Modulation and Convolution Theorems of F.T

We shall in this lecture prove the most important theorem regarding the Fourier Transform- the **Convolution Theorem**. It is this theorem that links the Fourier Transform to LSI systems, and opens up a wide range of applications for the Fourier Transform. We shall inspire its importance with an application example.

Modulation

Modulation refers to the process of embedding an information-bearing signal into a second carrier signal. Extracting the information-bearing signal is called demodulation. Modulation allows us to transmit information signals efficiently. It also makes possible the simultaneous transmission of more than one signal with overlapping spectra over the same channel. That is why we can have so many channels being broadcast on radio at the same time which would have been impossible without modulation

There are several ways in which modulation is done. One technique is amplitude modulation or AM in which the information signal is used to modulate the amplitude of the carrier signal. Another important technique is frequency modulation or FM, in which the information signal is used to vary the frequency of the carrier signal. Let us consider a very simple example of AM.

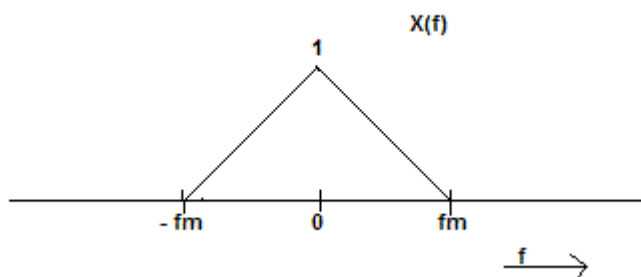Consider the signal **x(t)** which has the spectrum **X(f)** as shown :

Figure 3.4

Why such a spectrum ? Because it's the simplest possible multi-valued function. Also, it is band-limited (i.e.: the spectrum is non-zero in only a finite interval of the frequency axis), having a maximum frequency component f_m . Band-limited signals will be of interest to us later on.

x(t) is amplitude modulated with a carrier signal $cos(2\pi f_c t)$.

Thus   if $x(t) \xrightarrow{FT} X(f)$,   then   the   amplitude   modulated   signal

$x(t)\cos(2\pi f_c t) \xrightarrow{FT} \frac{1}{2}X(f-f_c)+\frac{1}{2}X(f+f_c)$ since $\cos(2\pi f_c t) = \frac{1}{2}e^{j2\pi f_c t} + \frac{1}{2}e^{-j2\pi f_c t}$

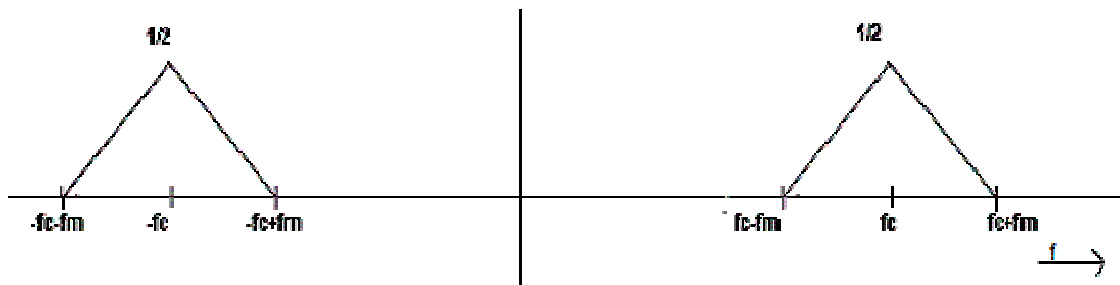If , the Fourier transform of the amplitude modulated signal is:

Figure 3.5

.

At the receiving end, in order to demodulate the signal, we multiply it again by $cos(\omega_c t)$

x(t) cos ω$_c$t ⟶ ⊗ ⟶ x(t) cos² ω$_c$t = ½ x(t) (1+ cos 2ω$_c$t)

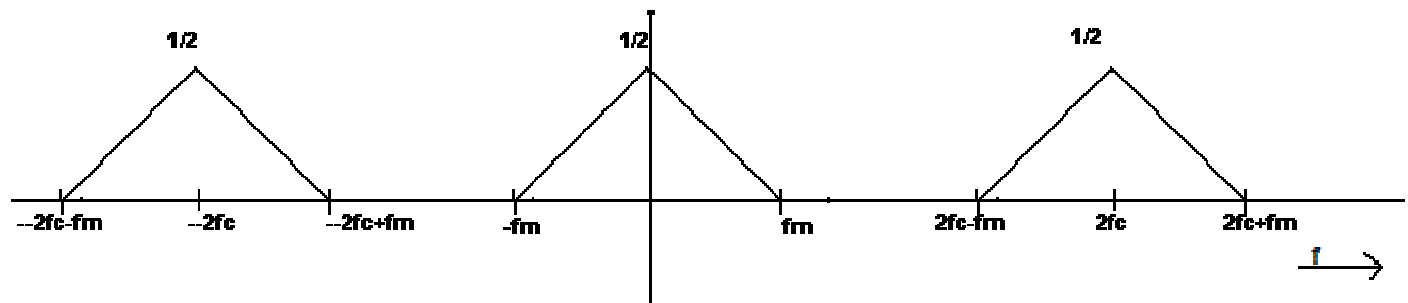cos ω$_c$t   = ½ x(t) + ½ x(t) cos 2ω$_c$t

Figure 3.6

Now, all we need is something that keeps the $-f_m$ to f_m part of the transmitted spectrum and simply chops away the rest of the spectrum. Such a device is called an ideal low-pass filter.

Filters :

The simplest ideal filters aim at retaining a portion of the spectrum of the input in some pre-defined region of the frequency axis and removing the rest.

A **LOWPASS FILTER** is a filter that passes low frequencies – i.e. around f = 0 and rejects the higher ones, i.e: it multiplies the input spectrum with the following:
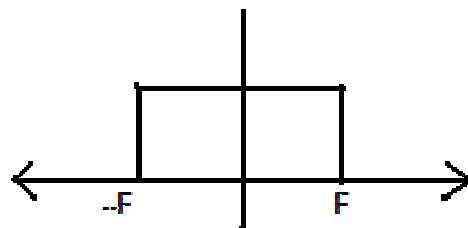

Figure 3.7

A **High pass filter** passes high frequencies and rejects low ones by multiplying the input spectrum by:
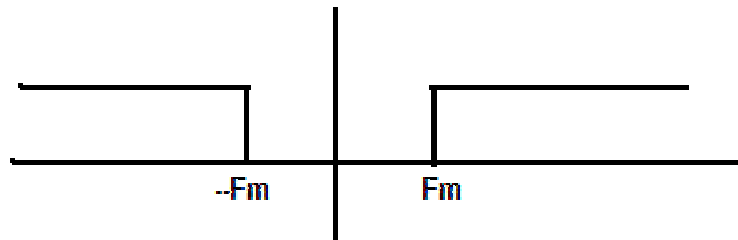
Figure 3.8

A **BANDPASS FILTER** passes a band of frequencies and rejects both higher and lower than those in the band that is passed, thus multiplying the input spectrum by:
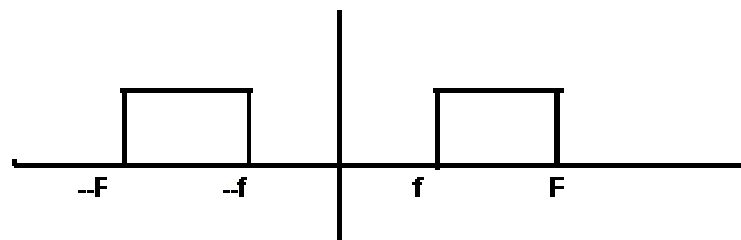

Figure 3.9

A **BANDSTOP FILTER** stops or rejects a band of frequencies and passes the rest of the spectrum, thus multiplying the input spectrum by:
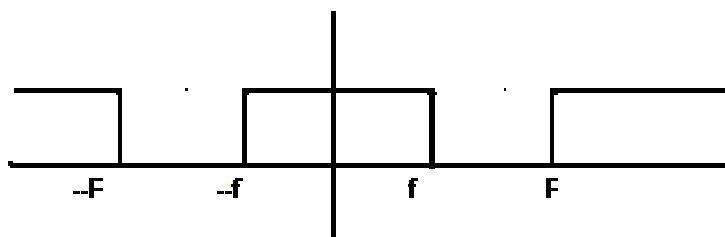

Figure 3.10

How do these filters work? That is, what does multiplication of two signals in the frequency domain imply in the time domain?

If we multiply two Fourier transforms **X(f)** and **H(f)**, let us see what the Inverse Fourier transform of this product is.

Consider the integral $\int_{-\infty}^{\infty} X(f)\, H(f)\, e^{j2\pi ft}\, df$

Let us replace **H(f)** by $\int_{-\infty}^{\infty} h(\lambda)\, e^{-j2\pi f\lambda}\, d\lambda$

This makes the integral, $\int_{-\infty}^{\infty} X(f)\, \left\{ \int_{-\infty}^{\infty} h(\lambda)\, e^{-j2\pi f\lambda}\, d\lambda \right\} e^{j2\pi ft}\, df$

We can interchange the order of integration, so long as the new double integral converges

$$\int_{-\infty}^{\infty} h(\lambda) \left\{ \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f(t-\lambda)}\, df \right\} d\lambda$$

we note that the term inside the bracket is just the inverse Fourier transform of **X(f)** evaluated at $(t-\lambda)$,

Thus the integral simplifies to $\int_{-\infty}^{\infty} h(\lambda)\, x(t-\lambda)\, d\lambda$ which is simply the convolution of **h(t)** with **x(t)** !

Convolution theorem for the Fourier Transform

What we have just proved is called the **Convolution theorem** for the Fourier Transform.

It states:

If two signals x(t) and y(t) are Fourier Transformable, and their convolution is also Fourier Transformable, then the Fourier Transform of their convolution is the product of their Fourier Transforms.

$$\text{If } x(t), y(t) \text{ and } (x^*y)(t) \text{ are Fourier Transformable and}$$
$$x(t) \xrightarrow{FT} X(F) \quad \& \quad y(t) \xrightarrow{FT} Y(f) \quad \text{then}$$
$$(x^*y)(t) \xrightarrow{FT} X(F)Y(f)$$

Dual of the convolution theorem

We now apply the Duality of the Fourier Transform to the Convolution Theorem to get another important theorem.

Let **x(t)** and **y(t)** be two Fourier transformable signals, with Fourier transforms **X(f)** and **Y(f)** respectively. Assume **X(f)*Y(f)** is Fourier Invertible. We now find its inverse.

What does Duality tell us? If $x(.) \xrightarrow{FT} X(.) \quad \text{then} \quad X(.) \xrightarrow{FT} x(-.)$.

Thus we know: $X(.) \xrightarrow{FT} x(-.) \quad \& \quad Y(.) \xrightarrow{FT} y(-.) \Rightarrow X(-.) \xrightarrow{FT} x(.) \quad \& \quad Y(-.) \xrightarrow{FT} y(.)$.

The Convolution theorem says: $X(-.)^*Y(-.) \xrightarrow{FT} x(.)\,y(.)$

Applying duality on this result,

$$x(.)\,y(.) \xrightarrow{FT} X(.)^*Y(.)$$

Thus we get the Dual version of the Convolution Theorem:

If **x(t)** and **y(t)** are Fourier Transformable, and **x(t) y(t)** is Fourier Transformable , then its Fourier Transform is the convolution of the Fourier Transforms of **x(t)** and **y(t)**. i.e:

$$
\boxed{
\begin{array}{l}
\text{If } x(t),\ y(t) \text{ and } x(t) \times y(t) \text{ are Fourier Transformable and} \\[4pt]
x(t) \xrightarrow{\;FT\;} X(f) \quad \&\quad y(t) \xrightarrow{\;FT\;} Y(f) \quad \text{then} \\[6pt]
\boxed{\,x(t) \cdot y(t) \xrightarrow{\;FT\;} X(f) * Y(f)\,}
\end{array}
}
$$