# Fourier Series representation of Periodic Signals

Consider a periodic signal **x(t)** with fundamental period **T**, i.e.

$$x(t+T) = x(t) \qquad \forall t$$

Then the **fundamental frequency** of this signal is defined as the reciprocal of the fundamental period, so that

$$f_0 = \frac{1}{T}$$

Under certain conditions, a periodic signal x(t) with period T can be expressed as a **linear combination of sinusoidal signals of discrete frequencies, which are multiples of the fundamental frequency of x(t)**. Further, sinusoidal signals are conveniently represented in terms of complex exponential signals. Hence, we can express the periodic signal in terms of complex exponentials, i.e.

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi f_0 kt}}$$

Such a representation of a periodic signal as a combination of complex exponentials of discrete frequencies, which are multiples of the fundamental frequency of the signal, is known as the **Fourier Series Representation of the signal.**

**Inner product**

The set of periodic signals with period T form a **vector space**.

We define the following inner product: $\boxed{\langle x_1(t), x_2(t)\rangle = \int_0^T x_1(t)\overline{x_2(t)}dt}$

And the norm or magnitude of the signal is defined as:- $\boxed{\left|x(t)\right| = \langle x(t), x(t)\rangle^{\frac{1}{2}}}$

Now we consider the set of vectors $e^{j2\pi f_0 kt}$, $(k \in \mathbb{Z})$, that belong to this vector space. (note $f_0 = \frac{1}{T}$)

We shall first show these vectors are **mutually orthogonal**. In other words we show that:-

$$\langle e^{j2\pi f_0 kt}, e^{j2\pi f_0 lt}\rangle = 0 \qquad \forall k \neq l$$

$$\begin{aligned}
\langle e^{j2\pi f_0 kt}, e^{j2\pi f_0 lt}\rangle &= \int_0^T e^{j2\pi f_0 kt} e^{-j2\pi f_0 lt} dt \\
&= \int_0^T e^{j2\pi f_0 (k-l)t} dt \\
&= \frac{\left[e^{j2\pi f_0 (k-l)t}\right]_0^T}{j2\pi f_0 (k-l)} \\
&= 0 \quad \forall k \neq l
\end{aligned}$$

$$\langle e^{j2\pi f_0 kt}, e^{j2\pi f_0 kt}\rangle = \left|e^{j2\pi f_0 kt}\right|^2 = T$$

Further, you may verify :

Thus, we have shown that this set of complex exponentials forms an **orthogonal set** in the vector space of all periodic signals with period T. Indeed, if we restrict ourselves to a certain class of signals in this vector space (those that satisfy the **Dirichlet Conditions**, which will be discussed in the next lecture), one can show that the above set of complex exponentials forms a basis for this class. i.e.: signals in this class can be expressed as a linear combination of these complex exponentials. In other words, such signals permit a Fourier Series representation.

Assuming the Fourier Series representation of a signal x(t), with period T exists, it is easy to find the Fourier Series coefficients, using the orthogonality of the basis set of complex exponentials.

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi f_0 kt}$$

Taking inner product with $e^{j2\pi f_0 kt}$ on both sides

$$\begin{aligned}
< x(t), e^{j2\pi f_0 kt} > &= < \sum_{k=-\infty}^{+\infty} c_k e^{j2\pi f_0 kt}, e^{j2\pi f_0 kt} > \\
&= \sum_{k=-\infty}^{+\infty} c_k < e^{j2\pi f_0 kt}, e^{j2\pi f_0 kt} > \\
&= c_k T \qquad (all\ other\ terms\ drop\ out)
\end{aligned}$$

$$\boxed{\therefore\ c_k = \frac{1}{T}\int_0^T x(t)\, e^{-j2\pi f_0 kt}\, dt}$$