Orthogonality in Signals

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The concept of vector comparison and orthogonality can be extended to signals. Let us consider two signals, $f_1(t)$ in terms of $f_2(t)$. Suppose we want to approximate $f_1(t)$ in terms of $f_2(t)$ over a certain interval $(t_1 < t < t_2)$

How shall we choose C_{12} in order to achieve the best approximation? Obviously, we must find C_{12} such that the error between the actual function and the approximated function is minimum over the interval $(t_1 < t < t_2)$. Let us define an error function $f_e(t)$ as

$$f_e(t) = f_1(t) - C_{12} f_2(t)$$

One possible criterion for minimizing the error $f_e(t)$ over the interval t_1 to t_2 is to minimize the average value of $f_e(t)$ over this interval; that is, to minimize

$$\frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} [f_e(t) - C_{12} f_2(t)] dt$$

However , this criterion is inadequate because there can be large positive and negative errors present that may cancel one another in the process of averaging and give the false indication that the error is zero. For example, if we approximate a function sing t with a null function $f\ (t) = 0$ over an interval 0 to 2π, the average error will be zero, indicating wrongly that sin t can be approximated to zero over the interval 0 to 2π without any error. This situation can be corrected if we choose to minimize the average (or the mean) of the square of the error instead of the error itself. Let us designate the average of $f_e^{\,2}(t)$ by ε.

$$\varepsilon = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} f_e(t) dt = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} \left[f_1(t) - C_{12} f_2(t) \right]^2 dt$$

To find the value of C_{12} which will minimize ε, we must have

$$\frac{d\varepsilon}{dC_{12}} = 0$$

That is,

$$\frac{d}{dC_{12}}\left\{\frac{1}{(t_2 - t_1)}\int_{t_1}^{t_2}\left[f_1(t) - C_{12} f_2(t)\right]^2 dt\right\} = 0$$

Changing the order of integration and differentiation, we get

$$\frac{1}{(t_2 - t_1)}\left[\int_{t_1}^{t_2}\frac{d}{dC_{12}} f_1^{\,2}(t)dt - 2\int_{t_1}^{t_2} f_1(t) f_2(t)dt + 2C_{12}\int_{t_1}^{t_2} f_2^{\,2}(t)dt\right] = 0$$

The first integral is obviously zero, and hence Eq. 3.9 yields

$$C_{12} = \frac{\int_{t_1}^{t_2} f_1(t) f_2(t)dt}{\int_{t_1}^{t_2} y f_2^{\,2}(t)dt}$$

By analogy with vectors, we say that $f\ (t)$ has a component of wave form $f_2(t)$, and this component has magnitude C_{12}. If C_{12} vanishes, then the signal $f\ (t)$ contains no component of signal $f_2(t)$ and we say that the two functions are orthogonal over the interval ($t_2 - t_1$). It therefore follows that the two functions $f_1(t)$ and $f_2(t)$ are orthogonal over an interval (t_1, t_2) if

$$\int_{t_1}^{t_2} f_1(t) f_2(t)dt = 0$$