Natural sampling

Our goal of achieving a sampled signal is possible by the multiplication of the original C.T. signal with the generated **train of pulses**. Now these two signals are multiplied practically with the help of a multiplier as shown in the schematic below. In our analysis so far, this is how we imagined sampling of a signal.

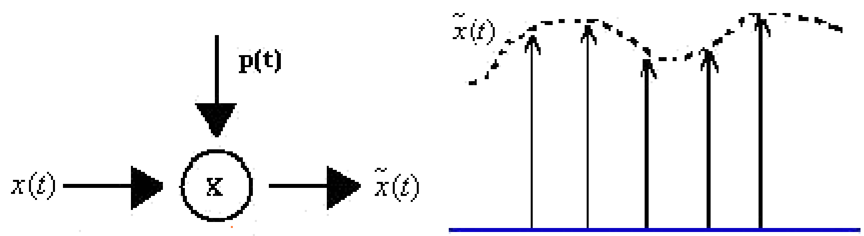

Figure 5.10

But impulses are a mathematical concept and they cannot be realized in a real system. In practice we can best obtain a train of pulses called a saw-tooth pulse. These pulses are generally used for creating a time-base for the operation of many electronic devices like the CRO (Cathode Ray Oscilloscope).

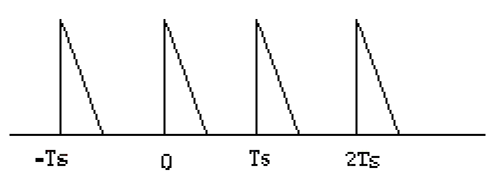

Figure 5.11

Practical Implementation:

Lets see how the train of pulses of the following kind can be multiplied by a signal 'x(t)'.

Consider the circuit below.

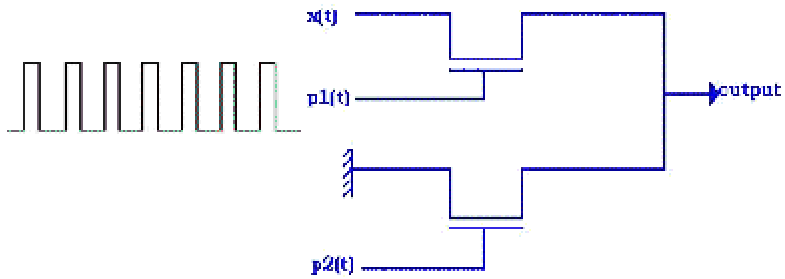

Figure 5.12

The two pulse trains **p_1(t)** and **p_2(t)** are synchronized so that when one is high the other is low and vice verse as shown in the figure below:

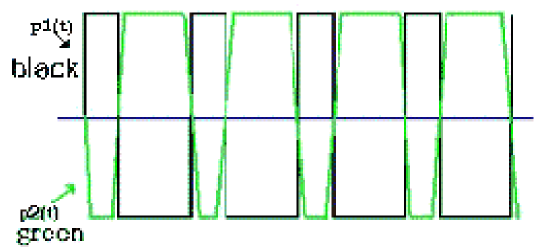

Figure 5.13

In the circuit when **x(t)** and **p_1(t)** are multiplied we get the output. Thus we get the output when **p_1(t)** is **ON** and it is **zero** when **p_2(t)** is **ON**.

You have just seen how we can multiply a signal **x(t)** with the following periodic pulse train **p(t)** to obtain the **sampled** signal $x_s(t)$. Now the train of pulses that we had used is shown below

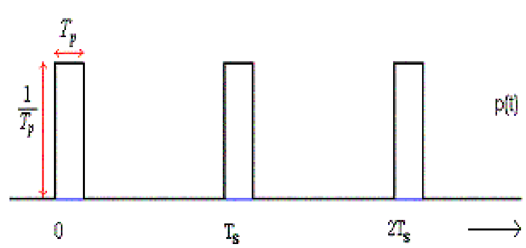

Figure 5.14

with respect to its amplitude and period.

Fourier series representation of p(t)

Now the Fourier Series Representation of 'p(t)' is given as:

$$p(t) = \sum_{k} C_k e^{j\frac{2\pi}{T_s}kt}$$

Where the Fourier Coefficients of the series are defined as:

$$C_k = \frac{1}{T_s}\int_{T_s} p(t)e^{-j\frac{2\pi}{T_s}kt}\,dt$$

For the constant term (**k = 0**) in the Fourier Series expansion is:

$$C_0 = \frac{1}{T_s} \times \frac{1}{T_p} \times T_p = \frac{1}{T_s}$$

In general we can represent k^{th} coefficient as:

$$C_k = \left.\frac{e^{-j\frac{2\pi}{T_s}kt}}{T_p T_s\left(-jk\frac{2\pi}{T_s}\right)}\right|_0^{T_p}$$

Simplifying the above term we get the envelope of the coefficients as a **sinc function**:

$$C_k = \frac{e^{-j\pi k\left(\frac{T_p}{T_s}\right)}}{T_s} \times \mathrm{Sinc}\left(k\left(\frac{T_p}{T_s}\right)\right)$$

$$\Rightarrow \qquad |C_k| = \frac{1}{T_s}\left|\mathrm{Sinc}\left(k\left(\frac{T_p}{T_s}\right)\right)\right|$$

Lets have a look at the envelope of $|C_k|$ which is shown as below:

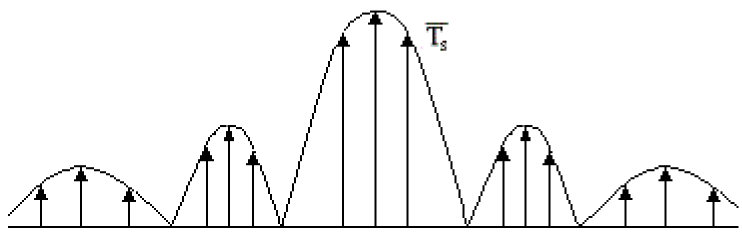

Figure 5.15

Looking at the expression for the coefficients of the Fourier Series Expansion we observe that:

If $\frac{T_p}{T_s}$ is large then there are few samples in the main lobe.

As $\frac{T_p}{T_s}$ increases then the main lobe broadens.

As $T_p \to 0,$ coefficients become constant (they tend to $\frac{1}{T_s}$) as the central lobe tends to infinity.

As $T_p \to 0,$ 'p(t)' tends to the train of impulses we had started our discussion on sampling with. Notice then that the observations above are consistent with this. The Fourier coefficients of the periodic train of impulses are indeed all constant

and equal to the reciprocal of the period of the impulse train.

The Fourier Transform of the Sampled Signal $x_s(t)$.

We now see what happens to the spectrum of continuous time signal on multiplication with the train of pulses. Having obtained the Fourier Series Expansion for the train of periodic pulses the expression for the sampled signal can be written as:

$$x_s(t) = x(t)\sum_k C_k e^{j\frac{2\pi}{T_s}kt}$$

$$= \sum_k C_k e^{j\frac{2\pi}{T_s}kt} x(t)$$

Taking Fourier transform on both sides and using the property of the Fourier transform with respect to translations in the frequency domain we get:

$$X_s(f) = \sum_k C_k X\left(f - \frac{k}{T_s}\right)$$

This is essentially the sum of displaced copies of the original spectrum modulated by the Fourier series coefficients of the pulse train. If 'x(t)' is **Band-limited** so long as the the displaced copies in the spectrum do not overlap. For this the condition that 'f_s' is greater than twice the bandwidth of the signal must be satisfied. The reconstruction is possible theoretically, using an I**deal low-pass filter** as shown below:

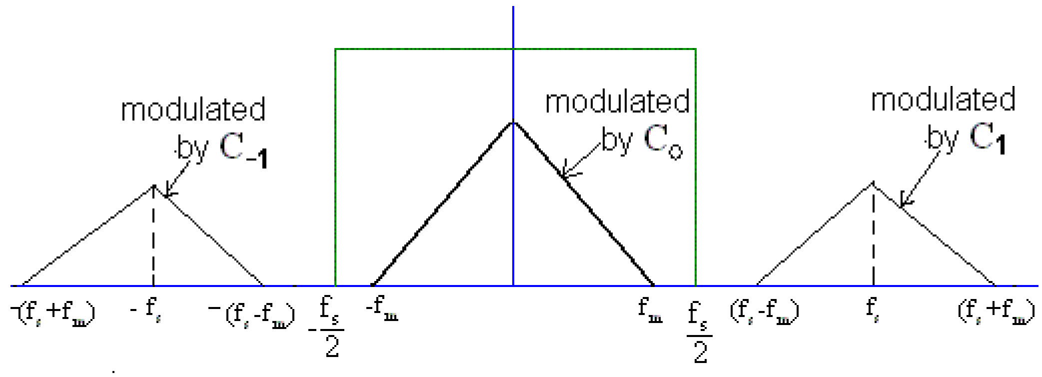

Figure 5.16

Thus the condition for faithful reconstruction of the original continuous time signal is : $f_s - f_m > f_m \Rightarrow f_s > 2f_m$ where f_m is the bandwidth of the original band-limited signal.