PHYS 595 - ASTROPHYSICAL FLUIDS 3

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Question 5

Throughout the following question we will be solving the 1-d linear advection equation using a variety of discretization methods. For all cases the methods will be done in a domain of $[0, 1]$ with velocity $u = 1$ and periodic boundary conditions. The initial distribution will be a top-hat distribution given by:

$$a = \begin{cases} 1, & \text{if } 0.4 \leq x \leq 0.6 \\ 0, & \text{otherwise} \end{cases}$$

The period of simulation is given as $T = 1/u$ and we will test periods of 0.1 and 1. We will also test a variety of CFL numbers $C = 0.1, 0.33(3), 0.7$ and resolutions $\Delta x = 0.05, 0.033(3), 0.01$. It is important to note that $C \equiv \frac{u\Delta t}{\Delta x}$ we can see that timestep is calculated as $\Delta t = \frac{C\Delta x}{u}$.

Upwind method

Using upwind discretization we can solve the 1-d linear advection equation using an equation of the form:

$$\frac{a_i^{n+1} - a_i^n}{\Delta t} = -u\frac{a_i^n - a_{i-1}^n}{\Delta x} \tag{1}$$

We can rearrange this equation to be in the form:

$$\begin{aligned} a_i^{n+1} &= \frac{-u\Delta t}{\Delta x}(a_i^n - a_{i-1}^n) + a_i^n \\ &= -C(a_i^n - a_{i-1}^n) + a_i^n \end{aligned} \tag{2}$$

Using this method we produce the figures 1, 2 and 3. From the results we can see that the solution is stable for all CFL numbers tested and the systems with higher resolution or lower Δx retain the initial top-hat distribution better. The stability can be found through a fourier analysis of equation 2. this will be done through a substitution of $a_j^n = A^n e^{ij\theta}$. Due to indexing from above we will replace our time index i with j and instead use i to represent imaginary values. The stability of the system is shown through:

$$
\begin{aligned}
A^{n+1}e^{ij\theta} &= -C(A^n e^{ij\theta} - A^n e^{i(j-1)\theta}) + A^n e^{ij\theta} \\
\frac{A^{n+1}}{A^n} &= 1 - C(1 - e^{-i\theta}) \\
&= 1 - C + C(\cos\theta - i\sin\theta) \\
&= 1 - C(1-\cos\theta) - iC\sin\theta \\
\left|\frac{A^{n+1}}{A^n}\right| &= (1 - C(1-\cos\theta))^2 - (iC\sin\theta)^2 \\
&= 1 - 2C(1-\cos\theta) + C^2(1+\cos^2\theta - 2\cos\theta) + C^2\sin^2\theta \\
&= 1 - 2C + 2C\cos\theta + 2C^2 - 2C^2\cos^2\theta \\
&= 1 - 2C(1-C)(1-\cos\theta)
\end{aligned}
\tag{3}
$$

Stability requires that $\left|\frac{A^{n+1}}{A^n}\right|^2 < 1$ so the criteria is:

$$
\begin{aligned}
|1 - 2C(1-C)(1-\cos\theta)| &\leq 1 \\
-2C(1-C)(1-\cos\theta) &\leq 0 \\
1 - C &\geq 0 \\
C &\leq 1
\end{aligned}
\tag{4}
$$

Diffusion can be shown by expanding equation 1:

$$
\begin{aligned}
\frac{a_i^{n+1} - a_i^n}{\Delta t} &= -u\frac{a_i^n - a_{i-1}^n}{\Delta x} \\
&= -u\left(\frac{a_{i+1}^n - a_{i-1}^n}{2\Delta x} - \Delta x\frac{(a_{i+1}^n - 2a_i^n + a_{i-1}^n)}{2\Delta x^2}\right) \\
&= -u\frac{(a_{i+1}^n - a_{i-1}^n)}{2\Delta x} + \frac{u\Delta x}{2}\frac{(a_{i+1}^n - 2a_i^n + a_{i-1}^n)}{\Delta x^2} \\
&= -u\frac{(a_{i+1}^n - a_{i-1}^n)}{2\Delta x} + D\frac{(a_{i+1}^n - 2a_i^n + a_{i-1}^n)}{\Delta x^2}
\end{aligned}
\tag{5}
$$

From equation 5 we can see that the upwind method can be broken into two portions. The left portion is identical to the FTCS method, while the right portion is a diffusive term that prevents the upwind solution from being unstable. This numerical diffusion is given by $D = \frac{u\Delta x}{2}$. From this we see that with higher Δx or lower resolution the diffusion is greater. The diffusion is more apparent when the system has taken more steps. This can be seen when the simulation period $T = 1.0$ as a has decreased in comparison to $T = 0.1$. The initial distribution has diffused outwards more as the system has updated more times. In general it appears that a higher CFL number results in less diffusion. This is due to higher CFL numbers resulting in a larger Δt when u and Δx are constant. With larger timesteps the system is updated fewer times resulting in less numeric diffusion.

FTCS Method

Using FTCS discretization we can solve the 1-d linear advection equation using an equation of the form:

$$\frac{a_i^{n+1} - a_i^n}{\Delta t} = -u\frac{a_{i+1}^n - a_{i-1}^n}{2\Delta x} \tag{6}$$

We can rearrange this equation to be in the form:

$$\begin{aligned} a_i^{n+1} &= \frac{-u\Delta t}{2\Delta x}(a_{i+1}^n - a_{i-1}^n) + a_i^n \\ &= -\frac{C}{2}(a_{i+1}^n - a_{i-1}^n) + a_i^n \end{aligned} \tag{7}$$

Using this method we produce the figures 4, 5 and 6. As expected using the FTCS method the solution is unstable. This instability can be shown if we take the fourier transform of equation 7.

$$\begin{aligned} A^{n+1}e^{ij\theta} &= -\frac{C}{2}(A^n e^{i(j+1)\theta} - A^n e^{i(j-1)\theta}) + A^n e^{i(j)\theta} \\ A^{n+1} &= -\frac{C}{2}A^n(e^{i\theta} - e^{-i\theta}) + A^n \\ A^{n+1} &= A^n(1 - iC\sin\theta) \\ \left|\frac{A^{n+1}}{A^n}\right|^2 &= 1 + C^2 sin^2\theta \end{aligned} \tag{8}$$

We can see from equation 8 that $\left|\frac{A^{n+1}}{A^n}\right|^2 > 1$. This means the FTCS is unstable for any CFL number. From equation 8 we also see that the instability grows with C which explains why larger CFL number peak at larger values. With a larger period these instabilities are more pronounced as the magnitude of the instability is proportional to C^2.

Implicit in Time Method

The implicit-in-time discretization is given by the equation:

$$\frac{a_i^{n+1} - a_i^n}{\Delta t} = -u\frac{a_i^{n+1} - a_{i-1}^{n+1}}{\Delta x} \tag{9}$$

Which similar to the previous methods can be rearranged:

$$\begin{aligned} a_i^{n+1} &= \frac{-u\Delta t}{\Delta x}(a_i^{n+1} - a_{i-1}^{n+1}) + a_i^n \\ &= -C(a_i^{n+1} - a_{i-1}^{n+1}) + a_i^n \\ a_i^n &= -Ca_{i-1}^{n+1} + (1+C)a_i^{n+1} \end{aligned} \tag{10}$$

Which when written in matrix form is generally given as:

$$\begin{pmatrix} 1+C & & & & & & -C \\ -C & 1+C & & & & & \\ & -C & 1+C & & & & \\ & & -C & 1+C & & & \\ & & & \ddots & \ddots & & \\ & & & & -C & 1+C & \\ & & & & & -C & 1+C \end{pmatrix} \begin{pmatrix} u_1^{n+1} \\ u_2^{n+1} \\ u_3^{n+1} \\ u_4^{n+1} \\ \vdots \\ u_{N-2}^{n+1} \\ u_{N-1}^{n+1} \end{pmatrix} = \begin{pmatrix} u_1^n \\ u_2^n \\ u_3^n \\ u_4^n \\ \vdots \\ u_{N-2}^n \\ u_{N-1}^n \end{pmatrix} \tag{11}$$

To solve for B in an equation of the form $AB = X$ we must take the inverse of A and perform $A^{-1}AB = A^{-1}X$ which becomes $B = A^{-1}X$. We can find the inverse of our matrix using the Gauss-Jordan elimination Method. To do this we append an identity matrix to the matrix which we wish to invert. From here we use basic row operations to change our matrix A into row-echelon form. This will also apply the operations to the identity matrix which once our matrix A is in row-echelon form, the right hand matrix will be A^{-1}.

$$\left(\begin{array}{ccccc|ccccc}
1+C & & & & -C & 1 & & & & \\
-C & 1+C & & & & & 1 & & & \\
 & \ddots & \ddots & & & & & \ddots & & \\
 & & -C & 1+C & & & & & 1 & \\
 & & & -C & 1+C & & & & & 1
\end{array}\right)$$

By doing this we can produce figures 7, 8 and 9. Similar to the upwind method, the implicit-in-time method produces a stable solution. Again if we perform a similar analysis as with the upwind method we can see that a numeric diffusion term is present.

$$\begin{aligned}
\frac{a_i^{n+1} - a_i^n}{\Delta t} &= -u\frac{a_i^{n+1} - a_{i-1}^{n+1}}{\Delta x} \\
&= -u\left(\frac{a_{i+1}^{n+1} - a_{i-1}^{n+1}}{2\Delta x} - \Delta x\frac{(a_{i+1}^{n+1} - 2a_i^{n+1} + a_{i-1}^{n+1})}{2\Delta x^2}\right) \\
&= -u\frac{(a_{i+1}^{n+1} - a_{i-1}^{n+1})}{2\Delta x} + \frac{u\Delta x}{2}\frac{(a_{i+1}^{n+1} - 2a_i^{n+1} + a_{i-1}^{n+1})}{\Delta x^2} \\
&= -u\frac{(a_{i+1}^{n+1} - a_{i-1}^{n+1})}{2\Delta x} + D\frac{(a_{i+1}^{n+1} - 2a_i^{n+1} + a_{i-1}^{n+1})}{\Delta x^2}
\end{aligned} \tag{12}$$

The effect of the resolution is again apparent with lower resolution producing a more diffuse solution with the diffusion term being represented by $D = \frac{u\Delta x}{2}$. The key difference between the implicit-in-time solution and the upwind solution is that the CFL number has a smaller effect in the implicit-in-time method. While changes in the CFL number resulted in significant changes in a for the upwind method, there is a much smaller difference in the implicit-in-time method. Another difference is that the higher CFL number produces smaller a values in the implicit-in-time method instead of larger a values in the upwind method.

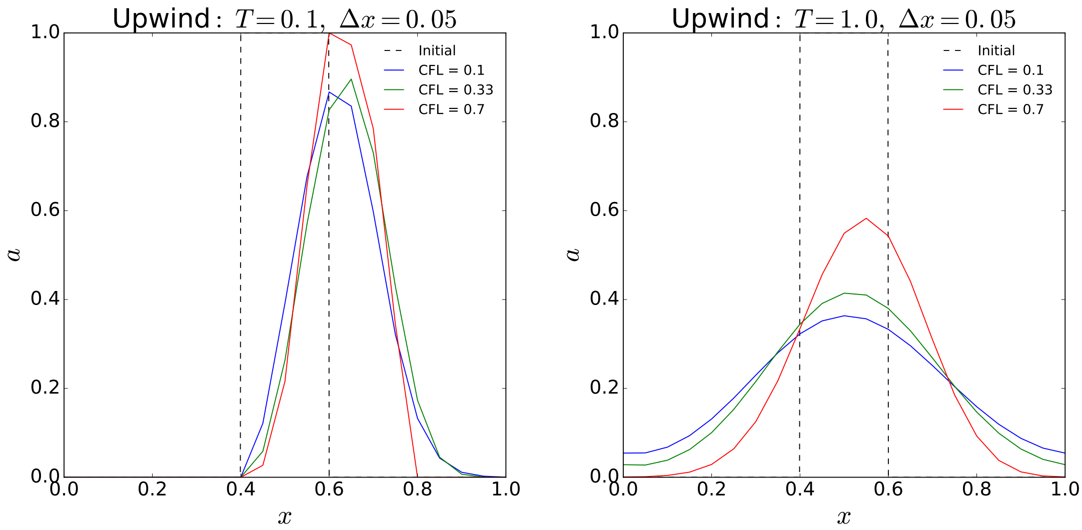

Figure 1: The system after $T = 0.1$ and $T = 1.0$ with resolution $\Delta x = 0.05$

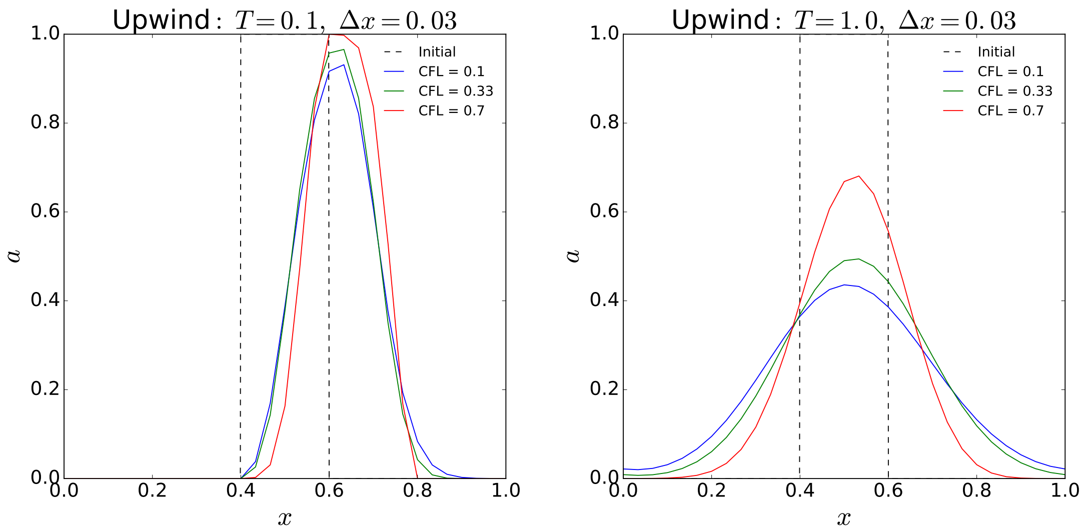

Figure 2: The system after $T = 0.1$ and $T = 1.0$ with resolution $\Delta x = 0.03$

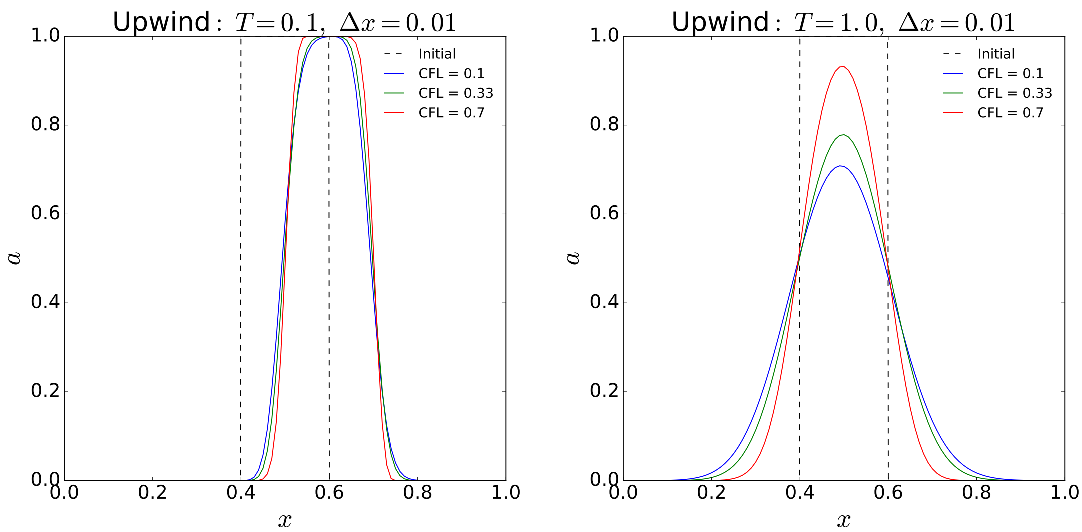

Figure 3: The system after $T = 0.1$ and $T = 1.0$ with resolution $\Delta x = 0.01$

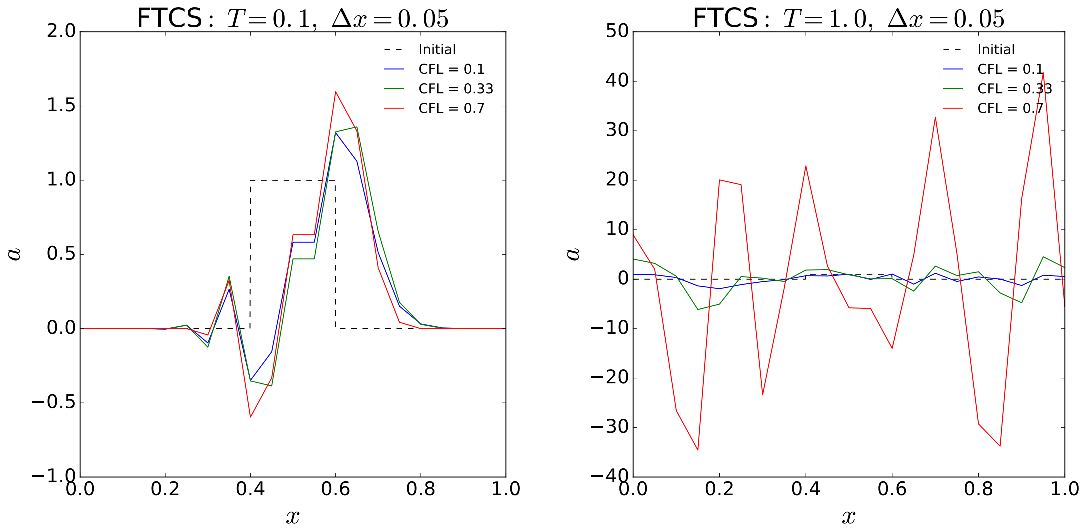

Figure 4: The system after $T = 0.1$ and $T = 1.0$ with resolution $\Delta x = 0.05$

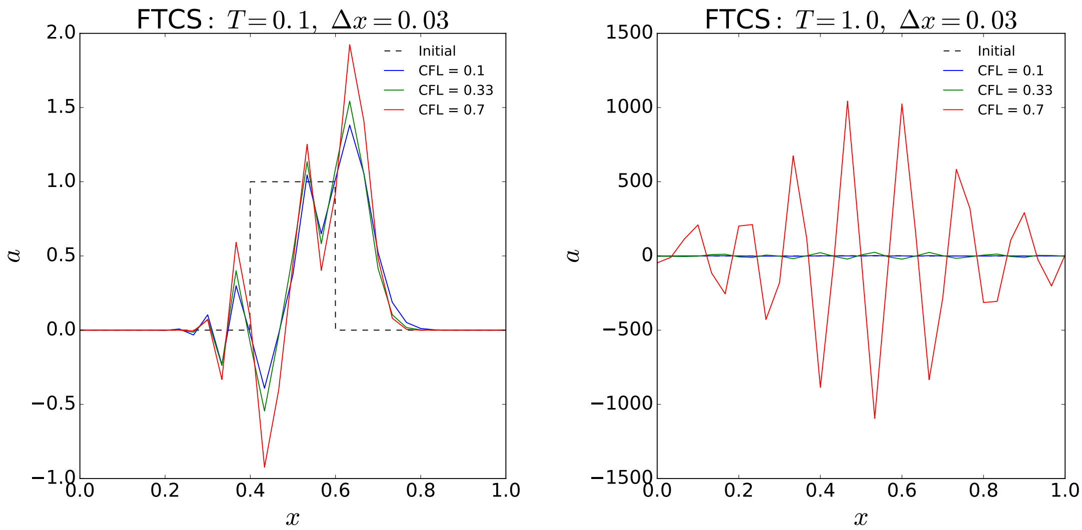

Figure 5: The system after $T = 0.1$ and $T = 1.0$ with resolution $\Delta x = 0.03$

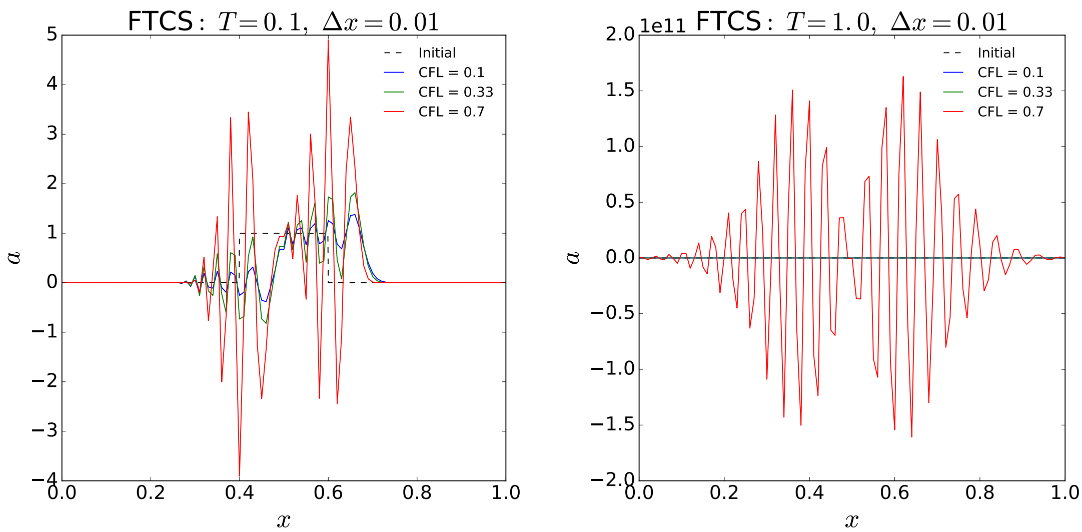

Figure 6: The system after $T = 0.1$ and $T = 1.0$ with resolution $\Delta x = 0.01$

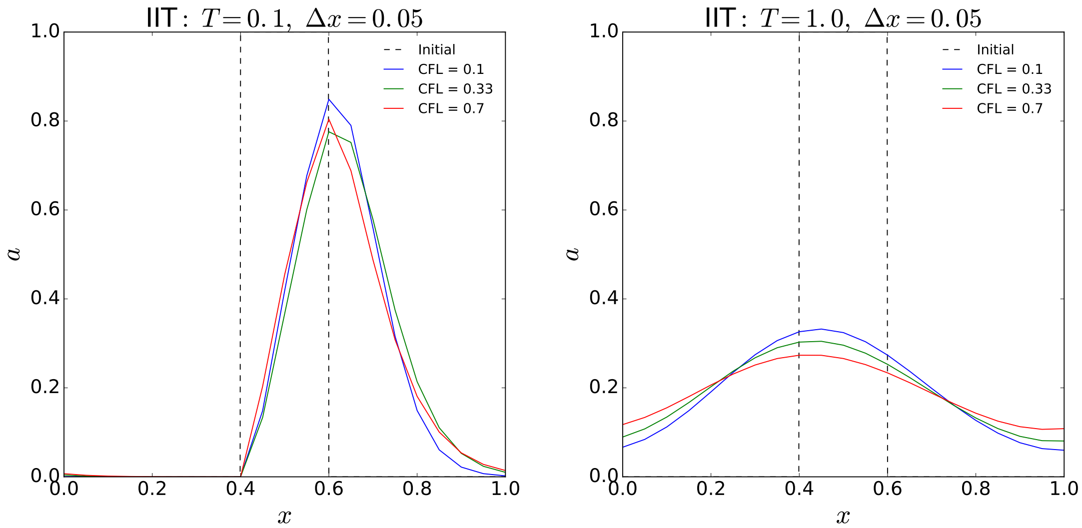

Figure 7: The system after $T = 0.1$ and $T = 1.0$ with resolution $\Delta x = 0.05$

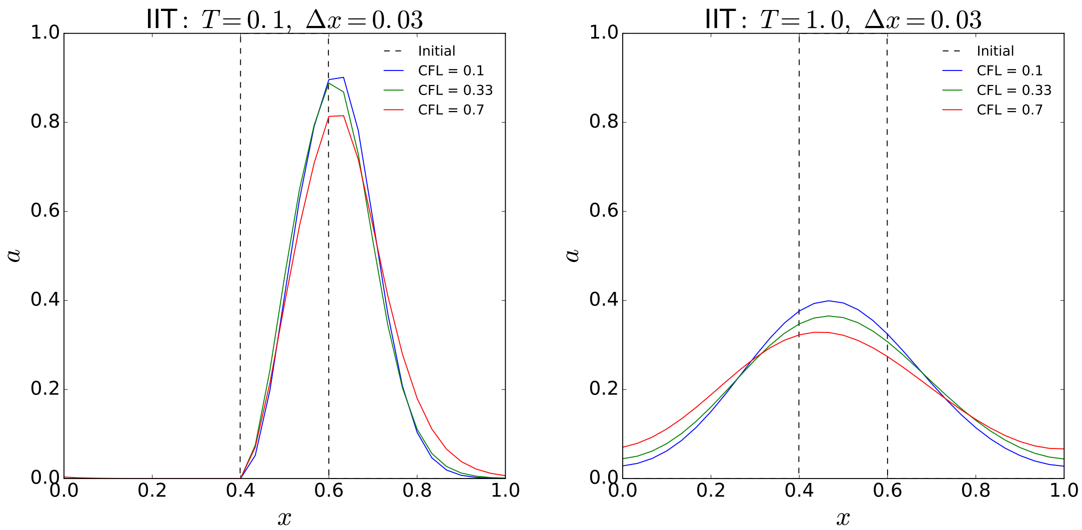

Figure 8: The system after $T = 0.1$ and $T = 1.0$ with resolution $\Delta x = 0.03$

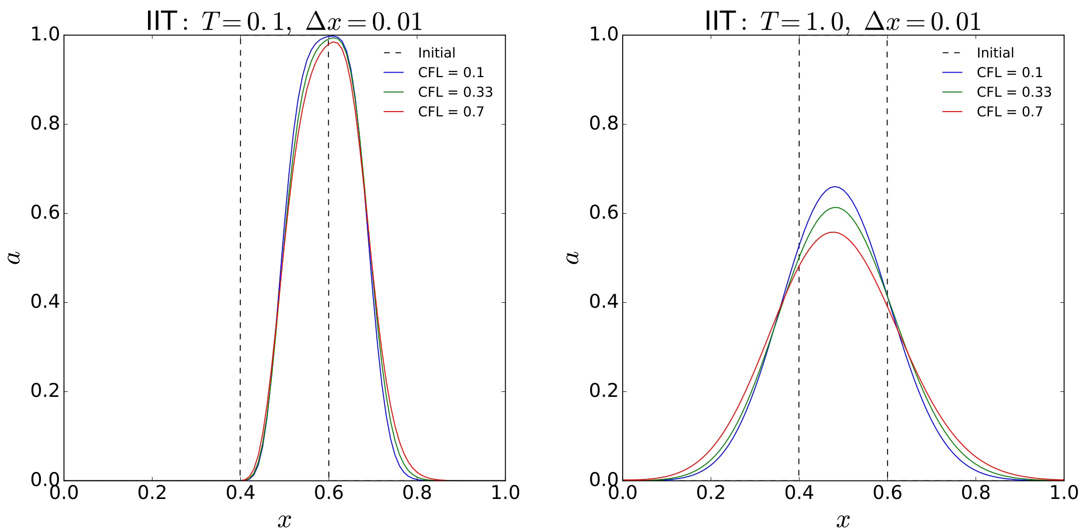

Figure 9: The system after $T = 0.1$ and $T = 1.0$ with resolution $\Delta x = 0.01$