Properties of Fourier Series

Using the properties we just proved for the Fourier Transform, we state now the corresponding properties for the Fourier series.

Time-shift

Recall, that if **x(t)** is periodic then **X(f)** is a train of impulses.

$$x(t) \xrightarrow{FT} X(f) \text{ where } X(f) = \sum c_k \, \delta(f - kf_0)$$

We know: $x(t - t_0) \xrightarrow{FT} e^{-j2\pi f t_0} X(f)$

Thus if x(t) is periodic with period T , x(t - t_0) has Fourier series coefficients $c_k e^{2\pi \frac{k}{T} f}$

Differentiation

If the periodic signal is differentiable then

$$x(t) \xrightarrow{FT} \sum_{-\infty}^{\infty} c_k \delta(f - kf_0)$$

$$\frac{dx(t)}{dt} \xrightarrow{FT} j2\pi f X(f)$$

$$\therefore \frac{dx(t)}{dt} \xrightarrow{FT} \sum_{-\infty}^{\infty} j2\pi \frac{k}{T} c_k \delta(f - kf_0)$$

Thus if x(t) is periodic with period T , x'(t) has Fourier Series coefficients $2\pi j \frac{k}{T} c_k$.

Scaling of the independent variable

$$x(t) = \sum_{-\infty}^{\infty} c_k e^{j\frac{2\pi}{T}kt}$$

$$x(at) = \sum_{-\infty}^{\infty} c_k e^{j\frac{2\pi}{T}kat}$$

If a > 0, **x(at)** is periodic with period **(T / a)** and now $\mathbf{c_k}$ becomes Fourier coefficient corresponding to frequency $\frac{k}{T/a}$.

If a < 0, **x(at)** is periodic with period **(T / -a)** and now $\mathbf{c_k}$ becomes Fourier coefficient corresponding to frequency $\frac{-k}{T/|a|}$

Multiplication by t

Multiplication by t of-course will not leave a periodic signal periodic. But what we can do is, multiply by t in one period

, and then consider a periodic extension. i.e: x(t) is periodic with period T, we see what the Fourier series coefficients of y(t),

defined as follows is:

$y(t) = tx(t)$ in $0 \le t \le T$ and $y(t+T) = y(t)$

Let $\tilde{x}(t) = x(t) \quad 0 \le t \le T$

$\tilde{x}(t) = 0$ otherwise

Then $\tilde{X}(f) = \int_0^T x(t) e^{-j2\pi ft} dt$

Note the k^th Fourier series co-efficient of x(t) is $c_k = \frac{1}{T} \tilde{X}\left(\frac{k}{T}\right)$

Similarly, let $\tilde{y}(t) = y(t)\{u(t) - u(t-T)\}$

$$\tilde{y}(t) = t\,\tilde{x}(t)$$

$$\tilde{Y}(f) = \frac{j}{2\pi}\frac{d\tilde{X}(f)}{df}$$

Therefore, kth Fourier series coefficient of $y = \frac{1}{T}\tilde{Y}\left(\frac{k}{T}\right)$

This idea is not of much use without knowledge of $\tilde{X}(f)$