National Institute of Technology, Rourkela

PH6123: An Introduction to General Relativity

Assignment 1
Due Date: September 18, 2024

All parts of a question should be answered at one place.

1. Show that an object of mass m moving with velocity v appears heavier compared to its rest mass m_0 by a factor of $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, i.e. $m = \gamma m_0$.

2. Show that the kinetic energy of a free particle moving with relativistic speed is given by: $K = (\gamma - 1)m_0c^2$. Show that when $v << c$, $K = m_0c^2 + \frac{1}{2}mv^2$.

3. Consider the transformation from rectangular Cartesian coordinates (x, y) to polar coordinates (r, θ), which corresponds to,

$$x = rcos\theta \qquad y = rsin\theta \tag{1}$$

Define $x^\mu = (x, y)$ and $\bar{x}^\mu = (r, \theta)$.

(i) Find out how the components of a vector V^μ in the x^μ coordinate system transforms to the $\bar{x}^\mu$ coordinate system. If the components of a vector are $V^1 = x^2y$ and $V^2 = x^3/y$. Find out its components in the polar coordinates.

(ii) In the x^μ coordinate system, the components of the velocity vector are $(\dot{x}, \dot{y})$. Find out the components in the polar coordinates.

4. Consider the coordinate system (x^0, x^1, x^2, x^3) , which gets transformed to another one (x'^0, x'^1, x'^2, x'^3), such that,

$$\begin{aligned} x^0 &= cosh\xi\ x'^0 + sinh\xi\ x'^1 \\ x^1 &= sinh\xi\ x'^0 + cosh\xi\ x'^1 \\ x^2 &= x'^2 \\ x^3 &= x'^3 \end{aligned} \tag{2}$$

Show that these represent Lorentz transformations. How is ξ related to the relative velocity V between the two coordinate systems.

5. Consider line element on a unit sphere $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$. Write down the metric and the Christoffel connections. Using the transformation $x = 2\tan\frac{\theta}{2}\cos(\phi)$ and $y = 2\tan\frac{\theta}{2}\sin(\phi)$ show that the line element on the two sphere in this coordinate system becomes $ds^2 = \cos^4\frac{\theta}{2}(dx^2 + dy^2)$.

6. Let u^μ be the four-velocity of an observer and let $h^\alpha_\beta = \delta^\alpha_\beta + u^\alpha u_\beta$ be the projection tensor. Show that u_μ is orthogonal to h^α_β, i.e., $u^\beta h^\alpha_\beta = 0$.

7. If $a^i = u^j \nabla_j u^i$, Show that,

$$u^i a_i = 0$$

8. Show that:

(i) If $A^\mu_{\nu\kappa} B^{\nu\kappa} = C^\mu$, where C^μ is a vector and $B^{\nu\kappa}$ is an anti-symmetric tensor then show that $A^\mu_{\nu\kappa} - A^\mu_{\kappa\nu}$ is a tensor.

(ii) Show that $\frac{\partial A^\mu}{\partial x^\nu}$ does not transform like a tensor but $\frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu}$ does.

9. Consider the line element,

$$ds^2 = dr^2 + r^2 d\phi^2 \tag{3}$$

Write down the metric and the Christoffel connections. Compute the Kretschmann scalar $I = R^{abcd} R_{abcd}$.

10. Show that covariant derivatives of a vector do not commute,

$$[\nabla_\mu, \nabla_\nu] A^\alpha = R^\alpha{}_{\sigma\mu\nu} A^\sigma \tag{4}$$

11. Consider the vector A^α on the surface of a unit sphere,

$$ds^2 = d\theta^2 + sin^2\theta d\phi^2$$

The vector A^α has components $A^\theta = A^\phi = 1$ at $\theta = \phi = \frac{\pi}{4}$. It is now parallely transported to $\theta = \phi = \pi/3$ along the two following paths:
(a) $(\pi/4, \pi/4) \rightarrow (\pi/4, \pi/3) \rightarrow (\pi/3, \pi/3)$
(b) $(\pi/4, \pi/4) \rightarrow (\pi/3, \pi/4) \rightarrow (\pi/3, \pi/3)$

Write down the change in the vector for these two paths.

12. Show that covariant divergence of a symmetric tensor is given by,

$$T^{\mu\nu}{}_{;\nu} = \frac{1}{\sqrt{-g}}(\sqrt{-g}T^{\mu\nu})_{,\nu} + \Gamma^{\mu}_{\alpha\beta}T^{\alpha\beta}$$

If $T^{\mu\nu}$ is anti-symmetric how will the above equation be modified?

13. The Weyl tensor can be written as,

$$W_{abcd} = R_{abcd} + \lambda P_{abcd}$$

where P_{abcd} is given by,

$$P_{abcd} = \frac{1}{2}\Big[g_{ac}L_{bd} - g_{ad}L_{bc} + L_{ac}g_{bd} - L_{ad}g_{bc}\Big]$$

where $L_{ab} = L_{ba}$. Show that:

(a) Show that P_{abcd} is antisymmetric on interchange of first two and last two indices.

(b) $P_{abcd} = P_{cdab}$

(c) $P_{abcd} + P_{acdb} + P_{adbc} = 0$

(d) $[P_{ijkl;m} + P_{ijlm;k} + P_{ijmk;l}]g^{kl} = 0$

(e) Demanding that W_{abcd} is completely tracefree show that:

$$L_{ij} = \frac{2}{\lambda(n-2)}R_{ij} - \frac{R}{\lambda(n-1)(n-2)}g_{ij}$$

Using the above result for L_{ij} show that

$$W_{abcd} = R_{abcd} - \frac{1}{(n-2)}\Big[g_{ac}R_{db} - g_{ad}R_{bc} + R_{ac}g_{bd} - R_{ad}g_{bc}\Big] + \frac{R}{(n-1)(n-2)}\Big[g_{ac}g_{bd} - g_{ad}g_{bc}\Big]$$

(f) Argue that $W_{abcd} = 0$ for $n < 4$.

14. A conformal transformation of the metric is given by,

$$\bar{g}_{\mu\nu} = \Omega^2(x)g_{\mu\nu}$$

Show that under such a transformation:

$$\bar{R}^a_{bcd} = R^a_{bcd} - 2\left(\delta^a_{[c}\delta^e_{d]}\delta^f_b - g_{b[c}\delta^e_{d]}g^{af}\right)\frac{1}{\Omega}\nabla_e\nabla_f\Omega + 2\left(2\delta^a_{[c}\delta^e_{d]}\delta^f_b - 2g_{b[c}\delta^e_{d]}g^{af} + g_{b[c}\delta^a_{d]}g^{ef}\right)\frac{1}{\Omega^2}\nabla_e\Omega\nabla_f\Omega,$$

$$\bar{R}_{ab} = R_{ab} - \Big[(N-2)\delta^e_a\delta^f_b + g_{ab}g^{ef}\Big]\frac{1}{\Omega}\nabla_e\nabla_f\Omega + \Big[2(N-2)\delta^e_a\delta^f_b - (N-3)g_{ab}g^{ef}\Big]\frac{1}{\Omega^2}\nabla_e\Omega\nabla_f\Omega$$

and

$$\bar{R} = \left[\frac{R}{\Omega^2} - 2\frac{(N-1)}{\Omega^3} g^{\mu\nu}\nabla_\nu \nabla_\mu \Omega - \frac{(N-1)(N-4)}{\Omega^4} g^{\mu\nu}\nabla_\nu \Omega \nabla_\mu \Omega\right]$$

while

$$\bar{W}_{abcd} = \Omega^2 W_{abcd}$$

Thus argue that W^a_{bcd} is conformally invariant in any dimension N and $W^a_{bcd} = 0$ if the metric is conformally flat. Here, N represents the number of spacetime dimensions and

$$\delta^a_{[c}\delta^e_{d]} = \frac{1}{2}\left[\delta^a_c \delta^e_d - \delta^a_d \delta^e_c\right]$$

15. Show that:

$$[\nabla_c, \nabla_d] g_{ab} = R^l_{acd} g_{lb} + R^l_{bcd} g_{al}$$

From above prove that $R_{abcd} = -R_{bacd}$. Comment on the role of the metricity condition on this result.