Deuteron

Introduction:

We discussed the size of the nucleus, the mass of the nucleus, and now the next phase is the forces within the nucleus. As you know, the nucleus is made of protons and neutrons, which we collectively call nucleons. The next phase involves understanding the forces between these nucleons, which ultimately bind the nucleus together.

So, first, let us consider what nucleons are. Protons and neutrons are not structureless particles; they do have internal structures. The process of revealing that structure is similar to how the alpha particle scattering experiment revealed the structure within an atom. That experiment answered the question, "What is inside an atom?" It showed that most of the atom is empty space, with a small, dense nucleus that strongly deflects alpha particles.

A similar question arises: what is inside a proton or neutron? The technique used to answer this is also similar: send highly energetic particles into the proton or neutron and observe how they scatter. By looking at the angular distribution of the scattered particles—such as high-energy electrons with energies in the range of giga-electron volts, where the De Broglie wavelength is much smaller than a femtometer (approximately 1/20 of a femtometer)—we can infer the structure inside the proton or neutron. These experiments reveal that protons and neutrons are not just point particles; they have internal structures, which we now know to be quarks. A proton is made of three quarks, and a neutron is also made of three quarks.

In our current understanding, all matter in the universe is composed of two types of particles: quarks and leptons. Nucleons (protons and neutrons) are made of quarks. Quarks come in six varieties, known as flavors: up, down, strange, charm, top, and bottom. Each quark has a corresponding antiparticle, so there are 12 different types of quarks. Alongside quarks, there are leptons, such as the electron, muon, and neutrino. Quarks and leptons form the building blocks of the universe. Quarks were first proposed as theoretical particles in 1964, and by 1968, experimental evidence began to confirm their existence within protons and neutrons. Today, all six quark flavors have been experimentally observed, and interestingly, quarks have fractional electric charges. For example, the up quark, charm quark, and top quark each have a charge of +2/3 e, while the down quark, strange quark, and bottom quark each have a charge of -1/3 e. The corresponding antiquarks have opposite charges.

A proton is composed of two up quarks and one down quark, giving it a total charge of +1 e (2/3 e + 2/3 e - 1/3 e = +1 e). A neutron is composed of two down quarks and one up quark, resulting in a total charge of 0 (-1/3 e - 1/3 e + 2/3 e = 0). However, this does not mean there is no charge within a neutron; rather, the individual quark charges sum to zero. Similarly, the proton's +1 e charge is the result of the sum of its quark components. The forces that arise between nucleons are essentially the forces between these quarks.

What is the nature of the strong force? The strong force, also known as the nuclear force, is a fundamental force just like the gravitational and electromagnetic forces. Gravitational force arises from mass—if there is mass, there is gravitational interaction. Similarly, electromagnetic interaction arises from electric charge—if you have charges, you have electromagnetic interactions. In the case of the strong force, the corresponding property is known as "color charge" or simply "color." This term is a technical one and doesn't refer to actual color as perceived by the human eye. Instead, color is an intrinsic property of quarks, similar to how mass and electric charge are intrinsic properties. Quarks carry color charge, and there are three types of color charges, conventionally labeled as red, green, and blue. These names are just labels and don't correspond to actual colors.

When the three quarks in a nucleon (like a proton or a neutron) each have a different color, the total color charge of the nucleon sums to zero. This "color neutrality" is why nucleons themselves do not exhibit a net color charge. The strong force arises due to the color charge on quarks. Although each quark in a nucleon has a color charge, the nucleon as a whole is color-neutral because the colors of the quarks cancel each other out. However, the strong force between nucleons is still present, similar to how neutral molecules can still interact with each other through electromagnetic forces.

Consider molecules: each molecule is electrically neutral overall, such as a water molecule (H_2O). In a substance like liquid water, these neutral molecules still exert electromagnetic forces on each other, which is why water maintains a definite volume. The molecules are confined within a particular space because they attract each other, even though each molecule is neutral. In gases, the interactions between molecules are much weaker, allowing them to spread out and fill the available volume. In solids and liquids, the molecular attractions are stronger, leading to more confined structures. Even neutral molecules can interact through electromagnetic forces, but these forces are only effective when the molecules are close to each other, making them short-range forces. For example, two charged particles separated by 10 angstroms experience significant electromagnetic force. However, if two neutral molecules are at the same distance, their interaction is much weaker because there is no net charge to create a strong force. Yet, when molecules are close enough that their internal charge distributions become relevant, these distributions can create what are known as residual forces. These residual forces, though weak, allow neutral molecules to interact. So, that is a short-range force. This distance defines the range, and the depth defines the strength of the attractive potential. Beyond this range, the attraction becomes very small. Similarly, in the case of nucleons, although the total color charge of two nucleons is zero, if the separation between them is small and comparable to the distribution size within the nucleon, then some color interaction will occur. This is akin to how molecules interact electromagnetically even when their total charge is zero.

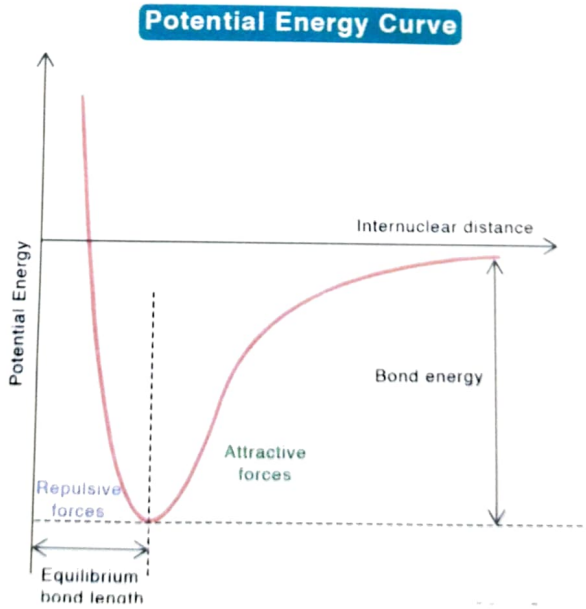

Fig. 1: Molecular Potential Energy

Similarly, here, even though each nucleon is color-neutral overall, if the two nucleons are close enough—such that the size of the color charge distribution inside each nucleon and the distance between them are of the same order—then the color interaction will become significant. This is the origin of the nucleon-nucleon force, which is short-range because it occurs between color-neutral entities. The nucleons must be brought close enough for this color interaction to be significant, which is why the force is short-range. Without delving too deeply into particle physics, we

recognize that nucleons do experience a strong interaction, commonly referred to as the nuclear force or nuclear interaction. This interaction corresponds to a potential energy that has both a certain strength (defined by the depth of the potential) and a range.

The study of nuclear forces involves understanding the shape and functional form of this potential energy, along with other details. With this basic understanding of why there is a nuclear force between nucleons, we can begin our study with the simplest possible case: the interaction between two nucleons in a bound system. The simplest system to study in this context is the deuteron, which consists of just two nucleons.

Deuteron:

The deuteron is the nucleus of two nucleons, consisting of one proton and one neutron, often written as ${}^{2}_{1}H_{1}$. Here, the mass number A is 2, Z (the number of protons) is 1, and N (the number of neutrons) is 1. Since Z is 1, indicating one proton, it is a hydrogen nucleus. The deuteron is the only nucleus with exactly two nucleons, as you do not have a bound system with two protons or two neutrons alone. However, the combination of one neutron and one proton does form a bound system.

Experimental measurements provide key parameters for the deuteron. One important parameter is the binding energy, which can be accurately determined from measured atomic masses. For the deuteron, the binding energy is 2.225 MeV, indicating that it is a weakly bound system. Why do we say it is weakly bound? With a binding energy of 2.225 MeV for two nucleons, the binding energy per nucleon is approximately 1.1 MeV. If you recall the binding energy per nucleon curve as a function of A, most nuclei have binding energies per nucleon around 7 to 8 MeV. This value is much higher compared to the deuteron's binding energy per nucleon, which underscores its weak binding.

Next, we consider the range of the nuclear interaction. This can be estimated from the measured radius of the deuteron, which is determined through various experiments. The radius is about 2.1 femtometers, which approximates the range of the nuclear interaction. The strength of the nuclear interaction has been studied extensively through numerous experiments, leading to the determination of the nucleon-nucleon interaction potential's functional form. However, this potential is quite complex and not easily expressed mathematically. To simplify the analysis, we can approximate the nucleon-nucleon interaction potential using a simple model, such as a square well potential, which is mathematically more manageable.

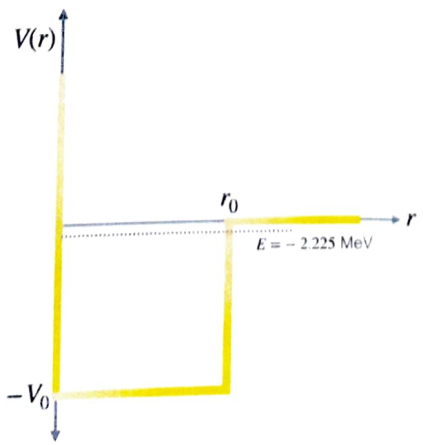

Fig. 2: Square well potential of deuteron

The square well potential is an assumed potential form with no strong theoretical basis behind it—it's simply a convenient approximation. This potential depends only on r, he separation between the two nucleons. In this model, the potential is defined as follows: when $r = 0$, the potential is at its maximum negative value, and when $r = r_0$ the potential is 0, representing an unbound system. Thus, beyond $r = r_0$, the nucleons are no longer bound, and the potential energy is zero. In this model, the depth of the potential well, V_0, represents the strength of the interaction, measured in mega-electron volts (MeV). A deeper potential well indicates a stronger interaction. The potential starts from $r = 0$ and and continues up to $r = r_0$, where r_0 represents the range of the interaction.

This simplified model helps us understand the strength of the nuclear potential. For example, molecular potential energies typically have depths of a few electron volts, whereas nuclear potentials are much deeper, on the order of MeV. The exact strength of the nuclear potential can be estimated using this square well model, the known range of 2.1 femtometers, and the binding energy of 2.225 MeV to determine the depth V_0. What does the binding energy represent in this diagram (**see Fig. 2**)? It means that the total energy E of the particle is -2.225 MeV. This negative value indicates that if you supply this amount of energy, the total energy would become zero, causing the two nucleons to separate and the deuteron to break apart. This is the meaning of binding energy: it is the energy required to disassemble the nucleus into its individual nucleons. The binding energy of 2.225 MeV reflects the nuclear interaction and the relative kinetic energy of the particles, not the mass energy of the proton and neutron, which are different (e.g., 938 MeV). The actual potential well depth V_0 would be greater than the binding energy.

In quantum mechanics, a square well potential with a certain depth V_0 allows bound states at specific energy levels. For the deuteron, there is only one bound state, the ground state, with no excited states. This is an important observation.

Assuming a three-dimensional square well potential, the potential energy V depends only on the distance r between the two nucleons, not on the angles θ and ϕ (**see Fig. 3**). This is known as a central potential, where V is a function of r alone. Here, r represents the separation between the two particles, and we work within a relative coordinate system to describe this interaction.

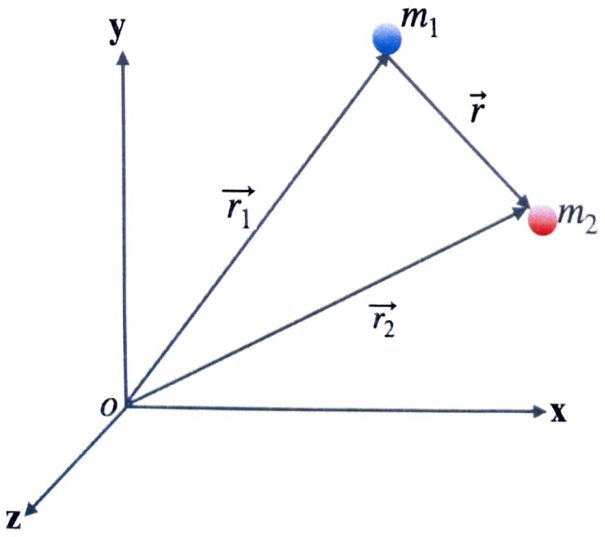

Fig. 3: 3D square well potential

In a two-particle system, consider two particles with position vectors $\vec{r_1}$ and $\vec{r_2}$. The relative position vector between them is $\vec{r} = \vec{r_1} + \vec{r_2}$. The potential energy and corresponding energy levels are functions of this relative position vector. The motion of the entire system, such as the deuteron moving as a whole, is not part of this potential. Instead, the potential concerns the interaction between the two nucleons, wherever they are.

The position $\vec{r}$ represents the position of the second particle relative to the first, implying that the origin is placed at the first particle. This setup can be viewed as a non-inertial frame because the first particle may experience acceleration due to the force from the second particle. To account for this, we introduce the concept of the reduced mass, $m = \dfrac{m_1 m_2}{m_1 + m_2}$. Using the reduced mass simplifies the equations and allows us to treat the system as if it were a single particle of mass m in an inertial frame.

With this understanding, we can apply the standard method for solving problems involving central potentials, where the energy eigenfunctions are straightforward to determine.

First, let us write the energy eigenvalue equations using the time-independent Schrödinger equation. The Hamiltonian operator, H, is given by:

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r})$$

The Schrödinger equation is: $H\psi(\vec{r}) = E\psi(\vec{r})$

If the potential $V(\vec{r})$ is central, meaning it depends only on the radial distance r and not on the angles θ and ϕ, the wave function $\psi(\vec{r})$ can be separated into a radial part $R(r)$ and angular part $Y_{lm}(\theta, \phi)$, known as spherical harmonics:

$$\psi(\vec{r}) = R(r)\, Y_{lm}(\theta, \phi)$$

The spherical harmonics $Y_{lm}(\theta, \phi)$ have standard expressions based on the quantum numbers l and m, where l can be 0,1,2, etc., and m ranges from $-l$ to $+l$.

The radial part $R(r)$ determines the energy levels. By defining $u(r) = rR(r)$, the Schrödinger equation for the radial part simplifies to:

$$-\frac{\hbar^2}{2m}\frac{d^2u(r)}{dr^2} + \left[V(r) + \frac{l(l+1)\hbar^2}{2mr^2}\right] u(r) = E\, u(r)$$

This equation depends only on the radial coordinate r. For the deuteron, which has only one bound state at the lowest energy level, we set $l = 0$ to minimize the energy, simplifying the equation further. In our analysis, we assume a central potential and a square well shape for simplicity, even though the actual potential is not purely central. With $l = 0$, the angular part Y_{00} is constant, and the wave function $\psi(\vec{r})$ is determined entirely by the radial part $R(r)$.

The equation is: $-\dfrac{\hbar^2}{2m}\dfrac{d^2u(r)}{dr^2} + V(r)\, u(r) = E\, u(r)$

There are certain boundary conditions on the wave function $\psi(r)$ that must be satisfied:

- The wave function $\psi(r)$ should be continuous everywhere.
- It must be finite everywhere, meaning it should not go to infinity at any point.
- The wave function should be square integrable, which means the integral of $|\psi(r)|^2$ over all space should be finite. This allows normalization, typically setting the integral to 1.
- The derivative of the wave function with respect to space should be continuous. This implies that the wave function should be smooth, with no cusps or corners.

When we express the wave function in terms of $u(r)$, these conditions translate to:

- $u(r)$ must be continuous and finite.
- $u(r)$ should be square integrable.
- The derivative $\frac{du}{dr}$ must be continuous.

Now, consider a potential of the following type: $V(r)$ is defined by two regions:

1. For $r < r_0$, $V(r) = V_0$
2. For $r > r_0$, $V(r) = 0$.

In the region $r < r_0$, the Schrödinger equation becomes:

$$-\frac{\hbar^2}{2m}\frac{d^2u(r)}{dr^2} - V_0\, u(r) = E\, u(r)$$

$$\frac{d^2u(r)}{dr^2} = \frac{2m(V_0 + |E|)}{\hbar^2}u(r)$$

$$\frac{d^2u(r)}{dr^2} = -k^2u(r)$$

where $k^2 = \frac{2m(V_0 + |E|)}{\hbar^2}$, k is real quantity. Since V_0 is positive and (with $|E| < V_0$), k^2 is positive, making k is real number. This leads to a sinusoidal solution for $u(r)$ in the region $r < r_0$.

Consider the first region, where $r < r_0$. The radial wave function $u(r)$ can be expressed:

$$u_l(r) = A\ \sin(kr) + B\ \cos(kr)$$

This is similar to simple harmonic oscillations. We then apply the boundary conditions. The wave function $\psi(r)$ must be continuous and finite everywhere, which implies that $u(r)$ must also be continuous and finite everywhere. Recall that $\psi(r) = \frac{u(r)}{r}$, with $\psi(r)$ being the radial part of the wave function, and the angular part being a constant in this case. For $\psi(r)$ to be finite everywhere, particularly at $r = 0$, $u(r)$ must be zero at $r = 0$.

Substituting $r = 0$ into the expression for $u_l(r)$:

$$u_I(0) = A \ \sin(0) + B \ \cos(0) = B$$

For $u_I(0) = 0$, we must have $B = 0$. Thus, the solution in the first region simplifies to:

$$u_I(r) = A \ \sin(kr)$$

Next, consider the second region, where $r > r_0$. Here, the potential energy $V(r) = 0$, so the Schrödinger equation simplifies to:

$$-\frac{\hbar^2}{2m}\frac{d^2u(r)}{dr^2} = E\ u(r)$$

$$\frac{d^2u(r)}{dr^2} = \gamma^2\ u(r),$$

where,

$$\gamma = \sqrt{\frac{-2mE}{\hbar^2}}.$$

The parameter γ is real because the total energy E is negative (since the system is bound), making γ^2 positive.

The general solution for $u(r)$ in this region is: $u_{II}(r) = C\ e^{\gamma r} + D\ e^{-\gamma r}$

However, the wave function must be square integrable, meaning that $\int u_{II}^2(r)\ dr$ over all space must be finite. For large r, $u_{II}(r)$ must approach zero; otherwise, the integral would diverge.

As r approaches infinity, $e^{\gamma r}$ grows exponentially, while $e^{-\gamma r}$ decays. To ensure that $u_{II}(r)$ vanishes at infinity, the coefficient C must be zero, leaving us with:

$$u_{II}(r) = D\ e^{-\gamma r}$$

Thus, in the second region, the wave function decays exponentially as $r$increases, ensuring that the wave function remains normalizable.

$$u_I(r_0) = u_{II}(r_0)$$

Since $u(r)$ should be continuous, this means that:

$$A \ \sin(kr_0) = D\ e^{-\gamma r_0}. \quad ---(1)$$

Similarly, the slope $\frac{du(r)}{dr}$ must also be continuous. So,

$$\left[\frac{du_I(r)}{dr}\right]_{r=r_0} = \left[\frac{du_{II}(r)}{dr}\right]_{r=r_0}$$

$$kA \ \cos(kr_0) = -D\ \gamma e^{-\gamma r_0} \quad ---(2)$$

Now, if you divide the equation (2) by (1), you get:

$$k \cot(k r_0) = -\gamma$$

$$\sqrt{\frac{2m(V_0 + |E|)}{\hbar^2}} \cot\left[\sqrt{\frac{2m(V_0 + |E|)}{\hbar^2}} r_0\right] = \sqrt{\frac{-2mE}{\hbar^2}} \quad \text{———(3)}$$

From experiments, we know that:

$$r_0 = 1.2 \text{ fm}, E = -2.225 \text{ MeV}, m = \frac{m_1 m_2}{m_1 + m_2}, \hbar = 197.326 \text{ MeV.fm}/c$$

The reduced mass V_0. Equation (3) can be solved, and the result for $V_0 = 36$ MeV.

The energy E is close to the top of the potential well (**see Fig. 2**), which is why the deuteron is weakly bound. If the energy were slightly higher, the system would be unbound. The binding energy of the deuteron, compared to the average binding energy per nucleon (which is around 7-8 MeV), is low, indicating that the deuteron is a weakly bound system.

To further analyze, we can calculate the wave number k. Using the formula:

$$k = \sqrt{\frac{2m(V_0 + |E|)}{\hbar^2}}$$

$\frac{\hbar^2}{2m} = 41.3 \textbf{ MeV . fm}^2$, $V_0 = 36$ MeV, and $E = -2.225$ MeV, so, $V_0 + E = 33.775$ MeV.

Now, let's calculate $\sin(k r_0)$ at the edge of the well. For $k = 0.90 \textbf{ fm}^{-1}$ and $r_0 = 1.2$ fm.

$$\sin(k r_0) = \sin(1.9) = 0.946$$

The maximum value of $\sin(kr)$ occurs when $kr = \pi/2$ (or 1.57 radians). Dividing 1.57 by $k = 0.90 \textbf{ fm}^{-1}$, we get $r \approx 1.74$ fm. On the diagram, this point corresponds to a location inside the well, near the maximum of the wave function.