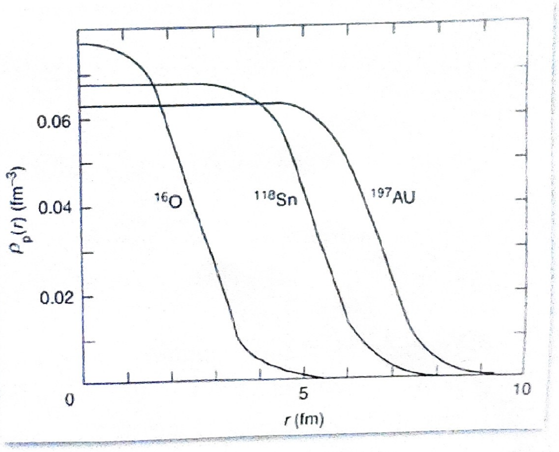
Nuclear radii and charge distribution:

In the previous lecture, we discussed the size of the nucleus, which is on the order of $10^{-15}\cancel{f}m$. How do we measure this size? As described, we use high-energy electrons to hit the sample material containing the nuclei, then measure the scattered electrons as a function of the scattering angle θ. By measuring the number of electrons scattered at each angle, we can estimate the radius of the nucleus. Not only can we estimate the radius, but we can also derive the charge distribution inside the nucleus.

Mass distribution of nucleus:

The measurement of $\rho_p(r)$ is based on the distribution of charge and hence the distribution of protons in the nucleus. It is also important to know the distribution of all nucleons (neutrons and protons) in the nucleus. Since neutrons are uncharged, Coulomb scattering experiments do not observe them. It is, however, fairly straightforward to infer the total nucleon density (or nuclear mass density), $\rho(r)$, from measurement of $\rho_p(r)$. A nucleus with N neutrons and Z protons has an overall neutron to proton ratio N/Z, and a total number of nucleons $A = N + Z$. It is usually assumed that the ratio of neutron density, $\rho_n(r)$, to proton density, of $\rho_p(r)$, is the same everywhere within the nucleus. This is written as



Charge distributions for some nuclei

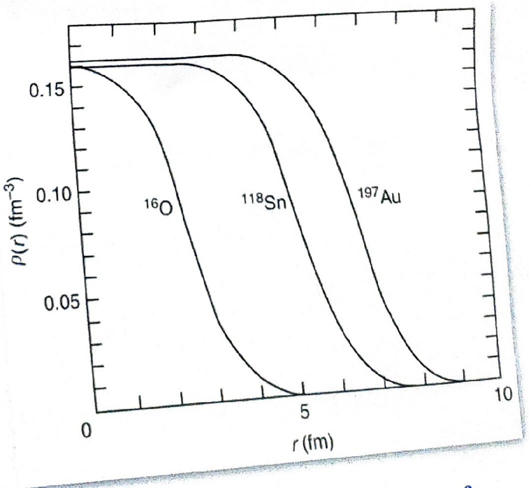
$$\frac{\rho_n(r)}{\rho_p(r)} = \frac{N}{Z}.$$

It is also known that the total density is the sum of the neutron and proton densities,

$$\rho(r) = \rho_n(r) + \rho_p(r).$$

These two equations can be easily solved to give,

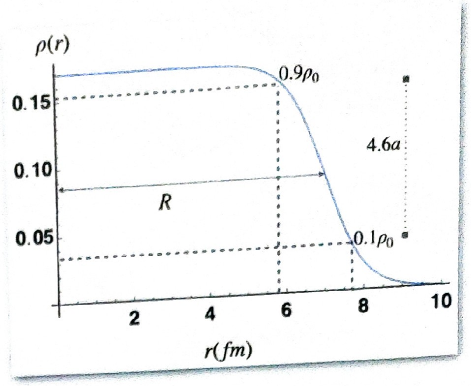
$$\rho(r) = \rho_p(r)\left[1 + \frac{N}{Z}\right].$$



Woods–Saxon mass distributions for some nuclei

Using this equation, data of the form proton density curve can be rescaled for different N and Z to give the total nucleon density, as illustrated in above figure. It is significant here that the mean nuclear radii and the width of the surface regions has not changed from that shown in proton density curve. It is also seen that all nuclei, regardless of total mass, have virtually the same nucleon density in their centers. This is a very important feature of nuclei and confirms that the strong interaction is very short ranged and is dominated by the interaction between nearest neighbor nucleons.

When plotting charge density as a function of distance from the nucleus's center (r), , the charge density appears almost constant up to a point, after which it tapers off. Different nuclei exhibit slightly varying values, but the overall trend remains similar. This observation arises because electrons only interact with protons, not neutrons, resulting in charge density data representing proton distribution. By scaling up to include neutrons, we derive the mass distribution, revealing that the overall distribution for different nuclei is consistent.



Woods–Saxon mass distributions for nucleus

This charge density (ρ) can be mathematically expressed as:

$$\rho = \frac{\rho_0}{1 + e^{(r-R)/a}}$$

In this function: ρ_0 is the central charge density, R is the radius where the density drops to half its central value and a is a parameter indicating how quickly the density falls off.

To understand these parameters better, let's delve into each one. The value R signifies the distance from the nucleus center where the density decreases to half of ρ_0. The parameter a describes how steeply this decrease occurs. If we were to measure points where the density drops to 90% and 10% of ρ_0, the distance between these points is related to a.

If we set $\rho = 0.9\rho_0$ and solve for r, we get:

$$0.9\rho_0 = \frac{\rho_0}{1 + e^{(r-R)/a}} \quad ==> e^{(r-R)/a} = 1/9 \quad ==> (r-R)/a = -2.3$$

$$r = R - 2.3a$$

Similarly, setting $\rho = 0.1\rho_0$, we get:

$$0.1\rho_0 = \frac{\rho_0}{1 + e^{(r-R)/a}} \quad ==> e^{(r-R)/a} = 9 \quad ==> (r-R)/a = 2.3$$

$$r = R + 2.3a$$

The distance between the 90% and 10% points is roughly $4.6a$, indicating the gradual decline in density.

Experimentally, a is found to be around 0.5 femtometers for nearly all nuclei, regardless of their size. This consistency means the "skin" or the surface layer of the nucleus, where the density decreases, is approximately 2.3×2, or about 2.5 femtometers thick.

The central core of the nucleus is uniformly populated with protons and neutrons up to a certain distance, beyond which the density decreases. The nucleus does not have a sharp boundary but a diffused one, where the density gradually becomes negligible.

The mean nuclear radius, R, increases with total nuclear mass and an analysis of the experimental results leads to the empirical expression, $R = R_0 A^{1/3}$, where $R_0 \approx 1.2\, fm$ and A is the mass number (total number of protons and neutrons).

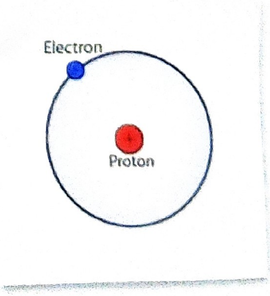
The nucleus exists due to nuclear attractive forces, which counteract the repulsion between protons. Protons naturally repel each other because of their positive charge, and without a counteracting force, the nucleus would disintegrate. However, nuclear forces are much stronger than Coulomb repulsion within the range of nuclear distances, ensuring the nucleus remains intact.

One might wonder why the nucleus isn't more crowded at the center if there is such a strong attractive force, similar to how the Earth's density is highest at its core due to gravitational attraction. The Earth's core is very dense, and as you move towards the surface, the density gradually decreases. In contrast, within the nucleus, the density of nucleons (protons and neutrons) remains relatively uniform up to the last couple of femtometers. This indicates that nuclear forces are short-ranged, meaning that nucleons only significantly interact with their immediate neighbors rather than all nucleons pulling towards the center.

This short-range nature of nuclear forces prevents overcrowding at the center of the nucleus. Essentially, each nucleon is primarily influenced by its closest neighbors, leading to a uniform density distribution rather than a central concentration.

Let's now explore another method for measuring the radius of a nucleus, which involves atomic electrons. While we are focused on the nucleus, this method leverages the behavior of electrons in atoms. In an atom, electrons orbit the nucleus, and their energy levels can be described by quantum mechanics. For a hydrogen atom or hydrogen-like ions (with one electron), the potential energy due to the nucleus is given by:

$$U(r) = -\frac{Ze^2}{4\pi\epsilon_0 r} \quad \text{(Assume that nucleus is point chage)}$$



Here, Z is the atomic number (number of protons), e is the electric charge, r is the distance from the nucleus, and ϵ_0 is the permittivity of free space. This potential energy term is crucial for solving the Schrödinger equation to find the energy levels and wave functions of the electron.

The Hamiltonian for the electron includes both kinetic and potential energy terms:

$$H = -\frac{\hbar^2}{2m}\nabla^2 + U(r) \quad \text{(For point charge)}$$

The ground state energy of a hydrogen atom, for example, is approximately $-13.6\ eV$. The corresponding wave function, particularly for the ground state, is not zero at $r = 0$, indicating a probability of finding the electron within the nucleus. This wave function, given by:

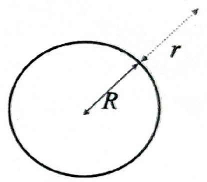
$$\psi_{100}(r) = \sqrt{\frac{Z^3}{\pi a_0^3}}\ e^{Zr/a_0}\ (a_0 \text{ is Bohr radius})$$

is not zero at $r = 0$, implying that the electron can penetrate the nucleus. If the nucleus were a mathematical point, this would be straightforward, but if the nucleus has a finite size, this interaction can affect the energy levels of the atom.

To estimate the nuclear radius, we consider the potential energy assuming a spherical nucleus with uniform charge distribution. For a point nucleus, the potential energy is straightforward. However, for an extended spherical nucleus of radius R, the potential energy changes.

Outside the sphere (for $r > R$), the potential remains the same as for a point charge:

$$V(r) = \frac{Ze}{4\pi\epsilon_0 r}\ \text{(Here, Ze is the charge of the spherical nucleus)}$$



Spherical nucleus

Inside the sphere (for $r \leq R$), the potential is derived by considering the uniform charge distribution. The electric field inside a uniformly charged sphere at a distance r from the center is:

$$E(r) = \frac{Ze}{4\pi\epsilon_0}\frac{r}{R^3}$$

Integrating this electric field from r to R gives the potential difference, and thus the potential inside the sphere:

$$V(r) = V(R) - \int_R^r E(r')dr'$$

$$V(r) = \frac{Ze}{4\pi\epsilon_0 R} - \int_R^r \frac{Ze}{4\pi\epsilon_0}\frac{r'}{R^3}dr'$$

$$V(r) = \frac{Ze}{4\pi\epsilon_0 R} - \frac{Ze}{4\pi\epsilon_0}\left[\frac{r'^2}{2R^3}\right]_R^r$$

$$V(r) = \frac{Ze}{4\pi\epsilon_0 R} - \frac{Ze}{4\pi\epsilon_0}\left[\frac{r^2 - R^2}{2R^3}\right]$$

$$V(r) = \frac{Ze}{4\pi\epsilon_0 R}\left(1 - \frac{r^2}{2R^2} + \frac{R^2}{2R^2}\right)$$

$$V(r) = \frac{Ze}{4\pi\epsilon_0}\left(\frac{3R^2 - r^2}{2R^3}\right).$$

If we account for the finite size of the nucleus, the potential energy changes. For a nucleus with a finite radius R ~~and~~ and a uniformly distributed charge, the potential energy inside the nucleus becomes:

$$U(r) = -\frac{Ze^2}{8\pi\epsilon_0 R}\left(3 - \frac{r^2}{R^2}\right).$$

Several assumptions are made here: the nucleus is spherical and its charge is uniformly distributed.

For the Hamiltonian of an electron around a point nucleus, we have:

$$H = -\frac{\hbar^2}{2m}\nabla^2 - \frac{Ze^2}{4\pi\epsilon_0 r},$$

where the first term represents the kinetic energy and the second term represents the potential energy.

When the finite size of the nucleus is considered, the Hamiltonian changes to: $H = H_0 + H_1$,

where H_0 is unperturbed Hamiltonian (point nucleus): $H_0 = -\frac{\hbar^2}{2m}\nabla^2 - \frac{Ze^2}{4\pi\epsilon_0 r}$

and H_1 is the perturbation due to the finite size of the nucleus:

$$H_1 = \frac{Ze^2}{8\pi\epsilon_0 r} - \frac{Ze^2}{8\pi\epsilon_0 R}\left(3 - \frac{r^2}{R^2}\right).$$

To find the energy difference caused by this perturbation, we use perturbation theory. The first-order correction to the energy ΔE is given by:

$\Delta E = <\psi_0 | H_1 | \psi_0>$, where ψ_0 is the wave function of the unperturbed Hamiltonian H_0.