the **energy** of the signal.